Rappel de cours:

- $\int_a^b f(u(x))u'(x)\,dx = \int_{x(a)}^{u(b)} u\,du$

Exercice 1

Exercice 1.a

$$\int tan(x)\,dx = \int \frac{sin(x)}{cos(x)}\,dx$$

Soit $u(x) = cos(x)$, $u'(x) = -sin(x)$ et $f(u) = -\frac{1}{u}$ donc

$$\int \frac{sin(x)}{cos(x)}\,dx = \int f(cos(x))sin(x)\,dx = \int -\frac{1}{u}\,du = -ln(u) = -ln(cos(x))$$

Exercice 1.b

$$\int \frac{ln(x+1)}{x+1}\,dx$$

Soit $u(x) = ln(x+1)$, $u'(x) = \frac{1}{x+1}$ et $f(u) = u$ donc

$$\int \frac{ln(x+1)}{x+1}\,dx = \int f(ln(x+1)).\frac{1}{x+1}\,dx = \int u\,du = \frac{u^2}{2} = \frac{ln^2(x+1)}{2}$$

Exercice 1.c

$$\int \frac{x}{\sqrt{2x^2+3}}$$

Soit $u(x) = 2x^2+3$, $u'(x) = 4x$ et $f(u) = \frac{1}{4\sqrt{u}}$ donc

$$\int \frac{x}{\sqrt{2x^2+3}}\,dx = \int f(2x^2+3).4x\,dx = \int \frac{1}{4\sqrt{u}}\,du = \frac{1}{4}\int \frac{1}{\sqrt{u}\,du} = \frac{\sqrt{u}}{2} = \frac{\sqrt{2x^2+3}}{2}$$

Exercice 1.d

$$\int \frac{cos(x)}{sin^2(x)}\,dx$$

Soit $u(x) = sin(x)$, $u'(x) = cos(x)$ et $f(u) = \frac{1}{u^2}$ donc

$$\int \frac{cos(x)}{sin^2(x)}\,dx = \int f(sin(x)).cos(x)\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{sin(x)}$$

Exercice 1.e

$$\int x\sqrt{x^2+1}\,dx$$

Soit $u(x) = x^2+1$, $u'(x) = 2x$ et $f(u) = \frac{\sqrt{(u)}}{2}$ donc

$$\int x\sqrt{x^2+1}\,dx = \int f(x^2+1).2x\,dx = \int \frac{\sqrt{(u)}}{2}\,du = \frac{1}{2}\int \sqrt{(u)}\,du = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} = \frac{1}{3}u^{\frac{3}{2}} = \frac{(x^2+1)^{\frac{3}{2}}}{3}$$

Exercice 1.f

$$\int \frac{1}{xln(x)}\,dx$$

Soit $u(x) = ln(x)$, $u'(x) = \frac{1}{x}$ et $f(u) = \frac{1}{u}$ donc

$$\int \frac{1}{xln(x)}\,dx = \int f(ln(x)).\frac{1}{x}\,dx = \int \frac{1}{u}\,du = ln(|u|) = ln(|ln(x)|)$$

Exercice 1.g

$$\int sin^2(x)cos(x)\,dx$$

Soit $u(x) = sin(x)$, $u'(x) = cos(x)$ et $f(u) = u^2$ donc

$$\int sin^2(x)cos(x)\,dx = \int f(sin(x)).cos(x)\,dx = \int u^2\,du = \frac{u^3}{3} = \frac{sin^3(x)}{3}$$

Exercice 1.h

$$\int e^{sin(x)}cos(x)\,dx$$

Soit $u(x) = sin(x)$, $u'(x) = cos(x)$ et $f(u) = e^u$ donc

$$\int e^{sin(x)}cos(x)\,dx = \int f(sin(x)).cos(x)\,dx = \int e^u\,du = e^u = e^{sin(x)}$$

Exercice 1.i

$$\int \frac{ln^2(x)}{x}\,dx$$

Soit $u(x) = ln(x)$, $u'(x) = \frac{1}{x}$ et $f(u) = u^2$ donc

$$\int \frac{ln^2(x)}{x}\,dx = \int f(ln(x)).\frac{1}{x}\,dx = \int u^2\,du = \frac{u^3}{3} = \frac{ln^3(x)}{3}$$

