Lectures on Random Algebraic Geometry

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Contents

1 How many zeros of a polynomial are real?

How many zeros has a polynomial? The answer to this question is taught in a basic algebra course: it is equal to the degree of the polynomial. This is known as the fundamental theorem of algebra.

However, this assumes that the question was stated as *How many complex zeros does a polynomial have?*. Yet, if the person, who asked that question, had in mind a real polynomial and real zeros, the answer is less clear. For instance, the polynomial $x^2 + ax + b$ has two real zeros, if $a^2 - 4b > 0$, it has one real zero, if $a^2 - 4b = 0$, and in the case $a^2 - 4b < 0$ it has no real zeros. The situation is depicted in Figure 1.1. This simple yet important example shows already that we can not give an answer to the above question in terms of the degree of the polynomial. Instead, we have to use a list of algebraic equalities and inequalities. While for polynomials of degree 2 this was simple enough for us to understand, more complicated counting problems pose uncomparably harder challenges. Think of the number of eigenvalues of an $n \times n$ matrix. The number of complex eigenvalues is always n (counted with multiplicity). But the algebraic constraints for the number of real solutions are already so complicated, that it is very difficult just to compute this number without computing all eigenvalues in the first place.

In this book we want to lay out an alternative perspective on counting problems like the ones above. Instead of computing a deterministic real picture, we want to understand its *statistical properties*. This thinking is not new: already in the 1930s and 1940s Littlewood, Offord [14] and Kac [10, 11] considered real zeros of random polynomials. Later, in 1973, Montgomery [15] introduced randomness to number therory. In the 1950s, Wigner [18], Dyson [5] and others proposed using probability for understanding models in theoretical physics. Ginibre [9] summarizes their motivation as follows.

> *"In the absence of any precise knowledge [...], one assumes a reasonable probability distribution [...], from which one deduces statistical properties [...]. Apart from the intrinsic interest of the problem, one may hope that the methods and results will provide further insight in the cases of physical interest or suggest as yet lacking applications."*

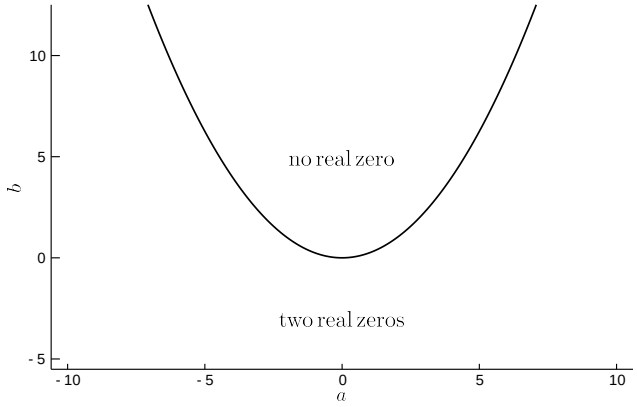


Figure 1.1: The configuration space for real zeros of the polynomial $f = x^2 + ax + b$. The blue curve $a^2 - 4b = 0$ is called the discriminant. If (a, b) is below the discriminant, then f has two real zeros. If it is above, it has no real zero. Polynomials on the discriminant have one real zero.

Although written in the context of statistical physics, Ginibre's words perfectly outline the ideas we wish to present with this book: we want to use tools from probability theory to understand the nature of algebraic–geometric objects.

Edelman and Kostlan [6] condense the probabilistic approach in the title of their seminal paper "How many zeros of a random polynomial are real?" (the answer is in Example 1.12 below). We chose the title of this introductory section as a homage of their work. Starting from their results, we explore in this book algebraic geometry from a probabilistic point of view. Our name for this new field of research is *Random Algebraic Geometry*.

