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Eye Pupil Control Analysis

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Abstract

The human eye, a pivotal sense organ, plays a crustal role in shaping our perception of the world. Researchers have consistently delved into its functional aspects, reflecting its paramount significance. Comprising intricate components, the human eye collectively orchestrates the intricate symphony of human visual perception. This study primarily focuses on the human pupil—a pivotal component of this sensory apparatus. In this paper, we rigorously investigate the dynamics of the human pupil through a comprehensive linear control analysis. Our approach involves deriving a linearized model based on prior research findings. Subsequently, we apply advanced control analysis techniques to gain deeper insights into its behavior. The obtained results provide valuable contributions to our understanding of the intricate dynamics governing the human pupil.

This research sheds light on the complex interplay of factors that govern the behavior of the human pupil, offering insights with potential applications in fields such as ophthalmology, optometry, and human-computer interaction.

Keywords: pupil, linear control analysis, linearized equation

INTRODUCTION

The human eye is a complex and delicate organ that is responsible for vision and plays an important role in everyday life. The pupil, located in the center of the iris, controls the amount of light entering the eye and plays an important role in regulating vision. The pupil controls amount light flux that enters inner part human eye, by changing it's area. This system preventing damage to the sensitive cells of the eye, such as the retina, and also improving visual perception.

This intricate system has undergone extensive evolutionary development within the human species, and comprehending its intricate mechanisms holds significant potential. Undoubtedly, a robust and intricate control system plays a pivotal role in modulating pupil size, offering valuable insights for the design and safeguarding of optical systems. Given the limited understanding of the eye's control processes and the relative scarcity of research in this

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domain, this article presents a comprehensive examination of the human eye's pupil control system. In this work, we investigate the pupil control systems through the lens of linear control theory, recognizing that grasping the principles of linear control is a fundamental and essential aspect of system control analysis.

In the context of optical stimulation of the eye pupil, a linear control system can be used to regulate the amount of light entering the eye by adjusting the size of the pupil. The control system can receive input signals from various sources, such as light sensors, and can use feedback mechanisms to adjust the output signal, which controls pupil size.

In this paper, we investigate the use of linear control processes for optical stimulation of the eye pupil in humans. We analyze the pupil's dynamics and develop a mathematical model for the pupil's linear behavior.

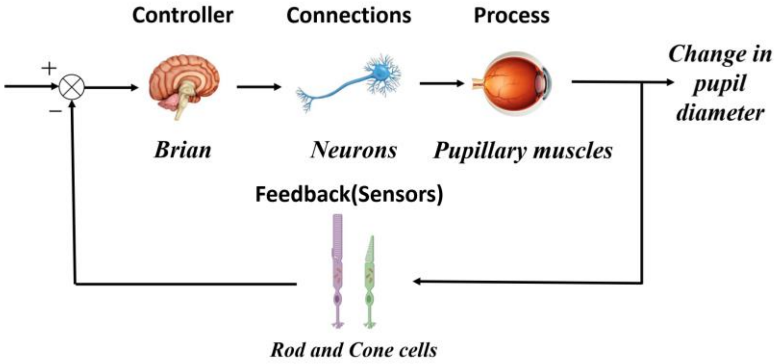

Fig 1. Diagram of the system: All main parts in the visual system control loop have been illustrated in this diagram.

RELATED WORKS

In order to perform control analysis on a system, we need to have an understanding of the equations describing the behavior of the system, for the purpose of this study, we require the pupil light reflex/response (PLR) equations, so in this section, we will cover some of the studies in this area. In order to perform control analysis on a system, we need to have an understanding of the equations describing the behavior of the system, for the purpose of this study, we require the pupil light reflex/response (PLR) equations, so in this section, we will cover some of the studies in this area. Many models have been proposed by researchers to estimate the steady-state diameter of the pupil under various luminance. For example, [1] derived steady state formula for light- adapted pupil diameter that includes the effects of luminance, adapting field size, age of the observer, and whether one eye or two are used. Later on, the field area of the target was introduced as a parameter by Stanley and Davies as shown in Eq. 1.

