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The Material Tools of Algorithmic Thinking

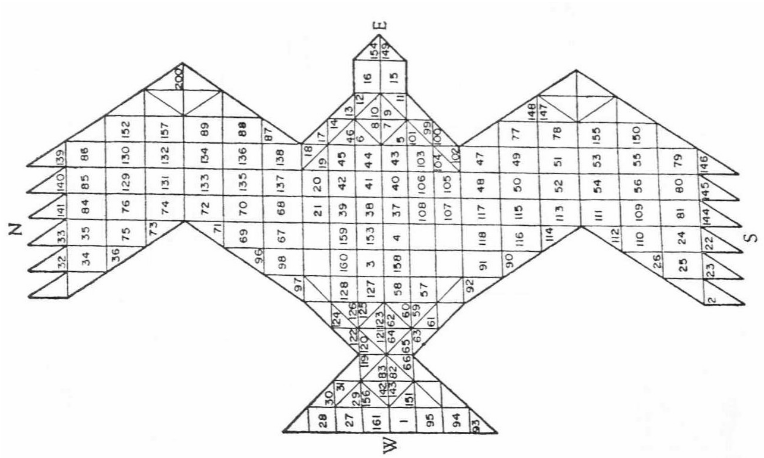

Figure 1.1. Diagram of the Agnicayana fire altar. Frits Staal, 'Greek and Vedic Geometry', *Journal of Indian Philosophy* 27, no. 1 (1999): 111 (image rotated).

The power of our 'mental' tools is amplified by the power of our 'metal' tools.

Jeannette Wing, 'Computational Thinking', 2008[1]

1 Jeannette M. Wing, 'Computational Thinking and Thinking about Computing', *Philosophical Transactions of the Royal Society A: Mathematical, Physical, and Engineering Sciences* 366, no. 1881 (2008): 3718.

> *When using a material tool, more can always be learned than the knowledge invested in its invention.*
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> Peter Damerow and Wolfgang Lefèvre, 'Tools of Science', 1981[2]

> *Rules became mechanical before they could actually be executed by machines.*
>
> Lorraine Daston, 'Algorithms before Computers', 2017[3]

Recomposing a dismembered god

In a myth of cosmogenesis of the Vedas, it is narrated that the supreme god Prajapati is shattered into pieces in the act of creating the universe. In the aftermath of creation, counter-intuitive to Western narratives of mastery and principles of non-contradiction, the creator's body is found unstrung, dismembered. This ancient myth is still re-enacted today, in India, in the Agnicayana ritual, in which Hindu devotees symbolically recompose the fragmented body of the god by building the fire altar Syenaciti (see fig. 1.1). The Syenaciti altar is laid down by aligning a thousand bricks of precise shape and size according to an elaborate geometric plan that draws the profile of a falcon. Workers compose five layers of 200 bricks each while reciting dedicated mantra and following step-by-step instructions. Solving a riddle that is the key to the ritual, each layer must maintain the same area and shape but a different configuration.[4] Finally, the falcon altar must face east, in prelude to a symbolic flight of the reconstructed god towards the rising sun – a unique example of divine reincarnation by geometric means.

Agnicayana is meticulously described in the appendices to the Vedas dedicated to geometry, the Shulba Sutras, which were composed around

2 Peter Damerow and Wolfgang Lefèvre, 'Tools of Science', in Peter Damerow, *Abstraction and Representation: Essays on the Cultural Evolution of Thinking*, Berlin: Springer, 2013, 401.

3 Lorraine Daston, 'Algorithms before Computers: Patterns, Recipes, and Rules', Katz Distinguished Lecture in the Humanities, Simpson Center for the Humanities, University of Washington, 19 April 2017. See also: Lorraine Daston, *Rules: A Short History of What We Live By,* Princeton, NJ: Princeton University Press, 2022.

4 K. Ramasubramanian, 'Glimpses of the History of Mathematics in India', in *Mathematics Education in India: Status and Outlook*, ed. R. Ramanujam and K. Subramaniam, Mumbai: Homi Bhaba Centre for Science Education (TIFR), 2012.

800 BCE in India, yet recording a much-older oral tradition.[5] They narrate that the *rishi* (vital spirits) created seven square-shaped *purusha* (cosmic beings) which together composed a single body, and it is from this simple configuration that the complex body of Prajapati evolved.[6] The Shulba Sutras teach the construction of other altars of specific geometric forms to secure the auspices of gods. They suggest, for instance, that 'those who wish to destroy existing and future enemies should construct a fire-altar in the form of a rhombus'.[7] Beyond religious symbolism, the Agnicayana ritual and the Shulba Sutras in general had, in fact, the function of transmitting useful techniques for the society of the time, such as how to plan a construction and to enlarge existing buildings while maintaining their original proportions.[8] Agnicayana exemplifies the originary social materiality of mathematical knowledge but also the hierarchies of manual and mental labour typical of a caste system. In the construction of the altar, the workers are driven by rules which are traditionally possessed and transmitted only by a specific group of masters. Aside from geometric exercises, rituals such as Agnicayana taught a kind of procedural knowledge which is not just abstract but based on continuous 'mechanical' drill, pointing once again to the role of religion as a motivation for exactness and, at the same time, to spiritual exercises as a way of disciplining labour.[9]

5 Agnicayana has been reported by the Dutch indologist Frits Staal in two volumes and a documentary about an expedition in Kerala, India, in 1975: Frits Staal, *Agni: The Vedic Ritual of the Fire Altar*, two vols., Berkeley: Asian Humanities Press, 1983. Staal argued that abstract cultural forms emerge as nonconscious, and that language, numerals, and geometry are first collective practices. See Frits Staal, *Rules without Meaning: Ritual, Mantras, and the Human Sciences*, New York: Peter Lang, 1989, 71.

6 Paolo Zellini, *La matematica degli dèi e gli algoritmi degli uomini*, Milano: Adelphi, 2016, 41. Translated as *The Mathematics of the Gods and the Algorithms of Men*, London: Penguin, 2020.

7 Kim Plofker, 'Mathematics in India', in *The Mathematics of Egypt, Mesopotamia, China, India, and Islam*, ed. Victor Katz, Princeton, NJ: Princeton University Press, 2007.

