⇛ **Problem Example:**

Sample for ω_1: ⇛ $X_1 = (x_1, x_2) = \{(4,2), (2,4), (2,3), (3,6), (4,4)\}$

Sample for ω_2: ⇛ $X_2 = (x_1, x_2) = \{(9,10), (6,8), (9,5), (8,7), (10,8)\}$

now according to the steps:

① two class means are represented by μ_1 & μ_2.

so,
$$\mu_1 = \frac{1}{5}\left[\begin{pmatrix}4\\2\end{pmatrix} + \begin{pmatrix}2\\4\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} + \begin{pmatrix}3\\6\end{pmatrix} + \begin{pmatrix}4\\4\end{pmatrix}\right] = \begin{bmatrix}3\\3.8\end{bmatrix}$$

$$\mu_2 = \frac{1}{5}\left[\begin{pmatrix}9\\10\end{pmatrix} + \begin{pmatrix}6\\8\end{pmatrix} + \begin{pmatrix}9\\5\end{pmatrix} + \begin{pmatrix}8\\7\end{pmatrix} + \begin{pmatrix}10\\8\end{pmatrix}\right] = \begin{bmatrix}8.4\\7.6\end{bmatrix}$$

T.O

1st class

$$m_1 = \left[\begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}4\\2\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]^T \quad m_4 = \left[\begin{pmatrix}3\\6\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}3\\6\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]^T$$

$$m_2 = \left[\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]^T \quad m_5 = \left[\begin{pmatrix}4\\4\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}4\\4\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]^T$$

$$m_3 = \left[\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]\left[\begin{pmatrix}2\\3\end{pmatrix} - \begin{pmatrix}3\\3.8\end{pmatrix}\right]^T$$

so sample covariance will be $\Rightarrow S_1 = \frac{1}{N-1}\left[m_1 + m_2 + m_3 + m_4 + m_5\right]$

$$S_1 = \frac{1}{4}\left[\ \text{"}\ \right] = \begin{bmatrix}1 & -0.25\\ -0.25 & 2.2\end{bmatrix}$$

similarly $S_2 = \begin{bmatrix}2.3 & -0.05\\ -0.05 & 3.3\end{bmatrix}$

so $\underset{\substack{\downarrow\\ \text{within class scatter matrix.}}}{S_W} = S_1 + S_2 = \begin{bmatrix}1 & -0.25\\ -0.25 & 2.2\end{bmatrix} + \begin{bmatrix}2.3 & -0.05\\ -0.05 & 3.3\end{bmatrix} = \begin{bmatrix}3.3 & -0.3\\ -0.3 & 5.5\end{bmatrix}$

now for between class scatter matrix: $S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$

$$= \left[\begin{pmatrix}3\\3.8\end{pmatrix} - \begin{pmatrix}8.4\\7.6\end{pmatrix}\right]\left[\begin{pmatrix}3\\3.8\end{pmatrix} - \begin{pmatrix}8.4\\7.6\end{pmatrix}\right]^T$$

$$= \begin{pmatrix}-5.4\\-3.8\end{pmatrix}\begin{pmatrix}-5.4 & -3.8\end{pmatrix}$$

$$= \begin{pmatrix}29.16 & 20.52\\ 20.52 & 14.44\end{pmatrix}$$

$\Rightarrow$ Now find Eigen values:

$$S_W^{-1} S_B w = \lambda w$$

$$\Rightarrow |S_W^{-1} S_B - \lambda I| = 0$$

$$\Rightarrow \left| \begin{pmatrix}3.3 & -0.3\\ -0.3 & 5.5\end{pmatrix}^{-1}\begin{pmatrix}29.16 & 20.52\\ 20.52 & 14.44\end{pmatrix} - \lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right| = 0$$

$\vdots$

$$\Rightarrow \left|\begin{pmatrix}9.2213 - \lambda & 6.489\\ 4.2339 & 2.9794 - \lambda\end{pmatrix}\right| = 0$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_2 = 12.2007.$$

$\Rightarrow$ Eigen vector: $\left(S_W^{-1} S_B - \lambda I\right)\begin{pmatrix}w_1\\ w_2\end{pmatrix} = 0$

$$w_1 = \begin{pmatrix}-0.5755\\ 0.8178\end{pmatrix} \qquad w_2 = \begin{pmatrix}0.9088\\ 0.4173\end{pmatrix} = w^*$$

when $\lambda = 0$ $\qquad$ when $\lambda = 12.20$

Inv. of Matrix

$\frac{\text{Adj } A}{|A|}$.

