Dynamic Programming

Table of Contents

Theory

What is Dynamic Programming?

Dynamic programming (DP) is an optimization technique used to solve complex problems by breaking them down into simpler subproblems and storing the results of these subproblems to avoid redundant computations. It is particularly useful for problems with overlapping subproblems and optimal substructure properties.

Key Concepts of Dynamic Programming:

1. **Overlapping Subproblems**: Solving the same subproblems multiple times.
2. **Optimal Substructure**: The optimal solution to a problem can be constructed from optimal solutions of its subproblems.
3. **Memoization**: Storing results of subproblems to reuse them, thus saving computation time. Also called top-down approach.
4. **Tabulation**: Building a table in a bottom-up manner to store solutions of subproblems.

Steps to approach Dynamic Programming problems:

1. Find the recurrence relationship
2. Find the base case
3. Find way to store solutions of subproblems

Sample Code

Fibonacci Series – Recursion:

```python
def fib(n):
    if n<=1:
        return n
    else:
        return fib(n-1) + fib(n-2)

# fib : 0,1,1,2,3,5,8

print(fib(6)) #8
```

Here we see, our solution finding solution of same subproblems several time, which cause slowness. We can prevent this using DP, by saving solution of subproblems in an array.

Fibonacci Series –Top down:

Here we built solution from n to i, normally via recursion.

```
def fib_top_down(n, dp):
    if n<=1:
        return n
    if dp[n]!= -1:
        return dp[n]
    else:
        dp[n]= fib_top_down(n-1, dp) + fib_top_down(n-2, dp)
        return dp[n]

n=6
dp = [-1]*(n+1)
print(fib_top_down(n, dp)) #8
```

Fibonacci Series – Bottom Up:

Here we built solution from i to n, normally via loop.

```
def fib_bottom_up(n):
    dp = [-1]*(n+1)
    dp[0], dp[1] = 0, 1
    for i in range(2,n+1):
        dp[i] = dp[i-1] + dp[i-2]
    return dp[n]

print(fib_bottom_up(6)) #8
```

Fibonacci Series – Bottom Up – Space Optimized:

In place of dp array we can use 2 variables for same purpose

```
def fib_bottom_up(n):
    prev = 1
    prev1 = 0
    curr = 0

    for i in range(2,n+1):
        curr = prev + prev1
        prev1 = prev
        prev = curr

    return prev

print(fib_bottom_up(6)) #8
```

LEVEL 1: EASY

1. Climbing Stairs
   Link: https://leetcode.com/problems/climbing-stairs/description/

2. House Robber
   You are given money present in n adjacent houses, there is robber who wants to rob the houses. But he cannot rob from 2 adjacent houses. Find max loot of robber.

3. Min cost climbing stairs
   Link: https://leetcode.com/problems/min-cost-climbing-stairs/

4. Reach to one
   Given a number x, you can do 3 different operations on x: #1. Subtract 1 from it. #2 If it is divisible by 2, divide by 2. #3 If it is divisible by 3, divide by 3 .Find the minimum number of steps that it takes to get to 1 using only the above operations.

5. Coin change
   Link: https://leetcode.com/problems/coin-change/description/

6. Longest Increasing Subsequence
   Link: https://leetcode.com/problems/longest-increasing-subsequence/description/

7. Stones
   Link: https://atcoder.jp/contests/dp/tasks/dp_k

8. Unique paths
   Link: https://leetcode.com/problems/unique-paths/description/

LEVEL 2: Medium

1. Longest common subsequence
   Link: https://leetcode.com/problems/longest-common-subsequence/description/

2. Longest repeating subsequence
   Link: https://www.naukri.com/code360/problems/longest-repeating-subsequence

3. Minimum path sum
   Link: https://leetcode.com/problems/minimum-path-sum/

4. Ninja training
   Link: https://www.naukri.com/code360/problems/ninja-s-training_3621003

LEVEL 3: Difficult

1. K-ordered longest common subsequence
   A k-ordered LCS is defined to be the LCS of two sequences if you are allowed to change at most k elements in the first sequence to any value you wish to. You are given 2 integer sequences and a number k. You can make max k changes in sequence 1 to get maximum LCS, find the max length of LCS.

