Assignment 1

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Google drive link: https://drive.google.com/drive/folders/170cx7JQqDXPP-MnM9p8Fs250gswOngJN?usp=sharing

The program can be executed using the following command:

`python3 A1.py <mode>`

`1D,mech_fail, GPS, IMU, BS, SIM_GPS, SIM_IMU, GPS_ELLIPSE, 2D_BS`. If an invalid mode is provided, the program prints the list of available modes.

1 State Estimation of 1D motion using Kalman Filters

Environment Setup and Filtering

a) State Representation

$$X_t = \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}$$

Parameters

$$\begin{aligned} V_s &= 3000 \text{ kmph} \\ Q &= \text{diag}(0.01, 0.25) \quad \text{(Motion noise)} \\ N &= 326 \\ \Delta t &= 0.01 \end{aligned}$$

Initial Belief

$$X_0 = 0$$
$$P_0 = 10^{-4} \ * \ I_2$$

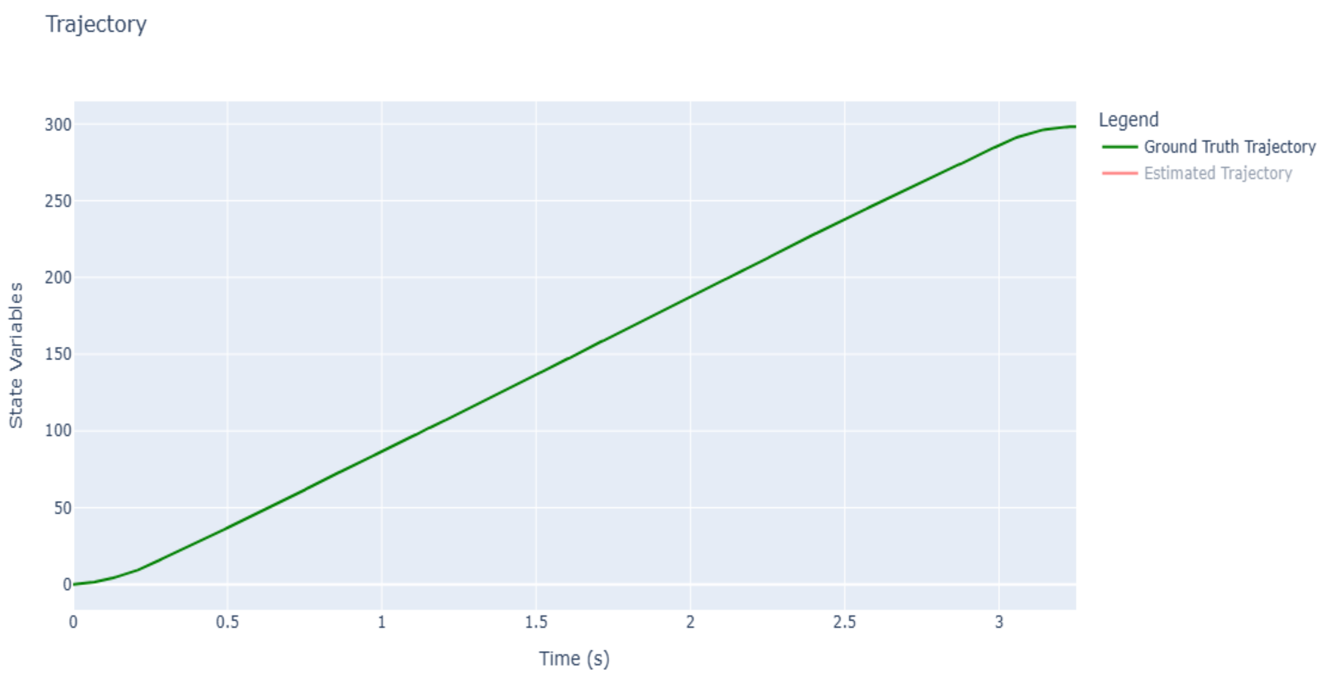
Motion Model

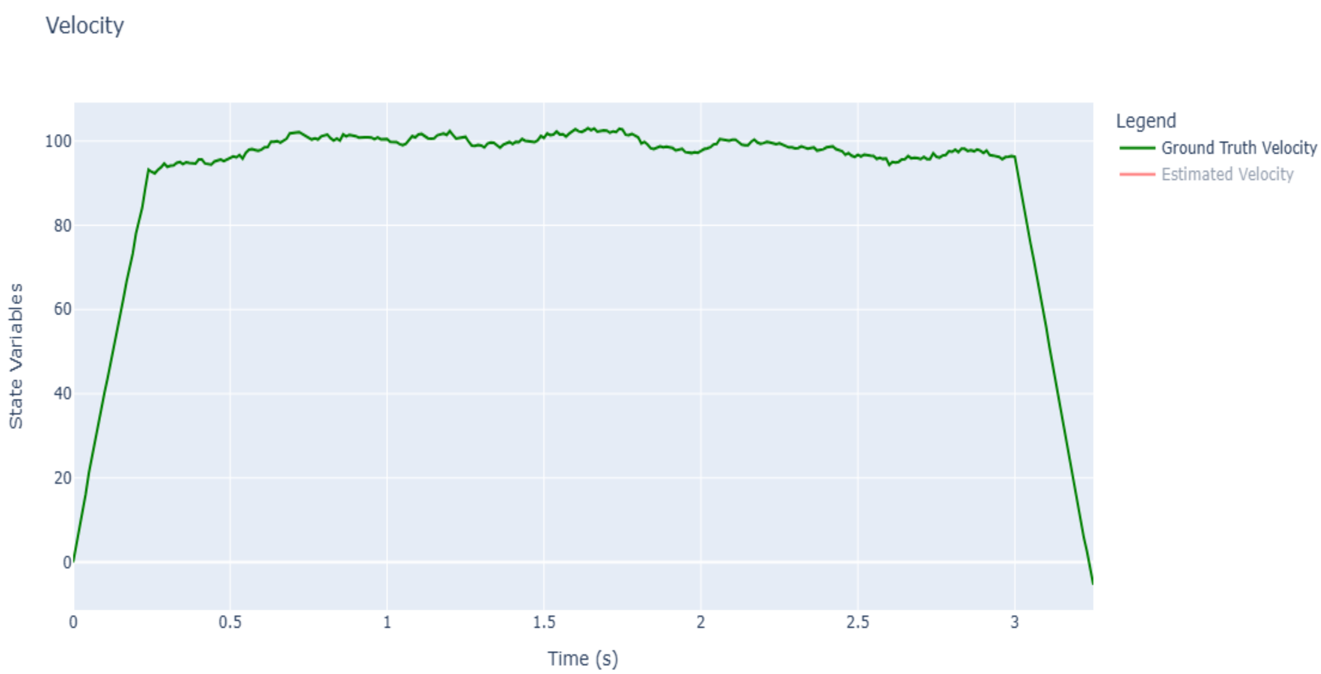
$$X_{t+1} = A_t X_t + B_t U_t + Motion\ Noise$$
$$A_t = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}$$
$$B_t = \begin{bmatrix} \frac{1}{2}\Delta t^2 \\ \Delta t \end{bmatrix}$$
$$U_t = \begin{cases} 400, & \text{if } t < 0.25 \\ -400, & \text{if } 3 < t < 3.25 \\ 0, & \text{otherwise} \end{cases}$$
$$P_{t+1} = A_t P_t A_t^T + Q$$

Observation Model

$$\text{Observation Noise} \sim \mathcal{N}(0, R), \quad \text{where } R = 0.0001$$

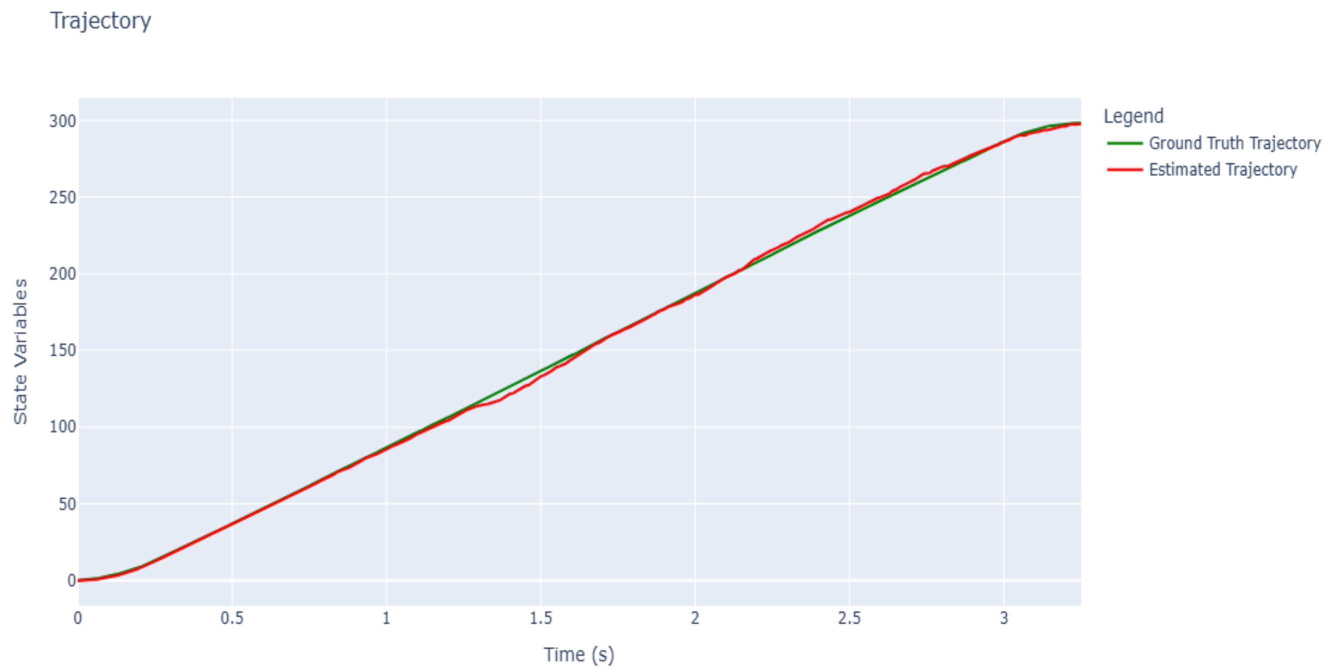
$$Z_t = C_t X_t + \text{Measurement Noise}$$
$$C_t = \begin{bmatrix} \frac{2}{V_s} \\ 0 \end{bmatrix}$$

