Generalized Hopf (GH) Bifurcation Proof

- **Article:** Study on Fear-Induced Group Defense and Cooperative Hunting: Bifurcation, Uncertainty, Seasonality, and Spatio-temporal Analysis in Predator-Prey System
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Our proposed system is:

$$\begin{aligned}\frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1H} - \frac{(h+\alpha P)HP}{1+b_2H+b_3\beta H^2} \equiv F_1(H,P) = Hf_1(H,P), \quad \text{(say)}\\ \frac{dP}{dt} &= \frac{e(h+\alpha P)HP}{1+b_2H+b_3\beta H^2} - d_2 P \equiv F_2(H,P) = Pf_2(H,P), \quad \text{(say)}\end{aligned} \tag{1}$$

Theorem 1. *System (1) undergoes a Generalized Hopf bifurcation (Bautin bifurcation) at the interior equilibrium point E_I when it reaches the bifurcation threshold $(d_1^{[GH]}, b_3^{[GH]})$, whenever the value of E_I fulfills following criteria:*
(GH1): $T = tr(J(E_I;(d_1^{[GH]}, b_3^{[GH]}))) = 0$
(GH2): $D = det(J(E_I;(d_1^{[GH]}, b_3^{[GH]}))) > 0$
(GH3): $L(E_I;(d_1^{[GH]}, b_3^{[GH]})) = 0$
where L is the first Lyapunov number.

Proof. Let the nontrivial equilibrium point E_I fulfills the aforementioned three conditions. The Jacobian matrix evaluated at the point E_I is

$$\begin{bmatrix} H\dfrac{\partial f_1}{\partial H} & H\dfrac{\partial f_1}{\partial P} \\ P\dfrac{\partial f_2}{\partial H} & P\dfrac{\partial f_2}{\partial P}\end{bmatrix} = \begin{bmatrix}\tilde{a} & \tilde{b}\\ \tilde{c} & \tilde{d}\end{bmatrix} \text{ (say)}$$

where,

$$\tilde{a} = \frac{b}{1+\beta P^*} - d_1 - \frac{rH^*\left(K_1H^*+2K\right)}{\left(K_1H^*+K\right)^2} + \frac{\left(\alpha P^*+h\right)P^*\left(b_3\beta H^{*2}-1\right)}{\left(b_3\beta H^{*2}+b_2H^*+1\right)^2},$$

$$\tilde{b} = -\frac{bH^*\beta}{(1+\beta P^*)^2} - \frac{H^*\left(2\alpha P^*+h\right)}{b_3\beta H^{*2}+b_2H^*+1}, \tilde{c} = -\frac{e\left(\alpha P^*+h\right)P^*\left(b_3\beta H^{*2}-1\right)}{\left(b_3\beta H^{*2}+b_2H^*+1\right)^2}, \tilde{d} = \frac{eH^*\left(2\alpha P^*+h\right)}{b_3\beta H^{*2}+b_2H^*+1} - d_2$$

Now, from above mentioned conditions (GH1) and (GH2) we get,

$$d_{1_{GH}} = \frac{b}{P\beta+1} - \frac{H\left(HK_1+2K\right)r}{\left(HK_1+K\right)^2} + \frac{\left(P\alpha+h\right)P\left(b_3\beta H^2-1\right)}{\left(b_3\beta H^2+b_2H+1\right)^2} + \frac{eH\left(2P\alpha+h\right)}{b_3\beta H^2+b_2H+1} - d_2$$

To obtain the first Lyapunov number L at E_I, we shift E_I to origin by taking the transformation $x_1 = H - H^*$ and $x_2 = P - P^*$. So, system (1) becomes

$$\begin{aligned}\frac{dx_1}{dt} &= \tilde{a}x_1 + \tilde{b}x_2 + S(x_1,x_2)\\ \frac{dx_2}{dt} &= \tilde{c}x_1 + \tilde{d}x_2 + Q(x_1,x_2)\end{aligned}$$

