CPSC 340: Machine Learning and Data Mining

Non-Parametric Models

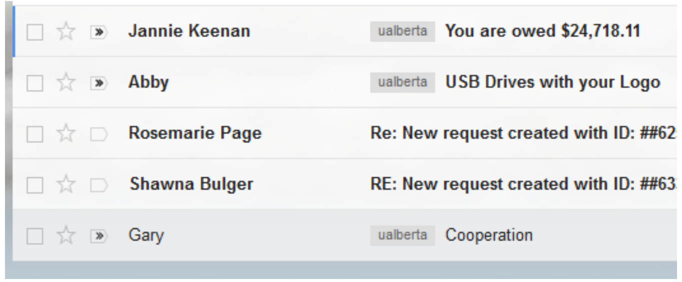
Fall 2019

Admin

- Course webpage:
  – https://www.cs.ubc.ca/~schmidtm/Courses/340-F19/
- Assignment 1:
  – 1 late day to hand in tonight, 2 for Wednesday.
- Assignment 2 is out.
  – Due Friday of next week. It's long so start early.
- Add/drop deadline is tomorrow.
- Auditing/Exchange:
  – Bring your form to me after class.

Last Time: E-mail Spam Filtering

- Want a build a system that filters spam e-mails:



- We formulated as supervised learning:
  - (y_i = 1) if e-mail 'i' is spam, (y_i = 0) if e-mail is not spam.
  - (x_{ij} = 1) if word/phrase 'j' is in e-mail 'i', (x_{ij} = 0) if it is not.

$	Hi	CPSC	340	Vicodin	Offer	…
1	1	0	0	1	0	…
0	0	0	0	1	1	…
0	1	1	1	0	0	…
…	…	…	…	…	…	…

Spam?
1
1
0
…

Last Time: Naïve Bayes

- We considered spam filtering methods based on naïve Bayes:

$$p(y_i = \text{"spam"} \mid x_i) = \frac{p(x_i \mid y_i = \text{"spam"})\,p(y_i = \text{"spam"})}{p(x_i)}$$

- Makes conditional independence assumption to make learning practical:

$$\underbrace{p(\text{hello}=1, \text{vicodin}=0, 340=1 \mid \text{spam})}_{\text{HARD}} \approx \underbrace{p(\text{hello}=1 \mid \text{spam})}_{\text{easy}}\,\underbrace{p(\text{vicodin}=0 \mid \text{spam})}_{\text{easy}}\,\underbrace{p(340=1 \mid \text{spam})}_{\text{easy}}$$

- Predict "spam" if $p(y_i = \text{"spam"} \mid x_i) > p(y_i = \text{"not spam"} \mid x_i)$.
  - We don't need $p(x_i)$ to test this.

Naïve Bayes

- Naïve Bayes formally:

$$p(y_i \mid x_i) = \frac{p(x_i \mid y_i)\,p(y_i)}{p(x_i)}$$ (first use Bayes rule)

$$\propto \underbrace{p(x_i \mid y_i)\,p(y_i)}$$ ("denominator doesn't matter") same for all y_i values

$$\approx \prod_{j=1}^{d}\underbrace{\left[p(x_{ij} \mid y_i)\right]p(y_i)}$$ (conditional independence assumption)

Only needs easy probabilities.

- Post-lecture slides: how to train/test by hand on a simple example.

Laplace Smoothing

- Our estimate of p('lactase' = 1| 'spam') is:

$$\frac{\#\text{ spam messages with lactase}}{\#\text{ spam messages}}$$

  - But there is a problem if you have no spam messages with lactase:
    - p('lactase' | 'spam') = 0, so spam messages with lactase automatically get through.
  - Common fix is Laplace smoothing:
    - Add 1 to numerator,
      and 2 to denominator (for binary features).
      - Acts like a "fake" spam example that has lactase,
        and a "fake" spam example that doesn't.