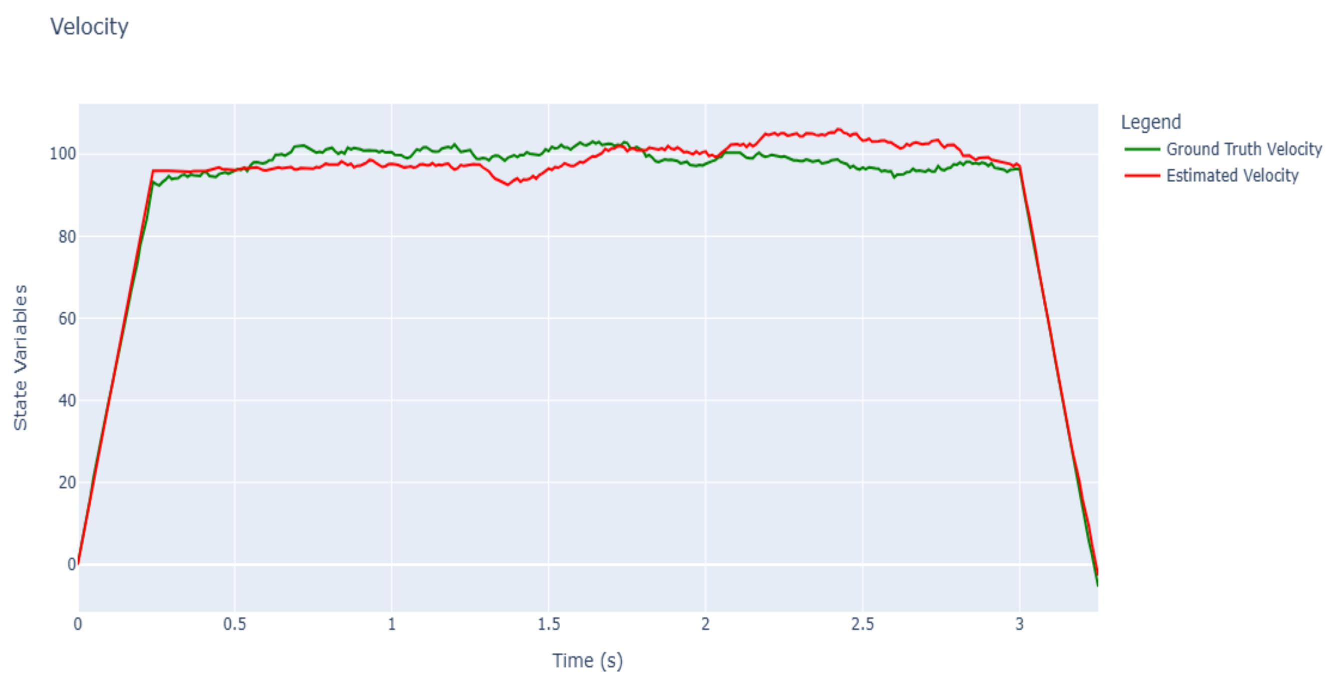




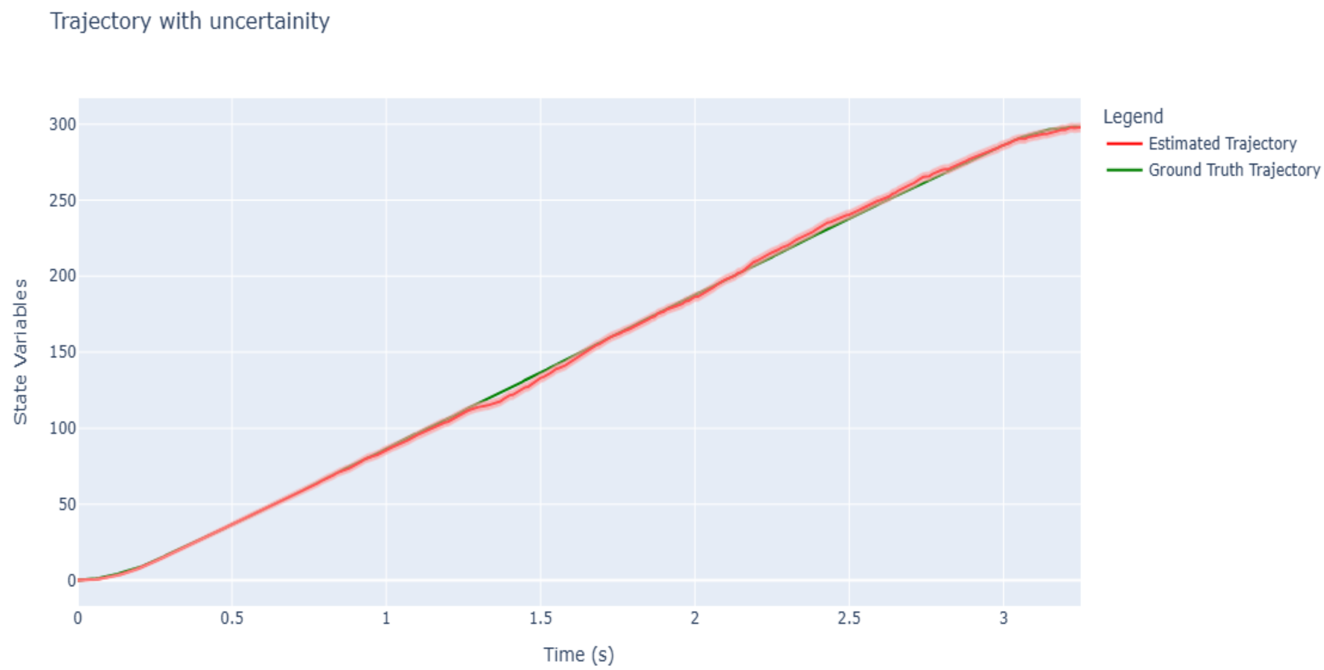
b) Kalman Gain

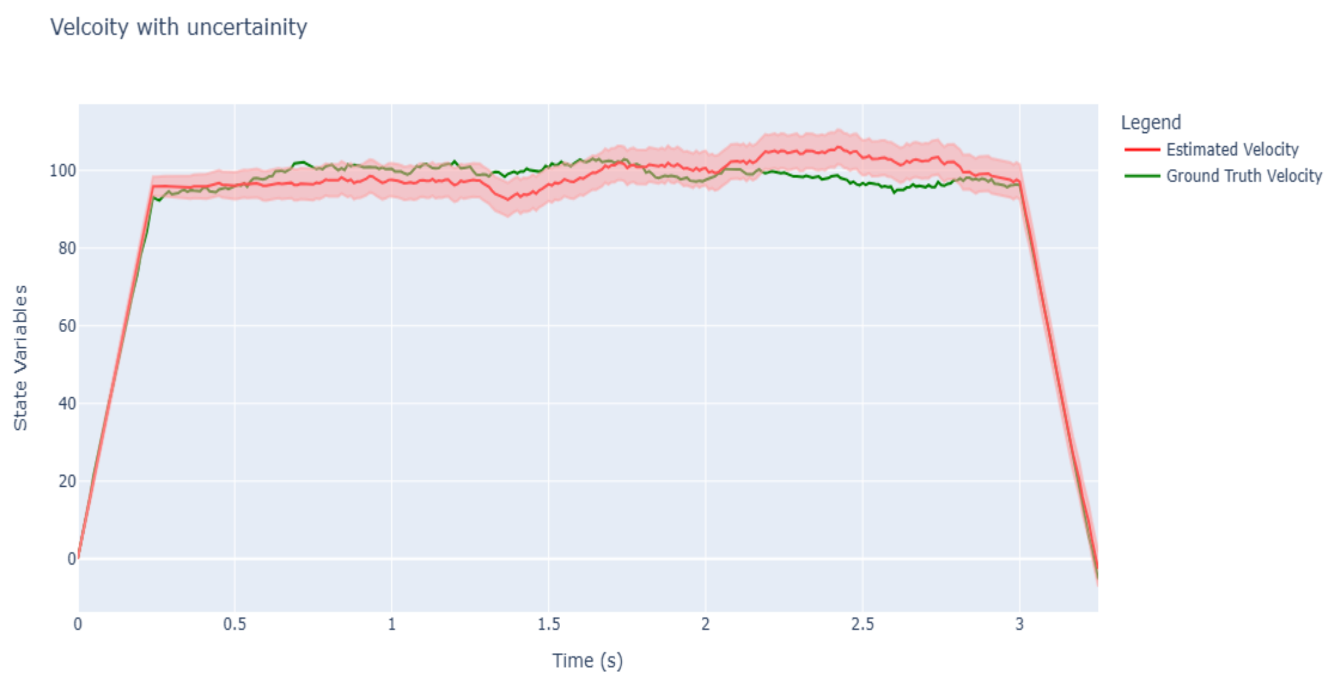
$$K_{\text{gain}} = P_{\text{pred}} C_t^T \left(C_t P_{\text{pred}} C_t^T + R \right)^{-1}$$





