Mathematical preliminaries

- **Article:** Dynamics of a Gestation-Delayed Predator-Prey System in Toxic Habitat with Auxiliary Food Resource and Weak Allee Effect.
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The system (2.2) is as follows:

$$\dot{x} = rx\left(1-\frac{x}{k}\right)\left(\frac{x}{\alpha+x}\right) - \frac{ax^2y}{b+x^2} - \beta_1 x^3 \equiv F_1(x,y) = xf_1(x,y) \quad , \quad x(0)>0$$
$$\dot{y} = \frac{cax^2y}{b+x^2} + Ay - \beta_2 y^2 \equiv F_2(x,y) = yf_2(x,y) \quad , \quad y(0)>0$$

Here,

$$f_1(x,y) = r\left(1-\frac{x}{k}\right)\left(\frac{x}{\alpha+x}\right) - \frac{axy}{b+x^2} - \beta_1 x^2 \text{ and } f_2(x,y) = \frac{cax^2}{b+x^2} + A - \beta_2 y$$

1 Boundedness Proof of System (2.2)

Theorem 1.1. *For any initial conditions where $x(0)>0$ and $y(0)>0$, it follows that all solutions of equation (2.2) stay bounded for all $t>0$.*

Proof. In order to establish the boundedness of system (2.2), it is necessary to first demonstrate the boundedness of x. To achieve this, we have analyzed two scenarios:

Case I: Let $x(0) \le k$. Now we claim that $x(t) \le k \ \forall \ t > 0$. Suppose that it is not true. Then $\exists \ t_1, \ t_2 > 0$ with $t_1 < t_2$ such that $x(t_1) = k$ and $x(t) > k \ \forall \ t \in (t_1, t_2)$. Then for any $t \in (t_1, t_2)$

$$x(t) = x(0)\cdot \exp\left(\int_0^{t_1} f_1(x(s),y(s))ds + \int_{t_1}^{t} f_1(x(s),y(s))ds\right)$$

$$\Rightarrow x(t) = k\cdot\exp\left(\int_{t_1}^{t} f_1(x(s),y(s))ds\right) \qquad \text{...}\mathbf{(1)}$$

Since $x(t) > k$ for all t within the interval (t_1,t_2). It follows that $f_1(x,y) < 0$ for all t in the interval (t_1,t_2). Consequently, from equation (1), we have $x(t) < k$ for all t in (t_1,t_2). This result is in direct contradiction with our initial hypothesis that $x(t) > k$ for every t within the interval (t_1,t_2).
Thus, we must have $x(t) \le k \ \forall \ t > 0$ ...**(2)**

Case II: Assume $x(0) > k$. In this case, two scenarios can occur:

Subcase I $\exists \ \psi > 0$ such that $x(t) > k$ for $t \in [0,\psi)$ with $x(\psi) = k$, and $x(t) \le k$, for $t > \psi$. For $t \in (0,\psi)$ we have, $f_1(x,y) < 0$.
Then,

$$x(t) = x(0)\cdot\exp\left(\int_0^{t} f_1(x(s),y(s))ds\right) < x(0)$$

$\Rightarrow x(t) < x(0) \ \forall \ t \in (0,\psi)$

Subcase II $x(t) > k \ \forall \ t \ge 0$

For $t \ge 0$, $f_1(x,y) < 0$. Then, $x(t) = x(0)\cdot\exp\left(\int_0^{t} f_1(x(s),y(s))ds\right) < x(0)$. Therefore, $x(t) <$

$x(0)\;\forall\, t > 0$. Thus, in case II, we can say, $x(t) < x(0)\;\forall\, t > 0$ ...**(3)**
Using **(2)** and **(3)** we get, $x(t) \le Max\{x(0), k\} = M_1$ (say), $\forall\; t > 0 \Rightarrow\; x(t) \le M_1\;\forall\; t > 0$.
Therefore, $x(t)$ is bounded, i.e., $0 < x(t) \le M_1\;\forall\; t > 0$ where, $M_1 = \text{Max}\{x(0), k\}$.
Now, let us take, $\upsilon = x(t) + \dfrac{y(t)}{e}$.

