Bogdanov-Takens (BT) Bifurcation Proof

- **Article:** Study on Fear-Induced Group Defense and Cooperative Hunting: Bifurcation, Uncertainty, Seasonality, and Spatio-temporal Analysis in Predator-Prey System
- **Authors:** Parvez Akhtar, Nirapada Santra, Guruprasad Samanta

Our proposed system is:

$$\begin{aligned}
\frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1 H} - \frac{(h+\alpha P)HP}{1+b_2 H + b_3 \beta H^2} \equiv F_1(H,P) = Hf_1(H,P), \quad \text{(say)} \\
\frac{dP}{dt} &= \frac{e(h+\alpha P)HP}{1+b_2 H+b_3\beta H^2} - d_2 P \equiv F_2(H,P) = Pf_2(H,P), \quad \text{(say)}
\end{aligned} \tag{1}$$

Theorem 1. *System (1) experiences a Bogdanov–Takens bifurcation around the interior equilibrium point $E_I(H^*, P^*)$ with respect to the bifurcation parameters d_1, b_3, whenever $E_I(H^*, P^*)$ satisfies the following conditions:*
(BT1): $tr(J(E_I; (d_1^{[BT]}, b_3^{[BT]}))) = 0$
(BT2): $det(J(E_I; (d_1^{[BT]}, b_3^{[BT]}))) = 0$

Proof. Conditions (BT1) and (BT2) are equivalent to the followings:

$$\frac{b}{P\beta+1} - d_1 - \frac{H\left(HK_1+2K\right)r}{\left(HK_1+K\right)^2} + \frac{\left(\alpha P+h\right)P\left(b_3\beta H^2-1\right)}{\left(b_3\beta H^2+Hb_2+1\right)^2} + \frac{eH\left(2\alpha P+h\right)}{b_3\beta H^2+Hb_2+1} - d_2 = 0$$

$$\begin{aligned}
&\left(\frac{b}{P\beta+1} - d_1 - \frac{H\left(HK_1+2K\right)r}{\left(HK_1+K\right)^2} + \frac{\left(\alpha P+h\right)P\left(b_3\beta H^2-1\right)}{\left(b_3\beta H^2+Hb_2+1\right)^2}\right)\left(\frac{eH\left(2\alpha P+h\right)}{b_3\beta H^2+Hb_2+1} - d_2\right) \\
&\qquad + \frac{\left(-\frac{bH\beta}{(P\beta+1)^2} - \frac{H(2\alpha P+h)}{b_3\beta H^2+Hb_2+1}\right)e\left(b_3\beta H^2-1\right)P\left(\alpha P+h\right)}{\left(b_3\beta H^2+Hb_2+1\right)^2} = 0
\end{aligned}$$

From above two equations we can get the value of $d_1^{[BT]}$ and $b_3^{[BT]}$.
Now, we have taken a small perturbation around the bifurcation threshold $(d_1^{[BT]}, b_3^{[BT]})$, say $(d_1^{[BT]} + \lambda_1, b_3^{[BT]} + \lambda_2)$, where $\{\lambda_i; i = 1, 2\}$ are sufficiently small.
Thus, the system becomes:

$$\begin{aligned}
\frac{dH}{dt} &= \frac{bH}{1+\beta P} - \left(d_1^{[BT]}+\lambda_1\right)H - \frac{rH^2}{K+K_1H} - \frac{(h+\alpha P)HP}{1+b_2H+\left(b_3^{[BT]}+\lambda_2\right)\beta H^2} \equiv G_1(H,P,\lambda_1,\lambda_2) \\
\frac{dP}{dt} &= \frac{e(h+\alpha P)HP}{1+b_2H+\left(b_3^{[BT]}+\lambda_2\right)\beta H^2} - d_2P \equiv G_2(H,P,\lambda_2)
\end{aligned} \tag{2}$$