C) Joint Plots with Uncertainity

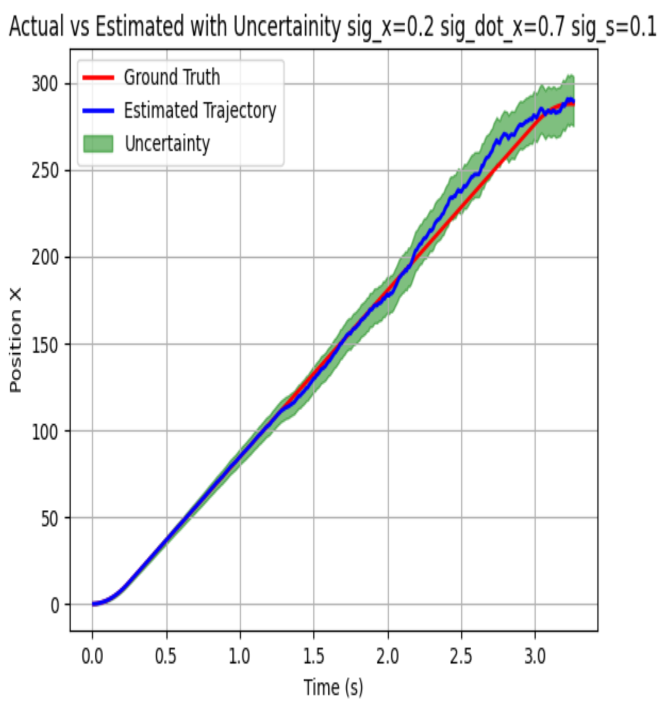


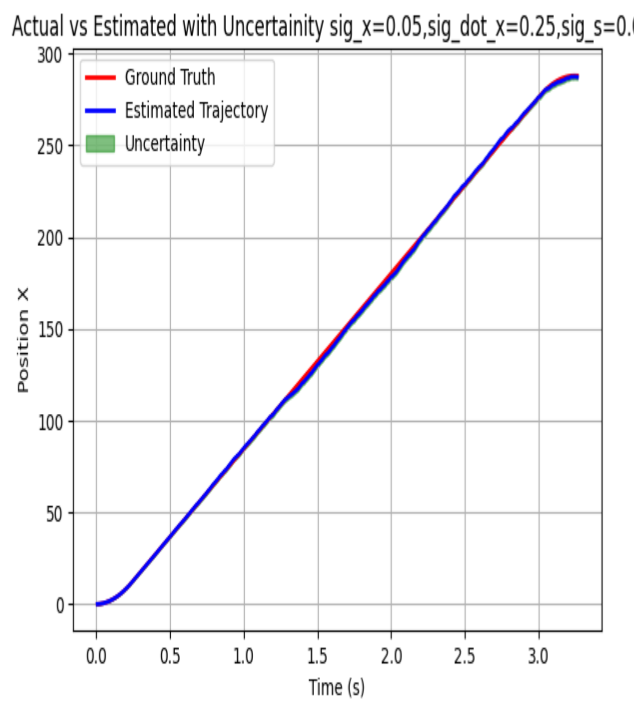


Experiments

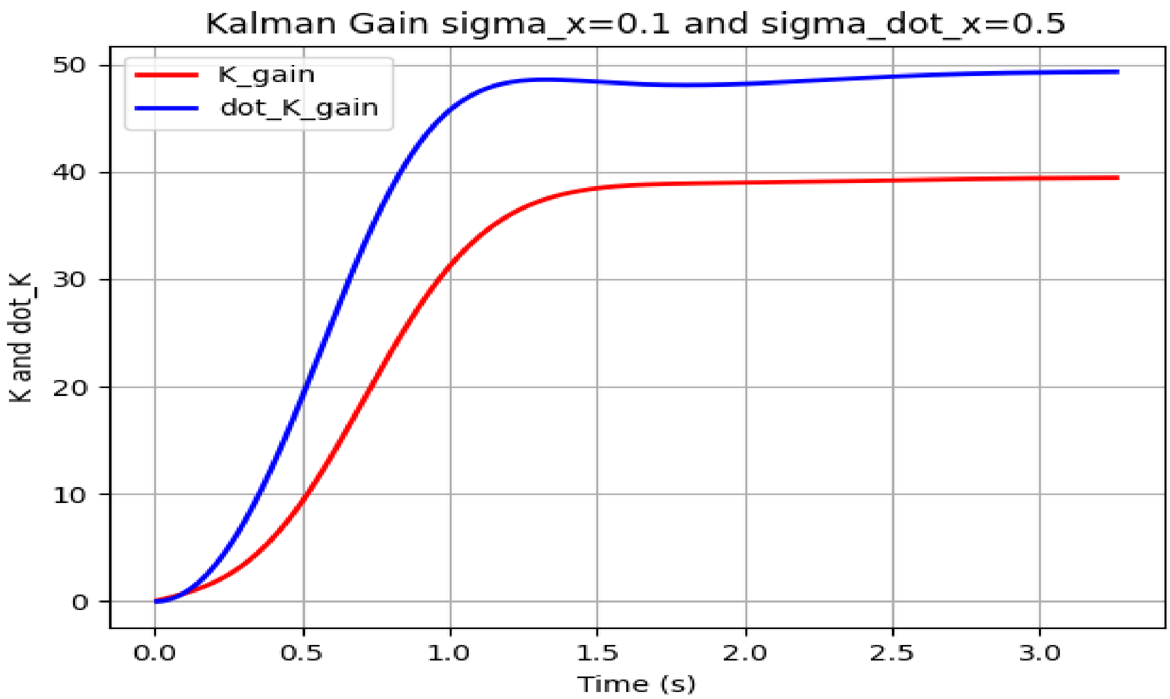
d)

- Increasing σ_x and $\sigma_{\dot{x}}$: The estimated trajectory deviates from the actual trajectory with high disturbances due to increased process noise.
- Increasing σ_x, $\sigma_{\dot{x}}$, and σ_s: The estimated trajectory deviates further, and uncertainty increases significantly due to both process and measurement noise being high.
- Decreasing σ_x, $\sigma_{\dot{x}}$, and σ_s: The estimated trajectory follows the actual trajectory closely with very minimal disturbances, as both process and sensor noises are reduced.
- Decreasing only σ_x and $\sigma_{\dot{x}}$: The estimated trajectory becomes smoother but is still affected by sensor noise, leading to moderate deviations.
- Increasing only σ_s: The estimated trajectory fluctuates more, and uncertainty increases, but the mean trajectory might still follow the actual path reasonably well.
- Decreasing only σ_s: The estimated trajectory becomes more accurate as measurement noise is lower, even if process noise remains constant.