8 For a study of knowledge and technology transfer in antiquity, see Jürgen Renn (ed.), *The Globalization of Knowledge in History*, Princeton, NJ: Princeton University Press, 2020.

9 The division of labour of Agnicayana is also reminiscent of the elaborate *chaîne opératoire* (operational chain) that the French anthropologist André Leroi-Gourhan has identified in many ancestral practices of tool-making which originally are not hierarchical but spontaneous and cooperative. See Frederic Sellet, 'Chaîne opératoire: The Concept and its Applications', *Lithic Technology* 18, nos. 1–2 (1993): 106–12.

Agnicayana is a unique artefact in the history of human civilisation: it is the most ancient documented ritual of humankind that is still practised today – although, due to its complexity, it is performed only a few times in a century.[10] Across all this time, it has transmitted and preserved sophisticated paradigms of knowledge, and because of its combinatorial mechanism, it can be defined as a primordial example of algorithmic culture. But how can one possibly interpret a ritual as ancient as Agnicayana as *algorithmic*? One of the most common definitions of algorithm in computer science is the following: a finite procedure of step-by-step instructions to turn an input into an output, independently of the data, and making the best use of the given resources.[11] The recursive mantras which guide workers in the construction site of the fire altar may indeed resemble the rules of a computer program: independently of the context, the Agnicayana algorithm organises a precise distribution of bricks which results every time in the construction of the Syenaciti. Historians have found that Indian mathematics has been predominantly algorithmic since ancient times, meaning that the solution to a problem was proposed via a step-by-step procedure rather than a logical demonstration.[12]

Similarly, the Italian mathematician Paolo Zellini has argued that the Agnicayana ritual evidences a more sophisticated technique than simple obedience to a rigid rule, namely the *heuristic technique of incremental approximation*. It is known that Vedic mathematics, before other civilisations, was familiar with infinitely large and infinitesimally small numbers: ancient sutras already multiplied the positional numerals of the Hindu system to large scales to signify the vast dimensions of the universe (a speculative exercise that would be impracticable with the additive systems of Sumerian, Greek, and Roman numerals, for instance). Vedic mathematics was also familiar with irrational numbers, such as the square root, which in many cases (such as $\sqrt{2}$) can only be

10 The last were in 1955, 1975 (the ceremony documented by Frits Staal), and 2011.

11 See also: 'An algorithm is a finite sequence of rules to apply in a determined order to a finite set of data to arrive, in a finite number of steps, at a certain result, independently of the data'. Jean-Luc Chabert (ed.), *A History of Algorithms: From the Pebble to the Microchip*, Berlin: Springer, 1999, 2. My translation: the French original provides a more precise definition as the English edition removes 'independently of the data'. Jean-Luc Chabert, *Histoire d'algorithmes: Du caillou à la puce*, Paris: Belin, 1994, 6.

12 M. S. Sriram, 'Algorithms in Indian Mathematics', in *Contributions to the History of Indian Mathematics*, Gurgaon: Hindustan Book Agency, 2005, 153–82.

calculated by approximation. The mantras of the Shulba Sutras intone the most ancient (and pedantic) explications of computational procedures (like to the so-called Babylonian algorithm) to approximate square root results. Procedures of approximation may appear cumbersome, weak, and imprecise compared to the exactitude of our mathematical functions and geometric theorems, but their role within the history of mathematics and technology is more important than is commonly thought. In his history of the techniques of incremental growth (which include the ancient method of gnomon, among others), Zellini has argued that the ancient Hindu techniques of incremental approximation are equivalent to the modern algorithms of Leibniz and Newton's calculus, and even to the error-correction techniques that are found at the core of artificial neural networks and machine learning, which constitute the current paradigm of AI (see chapter 9).[13]

To some, it may appear an act of misappropriation to read ancient cultures through the paradigm of the latest technologies from Silicon Valley or to study the mathematical component of religious rituals in an age of rampant nationalism. However, to claim that abstract techniques of knowledge and artificial metalanguages belong uniquely to the modern industrial West is not only historically inaccurate but an act of implicit *epistemic colonialism* towards the cultures of other places and other times.[14] Thanks to the contribution of ethnomathematics, decolonial studies, and the history of science and technology, alternative forms of computation are now recognised and investigated outside the Global North hegemony and its regime of knowledge extractivism. Because of their role in computer programming, algorithms are usually perceived as the application of complex sets of rules in the abstract; on the contrary, I argue here that algorithms, even the complex ones of AI and machine

13 Zellini, *La matematica degli dèi*, 51. For a contested but influential history of calculus, see Hermann Cohen, *Das Prinzip der Infinitesimal-Methode und seine Geschichte: Ein Kapitel zur Grundlegung der Erkenntniskritik* (1883).

14 The historian of mathematics Senthil Babu remarks: 'The history of mathematics in India has thus far primarily been an engagement with a corpus of texts recorded in Sanskrit . . . Indology recognized and canonized only the dignified Sanskritic tradition. The knowledge of many practitioners of mathematics was rendered invisible.' Senthil Babu, *Mathematics and Society: Numbers and Measures in Early Modern South India*, Oxford: Oxford University Press, 2022, 2–5. See also Senthil Babu, 'Indigenous Traditions and the Colonial Encounter: A Historical Perspective on Mathematics Education in India', in Ramanujam and Subramaniam, *Mathematics Education in India*.

learning, have their genesis in social and material activities. Algorithmic thinking and algorithmic practices, broadly understood as rule-based problem-solving, have been part of all cultures and civilisations.

Along these lines of inquiry, this chapter sketches a provisional history of algorithms, broadly examining in turn (1) *social algorithms*, that is, procedures that were embodied in rituals and practices, often transmitted orally and not formalised into symbolic language; (2) *formal algorithms*, that is, mathematical procedures to help calculation and administrative operations as they are found, for instance, in Europe since the Middle Ages and before that in India; and (3) *automated algorithms*, that is, the implementation of formal algorithms in machines and electronic computers starting with the industrial age in the West.