By using the transformations $\mathrm{x}_1 = H - H^*$ and $\mathrm{x}_2 = P - P^*$ we shift the equilibrium point $E_I(H^*, P^*)$ to the origin. Therefore, system (2) becomes

$$\begin{aligned}
\frac{d\mathrm{x}_1}{dt} &= \mathfrak{p}_{00} + \mathfrak{p}_{10}\mathrm{x}_1 + \mathfrak{p}_{01}\mathrm{x}_2 + \frac{\mathfrak{p}_{11}}{2}{\mathrm{x}_1}^2 + \mathfrak{p}_{12}\mathrm{x}_1\mathrm{x}_2 + \frac{\mathfrak{p}_{22}}{2}{\mathrm{x}_2}^2 + ... \\
\frac{d\mathrm{x}_2}{dt} &= \mathfrak{q}_{00} + \mathfrak{q}_{10}\mathrm{x}_1 + \mathfrak{q}_{01}\mathrm{x}_2 + \frac{\mathfrak{q}_{11}}{2}{\mathrm{x}_1}^2 + \mathfrak{q}_{12}\mathrm{x}_1\mathrm{x}_2 + \frac{\mathfrak{q}_{22}}{2}{\mathrm{x}_2}^2 + ...
\end{aligned} \tag{3}$$

where, $\mathfrak{p}_{00} = G_1(H^*, P^*, \lambda_1, \lambda_2)$, $\mathfrak{q}_{00} = G_2(H^*, P^*, \lambda_1, \lambda_2)$,

$$\mathfrak{p}_{10} = \frac{\partial G_1}{\partial H}(H^*, P^*, \lambda_1, \lambda_2) = \frac{b}{P^*\beta+1} - d_1 - \lambda_1 - \frac{H^*(H^*K_1+2K)r}{(H^*K_1+K)^2} + \frac{\left(-1+(b_3+\lambda_2)\beta H^{*2}\right)P^*(\alpha P^*+h)}{(1+H^*(b_2+(b_3+\lambda_2)\beta H^*))^2},$$

$$\mathfrak{q}_{10} = \frac{\partial G_2}{\partial H}(H^*, P^*, \lambda_1, \lambda_2) = \frac{e\left(\alpha P^* + h\right) P^* \left(H^{*2}\beta\,\lambda_2 + b_3\beta H^{*2} - 1\right)}{\left(H^{*2}\beta\,\lambda_2 + b_3\beta H^{*2} + H^* b_2 + 1\right)^2},$$

$$\mathfrak{p}_{01} = \frac{\partial G_1}{\partial P}(H^*, P^*, \lambda_1, \lambda_2) = \frac{bH^*\beta}{\left(P^*\beta+1\right)^2} - \frac{H^*\left(2\alpha P^* + h\right)}{1 + H^*\left(b_2 + \left(b_3+\lambda_2\right)\beta H^*\right)},$$

$$\mathfrak{q}_{01} = \frac{\partial G_2}{\partial P}(H^*, P^*, \lambda_1, \lambda_2) = \frac{eH^*\left(2\alpha P^* + h\right)}{1 + H^*\left(b_2 + \left(b_3+\lambda_2\right)\beta H^*\right)} - d_2,$$

$$\mathfrak{p}_{11} = \frac{\partial^2 G_1}{\partial H^2}(H^*, P^*, \lambda_1, \lambda_2) = \frac{2K^2 r}{(H^*K_1+K)^3} + \frac{2\left(b_2-\beta H^*(b_3+\lambda_2)\left(-3+(b_3+\lambda_2)\beta H^{*2}\right)\right)P^*(\alpha P^*+h)}{(1+H^*(b_2+(b_3+\lambda_2)\beta H^*))^3},$$

$$\mathfrak{p}_{22} = \frac{\partial^2 G_1}{\partial P^2}(H^*, P^*, \lambda_1, \lambda_2) = \frac{2bH^*\beta^2}{\left(P^*\beta+1\right)^3} - \frac{2H^*\alpha}{1 + H^*\left(b_2 + \left(b_3+\lambda_2\right)\beta H^*\right)},$$

$$\mathfrak{q}_{11} = \frac{\partial^2 G_2}{\partial H^2}(H^*, P^*, \lambda_1, \lambda_2) = \frac{2e\left(b_2 - \beta H^*\left(b_3+\lambda_2\right)\left(-3 + \left(b_3+\lambda_2\right)\beta H^{*2}\right)\right)P^*\left(\alpha P^* + h\right)}{\left(1 + H^*\left(b_2 + \left(b_3+\lambda_2\right)\beta H^*\right)\right)^3},$$

$$\mathfrak{q}_{22} = \frac{\partial^2 G_2}{\partial P^2}(H^*, P^*, \lambda_1, \lambda_2) = \frac{2eH^*\alpha}{1 + H^*\left(b_2 + \left(b_3+\lambda_2\right)\beta H^*\right)},$$