SOLUTIONS:

LEVEL 1:

1. Climbing Stairs

Think of problem in terms of index,
and built solution step by step

for i=0; steps=1
for counting problems
keep base case 1

for i=1 (1) 0→1
i=2 (2) 0→1→2
0→2

i=3(3) (1+1)
{0→1→3
{0→1→2→3
{0→2→3

number of ways to go to 2 (and take 1 step)
+
number of ways to go to 1(and take 2 stairs

```
# for n = number of ways to go to n-1 + number of ways to go to n-2
# 0->1 for making code easy
# 1->1
# 2 -> 2  (0->2 , 0->1->2)

class Solution:
    def climbStairs(self, n: int) -> int:
        dp = [1]*(n+1)
        for i in range(2,n+1):
            dp[i] = dp[i-1] + dp[i-2]

        return dp[n]
```

```
class Solution:
    def climbStairs(self, n: int) -> int:
        def helper(n):
            if n==0: return 1
            if n==1: return 1
            if dp[n]!=-1 : return dp[n]

            dp[n] = helper(n-1) + helper(n-2)
            return dp[n]

        dp = [-1]*(n+1)
        return helper(n)
```

2. Loot house

F[i] = max loot done till i^{th} house, so F[i] = max (arr[i] + F[i-2] , F[i-1])

```
#Loot HOUSE
def lootBU(n,arr):
    dp=[0]*(n)
    dp[0],dp[1] = arr[0],max(arr[0],arr[1])
    for i in range(2,n):
        dp[i] = max(arr[i]+dp[i-2] ,dp[i-1])
    print(dp)
    return dp[n-1]

arr = [6,2,3,9]
print(lootBU(len(arr),arr))
```

3. Min cost climbing stairs

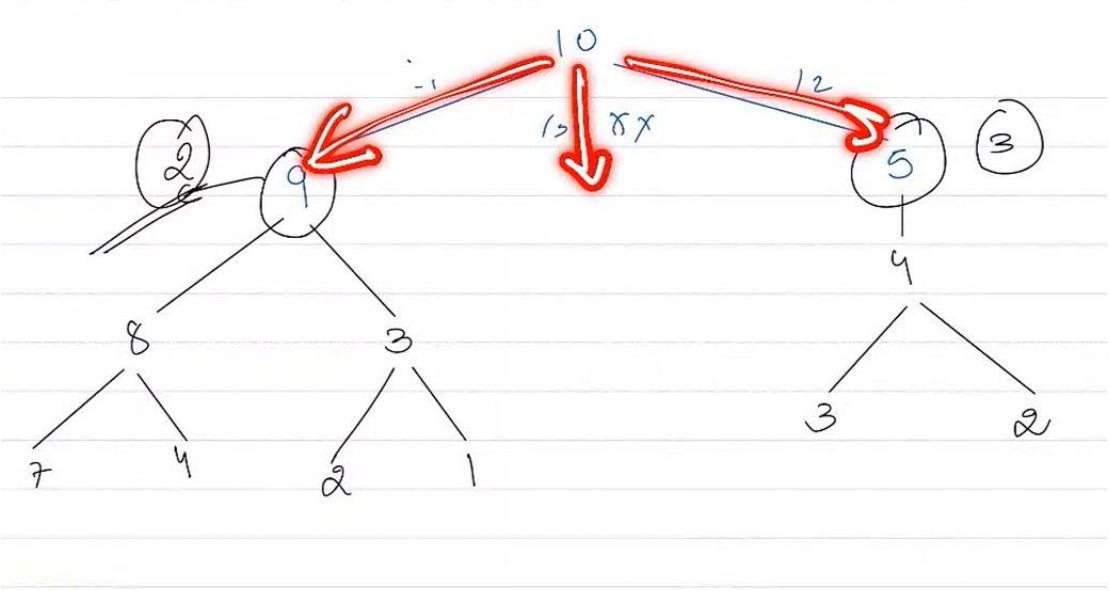
Here dp[i] = cost of reaching at i^{th} step.

```
#Bottom-up
class Solution:
    def minCostClimbingStairs(self, cost: List[int]) -> int:
        dp = [0]*len(cost)
        dp[0] = cost[0]
        dp[1] = cost[1]
        for i in range(2,len(cost)):
            dp[i] = cost[i]+min(dp[i-1],dp[i-2])

        return min(dp[-1],dp[-2])
```

```
#Top-down
class Solution:
    def minCostClimbingStairs(self, cost: List[int]) -> int:
        n=len(cost)
        self.dp = [-1]*n
        self.dp[0]=cost[0]
        self.dp[1]=cost[1]
        def helper(n):
            if self.dp[n]!=-1:
                return self.dp[n]
            else:
                self.dp[n] = cost[n] + min(helper(n-1),helper(n-2))
                return self.dp[n]

        helper(n-1)
        return min(self.dp[-1],self.dp[-2])
```