where, $\tilde{a} = \left(\dfrac{\partial F_1}{\partial H}\right)_{E_I}$, $\tilde{b} = \left(\dfrac{\partial F_1}{\partial P}\right)_{E_I}$, $\tilde{c} = \left(\dfrac{\partial F_2}{\partial H}\right)_{E_I}$, $\tilde{d} = \left(\dfrac{\partial F_2}{\partial P}\right)_{E_I}$ and $S(x_1,x_2)$, $Q(x_1,x_2)$ are analytic functions, defined by

$$S(x_1,x_2)=\sum_{i+j\geq 2}\tilde{a}_{ij}{x_1}^i{x_2}^j$$
$$Q(x_1,x_2)=\sum_{i+j\geq 2}\tilde{b}_{ij}{x_1}^i{x_2}^j$$

Here, $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ are defined by, $\tilde{a}_{ij}=\dfrac{1}{i!\ j!}\Big(\dfrac{\partial^{i+j}F_1}{\partial x^i\partial y^j}\Big)_{E_I}$ and $\tilde{b}_{ij}=\dfrac{1}{i!\ j!}\Big(\dfrac{\partial^{i+j}F_2}{\partial x^i\partial y^j}\Big)_{E_I}$.

Now, the first Lyapunov number is as follows;

$$L=-\frac{3\pi}{2\tilde{b}D^{\frac{3}{2}}}\Big[\Big\{\tilde{a}\tilde{c}(\tilde{a}_{11}^2+\tilde{a}_{11}\tilde{b}_{02}+\tilde{a}_{02}\tilde{b}_{11})+\tilde{a}\tilde{b}(\tilde{b}_{11}^2+\tilde{a}_{20}\tilde{b}_{11}+\tilde{a}_{11}\tilde{b}_{02})+\tilde{c}^2(\tilde{a}_{11}\tilde{a}_{02}+2\tilde{a}_{02}\tilde{b}_{02})-2\tilde{a}\tilde{c}(\tilde{b}_{02}^2-\tilde{a}_{20}\tilde{a}_{02})-2\tilde{a}\tilde{b}(\tilde{a}_{20}^2-\tilde{b}_{20}\tilde{b}_{02})-\tilde{b}^2(2\tilde{a}_{20}\tilde{b}_{20}+\tilde{b}_{11}\tilde{b}_{20})+(\tilde{b}\tilde{c}-2\tilde{a}^2)(\tilde{b}_{11}\tilde{b}_{02}-\tilde{a}_{11}\tilde{a}_{20})\Big\}-(\tilde{a}^2+\tilde{b}\tilde{c})\Big\{3(\tilde{c}\tilde{b}_{03}-\tilde{b}\tilde{a}_{30})+2\tilde{a}(\tilde{a}_{21}+\tilde{b}_{12})+(\tilde{c}\tilde{a}_{12}-\tilde{b}\tilde{b}_{21})\Big\}\Big]$$

Let us determine the coefficients $\tilde{a}_{ij}$, $\tilde{b}_{ij}$ that are needed for the calculation of the first Lyapunov number.

