Positivity and boundedness

- **Article:** Study on Fear-Induced Group Defense and Cooperative Hunting: Bifurcation, Uncertainty, Seasonality, and Spatio-temporal Analysis in Predator-Prey System
- **Authors:** Parvez Akhtar, Nirapada Santra, Guruprasad Samanta

Our proposed system is:

$$\begin{aligned} \frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1 H} - \frac{(h+\alpha P)HP}{1+b_2 H + b_3 \beta H^2} \equiv F_1(H,P) = Hf_1(H,P), \quad \text{(say)} \\ \frac{dP}{dt} &= \frac{e(h+\alpha P)HP}{1+b_2 H + b_3 \beta H^2} - d_2 P \equiv F_2(H,P) = Pf_2(H,P), \quad \text{(say)} \end{aligned} \tag{1}$$

Theorem 1. *For any time $t > 0$ every solution of system (1) remains positive and uniformly bounded with initial condition $H(0) > 0,\ P(0) > 0$.*

Proof.

$$\begin{cases} \dfrac{dH}{dt} = Hf_1(H,P) \\[2ex] \dfrac{dP}{dt} = Pf_2(H,P) \end{cases} \tag{2}$$

where, $f_1(H,P) = \dfrac{b}{1+\beta P} - d_1 - \dfrac{rH}{K+K_1 H} - \dfrac{(h+\alpha P)P}{1+b_2 H + b_3 \beta H^2}$ and
$f_2(H,P) = \dfrac{e(h+\alpha P)H}{1+b_2 H + b_3 \beta H^2} - d_2$.
From (2) we get,

$$H(t) = H(0)\exp\left(\int_0^t f_1(H,P)dt\right) > 0, \text{ for } H(0) > 0.$$

$$P(t) = P(0)\exp\left(\int_0^t f_2(H,P)dt\right) > 0, \text{ for } P(0) > 0.$$

Therefore, $H(t)$ and $P(t)$ both are positive whenever $H(0) > 0$ and $P(0) > 0\ \forall\ t > 0$.

Now, we have to prove the boundedness of system (1).

$$\begin{aligned} \frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1 H} - \frac{(h+\alpha P)HP}{1+b_2 H + b_3 \beta H^2} \\ &\le bH - d_1 H - \frac{rH^2}{K+K_1 H} \end{aligned}$$

After some basic calculations we get, (the calculation is provided in Appendix **??**)

$$\frac{dH}{dt} + \mu H \le \frac{rK}{K_1^2}, \text{ where } \mu = \left(\frac{r}{K_1} + d_1 - b\right), \text{ provided } \left(\frac{r}{K_1} + d_1\right) > b$$

Now, with the help of the differential inequality theory [Ross & Ross, 2004] for $H(t)$, we get

$$0 \le H(t) \le \frac{rK}{K_1^2} \cdot \frac{1}{\mu}(1 - e^{-\mu t}) + H(0)e^{-\mu t} \tag{3}$$

Taking $t \to \infty$, we get, $0 < H(t) \le \left(\frac{rK}{K_1^2} \cdot \frac{1}{\mu} + \epsilon_1\right) = M_1$, for any $\epsilon_1 > 0$.

Next, let us take $w(t) = H(t) + \frac{1}{e}P(t)$.
Therefore,

$$\begin{aligned}
\frac{dw}{dt} &= \frac{dH}{dt} + \frac{1}{e}\frac{dP}{dt} \\
\Rightarrow \frac{dw}{dt} &\le bH - d_1 H - \frac{d_2}{e}P = bH - \tau\left(H + \frac{1}{e}P\right), \quad \text{where } \tau = \min\{d_1, d_2\} \\
&= bH - \tau w(t)
\end{aligned}$$

Therefore,

$$\frac{dw}{dt} \le bM_1 - \tau w(t)$$

Again, with the help of the differential inequality theory [Ross & Ross, 2004] for $w(t)$, we get

$$0 \le w(t) \le \frac{bM_1}{\tau}(1 - e^{-\tau t}) + w(0)e^{-\tau t} \tag{4}$$

Taking $t \to \infty$, we get, $0 < w(t) \le \frac{bM_1}{\tau} + \epsilon_2$, for any $\epsilon_2 > 0$. Therefore, all the solutions of the system (1), starting from the domain $\mathbb{R}_+^2$ are remain in $\left\{(H, P) \in \mathbb{R}_+^2 : 0 < H(t) + \frac{1}{e}P(t) \le \frac{bM_1}{\tau} + \epsilon_2\right\}$. This proves that every solutions of system (1) are uniformly bounded.

■

References

Ross, C. C. & Ross, C. C. [2004] *About Differential Equations* (Springer), doi:10.1007/978-1-4757-3949-7_1.