4. Minimum steps to reach one

If see local minimum, going from 10 to 5 is preferred than 10 to 9, however 5 takes more time to go to 1 than 9. Therefore in global way going to 9 is preferred.




```python
#Top down
def minStepsToOneTD(n,dp):
    if n==1: return 0
    if n==2 or n==3: return 1
    if dp[n]!=0: return dp[n]
    div_by_3, div_by_2, less_by_1 = float('inf'),float('inf'),float('inf')

    if(n%3==0):
        div_by_3 = 1+minStepsToOneTD(n//3,dp)
    if(n%2==0):
        div_by_2 = 1+minStepsToOneTD(n//2,dp)
    less_by_1 = 1+minStepsToOneTD(n-1,dp)

    dp[n]=min(div_by_3, div_by_2, less_by_1)
    return dp[n]

n=7
dp=[0]*(n+1)
print(minStepsToOneTD(n,dp))
```

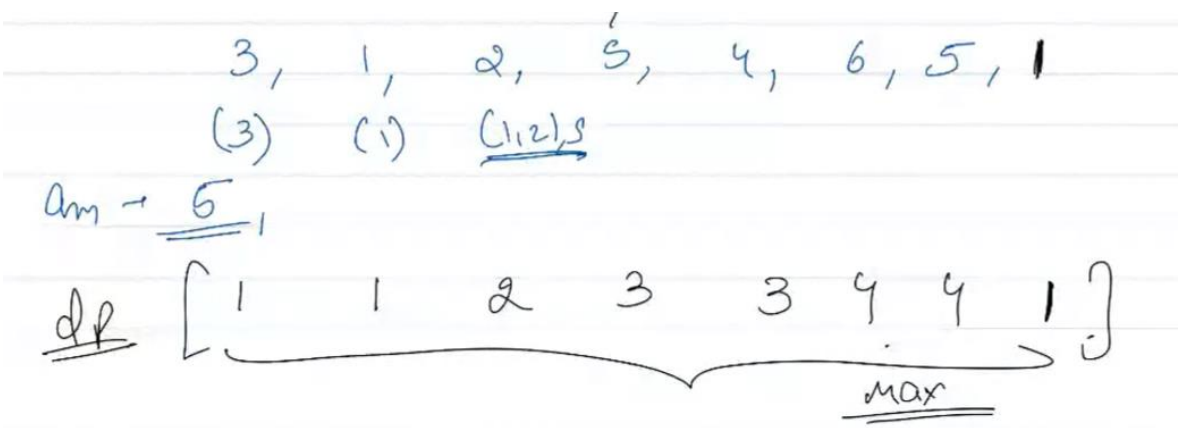
```python
#Bottom Up
def min_steps_to_one(x):
    dp = [1]*(x+1)
    dp[1]=0
    dp[2],dp[3]=1,1
    for i in range(4,x+1):
        dp[i]=1+min(dp[i-1], dp[i//2] if i%2==0 else x, dp[i//3] if i%3==0 else x)
    return dp[x]
print(min_steps_to_one(10))
```

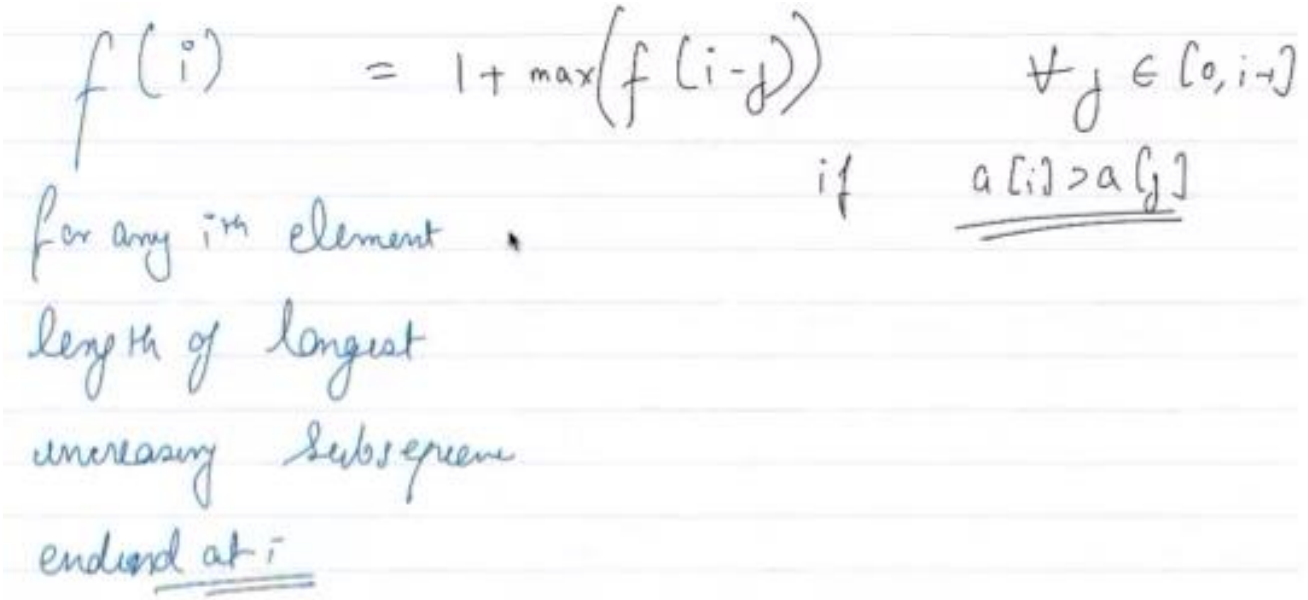
5. Coin change

```python
#Top down
class Solution:
    def coinChange(self, coins: List[int], amount: int) -> int:
        def helper_td(n , coins , dp):
            if n==0: return 0
            if dp[n]!=-1:return dp[n]

            temp = float('inf')
            for i in coins:
                if i <= n:
                    temp = min(temp,1+helper_td(n-i,coins,dp))
            dp[n] = temp
            return dp[n]

        dp = [-1]*(amount+1)
        ans=helper_td(amount , coins, dp)
```

```python
#Bottom up  (performs better both space and time wise)
class Solution:
    def coinChange(self, coins: List[int], amount: int) -> int:
        dp = [float('inf')]*(amount+1)
        dp[0]=0
        for i in range(1,amount+1):
            for j in coins:
                if j<=i:
                    dp[i] = min(dp[i], 1+dp[i-j])
        return dp[amount] if dp[amount]!=float('inf') else -1
```

