Rappel de cours:

- $\int_a^b f(u(x))u'(x)\,dx = \int_{x(a)}^{u(b)} u\,du$

Exercice 1

Exercice 1.a

$$\int tan(x)\,dx = \int \frac{sin(x)}{cos(x)}\,dx$$

Soit $u(x) = cos(x)$, $u'(x) = -sin(x)$ et $f(u) = -\frac{1}{u}$ donc

$$\int \frac{sin(x)}{cos(x)}\,dx = \int f(cos(x))sin(x)\,dx = \int -\frac{1}{u}\,du = -ln(u) = -ln(cos(x))$$

Exercice 1.b

$$\int \frac{ln(x+1)}{x+1}\,dx$$

Soit $u(x) = ln(x+1)$, $u'(x) = \frac{1}{x+1}$ et $f(u) = u$ donc

$$\int \frac{ln(x+1)}{x+1}\,dx = \int f(ln(x+1)).\frac{1}{x+1}\,dx = \int u\,du = \frac{u^2}{2} = \frac{ln^2(x+1)}{2}$$

Exercice 1.c

$$\int \frac{x}{\sqrt{2x^2+3}}$$

Soit $u(x) = 2x^2+3$, $u'(x) = 4x$ et $f(u) = \frac{1}{4\sqrt{u}}$ donc

$$\int \frac{x}{\sqrt{2x^2+3}}\,dx = \int f(2x^2+3).4x\,dx = \int \frac{1}{4\sqrt{u}}\,du = \frac{1}{4}\int \frac{1}{\sqrt{u}\,du} = \frac{\sqrt{u}}{2} = \frac{\sqrt{2x^2+3}}{2}$$

Exercice 1.d

$$\int \frac{cos(x)}{sin^2(x)}\,dx$$

Soit $u(x) = sin(x)$, $u'(x) = cos(x)$ et $f(u) = \frac{1}{u^2}$ donc

$$\int \frac{cos(x)}{sin^2(x)}\,dx = \int f(sin(x)).cos(x)\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{sin(x)}$$

Exercice 1.e

$$\int x\sqrt{x^2+1}\,dx$$

Soit $u(x) = x^2+1$, $u'(x) = 2x$ et $f(u) = \frac{\sqrt{(u)}}{2}$ donc

$$\int x\sqrt{x^2+1}\,dx = \int f(x^2+1).2x\,dx = \int \frac{\sqrt{(u)}}{2}\,du = \frac{1}{2}\int \sqrt{(u)}\,du = \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}} = \frac{1}{3}u^{\frac{3}{2}} = \frac{(x^2+1)^{\frac{3}{2}}}{3}$$

Exercice 1.f

$$\int \frac{1}{x\ln(x)}\, dx$$

Soit $u(x) = \ln(x)$, $u'(x) = \frac{1}{x}$ et $f(u) = \frac{1}{u}$ donc

$$\int \frac{1}{x\ln(x)}\, dx = \int f(\ln(x)).\frac{1}{x}\, dx = \int \frac{1}{u}\, du = \ln(|u|) = \ln(|\ln(x)|)$$

Exercice 1.g

$$\int \sin^2(x)\cos(x)\, dx$$

Soit $u(x) = \sin(x)$, $u'(x) = \cos(x)$ et $f(u) = u^2$ donc

$$\int \sin^2(x)\cos(x)\, dx = \int f(\sin(x)).\cos(x)\, dx = \int u^2\, du = \frac{u^3}{3} = \frac{\sin^3(x)}{3}$$

Exercice 1.h

$$\int e^{\sin(x)}\cos(x)\, dx$$

Soit $u(x) = \sin(x)$, $u'(x) = \cos(x)$ et $f(u) = e^u$ donc

$$\int e^{\sin(x)}\cos(x)\, dx = \int f(\sin(x)).\cos(x)\, dx = \int e^u\, du = e^u = e^{\sin(x)}$$

Exercice 1.i

$$\int \frac{\ln^2(x)}{x}\, dx$$

Soit $u(x) = \ln(x)$, $u'(x) = \frac{1}{x}$ et $f(u) = u^2$ donc

$$\int \frac{\ln^2(x)}{x}\, dx = \int f(\ln(x)).\frac{1}{x}\, dx = \int u^2\, du = \frac{u^3}{3} = \frac{\ln^3(x)}{3}$$