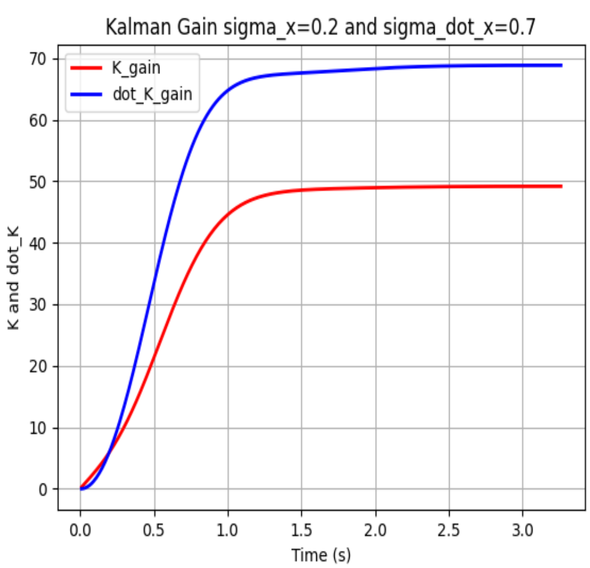


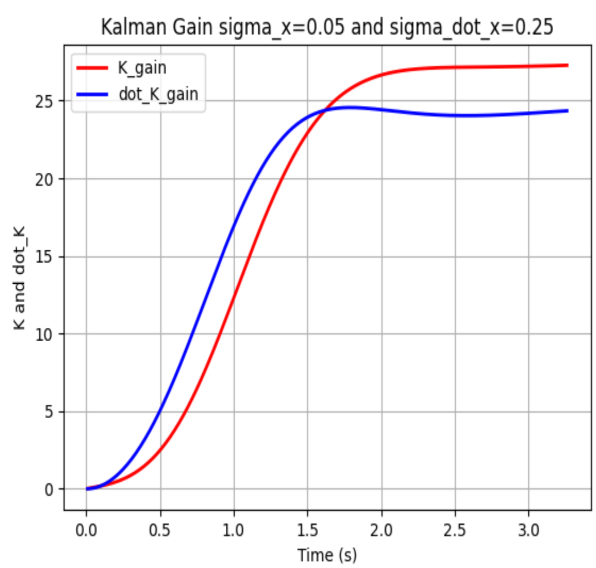


e) Kalman Gain Plots

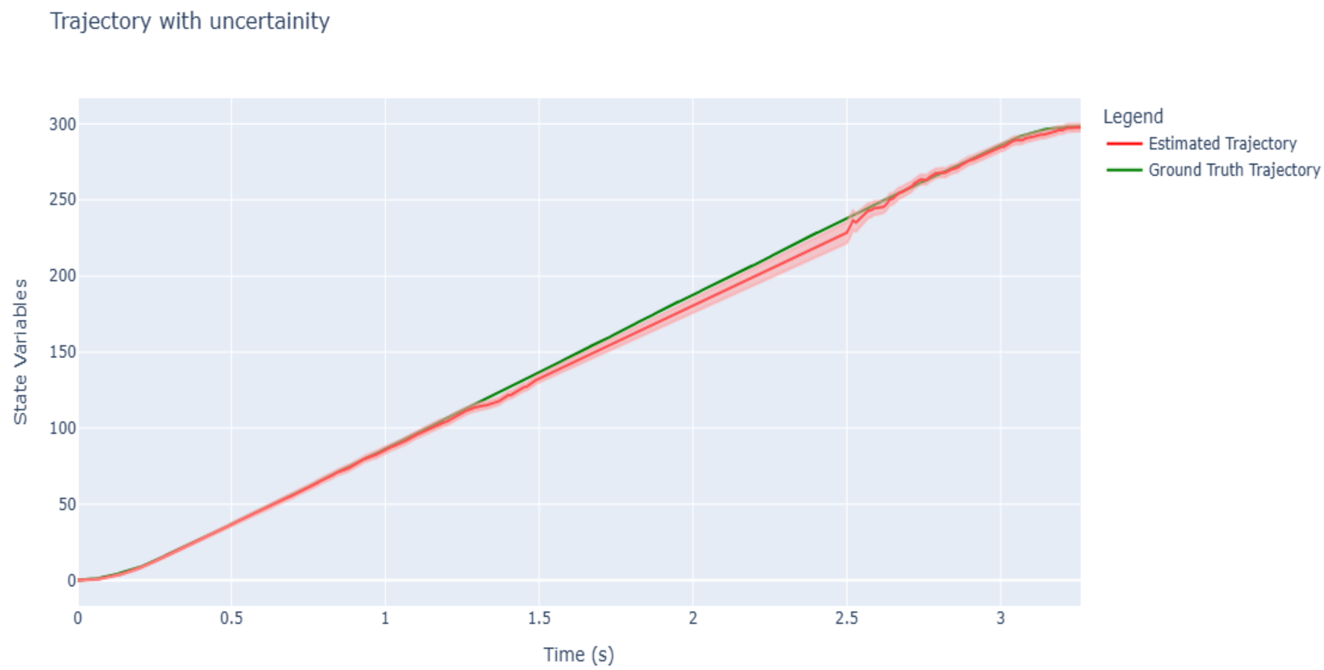


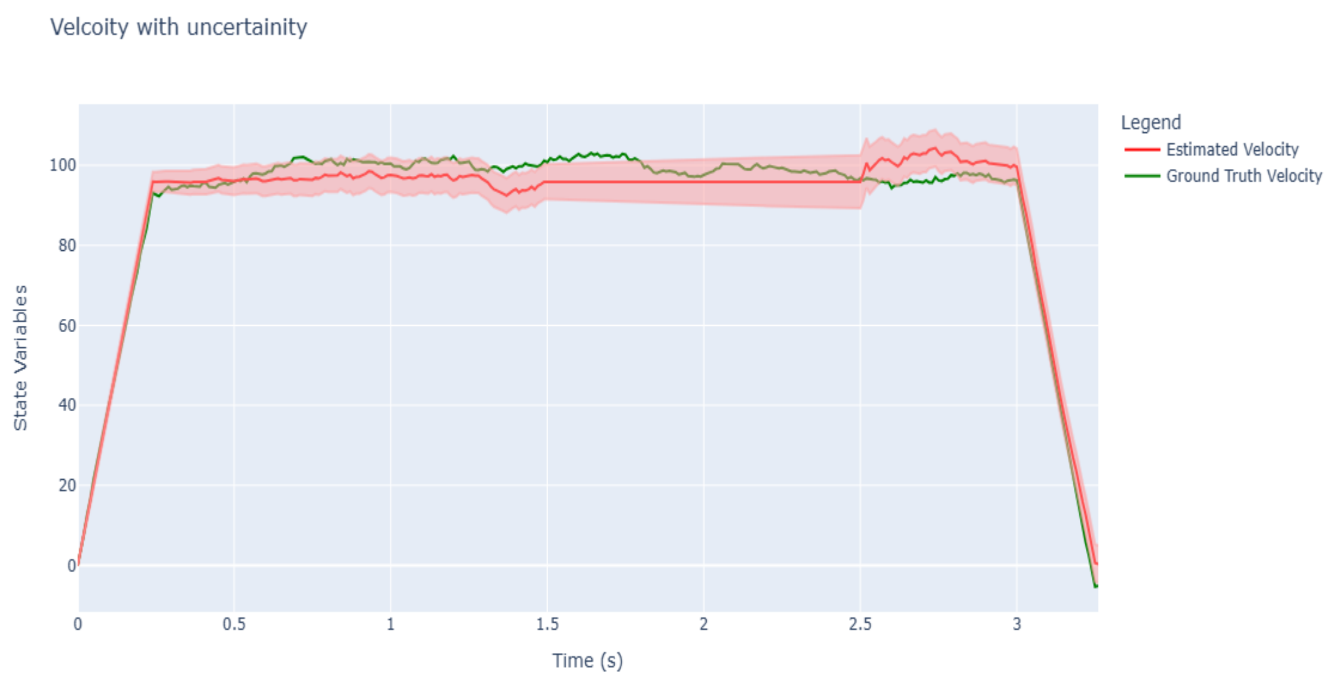
- Kalman gain increases with an increase in variance. With more noise, stronger correction is required. So, K_{gain} will be relatively high.
- Kalman gain decreases with a decrease in variance. Less noise means it is closer to the ground truth, which implies that a smaller correction is required. So, K_{gain} will be relatively low.





f) Mechanical Fail in [1.5,2.5]





2 State Estimation of 3D motion using Kalman Filters

Environment Setup and Filtering

a)

$$\mathbf{X} = [x, y, z, v_x, v_y, v_z]^T$$

Assuming a time step of $\Delta t = 0.01$ seconds (100 Hz update rate), the state transition model is given by:

$$A_t = \begin{bmatrix} 1 & 0 & 0 & \Delta t & 0 & 0 \\ 0 & 1 & 0 & 0 & \Delta t & 0 \\ 0 & 0 & 1 & 0 & 0 & \Delta t \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The control input matrix is:

$$B_t = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}\Delta t^2 \\ 0 \\ 0 \\ \Delta t \end{bmatrix}$$

where U_t is the control input:

$$U_t = \begin{bmatrix} g \end{bmatrix}$$

The predicted state is computed as:

$$X_{\text{pred}} = A_t X + B_t U_t + \text{motion noise}$$

The predicted covariance is:

$$P_{\text{pred}} = A_t P A_t^T + Q$$

b)

Observation Model - GPS

The observation matrix for GPS is:

$$C_t^{\text{GPS}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The measurement noise covariance matrix is:

$$R_{\text{GPS}} = 0.01 I_3$$

where I_3 is a 3×3 identity matrix.
The GPS measurement update equation is:

$$Z_t^{\text{GPS}} = C_t^{\text{GPS}} X_{\text{pred}} + \text{measurement noise}$$

where the measurement noise is drawn from a multivariate normal distribution:

$$\text{measurement noise} \sim \mathcal{N}(0, R_{\text{GPS}})$$

The innovation covariance matrix is:

$$S = C_t^{\text{GPS}} P_{\text{pred}} C_t^{\text{GPS}T} + R_{\text{GPS}}$$

The Kalman gain for GPS is computed as:

$$K_{\text{gain}}^{\text{GPS}} = P_{\text{pred}} C_t^{\text{GPS}T} S^{-1}$$

Observation Model - IMU

The observation matrix for IMU is:

$$C_t^{\text{IMU}} = I_6$$

where I_6 is a 6×6 identity matrix, meaning that the IMU provides full-state measurements.
The measurement noise covariance for IMU is:

$$R_{\text{IMU}} = 0.01 I_6$$

The IMU measurement update equation is:

$$Z_t^{\text{IMU}} = C_t^{\text{IMU}} X_{\text{pred}} + \text{measurement noise}$$

where the measurement noise is drawn from:

$$\text{measurement noise} \sim \mathcal{N}(0, R_{\text{IMU}})$$

The innovation covariance matrix for IMU is:

$$S = C_t^{\text{IMU}} P_{\text{pred}} C_t^{\text{IMU}T} + R_{\text{IMU}}$$

The Kalman gain for IMU is:

$$K_{\text{gain}}^{\text{IMU}} = P_{\text{pred}} C_t^{\text{IMU}T} S^{-1}$$

Observation Model for Base Station Case

We have four base stations that provide range measurements to estimate the trajectory of a moving object (e.g., a football). The measurement model is nonlinear because the measurements are based on Euclidean distances. Therefore, we use the **Extended Kalman Filter (EKF)**, which linearizes the model using the Jacobian matrix.

Measurement Model

The true range measurement from the i-th base station located at $\mathbf{p}_i = (x_i, y_i, z_i)$ to the object with state

$$\mathbf{X} = [x, y, z, v_x, v_y, v_z]^T$$

is given by:

$$D_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2} \tag{1}$$

where D_i is the expected (true) distance from the base station to the object.
Since we have four base stations, the measurement vector $\mathbf{Z}_t$ is:

$$\mathbf{Z}_t = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} + \mathbf{v} \tag{2}$$

where $\mathbf{v} \sim \mathcal{N}(0, \mathbf{R})$ represents measurement noise modeled as Gaussian with covariance $\mathbf{R}$.