6. Longest Increasing subsequence






```python
class Solution:
    def lengthOfLIS(self, nums: List[int]) -> int:
        n = len(nums)
        dp =[1]*n
        for i in range(n):
            for j in range(i):
                if nums[i]>nums[j]:
                    dp[i] = max(dp[i] , 1+dp[j])
        return max(dp)
```

7. Stone

If k==0 : any player who reaches this state will lose

If k<min(a): here also if any player reaches this state will lose

So winning or losing depends on state and independent of who plays. So if state=k is winning state, player one wins else player 2.

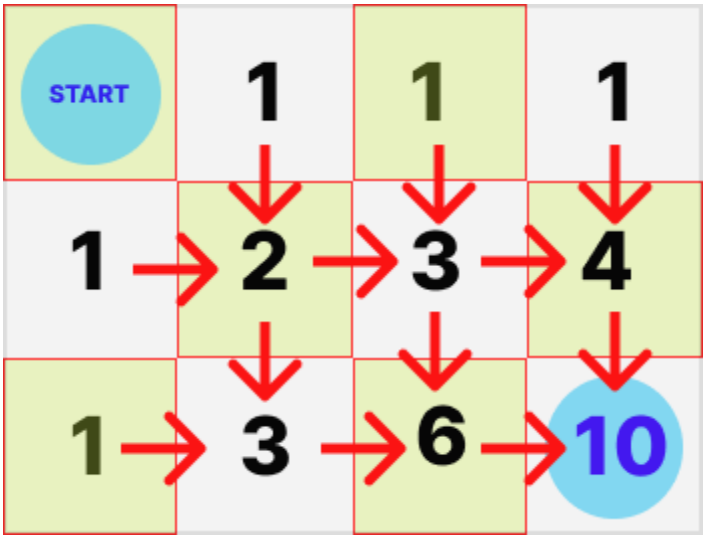
State K is winning state if any state (k-a[i]) is losing state for all a[i] in array a. Meaning if first player can push second player to any losing state then first player can win. But if all (k-a[i]) states are winning states, then kth state is losing state.

```python
n , k = map(int,input().split())
a = list(map(int,input().split()))

dp = [-1]*(k+1)
#at state 0 all will lose
dp[0]=0

#if a=[2,3] so at 2 and 3, player takes all stone and next player won't have any stone to pick.
#so that will be winning state.

for i in range(1,k+1):
    flag=0
    for j in a:
        if j<=i and dp[i-j]==0: #can send next player to any losing state
            flag=1
    dp[i]=flag

print("First" if dp[k]==1 else "Second")
```

8. Unique Paths

To go to m,n. we have 2 options. (1) go down from n-,m (2)go right from n,m-1.

By this logic. For all m=0 or n=0 there is only one way. So value=1.



```python
class Solution:
    def uniquePaths(self, m: int, n: int) -> int:
        dp = [[1] * n for i in range(m)]

        for i in range(1, m):
            for j in range(1, n):
                dp[i][j] = dp[i - 1][j] + dp[i][j - 1]

        return dp[m - 1][n - 1]
```

Using combination

For any given M x N grid, each unique path (no matter which one it is) requires you to move right from the starting point N - 1 times and move down from the starting point M - 1 times. Hence, regardless of the order you choose to move right or down, you need to make a total of (M - 1) + (N - 1) = M + N - 2 moves.

