

Rappel de cours:

- $\int_a^b f(u(x))u'(x) dx = \int_{x(a)}^{u(b)} u du$

Exercice 1

Exercice 1.a

$$\int \tan(x) dx = \int \frac{\sin(x)}{\cos(x)} dx$$

Soit $u(x) = \cos(x)$, $u'(x) = -\sin(x)$ et $f(u) = -\frac{1}{u}$ donc

$$\int \frac{\sin(x)}{\cos(x)} dx = \int f(\cos(x))\sin(x) dx = \int -\frac{1}{u} du = -\ln(u) = -\ln(\cos(x))$$

Exercice 1.b

$$\int \frac{\ln(x+1)}{x+1} dx$$

Soit $u(x) = \ln(x+1)$, $u'(x) = \frac{1}{x+1}$ et $f(u) = u$ donc

$$\int \frac{\ln(x+1)}{x+1} dx = \int f(\ln(x+1)) \cdot \frac{1}{x+1} dx = \int u du = \frac{u^2}{2} = \frac{\ln^2(x+1)}{2}$$

Exercice 1.c

$$\int \frac{x}{\sqrt{2x^2+3}}$$

Soit $u(x) = 2x^2 + 3$, $u'(x) = 4x$ et $f(u) = \frac{1}{4\sqrt{u}}$ donc

$$\int \frac{x}{\sqrt{2x^2+3}} dx = \int f(2x^2+3) \cdot 4x dx = \int \frac{1}{4\sqrt{u}} du = \frac{1}{4} \int \frac{1}{\sqrt{u}} du = \frac{\sqrt{u}}{2} = \frac{\sqrt{2x^2+3}}{2}$$

Exercice 1.d

$$\int \frac{\cos(x)}{\sin^2(x)} dx$$

Soit $u(x) = \sin(x)$, $u'(x) = \cos(x)$ et $f(u) = \frac{1}{u^2}$ donc

$$\int \frac{\cos(x)}{\sin^2(x)} dx = \int f(\sin(x)) \cdot \cos(x) dx = \int \frac{1}{u^2} du = -\frac{1}{u} = -\frac{1}{\sin(x)}$$

Exercice 1.e

$$\int x\sqrt{x^2+1} dx$$

Soit $u(x) = x^2 + 1$, $u'(x) = 2x$ et $f(u) = \frac{\sqrt{u}}{2}$ donc

$$\int x\sqrt{x^2+1} dx = \int f(x^2+1) \cdot 2x dx = \int \frac{\sqrt{u}}{2} du = \frac{1}{2} \int \sqrt{u} du = \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} = \frac{1}{3} u^{\frac{3}{2}} = \frac{(x^2+1)^{\frac{3}{2}}}{3}$$

Exercice 1.f

$$\int \frac{1}{x \ln(x)} dx$$

Soit $u(x) = \ln(x)$, $u'(x) = \frac{1}{x}$ et $f(u) = \frac{1}{u}$ donc

$$\int \frac{1}{x \ln(x)} dx = \int f(\ln(x)) \cdot \frac{1}{x} dx = \int \frac{1}{u} du = \ln(|u|) = \ln(|\ln(x)|)$$

Exercice 1.g

$$\int \sin^2(x) \cos(x) dx$$

Soit $u(x) = \sin(x)$, $u'(x) = \cos(x)$ et $f(u) = u^2$ donc

$$\int \sin^2(x) \cos(x) dx = \int f(\sin(x)) \cdot \cos(x) dx = \int u^2 du = \frac{u^3}{3} = \frac{\sin^3(x)}{3}$$

Exercice 1.h

$$\int e^{\sin(x)} \cos(x) dx$$

Soit $u(x) = \sin(x)$, $u'(x) = \cos(x)$ et $f(u) = e^u$ donc

$$\int e^{\sin(x)} \cos(x) dx = \int f(\sin(x)) \cdot \cos(x) dx = \int e^u du = e^u = e^{\sin(x)}$$

Exercice 1.i

$$\int \frac{\ln^2(x)}{x} dx$$

Soit $u(x) = \ln(x)$, $u'(x) = \frac{1}{x}$ et $f(u) = u^2$ donc

$$\int \frac{\ln^2(x)}{x} dx = \int f(\ln(x)) \cdot \frac{1}{x} dx = \int u^2 du = \frac{u^3}{3} = \frac{\ln^3(x)}{3}$$

Exercice 2**Exercice 2.a**

$t = u(x) = \tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2\arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.