Archaeology of the algorithm

The idea of investigating 'algorithms before computers' first came, unsurprisingly, from the field of computer science. In the late 1960s, the US mathematician Donald Knuth authored the influential book *The Art of Computer Programming* and gave important contributions to excavating the deep time of mathematical techniques in essays such as 'Ancient Babylonian Algorithms'. In those years, Knuth's mission was to systematise the field of computer science and to make it into a respectable academic discipline. The evidence of ancient algorithms was mobilised to stress that computer science was not about obscure electronic apparatuses but part of a long tradition of cultural techniques of symbolic manipulation. In this case, however, the archaeology of the algorithm was pursued not to demonstrate universalistic principles of thinking or the emancipatory potential of learning across the history of civilisation, but for the specific interests of the new classes of computer programmers and manufacturers:

> One of the ways to help make computer science respectable is to show that it is deeply rooted in history, not just a short-lived phenomenon. Therefore it is natural to turn to the earliest surviving documents which deal with computation, and to study how people approached the subject nearly 4000 years ago. Archeological expeditions in the Middle East have unearthed a large number of clay tablets which contain

> mathematical calculations, and we shall see that these tablets give many interesting clues about the life of early 'computer scientists'.[15]

Knuth observed that mathematical formulas that today would be defined as algebraic or analytical were already described by the Babylonians through step-by-step procedures, namely algorithms. These procedures were, of course, formulated in the words of the common language and not yet in the symbolic metalanguage of mathematics. Knuth's research confirms the hypothesis that procedure-based methods (what he called a 'machine language') predated the consolidation of mathematics as a metalanguage of symbolic representations:

> The Babylonian mathematicians were not limited simply to the processes of addition, subtraction, multiplication, and division; they were adept at solving many types of algebraic equations. But they did not have an algebraic notation that is quite as transparent as ours; they represented each formula by a step-by-step list of rules for its evaluation, i.e. by an algorithm for computing that formula. In effect, they worked with a 'machine language' representation of formulas instead of a symbolic language.[16]

Knuth intended to liberate the algorithm from the age of computer science and engineering in order to make it, retroactively, a broad subject for the history of culture. This happened in the 1960s, when computer science was still struggling, as the historian Nathan Ensmenger has highlighted, to achieve the status of proper discipline in the United States. This qualification became possible by establishing as its central concept the algorithm, rather than information as happened in Europe (see the German *Informatik*, the French *informatique*, and the Italian *informatica* as names for computer science).[17] This canonization of the algorithm is particularly significant for the historians of science and technology because it proceeded from within its original professional milieu: the operators of computing machines, a new

15 Donald E. Knuth, 'Ancient Babylonian Algorithms', *Communications of the ACM* 15, no. 7 (1972): 671.

16 Ibid., 672.

17 Nathan Ensmenger, *The Computer Boys Take Over: Computers, Programmers, and the Politics of Technical Expertise*, Cambridge, MA: MIT Press, 2010, 131.

generation of mental workers, were up to write their own history of technology – and obviously they did it according to the logical form their work embodied.

The reconstruction of the prehistory of the algorithm (one may say its 'archaeology') has also been a resurgent concern in mathematics. Notably, the French mathematician Jean-Luc Chabert has contributed an exemplary synthesis that also ventures beyond the disciplinary borders of computer science:

> Algorithms have been around since the beginning of time and existed well before a special word had been coined to describe them. Algorithms are simply a set of step by step instructions, to be carried out quite mechanically, so as to achieve some desired result . . . Algorithms are not confined to mathematics . . . The Babylonians used them for deciding points of law, Latin teachers used them to get the grammar right, and they have been used in all cultures for predicting the future, for deciding medical treatment, or for preparing food . . . We therefore speak of recipes, rules, techniques, processes, procedures, methods, etc., using the same word to apply to different situations. The Chinese, for example, use the word *shu* (meaning rule, process or stratagem) both for mathematics and in martial arts . . . In the end, the term algorithm has come to mean any process of systematic calculation, that is a process that could be carried out automatically. Today, principally because of the influence of computing, the idea of finiteness has entered into the meaning of algorithm as an essential element, distinguishing it from vaguer notions such as process, method or technique.[18]

Also in this reading, the algorithm does not appear to be the most recent technological abstraction but a very ancient technique – one that predates many tools and machines that the human mind has designed. These efforts of historicisation invite a reconsideration of the algorithm, then, as a fundamental *cultural technique* of humankind, which gradually emerged from collective practices and rituals temporally very close to the constituent and primordial traits of all civilisations. The algorithm should be added, in summary, to the list of techniques that the historian of culture Thomas Macho compiled in an often-quoted passage:

18 Chabert, *A History of Algorithms*, 1.

> Cultural techniques – such as writing, reading, painting, counting, making music – are always older than the concepts that are generated from them. People wrote long before they conceptualized writing or alphabets; millennia passed before pictures and statues gave rise to the concept of the image; and until today, people sing or make music without knowing anything about tones or musical notation systems. Counting, too, is older than the notion of numbers. To be sure, most cultures counted or performed certain mathematical operations; but they did not necessarily derive from this a concept of number.[19]

The research on cultural techniques (in German, *Kulturtechniken*, which can also be translated as 'techniques of civilisation') has stressed the role of material practices in the making of all the symbolic forms of civilisations. This open-minded and pluralistic view, however, often neglects to study the causes of this evolution towards abstraction, resulting in a culturalist interpretation of what are more profound phenomena. Chabert, in his history of algorithms, for example, relates the rise of techniques of calculation to economic needs: 'The basic arithmetic operations of the elementary school, multiplying and dividing, appear to have derived from extremely early economic needs, certainly earlier than the emergence of civilisation using writing.'[20] Although it is always difficult to generalise historical findings about the remote past, economic problems – such as conditions of lack or surplus of resources – appear to be at the origins of counting and mathematical techniques.[21] It is worth remembering, with no intention of reviving ancestral famines, that the word 'number' comes from the Latin *numerus*, or 'portion of food'.

Well before the institution of mathematical and geometric disciplines, ancient civilisations were already large 'machines' of social segmentation, marking human bodies and territories with abstractions that would remain operative for millennia. It is known and repeated that one of the first recorded censuses of the population, organised by the

19 Thomas Macho, 'Zeit und Zahl: Kalender- und Zeitrechnung als Kulturtechniken', in *Bild – Schrift – Zahl*, ed. Sybille Krämer and Horst Bredekamp, Munich: Wilhelm Fink, 179. Quoted in translation in Geoffrey Winthrop-Young, 'Cultural Techniques: Preliminary Remarks', *Theory, Culture, and Society* 30, no. 6 (2013): 8.