Exercice 2

Exercice 2.1

$t = u(x) = tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.
On a

$$sin(x) = \frac{2tan(\frac{x}{2})}{1+tan^2(\frac{x}{2})}$$

$$1 - sin(x) = 1 - \frac{2tan(\frac{x}{2})}{1+tan^2(\frac{x}{2})} = \frac{tan^2(\frac{x}{2}) - 2tan^2(\frac{x}{2}) + 1}{1+tan^2(\frac{x}{2})} = \frac{(tan(\frac{x}{2})-1)^2}{1+tan^2(\frac{x}{2})}$$

Donc

$$\frac{1}{1-sin(x)} = \frac{1+tan^2(\frac{x}{2})}{(tan(\frac{x}{2})-1)^2}$$

Et

$$\int_0^{\pi/3} \frac{1}{1-sin(x)}\,dx = \int_0^{\pi/3} \frac{1+tan^2(\frac{x}{2})}{(tan(\frac{x}{2})-1)^2}\,dx$$

$$\int_{tan(0)}^{tan(\pi/6)} \frac{1+t^2}{(t-1)^2}.\frac{2}{1+t^2}\,dt = \int_{tan(0)}^{tan(\pi/6)} \frac{2}{(t-1)^2}\,dt$$

Soit $u(x) = x - 1$, $u'(x) = 1$ et $f(u) = \frac{1}{u^2}$ donc

$$\int_{tan(0)}^{tan(\pi/6)} \frac{2}{(t-1)^2}\,dt = 2\int_{tan(0)}^{tan(\pi/6)} f(t-1).1\,dt = 2\int_{tan(0)-1}^{tan(\pi/6)-1} \frac{1}{u^2}\,du = \left[-\frac{2}{u}\right]_{tan(0)-1}^{tan(\pi/6)-1} = -\frac{2}{tan(\pi/6)-1}+2$$

Exercice 2.2.a

$t = u(x) = tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.
On a

$$cos(x) = \frac{1 - tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})}$$

$$5 - 3cos(x) = 5 - 3.\frac{1 - tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})} = \frac{2 + 8tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})}$$

Donc

$$\frac{1}{5 - 3cos(x)} = \frac{1 + tan^2(\frac{x}{2})}{2(1 + 4tan^2(\frac{x}{2}))}$$

Et

$$\int_0^{\pi/2} \frac{1}{5 - 3cos(x)}\,dx = \int_0^{\pi/2} \frac{1 + tan^2(\frac{x}{2})}{2(1 + 4tan^2(\frac{x}{2}))}\,dx$$

$$\int_{tan(0)}^{tan(\pi/4)} \frac{1+t^2}{2(1+4t^2)}.\frac{2}{1+t^2}\,dt = \int_{tan(0)}^{tan(\pi/4)} \frac{1}{1+4t^2}\,dt$$

Soit $u(t) = 2t$, $u'(t) = 2$ et $f(u) = \frac{1}{2(u^2+1)}$ donc

$$\int_{tan(0)}^{tan(\pi/4)} \frac{1}{1+4t^2}\,dt = \int_{tan(0)}^{tan(\pi/4)} f(2t).2\,dt = \int_{2tan(0)}^{2tan(\pi/4)} \frac{1}{2(u^2+1)}\,du = \frac{1}{2}\left[arctan(u)\right]_{2tan(0)}^{2tan(\pi/4)} = \frac{\text{arctan}(2)}{2}$$

Exercice 2.2.b

$t = u(x) = tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.
On a

$$cos(x) = \frac{1 - tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})}$$

$$1 + cos(x) = 1 + \frac{1 - tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})} = \frac{2}{1 + tan^2(\frac{x}{2})}$$

$$(1 + cos(x))^2 = \frac{4}{(1 + tan^2(\frac{x}{2}))^2}$$

Donc

$$\frac{1}{(1 + cos(x))^2} = \frac{(1 + tan^2(\frac{x}{2}))^2}{4}$$

Et

$$\int_0^{\pi/2} \frac{1}{(1 + cos(x))^2}\,dx = \int_0^{\pi/2} \frac{(1 + tan^2(\frac{x}{2}))^2}{4}\,dx$$

$$\int_{tan(0)}^{tan(\pi/4)} \frac{(1+t^2)^2}{4}.\frac{2}{1+t^2}\,dt = \int_{tan(0)}^{tan(\pi/4)} \frac{1}{2}(1+t^2)\,dt = \frac{1}{2}\left[t + \frac{t^3}{3}\right]_{tan(0)}^{tan(\pi/4)} = \frac{1}{2}\left[t + \frac{t^3}{3}\right]_0^1 = \frac{2}{3}$$

Exercice 2.3

$t = u(x) = \tan(x)$ donc $x = u^{-1}(t) = arctan(x)$ et $dx = \frac{1}{1+t^2}\,dt$.

