Transcritical Bifurcation Proof

- **Article:** Study on Fear-Induced Group Defense and Cooperative Hunting: Bifurcation, Uncertainty, Seasonality, and Spatio-temporal Analysis in Predator-Prey System
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Our proposed system is:

$$\begin{aligned}\frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1H} - \frac{(h+\alpha P)HP}{1+b_2H+b_3\beta H^2} \equiv F_1(H,P) = Hf_1(H,P), \quad \text{(say)} \\ \frac{dP}{dt} &= \frac{e(h+\alpha P)HP}{1+b_2H+b_3\beta H^2} - d_2P \equiv F_2(H,P) = Pf_2(H,P), \quad \text{(say)}\end{aligned} \tag{1}$$

Theorem 1. *At the bifurcation threshold* $d_2^{[TC]} = \dfrac{eh\hat{a}}{1+b_2\hat{a}+b_3\beta\hat{a}^2}$*, the system (1) undergoes a transcritical bifurcation around the predator-free equilibrium point* $E_a(\hat{a},0)$ *where,* $\hat{a} = \dfrac{K}{1-K_1}$.

Proof. We will use Sotomayor's theorem [Perko, 2013] to prove the presence of transcritical bifurcation in system (1). The Jacobian matrix at $E_a(\hat{a},0)\Big|_{d_2=d_2^{[TC]}}$ is

$$J(E_a)\big|_{d_2=d_2^{[TC]}} = \begin{bmatrix} r - \dfrac{2r\hat{a}}{K+K_1\hat{a}} + \dfrac{r\hat{a}^2K_1}{(K_1\hat{a}+K)^2} & -\left(b\hat{a}\beta + \dfrac{h\hat{a}}{1+b_2\hat{a}+b_3\beta\hat{a}^2}\right) \\ 0 & 0 \end{bmatrix} = M$$

It follows that 0 is an eigenvalue of the matrix M. Thus, 0 is also an eigenvalue of M^T, and let W and Z denote the eigenvectors corresponding to 0 of the matrices M and M^T, respectively. Here, $W = \begin{pmatrix} 1 \\ w_1 \end{pmatrix}$, and $Z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where, $w_1 = \dfrac{r\Big((K_1^2-K_1)\hat{a}^2 + 2K(K_1-1)\hat{a} + K^2\Big)(1+b_2\hat{a}+b_3\beta\hat{a}^2)}{(K_1\hat{a}+K)^2\hat{a}(b\beta - bb_2\beta\hat{a} + bb_3\beta^2\hat{a}^2 + h)}$.

Let $\mathscr{F}(H,P) = \begin{bmatrix} F_1(H,P) \\ F_2(H,P) \end{bmatrix}$.

So the transversality conditions for transcritical bifurcation are,

$$\begin{aligned} &Z^T\left[\mathscr{F}_{d_2}\left(E_a; d_2 = d_2^{[TC]}\right)\right] = 0 \\ &Z^T\left[D\mathscr{F}_{d_2}\left(E_a; d_2 = d_2^{[TC]}\right)W\right] = \frac{r\Big((K_1^2-K_1)\hat{a}^2 + 2K(K_1-1)\hat{a} + K^2\Big)(1+b_2\hat{a}+b_3\beta\hat{a}^2)}{(K_1\hat{a}+K)^2\hat{a}(b\beta - bb_2\beta\hat{a} + bb_3\beta^2\hat{a}^2 + h)} \neq 0 \\ &Z^T\left[D^2\mathscr{F}\left(E_a; d_1 = d_1^{[TC]}\right)(W,W)\right] = \mathcal{A} + 2\cdot w_1\cdot\mathcal{B} + w_1^2\cdot\mathcal{C} \neq 0 \end{aligned}$$

where,

$\mathcal{A} = -\dfrac{2e(\alpha P+h)P(b_2+3b_3\beta H - b_3^2\beta^2H^3)}{(1+b_2H+b_3\beta H^2)^3}$, $\mathcal{B} = -\dfrac{e(b_3\beta H^2-1)(2\alpha P+h)}{(1+b_2H+b_3\beta H^2)^2}$, $\mathcal{C} = \dfrac{2e\alpha H}{1+b_2H+b_3\beta H^2}$ and

$$\left(\frac{2e(\alpha P+h)P(b_2+3b_3\beta H - b_3^2\beta^2H^3)}{(1+b_2H+b_3\beta H^2)^3} + 2\cdot w_1\cdot\frac{e(b_3\beta H^2-1)(2\alpha P+h)}{(1+b_2H+b_3\beta H^2)^2}\right) \neq w_1^2\cdot\frac{2e\alpha H}{1+b_2H+b_3\beta H^2}$$

Thus, by the help of Sotomayor's theorem we can say that the system has transcritical bifurcation around $E_a\left(\hat{a},0\right)$ at the bifurcation threshold $d_2^{[TC]}=\dfrac{eh\hat{a}}{1+b_2\hat{a}+b_3\beta\hat{a}^2}$. ■

References

Perko, L. [2013] *Differential Equations and Dynamical Systems*, Vol. 7 (Springer Science & Business Media), doi:10.1007/978-1-4613-0003-8.