Rappel de cours

Definition 1. Deux suites $(u_n)_{n\geq 0}$ et $(v_n)_{n\geq 0}$ sont adjacentes ssi:

- $(u_n)_{n\geq 0}$ est croisssante et $(v_n)_{n\geq 0}$ est décroissante
- $\forall n \in \mathbb{N}, u_n \leq v_n$
- $\lim_{n\to\infty}(v_n - u_n)_{n\geq 0} = 0$

Definition 2. Le coefficient binomial est donnée par $\binom{n}{k} = k\frac{n!}{k!(n-k)!}$.

Definition 3. Quelques propriétés

- $\binom{n}{0} = 1$
- $\binom{n}{n} = 1$
- $\binom{n}{k} = \binom{n}{n-k}$
- $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n$
- $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Exercice 3

Pour que $\sum c_n z^n$ converge, il suffit de montrer, par le critère d'Abel, que $\exists M, \forall n, |\sum_{k=0}^{n} z^k| \leq M$. On a

$$|\sum_{k=0}^{n} z^k| = \left|1.\frac{1-z^{n+1}}{1-z}\right| \leq \frac{1+|z^{n+1}|}{|1-z|} < \frac{2}{|1-z|}$$

car $|z| \leq 1$ et $|z^n| \leq 1$. On a trouvé un $M = \frac{2}{1-|z|}$, ce qui permet de montrer que $\sum c_n z^n$ converge.

Exercice 4

Exercice 4.1.a

Calculons

$$\frac{v_n}{u_n} = \frac{\frac{(-1)^n}{\sqrt{n}}}{\frac{(-1)^n}{\sqrt{n}-(-1)^n}} = \frac{\sqrt{n}-(-1)^n}{\sqrt{n}} = 1 - \frac{(-1)^n}{\sqrt{n}}$$

On a $\lim_{n\to\infty} \frac{v_n}{u_n} = 1$ donc $u_n \ {}_{n\to\infty} v_n$

Exercice 4.1.b

$v_n = \frac{(-1)^n}{\sqrt{n}}$ converge?

1. Y a-t-il Convergence absolue? $\sum |\frac{(-1)^n}{\sqrt{n}}| = \sum \left|\frac{1}{\sqrt{n}}\right|$ Cette suite diverge. Donc il n'y a pas de convergence absolue.
2. Cas Special Série Alternée? La série est alternée car $(-1)^n$ est alternée et $\frac{1}{\sqrt{n}}$ est positif. Il faut montrer que $\frac{1}{\sqrt{n}}$ converge vers 0. Ce qui est vrai quand $n \to \infty$. Donc, la série de terme général $v_n = \frac{(-1)^n}{\sqrt{n}}$ converge.

Exercice 4.2

Exercice 4.3

Exercice 4.4

Exercice 5

Exercice 5.1

$$\begin{aligned}
u_n - \frac{(-1)^n}{n} &= \frac{1}{\ln(n)+(-1)^n n} - \frac{(-1)^n}{n} = \frac{n}{n(\ln(n)+(-1)^n n)} - \frac{(-1)^n(\ln(n)+(-1)^n n)}{n(\ln(n)+(-1)^n n)} \\
&= \frac{n-((-1)^n(\ln(n)+(-1)^n n))}{n(\ln(n)+(-1)^n n)} = \frac{n-(-1)^n\ln(n)-(-1)^n(-1)^n n}{n(\ln(n)+(-1)^n n)} = \frac{-(-1)^n\ln(n)}{n(\ln(n)+(-1)^n n)} \\
&= \frac{-\ln(n)}{(-1)^n n\ln(n)+n^2)} = \frac{\ln(n)}{n}\frac{-1}{(-1)^n\ln(n)+n}
\end{aligned}$$

??

Exercice 5.2

On a

$$u_n = \left(u_n - \frac{(-1)^n}{n}\right) - \frac{(-1)^n}{n}$$

Avec $u_n - \frac{(-1)^n}{n}$ qui converge et $\frac{(-1)^n}{n}$ qui converge aussi (C.S.S.A avec $v_n = \frac{1}{n}$). Donc la série de terme général u_n converge (somme de 2 séries qui convergent).

