

Exercice 17

Une suite réelle u_n converge vers le réel l si

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n \geq N_\epsilon \implies |u_n - l| < \epsilon \text{ [P1]}$$

Une suite réelle u_n diverge vers $+\infty$ si

$$\forall A \in \mathbb{R} \exists N_A \in \mathbb{N}, \forall n \geq N_A \implies U_A \geq A \text{ [P2]}$$

Une suite réelle u_n diverge vers $-\infty$ si

$$\forall B \in \mathbb{R} \exists N_B \in \mathbb{N}, \forall n \geq N_B \implies U_B \leq B \text{ [P3]}$$

Exercice 17.1

Supposons que $l = 2$.

Prenons un $\epsilon > 0$, trouvons un N_ϵ tel que $|u_{N_\epsilon} - 2| < \epsilon$. Par exemple, $N_\epsilon = 4$, car $|u_4 - 2| = |2 - 2| = 0 < \epsilon$. Maintenant, vérifions [P1] pour $l = 2$. $\forall \epsilon > 0, \forall n > 4, |u_n - 2| < \epsilon$, calculons $u_{n>4} = 2 = u_4$, la propriété [P1] est vérifiée pour tous les $n > N_\epsilon$.

Exercice 17.2

- $a = 1$, $\lim_{n \rightarrow \infty} u_n = 1$
- $|a| < 1$, $\lim_{n \rightarrow \infty} u_n = 0$
- $a \leq -1$, pas de limite
- $a \geq 1$, $\lim_{n \rightarrow \infty} u_n = +\infty$

Pour le second cas, posons $l = 0$, trouvons un N_ϵ tel que $|u_{N_\epsilon} - 0| < \epsilon$. Par exemple, $N_\epsilon, |a^{N_\epsilon}| < \epsilon$. N_ϵ existe car $|a| < 1$. On a bien $|u_{N_\epsilon} - 0| = |a^{N_\epsilon}| < \epsilon$. Maintenant, vérifions [P1] pour $l = 0$. $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, |u_{N_\epsilon} - 0| < \epsilon$. Calculons $u_{N_\epsilon+1} = a^{N_\epsilon+1} < a^{N_\epsilon}$, la propriété [P1] est vérifiée pour tous les $n > N_\epsilon$. Même raisonnement pour les autres cas.

Exercice 17.3

La suite diverge. Prenons un A , et calculons N_A tel que $u_{N_A} > A$. Calculons u_{N_A+1} .

$u_{N_A+1} = \frac{(N_A+1)^{N_A+1}}{(N_A+1)!} = \frac{(N_A+1) \dots (N_A+1) \cdot (N_A+1)}{N_A! (N_A+1)} = \frac{(N_A+1) \dots (N_A+1)}{N_A!}$, Les nombres au numérateur sont toujours plus grand que ceux du numérateur pour u_{N_A} , donc la suite $u_{N_A+1} > u_{N_A} > A$.

La propriété [P2] est vérifiée.

La suite diverge. Prenons un A , et calculons N_A tel que $u_{N_A} > A$. Calculons u_{N_A+1} .

$u_{N_A+1} = \frac{(N_A+1)!}{2^{N_A+1}} = \frac{N_A! (N_A+1)}{2 \cdot 2 \cdot 2 \dots 2} = U_{N_A} \cdot \frac{(N_A+1)}{2}$. Donc la suite $u_{N_A+1} > u_{N_A} > A$.

La propriété [P2] est vérifiée.

Exercice 18

On a $\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N}, \forall n \geq N_\epsilon \implies |u_n - l| < \epsilon$

Exercice 39

Exercice 39.1

$$\lim_{n \rightarrow +\infty} u_{2n} = (-1)^{2n} = 1 \text{ et } \lim_{n \rightarrow +\infty} u_{2n+1} = (-1)^{2n+1} = -1$$

Exercice 39.2

$$\lim_{n \rightarrow +\infty} v_{4n} = \sin(4n \frac{\pi}{2}) = 0 \text{ et } \lim_{n \rightarrow +\infty} v_{4n+1} = \sin((4n+1) \frac{\pi}{2}) = 1$$

$$\lim_{n \rightarrow +\infty} v_{4n+2} = \sin((4n+2) \frac{\pi}{2}) = 0 \text{ et } \lim_{n \rightarrow +\infty} v_{4n+3} = \sin((4n+3) \frac{\pi}{2}) = -1$$