On a $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$\tan^2(x) = \frac{\sin^2(x)}{\cos^2(x)} = \frac{1-\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1$$

Donc

$$\int_0^{\pi/4} \frac{1}{\cos^4(x)}\,dx = \int_0^{\pi/4} \left(\frac{1}{\cos^2(x)}\right)^2 dx = \int_0^{\pi/4} (1+\tan^2(x))^2\,dx$$

$$\int_{\tan(0)}^{\tan(\pi/4)} (1+t^2)^2 . \frac{1}{1+t^2}\,dt = \int_{\tan(0)}^{\tan(\pi/4)} 1+t^2\,dt = [t+\frac{t^3}{3}]_{\tan(0)}^{\tan(\pi/4)} = \left[t+\frac{t^3}{3}\right]_0^1 = 1+\frac{1}{3} = \frac{4}{3}$$

Exercice 3

Exercice 3.a

$$\int \arcsin(x)\,dx$$

Par partie avec

$$\begin{array}{ll} f(x) = \arcsin(x) & f'(x) = \frac{1}{\sqrt{1-x^2}} \\ g'(x) = 1 & g(x) = x \end{array}$$

Donc

$$\int \arcsin(x)\,dx = x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

Soit $u(x) = 1-x^2$, $u'(x) = -2x$ et $f(u) = -\frac{1}{2\sqrt{u}}$ Donc

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = \int f(1-x^2).2x\,dx = \int -\frac{1}{2\sqrt{u}}\,du = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}.2\sqrt{u} = -\sqrt{u}$$

Donc

$$\int \arcsin(x)\,dx = x\arcsin(x) + \sqrt{1-x^2}$$

Exercice 3.b

$$\int x^n \ln(x)\,dx$$

Par partie avec

$$\begin{array}{ll} f(x) = \ln(x) & f'(x) = \frac{1}{x} \\ g'(x) = x^n & g(x) = \frac{x^{n+1}}{n+1} \end{array}$$

Donc

$$\int x^n \ln(x)\,dx = \frac{x^{n+1}\ln(x)}{n+1} - \int \frac{1}{x}.\frac{x^{n+1}}{n+1}\,dx = \frac{x^{n+1}\ln(x)}{n+1} - \frac{1}{n+1}\int x^n\,dx = \frac{x^{n+1}\ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2}$$

Exercice 3.c

$$\int x \arcsin(x)\, dx$$

Par partie avec

$$\begin{array}{ll} f(x) = \arcsin(x) & f'(x) = \frac{1}{\sqrt{1-x^2}} \\ g'(x) = x & g(x) = \frac{x^2}{2} \end{array}$$

Donc

$$\int x \arcsin(x)\, dx = \frac{1}{2}x^2 . \arcsin(x) - \int \frac{x^2}{2\sqrt{1-x^2}}\, dx$$

Soit $x = \sin(t)$, $\frac{dx}{dt} = \cos(t)$. Donc

$$\int \frac{x^2}{2\sqrt{1-x^2}}\, dx = \int \frac{\sin^2(t)\cos(t)}{2\sqrt{1-\sin^2(t)}}\, dt = \int \frac{\sin^2(t)\cos(t)}{2\sqrt{\cos^2(t)}}\, dt = \int \frac{1}{2}\sin^2(t)\, dt$$

Exercice 3.d

$$\int \ln(x^2+1)\, dx$$

Par partie avec

$$\begin{array}{ll} f(x) = \ln(x^2+1) & f'(x) = \frac{2x}{1+x^2} \\ g'(x) = 1 & g(x) = x \end{array}$$

Donc

$$\int \ln(x^2+1)\, dx = x\ln(x^2+1) - \int \frac{2x^2}{1+x^2}\, dx$$

$$\int \frac{2x^2}{1+x^2}\, dx = 2\int \frac{x^2+1-1}{1+x^2}\, dx = 2\int \frac{x^2+1}{1+x^2} - \frac{1}{1+x^2}\, dx$$

$$= 2\int 1\, dx - 2\int \frac{1}{1+x^2}\, dx = 2x - 2\arctan(x)$$

Enfin

$$\int \ln(x^2+1)\, dx = x\ln(x^2+1) - 2x + 2\arctan(x)$$

QED