Exercice 6

Exercice 6.a

$$a_n = \sum_{k=n}^{2n} \frac{1}{n+k} = \sum_{k=n}^{2n} \frac{1}{n} \frac{1}{1+\frac{k}{n}}$$

Prenons $x = \frac{k}{n}$, on a $dx = \frac{1}{n}$ donc

$$\sum_{k=n}^{2n} \frac{1}{n} \frac{1}{1+\frac{k}{n}} = \int_1^2 \frac{1}{1+x} dx = [\ln(|1+x|]_1^2 = \ln(3) - \ln(2) = \ln(\frac{3}{2})$$

Exercice 6.b

$$b_n = \sqrt[n]{\frac{(2n)!}{n!n^n}}$$

Calcul de

$$\ln\left(\sqrt[n]{\frac{(2n)!}{n!n^n}}\right) = \frac{1}{n}\ln\left(\frac{(2n)!}{n!n^n}\right) = \frac{1}{n}(\ln((2n)!) - \ln(n!) - n\ln(n))$$

On a

$$\ln(n!) = \ln(1*2*3*\ldots*n) = \ln(1) + \ln(2) + \ln(3) + \ldots + \ln(n) = \sum_{k=1}^{n} \ln(k)$$

$$\ln(2n!) = \ln(1*2*3*\ldots*2n) = \ln(1) + \ln(2) + \ln(3) + \ldots + \ln(2n) = \sum_{k=1}^{2n} \ln(k)$$

Donc

$$\ln((2n)!) - \ln(n!) = \sum_{k=n+1}^{2n} \ln(k)$$

$$\ln((2n)!) - \ln(n!) - n\ln(n) = \sum_{k=n+1}^{2n} \ln(k) - \sum_{k=1}^{n} \ln(n) = \sum_{k=n+1}^{2n} \ln(k) - \sum_{k=n+1}^{2n} \ln(n) = \sum_{k=n+1}^{2n} \ln(k) - \ln(n) = \sum_{k=n+1}^{2n} \ln\left(\frac{k}{n}\right)$$

donc

$$\frac{1}{n}(\ln((2n)!) - \ln(n!) - n\ln(n)) = \frac{1}{n}\sum_{k=n+1}^{2n} \ln\left(\frac{k}{n}\right) = \int_{\frac{n+1}{n}}^{2} \ln(x) = [x(\ln(x)-1)]_{\frac{n+1}{n}}^{2}$$

Exercice 7

Exercice 7.a

$$a_n = \sum_{k=0}^{n} \frac{1}{n+k} = \sum_{k=0}^{n} \frac{1}{n} \frac{1}{1+\frac{k}{n}}$$

Prenons $x = \frac{k}{n}$, on a $dx = \frac{1}{n}$ donc

$$\sum_{k=0}^{n} \frac{1}{n} \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x} dx = [\ln(|1+x|]_0^1 = \ln(2) - \ln(1) = \ln(2)$$

Exercice 7.b

$$b_n = \sum_{k=0}^{n} \frac{n}{n^2+k^2} = \sum_{k=0}^{n} \frac{n}{n^2} \frac{1}{1+\frac{k^2}{n^2}}$$

Prenons $x = \frac{k}{n}$, on a $dx = \frac{1}{n}$ donc

$$\sum_{k=0}^{n} \frac{1}{n} \frac{1}{1+\left(\frac{k}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2} dx = [\arctan(x)]_0^1 = \arctan(1) - \arctan(0) = \arctan(1)$$

Exercice 7.c

$$c_n = \frac{1}{n^2}\prod_{k=1}^{n}(n^2+k^2)^{1/n}$$

Exercice 8

Exercice 8.a

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = k\frac{n!}{k(k-1)!(n-k)!} = \frac{n(n-1)!}{(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = n\binom{n-1}{k-1}$$

Exercice 8.b

$$k(k-1)\binom{n}{k} = k(k-1)\frac{n!}{k!(n-k)!} = k(k-1)\frac{n!}{k(k-1)(k-2)!((n-2)-(k-2))!}$$
$$= \frac{n(n-1)(n-2)!}{(k-2)!((n-2)-(k-2))!} = n(n-1)\binom{n-2}{k-2}$$

Exercice 9

Exercice 9.1

On sait que $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$. En prenant $y = 1-x$, on a

$$\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = (x+(1-x))^n = 1^n = 1$$

Exercice 9.2

Le premier terme est toujours égal à 0.

$$E(X) = \sum_{k=0}^{n}kp(X=k) = \sum_{k=1}^{n}kp(X=k) = \sum_{k=1}^{n}k\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=1}^{n}n\binom{n-1}{k-1}p^k(1-p)^{n-k} =$$

$$= \sum_{k=1}^{n}n\binom{n-1}{k-1}pp^{(k-1)}(1-p)^{n-k} = np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{(k-1)}(1-p)^{(n-1)-(k-1)}$$