Jacobian Matrix H

Since the measurement function is nonlinear, we compute the Jacobian matrix H, which represents the partial derivatives of the range measurements with respect to the state:

$$H_i = \begin{bmatrix} \frac{\partial D_i}{\partial x} & \frac{\partial D_i}{\partial y} & \frac{\partial D_i}{\partial z} & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

For each base station i, the derivatives are:

$$\frac{\partial D_i}{\partial x} = \frac{x - x_i}{D_i}, \quad \frac{\partial D_i}{\partial y} = \frac{y - y_i}{D_i}, \quad \frac{\partial D_i}{\partial z} = \frac{z - z_i}{D_i} \tag{4}$$

Thus, the full Jacobian matrix H for all four base stations is:

$$H = \begin{bmatrix} \frac{x-x_1}{D_1} & \frac{y-y_1}{D_1} & \frac{z-z_1}{D_1} & 0 & 0 & 0 \\ \frac{x-x_2}{D_2} & \frac{y-y_2}{D_2} & \frac{z-z_2}{D_2} & 0 & 0 & 0 \\ \frac{x-x_3}{D_3} & \frac{y-y_3}{D_3} & \frac{z-z_3}{D_3} & 0 & 0 & 0 \\ \frac{x-x_4}{D_4} & \frac{y-y_4}{D_4} & \frac{z-z_4}{D_4} & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

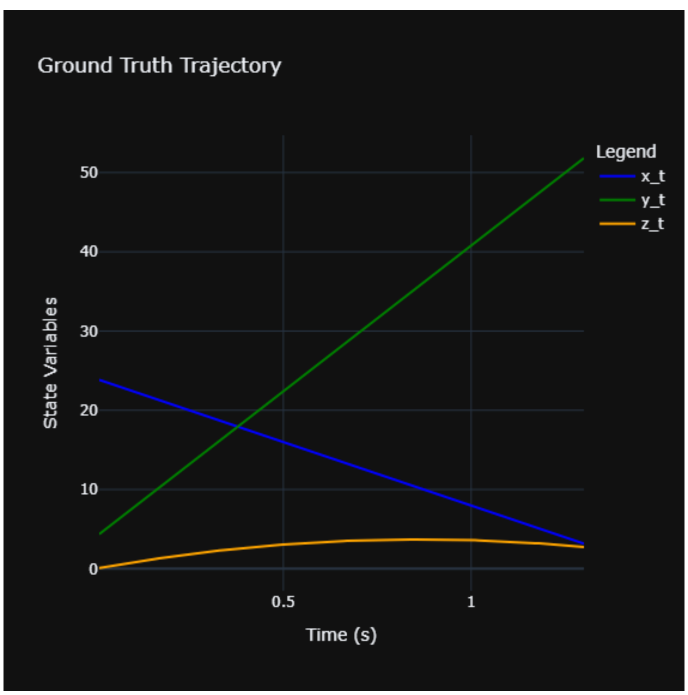
EKF Correction Step

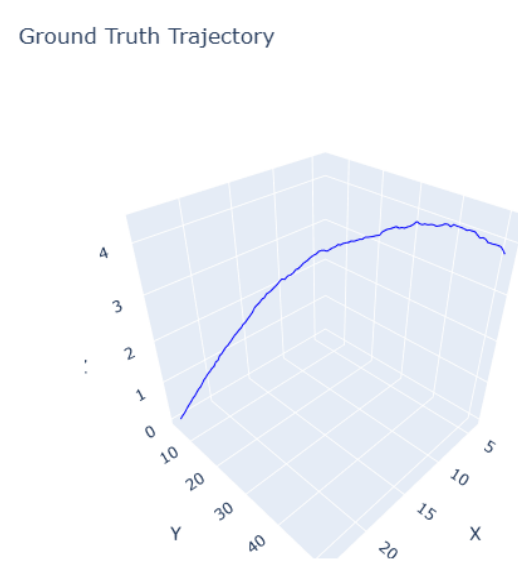
The EKF correction step involves three main computations:

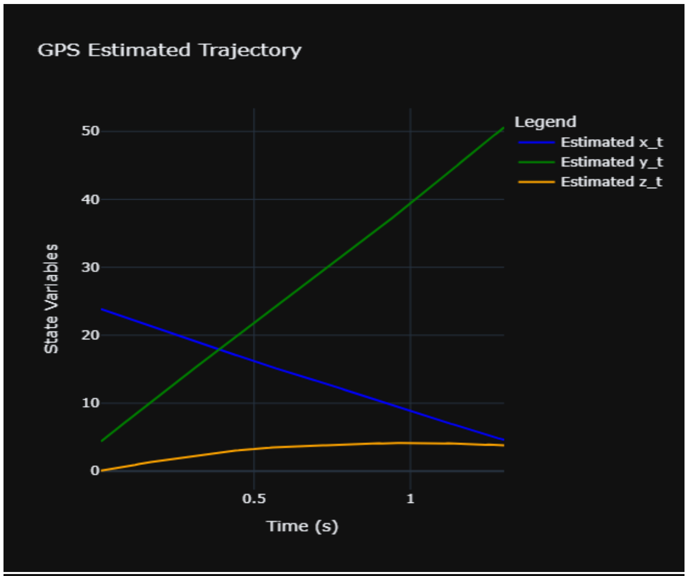
1. **Compute Kalman Gain:**
$$S = HP_{\text{pred}}H^T + R \tag{6}$$
$$K = P_{\text{pred}}H^T S^{-1} \tag{7}$$

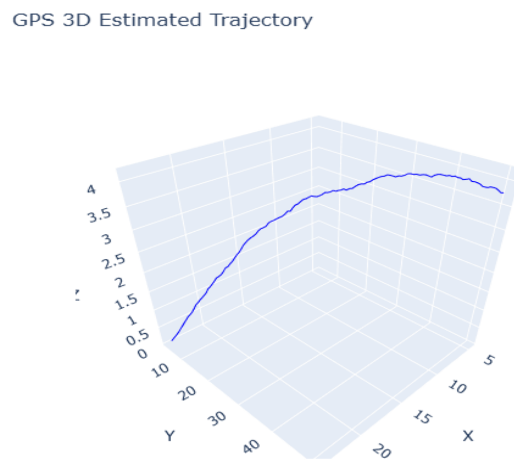
2. **Update State Estimate:**
$$X_{\text{corrected}} = X_{\text{pred}} + K(Z_{\text{BS}} - D) \tag{8}$$

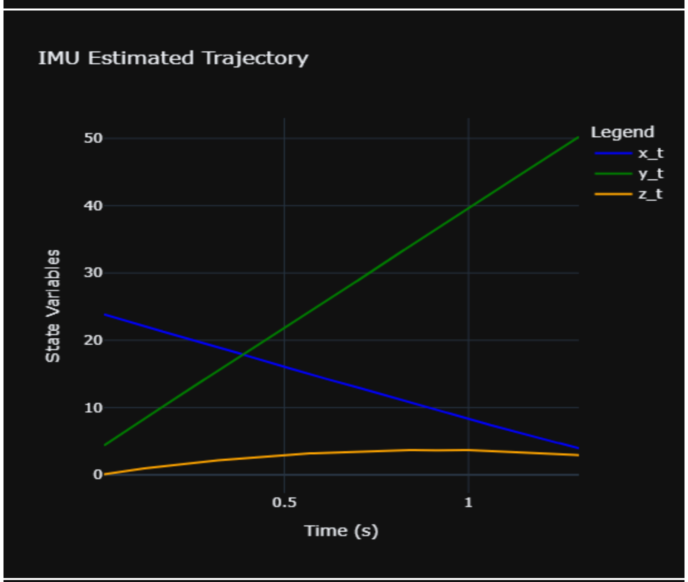
3. **Update Covariance:**
$$P_{\text{corrected}} = (I - KH)P_{\text{pred}} \tag{9}$$

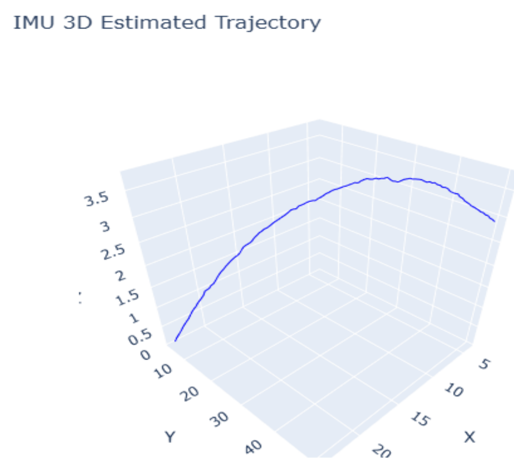


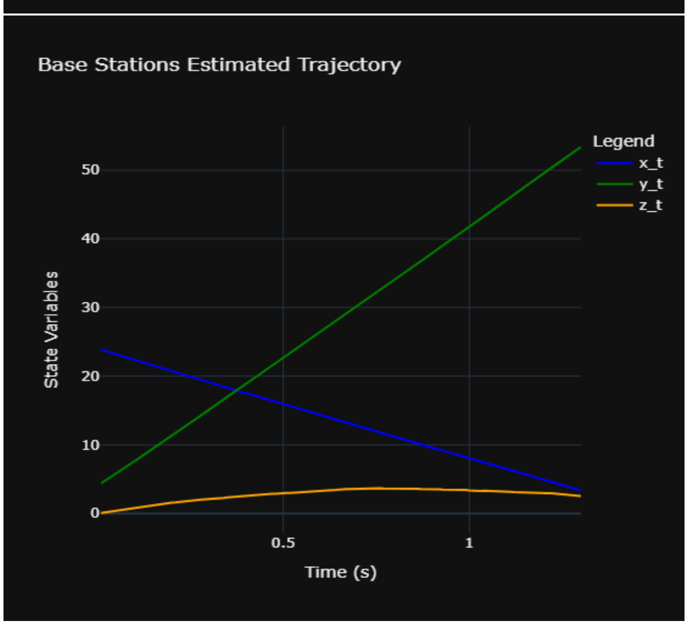


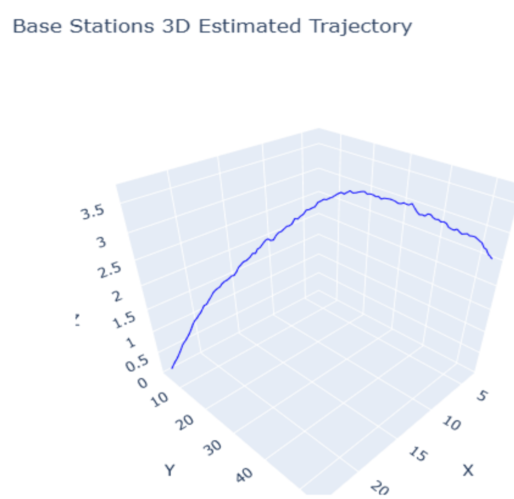

GPS Estimated Trajectory
Legend
Estimated x_t
Estimated y_t
Estimated z_t
State Variables
Time (s)
0
10
20
30
40
50
0.5
1


