Math 3C Homework 9 Solutions

Ilhwan Jo and Akemi Kashiwada

ilhwanjo@math.ucla.edu, akashiwada@ucla.edu

Assignment: Section 12.6 Problems 15, 16, 17(a), 18(a), 19(a), 20(a), 23-26

15. Toss a fair coin 400 times. Use the central limit theorem to find an approximation for the probability of at most 190 heads.

Solution

Let

$$X_i = \begin{cases} 1, & \text{if } i\text{th toss is heads} \\ 0, & \text{otherwise} \end{cases}.$$

Since the X_i are i.i.d. and binomially distributed we know

$$\mu = EX_i = np = (1)\left(\frac{1}{2}\right) = \frac{1}{2}, \quad \sigma^2 = \text{var}(X_i) = np(1-p) = (1)\left(\frac{1}{2}\right)\left(1 - \frac{1}{2}\right) = \frac{1}{4}.$$

Now let $S_{400} = \sum_{i=1}^{400} X_i$ count the number of heads we get out of 400 coin tosses. By the central limit theorem we get

$$\begin{aligned} P(S_{400} \leq 190) &= P\left(\frac{S_{400} - 400\mu}{\sqrt{400\sigma^2}} \leq \frac{190 - 200}{10}\right) \\ &= P\left(\frac{S_{400} - 400\mu}{\sqrt{400\sigma^2}} \leq -1\right) \\ &\approx 1 - \Phi(1) \\ &= 1 - 0.8413 \\ &= 0.1587 \end{aligned}$$

16. Toss a fair coin 150 times. Use the central limit theorem to find an approximation for the probability that the number of heads is at least 70.

Solution

Let X_i be defined as in the previous problem and let $S_{150} = \sum_{i=1}^{150} X_i$. By the central limit theorem

$$\begin{aligned} P(S_{150} \geq 70) &= P\left(\frac{S_{150} - 150\mu}{\sqrt{150\sigma^2}} \geq \frac{70 - 75}{\sqrt{37.5}}\right) \\ &= 1 - P\left(\frac{S_{150} - 150\mu}{\sqrt{150\sigma^2}} \leq -0.82\right) \\ &\approx 1 - \Phi(-0.82) \\ &= \Phi(0.82) \\ &= 0.7939 \end{aligned}$$

17a. Toss a fair coin 200 times. Use the central limit theorem to find an approximation for the probability that the number of heads is at least 120.

Solution

Let X_i be defined above and let $S_{200} = \sum_{i=1}^{200} X_i$. By the central limit theorem

$$\begin{aligned}
P(S_{200} \geq 120) &= P\left(\frac{S_{200} - 200\mu}{\sqrt{200\sigma^2}} \geq \frac{120 - 100}{\sqrt{50}}\right) \\
&= 1 - P\left(\frac{S_{200} - 200\mu}{\sqrt{200\sigma^2}} \leq 2.83\right) \\
&\approx 1 - \Phi(2.83) \\
&= 1 - 0.9977 \\
&= 0.0023
\end{aligned}$$

18a. Toss a fair coin 300 times. Use the central limit theorem to find an approximation for the probability that the number of heads is between 140 and 160.

Solution

Let X_i be defined above and let $S_{300} = \sum_{i=1}^{300} X_i$. By the central limit theorem

$$\begin{aligned}
P(S_{300} \in [140, 160]) &= P\left(\frac{S_{300} - 300\mu}{\sqrt{300\sigma^2}} \in \left[\frac{140 - 150}{\sqrt{75}}, \frac{160 - 150}{\sqrt{75}}\right]\right) \\
&= P\left(\frac{S_{300} - 300\mu}{\sqrt{300\sigma^2}} \leq 1.15\right) - P\left(\frac{S_{300} - 300\mu}{\sqrt{300\sigma^2}} \leq -1.15\right) \\
&\approx \Phi(1.15) - \Phi(-1.15) \\
&= 0.8749 - (1 - 0.8749) \\
&= 0.7498
\end{aligned}$$

19a. Suppose S_n is binomially distributed with parameters $n = 200$ and $p = 0.3$. Use the central limit theorem to find an approximation for $P(99 \leq S_n \leq 101)$ without the histogram correction.