$$D_{SD}(L,a) = 7.75 - 5.75\left(\frac{(La/846)^{0.41}}{(La/846)^{0.41}+2}\right) \quad [1]$$

Where L is the luminance measured in candelas per meter square ($\frac{cd}{m^2}$), and a is the field area in degrees squared (deg2). The field area describes the size of the source of luminance in degrees of the field of view of the subject. Then define an adaptation of pupil size based on the age of the subject:

$$A(L,a,y,y_0,e) = (y - y_0) \times \left(0.021323 \; - \; 0.0095623 \, D_{SD}\,(LM(e),a)\right) \quad [2]$$

Based on the Stanley and Davies dataset [2], y0 is the reference age, and the age of the population is 20 < y < 83. Appendix 1 of [1] proposes an adjustment for individuals below the age of 20. The unified formula for light-adapted pupil size is obtained by combining Eqs 1, 2:

$$D_U(L,a,y,y_0,e) = D_{SD}(LM(e),a) + A(L,a,y,y_0,e) \text{ [3]}$$

Although this model can predict PLR well in a steady state, pupil movement in the real world is dynamic and these steady-state models cannot accurately obtain PLR.

[3]provide the first physiologically based model of the pupil light reflex which is a non-linear delay-differential equation that describes the changes in pupil diameter as a function of the environment lighting which correctly simulates the actual behavior of the human pupil, this model can predict PLR more accurately than other steady-states models.

To estimate the PLR, data-driven methods could be utilized. Neural networks, which are entirely data-driven, have also been successfully used to model different biological processes including pupil responses. Consider the research of [4] as an example. Based on gaze data (fixations, saccades, etc.) and previous pupil size, they trained an LSTM model to predict pupil size. In this study, participants watched five-minute academic video lectures, monitored their gaze and pupil data, and then answered to comprehension. [4]

Kasthurirangan and Glasser have derived PLR latency considering dynamic accommodative and pupillary responses to step stimuli were recorded in 66 subjects (ages: 14–45 years). Exponential fits data provided amplitude, peak velocity, and time constants. Finally, they reported the mean latency of accommodation is 256 ms [5] which is slightly lower than previous reports ranging from 285 to 370 ms [6] [7] [8].

In summary, the PLR has been extensively studied, and various steady-state and dynamic models have been proposed to explain pupil light reflex/response. Considering that PPR, in addition to PLR, affects the reaction of the pupil, PPR should also be considered for a more accurate prediction. Although the effect of PPR has been proven, researchers have not yet managed to obtain a model that accurately predicts PRR [9], thus in this study, we only consider the effect of PLR. As mentioned, many studies have been conducted on the reaction of the pupil to light and emotions in various conditions, but not many studies have been conducted on the control processes of the pupil. Our goal in this study is to use the dynamic equations obtained by previous researchers and implement linear control topics after linearizing the equations.

SYSTEM EQUATION

A) Differential Equation

In real-world conditions, the amount of light is dynamic and changes with time, also the pupil size has a very complex be-havior in these dynamic conditions. In fact, several researchers have studied the transient and dynamic behavior of the pupil and some have developed dynamic models. which is often based on the research of neural control systems. One of these dynamic models developed by [3]Using the delay differential equation model that combines the dynamic model of the eye pupil, Longtin and Milton [10] and the steady state model of Moon and Spencer [11], modeled the pupil dynamics to light changes.

$$\frac{dg}{dA}\left(\frac{dA}{dt}\right) + ag(A) = \gamma\ ln\left[\frac{\phi(t-\tau)}{\bar{\phi}}\right]\ [4]$$

Where A describes the pupil area ($\pi\left(\frac{D}{2}\right)^2$ where D is the pupil diameter), $\varphi(t)$ describes the retinal light flux ($L_1 A$ where L_1 is the luminance in lumens), and $\bar{\phi}$ which is the retinal flux at the luminance below which there is no longer any change in the pupil size.