$$\mathfrak{p}_{12} = \frac{\partial^2 G_1}{\partial P^*\partial H}(H^*, P^*, \lambda_1, \lambda_2) = \frac{b\beta}{\left(P^*\beta+1\right)^2} + \frac{\left(\left(b_3+\lambda_2\right)\beta H^{*2} - 1\right)\left(2\alpha P^* + h\right)}{\left(1 + H^*\left(b_2 + \left(b_3+\lambda_2\right)\beta H^*\right)\right)^2},$$

$$\mathfrak{q}_{12} = \frac{\partial^2 G_2}{\partial P^*\partial H}(H^*, P^*, \lambda_1, \lambda_2) = \frac{e\left(\left(b_3+\lambda_2\right)\beta H^{*2} - 1\right)\left(2\alpha P^* + h\right)}{\left(1 + H^*\left(b_2 + \left(b_3+\lambda_2\right)\beta H^*\right)\right)^2}$$

Let us assume

$$\tilde{a} = \frac{b}{1+\beta P^*} - d_1 - \frac{rH^*\left(K_1H^* + 2K\right)}{\left(K_1H^* + K\right)^2} + \frac{\left(\alpha P^* + h\right)P^*\left(b_3\beta H^{*2} - 1\right)}{\left(b_3\beta H^{*2} + b_2H^* + 1\right)^2},$$

$$\tilde{b} = -\frac{bH^*\beta}{(1+\beta P^*)^2} - \frac{H^*\left(2\alpha P^* + h\right)}{b_3\beta H^{*2} + b_2H^* + 1},\ \tilde{c} = -\frac{e\left(\alpha P^* + h\right)P^*\left(b_3\beta H^{*2} - 1\right)}{\left(b_3\beta H^{*2} + b_2H^* + 1\right)^2},\ \tilde{d} = \frac{eH^*\left(2\alpha P^* + h\right)}{b_3\beta H^{*2} + b_2H^* + 1} - d_2$$

Now, we introduce the affine transformation [Perko, 2013] defined as $z_1 = \mathrm{x}_1$, $z_2 = \tilde{a}\mathrm{x}_1 + \tilde{b}\mathrm{x}_2$, which transforms the aforementioned system into

$$\begin{aligned}\frac{dz_1}{dt} &= z_2 + \zeta_{00}(\lambda) + \zeta_{10}(\lambda)z_1 + \zeta_{01}(\lambda)z_2 + \frac{\zeta_{20}(\lambda)}{2}{z_1}^2 + \zeta_{11}z_1z_2 + \frac{\zeta_{02}(\lambda)}{2}{z_2}^2 + B_1(z_1,z_2)\\ \frac{dz_2}{dt} &= \eta_{00}(\lambda) + \eta_{10}(\lambda)z_1 + \eta_{01}(\lambda)z_2 + \frac{\eta_{20}(\lambda)}{2}{z_1}^2 + \eta_{11}z_1z_2 + \frac{\eta_{02}}{2}{z_2}^2 + B_2(z_1,z_2)\end{aligned} \tag{4}$$

where, $\lambda = (\lambda_1, \lambda_2)$ and $B_1(z_1, z_2)$, $B_2(z_1, z_2)$ are $\mathbb{C}^\infty$ functions at least of third order with respect to (z_1, z_2).

$$\zeta_{00}(\lambda) = G_1(H^*, P^*, \lambda)\ ,\ \ \eta_{00}(\lambda) = \tilde{a}G_1(H^*, P^*, \lambda) + \tilde{b}G_2(H^*, P^*, \lambda)\ ,\ \ \zeta_{10}(\lambda) = (\mathfrak{p}_{10} - \frac{\tilde{a}}{\tilde{b}}\mathfrak{p}_{01}),$$

$$\eta_{10}(\lambda) = \tilde{b}\mathfrak{q}_{10} - \tilde{a}\mathfrak{q}_{01} + \tilde{a}\mathfrak{p}_{10} - \frac{\tilde{a}^2}{\tilde{b}}\mathfrak{p}_{01}\ ,\ \ \zeta_{01}(\lambda) = \frac{1}{\tilde{b}}\mathfrak{p}_{01} - 1\ ,\ \ \eta_{01}(\lambda) = \mathfrak{q}_{01} + \frac{\tilde{a}}{\tilde{b}}\mathfrak{p}_{01}\ ,\ \ \zeta_{02}(\lambda) = \frac{\mathfrak{p}_{22}}{\tilde{b}^2},$$