20 Chabert, *A History of Algorithms*, 7.

21 For an alternative analysis of primitive economies, see Marshall Sahlins, *Stone Age Economics*, Chicago: Aldine-Atherton, 1972.

Babylonians, took place around 3800 BCE, but history records that these 'cultural techniques' were also inhuman and ruthless. Drawing on historian Lewis Mumford and his account of ancient societies as 'megamachines', Gilles Deleuze and Félix Guattari enumerated other techniques of abstraction than number on which social order was based. They argue that in ancient civilisation, the power to control 'productive forces . . . resides in these operations: tattooing, excising, incising, carving, scarifying, mutilating, encircling, and initiating'.[22] Numbers and counting tools were components of these *primitive abstract machines* that forged human civilisations through territorialisation and segmentation. Numbers, as much as abstract rules and heuristic practices, were key tools in the administration of ancient societies, but they were not invented from nothing: they materially emerged as a form of power through labour and rituals, through discipline and drill.

This intrinsic relation between mathematical abstractions and material life was not overlooked even by a neo-Kantian philosopher like Ernst Cassirer, who has exercised quite an influence on cultural studies in German-speaking countries. According to Cassirer, the 'symbolic form' of number emerged from the relation of the human body with its environment and the contingent use of the body as the first medium of calculation: 'It is through material enumerable things, however sensuous, concrete and limited its first representation of these things may be, that language develops the new form and the new logical force that are contained in number.'[23] Analysing the perception of space and time, Cassirer traced the origin of numerical abstractions to the rhythmical activities of work. Following Karl Bücher's seminal book *Arbeit und Rhythmus* (1896) and other anthropological studies, Cassirer remarked that the symbolic form of number grew out of the custom of work songs – that is, singing to sustain the rhythm of work:

> Attempts have been made to trace the beginnings of poetry back to those first primitive *work songs* in which for the first time the rhythm felt by man in his own physical movements was, as it were,

22 Gilles Deleuze and Felix Guattari, *Anti-Oedipus: Capitalism and Schizophrenia*, New York: Viking, 1977, 145.

23 Ernst Cassirer, *The Philosophy of Symbolic Forms*, vol. 1, New Haven, CT: Yale University Press, 1965, 228. Based on Max Wertheimer, *Über das Denken der Naturvölker*, Leipzig: Barth, 1910.

> objectified . . . Every form of physical labor, particularly when performed by a group, occasions a specific coordination of movements, which leads in turn to a rhythmic organization and punctuation of work phases . . . Grinding and rubbing, pushing and pulling, pressing and trampling: each is distinguished by a rhythm and tone quality of its own. In all the vast variety of work songs, in the songs of spinners and weavers, threshers and oarsmen, millers and bakers, etc., we can still hear with a certain immediacy how a specific rhythmic sense, determined by the character of the task, can only subsist and enter into the work if it is at the same time objectified in sound . . . In any case, language could acquire consciousness of the pure forms of time and number only through association with certain contents, certain fundamental rhythmic experiences, in which the two forms seem to be given in immediate concretion and fusion.[24]

This study can be taken as a rejoinder to the Platonic numerology that is central to the history of music: before numbers were used to measure the proportions of rhythm, the rhythm of work contributed to the invention of numbers. At the end, these findings cast a different light on the history of mathematics, so much that one could suspect, at this point, that algorithmic practices are even older than the concept of number itself.

Tools for the construction of mathematical ideas

Numbers are often considered as something given, originary and elemental, not composed of anything else and not resulting from any prior conceptual fabrication. Numbers appears to be self-explanatory, eternal, and not constructed. Such a Platonic and intuitionist view of the concept of number has been criticised by historians of mathematics, who are particularly concerned with explaining how techniques of numeration arose and evolved. Archaeologists, especially, are inclined to suggest that the institution of number cannot be an a priori category, as human activity with materials and symbolic tools testifies to its gradual evolution: counting appeared to have emerged, as already mentioned,

24 Ibid., 240.

from the need to calculate and solve practical problems, such as the equal distribution of land and natural resources in the population.

Among the *archaeologists of abstraction* we encounter the German historian of science Peter Damerow, who extensively studied, among other artefacts, ancient Babylonian clay tablets that were used as counting tools. Damerow came to the conclusion that the idea of number is not a form of a priori knowledge but 'subject to historical development'.

> Reflections on numbers and their properties led already in antiquity to the belief that propositions concerning numbers have a special status, since their truth is dependent neither on empirical experience nor on historical circumstances. In a historical tradition extending from the Pythagorean through the Platonic tradition of Antiquity, Late Antiquity and the Middle Ages, further through the rationalism and the critical idealism of Kantian and neo-Kantian philosophy to the logical positivism and constructivism of the present, this belief has been considered proof that there are objects of which we can gain knowledge *a priori*. Like a recurring leitmotif, the conviction that numbers are by nature ahistorical and universal is woven through the history of philosophy. A variety of reasons have been proposed to explain this puzzling phenomenon. The historian, on the other hand, is confronted with the fact that numerical techniques and arithmetic insights have a history that is, at least on its surface, in no way different from other achievements of our culture. In view of the variety of historically documented arithmetical techniques, it is scarcely possible to dismiss the assumption that the concept of number – in the same way as most structures of human cognition – is subject to historical development, which in the course of history exposes it to substantial change.[25]

Engaging with the findings of archaeology, moreover, Damerow realised that 'the emergence of numbers appears as the result of manifold

25 Peter Damerow, 'The Material Culture of Calculation: A Theoretical Framework for a Historical Epistemology of the Concept of Number', in *Mathematisation and Demathematisation: Social, Philosophical, and Educational Ramifications*, ed. Uwe Gellert and Eva Jablonka, Leiden: Brill, 2008, 19.

learning processes'.[26] Learning became a central notion in Damerow's research, through which he explained the making of human civilisation and its evolution. For Damerow, learning is a process of interaction of humankind with nature and the world, mediated by labour, tools, and language in a continuous process of abstraction. Learning, however, is not a process of abstraction for the sake of abstraction but a collective means of emancipation and empowerment. How does this social process of learning take place?