$$\Rightarrow \frac{d\upsilon}{dt} = \frac{dx}{dt} + \frac{1}{e}\frac{dy}{dt}$$

$$\Rightarrow \frac{d\upsilon}{dt} = rx\left(1 - \frac{x}{k}\right)\left(\frac{x}{\alpha + x}\right) - \beta_1 x^3 + \frac{A}{e}y - \frac{\beta_2}{e}y^2$$

$$\le rx\left(1 - \frac{x}{k}\right)\left(\frac{x}{\alpha + x}\right) + x - \frac{\beta_2}{e}y^2 + \frac{2A}{e}y - \Omega\upsilon \quad \text{where} \quad \Omega = \text{Min}\;\{1, A\}$$

$$\Rightarrow \frac{d\upsilon}{dt} + \Omega\upsilon \le (r+1)x + \frac{A^2}{e\beta_2} - \frac{\beta_2}{e}\left(y - \frac{A}{\beta_2}\right)^2 \le (r+1)x + \frac{A^2}{e\beta_2} = g(x)$$

Now, $g'(x) = (r+1) > 0\;\forall x \in (0, M_1]$. Thus $g(x)$ gives its maximum value at $x = M_1$.
So, $g(M_1) = (r+1)M_1 + \dfrac{A^2}{e\beta_2} = B_1$ (say).
Therefore, $\dfrac{d\upsilon}{dt} + \Omega\upsilon \le B_1$
Now, using the theory of differential inequality for $\Omega(t)$ we have,

$$0 < \upsilon(t) \le \frac{B_1}{\Omega} + \left(\upsilon(0) - \frac{B_1}{\Omega}\right)\cdot \exp(-\Omega t).$$

Taking limit $t \to \infty$ we get, $0 < \upsilon(t) \le \dfrac{B_1}{\Omega} + \epsilon$, for any $\epsilon > 0$. Hence all solutions are entering in the region: $\left\{(x, y) \in \mathbb{R}^2_+ : 0 < x(t) \le M_1,\; 0 < \upsilon(t) \le \dfrac{B_1}{\Omega} + \epsilon\right\}$, where $M_1 = \text{Max}\{x(0), k\}$ and $B_1 = (r+1)M_1 + \dfrac{A^2}{e\beta_2}$.

□

2 Saddle-node Bifurcation Proof

Theorem 2.1. *System (2.2) undergoes saddle-node bifurcation with respect to bifurcation parameter β_2.*

Proof. Let x' be the double root of $\Phi(x) = 0$, where $\Phi(x) = a_0x^6 + a_1x^5 + a_2x^4 + a_3x^3 + a_4x^2 + a_5x + a_6$ at $\beta_2 = \beta_2^{(SN_1)}$. The coefficients $a_0, a_1, a_2, a_3, a_4, a_5$ and a_6 are provided in section 4.2. If the two nontrivial nullclines, $f_1(x, y) = 0$ and $f_2(x, y) = 0$, intersect tangentially at point $E_{I1}(x', y')$, then $\left.\dfrac{dy^{(f_1)}}{dx}\right|_{E_{I1}} = \left.\dfrac{dy^{(f_2)}}{dx}\right|_{E_{I1}}$.

The Jacobian matrix $J(E_{I1})$ at $\beta_2 = \beta_2^{(SN_1)}$ is given by

$$J(E_{I1}) = \begin{bmatrix} x\dfrac{\partial f_1}{\partial x} & x\dfrac{\partial f_1}{\partial y} \\ \\ y\dfrac{\partial f_2}{\partial x} & y\dfrac{\partial f_2}{\partial y} \end{bmatrix}_{E_{I1},\beta_2^{(SN_1)}}$$

$$= \begin{bmatrix} -\dfrac{rx^2}{k(\alpha+x)} + \dfrac{r\alpha x\left(1-\frac{x}{k}\right)}{(\alpha+x)^2} - 2\beta_1 x^2 - \dfrac{axy(b-x^2)}{(b+x^2)^2} & -\dfrac{ax^2}{(b+x^2)} \\ \\ \dfrac{2cabxy}{(b+x^2)^2} & -\beta_2 y \end{bmatrix}_{E_{I1},\beta_2^{(SN_1)}}$$