On a

$$\sin(x) = \frac{2\tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}$$

$$1 - \sin(x) = 1 - \frac{2\tan(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} = \frac{\tan^2(\frac{x}{2}) - 2\tan(\frac{x}{2}) + 1}{1 + \tan^2(\frac{x}{2})} = \frac{(\tan(\frac{x}{2}) - 1)^2}{1 + \tan^2(\frac{x}{2})}$$

Donc

$$\frac{1}{1 - \sin(x)} = \frac{1 + \tan^2(\frac{x}{2})}{(\tan(\frac{x}{2}) - 1)^2}$$

Et

$$\begin{aligned} \int_0^{\pi/3} \frac{1}{1 - \sin(x)} dx &= \int_0^{\pi/3} \frac{1 + \tan^2(\frac{x}{2})}{(\tan(\frac{x}{2}) - 1)^2} dx \\ \int_{\tan(0)}^{\tan(\pi/6)} \frac{1+t^2}{(t-1)^2} \cdot \frac{2}{1+t^2} dt &= \int_{\tan(0)}^{\tan(\pi/6)} \frac{2}{(t-1)^2} dt \end{aligned}$$

Soit $u(x) = x - 1$, $u'(x) = 1$ et $f(u) = \frac{1}{u^2}$ donc

$$\int_{\tan(0)}^{\tan(\pi/6)} \frac{2}{(t-1)^2} dt = 2 \int_{\tan(0)}^{\tan(\pi/6)} f(t-1) \cdot 1 dt = 2 \int_{\tan(0)-1}^{\tan(\pi/6)-1} \frac{1}{u^2} du = \left[-\frac{2}{u} \right]_{\tan(0)-1}^{\tan(\pi/6)-1} = -2 - \frac{2}{\tan(\pi/6) - 1}$$

Exercice 2.b

$t = u(x) = \tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2\arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.

On a

$$\cos(x) = \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}$$

$$5 - 3\cos(x) = 5 - 3 \cdot \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} = \frac{2 + 8\tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}$$

Donc

$$\frac{1}{5 - 3\cos(x)} = \frac{1 + \tan^2(\frac{x}{2})}{2(1 + 4\tan^2(\frac{x}{2}))}$$

Et

$$\begin{aligned} \int_0^{\pi/2} \frac{1}{5 - 3\cos(x)} dx &= \int_0^{\pi/2} \frac{1 + \tan^2(\frac{x}{2})}{2(1 + 4\tan^2(\frac{x}{2}))} dx \\ \int_{\tan(0)}^{\tan(\pi/4)} \frac{1 + t^2}{2(1 + 4t^2)} \cdot \frac{2}{1 + t^2} dt &= \int_{\tan(0)}^{\tan(\pi/4)} \frac{1}{1 + 4t^2} dt \end{aligned}$$

Soit $u(t) = 2t$, $u'(t) = 2$ et $f(u) = \frac{1}{2(u^2+1)}$ donc

$$\int_{\tan(0)}^{\tan(\pi/4)} \frac{1}{1 + 4t^2} dt = \int_{\tan(0)}^{\tan(\pi/4)} f(2t) \cdot 2 dt = \int_{2\tan(0)}^{2\tan(\pi/4)} \frac{1}{2(u^2 + 1)} du = \frac{1}{2} [\arctan(u)]_{2\tan(0)}^{2\tan(\pi/4)}$$

Exercice 2.c

$t = u(x) = \tan(\frac{x}{2})$ donc $x = u^{-1}(t) = 2\arctan(t)$ et $(u^{-1})'(t) = \frac{dx}{dt} = \frac{2}{1+t^2}$.