Damerow argued that learning is based on the construction of 'mental models' that fundamentally represent and internalise external actions.[27] On top of these internalised mental models, further levels of abstraction can be built in a progressive scaffolding of 'meta-cognitive constructs'.[28] This continuous scaffolding of abstractions is a form of the emancipation of reason, but it happens that some levels are eventually perceived as metaphysical and separated from others. According to Damerow, the higher levels of the cognitive scaffolding create the illusion of dematerialised abstractions and a priori categories such as the concept of number. However, what is decisive in this theory is not simply the explanation of the a priori illusion but rather how 'mental operations . . . reflect actions on real objects' and, vice versa, how tools help constructing mental models:

> Logico-mathematical concepts are abstracted not directly from the objects of cognition, but from the coordination of the actions that they are applied to and by which they are somehow transformed. According to this assumption the emergence of mental operations of logico-mathematical thought is based on the internalisation of systems of real actions. The internalised actions are the starting-point for meta-cognitive constructions, through which they become elements of systems of reversible mental transformations which, following Piaget's terminology, we will call here 'operations'. Meta-cognitive constructs such as the concept of number that are generated by reflective abstractions can thus be understood as internally represented invariables of mental operations which reflect actions on real

26 Ibid., 20.
27 Ibid.
28 Ibid., 22.

> objects. This explains the puzzling *a priori* nature of constructions such as the number concept.[29]

To explain the formation of the concept of number throughout history, Damerow suggested a scaffolding of semiotic and cognitive models that progressively unfolded from *practices of counting* (which are heuristic and non-formalised, such as reckoning with fingers), to *systems of numeration* (which represent quantities in a matrix of symbols), to *techniques of computation* (which express algorithms or procedure to solve problems by manipulating symbols), and eventually to *number theory* (namely arithmetic as a formal discipline). This process is not linear but unfolds, according to Damerow, through an alternate movement of *representation* (the use of objects and signs as symbols of other objects, signs, and ideas) and *abstraction* (problem-solving).

Applying the idea of *reflective abstraction* that combines both Hegel's dialectical logic and Jean Piaget's genetic epistemology, Damerow sketched progressive stages of symbolic representation (see fig 1.2), in which the passage from one order of representation to the following occurs via the solution of a problem. According to Damerow,

> *First order representations* are representations of real objects by symbols or models which permit the performance of essentially the same actions or operations with these symbols as can be performed with the real objects themselves . . . *Second and higher order representations* are representations of mental objects by symbols and symbol transformation rules which correspond to mental operations belonging to the cognitive structures constituting the mental objects.[30]

The concept of number developed, then, through cycles of symbolic representation and abstraction. First, processes of quantification and comparison that were based on equivalence without involving counting. Counting then emerged as a *context-dependent activity* that utilised aids such as fingers, stones, and so forth. Thereafter, these counting devices

29 Ibid.

30 Peter Damerow, 'Abstraction and Representation', in *Abstraction and Representation: Essays on the Cultural Evolution of Thinking*, Berlin: Springer, 2013, 373.

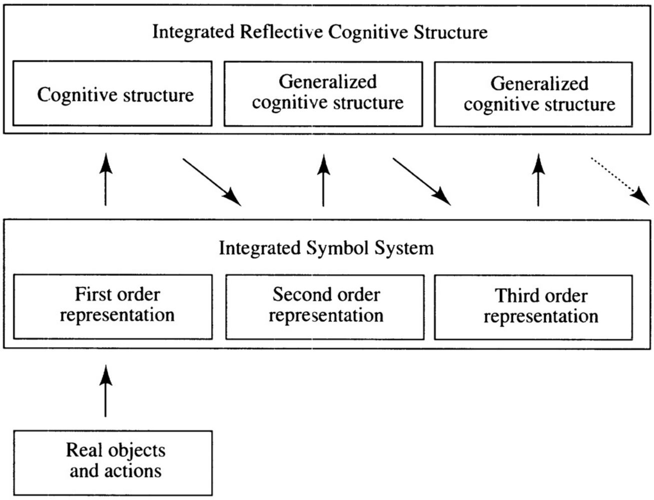

Figure 1.2. The reflective structure of abstraction. Peter Damerow, *Abstraction and Representation* Berlin: Springer, 2013, 379.

were replaced by *context-dependent symbols* (such as the signs on the bullae for trading in ancient Mesopotamia). Subsequently, *context-free symbols* were introduced, namely numbers in the modern sense. Finally, *arithmetic* emerged as a discipline to describe numbers and operations with natural language words, eventually to be replaced by new symbols themselves.[31]

To understand if such analysis can also be applied to the algorithmic form as a practice of problem-solving, it is necessary, at this point, to clarify Damerow's idea of abstraction. Following Hegel and Piaget, Damerow understood abstraction as a process in which materiality and reflection, that is tools and cognition, are mutually imbricated and mutually evolve:

> The concept [of reflection] was introduced [by Hegel] in the *Jenaer Realphilosophie* to distinguish labor as 'reflective activity' from activity as 'pure mediation', as the mere satisfaction of a desire by means of the destruction of its object. What distinguishes labor as 'reflective activity' from activity as 'pure mediation' is the endurance of its material means, of its tools, in which activity as the unity of ideal purpose

31 Damerow, 'Material Culture', 34–47.

> and material object has materially objectified itself . . . The unity of the sensory given and mediating activity constructed in Hegel's logic, the mediated immediacy as the result of reflection, not only constitutes a hypothetico-theoretical construct, but this unity is actually created in the material means of the objective activity in a myriad of forms.[32]

For Damerow abstraction is not about isolating the most prominent features of a given structure but about producing new knowledge in relation to a problem to solve: abstraction is not just an 'elegant solution' to a problem but 'an activity directed towards some end or goal', which includes the contingent understanding of the environment:[33]

> It is a common view that abstraction means refraining from using the information available on a given real object, and instead isolating certain properties and dealing with these independently. But this concept of abstraction reveals itself as unsatisfactory if it is used to conceptualize the development of mathematical thinking. Abstraction in this sense does not explain that outcome of new knowledge which obviously does result from mathematical thinking. Furthermore, this concept of abstraction makes it impossible or at least difficult to understand why certain abstractions turn out to be very useful but the huge mass which might be produced by arbitrarily isolating properties of mathematical objects would only result in nonsense . . . To understand abstraction essentially means understanding *what* has to be abstracted rather than merely knowing *how* it has to take place. To understand the abstraction leading to an elegant solution of the problem means understanding how the solution can really be found.[34]

Abstraction always operates within given material constraints and through them: symbols, tools, techniques, and technologies are conceived and realised in relation to limited resources of matter, energy, space, time, and so on. The reality which abstraction is struggling with

32 Damerow, 'Action and Cognition in Piaget's Genetic Epistemology and in Hegel's Logic', in Damerow, *Abstraction and Representation*, 8.