Then, out of the M + N - 2 moves, we need to select M - 1 moves to move right and the remaining N - 1 moves to move down. This essentially is why this problem boils down to combinatorics, because we need to calculate how many different ways we can select M - 1 moves from M + N - 2 moves (or equivalently, N - 1 moves from M + N - 2 moves).

```python
class Solution:
    def uniquePaths(self, m: int, n: int) -> int:
        return math.comb(m+n-2, m-1)  # or math.comb(m+n-2, n-1)
```

LEVEL 2:

1. Longest common subsequence

Here recursive relation is:

Eg:- st1= “abc” , st2=“adb” , I know st1[0] == st2[0],
So to get lcs, do **lcs(“abc”, “adb”) = 1 + lcs(“bc”,”db”)**

Eg2:- st1= “pqrs” , st2= “xqor” , I know st1[0] != st2[0]
So to get lcs, do **lcs(“pqrs”, “xqor”) = max(lcs(“qrs” , “xqor”) , lcs(“pqrs”, “qor”))**

text1 = 'bcead', text2 = 'abcde'

		a	b	c	d	e
	0	0	0	0	0	0
b	0	0	1	1	1	1
c	0	0	1	2	2	2
e	0	1	1	2	2	
a	0					
d	0					

For this red box:
Till now, text1=”bce” and text2=”abcde”
Now since last values are equal, it is equal to 1+lcs(“bc”, “abcd”) =3

text1 = 'bcead', text2 = 'abcde'

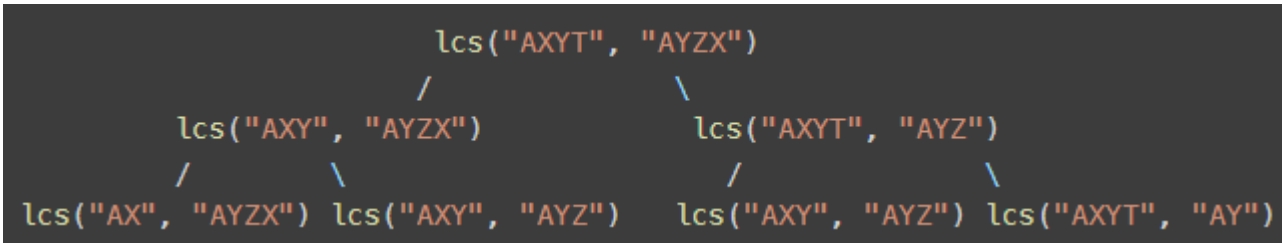
		a	b	c	d	e
	0	0	0	0	0	0
b	0	0	1	1	1	1
c	0	0	1	2	2	2
e	0	1	1	2	2	3
a	0	1	1	2	2	3
d	0	1	1	2	3	3

```python
class Solution:
    def longestCommonSubsequence(self, text1: str, text2: str) -> int:
        n,m=len(text1),len(text2)
        dp = [[0]*(m+1) for i in range(n+1)] #m+1 rows and n+1 columns

        for i in range(1,n+1):
            for j in range(1,m+1):
                if text1[i-1]==text2[j-1]:
                    dp[i][j] = 1+dp[i-1][j-1]
                else:
                    dp[i][j] = max(dp[i-1][j] ,dp[i][j-1])
        return dp[n][m]
```

```
## Demo run
      ''  a  c  e
 ''    [0, 0, 0, 0]
  a    [0, 1, 1, 1]
  b    [0, 1, 1, 1]
  c    [0, 1, 2, 2]
  d    [0, 1, 2, 2]
  e    [0, 1, 2, 3]
```

For top down

Will start looking from last values str[-1] and str[-2]




```python
#Normal Recursion
class Solution:
    def longestCommonSubsequence(self, text1: str, text2: str) -> int:
        n = len(text1)
        m = len(text2)
        def helper(i, j):
            if i==0 or j==0: return 0
            if text1[i-1] == text2[j-1]:
                return helper(i-1,j-1)+1
            else:
                return  max(helper(i-1, j), helper(i, j-1))

        return helper(n,m)
```