$$\frac{(\#\text{ spam messages with lactase}) + 1}{(\#\text{ spam messages}) + 2}$$

Laplace Smoothing

- Laplace smoothing: $\dfrac{(\#\text{spam messages with lactase}) + 1}{(\#\text{spam messages}) + 2}$
  - Typically you do this for all features.
    - Helps against overfitting by biasing towards the uniform distribution.
- A common variation is to use a real number β rather than 1.
  - Add 'βk' to denominator if feature has 'k' possible values (so it sums to 1).

$$p(x_{ij} = c \mid y_i = \text{class}) \approx \frac{(\text{number of examples in class with } x_{ij} = c) + \beta}{(\text{number of examples in class}) + \beta k}$$

This is a "maximum a posteriori" (MAP) estimate of the probabiliy. We'll discuss MAP and how to derive this formula later.

Decision Theory

- Are we equally concerned about "spam" vs. "not spam"?
- True positives, false positives, false negatives, true negatives:

Predict / True	True 'spam'	True 'not spam'
Predict 'spam'	True Positive	False Positive
Predict 'not spam'	False Negative	True Negative

- The costs mistakes might be different:
  - Letting a spam message through (false negative) is not a big deal.
  - Filtering a not spam (false positive) message will make users mad.

Decision Theory

- We can give a cost to each scenario, such as:

Predict / True	True 'spam'	True 'not spam'
Predict 'spam'	0	100
Predict 'not spam'	10	0

- Instead of most probable label, take $\hat{y}_i$ minimizing expected cost:

$$\underbrace{\mathbb{E}}_{\text{expectation of model with respect to } \tilde{y}_i}\Big[\underbrace{\text{cost}(\hat{y}_i, \tilde{y}_i)}_{\text{cost of predicting } \hat{y}_i \text{ if it's really } \tilde{y}_i}\Big]$$

- Even if "spam" has a higher probability, predicting "spam" might have a expected higher cost.

Decision Theory Example

Predict / True	True 'spam'	True 'not spam'
Predict 'spam'	0	100
Predict 'not spam'	10	0

- Consider a test example we have $p(\tilde{y}_i = \text{"spam"} \mid \tilde{x}_i) = 0.6$, then:

$$\mathbb{E}[\text{cost}(\hat{y}_i = \text{"spam"}, \tilde{y}_i)] = p(\tilde{y}_i = \text{"spam"} \mid \tilde{x}_i)\,\text{cost}(\hat{y}_i = \text{"spam"}, \tilde{y}_i = \text{"spam"}) + p(\tilde{y}_i = \text{"not spam"} \mid \tilde{x}_i)\,\text{cost}(\hat{y}_i = \text{"spam"}, \tilde{y}_i = \text{"not spam"})$$

$$= (0.6)(0) + (0.4)(100) = 40$$

$$\mathbb{E}[\text{cost}(\hat{y}_i = \text{"not spam"}, \tilde{y}_i)] = (0.6)(10) + (0.4)(0) = 6$$

- Even though "spam" is more likely, we should predict "not spam".

Naïve Bayes Training Phase

- Training a naïve Bayes model:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Naïve Bayes Training Phase

- Training a naïve Bayes model:

1. Set n_c to the number of times $(y_i = c)$.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$n_1 = 6$

$n_0 = 4$

Naïve Bayes Training Phase

- Training a naïve Bayes model:

1. Set n_c to the number of times $(y_i = c)$.
2. Estimate $p(y_i = c)$ as $\frac{n_c}{n}$.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$p(y_i = 1) = \frac{6}{10} \leftarrow n_1 = 6$

$p(y_i = 0) = \frac{4}{10} \leftarrow n_0 = 4$

Naïve Bayes Training Phase

- Training a naïve Bayes model:

1. Set n_c to the number of times $(y_i = c)$.
2. Estimate $p(y_i = c)$ as $\frac{n_c}{n}$.
3. Set n_{cjk} as the number of times $(y_i = c, x_{ij} = k)$