Exercice 2

Exercice 2.a

$t = u(x) = \tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2\arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.
On a

$$\sin(x) = \frac{2\tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}$$

$$1 - \sin(x) = 1 - \frac{2\tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} = \frac{\tan^2(\frac{x}{2}) - 2\tan^2(\frac{x}{2}) + 1}{1 + \tan^2(\frac{x}{2})} = \frac{(\tan(\frac{x}{2}) - 1)^2}{1 + \tan^2(\frac{x}{2})}$$

Donc

$$\frac{1}{1 - \sin(x)} = \frac{1 + \tan^2(\frac{x}{2})}{(\tan(\frac{x}{2}) - 1)^2}$$

Et

$$\int_0^{\pi/3} \frac{1}{1 - \sin(x)}\, dx = \int_0^{\pi/3} \frac{1 + \tan^2(\frac{x}{2})}{(\tan(\frac{x}{2}) - 1)^2}\, dx$$

$$\int_{\tan(0)}^{\tan(\pi/6)} \frac{1 + t^2}{(t - 1)^2}.\frac{2}{1 + t^2}\, dt = \int_{\tan(0)}^{\tan(\pi/6)} \frac{2}{(t - 1)^2}\, dt$$

Soit $u(x) = x - 1$, $u'(x) = 1$ et $f(u) = \frac{1}{u^2}$ donc

$$\int_{tan(0)}^{tan(\pi/6)} \frac{2}{(t-1)^2}\, dt = 2\int_{tan(0)}^{tan(\pi/6)} f(t-1).1\, dt = 2\int_{tan(0)-1}^{tan(\pi/6)-1} \frac{1}{u^2}\, du = [\frac{2}{u}]_{tan(0)-1}^{tan(\pi/6)-1} = -2 - \frac{2}{tan(\pi/6)-1}$$

Exercice 2.b

$t = u(x) = tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.
On a
$$cos(x) = \frac{1 - tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})}$$

$$5 - 3cos(x) = 5 - 3.\frac{1 - tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})} = \frac{2 + 8tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})}$$

Donc
$$\frac{1}{5 - 3cos(x)} = \frac{1 + tan^2(\frac{x}{2})}{2(1 + 4tan^2(\frac{x}{2}))}$$

Et
$$\int_0^{\pi/2} \frac{1}{5 - 3cos(x)}\, dx = \int_0^{\pi/2} \frac{1 + tan^2(\frac{x}{2})}{2(1 + 4tan^2(\frac{x}{2}))}\, dx$$

$$\int_{tan(0)}^{tan(\pi/4)} \frac{1 + t^2}{2(1 + 4t^2)}.\frac{2}{1 + t^2}\, dt = \int_{tan(0)}^{tan(\pi/4)} \frac{1}{1 + 4t^2}\, dt$$

Soit $u(t) = 2t$, $u'(t) = 2$ et $f(u) = \frac{1}{2(u^2+1)}$ donc

$$\int_{tan(0)}^{tan(\pi/4)} \frac{1}{1 + 4t^2}\, dt = \int_{tan(0)}^{tan(\pi/4)} f(2t).2\, dt = \int_{2tan(0)}^{2tan(\pi/4)} \frac{1}{2(u^2 + 1)}\, du = \frac{1}{2}[arctan(u)]_{2tan(0)}^{2tan(\pi/4)}$$

Exercice 2.c

$t = u(x) = tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.
On a
$$cos(x) = \frac{1 - tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})}$$

$$1 + cos(x) = 1 + \frac{1 - tan^2(\frac{x}{2})}{1 + tan^2(\frac{x}{2})} = \frac{2}{1 + tan^2(\frac{x}{2})}$$

$$(1 + cos(x))^2 = \frac{4}{(1 + tan^2(\frac{x}{2}))^2}$$

Donc
$$\frac{1}{(1 + cos(x))^2} = \frac{(1 + tan^2(\frac{x}{2}))^2}{4}$$

Et
$$\int_0^{\pi/2} \frac{1}{(1 + cos(x))^2}\, dx = \int_0^{\pi/2} \frac{(1 + tan^2(\frac{x}{2}))^2}{4}\, dx$$

$$\int_{tan(0)}^{tan(\pi/4)} \frac{(1 + t^2)^2}{4}.\frac{2}{1 + t^2}\, dt = \int_{tan(0)}^{tan(\pi/4)} \frac{1}{2}(1 + t^2)\, dt = \frac{1}{2}[t + \frac{t^3}{3}]_{tan(0)}^{tan(\pi/4)}$$

Exercice 3

QED