Changement de variables $j = k-1$ et $m = n-1$ on a

$$np\sum_{j=0}^{m}\binom{m}{j}p^j(1-p)^{(m-j)} = np.1 = np$$

Exercice 10

$$V(X) = \sum_{k=0}^{n}(k-np)^2P(X=k) = \sum_{k=0}^{n}(k^2-2knp+n^2p^2)P(X=k)$$

$$= \sum_{k=0}^{n}k^2P(X=k) - 2np\sum_{k=0}^{n}kP(X=k) + n^2p^2\sum_{k=0}^{n}P(X=k) = \sum_{k=0}^{n}k^2P(X=k) - 2npnp + n^2p^2 = \sum_{k=0}^{n}k^2P(X=k) - n^2p^2$$

On repart de

$$\sum_{k=0}^{n}k^2P(X=k) = \sum_{k=0}^{n}k.k.P(X=k) = \sum_{k=0}^{n}k.k.\binom{n}{k}p^k(1-p)^{n-k} = \sum_{k=0}^{n}k.n.\binom{n-1}{k-1}p^k(1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k.n. \binom{n-1}{k-1} pp^{k-1}(1-p)^{(n-1)-(k-1)} = np \sum_{k=0}^{n} k. \binom{n-1}{k-1} p^{k-1}(1-p)^{(n-1)-(k-1)}$$

Le premier terme est égal à 0.

$$np \sum_{k=1}^{n} k. \binom{n-1}{k-1} p^{k-1}(1-p)^{(n-1)-(k-1)}$$

Changement de variables $j = k-1$, $m = n-1$

$$np \sum_{j=0}^{m} (j+1). \binom{m}{j} p^{j}(1-p)^{m-j} = np \left(\sum_{j=0}^{m} j \binom{m}{j} p^{j}(1-p)^{m-j} + \sum_{j=0}^{m} \binom{m}{j} p^{j}(1-p)^{m-j} \right)$$

$$= np(mp+1) = np((n-1)p+1) = n^2p^2 - np^2 + np = n^2p^2 + np(1-p)$$

Donc

$$V(X) = \sum_{k=0}^{n} k^2 P(X=k) - n^2p^2 = n^2p^2 + np(1-p) - n^2p^2 = np(1-p)$$

Exercice 11

On a

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Vérifions pour $n = 1$, $\sum_{k=1}^{1} \frac{(-1)^{k+1}}{k} \binom{1}{k} = 1 = \sum_{k=1}^{1} \frac{1}{k}$. Supposons $\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k}$ vrai au rang n, calculons

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} = \sum_{k=1}^{n+1} \left(\frac{(-1)^{k+1}}{k} \binom{n}{k} + \binom{n}{k-1} \right)$$

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k} + \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k-1}$$

Premiere partie

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} + \frac{(-1)^{k+2}}{n+1} \binom{n}{n+1} = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k}$$

Car $\binom{n}{n+1} = 0$ par définition et par hypothèse de récurrence.

Seconde partie, commencons par calculer

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k-1} = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{(n+1)-1}{k-1} = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{n+1} \binom{n+1}{k} = \frac{1}{n+1} \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k}$$

On a $(x+y)^n = \sum_{k=0}^{n} x^n y^{n-k} \binom{n}{k}$, en prenant $x = -1$ et $y = 1$ on a

$$((-1)+1)^n = \sum_{k=0}^{n} (-1)^k 1^{n-k} \binom{n}{k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 1 + \sum_{k=1}^{n} (-1)^k \binom{n}{k}$$

Donc $\sum_{k=1}^{n} (-1)^k \binom{n}{k} = -1$ ce qui fait $\sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} = 1$ On a pour finir

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k} + \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k-1} = \sum_{k=1}^{n} \frac{1}{k} + \frac{1}{n+1} = \sum_{k=1}^{n+1} \frac{1}{k}$$

Fini.