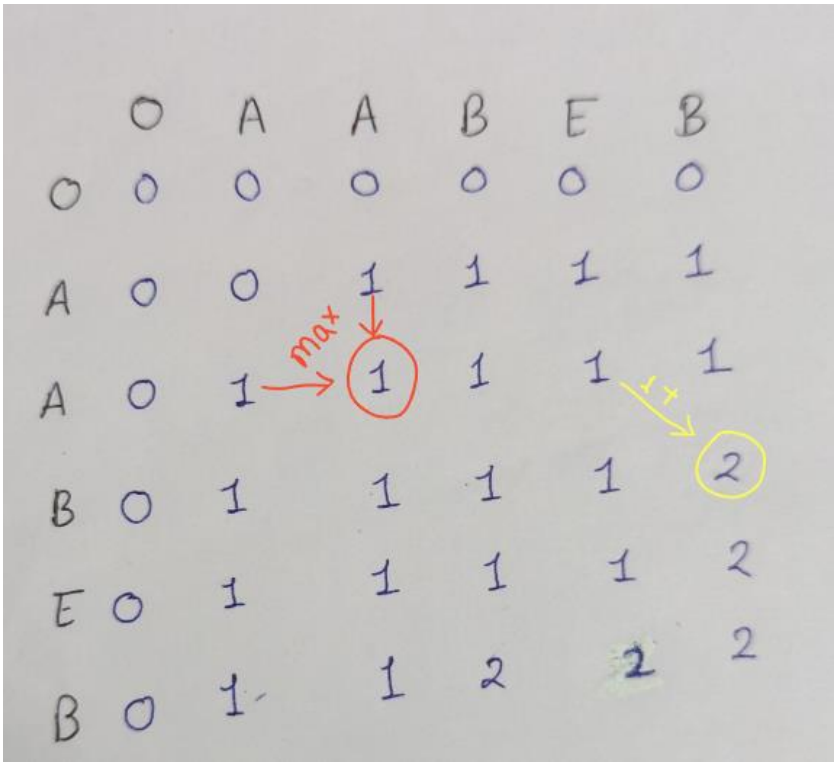
Top down approach

```python
class Solution:
    def longestCommonSubsequence(self, text1: str, text2: str) -> int:
        n = len(text1)
        m = len(text2)

        def helper(i, j, dp):
            if i==0 or j==0: return 0
            if dp[i][j]!=-1: return dp[i][j]

            if text1[i-1] == text2[j-1]:
                dp[i][j] = helper(i-1,j-1, dp)+1
            else:
                dp[i][j] =  max(helper(i-1, j, dp), helper(i, j-1, dp))
            return dp[i][j]

        dp = [[-1]*(m+1) for i in range(n+1)]
        return helper(n,m,dp)
```

