Exercice 1

Exercice 1.1

$$\frac{\partial}{\partial x} f(x,y) = 8x - 4y - 4$$

$$\frac{\partial}{\partial y} f(x,y) = 20y - 4x - 16$$

Exercice 1.2

Point critique est le point où les 2 dérivées s'annulent.

$$\left\{\begin{array}{l} 8x - 4y - 4 = 0 \\ 20y - 4x - 16 = 0 \end{array}\right. \implies \left\{\begin{array}{l} 2x - y - 1 = 0 \\ 10y - 2x - 8 = 0 \end{array}\right. \implies \left\{\begin{array}{l} 9y - 9 = 0 \\ 2x - y - 1 = 0 \end{array}\right. \implies \left\{\begin{array}{l} y = 1 \\ 2x - 2 = 0 \end{array}\right.$$

Le point critique est $A = (1,1)$.

Exercice 1.3.a

$f(x,0) = 4x^2 - 4x + 11$ on a $f(x,0) < 4x^2 + 11$ donc $f(x,0) > C$ pour tout $x > M$ avec $M = \sqrt{|C - 11|/4}$

Exercice 1.3.b

Le point A est un maximum global si $\forall (x,y) \in \mathbb{R}^2, f(x,y) < f(A) = f(1,1)$. Il suffit de trouver un contre exemple, ie un point (x,y) tel que $f(x,y) > f(1,1)$. On a $f(1,1) = 1$, il suffit de prendre le point $(0,0)$ car $f(0,0) = 11$.

Exercice 1.4.a

$$g(x,y) = f(x,y) - 4x^2 + 4xy + 4x - y^2 - 2y - 1 = 9y^2 - 18y + 10$$

$$\frac{\partial}{\partial x} g(x,y) = \frac{\partial}{\partial x}(9y^2 - 18y + 10) = 0$$

Exercice 1.4.b

$$g(x,y) = (ay+b)^2 + 1 = a^2y^2 + 2aby + b^2 + 1 = 9y^2 - 18y + 10$$

Donc $a^2 = 9$, $2ab = -18$ et $b^2 = 9$. Ce qui fait $a = 3, b = -3$ ou $a = -3, b = 3$.

Exercice 1.4.b

Le point A est un minimum global si $\forall (x,y) \in \mathbb{R}^2, f(x,y) \geq f(A) = f(1,1)$. On a $g(x,y) = f(x,y) - (2x - y - 1)^2 = (ay+b)^2 + 1$. Donc $f(x,y) = (ay+b)^2 + 1 + (2x-y-1)^2$ et $f(x,y) = 1$. On a $(ay+b)^2 \geq 0$ et $(2x-y-1)^2 \geq 0$ donc $(ay+b)^2 + (2x-y-1)^2 + 1 \geq 1$. Donc le point A est un minimum global.

Exercice 2

Exercice 2.1

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}(1 + x + \sin(y))e^{-x} = (x + \sin(y))e^{x}$$

$$\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}(1 + x + \sin(y))e^{-x} = e^{-x}\cos(y)$$

Exercice 3

Exercice 3.1

??

Exercice 3.2

Gradient en tout point $(x, y) \in \mathbb{R}^2$.

$$\nabla f : (x, y) \to \left\{ \begin{array}{l} \frac{\partial}{\partial x} f(x, y) = 8x - 4y - 4 \\ \frac{\partial}{\partial y} f(x, y) = 2y - 4x - 2 \end{array} \right.$$

Exercice 3.3

$f(0, 2) = 0$ donc le point $(0, 2) \in Ł$

Gradient au point $(0, 2)$.

$$\nabla_{(0,2)} f = (-12, 2)$$

Tangente en un point (x_0, y_0) est

$$\{(x, y) \in \mathbb{R}^2, \frac{\partial}{\partial x} f(x_0, y_0)(x - x_0) + \frac{\partial}{\partial y} f(x_0, y_0)(y - y_0) = 0\}$$

$$\{(x, y) \in \mathbb{R}^2, -12(x - 0) + 2(y - 2) = -12x + 2y - 4 = 0\}$$

$$\{(x, y) \in \mathbb{R}^2, y = 6x + 2\}$$

Coefficent directeur est 6.

Exercice 3.4.a

$f(x, h_+(x)) = 0$?

$$f(x, h_+(x)) = 4x^2 - 4x(1 + 2x + \sqrt{1 + 8x}) + (1 + 2x + \sqrt{1 + 8x})^2 - 4x - 2(1 + 2x + \sqrt{1 + 8x})$$

$= 4x^2 - 4x - 8x^2 - 4x\sqrt{1 + 8x} + 1 + 2x + \sqrt{1 + 8x} + 2x + 4x^2 + 2x\sqrt{1 + 8x} + \sqrt{1 + 8x} + 2x\sqrt{1 + 8x} + 1 + 8x - 4x - 2 + 4x - 2\sqrt{1 + 8x} = 0$

ou

$f(x, h_+(x)) = (2x - (1 + 2x + \sqrt{1 + 8x}))^2 - 4x - 2(1 + 2x + \sqrt{1 + 8x}) = (-1 - \sqrt{1 + 8x})^2 - 4x - 2 - 4x - 2\sqrt{1 + 8x}$

$$= 1 + 2\sqrt{1 + 8x} + 1 + 8x - 8x - 2 - 2\sqrt{1 + 8x} = 0$$

$f(x, h_-(x)) = 0$?

$$f(x, h_-(x)) = 4x^2 - 4x(1 + 2x - \sqrt{1 + 8x}) + (1 + 2x - \sqrt{1 + 8x})^2 - 4x - 2(1 - 2x - \sqrt{1 + 8x})$$

$= 4x^2 - 4x - 8x^2 + 4x\sqrt{1 + 8x} + 1 + 2x - \sqrt{1 + 8x} + 2x + 4x^2 - 2x\sqrt{1 + 8x} - \sqrt{1 + 8x} - 2x\sqrt{1 + 8x} + 1 + 8x - 4x - 2 + 4x + 2\sqrt{1 + 8x} = 0$

ou

$f(x, h_+(x)) = (2x - (1 + 2x - \sqrt{1 + 8x}))^2 - 4x - 2(1 + 2x - \sqrt{1 + 8x}) = (-1 + \sqrt{1 + 8x})^2 - 4x - 2 - 4x + 2\sqrt{1 + 8x}$

$$= 1 - 2\sqrt{1 + 8x} + 1 + 8x - 8x - 2 + 2\sqrt{1 + 8x} = 0$$

Exercice 3.4.b

$$h'_+(x) = 2 + \frac{4}{\sqrt{1 + 8x}}$$

$$h'_+(0) = 2 + \frac{4}{\sqrt{1 + 8.0}} = 6$$

On a $h_+(0) = 2$, donc le point $(0, h_+(0)) = (0, 2)$. On sait que $h_+ \in Ł$. Donc la ...

QED