

Rappel de cours

Definition 1. Deux suites $(u_n)_{n \geq 0}$ et $(v_n)_{n \geq 0}$ sont adjacentes ssi:

- $(u_n)_{n \geq 0}$ est croissante et $(v_n)_{n \geq 0}$ est décroissante
- $\forall n \in \mathbb{N}, u_n \leq v_n$
- $\lim_{n \rightarrow \infty} (v_n - u_n)_{n \geq 0} = 0$

Exercice 3

Pour que $\sum c_n z^n$ converge, il suffit de montrer, par le critère d'Abel, que $\exists M, \forall n, |\sum_{k=0}^n z^k| \leq M$. On a

$$\left| \sum_{k=0}^n z^k \right| = \left| 1 \cdot \frac{1 - z^{n+1}}{1 - z} \right| \leq \frac{1 + |z|^{n+1}}{|1 - z|} < \frac{2}{|1 - z|}$$

car $|z| \leq 1$ et $|z^n| \leq 1$. On a trouvé un $M = \frac{2}{1-|z|}$, ce qui permet de montrer que $\sum c_n z^n$ converge.

Exercice 4

Exercice 4.1.a

Calculons

$$\frac{v_n}{u_n} = \frac{\frac{(-1)^n}{\sqrt{n}}}{\frac{(-1)^n}{\sqrt{n} - (-1)^n}} = \frac{\sqrt{n} - (-1)^n}{\sqrt{n}} = 1 - \frac{(-1)^n}{\sqrt{n}}$$

On a $\lim_{n \rightarrow \infty} \frac{v_n}{u_n} = 1$ donc $u_n \sim v_n$

Exercice 4.1.b

$v_n = \frac{(-1)^n}{\sqrt{n}}$ converge?

1. Y a-t-il Convergence absolue? $\sum \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum \left| \frac{1}{\sqrt{n}} \right|$ Cette suite diverge. Donc il n'y a pas de convergence absolue.
2. Cas Special Série Alternée? La série est alternée car $(-1)^n$ est alternée et $\frac{1}{\sqrt{n}}$ est positif. Il faut montrer que $\frac{1}{\sqrt{n}}$ converge vers 0. Ce qui est vrai quand $n \rightarrow \infty$. Donc, la série de terme général $v_n = \frac{(-1)^n}{\sqrt{n}}$ converge.

Exercice 4.2

Exercice 4.3

Exercice 4.4

Exercice 5

Exercice 5.1

$$\begin{aligned} u_n - \frac{(-1)^n}{n} &= \frac{1}{\ln(n) + (-1)^n n} - \frac{(-1)^n}{n} = \frac{n}{n(\ln(n) + (-1)^n n)} - \frac{(-1)^n(\ln(n) + (-1)^n n)}{n(\ln(n) + (-1)^n n)} \\ &= \frac{n - ((-1)^n(\ln(n) + (-1)^n n))}{n(\ln(n) + (-1)^n n)} = \frac{n - (-1)^n \ln(n) - (-1)^n (-1)^n n}{n(\ln(n) + (-1)^n n)} = \frac{-(-1)^n \ln(n)}{n(\ln(n) + (-1)^n n)} \\ &= \frac{-\ln(n)}{(-1)^n n \ln(n) + n^2} = \frac{\ln(n)}{n} \frac{-1}{(-1)^n \ln(n) + n} \end{aligned}$$

??

Exercice 5.2

On a

$$u_n = \left(u_n - \frac{(-1)^n}{n} \right) - \frac{(-1)^n}{n}$$

Avec $u_n - \frac{(-1)^n}{n}$ qui converge et $\frac{(-1)^n}{n}$ qui converge aussi (C.S.S.A avec $v_n = \frac{1}{n}$). Donc la série de terme général u_n converge (somme de 2 séries qui convergent).

Exercice 6

Exercice 6.a

$$a_n = \sum_{k=n}^{2n} \frac{1}{n+k} = \sum_{k=n}^{2n} \frac{1}{n} \frac{1}{1+\frac{k}{n}}$$

Prenons $x = \frac{k}{n}$, on a $dx = \frac{1}{n}$ donc

$$\sum_{k=n}^{2n} \frac{1}{n} \frac{1}{1+\frac{k}{n}} = \int_1^2 \frac{1}{1+x} dx = [\ln(|1+x|)]_1^2 = \ln(3) - \ln(2) = \ln\left(\frac{3}{2}\right)$$

Exercice 6.b

$$b_n = \sqrt[n]{\frac{(2n)!}{n!n^n}}$$

Calcul de

$$\ln \left(\sqrt[n]{\frac{(2n)!}{n!n^n}} \right) = \frac{1}{n} \ln \left(\frac{(2n)!}{n!n^n} \right) = \frac{1}{n} (\ln((2n)!) - \ln(n!) - n \ln(n))$$