2. Longest repeating subsequence

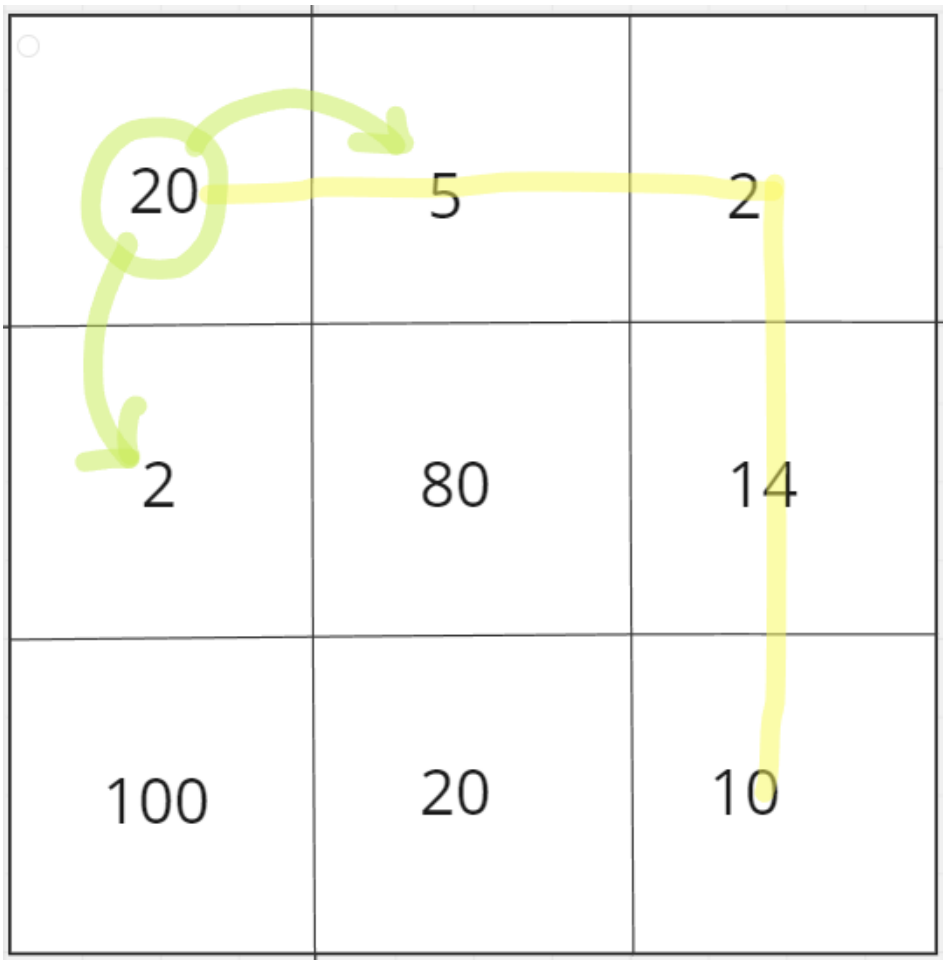
Treat it as same as LCS on same string

But equal condition will have extra check that i != j,



```python
def longestRepeatingSubsequence(st, n):
    dp = [[0]*(n+1) for _ in range(n+1)]
    for i in range(1,n+1):
        for j in range(1,n+1):
            if i!=j and st[i-1]==st[j-1]:
                dp[i][j] = 1 + dp[i-1][j-1]
            else:
                dp[i][j] = max(dp[i][j-1], dp[i-1][j])

    return dp[n][n]
```

3. Minimum Path Sum

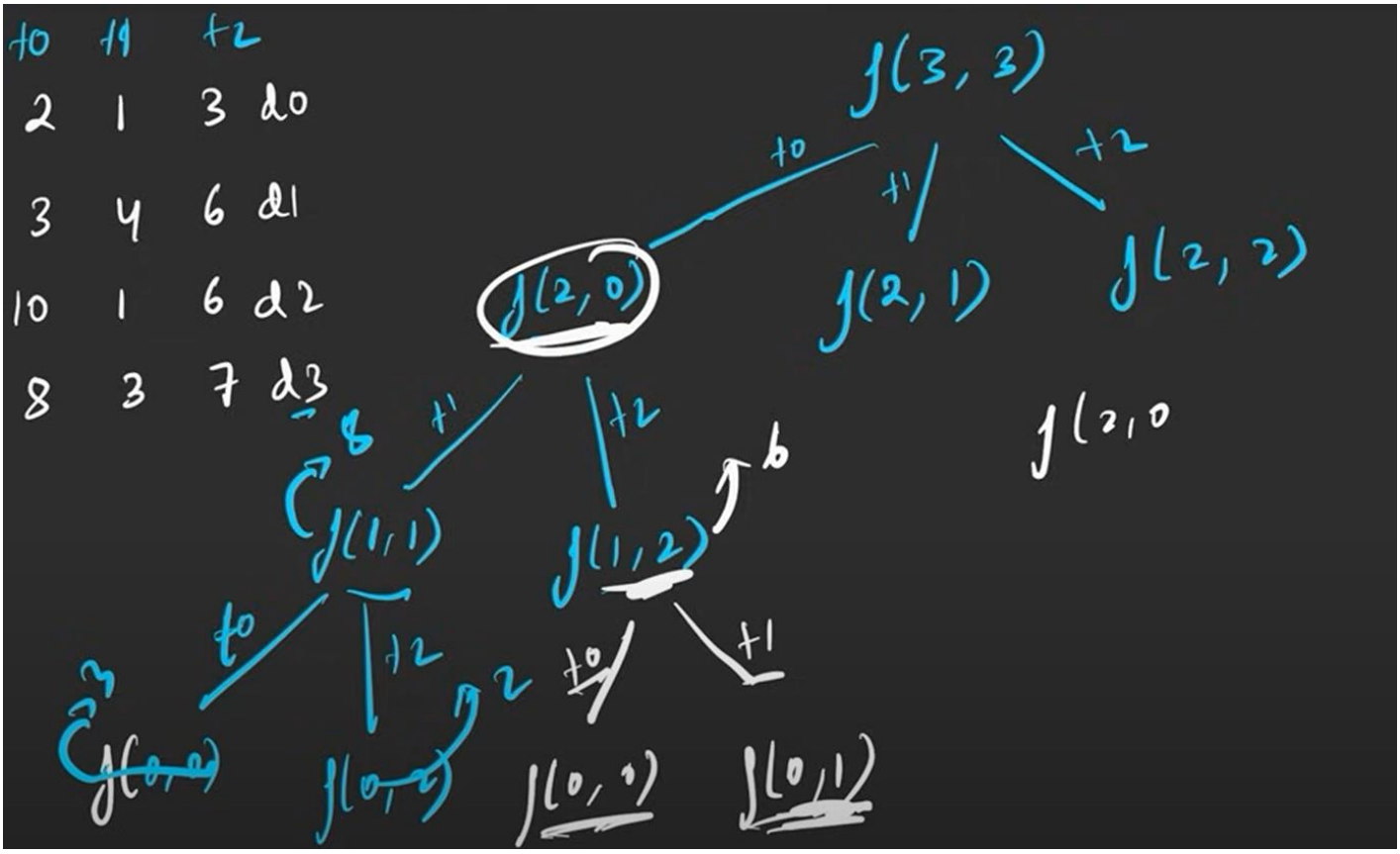


If we use greedy strategy here. Then we must be going to 2, instead of 5.