Exercice 12

$$\sum_{k=p}^{n+1}\binom{k}{p}=\sum_{k=p}^{n}\binom{k}{p}+\binom{n+1}{p}=\binom{n+1}{p+1}+\binom{n+1}{p}$$

On a

$$\binom{n+1}{p+1}=\frac{(n+1)!}{(p+1)!(n-p)!}=\frac{(n+1)!(n-p+1)}{(p+1)!(n-p+1)!}$$

$$\binom{n+1}{p}=\frac{(n+1)!}{(p)!(n+1-p)!}=\frac{(n+1)!(p+1)}{(p+1)!(n+1-p)!}$$

Donc

$$\binom{n+1}{p+1}+\binom{n+1}{p}=\frac{(n+1)!(n+1-p)}{(p+1)!(n+1-p)!}+\frac{(n+1)!(p+1)}{(p+1)!(n+1-p)!}=\frac{(n+1)!(n-p+1+p+1)}{(p+1)!(n+1-p)!}$$

$$=\frac{(n+2)!}{(p+1)!((n+2)-(p+1))!}=\binom{n+2}{p+1}$$

Exercice 13

Exercice 13.1

$$\sum_{k=0}^{m}\binom{2m}{2k}-\sum_{k=0}^{m-1}\binom{2m}{2k+1}=\binom{2m}{0}+\binom{2m}{2}+\ldots+\binom{2m}{2m}-(\binom{2m}{1}+\binom{2m}{3}+\ \ldots+\binom{2m}{2m-1})$$

$$\binom{2m}{0}-\binom{2m}{1}+\binom{2m}{2}-\binom{2m}{3}\cdots-\binom{2m}{2m-1}+\binom{2m}{2m}=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}=0$$

Pour $n=2m$ (n pair), on a

$$S_n=\sum_{k=0}^{m}\binom{2m}{2k},T_n=\sum_{k=0}^{m-1}\binom{2m}{2k+1}$$

Calculons S_n-T_n

$$\sum_{k=0}^{m}\binom{2m}{2k}-\sum_{k=0}^{m-1}\binom{2m}{2k+1}=\binom{2m}{0}+\binom{2m}{2}+\ldots+\binom{2m}{2m}-\left(\binom{2m}{1}+\binom{2m}{3}+\ \ldots+\binom{2m}{2m-1}\right)$$

$$\binom{2m}{0}-\binom{2m}{1}+\binom{2m}{2}-\binom{2m}{3}\cdots-\binom{2m}{2m-1}+\binom{2m}{2m}=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}=0$$

Pour $n=2m+1$ (n impair) on a

$$S_n=\sum_{k=0}^{m}\binom{2m+1}{2k},T_n=\sum_{k=0}^{m}\binom{2m+1}{2k+1}$$

Calculons S_n-T_n

$$\sum_{k=0}^{m}\binom{m}{2k}-\sum_{k=0}^{m}\binom{2m+1}{2k+1}=\binom{2m+1}{0}+\binom{2m+1}{2}+\ldots+\binom{2m+1}{2m}-\left(\binom{2m+1}{1}+\binom{2m+1}{3}+\ \ldots+\binom{2m+1}{2m+1}\right)$$

$$\binom{2m+1}{0}-\binom{2m+1}{1}+\binom{2m+1}{2}-\binom{2m+1}{3}\cdots+\binom{2m+1}{2m}-\binom{2m+1}{2m+1}=\sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}=0$$

Exercice 13.2

Calculons $S_n + T_n$ pour $n = 2m$

$$\sum_{k=0}^{m} \binom{2m}{2k} + \sum_{k=0}^{m-1} \binom{2m}{2k+1} = \binom{2m}{0} + \binom{2m}{2} + \ldots + \binom{2m}{2m} + \left(\binom{2m}{1} + \binom{2m}{3} + \ \ldots + \binom{2m}{2m-1}\right)$$

$$\binom{2m}{0} + \binom{2m}{1} + \binom{2m}{2} + \binom{2m}{3} \ldots + \binom{2m}{2m-1} + \binom{2m}{2m} = \sum_{k=0}^{2m} \binom{2m}{k} = 2^{2m} = 2^n$$

car $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$ et en prenant $x = y = 1$ on a

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^n 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$$

Calculons $S_n + T_n$ pour $n = 2m + 1$

$$\sum_{k=0}^{m} \binom{m}{2k} + \sum_{k=0}^{m} \binom{2m+1}{2k+1} = \binom{2m+1}{0} + \binom{2m+1}{2} + \ldots + \binom{2m+1}{2m} + \left(\binom{2m+1}{1} + \binom{2m+1}{3} + \ \ldots + \binom{2m+1}{2m+1}\right)$$

$$\binom{2m+1}{0} + \binom{2m+1}{1} + \binom{2m+1}{2} + \binom{2m+1}{3} \ldots + \binom{2m+1}{2m} + \binom{2m+1}{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} = 2^{2m+1} = 2^n$$

Exercice 13.3

On a $S_n - T_n = 0$ et $S_n + T_n = 2^n$, donc $2T_n = 2^n$, $T_n = 2^{n-1}$ et $S_n = T_n = 2^{n-1}$.

QED