$$D\ =\ 4.9 - 3tanh[0.4(\log(L_b) -\ 0.5\ )]\quad [5]$$

Where L_b is the luminance expressed in blondels. They first combine the above two models for steady-state conditions (Eq.7) and use Moon's [11] data (steady-state data) to evaluate the unknown parameter γ (0.45) and add an additional constant for vertical adjustment (5.2). They are able to combine the two models by approximating ag(A) as $-2.3026\,\text{atanh}\left(\frac{D-4.9}{3}\right)$ after some manipulations on Eq.5. Then they add back the dynamic terms for the final dynamic model (Eq.8).

$$M(D) = atanh\left(\frac{D-4.9}{3}\right)\ [6]$$

$$2.3026\ M(D) = 5.2 - 0.45\ ln\left[\frac{\phi}{\bar{\phi}}\right]\quad [7]$$

$$\frac{dM}{dD}\frac{dD}{dt} + 2.3026\ M(D) = 5.2 - 0.45\ ln\left[\frac{\phi(t-\tau)}{\bar{\phi}}\right]\ [8]$$

They compare their model to real pupil measurements and show that it realistically predicts the dynamic pupil size given the luminance. However, only a few of the model's parameters can be adapted to subjects and there is no mechanism to optimally fit the parameters to pupil data recorded under dynamic luminance conditions. [12] has also developed a dynamic model using a second-order differential equation and have applied a pattern search method for finding optimal parameters using experimental data [12]. However, this model is optimized for short flashes at a specific wavelength (100ms pulse of 530 nm wavelength light) under otherwise light-adapted conditions in order to isolate the transient responses of the pupil and may not be suitable for longer pulses or more dynamic luminance conditions.

B) Linearization

In order to check the control characteristics of the system linearly, the system must be linearized first. To linearize the system, we calculate Taylor's series of the transformation function around the equilibrium point. In the first step in linearizing the dynamic equation of the system, the equations should be simplified. To simplify, first, the derivative of M is computed and the derivative term in Eq.8 has been computed and combined with Eq.7 then the result is Eq.9.

$$\frac{dD}{dt} = \frac{5.2 - ln\left[\frac{\phi(t-\tau)}{\bar{\phi}}\right] - 2.302\ M(D)}{\frac{1}{2}asech^2\left(\frac{D-4.9}{3}\right)}$$

[9]

then we use Eq.7. in Eq.9 the result is Eq.10.

$$\frac{dD}{dt} = \frac{10.4 - 0.9\, ln\left[\frac{\phi(t-\tau)}{\bar{\phi}}\right] - 4.6052\, atanh\left[\frac{D-4.9}{3}\right]}{asech^2\left[\frac{D-4.9}{3}\right]}$$

[10]

To find the equilibrium point of the system, we set differential term ($\frac{dD}{dt}$) equal to zero. To establish equilibrium, both sides of the equation must be equal, so this equilibrium occurs as follows. First, we recall equation 8 and set its differential term equal to zero. Considering that according to [3], the eye is in a state of equilibrium where the input flux is the lowest state (i.e., $\phi(t)$ is equal to ($\bar{\phi}$) [3] [13]).

$$2.3026M(D) = 5.2 - 0.45\, ln\left[\frac{\phi(t-\tau)}{\bar{\phi}}\right]$$

[11]

The Ln term will be equal to zero considering the fact that $\varphi(t-\tau)$ is equal to $\bar{\phi}$. The result of combining Eq.6 and 11 will be Eq.12.

$$D = 3\, atanh\left(\frac{5.2}{2.3026}\right) + 4.9$$

[12]

In the next step, we linearize the nonlinear relationships using the first terms of the Taylor series. The linearization of the differential equation has been done and (Eq. 13) is the linearized equation of (Eq. 8).

$$\frac{dD}{dt} = -4.3\delta\phi - 1.876\delta D$$

[13]

CONTROL ANALYSIS

In this section of the article, we delve into a comprehensive analysis of the control system. Initially, we derive the system's transfer function, followed by an in-depth examination of its damping characteristics. Subsequently, we investigate the system's response to a diverse range of input signals, assessing its behavior under different conditions. We then ascertain the stability of the linearized system equation. Additionally, we explore the root loci of the poles and zeros, culminating in a thorough frequency analysis. Finally, we present Nyquist and Bode diagrams to visualize the system's frequency response.