$$D=det(J(E_I)),\quad \tilde{a}_{11}=\frac{\partial^2F_1}{\partial P\partial H}(H^*,P^*)=\frac{b\beta}{(\beta P^*+1)^2}+\frac{\left(b_3\beta H^{*2}-1\right)(2\alpha P^*+h)}{\left(1+H^*\left(b_3\beta H^*+b_2\right)\right)^2},$$
$$\tilde{b}_{03}=\frac{1}{6}\frac{\partial^3F_2}{\partial P^3}(H^*,P^*)=0,\quad \tilde{a}_{02}=\frac{1}{2}\frac{\partial^2F_1}{\partial P^2}(H^*,P^*)=\frac{bH^*\beta^2}{(\beta P^*+1)^3}-\frac{\alpha H^*}{b_3\beta H^{*2}+b_2H^*+1}$$
$$\tilde{a}_{20}=\frac{1}{2}\frac{\partial^2F_1}{\partial H^2}(H^*,P^*)=\frac{\left(b_2+H^*b_3\beta\left(-b_3\beta H^{*2}+3\right)\right)P\left(\alpha P+h\right)}{\left(1+H\left(Hb_3\beta+b_2\right)\right)^3}-\frac{K^2r}{\left(K_1H+K\right)^3},$$
$$\tilde{a}_{21}=\frac{1}{2}\frac{\partial^3F_1}{\partial H^2\partial P}(H^*,P^*)=\frac{(2\alpha P^*+h)\left(H^{*3}b_3^2\beta^2-3H^*b_3\beta-b_2\right)}{\left(b_3\beta H^{*2}+b_2H^*+1\right)^3},$$
$$\tilde{a}_{12}=\frac{1}{2}\frac{\partial^3F_1}{\partial H\partial P^2}(H^*,P^*)=\frac{\alpha\left(b_3\beta H^{*2}-1\right)}{\left(1+H^*\left(H^*b_3\beta+b_2\right)\right)^2}+\frac{b\beta^2}{\left(\beta P^*+1\right)^3},$$
$$\tilde{a}_{30}=\frac{1}{6}\frac{\partial^3F_1}{\partial H^3}(H^*,P^*)=\frac{K^2K_1r}{\left(K_1H^*+K\right)^4}-\frac{\left(4b_2b_3\beta H^*+b_2^2-b_3\beta\left(1+b_3\beta H^{*2}\left(b_3\beta H^{*2}-6\right)\right)\right)P^*\left(\alpha P^*+h\right)}{\left(1+H^*\left(H^*b_3\beta+b_2\right)\right)^4},$$
$$\tilde{a}_{03}=\frac{1}{6}\frac{\partial^3F_1}{\partial P^3}(H^*,P^*)=\frac{bH^*\beta^3}{\left(\beta P^*+1\right)^4},$$
$$\tilde{b}_{11}=\frac{\partial^2F_2}{\partial P\partial H}(H^*,P^*)=-\frac{(2\alpha P^*+h)\left(b_3\beta H^{*2}-1\right)e}{\left(b_3\beta H^{*2}+b_2H^*+1\right)^2},$$
$$\tilde{b}_{20}=\frac{1}{2}\frac{\partial^2F_2}{\partial H^2}(H^*,P^*)=\frac{e\left(\alpha P^*+h\right)P^*\left(H^{*3}b_3^2\beta^2-3H^*b_3\beta-b_2\right)}{\left(b_3\beta H^{*2}+b_2H^*+1\right)^3},$$
$$\tilde{b}_{02}=\frac{1}{2}\frac{\partial^2F_2}{\partial P^2}(H^*,P^*)=\frac{e\alpha H^*}{b_3\beta H^{*2}+b_2H^*+1},$$
$$\tilde{b}_{21}=\frac{1}{2}\frac{\partial^3F_2}{\partial H^{*2}\partial P^*}(H^*,P^*)=\frac{\left(H^{*3}b_3^2\beta^2-3H^*b_3\beta-b_2\right)(2\alpha P^*+h)\,e}{\left(b_3\beta H^{*2}+b_2H^*+1\right)^3},$$
$$\tilde{b}_{12}=\frac{1}{2}\frac{\partial^3F_2}{\partial H\partial P^2}(H^*,P^*)=\frac{e\alpha\left(b_3\beta H^{*2}-1\right)}{\left(b_3\beta H^{*2}+b_2H^*+1\right)^2},$$

$$\tilde{b}_{30} = \frac{1}{6}\frac{\partial^3 F_2}{\partial H^3}(H^*, P^*) = \frac{\left(H^{*4} b_3^3 \beta^3 - 6H^{*2} b_3^2 \beta^2 + \left(-4b_2 H^* + 1\right) b_3 \beta - b_2^2\right) eP^* \left(\alpha P^* + h\right)}{\left(b_3 \beta H^{*2} + b_2 H^* + 1\right)^4}$$

By substituting the values from the above expressions into the initial Lyapunov number and conducting algebraic computations, we get L, the first Lyapunov number. If $L = 0$, then system (1) experiences a generalized Hopf bifurcation. However, demonstrating that $L = 0$ is laborious and challenging; the presence of generalized Hopf bifurcation can be confirmed numerically for certain parameter values. ■