Exercice 39.3

$$\lim_{n \rightarrow +\infty} w_{6n} = \sin(6n \frac{\pi}{3}) = 0 \text{ et } \lim_{n \rightarrow +\infty} w_{6n+1} = \sin((6n+1) \frac{\pi}{3}) = \frac{\sqrt{3}}{2}$$

$$\lim_{n \rightarrow +\infty} w_{6n+2} = \sin((6n+2) \frac{\pi}{3}) = \frac{\sqrt{3}}{2} \text{ et } \lim_{n \rightarrow +\infty} w_{6n+3} = \sin((6n+3) \frac{\pi}{3}) = -1$$

$$\lim_{n \rightarrow +\infty} w_{6n+4} = \sin((6n+4) \frac{\pi}{3}) = -\frac{\sqrt{3}}{2} \text{ et } \lim_{n \rightarrow +\infty} w_{6n+5} = \sin((6n+5) \frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$$

Exercice 39.4

$$\lim_{n \rightarrow +\infty} x_{4n} = \sin^{4n}(4n \frac{\pi}{2}) = 0 \text{ et } \lim_{n \rightarrow +\infty} x_{4n+1} = \sin^{4n+1}((4n+1) \frac{\pi}{2}) = 1$$

$$\lim_{n \rightarrow +\infty} x_{4n+2} = \sin^{4n+2}((4n+2) \frac{\pi}{2}) = 0 \text{ et } \lim_{n \rightarrow +\infty} x_{4n+3} = \sin^{4n+3}((4n+3) \frac{\pi}{2}) = -1$$

Exercice 39.5

$$\lim_{n \rightarrow +\infty} x_{4n} = |\sin(4n \frac{\pi}{2})|^{\frac{4n}{2}} = 0 \text{ et } \lim_{n \rightarrow +\infty} x_{4n+1} = |\sin((4n+1) \frac{\pi}{2})|^{\frac{4n+1}{2}} = 1$$

$$\lim_{n \rightarrow +\infty} x_{4n+2} = |\sin((4n+2) \frac{\pi}{2})|^{\frac{4n+2}{2}} = 0 \text{ et } \lim_{n \rightarrow +\infty} x_{4n+3} = |\sin((4n+3) \frac{\pi}{2})|^{\frac{4n+3}{2}} = 1$$

Exercice 39.6

$$\lim_{n \rightarrow +\infty} v_{4n} = 4n \sin(4n \frac{\pi}{2}) = 0 \text{ et } \lim_{n \rightarrow +\infty} v_{4n+1} = (4n+1) \sin((4n+1) \frac{\pi}{2}) = 4n+1$$

$$\lim_{n \rightarrow +\infty} v_{4n+2} = \sin((4n+2) \frac{\pi}{2}) = 0 \text{ et } \lim_{n \rightarrow +\infty} v_{4n+3} = (4n+3) \sin((4n+3) \frac{\pi}{2}) = -4n-3$$

Exercice 40**Exercice 40.1**

$$\lim_{n \rightarrow +\infty} u_{2n} = \frac{1}{2^{2n}} = 0 \text{ et } \lim_{n \rightarrow +\infty} u_{2n+1} = 2n+1 = +\infty$$

Exercice 40.2

$$\lim_{n \rightarrow +\infty} u_{2n} = \frac{2n+4}{2^{2n}} = 0 \text{ et } \lim_{n \rightarrow +\infty} u_{2n+1} = \frac{2n+7}{e^{2n+1}} = 0$$

Exercice 40.3

Valeur $a = 0$, u_{2n} n'est pas définie.

Valeur $|a| < 1$, u_{2n}

$$\lim_{n \rightarrow +\infty} u_{2n} = \frac{2n+4}{a^{2n}} = +\infty$$

Valeur $|a| = 1$, u_{2n}

$$\lim_{n \rightarrow +\infty} u_{2n} = \frac{2n+4}{1^{2n}} = 2n+4$$

Valeur $|a| > 1$, u_{2n}

$$\lim_{n \rightarrow +\infty} u_{2n} = \frac{2n+4}{a^{2n}} = 0$$

Et

$$\lim_{n \rightarrow +\infty} u_{2n+1} = 2n+1 = +\infty$$

Exercice 40.4

$$\lim_{n \rightarrow +\infty} u_{3n} = \frac{3n+4}{3n} = 1 + \frac{4}{3n} = 1$$

Et

$$\lim_{n \rightarrow +\infty} u_{3n+1} = 3$$

Et

$$\lim_{n \rightarrow +\infty} u_{3n+2} = \frac{(3n+2)^2+1}{3n^2} = \frac{9n^2+12n+5}{3n^2} = 3 + \frac{4}{n} + \frac{5}{3n^2} = 3$$

