Positivity and boundedness

- **Article:** Study on Fear-Induced Group Defense and Cooperative Hunting: Bifurcation, Uncertainty, Seasonality, and Spatio-temporal Analysis in Predator-Prey System
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Our proposed system is:

$$\begin{aligned}\frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1 H} - \frac{(h+\alpha P)HP}{1+b_2 H + b_3 \beta H^2} \equiv F_1(H,P) = Hf_1(H,P), \quad \text{(say)}\\ \frac{dP}{dt} &= \frac{e(h+\alpha P)HP}{1+b_2 H + b_3 \beta H^2} - d_2 P \equiv F_2(H,P) = Pf_2(H,P), \quad \text{(say)}\end{aligned} \tag{1}$$

In order to determine the well-posedness of the system, we analyze the positivity and boundedness of the state variables, ensuring that every solution remains positively invariant in the first quadrant and does not exhibit abrupt increment over extended time intervals.

Theorem 1. *For any time $t > 0$ every solution of system (1) remains positive and uniformly bounded with initial condition $H(0) > 0,\ P(0) > 0$.*

Proof.

$$\begin{cases} \dfrac{dH}{dt} = Hf_1(H,P) \\ \\ \dfrac{dP}{dt} = Pf_2(H,P) \end{cases} \tag{2}$$

where, $f_1(H,P) = \dfrac{b}{1+\beta P} - d_1 - \dfrac{rH}{K+K_1 H} - \dfrac{(h+\alpha P)P}{1+b_2 H + b_3 \beta H^2}$ and
$f_2(H,P) = \dfrac{e(h+\alpha P)H}{1+b_2 H + b_3 \beta H^2} - d_2$.
From (2) we get,

$$H(t) = H(0)\exp\left(\int_0^t f_1(H,P)dt\right) > 0, \text{ for } H(0) > 0.$$

$$P(t) = P(0)\exp\left(\int_0^t f_2(H,P)dt\right) > 0, \text{ for } P(0) > 0.$$

Therefore, $H(t)$ and $P(t)$ both are positive whenever $H(0) > 0$ and $P(0) > 0\ \forall\ t > 0$.

Now, we have to prove the boundedness of system (1).

$$\begin{aligned}
\frac{dH}{dt} &= \frac{bH}{1+\beta P} - d_1 H - \frac{rH^2}{K+K_1H} - \frac{(h+\alpha P)HP}{1+b_2H+b_3\beta H^2}\\
&\leq bH - d_1H - \frac{rH^2}{K+K_1H}\\
&= bH - d_1H - \frac{r}{K_1^2}\frac{(K_1H+K-K)^2}{(K+K_1H)}\\
&= bH - d_1H - \frac{r}{K_1^2}(K_1H+K) + \frac{2Kr}{K_1^2} - \frac{K^2r}{K_1^2(K+K_1H)}\\
&\leq \frac{2Kr}{K_1^2} + bH - \frac{r}{K_1^2}(K_1H+K) - d_1H\\
&= \frac{Kr}{K_1^2} + bH - \frac{rH}{K_1} - d_1H\\
&\leq \frac{Kr}{K_1^2} + \left(b - \frac{r}{K_1} - d_1\right)H\\
&= \frac{Kr}{K_1^2} - \left(\frac{r}{K_1} + d_1 - b\right)H
\end{aligned}$$

$$\therefore \frac{dH}{dt} + \mu H \leq \frac{rK}{K_1^2}, \text{ where } \mu = \left(\frac{r}{K_1} + d_1 - b\right), \text{ provided } \left(\frac{r}{K_1} + d_1\right) > b$$

Now, with the help of the differential inequality theory [Ross & Ross, 2004] for $H(t)$, we get

$$0 \leq H(t) \leq \frac{rK}{K_1^2}\cdot\frac{1}{\mu}(1-e^{-\mu t}) + H(0)e^{-\mu t} \tag{3}$$

Taking $t \to \infty$, we get, $0 < H(t) \leq \left(\frac{rK}{K_1^2}\cdot\frac{1}{\mu} + \epsilon_1\right) = M_1$, for any $\epsilon_1 > 0$.

Next, let us take $w(t) = H(t) + \frac{1}{e}P(t)$.
Therefore,

$$\begin{aligned}
\frac{dw}{dt} &= \frac{dH}{dt} + \frac{1}{e}\frac{dP}{dt}\\
\Rightarrow \frac{dw}{dt} &\leq bH - d_1H - \frac{d_2}{e}P\\
&= bH - \tau\left(H + \frac{1}{e}P\right), \quad \text{where } \tau = \min\{d_1, d_2\}\\
&= bH - \tau w(t)
\end{aligned}$$

Therefore,

$$\frac{dw}{dt} \leq bM_1 - \tau w(t)$$

Again, with the help of the differential inequality theory [Ross & Ross, 2004] for $w(t)$, we get

$$0 \leq w(t) \leq \frac{bM_1}{\tau}(1-e^{-\tau t}) + w(0)e^{-\tau t} \tag{4}$$

Taking $t \to \infty$, we get, $0 < w(t) \leq \frac{bM_1}{\tau} + \epsilon_2$, for any $\epsilon_2 > 0$. Therefore, all the solutions of the system (1), starting from the domain $\mathbb{R}_+^2$ are remain in $\left\{(H,P) \in \mathbb{R}_+^2 : 0 < H(t) + \frac{1}{e}P(t) \leq \frac{bM_1}{\tau} + \epsilon_2\right\}$. This proves that every solutions of system (1) are uniformly bounded.

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References

Ross, C. C. & Ross, C. C. [2004] *About Differential Equations* (Springer), doi:10.1007/978-1-4757-3949-7_1.