GPS 3D Estimated Trajectory
Y
X
0
0.5
1
1.5
2
2.5
3
3.5
4
10
20
30
40
5
10
15
20


IMU Estimated Trajectory
Legend
x_t
y_t
z_t
State Variables
Time (s)
0
10
20
30
40
50
0.5
1


IMU 3D Estimated Trajectory
Y
X
0
0.5
1
1.5
2
2.5
3
3.5
10
20
30
40
5
10
15
20

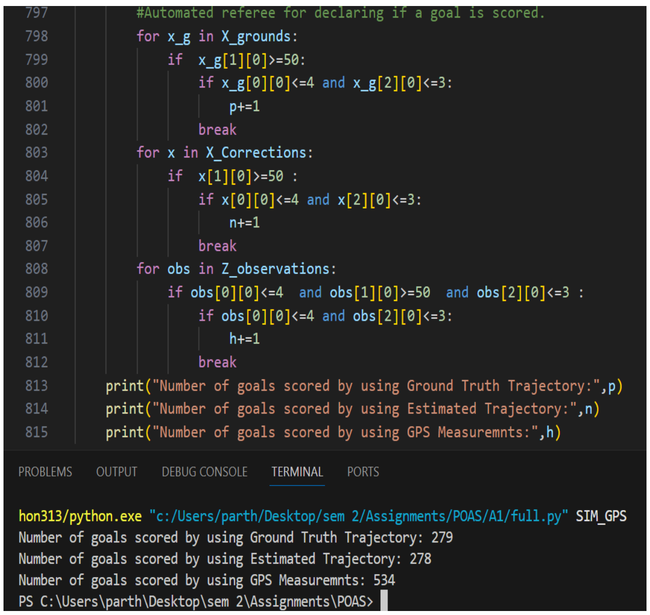

Base Stations Estimated Trajectory
Legend
x_t
y_t
z_t
State Variables
Time (s)
0
10
20
30
40
50
0.5
1

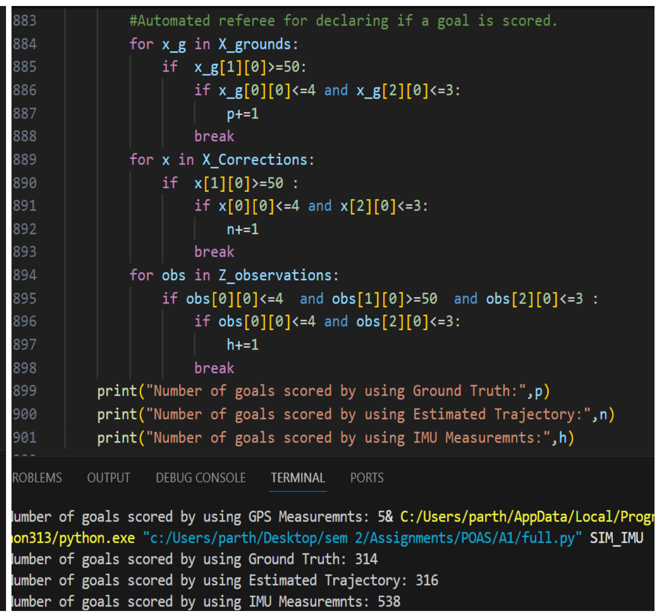

Base Stations 3D Estimated Trajectory
Y
X
0
0.5
1
1.5
2
2.5
3
3.5
10
20
30
40
5
10
15
20

c) Fraction of Goals scored

Method	Sensor	Goals Scored
Ground Truths	GPS	279
Estimates		278
Measurements		534
Ground Truths	IMU	314
Estimates		316
Measurements		538

Table 1: Goals Scored out of 1000 simulations





d) Change in σ's

Impact of Noise on State Estimation and Goal fraction

The effect of varying noise parameters on trajectory estimation and goal-scoring performance was analyzed through 1,000 simulations. The results indicate a strong correlation between noise levels and the accuracy of state estimation, directly influencing trajectory prediction and overall success.

Key Observations

- **Lower Position and Velocity Noise Improves Performance:** Reducing position and velocity noise leads to more accurate trajectory estimates, resulting in improved goal-scoring performance. The best results were observed when $\sigma_x = \sigma_y = \sigma_z = 0.005$ and $\dot{\sigma}_x = \dot{\sigma}_y = \dot{\sigma}_z = 0.05$, yielding the highest goal fraction.

- **Excessive Noise Degrades Performance:** Increasing noise in position and velocity estimates reduces the reliability of trajectory predictions, leading to a lower goal fraction. Higher values, such as $\sigma_x = \sigma_y = \sigma_z = 0.02$ and $\dot{\sigma}_x = \dot{\sigma}_y = \dot{\sigma}_z = 0.2$, resulted in significant performance degradation.

Noise Parameters	Method	Goal Fraction
$\sigma_x = \sigma_y = \sigma_z = 0.01$	Ground Truths	279
$\sigma_{\dot{x}} = \sigma_{\dot{y}} = \sigma_{\dot{z}} = 0.1$	Estimates	278
$\sigma_s = 0.1$	Measurements	534
$\sigma_x = \sigma_y = \sigma_z = 0.005$	Ground Truths	341
$\sigma_{\dot{x}} = \sigma_{\dot{y}} = \sigma_{\dot{z}} = 0.05$	Estimates	342
$\sigma_s = 0.05$	Measurements	730
$\sigma_x = \sigma_y = \sigma_z = 0.02$	Ground Truths	200
$\sigma_{\dot{x}} = \sigma_{\dot{y}} = \sigma_{\dot{z}} = 0.2$	Estimates	198
$\sigma_s = 0.2$	Measurements	343
$\sigma_x = \sigma_y = \sigma_z = 0.02$	Ground Truths	200
$\sigma_{\dot{x}} = \sigma_{\dot{y}} = \sigma_{\dot{z}} = 0.2$	Estimates	199
$\sigma_s = 0.1$	Measurements	324
$\sigma_x = \sigma_y = \sigma_z = 0.01$	Ground Truths	279
$\sigma_{\dot{x}} = \sigma_{\dot{y}} = \sigma_{\dot{z}} = 0.1$	Estimates	296
$\sigma_s = 0.5$	Measurements	657
$\sigma_x = \sigma_y = \sigma_z = 0.01$	Ground Truths	279
$\sigma_{\dot{x}} = \sigma_{\dot{y}} = \sigma_{\dot{z}} = 0.1$	Estimates	282
$\sigma_s = 0.05$	Measurements	518

Table 2: Effect of Noise Parameters on Goal Fraction

- **Significance of Sensor Noise:** The sensor noise parameter σ_s has a notable impact on state estimation accuracy. Larger values introduce additional uncertainty, leading to incorrect trajectory predictions. For instance, an increase in σ_s to 0.5 led to a decline in performance, even with low position and velocity noise.

- **Optimized Noise Levels:** The best overall performance was achieved with $\sigma_x = \sigma_y = \sigma_z = 0.005$, $\dot{\sigma}_x = \dot{\sigma}_y = \dot{\sigma}_z = 0.05$, and $\sigma_s = 0.05$, resulting in the highest goal fraction.

Maintaining low noise levels in position and velocity estimates is essential for accurate state estimation and optimal performance. Additionally, minimizing sensor noise is crucial to prevent measurement distortions. The results emphasize the importance of carefully tuning noise parameters to achieve reliable trajectory estimation and improved success in dynamic environments.
Other images Uploaded in drive: Google Drive Link

e)Effect of Process and Observation Noise on Uncertainty Ellipses

Effect of Process Noise ($\sigma_x, \sigma_y, \sigma_z$ in Q)

- **Low Process Noise:** The uncertainty ellipses remain small and tight, indicating high confidence in the estimates.

- **High Process Noise:** The ellipses become larger and elongated, representing increased uncertainty. The system relies more on observations for correction.

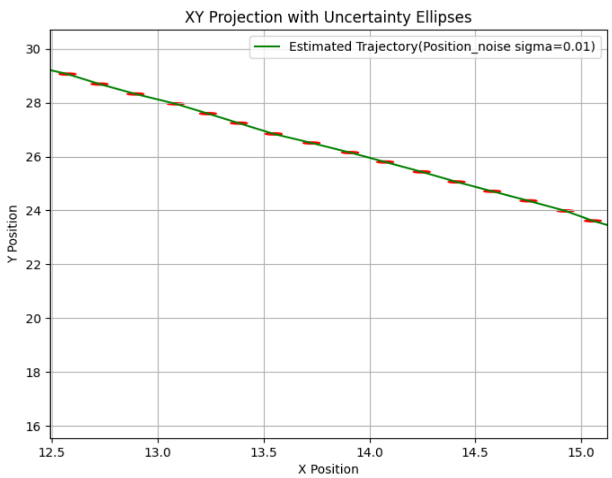
Effect of Observation Noise (σ_{GPS} in R_{GPS})

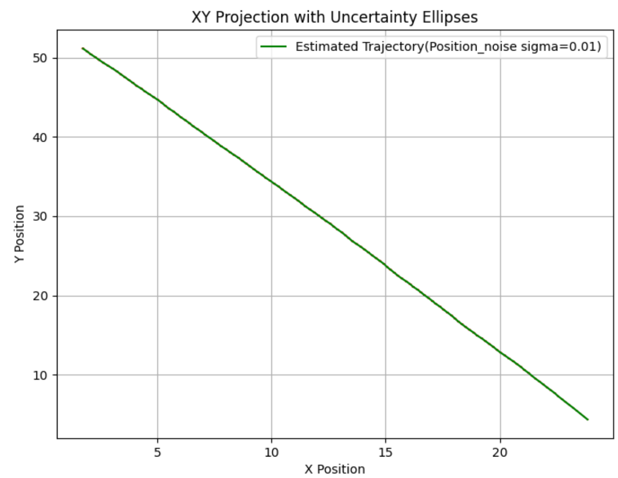
- **Low Observation Noise:** The uncertainty ellipses remain small since the measurements are reliable.
- **High Observation Noise:** The ellipses become larger, reflecting lower confidence in the measurements, making the filter depend more on predictions.