for $2\times 2 \Rightarrow$ $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$

change sign $\qquad$ change place

for $3\times 3 \Rightarrow$ find cofactor

put $[\text{cofactor}]^T = \text{adjoint}$

$(-1)^{i+j}\begin{bmatrix}- & -\\ - & -\end{bmatrix}$

The optimal projection is the one that gave maximum $\lambda = J(w)$.

or directly $w^* = S_W^{-1}(\mu_1 - \mu_2) = \begin{bmatrix}3.3 & -0.3\\ -0.3 & 5.5\end{bmatrix}^{-1}\left[\begin{pmatrix}3\\3.8\end{pmatrix} - \begin{pmatrix}8.4\\7.6\end{pmatrix}\right]$

$$= \begin{pmatrix}0.3048 & 0.0166\\ 0.0166 & 0.1827\end{pmatrix}\begin{pmatrix}-5.4\\ -3.8\end{pmatrix}$$

$$= \begin{pmatrix}0.9088\\ 0.4173\end{pmatrix}$$

→ so we got $\omega^* = \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix}$

→ So, now to obtain the ~~eigen~~ LDA, we need to perform the dot product between the eigen vector and original data.

hence $\omega_1 : X_1 = (x_1, x_2) = \{(4,2), (2,4), (2,3), (3,6), (4,4)\}$

$\omega_2 : X_2 = (x_1, x_2) = \{(9,10), (6,8), (9,5), (8,7), (10,8)\}$

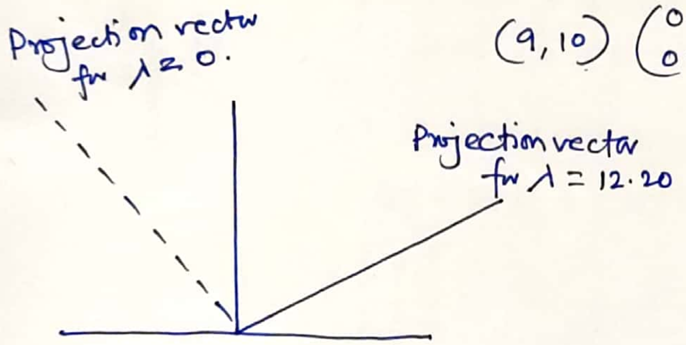
x_1	4	2	2	3	4	9	6	9	8	10
x_2	2	4	3	6	4	10	8	5	7	8
LD	4.47	3.49	3.06	5.2	5.3	12.35	8.8	10.2	10.19	12.42

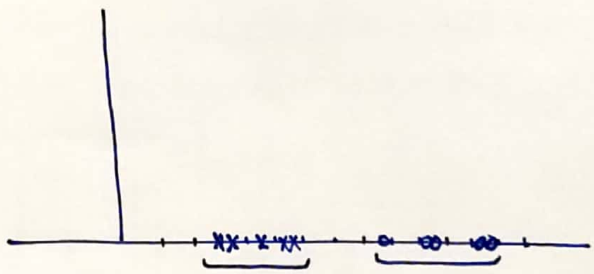
so for dot product $\begin{pmatrix} 4 & 2 \end{pmatrix} \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix} = 4.47.$

$\begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix} = 3.49$

$\vdots$

$(9, 10) \begin{pmatrix} 0.9088 \\ 0.4173 \end{pmatrix} = 12.35$





Calculation in detail.

$$\Rightarrow \quad S_W^{-1} S_B w = \lambda w.$$

$$\Rightarrow \quad |S_W^{-1} S_B - \lambda I| = 0$$

$$\Rightarrow \quad \left| \begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix}^{-1} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \left| \frac{\mathrm{adj}\begin{pmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{pmatrix}}{\begin{vmatrix} 3.3 & -0.3 \\ -0.3 & 5.5 \end{vmatrix}} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \left| \frac{\begin{pmatrix} 5.5 & 0.3 \\ 0.3 & 3.3 \end{pmatrix}}{18.15 - 0.09} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \left| \frac{\begin{pmatrix} 5.5 & 0.3 \\ 0.3 & 3.3 \end{pmatrix}}{18.06} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \left| \begin{pmatrix} 0.3045 & 0.0166 \\ 0.0166 & 0.1827 \end{pmatrix} \begin{pmatrix} 29.16 & 20.52 \\ 20.52 & 14.44 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \left| \begin{matrix} 0.3045 \times 29.16 + 0.0166 \times 20.52 & 0.3045 \times 20.52 + 0.0166 \times 14.44 \\ 0.0166 \times 29.16 + 0.1827 \times 20.52 & 0.0166 \times 20.52 + 0.1827 \times 14.44 \end{matrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \left| \begin{pmatrix} 9.22 & 6.48 \\ 4.23 & 2.97 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 9.22 - \lambda & 6.48 \\ 4.23 & 2.97 - \lambda \end{vmatrix} = 0$$