33 Damerow, *Abstraction and Representation*, 372.

34 Ibid., 371.

is not the idealised space of Platonic ideas but the actual living world, made of force fields and conflicts. In this sense, abstraction is also part of the larger social antagonism.

Importantly, material constraints give an impetus to expand the reach of abstraction beyond its original field. Together with his colleague Wolfgang Lefèvre, Damerow extended the historical epistemology of mathematics to the relation of science in general with tools and instruments. Their understanding of tools is, at one and the same time, contingent and speculative – in short, dialectical. Tools are not just means to an end but means that exceed the purpose of their initial design:

> Tools determine whether goals anticipated mentally can be realised. In that sense tools are never just what they actually are. Rather, they represent the potential of realizing intellectually anticipated goals, which is to say, they represent ideas as real possibilities. Their application mediates between possibility and reality. The use of tools primarily serves the purpose for which they were produced. But tools are more general than particular purposes, and the accumulated experience acquired in the course of their use leads to knowledge about possibilities capable of being realized and about relationships between goals and means under various conditions of realization. Thus, the primary form in which knowledge about natural and social relationships arising from the labor process is represented is the form of rules for the appropriate use of tools.[35]

In this understanding, the speculative process starts with labour that invents tools and technologies which, subsequently, project new ontological dimensions and scientific fields (a canonical example is the invention of the steam engine that engendered the discipline of thermodynamics, rather than the other way around; see chapter 3). Damerow and Lefèvre advance a political epistemology that acknowledges the constraints of historical forces, namely the control of resources and population, economic production and capital accumulation, the rise of wars and social conflicts, and, because of all of this, the development of new tools, techniques, technologies, and eventually science. They acknowledge all these forces within the category of labour, through

35 Damerow and Lefèvre, 'Tools of Science', 395.

which humans transform nature and produce new knowledge about it.[36] Science in general, as much as the concept of number in particular, is a projection of the use of material tools:

> The development of science depends on the development of its material tools . . . The key of understanding the growth of scientific knowledge consists in the fact that the knowledge to be gained by using a new tool exceeds the cognitive preconditions of its invention. The reason for this is due to the fact that the tools of science like tools in general are material tools: When using a material tool, more can always be learned than the knowledge invested in its invention.[37]

Along this historical overview of material abstractions, one can easily imagine also the concept of algorithm emerging as the result of a dialectical process of reflection with objects and tools. The method of the algorithm – the resolution of a problem by step-by-step instructions – is an abstraction that like many others emerged from the troubles of this world.[38]

From counting tools to algorithms for calculation

The English term 'algorithm' is circa eight centuries old. It derives from the medieval Latin term *algorismus*, which referred to the procedures for executing the basic mathematical operations with the Hindu–Arabic numerals. In the Europe of the Middle Ages, thanks to the trading routes with the Arab world, the limited system of Roman additive numerals came gradually to be replaced by the more versatile Hindu–Arabic positional system, which was more practical for complex operations on large numbers and has since become the planetary standard. The Latin term

36 '[The] basic structures of logico-mathematical thought are . . . developed by the individual growing up in confrontation with culture-specific challenges and constraints under which the systems of action have to be internalised.' Damerow, 'Material Culture', 22.

37 Damerow and Lefèvre, 'Tools of Science', 400–1.

38 For a discourse analysis of the algorithm concept that does not consider the economic matrix, see Yu Mingyi, 'The Algorithm Concept, 1684–1958', *Critical Inquiry* 47, no. 3 (2021): 592–609.

algorismus is found, for instance, in the 1240 poem 'Carmen de Algorismo' by Alexandre de Villedieu – a manual of calculation techniques that was composed in rhyming verse as an aid to memorise such procedures. A book printed in Venice in 1501 and attributed to the thirteenth-century monk Johannes de Sacrobosco bears the title *Algorismus Domini* and explicates hand calculation using Hindu numerals also with diagrams.

Only recently has it been established that *algorismus* is a Latinisation of the name of the Persian scholar Muhammad ibn Musa al-Khwarizmi, head librarian at the House of Wisdom in Baghdad, who authored a book on calculation with Hindu numerals around 825 CE. Al-Khwarizmi's original in Arabic has been lost, but in the twelfth century at least four Latin translations circulated under different titles: a manuscript at Cambridge University Library bears the incipit 'DIXIT algorizmi' (meaning: 'so spoke Al-Khwarizmi'), while another was given in 1857 the title *Algoritmi de numero Indorum* by Italian mathematician Baldassarre Boncompagni.[39] It is through various transliterations in Romance languages, such as French and Spanish, that the English term 'algorithm' has reached contemporary mathematics and computer science. Al-Khwarizmi's book helped to introduce the positional Hindu numerals to the West, yet merchants, such as the Italian mathematician Fibonacci, who travelled frequently across the Mediterranean, probably learned the system more through commercial exchanges and practice than through books.

In terms of mathematical conventions, the adoption of the term 'algorithm' in the West marked the shift from the additive to the positional system of numeration. This shift was both technical and economic, as it was related to the acceleration of commercial exchanges across Europe and the Mediterranean that demanded a better system of accounting. The decimal positional system made it possible to write numbers more concisely and sped-up calculations. In Italy, Florentine and Venetian merchants were the first to adopt the Hindu numerals, favoured for their greater versatility in commercial transactions and in handling capital's increasingly large figures. A drawing from the 1503

39 John N. Crossley and Alan S. Henry, 'Thus Spake al-Khwârizmî: A Translation of the Text of Cambridge University Library Ms. Ii. vi. 5', *Historia Mathematica* 17, no. 2 (1990): 103–31.