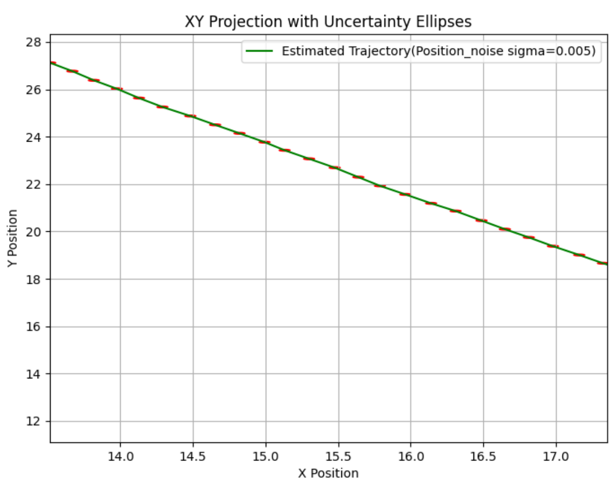
Combined Effects (High Q & High R_{GPS})

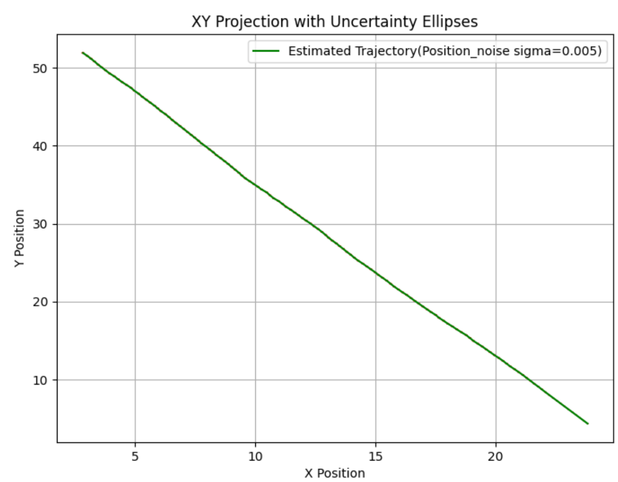
- When both process and observation noise are high, the uncertainty ellipses become significantly larger and erratic, indicating reduced confidence in state estimation.

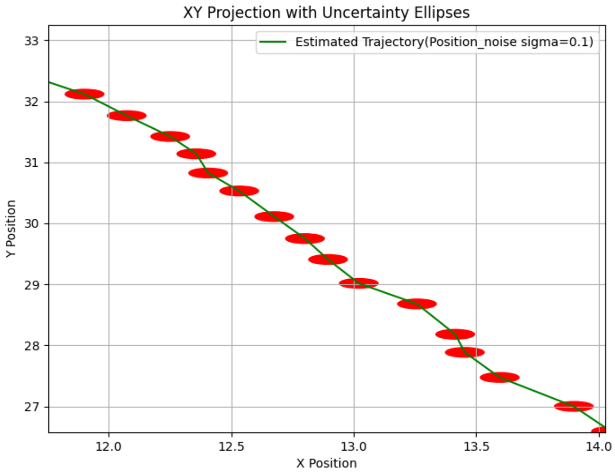
Summary: Larger ellipses indicate higher uncertainty, while smaller ellipses show greater confidence in estimates. Increasing process noise (Q) reduces the goal fraction, whereas increasing observation noise (R_{GPS}) reduces the reliability of corrections.

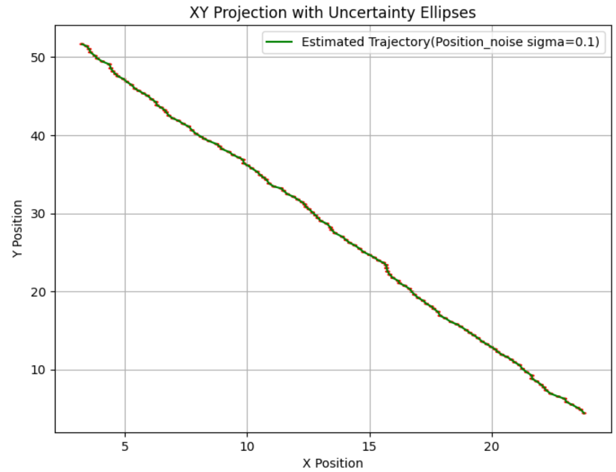


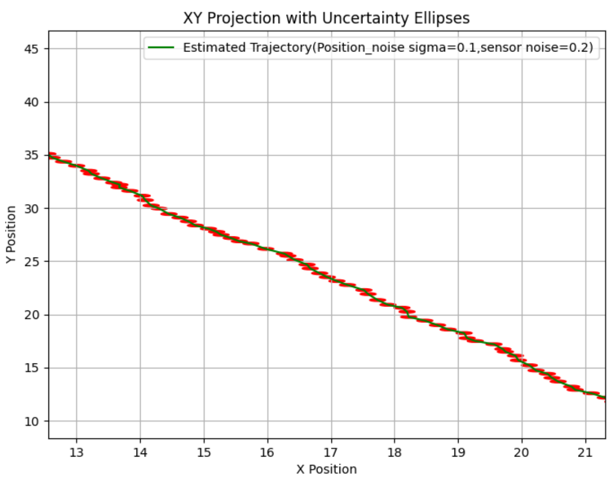


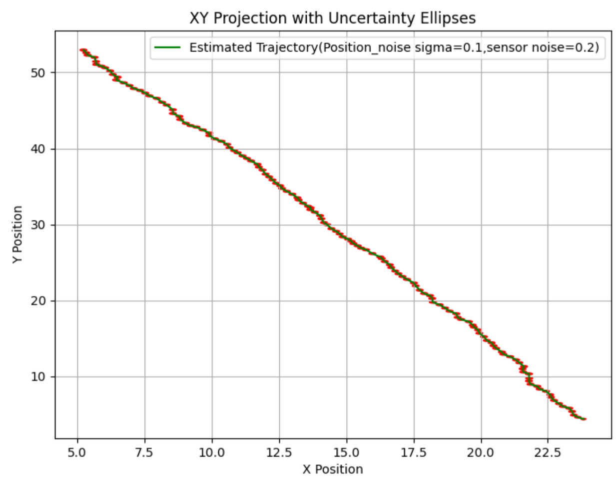





XY Projection with Uncertainty Ellipses
Estimated Trajectory(Position_noise sigma=0.1)
X Position
Y Position

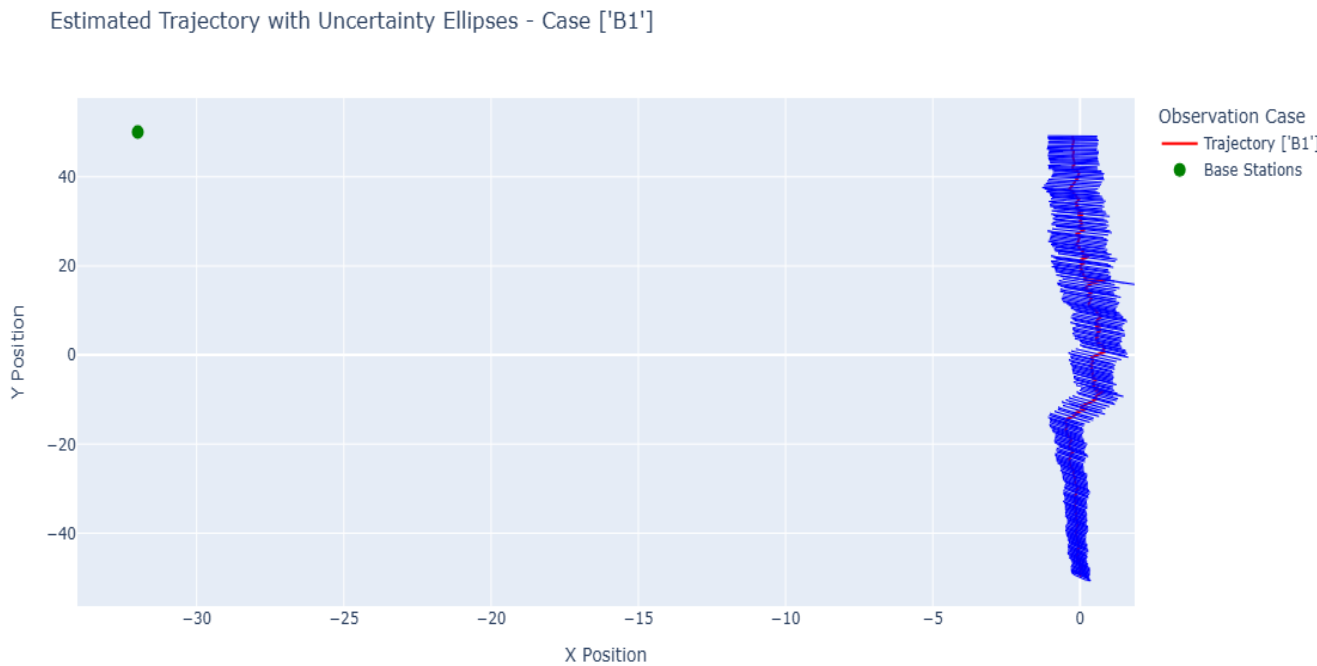

XY Projection with Uncertainty Ellipses
Estimated Trajectory(Position_noise sigma=0.1)
X Position
Y Position