$$\zeta_{20}(\lambda) = \left[\mathfrak{p}_{11} - \frac{2\tilde{a}\mathfrak{p}_{12}}{\tilde{b}} + \frac{\tilde{a}^2\mathfrak{p}_{22}}{\tilde{b}^2}\right]\ ,\ \ \eta_{20}(\lambda) = \left[\tilde{a}\mathfrak{p}_{11} + \tilde{b}\mathfrak{q}_{11} - \frac{2\tilde{a}(\tilde{a}\mathfrak{p}_{12} + \tilde{b}\mathfrak{q}_{12})}{\tilde{b}} + \frac{\tilde{a}^2(\tilde{a}\mathfrak{p}_{22} + \tilde{b}\mathfrak{q}_{22})}{\tilde{b}^2}\right],$$

$$\zeta_{11}(\lambda) = \left[\frac{\mathfrak{p}_{12}}{\tilde{b}} - \frac{\tilde{a}\mathfrak{p}_{22}}{\tilde{b}^2}\right]\ ,\ \ \eta_{11}(\lambda) = \left[\frac{(\tilde{a}\mathfrak{p}_{12} + \tilde{b}\mathfrak{q}_{12})}{\tilde{b}} - \frac{\tilde{a}(\tilde{a}\mathfrak{p}_{22} + \tilde{b}\mathfrak{q}_{22})}{\tilde{b}^2}\right]\ ,\ \ \eta_{02}(\lambda) = \frac{(\tilde{a}\mathfrak{p}_{22} + \tilde{b}\mathfrak{q}_{22})}{\tilde{b}^2}.$$

The degeneracy conditions [Kuznetsov *et al.*, 1998] of the Bogdanov-Takens bifurcations at $\left(d_1^{[BT]}, b_3^{[BT]}\right)$ are

I. $\begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

II. $\zeta_{20}(0) + \eta_{11}(0) = \dfrac{2\left(b_2 - \beta H^* b_3\left(b_3\beta H^{*2} - 3\right)\right)P^*\left(\alpha P^* + h\right)}{\left(1 + H^*\left(\beta H^* b_3 + b_2\right)\right)^3} - R_1 \neq 0$, the value of R_1 is given in Remark 1.

III.

$$\eta_{20} = \left(\frac{b}{P^*\beta+1} - \frac{eH^*\left(2P^*\alpha+h\right)}{H^{*2}b_3\beta + H^*b_2 + 1} + d_2 - \frac{H^*\left(H^*K_1 + 2K\right)r}{\left(H^*K_1+K\right)^2} + \frac{\left(P^*\alpha+h\right)P^*\left(H^{*2}b_3\beta - 1\right)}{\left(H^{*2}b_3\beta + H^*b_2 + 1\right)^2}\right)$$
$$\left(-\frac{2K^2r}{\left(H^*K_1+K\right)^3} + \frac{2\left(b_2 - \beta H^* b_3\left(H^{*2}b_3\beta - 3\right)\right)P^*\left(P^*\alpha+h\right)}{\left(1+H^*\left(\beta H^* b_3 + b_2\right)\right)^3}\right) - \mathscr{M} - \mathscr{N} + \mathscr{T}$$

The expressions of $\mathscr{M}$, $\mathscr{N}$ and $\mathscr{T}$ are given in Remark 2.
Therefore, $\zeta_{20}(0) + \eta_{11}(0) \neq 0$ and $\eta_{20}(0)$ may or may not be 0. When $\eta_{20}(0) \neq 0$, it follows that **sign** $\left[\eta_{20}(0)\left(\zeta_{20}(0) + \eta_{11}(0)\right)\right]$, if $= +1$, the predator-prey model experiences a subcritical BT bifurcation; if $= -1$, it undergoes a supercritical BT bifurcation. Demonstrating that $\eta_{20}(0) \neq 0$ is challenging; nevertheless, we can confirm the numerical presence of a Bogdanov-Takens bifurcation for certain parameter selections and it is shown in section 'Numerical analysis'. ■