Solution

Since S_n is binomially distributed, $S_n = \sum_{i=1}^{n} X_i$ where

$$X_k = \begin{cases} 1 & \text{if the } k\text{-th trial is successful} \\ 0 & \text{otherwise} \end{cases}$$

We have

$$\begin{aligned}
\mu &= EX_i = 1 \cdot p + 0 \cdot (1 - p) = p = 0.3, \\
\sigma^2 &= \mathrm{var}(X_i) = EX_i^2 - (EX_i)^2 = 1^2 \cdot p + 0^2 \cdot (1 - p) - p^2 = p(1 - p) = 0.3 \cdot 0.7 = 0.21.
\end{aligned}$$

By the central limit theorem,

$$\begin{aligned}
P(99 \leq S_n \leq 101) &= P\left(\frac{99 - 200 \cdot 0.3}{\sqrt{200 \cdot 0.21}} \leq \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq \frac{101 - 200 \cdot 0.3}{\sqrt{200 \cdot 0.21}}\right) \\
&= P\left(6.02 \leq \frac{S_n - n\mu}{\sqrt{n\sigma^2}} \leq 6.33\right) \\
&\approx \Phi(6.33) - \Phi(6.02) \\
&= 0.
\end{aligned}$$

20a. Suppose S_n is binomially distributed with parameters $n = 150$ and $p = 0.4$. Use the central limit theorem to find an approximation for $P(S_n = 60)$ without the histogram correction.

Solution

Since S_n is binomially distributed, we know

$$\mu = EX_i = p = 0.4, \quad \sigma^2 = \mathrm{var}(X_i) = p(1-p) = (0.4)(0.6) = 0.24.$$

By the central limit theorem,

$$\begin{aligned} P(S_n = 60) &= P\left(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{60 - 150 \cdot 0.4}{\sqrt{150 \cdot 0.24}}\right) \\ &= P\left(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} = 0\right) \\ &\approx P(Z = 0) \\ &= 0 \end{aligned}$$

where Z is the standard normally distributed random variable.

23. How often should you toss a coin to be at least 90% certain that your estimate of $P(\text{heads})$ is within 0.1 of its true value?

Solution

Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th toss is heads} \\ 0 & \text{otherwise} \end{cases}$$

Then $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is an estimate of the proportion of tosses resulting in heads. If we let $S_n = \sum_{i=1}^n X_i$, then $\sqrt{n}\frac{\overline{X}_n - p}{\sqrt{p(1-p)}} = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}$ is approximately standard normally distributed with $\mu = p = EX_i, \sigma^2 = \mathrm{var}(X_i) = p(1-p)$. We want to find n so that

$$P(|\overline{X}_n - p| \leq 0.1) \geq 0.9,$$

or

$$P\left(\sqrt{n}\frac{-0.1}{\sqrt{p(1-p)}} \leq \sqrt{n}\frac{\overline{X}_n - p}{\sqrt{p(1-p)}} \leq \sqrt{n}\frac{0.1}{\sqrt{p(1-p)}}\right) \geq 0.9.$$

Since $\sqrt{n}\frac{\overline{X}_n - p}{\sqrt{p(1-p)}}$ is approximately standard normally distributed,

$$2\Phi\left(\sqrt{n}\frac{0.1}{\sqrt{p(1-p)}}\right) - 1 \geq 0.9$$

or

$$\Phi\left(\sqrt{n}\frac{0.1}{\sqrt{p(1-p)}}\right) \geq 0.95$$

which is true if

$$\sqrt{n}\frac{0.1}{\sqrt{p(1-p)}} \geq 1.65$$

So we have

$$n \geq (16.5)^2 p(1-p)$$

where $p(1-p)$ attains its maximum $\frac{1}{4}$ when $p = \frac{1}{2}$. This gives us $n \geq 68.06$, which implies $n = 69$ would be sufficient.