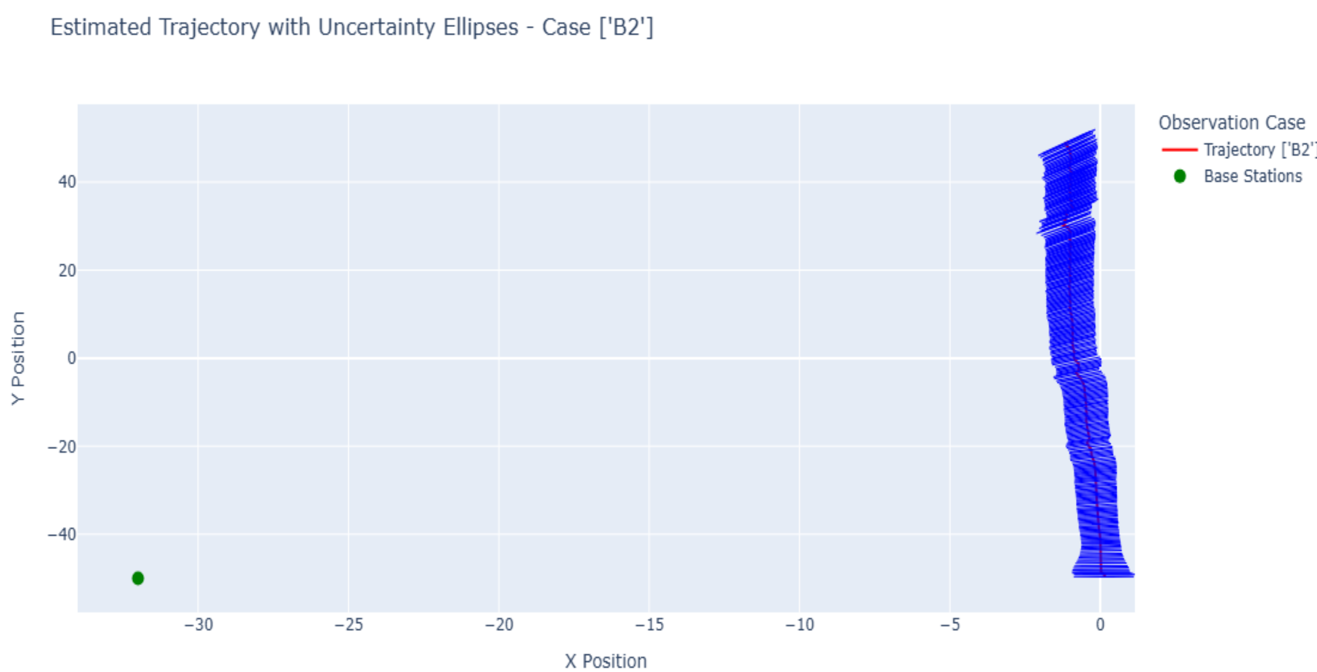

XY Projection with Uncertainty Ellipses
Estimated Trajectory(Position_noise sigma=0.1,sensor noise=0.2)
X Position
Y Position

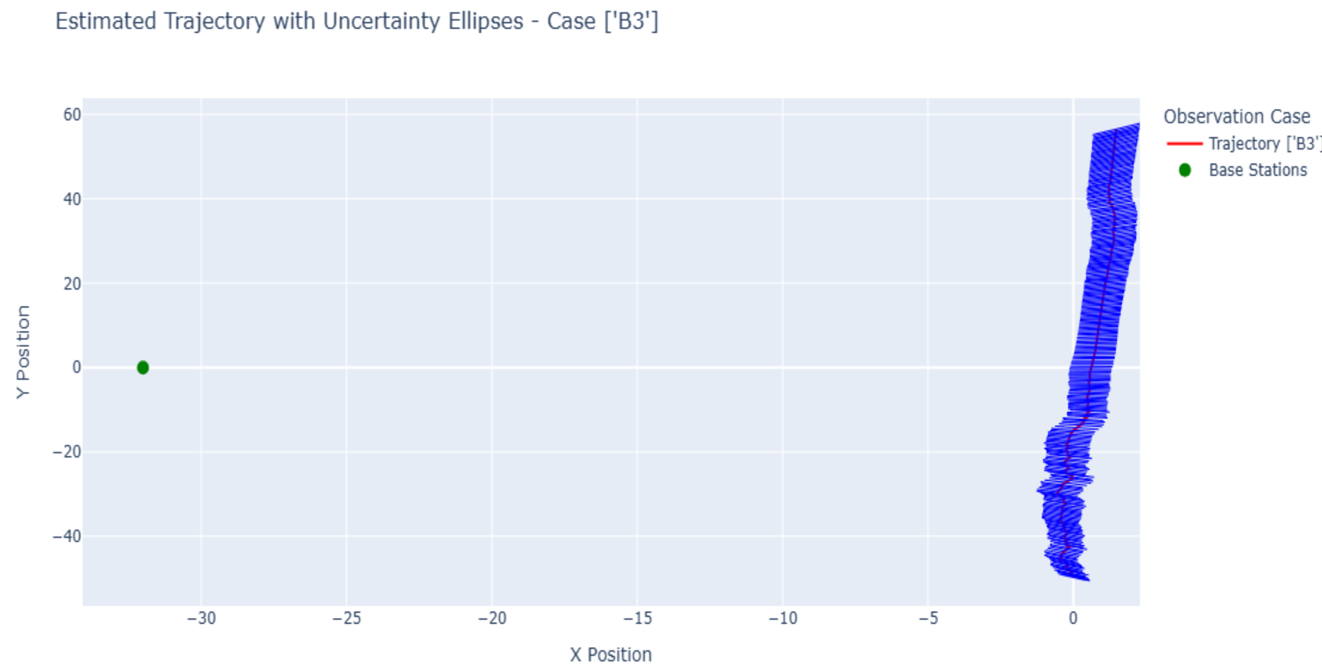

XY Projection with Uncertainty Ellipses
Estimated Trajectory(Position_noise sigma=0.1,sensor noise=0.2)
X Position
Y Position

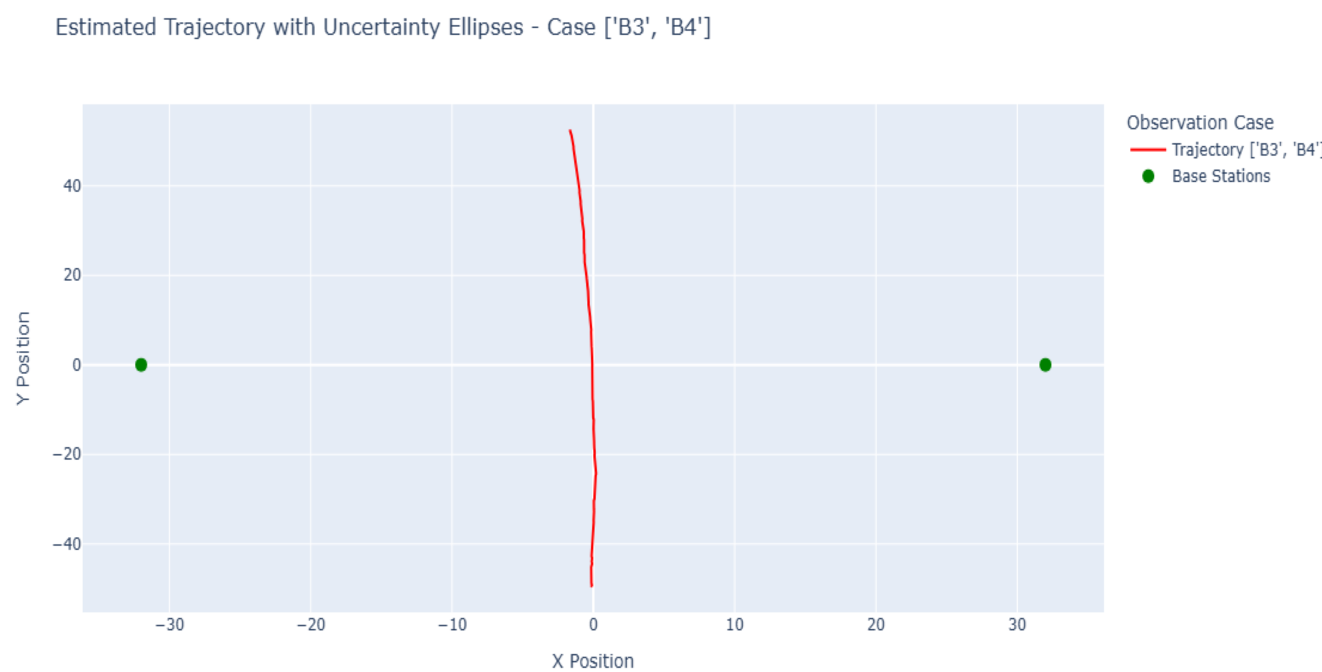
f) 2D Case of Base Stations

1)









2)

When measurements are taken from two base stations, the uncertainty ellipses are small, indicating higher accuracy. However, when only one base station is used, the ellipses become elongated perpendicular to the trajectory due to uncertainty in the unmeasured direction. They are oriented along the lines drawn from the base station to the point on the trajectory.

3)


Major and Minor Axis Lengths of Uncertainty Ellipses vs Time
Axis Length
Time (s)
0
1
2
3
4
5
6
0
0.5
1
1.5
2
Observation Cases
Major Axis ['B1']
Minor Axis ['B1']
Major Axis ['B2']
Minor Axis ['B2']
Major Axis ['B3']
Minor Axis ['B3']
Major Axis ['B3', 'B4']
Minor Axis ['B3', 'B4']