A) Transfer Function

Obtaining the system transformation function is another step that is performed first in any linear control study and provides a lot of information to researchers. The transfer function of a system is a mathematical function that theoretically models the system's output for each possible input. In order to calculate the transformation function of the system, the Laplace transform of the equation has been calculated.

$$\begin{cases} D = Y \\ \phi = X \end{cases} then\ \dot{Y} = -4.3\, X - 1.876\, Y$$

[14]

$$\frac{Y}{X} = \frac{-4.3}{S + 1.876}\, e^{-\tau S}$$

[15]

Considering that eye pupil reaction time is an average of (200, 350) ms [13] [14] [5] we consider pupil reflex delay of 350ms in this study.

$$\frac{Y}{X} = \frac{-4.3}{S + 1.876} e^{-350\times 10^{-3} s}$$

[16]

Then we approximate the delay as follows by adding a pole and a zero to the system.

$$\frac{Y}{X} = \frac{-4.3(S - 0.175)}{(S + 1.876)(S + 0.175)}$$

[17]

Understanding of damping behavior of systems is crucial information for researchers. Considering that Eq.17 the system is second-order and analyzing the damping behavior of the system gives us valuable information about how human pupils reacted to the incoming flow of light. To figure out the damping behavior of the system we used MATLAB(R2021b) for simulations.

The damping behavior of the system can be obtained from the characteristic equation of the system. The denominator of the transformation function can be considered the system characteristic equation.

$$\Delta(S) = S^2 + 2.05S + 0.32 \rightarrow \begin{cases} \omega_n = 0.57, & Natural\ frequncy \\ \zeta = 1.8, & Damping\ ratio \end{cases}$$

[18]

The characteristic equation shows the system should be over damped and it slowly converges.

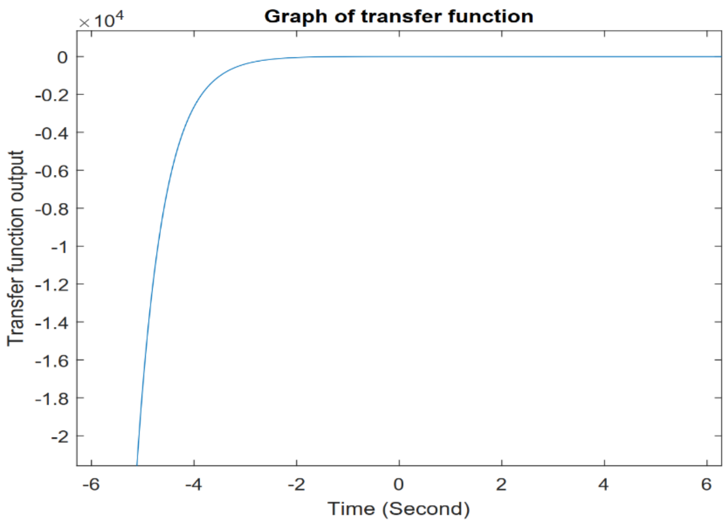

Fig 2. Transfer function figure

As expected, the simulation results do not show any oscillations in the transform function graph because the transform function of the system is overdamped.

B) System Responses

According to the human eye transfer function (Eq .17), the pupil diameter decreases with increasing input flux, and the pupil has an inverse relationship with changes in input flux. Matlab's Simulink was used to simulate the human eye's functionality and its response to inputs. Different inputs were given to the system, and output graphs were received.