Here is an illustrative example of what we have in mind: consider the degree 8 polynomial $f_\epsilon(x) = 1 + \epsilon_1 x + \epsilon_2 x^2 + \epsilon_3 x^3 + \epsilon_4 x^4 + \epsilon_5 x^5 + \epsilon_6 x^6 + \epsilon_7 x^7 + \epsilon_8 x^8$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_8) \in \{-1, 1\}^8$. This polynomial can have $0, 2, 4, 6$ or 8 zeros, because complex zeros come in conjugate pairs. Instead of attempting to understand the equations separating the regions with a certain number of real solutions, we endow the coefficients of f_ϵ with a probability distribution. We assume that $\epsilon_1, \ldots, \epsilon_8$ are independent random variables with $\mathbb{P}\{\epsilon_i = 1\} = \frac{1}{2}$ for $1 \leq i \leq 8$, and we denote by $n(\epsilon)$ the random variable "number of real zeros of f_ϵ". Booth [3] showed that

$$\mathbb{P}\{n(\epsilon) = 0\} = \frac{58}{2^8}, \quad \mathbb{P}\{n(\epsilon) = 2\} = \frac{190}{2^8}, \quad \mathbb{P}\{n(\epsilon) = 4\} = \frac{8}{2^8}, \quad \text{and}$$
$$\mathbb{P}\{n(\epsilon) = 6\} = \mathbb{P}\{n(\epsilon) = 8\} = 0,$$

which shows that f_ϵ has at most 4 zeros, and having more than 2 zeros is unlikely.

In Booth's example we have access to the full probability law. However, during this book we will encounter many situations in which computing the probability law is too ambitious. Instead, it is often feasible to compute or estimate the expected value of a random geometric property. For instance, in Booth's example the expected value of the number of roots is $\mathbb{E}\, n(\epsilon) = 1.609375$. Just based on this information we can conclude that having a large number of zeros is unlikely.

Interestingly, many of the expected values we will meet later in this book obey what is called the "square-root law": the expected number of real solutions is roughly the square-root of the number of complex solutions. If this law holds, it immediately implies that instances, for which the number of real solutions equal the number of complex solutions, are *rarae aves*. This phenomenon, which is specific of a particular, but natural, probability distribution that we will work with, has several manifestation: from geometry (expectation of volumes of real algebraic sets) to topology (expectation of Betti numbers).

1.1 Discriminants

Let us have a closer look at the picture in Figure 1.1. We can see that the *discriminant* $\Sigma_{\mathbb{R}} := \{(a,b) \in \mathbb{R}^2 \mid a^2 - 4b = 0\}$ divides the real (a,b)-plane into two components – one, where the number of real zeros is two, and one, where there are no real zeros. This is because the discriminant is a curve of *real* codimension 1. The complex picture is different: here, the complex curve $\Sigma_{\mathbb{C}} = \{(a,b) \in \mathbb{C}^2 \mid a^2 - 4b = 0\}$ is of *complex* codimension one. In particular, it is of real codimesion two, and $\mathbb{C}^2 \backslash \Sigma_{\mathbb{C}}$ is path-connected! We show this in Lemma 1.5, but it can also be seen in Figure 1.2. This is the reason for why each polynomial of degree 2 outside $\Sigma_{\mathbb{C}}$ has two complex zeros: a function which is locally constant on a connected space is constant. We say that having two complex zeros is a *generic property*. We will give a more precise definition of this later in Definition 1.4.

In algebraic geometry, it is more appropriate to work with zeros of polynomials in *projective space* rather than with zeros in $\mathbb{C}^n$. The definition of projective space comes next.

Definition 1.1 (Complex projective Space)**.** The complex projective space $\mathbb{C}\mathrm{P}^n$ of dimension n is defined to be the set of lines through the origin in $\mathbb{C}^{n+1}$. That is, $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \backslash \{0\}) / \sim$, where $y \sim z$, if and only of there exists some $\lambda \in \mathbb{C} \backslash \{0\}$ with $y = \lambda z$. For a point $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ we denote by $[z_0, z_1, \ldots, z_n]$ its equivalence class in $\mathbb{C}\mathrm{P}^n$.