Now,

$$det(J(E_{I1})) = \left[xy\left(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}\right)\right]_{E_{I1},\beta_2^{(SN_1)}} = \left[xy\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial y}\left(\frac{dy^{(f_2)}}{dx} - \frac{dy^{(f_1)}}{dx}\right)\right]_{E_{I1},\beta_2^{(SN_1)}} = 0.$$

Let $J(E_{I1}) = B_2$ and $J^T(E_{I1}) = C_2$. Since $det(J(E_{I1})) = 0$, so 0 is an eigenvalue of both B_2 and C_2. Now the eigenvectors of B_2 and C_2 corresponding to this zero eigenvalue are given by $W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}_{E_{I1},\beta_2^{(SN_1)}}$ and $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}_{E_{I1},\beta_2^{(SN_1)}}$ respectively, where $w_1 = 1$, $w_2 = \dfrac{2cabx}{\beta_2(b+x^2)^2}$, $z_1 = -\dfrac{\beta_2 y(b+x^2)}{ax^2}$ and $z_2 = 1$.
Using the Sotomayor's theorem [1] at $\beta_2 = \beta_2^{(SN_1)}$, we have
$Z^T F_{\beta_2}(E_{I1}, \beta_2^{(SN_1)}) = -(y')^2 \neq 0$

$$Z^T\left[D^2F(E_{I1}, \beta_2^{(SN_1)})(W,W)\right] =$$
$$-\tfrac{2r\beta_2 A^2 y(B+x^2)}{bx^2(A+x)^3} + \tfrac{2r\beta_2 y(B+x^2)(x^2+3Ax+3A^2)}{bKx(A+x)^3} + \tfrac{6\beta_1\beta_2 y(B+x^2)}{bx} + \tfrac{2B\beta_2 y^2(B-3x^2)}{x^2(B+x^2)^2} + \tfrac{8cab^2y}{(b+x^2)^3} + \tfrac{2caby(b-3x^2)}{(b+x^2)^3}\Big|_{E_{I1},\beta_2^{(SN_1)}}$$
$\neq 0$,
provided, $c \neq \frac{(b+x^2)^3}{2aby(5b-3x^2)}\left[\frac{2r\beta_2\alpha^2 y(b+x^2)}{ax^2(\alpha+x)^3} - \frac{6\beta_1\beta_2 y(b+x^2)}{ax} - \frac{2r\beta_2 y(b+x^2)(x^2+3\alpha x+3\alpha^2)}{akx(\alpha+x)^3} + \frac{2b\beta_2 y^2(3x^2-a)}{x^2(a+x^2)^2}\right]_{E_{I1},\beta_2^{(SN_1)}}$
Here, $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, each of F_1 and F_2 is specified in equation (2.2). Hence, the necessary conditions for a saddle-node bifurcation are fulfilled and the system undergoes a saddle-node bifurcation at $\beta_2 = \beta_2^{(SN_1)}$. Similarly, it can be easily demonstrated that another saddle-node bifurcation occurs at $\beta_2 = \beta_2^{(SN_2)}$. □

3 Hopf Bifurcation Proof

Considering A as a varying parameter, we can express the characteristic equation of the system (2.2) for the Jacobian matrix $J(E_I(x_2, y_2))$ as

$$\lambda^2 - T(A)\lambda + D(A) = 0 \tag{3.1}$$

where $T(A)$, $D(A)$ are the trace and the determinant of the Jacobian matrix for the interior equilibrium point $E_I(x_2, y_2)$, respectively, which can be easily calculated from the Jacobian of the system 2.2 provided in the manuscript. Observing a change in the sign of the real component of λ from negative to positive leads to a change in stability for the interior equilibrium point $E_I(x_2, y_2)$, thus changing it from stable to unstable. The occurrence of this transition is the result of a Hopf bifurcation [2], which occurs when the characteristic equation (3.1) has a pair of roots that are entirely imaginary. Suppose that those purely imaginary roots occur at $A = A^{(H)}$, then $T(A^{(H)}) = 0$, $D(A^{(H)}) > 0$. In the following theorem, we demonstrate the occurrence of a Hopf bifurcation inside system (2.2) at $A = A^{(H)}$.