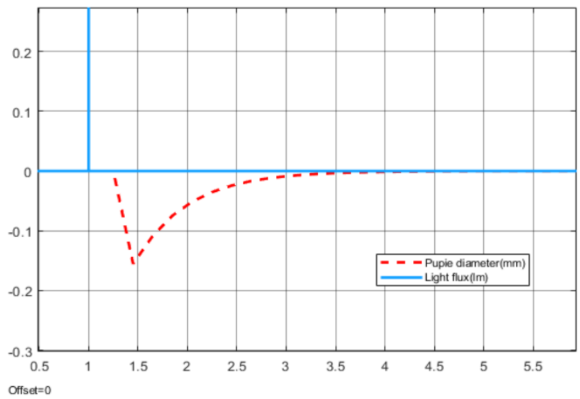

Fig. 3. Figure of the system response to impulse input

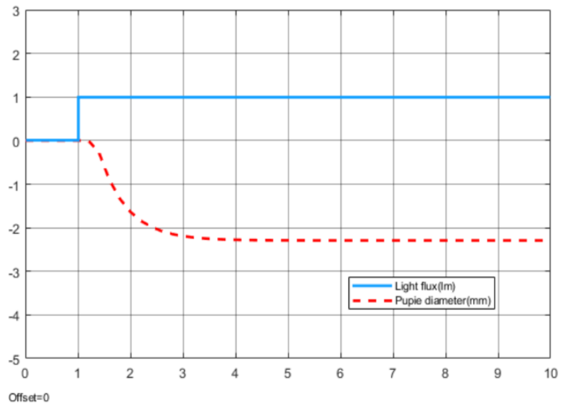

Fig 4. Figure of the system response to step input

It can easily be obtained from the system response to impulse input (Fig. 2) has an inverse relationship with changes in the incoming flux of light and the simulation results confirm the theoretical results.

system is stable if its output is under control, otherwise, it is said to be unstable. A stable system produces a limited output for a limited input. Clearly, the pupil of the human eye is a stable system that does not lose its stability under diverse levels of incoming light. In severe lighting conditions, when even the smallest diameter of the pupil is unable to reduce the amount of light flux entering the vitreous to a level that is suitable, other control systems of the human body like the eyelids react to prevent the vitreous from being overloaded with light. In this section, we perform the stability test on the linearized eye model. This test shows us how close the obtained model is to the real state of the human eye. To check the stability of our system, we performed two methods. First, with the Routh-Hurwitz stability criterion [15] and second, using the MATLAB program.

In the Routh-Hurwitz stability criterion, we make a table from the system characteristic equation (Eq. 18) and if there is no change in signs in the first column, we can claim that all poles are stable.

Table 1. ROUTH-HURWITZ TABLE. THESE TABLE ELEMENTS ARE FILLED WITH THE CHARACTERISTIC EQUATION PARAMETERS.

s^2	1	0.5033
s^1	6.351	0
s_0	0.5033	0

Table 1 shows the consistency of the stability of the obtained linear model with the reality of the human eye pupil. In [Modern Control Systems-12th Edition] there is code for evaluating the stability of the system if outputs of the following code are integers number, we can claim that all poles are stable.

```
% Stability analysis
numg=[1]; deng=[1 6.351 0.5033];
sysg= tf(numg,deng);
sys=feedback(sysg,[1]);
pole(sys)
```

Output code is −6.1047 and −0.2463 so the system is stable.

C) Root Locus

The position of the closed-loop roots of the characteristic equation in the s-plane is strongly connected to the relative stability and transient performance of a closed-loop control system. In order to get acceptable root locations, one or more system parameters must often be adjusted. Therefore, it is worthwhile to determine how the roots of the characteristic equation of a given system migrate about the s-plane as the parameters are varied; It is useful to determine the locus of roots in the s-plane as a parameter is varied. In this section, we obtained the system root locus from Matlab with the “rlocus” function, and the result is shown in (Fig 5).

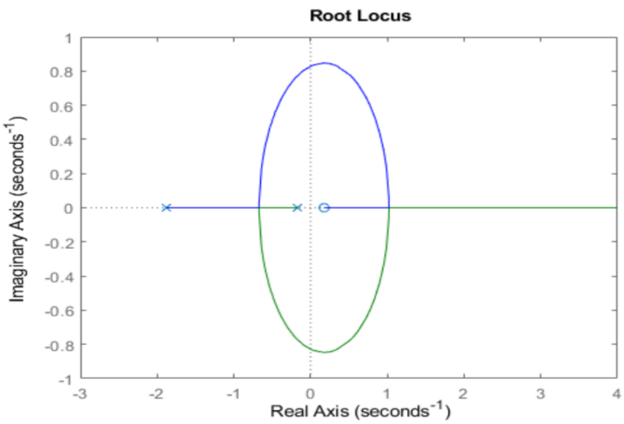

Fig 5. Figure of Root-Loucas diagram

(Fig 5) shows that there are two poles at left side of diagram and there is one zero at right side of diagram. Understanding location of poles and zeros is very important for designing controller for a system.