$p(y_i = 1) = \frac{6}{10} \leftarrow n_1 = 6$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$n_{121} = 4$

$n_0 = 4$

$p(y_i = 0) = \frac{4}{10} \leftarrow n_0 = 4$

Naïve Bayes Training Phase

- Training a naïve Bayes model:

1. Set n_c to the number of times $(y_i = c)$.
2. Estimate $p(y_i = c)$ as $\frac{n_c}{n}$.
3. Set n_{cjk} as the number of times $(y_i = c, x_{ij} = k)$
4. Estimate $p(x_{ij} = k, y_i = c)$ as $\frac{n_{cjk}}{n}$

$$p(y_i = 1) = \frac{6}{10} \leftarrow n_1 = 6$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$n_{121} = 4$$

$$p(x_{i2} = 1, y_i = 1) = \frac{4}{10}$$

$$p(y_i = 0) = \frac{4}{10} \leftarrow n_0 = 4$$

Naïve Bayes Training Phase

- Training a naïve Bayes model:

1. Set n_c to the number of times $(y_i = c)$.
2. Estimate $p(y_i = c)$ as $\frac{n_c}{n}$.
3. Set n_{cjk} as the number of times $(y_i = c, x_{ij} = k)$
4. Estimate $p(x_{ij} = k, y_i = c)$ as $\frac{n_{cjk}}{n}$
5. Use that $p(x_{ij} = k \mid y_i = c) = \frac{p(x_{ij} = k, y_i = c)}{p(y_i = c)}$

$$= \frac{n_{cjk}/n}{n_c/n} = \frac{n_{cjk}}{n_c}$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$p(y_i = 1) = \frac{6}{10} \leftarrow n_1 = 6$

$n_{121} = 4$

$p(x_{i2} = 1, y_i = 1) = \frac{4}{10}$

$p(y_i = 0) = \frac{4}{10} \leftarrow n_0 = 4$

$p(x_{i2} = 1 \mid y_i = 1) = \frac{4}{6} = \frac{2}{3}$

Naïve Bayes Prediction Phase

- Prediction in a naïve Bayes model:

Given a test example $\tilde{x}_i$ we set prediction $\hat{y}_i$ to the 'c' maximizing $p(\tilde{x}_i \mid \tilde{y}_i = c)$

Under the naive Bayes assumption we can maximize:

$$p(\tilde{y}_i = c \mid \tilde{x}_i) \propto \prod_{j=1}^{d} \left[p(\tilde{x}_{ij} \mid \tilde{y}_i = c) \right] p(\tilde{y}_i = c)$$

Naïve Bayes Prediction Phase

- Prediction in a naïve Bayes model:

Consider $\tilde{x}_i = [1 \ 1]$ in this data set

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Naïve Bayes Prediction Phase

- Prediction in a naïve Bayes model:

Consider $\tilde{x}_i = [1 \ 1]$ in this data set $\longrightarrow$

$$p(\tilde{y}_i = 0 \mid \tilde{x}_i) \propto p(\tilde{x}_{i1} = 1 \mid \tilde{y}_i = 0)\, p(\tilde{x}_{i2} = 1 \mid \tilde{y}_i = 0)\, p(\tilde{y}_i = 0)$$
$$= (1) \qquad (0.25) \qquad (0.4) = 0.1$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Naïve Bayes Prediction Phase

- Prediction in a naïve Bayes model:

Consider $\tilde{x}_i = [1\ 1]$ in this data set $\longrightarrow$

$$p(\tilde{y}_i = 0 \mid \tilde{x}_i) \propto p(\tilde{x}_{i1} = 1 \mid \tilde{y}_i = 0)\, p(\tilde{x}_{i2} = 1 \mid \tilde{y}_i = 0)\, p(\tilde{y}_i = 0)$$
$$= (1) \quad (0.25) \quad (0.4) = 0.1$$

$$p(\tilde{y}_i = 1 \mid \tilde{x}_i) \propto p(\tilde{x}_{i1} = 1 \mid \tilde{y}_i = 1)\, p(\tilde{x}_{i2} = 1 \mid \tilde{y}_i = 1)\, p(\tilde{y}_i = 1)$$
$$= (0.5) \quad (0.666\ldots) \quad (0.6) = 0.2$$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Naïve Bayes Prediction Phase