But 5 gives us best results globally. So for this case as we have global consideration to get best results, we use dp to check all possible paths and get best results.

```python
class Solution:
    def minPathSum(self, grid: List[List[int]]) -> int:
        n = len(grid)
        m = len(grid[0])
        dp = [[0]*m for _ in range(n)]

        for i in range(n):
            for j in range(m):
                if i==0 and j==0:
                    dp[i][j] = grid[i][j]
                else:
                    up,left = float('inf'),float('inf')
                    if i>0:
                        up = grid[i][j] + dp[i-1][j]
                    if j>0:
                        left = grid[i][j] + dp[i][j-1]
                    dp[i][j] = min(up,left)
        return dp[n-1][m-1]
```

4. Ninja Training



Can further do space optimization as in each step we just need to get the previous state. So in place of saving whole dp, just save the previous state.

```python
def ninjaTraining(n: int, points: List[List[int]]) -> int:

    # Write your code here.
    dp = [[0]*3 for i in range(n)]
    dp[0][0] = points[0][0]
    dp[0][1] = points[0][1]
    dp[0][2] = points[0][2]

    for i in range(1,n):
        dp[i][0] = points[i][0] + max(dp[i-1][1] , dp[i-1][2])
        dp[i][1] = points[i][1] + max(dp[i-1][0] , dp[i-1][2])
        dp[i][2] = points[i][2] + max(dp[i-1][0] , dp[i-1][1])

    return max(dp[n-1])
```

LEVEL 3:

1. K-ordered LCS

Same as like LCS, but one new case.

New case. Now we can change k values in seq1 to make it's i^{th} value equal to j^{th} value of seq2.

So when seq1[i] != seq2[j], then 2 cases, (1) we can use k and make both values equal, (2) don't use k and proceed as normal. Final ans is maximum of both.

```
def KOrderedLCS(seq1, seq2, k):

    def helper(i, j, k, dp):
        if i==0 or j==0: return 0
        if dp[i][j][k]!=-1:
            return dp[i][j][k]

        if seq1[i-1] == seq2[j-1]:
            dp[i][j][k] = 1+ helper(i-1,j-1,k,dp)
        else:
            if k>0:
                #we replacing a value in seq1,so will act as seq1[i]==seq2[j]
                temp = 1+ helper(i-1 ,j-1 , k-1, dp)
                #Case we not replacing any value
                temp2 =  max(helper(i-1,j ,k ,dp), helper(i, j-1, k, dp))
                dp[i][j][k] = max(temp, temp2)
            else:
                dp[i][j][k] = max(helper(i-1, j, k, dp), helper(i, j-1, k, dp))

        return dp[i][j][k]

    n,m=len(seq1),len(seq2)
    # dp of size n*m*k
    dp = [[[-1 for _ in range(k+1)] for _ in range(m+1)] for _ in range(n+1)]
    return helper(n,m,k,dp)



seq1 = [1, 2, 3, 4, 5]
seq2 = [5, 3, 1, 4, 2]
k=1
print(KOrderedLCS(seq1, seq2, k))
```

```
## Demo run
eg:-
seq1 = [1, 2, 3, 4, 5]
seq2 = [5, 3, 1, 4, 2]
k = 1
Ans = 3
You can change the first element of the first sequence to 5 to get the LCS comprising of
the sequence (5, 3, 4)
```

Bottom Up

```
def KOrderedLCS(seq1, seq2, k):
    n, m = len(seq1), len(seq2)
    # Initialize the dp table with zero values
    dp = [[[0 for _ in range(k+1)] for _ in range(m+1)] for _ in range(n+1)]

    # Iterate through all positions of seq1 and seq2
    for i in range(1, n+1):
        for j in range(1, m+1):
            for z in range(k+1):
                if seq1[i-1] == seq2[j-1]:
                    dp[i][j][z] = dp[i-1][j-1][z] + 1
                else:
                    if z > 0:
                    # Option 1: Replace seq1[i-1] with seq2[j-1]
                        replace = dp[i-1][j-1][z-1] + 1
                    # Option 2: Do not replace, just move in one of the sequences
                        dont_replace = max(dp[i-1][j][z], dp[i][j-1][z])
                        dp[i][j][z] = max(replace, dont_replace)
                    else:
                    # When no replacements are allowed
                        dp[i][j][z] = max(dp[i-1][j][z], dp[i][j-1][z])

    return dp[n][m][k]

# Read input
seq1 = [1, 2, 3, 4, 5]
seq2 = [5, 3, 1, 4, 2]
k=1
# Print the result
print(KOrderedLCS(seq1, seq2, k))
```