On a

$$\ln(n!) = \ln(1 * 2 * 3 * \dots * n) = \ln(1) + \ln(2) + \ln(3) + \dots + \ln(n) = \sum_{k=1}^n \ln(k)$$

$$\ln(2n!) = \ln(1 * 2 * 3 * \dots * 2n) = \ln(1) + \ln(2) + \ln(3) + \dots + \ln(2n) = \sum_{k=1}^{2n} \ln(k)$$

Donc

$$\ln((2n)!) - \ln(n!) = \sum_{k=n+1}^{2n} \ln(k)$$

$$\ln((2n)!) - \ln(n!) - n \ln(n) = \sum_{k=n+1}^{2n} \ln(k) - \sum_{k=1}^n \ln(n) = \sum_{k=n+1}^{2n} \ln(k) - \sum_{k=n+1}^{2n} \ln(n) = \sum_{k=n+1}^{2n} \ln(k) - \ln(n) = \sum_{k=n+1}^{2n} \ln\left(\frac{k}{n}\right)$$

donc

$$\frac{1}{n} (\ln((2n)!) - \ln(n!) - n \ln(n)) = \frac{1}{n} \sum_{k=n+1}^{2n} \ln\left(\frac{k}{n}\right) = \int_{\frac{n+1}{n}}^2 \ln(x) = [x(\ln(x) - 1)]_{\frac{n+1}{n}}^2$$

Exercice 7

Exercice 7.a

$$a_n = \sum_{k=0}^n \frac{1}{n+k} = \sum_{k=0}^n \frac{1}{n} \frac{1}{1+\frac{k}{n}}$$

Prenons $x = \frac{k}{n}$, on a $dx = \frac{1}{n}$ donc

$$\sum_{k=0}^n \frac{1}{n} \frac{1}{1+\frac{k}{n}} = \int_0^1 \frac{1}{1+x} dx = [\ln(|1+x|)]_0^1 = \ln(2) - \ln(1) = \ln(2)$$

Exercice 7.b

$$b_n = \sum_{k=0}^n \frac{n}{n^2 + k^2} = \sum_{k=0}^n \frac{n}{n^2} \frac{1}{1 + \frac{k^2}{n^2}}$$

Prenons $x = \frac{k}{n}$, on a $dx = \frac{1}{n}$ donc

$$\sum_{k=0}^n \frac{1}{n} \frac{1}{1 + \left(\frac{k}{n}\right)^2} = \int_0^1 \frac{1}{1+x^2} dx = [\arctan(x)]_0^1 = \arctan(1) - \arctan(0) = \arctan(1)$$

Exercice 7.c

$$c_n = \frac{1}{n^2} \prod_{k=1}^n (n^2 + k^2)^{1/n}$$

Exercice 8**Exercice 8.a**

$$k \binom{n}{k} = k \frac{n!}{k!(n-k)!} = k \frac{n!}{k(k-1)!(n-k)!} = \frac{n(n-1)!}{(k-1)!(n-k)!} = n \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = n \binom{n-1}{k-1}$$

Exercice 8.b

$$\begin{aligned} k(k-1) \binom{n}{k} &= k(k-1) \frac{n!}{k!(n-k)!} = k(k-1) \frac{n!}{k(k-1)(k-2)!((n-2)-(k-2))!} \\ &= \frac{n(n-1)(n-2)!}{(k-2)!((n-2)-(k-2))!} = n(n-1) \binom{n-2}{k-2} \end{aligned}$$

Exercice 9**Exercice 9.1**

On sait que $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$. En prenant $y = 1-x$, on a

$$\sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} = (x + (1-x))^n = 1^n = 1$$

Exercice 9.2

Le premier terme est toujours égal à 0.

$$\begin{aligned} E(X) &= \sum_{k=0}^n kp(X=k) = \sum_{k=1}^n kp(X=k) = \sum_{k=1}^n k \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=1}^n n \binom{n-1}{k-1} p^k (1-p)^{n-k} = \\ &= \sum_{k=1}^n n \binom{n-1}{k-1} pp^{(k-1)} (1-p)^{n-k} = np \sum_{k=1}^n \binom{n-1}{k-1} p^{(k-1)} (1-p)^{(n-1)-(k-1)} \end{aligned}$$