- Prediction in a naïve Bayes model:

Consider $\tilde{x}_i = [1 \ 1]$ in this data set $\longrightarrow$

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$p(\tilde{y}_i = 0 \mid \tilde{x}_i) \propto p(\tilde{x}_{i1} = 1 \mid \tilde{y}_i = 0)\, p(\tilde{x}_{i2} = 1 \mid \tilde{y}_i = 0)\, p(\tilde{y}_i = 0)$$
$$= (1) \quad (0.25) \quad (0.4) = 0.1$$

$$p(\tilde{y}_i = 1 \mid \tilde{x}_i) \propto p(\tilde{x}_{i1} = 1 \mid \tilde{y}_i = 1)\, p(\tilde{x}_{i2} = 1 \mid \tilde{y}_i = 1)\, p(\tilde{y}_i = 1)$$
$$= (0.5) \quad (0.666\ldots) \quad (0.6) = 0.2$$

Since $p(\tilde{y}_i = 1 \mid \tilde{x}_i)$ is bigger than $p(\tilde{y}_i = 0 \mid \tilde{x}_i)$, naive Bayes predicts $\hat{y}_i = 1$

(Don't sum to 1 because we're ignoring $p(\tilde{x}_i)$)

"Proportional to" for Probabilities

- When we say "p(y) ∝ exp(-y²)" for a function 'p', we mean:

$p(y) = \beta \exp(-y^2)$ for some constant 'β'.

- However, if 'p' is a probability then it must sum to 1.
  - If $y \in \{1,2,3,4\}$ then $p(1)+p(2)+p(3)+p(4) = 1$

- Using this fact, we can find β:

$$\beta\exp(-1^2)+\beta\exp(-2^2)+\beta\exp(-3^2)+\beta\exp(-4^2)=1$$

$$\Leftrightarrow \beta[\exp(-1^2)+\exp(-2^2)+\exp(-3^2)+\exp(-4^2)=1$$

$$\Leftrightarrow \beta = \frac{1}{\exp(-1^2)+\exp(-2^2)+\exp(-3^2)+\exp(-4^2)}$$

Probability of Paying Back a Loan and Ethics

- Article discussing predicting “whether someone will pay back a loan”:
  - https://www.thecut.com/2017/05/what-the-words-you-use-in-a-loan-application-reveal.html

- Words that increase probability of paying back the most:
  - *debt-free, lower interest rate, after-tax, minimum payment, graduate.*

- Words that decrease probability of paying back the most:
  - *God, promise, will pay, thank you, hospital.*

- Article also discusses an important issue: are all these features ethical?
  - Should you deny a loan because of religion or a family member in the hospital?
  - ICBC is limited in the features it is allowed to use for prediction.

Avoiding Underflow

- During the prediction, the probability can underflow:

$$p(y_i = c \mid x_i) \propto \underbrace{\prod_{j=1}^{d} [p(x_{ij} \mid y_i = c)]}\ \underbrace{p(y_i = c)}$$

All these are < 1 so the product gets very small!

