Hopf Bifurcation Proof

- **Article:** Study on Fear-Induced Group Defense and Cooperative Hunting: Bifurcation, Uncertainty, Seasonality, and Spatio-temporal Analysis in Predator-Prey System
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Our proposed system is:

$$\begin{aligned} \frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1H} - \frac{(h+\alpha P)HP}{1+b_2H+b_3\beta H^2} \equiv F_1(H,P) = Hf_1(H,P), \quad \text{(say)} \\ \frac{dP}{dt} &= \frac{e(h+\alpha P)HP}{1+b_2H+b_3\beta H^2} - d_2P \equiv F_2(H,P) = Pf_2(H,P), \quad \text{(say)} \end{aligned} \tag{1}$$

By treating β as a variable parameter, let us write the characteristic equation of system (1) for the Jacobian matrix $J(E_I)$ as follows:

$$\lambda^2 - T(\beta)\lambda + D(\beta) = 0 \tag{2}$$

In this context, $T(\beta)$ and $D(\beta)$ represent the trace and determinant of the Jacobian matrix associated with the interior equilibrium point E_I, respectively. Noticing a transition in the sign of the real component of λ from negative to positive indicates a shift in stability for the interior equilibrium point E_I, thereby altering its status from stable to unstable. This transition results in a Hopf bifurcation [Kuznetsov *et al.*, 1998], which takes place when the characteristic equation (2) provides a pair of roots that are purely imaginary. Assuming that the purely imaginary roots are located at $\beta = \beta^{(H)}$, it follows that $T(\beta^{[H]}) = 0$ and $D(\beta^{[H]}) > 0$. This theorem establishes the presence of a Hopf bifurcation within system (1) at the point where $\beta = \beta^{[H]}$.

Theorem 1. *(Hopf-bifurcation Theorem [Murray, 2007]). If $T(\beta)$ and $D(\beta)$ are the smooth functions of β in an open interval about $\beta^{[H]} \in \mathbb{R}$ such that the characteristic equation (2) has*

I. *A pair of imaginary eigen-values $\lambda = g_1(\beta) \pm ig_2(\beta)$ with $g_1(\beta)$ and $g_2(\beta) \in \mathbb{R}$ so that they become purely imaginary at $\beta = \beta^{[H]}$ and $\left[\frac{dT}{d\beta}\right]_{\beta=\beta^{[H]}} \neq 0$.*

II. *The other eigen-value is negative at $\beta = \beta^{[H]}$. Then a Hopf-bifurcation occurs around $E_I(H^*, P^*)$ at $\beta = \beta^{[H]}$ (i.e, a stability changes of $E_I(H^*, P^*)$ accompanied by the creation of a limit cycle at $\beta = \beta^{[H]}$).*

Theorem 2. *System (1) exhibits Hopf-bifurcation around $E_I(H^*, P^*)$ at $\beta = \beta^{[H]}$, when $T(\beta^{[H]}) = 0$, $D(\beta^{[H]}) > 0$ and $\left[\frac{dT}{d\beta}\right]_{\beta=\beta^{[H]}} \neq 0$.*

Proof. At $\beta = \beta^{[H]}$, $T(\beta^{[H]}) = 0$, $D(\beta^{[H]}) > 0$, the characteristic equation has purely imaginary roots, i.e. $\lambda_1 = i\sqrt{D(\beta^{[H]})}$ and $\lambda_2 = -i\sqrt{D(\beta^{[H]})}$. Therefore, $\lambda_1 = g_1(\beta) + ig_2(\beta)$ and $\lambda_2 = g_1(\beta) - ig_2(\beta)$ are the roots of the characteristic equation (2) in any open neighbourhood of $\beta^{(H)}$, where $g_1(\beta)$ and $g_2(\beta)$ are real valued functions of β. Using Hopf-Bifurcation Theorem [Murray & Murray, 2003], we may say that the stability of system (1) switches through a Hopf bifurcation, provided the following transversality condition

$$\left[\frac{d}{d\beta}(Re(\lambda_i(\beta)))\right]_{\beta=\beta^{[H]}} = \left[\frac{dg_1(\beta)}{d\beta}\right]_{\beta=\beta^{[H]}} \neq 0, \quad \text{for } i = 1, 2.$$

is satisfied. Let us put $\lambda(\beta) = g_1(\beta) + ig_2(\beta)$ in equation (2), then we get,

$$(g_1(\beta) + ig_2(\beta))^2 - T(\beta)(g_1(\beta) + ig_2(\beta)) + D(\beta) = 0$$

Now, differentiating both sides w.r.t. β we get,

$$2(g_1(\beta)+ig_2(\beta))(\dot{g}_1(\beta)+i\dot{g}_2(\beta))-T(\beta)(\dot{g}_1(\beta)+i\dot{g}_2(\beta))-\dot{T}(\beta)(g_1(\beta)+ig_2(\beta))+\dot{D}(\beta)=0$$

Now comparing the real and imaginary components on both sides, we get;

$$(2g_1-T)\dot{g}_1-2g_2\dot{g}_2-\dot{T}g_1+\dot{D}=0 \Rightarrow X_1\dot{g}_1-X_2\dot{g}_2+X_3=0 \tag{3}$$

$$(2g_2)\dot{g}_1+(2g_1-T)\dot{g}_2-\dot{T}g_2=0 \Rightarrow X_2\dot{g}_1+X_1\dot{g}_2+X_4=0 \tag{4}$$

where, $X_1=(2g_1-T)$, $X_2=2g_2$, $X_3=(\dot{D}-\dot{T}g_1)$, $X_4=-\dot{T}g_2$. Now solving equations (3) and (4) we get,

$$\dot{g}_1=-\frac{(X_1X_3+X_2X_4)}{X_1^2+X_2^2} \tag{5}$$

At $\beta=\beta^{[H]}$. Now two cases arise :

Case I: When $g_1=0$, $g_2=\sqrt{D}$ then, $X_1=0$, $X_2=2\sqrt{D}$, $X_3=\dot{D}$ and $X_4=-\dot{T}\sqrt{D}$. Hence from (5) we get,

$$[\dot{g}_1]_{\beta=\beta^{[H]}}=\left[\frac{dg_1(\beta)}{d\beta}\right]_{\beta=\beta^{[H]}}=\frac{1}{2}\left[\frac{dT(\beta)}{d\beta}\right]_{\beta=\beta^{[H]}}\neq 0$$

Case II: When $g_1=0$, $g_2=-\sqrt{D}$ then, $X_1=0$, $X_2=-2\sqrt{D}$, $X_3=\dot{D}$ and $X_4=\dot{T}\sqrt{D}$. Hence from (5) we get,

$$[\dot{g}_1]_{\beta=\beta^{[H]}}=\left[\frac{dg_1(\beta)}{d\beta}\right]_{\beta=\beta^{[H]}}=-\frac{1}{2}\left[\frac{dT(\beta)}{d\beta}\right]_{\beta=\beta^{[H]}}\neq 0$$

Therefore, $\frac{d}{d\beta}(Re(\lambda_i(\beta)))\Big|_{\beta=\beta^{[H]}}=-\frac{(X_1X_3+X_2X_4)}{X_1^2+X_2^2}\Big|_{\beta=\beta^{[H]}}\neq 0$ for $i=1,2$ and $T(\beta^{[H]})=0$, $D(\beta^{[H]})>0$.

Hence, theorem 2 is proved using theorem 1.

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References

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