Exercice 51**Exercice 51.1**

$$\lim_{x \rightarrow 1} x^2 + 1 = \lim_{x \rightarrow 0} (x+1)^2 + 1 = \lim_{x \rightarrow 0} x^2 + \lim_{x \rightarrow 0} 2x + \lim_{x \rightarrow 0} 1 + \lim_{x \rightarrow 0} 1 = 0 + 0 + 1 + 1 = 2$$

Exercice 51.2

$$\lim_{x \rightarrow +\infty} -x - \ln(x) = -\lim_{x \rightarrow +\infty} x - \lim_{x \rightarrow +\infty} \ln(x) = -\infty - \infty = -\infty$$

Exercice 51.3

$$\lim_{x \rightarrow +\infty} x - \ln(x) = x \cdot \left(1 - \frac{\ln(x)}{x}\right) = \lim_{x \rightarrow +\infty} x \cdot \lim_{x \rightarrow +\infty} \left(1 - \frac{\ln(x)}{x}\right) = +\infty \cdot (1 - 0) = +\infty$$

Car $\ln(x) \ll x$.

Exercice 51.4

$$\begin{aligned} \lim_{x \rightarrow 1^+} \frac{1}{x-1} + \ln(x-1) &= \lim_{x \rightarrow 0^+} \frac{1}{x} + \ln(x) = \lim_{x \rightarrow 0^+} \frac{1+x \ln(x)}{x} = \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot \lim_{x \rightarrow 0^+} (1+x \ln(x)) \\ &= \frac{1}{0^+} \cdot (1+0) = +\infty \end{aligned}$$

Voir exercice 17.

Exercice 51.5

$$\lim_{x \rightarrow -\infty} x^2 - x = \lim_{x \rightarrow -\infty} x(x-1) = \lim_{x \rightarrow -\infty} x \cdot \lim_{x \rightarrow -\infty} (x-1) = -\infty \cdot -\infty = +\infty$$

Exercice 52

On a $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$, calculons $\lim_{x \rightarrow 0} \cos(x)$.

En utilisant la règle de l'Hospital, on a $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \lim_{x \rightarrow 0} \frac{\sin'(x)}{x'} = \lim_{x \rightarrow 0} \frac{\cos(x)}{1} = \lim_{x \rightarrow 0} \cos(x) = 1$.

Exercice 51.1

$$\lim_{x \rightarrow 0} \frac{\cos(x)}{x - \frac{\pi}{2}} = \frac{\lim_{x \rightarrow 0} \cos(x)}{\lim_{x \rightarrow 0} x - \frac{\pi}{2}} = \frac{1}{-\frac{\pi}{2}} = -\frac{2}{\pi}$$

Exercice 52.2

On a $\tan(x) = \frac{\sin(x)}{\cos(x)}$. Calculons $\lim_{x \rightarrow 0} \tan(x) = \lim_{x \rightarrow 0} \frac{\sin(x)}{\cos(x)} = \lim_{x \rightarrow 0} \frac{x \sin(x)}{x \cos(x)}$.

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} \cdot \lim_{x \rightarrow 0} \frac{x}{\cos(x)} = 1 \cdot \frac{\lim_{x \rightarrow 0} x}{\lim_{x \rightarrow 0} \cos(x)} = 1 \cdot 0 = 0$$

Exercice 52.3

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\cos(x) - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{\cos(x) - (\cos^2(x) + \sin^2(x))}{x^2} = \lim_{x \rightarrow 0} \frac{\cos(x)}{x^2} - \frac{\cos^2(x)}{x^2} - \frac{\sin^2(x)}{x^2} \\ \lim_{x \rightarrow 0} \left(\frac{\cos(x)}{x^2} - \frac{\cos^2(x)}{x^2} \right) - \lim_{x \rightarrow 0} \frac{\sin^2(x)}{x^2} &= \lim_{x \rightarrow 0} \left(\frac{\cos(x)}{x^2} - \frac{\cos^2(x)}{x^2} \right) - \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \\ &= \lim_{x \rightarrow 0} \left(\frac{\cos(x)}{x^2} (1 - \cos(x)) \right) - 1 \end{aligned}$$