1. Remark 1:

$$R_1 = \frac{2K^2r}{\left(H^*K_1+K\right)^3} +$$
$$\frac{2\left(\frac{b}{P^*\beta+1} - \frac{eH^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1} + d_2 - \frac{H^*(H^*K_1+2K)r}{(H^*K_1+K)^2} + \frac{(\alpha P^*+h)P^*\left(b_3\beta H^{*2}-1\right)}{\left(b_3\beta H^{*2}+H^*b_2+1\right)^2}\right)\left(-\frac{b\beta}{(P^*\beta+1)^2} + \frac{\left(b_3\beta H^{*2}-1\right)(2\alpha P^*+h)}{(1+H^*(\beta H^*b_3+b_2))^2}\right)}{-\frac{bH^*\beta}{(P^*\beta+1)^2} - \frac{H^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1}} -$$
$$\frac{\left(\frac{b}{P^*\beta+1} - \frac{eH^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1} + d_2 - \frac{H^*(H^*K_1+2K)r}{(H^*K_1+K)^2} + \frac{(\alpha P^*+h)P^*\left(b_3\beta H^{*2}-1\right)}{\left(b_3\beta H^{*2}+H^*b_2+1\right)^2}\right)^2\left(\frac{2bH^*\beta^2}{(P^*\beta+1)^3} - \frac{2H^*\alpha}{1+H^*(\beta H^*b_3+b_2)}\right)}{\left(-\frac{bH^*\beta}{(P^*\beta+1)^2} - \frac{H^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1}\right)^2} -$$
$$\frac{\left(\frac{b}{P^*\beta+1} - \frac{eH^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1} + d_2 - \frac{H^*(H^*K_1+2K)r}{(H^*K_1+K)^2} + \frac{(\alpha P^*+h)P^*\left(b_3\beta H^{*2}-1\right)}{\left(b_3\beta H^{*2}+H^*b_2+1\right)^2}\right)\left(-\frac{b\beta}{(P^*\beta+1)^2} + \frac{\left(b_3\beta H^{*2}-1\right)(2\alpha P^*+h)}{(1+H^*(\beta H^*b_3+b_2))^2}\right) - \mathscr{S}}{-\frac{bH^*\beta}{(P^*\beta+1)^2} - \frac{H^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1}} +$$
$$\frac{\left(\frac{b}{P^*\beta+1} - \frac{eH^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1} + d_2 - \frac{H^*(H^*K_1+2K)r}{(H^*K_1+K)^2} + \frac{(\alpha P^*+h)P^*\left(b_3\beta H^{*2}-1\right)}{\left(b_3\beta H^{*2}+H^*b_2+1\right)^2}\right)\left(\mathscr{U}\left(\frac{2bH^*\beta^2}{(P^*\beta+1)^3} - \frac{2H^*\alpha}{1+H^*(\beta H^*b_3+b_2)}\right) + \mathscr{V}\right)}{\left(-\frac{bH^*\beta}{(P^*\beta+1)^2} - \frac{H^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1}\right)^2}$$

Here, $\mathscr{S} = \dfrac{\left(-\frac{bH^*\beta}{(P^*\beta+1)^2} - \frac{H^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1}\right)e\left(b_3\beta H^{*2} - 1\right)\left(2\alpha P^* + h\right)}{\left(1 + H^*\left(\beta H^* b_3 + b_2\right)\right)^2}$,

$$\mathscr{U} = \left(\frac{b}{P^*\beta+1} - \frac{eH^*\left(2\alpha P^*+h\right)}{b_3\beta H^{*2} + Hb_2 + 1} + d_2 - \frac{H^*\left(H^*K_1+2K\right)r}{\left(H^*K_1+K\right)^2} + \frac{\left(\alpha P^*+h\right)P^*\left(b_3\beta H^{*2}-1\right)}{\left(b_3\beta H^{*2} + H^*b_2 + 1\right)^2}\right)$$

$$\mathscr{V} = \frac{2\left(-\frac{bH^*\beta}{(P^*\beta+1)^2} - \frac{H^*(2\alpha P^*+h)}{b_3\beta H^{*2}+H^*b_2+1}\right)eH^*\alpha}{1+H^*\left(\beta H^* b_3 + b_2\right)}$$