- Standard fix is to (equivalently) maximize the logarithm of the probability:

Rember that $\log(ab) = \log(a) + \log(b)$ so $\log(\prod a_i) = \sum_i \log(a_i)$

Since log is monotonic the 'c' maximizing $p(y_i = c \mid x_i)$ also maximizes $\log p(y_i = c \mid x_i)$,

so maximize $\log\left(\prod_{j=1}^{d} [p(x_{ij} \mid y_i = c)]\, p(y_i = c)\right) = \sum_{j=1}^{d} \log(p(x_{ij} \mid y_i = c)) + \log(p(y_i = c))$

Less-Naïve Bayes

- Given features {x1,x2,x3,…,xd}, naïve Bayes approximates p(y|x) as:

$$p(y \mid x_1, x_2, \dots, x_d) \propto p(y)\,p(x_1, x_2, \dots, x_d \mid y)$$

product rule applied repeatedly

$$= p(y)\,p(x_1 \mid y)\,p(x_2 \mid x_1, y)\,p(x_3 \mid x_2, x_1, y) \cdots p(x_d \mid x_1, x_2, \dots, x_{d-1}, y)$$

$$\approx p(y)\,p(x_1 \mid y)\,p(x_2 \mid y)\,p(x_3 \mid y) \cdots p(x_d \mid y) \quad \text{(naive Bayes assumption)}$$

- The assumption is very strong, and there are "less naïve" versions:
  - Assume independence of all variables except up to 'k' largest 'j' where j < i.
    - E.g., naïve Bayes has k=0 and with k=2 we would have:

$$\approx p(y)\,p(x_1 \mid y)\,p(x_2 \mid x_1, y)\,p(x_3 \mid x_2, x_1, y)\,p(x_4 \mid x_3, x_2, y) \cdots p(x_d \mid x_{d-2}, x_{d-1}, y)$$

    - Fewer independence assumptions so more flexible, but hard to estimate for large 'k'.
  - Another practical variation is "tree-augmented" naïve Bayes.

Computing $p(x_i)$ under naïve Bayes

- Generative models don't need $p(x_i)$ to make decisions.
- However, it's easy to calculate under the naïve Bayes assumption:

$$p(x_i) = \sum_{c=1}^{K} p(x_i, y=c) \quad \text{(marginalization rule)}$$

$$= \sum_{c=1}^{K} p(x_i \mid y=c)\, p(y=c) \quad \text{(product rule)}$$

$$= \sum_{c=1}^{K} \underbrace{\left[\prod_{j=1}^{d} p(x_{ij} \mid y=c)\right] p(y=c)}_{\text{These are the quantities we compute during training}} \quad \text{(naïve Bayes assumption)}$$

Gaussian Discriminant Analysis

- Classifiers based on Bayes rule are called generative classifier:
  - They often work well when you have tons of features.
  - But they need to know $p(x_i \mid y_i)$, probability of features given the class.
    - How to "generate" features, based on the class label.

- To fit generative models, usually make BIG assumptions:
  - Naïve Bayes (NB) for discrete x_i:
    - Assume that each variables in x_i is independent of the others in x_i given y_i.

  - Gaussian discriminant analysis (GDA) for continuous x_i.
    - Assume that $p(x_i \mid y_i)$ follows a multivariate normal distribution.
    - If all classes have same covariance, it's called "linear discriminant analysis".