Exercice 53**Exercice 53.1**

$$\lim_{x \rightarrow 1} \frac{1}{x-1} - \frac{2}{x^2-1} = \lim_{x \rightarrow 1} \frac{x+1}{(x+1)(x-1)} - \frac{2}{(x+1)(x-1)} = \lim_{x \rightarrow 1} \frac{x-1}{(x+1)(x-1)} = \lim_{x \rightarrow 1} \frac{1}{x+1} = \frac{1}{2}$$

Exercice 53.2

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \rightarrow 0} \frac{(\sqrt{1+x} - 1)(\sqrt{1+x} + 1)}{x(\sqrt{1+x} + 1)} = \lim_{x \rightarrow 0} \frac{1+x-1}{x(\sqrt{1+x} + 1)} = \lim_{x \rightarrow 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{2}$$

Exercice 53.3

Calculer $\lim_{x \rightarrow 0} \frac{|x|}{x}$. 2 cas $\lim_{x \rightarrow 0^+} \frac{|x|}{x}$ et $\lim_{x \rightarrow 0^-} \frac{|x|}{x}$

$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1$$

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$$

Exercice 53.4

$$\lim_{x \rightarrow 0} \frac{\sin(4x)}{\tan(5x)}$$

Utilisation des développements limités de $\sin(x)$ et $\tan(x)$

$$\lim_{x \rightarrow 0} \frac{4x - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} + \epsilon(x)}{5x + \frac{(5x)^3}{3} + \frac{2(5x)^5}{15} + \epsilon(x)} = \lim_{x \rightarrow 0} \frac{4 - \frac{(4x)^2}{3!} + \frac{(4x)^4}{5!} + \epsilon(x)}{5 + \frac{(5x)^2}{3} + \frac{(25x)^4}{15} + \epsilon(x)} = \frac{\lim_{x \rightarrow 0} 4 - \frac{(4x)^2}{3!} + \frac{(4x)^4}{5!} + \epsilon(x)}{\lim_{x \rightarrow 0} 5 - \frac{(5x)^2}{3} + \frac{(25x)^4}{15} + \epsilon(x)} = \frac{4}{5}$$

Exercice 53.5

Calculer $\lim_{x \rightarrow 0} \frac{|\sin(4x)|}{\tan(5x)}$. 2 cas $\lim_{x \rightarrow 0+} \frac{|\sin(4x)|}{\tan(5x)}$ et $\lim_{x \rightarrow 0-} \frac{|\sin(4x)|}{\tan(5x)}$ Utilisation des développements limités de $\sin(x)$ et $\tan(x)$

$$\lim_{x \rightarrow 0+} \frac{|4x - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} + \epsilon(x)|}{5x + \frac{(5x)^3}{3} + \frac{2(5x)^5}{15} + \epsilon(x)} = \lim_{x \rightarrow 0} \frac{4 - \frac{(4x)^2}{3!} + \frac{(4x)^4}{5!} + \epsilon(x)}{5 + \frac{(5x)^2}{3} + \frac{(25x)^4}{15} + \epsilon(x)} = \frac{\lim_{x \rightarrow 0} 4 - \frac{(4x)^2}{3!} + \frac{(4x)^4}{5!} + \epsilon(x)}{\lim_{x \rightarrow 0} 5 - \frac{(5x)^2}{3} + \frac{(25x)^4}{15} + \epsilon(x)} = \frac{4}{5}$$

On sait que la fonction \sin est impaire donc $\sin(-x) = -\sin(x)$.

$$\lim_{x \rightarrow 0-} \frac{|4x - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} + \epsilon(x)|}{5x + \frac{(5x)^3}{3} + \frac{2(5x)^5}{15} + \epsilon(x)} = \frac{-4x + \frac{(4x)^3}{3!} - \frac{(4x)^5}{5!} + \epsilon(x)}{5x + \frac{(5x)^3}{3} + \frac{2(5x)^5}{15} + \epsilon(x)} = \lim_{x \rightarrow 0} \frac{-4 + \frac{(4x)^2}{3!} - \frac{(4x)^4}{5!} + \epsilon(x)}{5 + \frac{(5x)^2}{3} + \frac{(25x)^4}{15} + \epsilon(x)} = -\frac{4}{5}$$