Figure 1.3. Allegory of Arithmetic. Gregor Reisch, *Margarita Philosophica*, 1503.

book *Margarita philosophica*, edited by the German monk and polymath Gregor Reisch, shows the dispute between abacists (who were still using the Roman system and the abacus) and the new algorists (who adopted the Hindu system and its algorithms to make calculations on paper with a stylus). The allegory of *Arithmetica* supervises the dispute, clearly deciding in favour of the algorist, her cloth covered with the new numerals (see fig. 1.3). Later, the term 'algorithm' was adopted by the scholars of European high culture, such as Leibniz, who used it to define his method of differential calculus.[40] 'Algorithm' was broadly defined by D'Alembert's *Encyclopédie* as an

> Arab term, used by several authors, and particularly by the Spanish to mean the practice of algebra. It is also sometimes taken to mean arithmetic by digits . . . The same word is taken to mean, in general, the method and notation of all types of calculation. In this sense, we say the algorithm of the integral calculus, the algorithm of the exponential calculus, the algorithm of sines, etc.[41]

The techniques and tricks for doing calculations by hand that are still taught in school to this day are a set of algorithms for the manipulation of numerical signs. They possess a recursive structure that can handle infinite and approximate digits, as occurs in the simple division of the prime numbers: 2/3 = 0.666666666 . . . The simple continuous form of this fraction shows that even rational numbers cannot be calculated and expressed without the help of an algorithm. More precisely, even the way in which numbers are written in a system of numeration constitutes an algorithm – in this case, an algorithm to represent simple quantities. For instance, when we write the number 101 in Hindu numerals, this simple sign should be translated as:

> Consider a linear sequence of positions to be occupied by symbols of quantity running from right to left. Each position represents incrementally a power of ten and can be filled by one of the ten units: 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9. The first position represents ten to the power of zero (that is, normal units), the second position ten to the power of one (ten), the third position ten to the power of two (hundred), and so on. The value of a number represented in this way is given by the addition of each unit after being multiplied to the power of ten, represented by the occupied position. When numbers are expressed in this way, the scale of the powers of ten is not explicitly stated but remains implicit.

40 See also Sybille Krämer, 'Zur Begründung des Infinitesimalkalküls durch Leibniz', *Philosophia Naturalis* 28, no. 2 (1991): 117–46; Peter Damerow and Wolfgang Lefèvre, 'Wissenssysteme im geschichtlichen Wandel', in *Enzyklopädie der Psychologie. Themenbereich C: Theorie und Forschung*, ed. F. Klix and H. Spada, Serie II: Kognition, Band 6: Wissen, Göttingen: Hogrefe, 1998, 77–113.

41 Quoted in Chabert, *A History of Algorithms*, 2.

The number 101, therefore, is equal to: (1 × hundred) + (0 × ten) + (1 × one). This verbose explanation of the decimal system in natural language can be easily adapted to represent the binary system by simply switching the power of ten to the one of two – that is, by changing one rule in the general procedure of numeration. In the binary system, the number 101 comes to signify a different quantity:

> Consider a linear sequence of positions to be occupied by symbols of quantity running from right to left. Each position represents incrementally a power of two and can be filled by one of the two units: 0 or 1. The first position represents two to the power of zero (that is, normal units), the second position two to the power of one (two), the third position two to the power of two (four), and so on. The value of a number represented in this way is given by the addition of each unit after being multiplied to the power of two, represented by the occupied position. When numbers are expressed in this way, the scale of the powers of two is not explicitly stated but remains implicit.

In this case, the number 101 is equal to: (1 × four) + (0 × two) + (1 × one), that is, 5 in decimal notation. In both of these verbose paraphrases, the words of natural language are not used to explain but to *encode* rules for the construction of numbers with a procedure of step-by-step instructions. These paraphrases make visible the procedure of the systems of numeration which are taught at school mostly through exercise and usually remain unexpressed. Such pedantic rehearsal of decimal and binary numerals, however, is helpful to say something non-pedantic: all systems of numeration appear to be algorithmic by constitution. As any word implies a grammar, any number hides an algorithm – that is, a procedure for representing quantities and for performing operations with quantities. In conclusion, all numbers are *algorithmic numbers* as they are manufactured by those algorithms that are the systems of numerations. Numerals count nothing (so to speak); they are simply position holders in a procedure – an algorithm – of quantification.

The mechanisation of the algorithm

Algorithms for hand calculation were mechanised gradually. In seventeenth-century Europe, natural philosophers such as Pascal and Leibniz designed hand calculators to automate the four basic operations with the decimal system. These devices were not at all cabinet curiosities but signalled more profound epistemic changes. At the time, modern thought had already developed in close relation to machines, to the point that mechanical thinking can be recorded having an influence on philosophical thinking too. Descartes's famous 'Method' of reasoning, for instance, looked quite 'mechanical' in its emphasis upon the decomposition of a problem into simpler elements. According to the Polish economist Henryk Grossmann, it was not by accident that Descartes conceived his rational method while designing tooling machines himself. But Grossmann noted also a more profound relation between mathematics and machines: 'every mathematical rule has [a] mechanical character that spares intellectual work and much calculation.' This economic principle – to save time, work, and resources – remains a key aspect of algorithmic thinking and practices as they have been illustrated so far.[42]

As the following chapter will show, in the context of the industrial economy of the early nineteenth century, the first computing algorithm to be mechanised was Gaspard de Prony's method of difference to calculate large logarithmic tables, which Charles Babbage implemented in the Difference Engine. The Difference Engine was designed to embody only this type of algorithm, but Babbage's envisioned also a programmable machine – the Analytical Engine – which could express different kinds of equations (although it was never realised). The first computer algorithm or 'program' is considered to be Ada Lovelace's 'diagram for the computation of Bernoulli numbers' that was tentatively written for the Analytical Engine. Babbage's calculating engines represent the point of convergence of calculation algorithms and industrial automation, although they severely struggled among other difficulties, to represent the decimal system in mechanical gears.[43]

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42 Henryk Grossmann, 'Descartes and the Social Origins of the Mechanistic Concept of the World', in *The Social and Economic Roots of the Scientific Revolution: Texts by Boris Hessen and Henryk Grossmann*, Berlin: Springer, 2009, 181.