D) Frequency Response Methods

In this section, to create a better understanding of the frequency response of the system. We have drawn "Bode" and "Nyquist" diagrams. (Fig 6) shows the Nyquist diagram and (Fig 7) shows the Bode diagram. Having diagrams provides a very good understanding of the system's characteristics, and by using these diagrams, more advanced control analyzes can be performed and even suitable controllers can be created for this system. We have made all our code and results open source for researchers to use in their research.

Considering that the aim of this article is the linear analysis of the eye pupil, we did not go into deeper and more detailed analysis. More advanced control analyzes on the pupil can be an interesting topic to be carried out by researchers interested in this field.

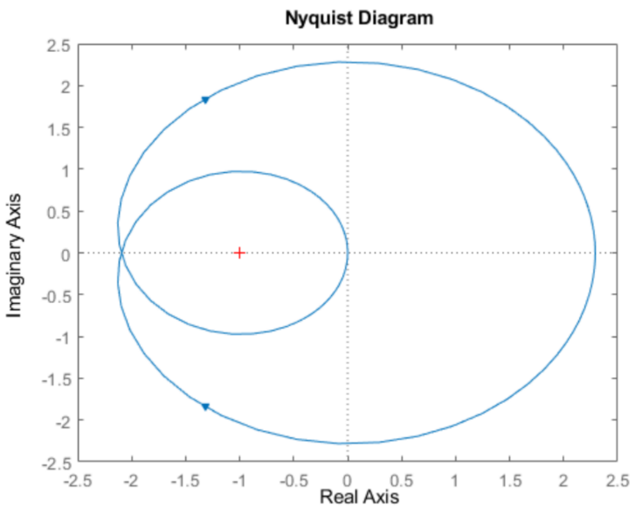

Fig 6. Nyquist diagram of system

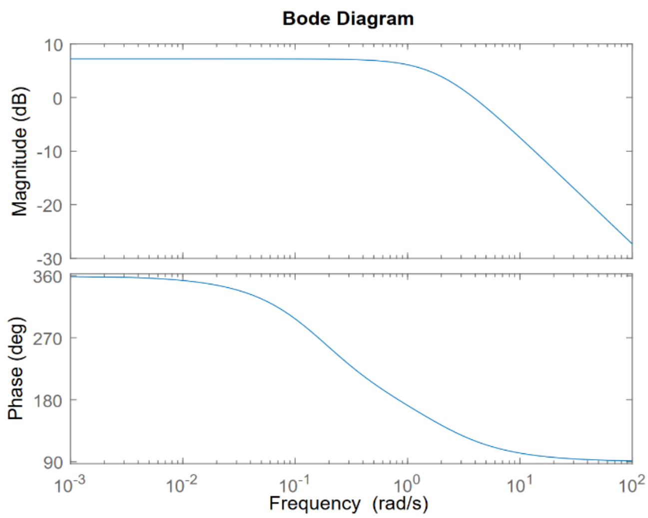

Fig 7. Bode diagram of system

CONCLUSION

In this study, our initial focus was on the examination of the mathematical expressions governing the reaction of the pupil of the eye to varying levels of illumination. Subsequently, we proceeded to linearize these equations, followed by conducting an analysis centered on linear control principles. This involved the determination of a transfer function, which, in turn, allowed us to assess the system's response to various input stimuli and ascertain its overall stability. Furthermore, we visually represented the system's behavior by generating root locus plots and performed frequency analysis utilizing Nyquist and Bode diagrams. In the spirit of open collaboration and knowledge sharing, we are committed to making all the codebase and associated resources for this project accessible to the community. We will be hosting the complete set of project codes on GitHub, ensuring that they are readily available for reference, contribution, and reuse by interested individuals and researchers.†

† *https://github.com/PARSA-MHMDI/Eye-Pupil-Control-Analysis.git*

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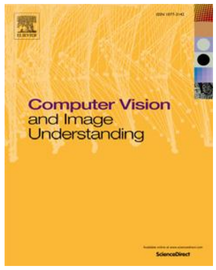

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