Exercice 53.6

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\cos(x)}{x - \frac{\pi}{2}} = \lim_{x \rightarrow 0} \frac{\cos(x + \frac{\pi}{2})}{x} = \lim_{x \rightarrow 0} -\frac{\sin(x)}{x} = -1$$

Exercice 54**Exercice 54.1.1**

$$\lim_{x \rightarrow +\infty} \frac{2x^3 - 3x^2 + 1}{-4x^3 + 3x + 1} = \lim_{x \rightarrow +\infty} \frac{2 - 3x^{-1} + x^{-3}}{-4 + 3x^{-2} + x^{-3}} = \frac{2}{-4} = -\frac{1}{2}$$

Exercice 54.1.2

$$\begin{aligned} \lim_{x \rightarrow 1} \frac{2x^3 - 3x^2 + 1}{-4x^3 + 3x + 1} &= \lim_{x \rightarrow 0} \frac{2(x+1)^3 - 3(x+1)^2 + 1}{-4(x+1)^3 + 3(x+1) + 1} \\ &= \lim_{x \rightarrow 0} \frac{2x^3 + 3x^2}{-4x^3 - 12x^2 - 9x} = \lim_{x \rightarrow 0} \frac{2x^2 + 3x}{-4x^2 - 12x - 9} = \frac{\lim_{x \rightarrow 0} 2x^2 + 3x}{\lim_{x \rightarrow 0} -4x^2 - 12x - 9} = \frac{0}{9} = 0 \end{aligned}$$

Exercice 54.2

$$\lim_{x \rightarrow +\infty} \frac{2x+3}{3x^4+2} e^x =$$

Calculons

$$\begin{aligned} \lim_{x \rightarrow +\infty} \ln \left(\frac{2x+3}{3x^4+2} e^x \right) &= \lim_{x \rightarrow +\infty} \ln \left(\frac{2x+3}{3x^4+2} \right) + \ln(e^x) = \lim_{x \rightarrow +\infty} \ln \left(\frac{2x^{-3} + 3x^{-4}}{3 + 2x^{-4}} \right) + x \\ &= \ln \left(\lim_{x \rightarrow +\infty} 2x^{-3} + 3x^{-4} \right) - \ln \left(\lim_{x \rightarrow +\infty} 3 + 2x^{-4} \right) + \lim_{x \rightarrow +\infty} x = \\ &= \ln \left(\lim_{x \rightarrow +\infty} 2x + 3 \right) - \ln \left(\lim_{x \rightarrow +\infty} x^4 \right) - \ln(3) + \lim_{x \rightarrow +\infty} x = +\infty \end{aligned}$$

Exercice 54.3

$$\lim_{x \rightarrow +\infty} \frac{2x+3}{3x^4+2} e^{\ln(x)} = \lim_{x \rightarrow +\infty} \frac{2x+3}{3x^4+2} x = \lim_{x \rightarrow +\infty} \frac{2x^2+3x}{3x^4+2} = \lim_{x \rightarrow +\infty} \frac{2x^{-2}+3x^{-3}}{3+2x^{-4}} = \frac{\lim_{x \rightarrow +\infty} 2x^{-2}+3x^{-3}}{\lim_{x \rightarrow +\infty} 3+2x^{-4}} = \frac{0}{3} = 0$$

Exercice 55**Exercice 55.2**

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{\ln(1+2x)}{\ln(1+x)} &= \lim_{x \rightarrow +\infty} \frac{\ln((x^{-1}+2)x)}{\ln(1+x)} = \lim_{x \rightarrow +\infty} \frac{\ln((x^{-1}+2) + \ln(x))}{\ln(1+x)} \\ \lim_{x \rightarrow +\infty} \frac{\ln((x^{-1}+2))}{\ln(1+x)} + \frac{\ln(x)}{\ln(1+x)} &= \lim_{x \rightarrow +\infty} \frac{\ln(2)}{\ln(1+x)} + \frac{\ln(x)}{\ln(1+x)} = 1 \end{aligned}$$

car à l'infini $x \approx 1+x$.

QED