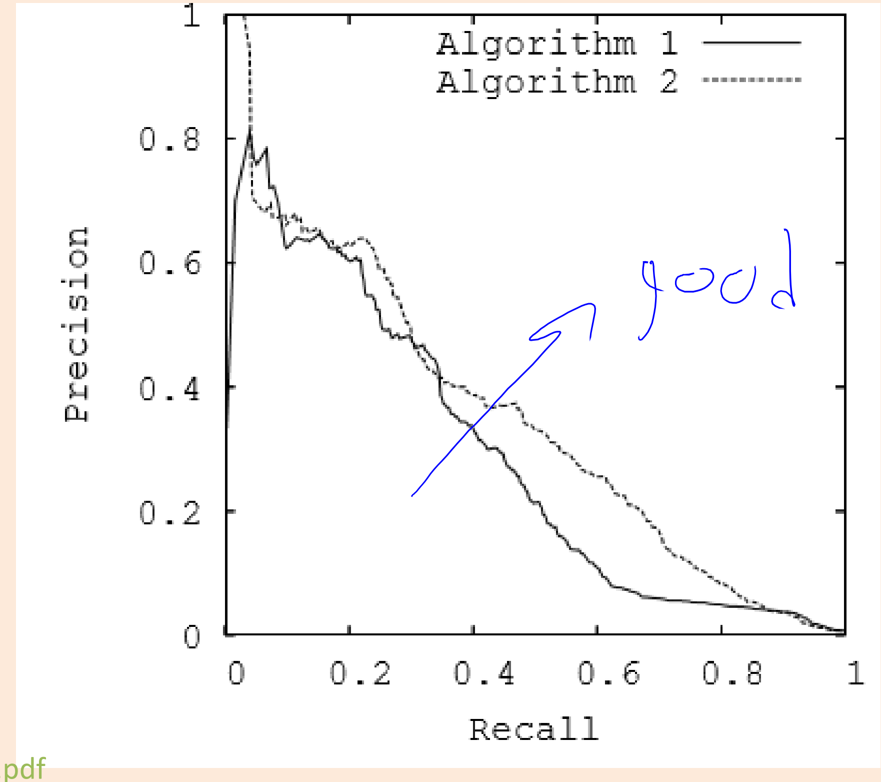
Other Performance Measures

- Classification error might be wrong measure:
  - Use weighted classification error if have different costs.
  - Might want to use things like Jaccard measure: TP/(TP + FP + FN).

- Often, we report precision and recall (want both to be high):
  - Precision: "if I classify as spam, what is the probability it actually is spam?"
    - Precision = TP/(TP + FP).
    - High precision means the filtered messages are likely to really be spam.
  - Recall: "if a message is spam, what is probability it is classified as spam?"
    - Recall = TP/(TP + FN)
    - High recall means that most spam messages are filtered.

Precision-Recall Curve

- Consider the rule $p(y_i = \text{'spam'} \mid x_i) > t$, for threshold 't'.
- Precision-recall (PR) curve plots precision vs. recall as 't' varies.

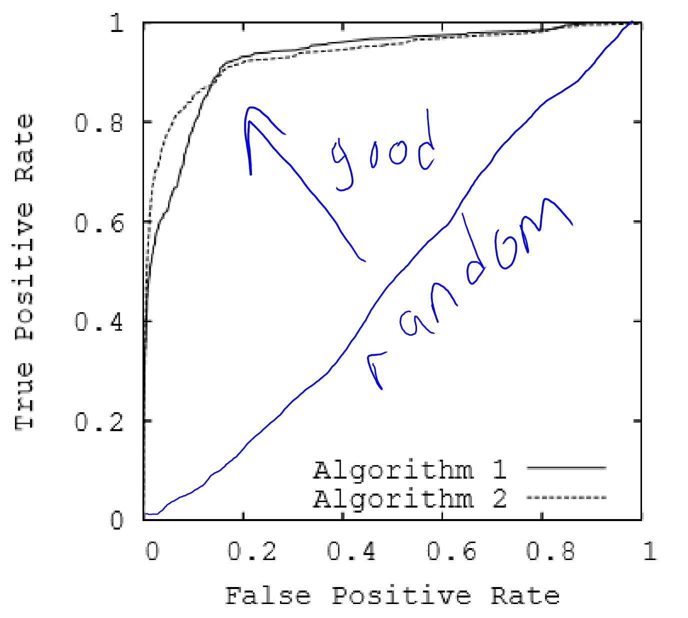


http://pages.cs.wisc.edu/~jdavis/davisgoadrichcamera2.pdf

ROC Curve

- Receiver operating characteristic (ROC) curve:
  - Plot true positive rate (recall) vs. false positive rate (FP/FP+TN).

    (negative examples classified as positive)



  - Diagonal is random, perfect classifier would be in upper left.
  - Sometimes papers report area under curve (AUC).
    - Reflects performance for different possible thresholds on the probability.

http://pages.cs.wisc.edu/~jdavis/davisgoadrichcamera2.pdf