Changement de variables $j = k-1$ et $m = n-1$ on a

$$np \sum_{j=0}^m \binom{m}{j} p^j (1-p)^{(m-j)} = np.1 = np$$

Exercice 10

$$\begin{aligned} V(X) &= \sum_{k=0}^n (k-np)^2 P(X=k) = \sum_{k=0}^n (k^2 - 2knp + n^2 p^2) P(X=k) \\ &= \sum_{k=0}^n k^2 P(X=k) - 2np \sum_{k=0}^n k P(X=k) + n^2 p^2 \sum_{k=0}^n P(X=k) = \sum_{k=0}^n k^2 P(X=k) - 2npnp + n^2 p^2 = \sum_{k=0}^n k^2 P(X=k) - n^2 p^2 \end{aligned}$$

On repart de

$$\sum_{k=0}^n k^2 P(X=k) = \sum_{k=0}^n k.k.P(X=k) = \sum_{k=0}^n k.k. \binom{n}{k} p^k (1-p)^{n-k} = \sum_{k=0}^n k.n. \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^n k \cdot n \cdot \binom{n-1}{k-1} p p^{k-1} (1-p)^{(n-1)-(k-1)} = np \sum_{k=0}^n k \cdot \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

Le premier terme est égal à 0.

$$np \sum_{k=1}^n k \cdot \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

Changement de variables $j = k - 1$, $m = n - 1$

$$\begin{aligned} np \sum_{j=0}^m (j+1) \cdot \binom{m}{j} p^j (1-p)^{m-j} &= np \left(\sum_{j=0}^m j \binom{m}{j} p^j (1-p)^{m-j} + \sum_{j=0}^m \binom{m}{j} p^j (1-p)^{m-j} \right) \\ &= np(mp+1) = np((n-1)p+1) = n^2 p^2 - np^2 + np = n^2 p^2 + np(1-p) \end{aligned}$$

Donc

$$V(X) = \sum_{k=0}^n k^2 P(X=k) - n^2 p^2 = n^2 p^2 + np(1-p) - n^2 p^2 = np(1-p)$$

Exercice 11

On a

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

Vérifions pour $n = 1$, $\sum_{k=1}^1 \frac{(-1)^{k+1}}{k} \binom{1}{k} = 1 = \sum_{k=1}^1 \frac{1}{k}$. Supposons $\sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{k=1}^n \frac{1}{k}$ vrai au rang n , calculons

$$\begin{aligned} \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n+1}{k} &= \sum_{k=1}^{n+1} \left(\frac{(-1)^{k+1}}{k} \binom{n}{k} + \binom{n}{k-1} \right) \\ &= \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k} + \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k-1} \end{aligned}$$

Première partie

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} + \frac{(-1)^{n+2}}{n+1} \binom{n}{n+1} = \sum_{k=1}^n \frac{(-1)^{k+1}}{k} \binom{n}{k} = \sum_{k=1}^n \frac{1}{k}$$

Car $\binom{n}{n+1} = 0$ par définition et par hypothèse de récurrence.

Seconde partie, commençons par calculer

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k-1} = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{(n+1)-1}{k-1} = \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{n+1} \binom{n+1}{k} = \frac{1}{n+1} \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k}$$

On a $(x+y)^n = \sum_{k=0}^n x^k y^{n-k} \binom{n}{k}$, en prenant $x = -1$ et $y = 1$ on a

$$((-1) + 1)^n = \sum_{k=0}^n (-1)^k 1^{n-k} \binom{n}{k} = \sum_{k=0}^n (-1)^k \binom{n}{k} = 1 + \sum_{k=1}^n (-1)^k \binom{n}{k}$$

Donc $\sum_{k=1}^n (-1)^k \binom{n}{k} = -1$ ce qui fait $\sum_{k=1}^n (-1)^{k+1} \binom{n}{k} = 1$ On a pour finir

$$\sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k} + \sum_{k=1}^{n+1} \frac{(-1)^{k+1}}{k} \binom{n}{k-1} = \sum_{k=1}^n \frac{1}{k} + \frac{1}{n+1} = \sum_{k=1}^{n+1} \frac{1}{k}$$

Fini.

Exercice 12

$$\sum_{k=p}^{n+1} \binom{k}{p} = \sum_{k=p}^n \binom{k}{p} + \binom{n+1}{p} = \binom{n+1}{p+1} + \binom{n+1}{p}$$

On a

$$\begin{aligned} \binom{n+1}{p+1} &= \frac{(n+1)!}{(p+1)!(n-p)!} = \frac{(n+1)!(n-p+1)}{(p+1)!(n-p+1)!} \\ \binom{n+1}{p} &= \frac{(n+1)!}{(p)!(n+1-p)!} = \frac{(n+1)!(p+1)}{(p+1)!(n+1-p)!} \end{aligned}$$