24. How often should you toss a fair coin to be at least 90% certain that your estimate of $P(\text{heads})$ is within 0.01 of its true value?

Solution This question is basically same as **23.** So we have

$$\Phi\left(\sqrt{n}\frac{0.01}{\sqrt{p(1-p)}}\right) \geq 0.95$$

which is true if

$$\sqrt{n}\frac{0.01}{\sqrt{p(1-p)}} \geq 1.65$$

So we have

$$n \geq (165)^2 p(1-p)$$

where $p(1-p)$ attains its maximum $\frac{1}{4}$ when $p = \frac{1}{2}$. This gives us $n \geq 6806.3$, which implies $n = 6807$ would be sufficient.

25. To forecast the outcome of a presidential election in which two candidates run for office, a telephone poll is conducted. How many people should be surveyed to be at least 95% sure that the estimate is within 0.05 of the true value?

Solution

We assume that *each individual votes for one and only one of the two candidates*. Let A be one of the two candidates and

$$X_i = \begin{cases} 1 & \text{if } i\text{th individual voted for } A \\ 0 & \text{otherwise} \end{cases}$$

We assume that X_i's are i.i.d. Then $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is an estimate of the proportion of individuals who voted for A. If we let $S_n = \sum_{i=1}^n X_i$, then $\sqrt{n}\frac{\overline{X}_n - p}{\sqrt{p(1-p)}} = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}$ is approximately standard normally distributed with $\mu = p = EX_i, \sigma^2 = \text{var}(X_i) = p(1-p)$. We want to find n so that

$$P(|\overline{X}_n - p| \leq 0.05) \geq 0.95,$$

As in previous questions, we get

$$2\Phi\left(\sqrt{n}\frac{0.05}{\sqrt{p(1-p)}}\right) - 1 \geq 0.95$$

or

$$\Phi\left(\sqrt{n}\frac{0.05}{\sqrt{p(1-p)}}\right) \geq 0.975$$

which is true if

$$\sqrt{n}\frac{0.05}{\sqrt{p(1-p)}} \geq 1.96$$

So we have

$$n \geq (39.2)^2 \cdot \frac{1}{4} = 384.2,$$

which implies $n = 385$ would be sufficient.

26. A medical study is conducted to estimate the proportion of people suffering from seasonal affected disorder. How many people should be surveyed to be at least 99% sure that the estimate is within 0.02 of the true value?

Solution

Let

$$X_i = \begin{cases} 1 & \text{if } i\text{th individual is suffering from the disorder} \\ 0 & \text{otherwise} \end{cases}$$

We assume that X_i's are i.i.d. Then $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ is an estimate of the proportion of individuals who is suffering from the disorder. If we let $S_n = \sum_{i=1}^n X_i$, then $\sqrt{n}\frac{\overline{X}_n - p}{\sqrt{p(1-p)}} = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}$ is approximately standard normally distributed with $\mu = p = EX_i, \sigma^2 = \mathrm{var}(X_i) = p(1-p)$. We want to find n so that

$$P(|\overline{X}_n - p| \leq 0.02) \geq 0.99,$$

Again, we get

$$2\Phi\left(\sqrt{n}\frac{0.02}{\sqrt{p(1-p)}}\right) - 1 \geq 0.99$$

or

$$\Phi\left(\sqrt{n}\frac{0.02}{\sqrt{p(1-p)}}\right) \geq 0.995$$

which is true if

$$\sqrt{n}\frac{0.02}{\sqrt{p(1-p)}} \geq 2.58$$

So we have

$$n \geq (129)^2 \cdot \frac{1}{4} = 4160.3,$$

which implies $n = 4161$ would be sufficient.