Saddle-node Bifurcation Proof

- **Article:** Study on Fear-Induced Group Defense and Cooperative Hunting: Bifurcation, Uncertainty, Seasonality, and Spatio-temporal Analysis in Predator-Prey System
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Our proposed system is:

$$\begin{aligned} \frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1 H} - \frac{(h+\alpha P)HP}{1+b_2 H + b_3\beta H^2} \equiv F_1(H,P) = Hf_1(H,P), \quad \text{(say)} \\ \frac{dP}{dt} &= \frac{e(h+\alpha P)HP}{1+b_2 H + b_3\beta H^2} - d_2 P \equiv F_2(H,P) = Pf_2(H,P), \quad \text{(say)} \end{aligned} \tag{1}$$

Theorem 1. *System (1) demonstrates a saddle-node bifurcation associated with the bifurcation parameter d_2 at $d_2 = d_2^{[SN]}$.*

Proof. If the nontrivial nullcline $f_1(H,P)=0$ intersects the nontrivial nullcline $f_2(H,P)=0$ tangentially at E_I, then $\left.\frac{dP^{(f_1)}}{dH}\right|_{E_I} = \left.\frac{dP^{(f_2)}}{dH}\right|_{E_I}$.
The Jacobian matrix $\mathcal{J}(E_I)$ at $d_2 = d_2^{[SN]}$ is:

$$\mathcal{J}(E_I) = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix}_{d_2=d_2^{[SN]}} = \begin{bmatrix} H\frac{\partial f_1}{\partial H} & H\frac{\partial f_1}{\partial P} \\ P\frac{\partial f_2}{\partial H} & P\frac{\partial f_2}{\partial P} \end{bmatrix}_{E_I, d_2^{[SN]}}$$

Now,

$$det(\mathcal{J}(E_I)) = \left[HP\left(\frac{\partial f_1}{\partial H}\frac{\partial f_2}{\partial P} - \frac{\partial f_1}{\partial P}\frac{\partial f_2}{\partial H}\right)\right]_{E_I, d_2^{[SN]}} = \left[HP\frac{\partial f_1}{\partial P}\frac{\partial f_2}{\partial P}\left(\frac{dP^{(f_2)}}{dH} - \frac{dP^{(f_1)}}{dH}\right)\right]_{E_I, d_2^{[SN]}} = 0.$$

Both $\mathcal{J}(E_I)$ and $\mathcal{J}^T(E_I)$ have 0 as an eigenvalue since $det(\mathcal{J}(E_I)) = 0$. Corresponding to this zero eigenvalue, the eigenvectors of $\mathcal{J}(E_I)$ and $\mathcal{J}^T(E_I)$ are provided by $W = \begin{bmatrix} \tilde{w_1} \\ \tilde{w_2} \end{bmatrix}_{E_I, d_2^{[SN]}}$ and $Z = \begin{bmatrix} \tilde{z_1} \\ \tilde{z_2} \end{bmatrix}_{E_I, d_2^{[SN]}}$ respectively, where $\tilde{w_1} = 1, \tilde{w_2} = \dfrac{H\Big(b\beta(1+H(b_2+b_3\beta H)) + (h+2\alpha P)(1+\beta P)^2\Big)}{(1+\beta P)^2\Big(d_2 + d_2 H(b_2+b_3+\beta H) - eH(h+2\alpha P)\Big)}$ and $\tilde{z_1} = \dfrac{\Big(1+H(b_2+b_3\beta H)\Big)\Big(d_2 + d_2 H(b_2+b_3\beta H) - eH(h+2\alpha P)\Big)}{e(b_3\beta H^2 - 1)P(h+\alpha P)}, \tilde{z_2} = 1$. Now, using Sotomayor's theorem [Perko, 2013] at $d_2 = d_2^{[SN]}$ we get,

$$Z^T \tilde{F}_{d_2}(E_I, d_2^{[SN]}) = -P^* \neq 0 \tag{2}$$

$$Z^T\left[D^2\tilde{F}(E_{I1},\eta_2^{(SN_1)})(W,W)\right]=\tilde{z}_1(\mathscr{A}\tilde{w_1}^2+2\mathscr{B}\tilde{w_1}\tilde{w_2}+\mathscr{C}\tilde{w_2}^2)+ \tilde{z}_2(\mathscr{D}\tilde{w_1}^2+2\mathscr{E}\tilde{w_1}\tilde{w_2}+\mathscr{F}\tilde{w_2}^2)\neq 0 \tag{3}$$

where, $\tilde{w_1}$, $\tilde{w_2}$, $\tilde{z_1}$ and $\tilde{z_2}$ are given above in this proof, while

$$\mathscr{A}=\frac{2P(\alpha P+h)\left(b_2+b_3\beta H\left(3-b_3\beta H^2\right)\right)}{(H(b_2+b_3\beta H)+1)^3}-\frac{2K^2r}{(K+K_1H)^3}$$
$$\mathscr{B}=\frac{(2\alpha P+h)\left(b_3\beta H^2-1\right)}{(H(b_2+b_3\beta H)+1)^2}-\frac{b\beta}{(\beta P+1)^2}$$
$$\mathscr{C}=\frac{2bH\beta^2}{(\beta P+1)^3}-\frac{2\alpha H}{b_3\beta H^2+b_2H+1}$$
$$\mathscr{D}=\frac{2eP(\alpha P+h)\left(b_2+b_3\beta H\left(b_3\beta H^2-3\right)\right)}{(H(b_2+b_3\beta H)+1)^3}$$
$$\mathscr{E}=\frac{e(2\alpha P+h)\left(1-b_3\beta H^2\right)}{(H(b_2+b_3\beta H)+1)^2}$$
$$\mathscr{F}=\frac{2e\alpha H}{b_3\beta H^2+b_2H+1}$$

whenever,

$$\frac{2K^2r}{(K+K_1H)^3}\neq\frac{2P(\alpha P+h)\left(b_2+b_3\beta H\left(3-b_3\beta H^2\right)\right)}{(H(b_2+b_3\beta H)+1)^3} \tag{4}$$

$$\frac{b\beta}{(\beta P+1)^2}\neq\frac{(2\alpha P+h)\left(b_3\beta H^2-1\right)}{(H(b_2+b_3\beta H)+1)^2} \tag{5}$$

$$\frac{2bH\beta^2}{(\beta P+1)^3}\neq\frac{2\alpha H}{b_3\beta H^2+b_2H+1} \tag{6}$$

Now, to do additional check if the equation (3) is non-zero, we have taken our parameter set mentioned in Table 1 (in the manuscript) and found that in condition (4), both values are not equal (that constitutes our required condition). We also found the same for remaining conditions (5) and (6). We have also seen that the value of main criteria (3) is 101.6184, which is not equal to zero. One can check by choosing other parameter values and will find the value non-zero and the terms are not equal to each other for conditions (4), (5) and (6), respectively.

Here, $\tilde{F}=\begin{pmatrix}F_1\\F_2\end{pmatrix}$, both F_1 and F_2 are defined in system (1). Consequently, the system fulfills the necessary criteria for a saddle-node bifurcation at the interior equilibrium E_I when $d_2=d_2^{[SN]}$. ■

References

Perko, L. [2013] *Differential Equations and Dynamical Systems*, Vol. 7 (Springer Science & Business Media), doi:10.1007/978-1-4613-0003-8.