Donc

$$\begin{aligned} \binom{n+1}{p+1} + \binom{n+1}{p} &= \frac{(n+1)!(n+1-p)}{(p+1)!(n+1-p)!} + \frac{(n+1)!(p+1)}{(p+1)!(n+1-p)!} = \frac{(n+1)!(n-p+1+p+1)}{(p+1)!(n+1-p)!} \\ &= \frac{(n+2)!}{(p+1)!(n+2-(p+1))!} = \binom{n+2}{p+1} \end{aligned}$$

Exercice 13

Exercice 13.1

$$\begin{aligned} \sum_{k=0}^m \binom{2m}{2k} - \sum_{k=0}^{m-1} \binom{2m}{2k+1} &= \binom{2m}{0} + \binom{2m}{2} + \dots + \binom{2m}{2m} - \left(\binom{2m}{1} + \binom{2m}{3} + \dots + \binom{2m}{2m-1} \right) \\ \binom{2m}{0} - \binom{2m}{1} + \binom{2m}{2} - \binom{2m}{3} \dots - \binom{2m}{2m-1} + \binom{2m}{2m} &= \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} = 0 \end{aligned}$$

Pour $n = 2m$ (n pair), on a

$$S_n = \sum_{k=0}^m \binom{2m}{2k}, T_n = \sum_{k=0}^{m-1} \binom{2m}{2k+1}$$

Calculons $S_n - T_n$

$$\begin{aligned} \sum_{k=0}^m \binom{2m}{2k} - \sum_{k=0}^{m-1} \binom{2m}{2k+1} &= \binom{2m}{0} + \binom{2m}{2} + \dots + \binom{2m}{2m} - \left(\binom{2m}{1} + \binom{2m}{3} + \dots + \binom{2m}{2m-1} \right) \\ \binom{2m}{0} - \binom{2m}{1} + \binom{2m}{2} - \binom{2m}{3} \dots - \binom{2m}{2m-1} + \binom{2m}{2m} &= \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} = 0 \end{aligned}$$

Pour $n = 2m + 1$ (n impair) on a

$$S_n = \sum_{k=0}^m \binom{2m+1}{2k}, T_n = \sum_{k=0}^m \binom{2m+1}{2k+1}$$

Calculons $S_n - T_n$

$$\begin{aligned} \sum_{k=0}^m \binom{2m+1}{2k} - \sum_{k=0}^m \binom{2m+1}{2k+1} &= \binom{2m+1}{0} + \binom{2m+1}{2} + \dots + \binom{2m+1}{2m} - \left(\binom{2m+1}{1} + \binom{2m+1}{3} + \dots + \binom{2m+1}{2m+1} \right) \\ \binom{2m+1}{0} - \binom{2m+1}{1} + \binom{2m+1}{2} - \binom{2m+1}{3} \dots + \binom{2m+1}{2m} - \binom{2m+1}{2m+1} &= \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} = 0 \end{aligned}$$

Exercice 13.2

Calculons $S_n + T_n$ pour $n = 2m$

$$\sum_{k=0}^m \binom{2m}{2k} + \sum_{k=0}^{m-1} \binom{2m}{2k+1} = \binom{2m}{0} + \binom{2m}{2} + \dots + \binom{2m}{2m} + \left(\binom{2m}{1} + \binom{2m}{3} + \dots + \binom{2m}{2m-1} \right)$$
$$\binom{2m}{0} + \binom{2m}{1} + \binom{2m}{2} + \binom{2m}{3} \dots + \binom{2m}{2m-1} + \binom{2m}{2m} = \sum_{k=0}^{2m} \binom{2m}{k} = 2^{2m} = 2^n$$

car $(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$ et en prenant $x = y = 1$ on a

$$(1+1)^n = \sum_{k=0}^n \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^n \binom{n}{k}$$

Calculons $S_n + T_n$ pour $n = 2m + 1$

$$\sum_{k=0}^m \binom{2m+1}{2k} + \sum_{k=0}^m \binom{2m+1}{2k+1} = \binom{2m+1}{0} + \binom{2m+1}{2} + \dots + \binom{2m+1}{2m} + \left(\binom{2m+1}{1} + \binom{2m+1}{3} + \dots + \binom{2m+1}{2m+1} \right)$$
$$\binom{2m+1}{0} + \binom{2m+1}{1} + \binom{2m+1}{2} + \binom{2m+1}{3} \dots + \binom{2m+1}{2m} + \binom{2m+1}{2m+1} = \sum_{k=0}^{2m+1} \binom{2m+1}{k} = 2^{2m+1} = 2^n$$

Exercice 13.3

On a $S_n - T_n = 0$ et $S_n + T_n = 2^n$, donc $2T_n = 2^n$, $T_n = 2^{n-1}$ et $S_n = T_n = 2^{n-1}$.

QED