2. Remark 2:

$$\mathscr{M} = \frac{2\left(-\frac{bH^*\beta}{(P^*\beta+1)^2}-\frac{H^*\left(2P^*\alpha+H^*\right)}{H^{*2}b_3\beta+H^*b_2+1}\right)e\left(b_2-\beta H^*(b_3)\left(-3+(b_3)\beta H^{*2}\right)\right)P^*(P^*\alpha+h)}{(1+H^*(b_2+\beta H^*(b_3)))^3}$$

$$\mathscr{N} = \frac{2\left(\frac{b}{P^*\beta+1}-\frac{eH^*\left(2P^*\alpha+h\right)}{H^{*2}b_3\beta+H^*b_2+1}+d_2-\frac{H^*\left(H^*K_1+2K\right)r}{(H^*K_1+K)^2}+\frac{(P^*\alpha+h)P^*\left(H^{*2}b_3\beta-1\right)}{\left(H^{*2}b_3\beta+H^*b_2+1\right)^2}\right)\left(\mathscr{N}_1\left(-\frac{b\beta}{(P^*\beta+1)^2}+\frac{\left(H^{*2}b_3\beta-1\right)\left(2P^*\alpha+h\right)}{(1+H^*(\beta H^*b_3+b_2))^2}\right)-\mathscr{N}_2\right)}{-\frac{bH^*\beta}{(P^*\beta+1)^2}-\frac{H^*(2P^*\alpha+h)}{H^{*2}b_3\beta+H^*b_2+1}}$$

where,

$$\mathscr{N}_1 = \left(\frac{b}{P^*\beta+1}-\frac{eH^*(2P^*\alpha+h)}{H^{*2}b_3\beta+H^*b_2+1}+d_2-\frac{H^*(H^*K_1+2K)r}{(H^*K_1+K)^2}+\frac{(P^*\alpha+h)P^*\left(H^{*2}b_3\beta-1\right)}{\left(H^{*2}b_3\beta+H^*b_2+1\right)^2}\right)$$

$$\mathscr{N}_2 = \frac{\left(-\frac{bH^*\beta}{(P^*\beta+1)^2}-\frac{H^*\left(2P^*\alpha+h\right)}{H^{*2}b_3\beta+H^*b_2+1}\right)e\left(H^{*2}b_3\beta-1\right)(2P^*\alpha+h)}{(1+H^*(\beta H^*b_3+b_2))^2}$$

$$\mathscr{T} = \frac{\left(\frac{b}{P^*\beta+1}-\frac{eH^*\left(2P^*\alpha+h\right)}{H^{*2}b_3\beta+H^*b_2+1}+d_2-\frac{H^*\left(H^*K_1+2K\right)r}{(H^*K_1+K)^2}+\frac{(P^*\alpha+h)P^*\left(H^{*2}b_3\beta-1\right)}{\left(H^{*2}b_3\beta+H^*b_2+1\right)^2}\right)^2\left(\mathscr{T}_1\left(\frac{2bH^*\beta^2}{(P^*\beta+1)^3}-\frac{2H^*\alpha}{1+H^*(\beta H^*b_3+b_2)}\right)+\mathscr{T}_2\right)}{\left(-\frac{bH^*\beta}{(P^*\beta+1)^2}-\frac{H^*(2P^*\alpha+h)}{H^{*2}b_3\beta+H^*b_2+1}\right)^2}$$

where,

$$\mathscr{T}_1 = \left(\frac{b}{P^*\beta+1}-\frac{eH^*(2P^*\alpha+h)}{H^{*2}b_3\beta+H^*b_2+1}+d_2-\frac{H^*(H^*K_1+2K)r}{(H^*K_1+K)^2}+\frac{(P^*\alpha+h)P^*\left(H^{*2}b_3\beta-1\right)}{\left(H^{*2}b_3\beta+H^*b_2+1\right)^2}\right)$$

$$\mathscr{T}_2 = \frac{2\left(-\frac{bH^*\beta}{(P^*\beta+1)^2}-\frac{H^*\left(2P^*\alpha+h\right)}{H^{*2}b_3\beta+H^*b_2+1}\right)eH^*\alpha}{1+H^*(\beta H^*b_3+b_2)}$$

References

Kuznetsov, Y. A., Kuznetsov, I. A. & Kuznetsov, Y. [1998] *Elements of applied bifurcation theory*, Vol. 112 (Springer), doi:10.1007/978-3-031-22007-4.

Perko, L. [2013] *Differential Equations and Dynamical Systems*, Vol. 7 (Springer Science & Business Media), doi:10.1007/978-1-4613-0003-8.