Theorem 3.1. *System (2.2) goes through a Hopf bifurcation around the interior equilibrium state $E_I(x_2, y_2)$ at $A = A^{(H)}$, provided $T(A^{(H)}) = 0$, $D(A^{(H)}) > 0$ and $\left[\frac{dT}{dA}\right]_{A=A^{(H)}} \neq 0$.*

Proof. At $AA = A^{(H)}$, $T(A^{(H)}) = 0$, $D(A^{(H)}) > 0$, the characteristic equation has purely imaginary roots, i.e. $\lambda_1 = i\sqrt{D(A^{(H)})}$ and $\lambda_2 = -i\sqrt{D(A^{(H)})}$. Therefore, $\lambda_1 = g_1(A) + ig_2(A)$ and $\lambda_2 = g_1(A) - ig_2(A)$ are the roots of the characteristic equation (3.1) in any open neighbourhood of $A^{(H)}$, where $g_1(A)$ and $g_2(A)$ are real valued functions of A. Using Hopf-Bifurcation Theorem [3], we may say that the stability of system (2.2) switches through a Hopf bifurcation, provided the following transversality condition

$$\left[\frac{d}{dA}(Re(\lambda_i(A)))\right]_{A=A^{(H)}} = \left[\frac{dg_1(A)}{dA}\right]_{A=A^{(H)}} \neq 0$$

is satisfied. Let us put $\lambda(A) = g_1(A) + ig_2(A)$ in equation (3.1), then we get,

$$(g_1(A) + ig_2(A))^2 - T(A)(g_1(A) + ig_2(A)) + D(A) = 0$$

Now, differentiating both sides w.r.t. A we get,

$$2(g_1(A) + ig_2(A))(\dot{g}_1(A) + i\dot{g}_2(A)) - T(A)(\dot{g}_1(A) + i\dot{g}_2(A)) - \dot{T}(A)(g_1(A) + ig_2(A)) + \dot{D}(A) = 0$$

Now comparing the real and imaginary components on both sides, we get;

$$(2g_1 - T)\dot{g}_1 - 2g_2\dot{g}_2 - \dot{T}g_1 + \dot{D} = 0 \Rightarrow X_1\dot{g}_1 - X_2\dot{g}_2 + X_3 = 0 \tag{3.2}$$

$$(2g_2)\dot{g}_1 + (2g_1 - T)\dot{g}_2 - \dot{T}g_2 = 0 \Rightarrow X_2\dot{g}_1 + X_1\dot{g}_2 + X_4 = 0 \tag{3.3}$$

where, $X_1 = (2g_1 - T)$, $X_2 = 2g_2$, $X_3 = (\dot{D} - \dot{T}g_1)$, $X_4 = -\dot{T}g_2$. Now solving equations (3.2) and (3.3) we get,

$$\dot{g}_1 = -\frac{(X_1X_3 + X_2X_4)}{X_1^2 + X_2^2} \tag{3.4}$$

At $m = m^{(H)}$. Now two cases arise :
Case I: When $g_1 = 0$, $g_2 = \sqrt{D}$ then, $X_1 = 0$, $X_2 = 2\sqrt{D}$, $X_3 = \dot{D}$ and $X_4 = -\dot{T}\sqrt{D}$. Hence from (3.4) we get,

$$[\dot{g}_1]_{A=A^{(H)}} = \left[\frac{dg_1(A)}{dA}\right]_{A=A^{(H)}} = \frac{1}{2}\left[\frac{dT(A)}{dA}\right]_{A=A^{(H)}} \neq 0$$

Case II: When $g_1 = 0$, $g_2 = -\sqrt{D}$ then, $X_1 = 0$, $X_2 = -2\sqrt{D}$, $X_3 = \dot{D}$ and $X_4 = \dot{T}\sqrt{D}$. Hence from (3.4) we get,

$$[\dot{g}_1]_{A=A^{(H)}} = \left[\frac{dg_1(A)}{dA}\right]_{A=A^{(H)}} = -\frac{1}{2}\left[\frac{dT(A)}{dA}\right]_{A=A^{(H)}} \neq 0$$

Hence, the theorem is proved. $\square$

References

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