hence $\lambda_1 = 0 \quad \lambda_2 = 12.20$.

as $(9.22 - \lambda)(2.97 - \lambda) - 6.48 \times 4.23 = 0$

$$\Rightarrow \quad 27.38 - 9.22\lambda - 2.97\lambda + \lambda^2 - 27.41 = 0$$

$$\Rightarrow \quad \lambda^2 - 12.19\lambda - 0.03 = 0$$

$$\lambda = \frac{-(-12.19) \pm \sqrt{(-12.19)^2 - 4.1.(-0.03)}}{2.1.}$$

$$= \frac{12.19 \pm \sqrt{148.596 + 0.12}}{2}$$

$$= \frac{12.19 \pm 12.19}{2} \Rightarrow \underline{\underline{0,\ 12.19}}$$

so eigen values $\lambda_1 = 0$ $\lambda_2 = 12.2$

Now to find eigen vector $Mx = \lambda x$. where M = covariance matrix.

in this case for Fisher's discriminant analysis →

$$M = S_W^{-1} S_B = \begin{pmatrix} 9.22 & 6.48 \\ 4.23 & 2.97 \end{pmatrix} \quad x = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

Hence ⇒

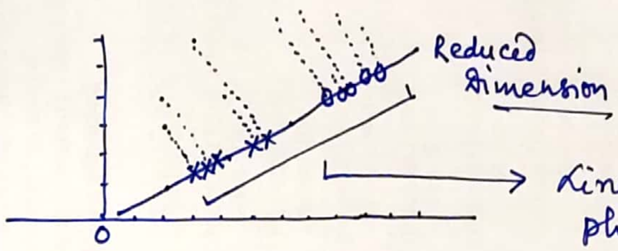
$$\begin{vmatrix} 9.22 & 6.48 \\ 4.23 & 2.97 \end{vmatrix} \begin{vmatrix} w_1 \\ w_2 \end{vmatrix} = 12.2 \begin{vmatrix} w_1 \\ w_2 \end{vmatrix}$$

$$\Rightarrow \quad 9.22 w_1 + 6.48 w_2 = 12.2 w_1 \quad — ①$$

$$4.23 w_1 + 2.97 w_2 = 12.2 w_2 \quad — ②$$

- from ① ⇒ $3.0 w_1 = 6.48 w_2$ ⇒ $w_1 = 2.16 w_2$ | $4.23 w_1 = 9.23 w_2$ ⇒ $w_1 = 2.18 w_2$

Hence $x_1 = 2.16 x_2$



→ Linearly discriminable points by orthographic projection.

Now according to the last step. dot product need to be done ...

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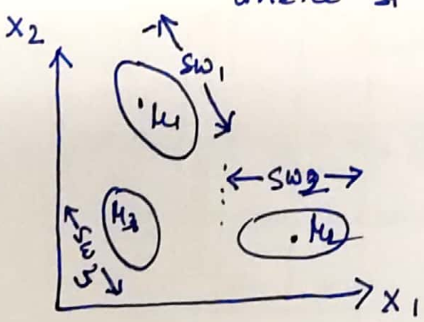
⇒ LDA for more than two classes: [Let's 'c']

/ consider you have been given more than two classes.

so in this case within class scatter plot will be $S_W = \sum_{i=1}^{c} S_i$

where S_i is nothing but $S_i = \sum_{x \in w_i} (x - \mu_i)(x - \mu_i)^T$

every μ_i is the corresponding mean.



three classes in two dimensional features.

– consider the between class scatter plot now.

it was $S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T$

for 'c' no of classes the between class scatter plot will be measured with respect to the mean of all classes as follows:

$$S_B = \sum_{i=1}^{c} N_i (\mu_i - \mu)(\mu_i - \mu)^T$$

μ = sample mean
μ_i = popular mean