43 See chapter 2.

In the twentieth century, algorithms for calculation were successfully automated thanks to the flexibility of the binary system.[44] Binary numerals are much easier to implement in an electric device than decimals into a mechanism, because an electric current's status that is on or off can directly represent the digits 0 and 1. In this way, the execution of addition and subtraction, for instance, is extremely simplified. Technically, binary operations started to be adopted and encoded in electric machines following the 1938 publication of US mathematician Claude Shannon's master's thesis 'A Symbolic Analysis of Relay and Switching Circuits'.[45] Shannon proposed for the first time to use the binary properties of electrical switches to represent not simply binary numbers and their operations, but propositional logic and, specifically, the Boolean logical operators AND, OR, and NOT.

After World War II, the binary code, the von Neumann architecture, and the engineering of efficient logic gates in microchips made possible the construction of fast computers and the formalisation of computer algorithms of larger size and higher complexity. For the first time in history, sequences of numerals came to represent not just quantities but instructions.[46] The so-called 'computer revolution' was not just about the use of *binary numerals* (binary digits, or bits) to encode human language and analogue content (digitisation) but about accelerating mechanical computation through *binary logic* (or Boolean logic). Contrary to the common view that stresses only the separation of hardware and software, digital computing is actually the imbrication, in the same medium of information *and* instruction, of binary numerals *and* Boolean logic – one as a complementary form of the other. In other words, with digital computing, the algorithm of numeration (binary numerals) and the algorithm of calculation (binary logic) have almost become one and the same thing.

In the digital age, the algorithm has risen to the role of an *abstract machine* (under the different denominations of program, software, code,

44 Within the Western tradition, Leibniz, inspired by the Chinese I Ching, already suggested binary numeration in his 1689 text *Explication de l'Arithmétique Binaire*.

45 Claude Shannon, 'A Symbolic Analysis of Relay and Switching Circuits', *Transactions of the American Institute of Electrical Engineers* 57, no. 12 (1938): 713–23.

46 Gödel introduced the idea of using numbers to represent mathematical functions (*Gödel numbering*) in his famous 1931 incompleteness theorems. Kurt Gödel, 'Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I', *Monatshefte für Mathematik und Physik* 38 (1931).

and so forth), which is used to control electronic computing machines. As mentioned at the beginning of this chapter, the definition of 'algorithm' which is the most familiar in contemporary times is the one of computer science: 'a finite procedure of step-by-step instructions to turn an input into an output, independently of the data, and making the best use of the given resources'.[47] The abstraction of logic from content is one of the key aspects of technical and cognitive development: as with other techniques of abstraction, an algorithm has to operate independently of environmental constraints and the origin of data. This chapter has questioned, however, this reading of abstraction as separation from the world and its historical developments. In fact, the advent of machine learning has turned this static definition of algorithm upside down: machine learning algorithms have became adaptive, and from rigid sets of rules now they 'learn' rules from data.

The canonical definition describes the algorithm as the application of rigid rules, top down, on some input data. Data do not affect the behaviour of the algorithm: they are simply *passive information* to be processed by rules. On the contrary, machine learning algorithms change their internal rules (called parameters) according to the input data. As such, data are no longer passive, so to speak, but become *active information* that influences the parameters of the step-by-step procedure which is, then, no longer strictly predetermined by the algorithm. The breakthrough of machine learning is exactly about this shift: algorithms for data analytics become dynamic and change their rigid inferential structure to adapt to further properties of data – usually logical and spatial relations. The canonical example is an artificial neural network for pattern recognition that changes the parameters of its nodes according to the relations among the elements of the visual matrix. In this respect, the structure of the most recent AI algorithms is not different and distant from ancient mathematical practices that emerged by the continuous imitation of configurations of space, time, labour, and social relations.

47 Chabert mentions another definition of algorithm by Robert McNaughton that can be used as example of the technical ossification of the social processes previously illustrated: 1). The algorithm must be capable of being written in a defined language. 2). The question that is posed is determined by some input data, called enter. 3). The algorithm is a procedure which is carried out step by step. 4). The action at each step is strictly determined by the algorithm. 5). The output or answer (called exit) is clearly specified. Chabert, *A History of Algorithms*, 455.

As the historian of science Jürgen Renn has noted, after Damerow, machine learning algorithms are nothing 'superhuman' but part of the cycle of internalisation and externalisation of cognitive functions that belongs to all cultural techniques:

> After all, machine learning algorithms . . . are simply a new form of the externalization of human thinking, even if they are a particularly intelligent form. As did other external representations before them, such as calculating machines, for example, they partly take over – in a different modality – functions of the human brain. Will they eventually supersede and even displace human thinking? The crucial point in answering this question is not that their overall intelligence still lags far behind human and even animal intelligence, but that they can play out their full potential only within the cycle of internalization and externalization that . . . is the hallmark and driving force of cultural evolution.[48]

In a similar way, this introductory chapter served to see the algorithm concept in perspective – in its historical context as well as in the long evolution of knowledge systems. In short, it was, firstly, the mercantile acceleration of the late Middle Age, and, secondly, the rise of the information society that contributed to formalise the algorithm as it is known today. For a linguistic coincidence, the medieval term *algorismus* marked the passage from the additive to the positional system of numeration, while the recent use of the term 'algorithm' has marked the passage from decimal to binary numerals. These were not simply formal and technical shifts but also economic ones; after all, Hindu–Arabic numerals and algorithms for hand calculation were adopted to simplify accounting and mercantile transactions, while binary numerals were adopted because they could be implemented in electrical circuits and logic gates to accelerate industrial automation and state administration. Just as the first transition is related to early mercantilism, so is the second to industrial capitalism – particularly in its demand to speed up communication technologies and automate mental labour.[49]

48 Renn, *The Evolution of Knowledge*, 398.

49 See also Matteo Pasquinelli, 'From Algorism to Algorithm: A Brief History of Calculation from the Middle Ages to the Present Day', *Electra* 15 (Winter 2021–22): 93–102.