Rappel de cours

Exercice 1

Exercice 1.1

Calculons $\det(A-\lambda.I)$

$$\det\left(\begin{vmatrix} 1-\lambda & 0 & 0 \\ a & -2-\lambda & 3 \\ 1 & -1 & 2-\lambda \end{vmatrix}\right)$$

$= (-2-\lambda)((2-\lambda)(1-\lambda)-0*1)-(-1)((1-\lambda)*3-0*a) = -(4-\lambda^2)(1-\lambda)+3(1-\lambda) = (1-\lambda)(-1+\lambda^2)$

donc

$$Sp(A) = \{1, -1\}$$

Exercice 1.2

Déterminons les vecteurs propres de A.
Calculons

$$E_1(A) = ker(A-I) = ker\left(\begin{vmatrix} 1-1 & 0 & 0 \\ a & -2-1 & 3 \\ 1 & -1 & 2-1 \end{vmatrix}\right)$$

Cherchons x, y, z tel que

$$\begin{vmatrix} 0 & 0 & 0 \\ a & -3 & 3 \\ 1 & -1 & 1 \end{vmatrix} . \begin{vmatrix} x \\ y \\ z \end{vmatrix} = 0$$

$$\begin{cases} 0x+0y+0z & = 0 \\ ax-3y+3z & = 0 \\ x-y+z & = 0 \end{cases}$$

$$\begin{cases} x+z & = y \\ ax-3x-3z+3z & = 0 \\ 0x+0y+0z & = 0 \end{cases}$$

$$\begin{cases} x+z & = y \\ x(a-3) & = 0 \\ 0x+0y+0z & = 0 \end{cases}$$

En fixant $x = c_1$ et $z = c_2$, on a

$$E_1(A) = \begin{cases} \{(1,1,0),(0,1,1)\} & a = 3 \\ \{0,1,1) & a \neq 3 \end{cases}$$

On a $dim\ E_1(A) = 2$ lorsque $a = 3$ et $dim\ E_1(A) = 1$ lorsque $a \neq 3$.

Calculons

$$E_{-1}(A) = ker(A+I) = ker\left(\begin{vmatrix} 1+1 & 0 & 0 \\ a & -2+1 & 3 \\ 1 & -1 & 2+1 \end{vmatrix}\right)$$

Cherchons x, y, z tel que

$$\begin{vmatrix} 2 & 0 & 0 \\ a & -1 & 3 \\ 1 & -1 & 3 \end{vmatrix} . \begin{vmatrix} x \\ y \\ z \end{vmatrix} = 0$$

$$\begin{cases} 2x+0y+0z & = 0 \\ ax-y+3z & = 0 \\ x-y+3z & = 0 \end{cases}$$

$$\begin{cases} x & = 0 \\ 3z & = y \end{cases}$$

En fixant $x = 0$ et $z = c_2$, on a

$$E_{-1}(A) = \{(0, 3, 1)\}$$

On a $dim\ E_{-1}(A) = 1$.

Pour que A soit diagonalisable il faut que $dim\ A = dim\ E_1 + dim\ E_{-1}$, donc on a $a = 3$.
La matrice

$$P = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

QED