On a

$$\cos(x) = \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})}$$

$$1 + \cos(x) = 1 + \frac{1 - \tan^2(\frac{x}{2})}{1 + \tan^2(\frac{x}{2})} = \frac{2}{1 + \tan^2(\frac{x}{2})}$$

$$(1 + \cos(x))^2 = \frac{4}{(1 + \tan^2(\frac{x}{2}))^2}$$

Donc

$$\frac{1}{(1 + \cos(x))^2} = \frac{(1 + \tan^2(\frac{x}{2}))^2}{4}$$

Et

$$\begin{aligned} \int_0^{\pi/2} \frac{1}{(1 + \cos(x))^2} dx &= \int_0^{\pi/2} \frac{(1 + \tan^2(\frac{x}{2}))^2}{4} dx \\ \int_{\tan(0)}^{\tan(\pi/4)} \frac{(1 + t^2)^2}{4} \cdot \frac{2}{1 + t^2} dt &= \int_{\tan(0)}^{\tan(\pi/4)} \frac{1}{2} (1 + t^2) dt = \frac{1}{2} \left[t + \frac{t^3}{3} \right]_{\tan(0)}^{\tan(\pi/4)} \end{aligned}$$

Exercice 2.c

Trop complexe.

Exercice 3

Exercice 3.a

$$\int \arcsin(x) dx$$

Par partie avec

$$\begin{aligned} f(x) &= \arcsin(x) & f'(x) &= \frac{1}{\sqrt{1-x^2}} \\ g'(x) &= 1 & g(x) &= x \end{aligned}$$

Donc

$$\int \arcsin(x) dx = x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} dx$$

Soit $u(x) = 1 - x^2$, $u'(x) = -2x$ et $f(u) = -\frac{1}{2\sqrt{u}}$ Donc

$$\int \frac{x}{\sqrt{1-x^2}} dx = \int f(1-x^2) \cdot 2x dx = \int -\frac{1}{2\sqrt{u}} du = -\frac{1}{2} \int \frac{1}{\sqrt{u}} du = -\frac{1}{2} \cdot 2\sqrt{u} = -\sqrt{u}$$

Donc

$$\int \arcsin(x) dx = x \arcsin(x) + \sqrt{1-x^2}$$

Exercice 3.b

$$\int x^n \ln(x) dx$$

Par partie avec

$$\begin{aligned} f(x) &= \ln(x) & f'(x) &= \frac{1}{x} \\ g'(x) &= x^n & g(x) &= \frac{x^{n+1}}{n+1} \end{aligned}$$

Donc

$$\int x^n \ln(x) dx = \frac{x^{n+1} \ln(x)}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1} dx = \frac{x^{n+1} \ln(x)}{n+1} - \frac{1}{n+1} \int x^n dx = \frac{x^{n+1} \ln(x)}{n+1} - \frac{x^{n+1}}{(n+1)^2}$$

Exercice 3.c

$$\int x \arcsin(x) dx$$

Par partie avec

$$\begin{aligned} f(x) &= \arcsin(x) & f'(x) &= \frac{1}{\sqrt{1-x^2}} \\ g'(x) &= x & g(x) &= \frac{x^2}{2} \end{aligned}$$

Donc

$$\int x \arcsin(x) dx = \frac{1}{2} x^2 \cdot \arcsin(x) - \int \frac{x^2}{2\sqrt{1-x^2}} dx$$

Soit $x = \sin(t)$, $\frac{dx}{dt} = \cos(t)$. Donc

$$\int \frac{x^2}{2\sqrt{1-x^2}} dx = \int \frac{\sin^2(t) \cos(t)}{2\sqrt{1-\sin^2(t)}} dt = \int \frac{\sin^2(t) \cos(t)}{2\sqrt{\cos^2(t)}} dt = \int \frac{1}{2} \sin^2(t) dt$$

Exercice 3.d

$$\int \ln(x^2 + 1) \, dx$$

Par partie avec

$$\begin{array}{ll} f(x) = \ln(x^2 + 1) & f'(x) = \frac{2x}{1+x^2} \\ g'(x) = 1 & g(x) = x \end{array}$$

Donc

$$\begin{aligned} \int \ln(x^2 + 1) \, dx &= x \ln(x^2 + 1) - \int \frac{2x^2}{1+x^2} \, dx \\ \int \frac{2x^2}{1+x^2} \, dx &= 2 \int \frac{x^2 + 1 - 1}{1+x^2} \, dx = 2 \int \frac{x^2 + 1}{1+x^2} - \frac{1}{1+x^2} \, dx \\ &= 2 \int 1 \, dx - 2 \int \frac{1}{1+x^2} \, dx = 2x - 2 \arctan(x) \end{aligned}$$

Enfin

$$\int \ln(x^2 + 1) \, dx = x \ln(x^2 + 1) - 2x + 2 \arctan(x)$$

QED