For completing the terminology, and distinguishing it from projective space, we say that $\mathbb{C}^n$ is an n-dimensional *affine complex space*.

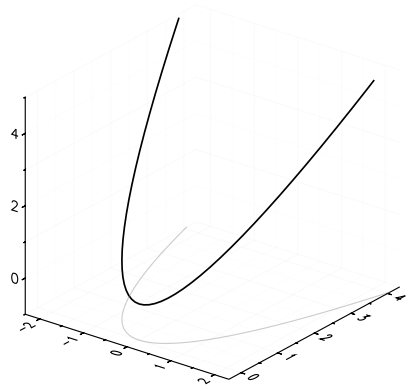


Figure 1.2: The picture shows the part of the complex discriminant $(a_1 + ia_2)^2 - 4(b_1 + ib_2) = 0$, where $a_1 = 2a_2$ As can be seen from the picture, the discriminant is of real codimension two. Because one can "go around" the discriminant without crossing it, a generic complex polynomial of degree 2 has two complex zeros.

The map

$$P : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}\mathrm{P}^n \, , (z_0, z_1, \ldots, z_n) \mapsto [z_0 : \ldots : z_n]$$

projects $(n+1)$-dimensional affine space onto n-dimensional projective space. On the other hand, the map $\psi : \mathbb{C}^n \to \mathbb{C}\mathrm{P}^n$, $(z_1, \ldots, z_n) \mapsto [1, z_1 : \ldots : z_n]$ embeds n-dimensional affine space into n-dimensional projective space. Using this embedding we can define the zero sets in Example 1.3 to be in $\mathbb{C}\mathrm{P}^n$.

Projective zero sets are defined by *homogeneous polynomials*. It is common to use the notation $f = \sum_{|\alpha|=d} f_\alpha \, z^\alpha$ for complex homogeneous polynomials of degree d in $n+1$ variables, where $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, $z^\alpha = \prod_{i=0}^n z_i^{\alpha_i}$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$. The space of homogeneous polynomials of degree d in $n+1$ many variables is

$$\mathbb{C}[x_0, \ldots, x_n]_{(d)} := \left\{ \sum_{|\alpha|=d} f_\alpha \, z^\alpha \;\middle|\; (f_\alpha) \in \mathbb{C}^N \right\}, \text{ where } N = \binom{n+d}{d},$$

and the projective space of polynomials is thus $\mathbb{C}\mathrm{P}^{N-1}$. The complex projective zero set of k polynomials $f = (f_1, \ldots, f_k)$, where the i-th polynomial is $f_i \in \mathbb{C}[z_0, \ldots, z_n]_{(d_i)}$, is

$$Z_{\mathbb{C}}(f) = \{[z] \in \mathbb{C}\mathrm{P}^n : f_1(z) = 0, \ldots, f_k(z) = 0\}.$$

For a simplified notation we also denote by $Z_{\mathbb{C}}(f)$ the zero set of f in $\mathbb{C}^{n+1}$.

Remark 1.2. A polynomial $f \in \mathbb{C}[z_0, \dots, z_n]_{(d)}$ is not a function on the complex projective space $\mathbb{C}\mathrm{P}^n$, but its zero set is still well defined.

Example 1.3. Here are a few more examples of generic properties. The first generalizes our introductory example to higher degrees.

1. A generic homogeneous polynomial $f \in \mathbb{C}[z_0, z_1]_{(d)}$ of degree d has d distinct zeros in $\mathbb{C}\mathrm{P}^1$ unless $\mathrm{Res}(f, f') = 0$ (i.e. the resultant of f and f' is zero). We define the polynomial map $\mathrm{disc}(f) := \mathrm{Res}(f, f')$ that associates to a polynomial f the resultant $\mathrm{Res}(f, f')$. Then, the zero set $\Sigma = Z_{\mathbb{C}}(\mathrm{disc})$ of this polynomial is a proper algebraic set in $\mathbb{C}[z_0, z_1]_d$, which we again call the discriminant. By Lemma 1.5 below, $\mathbb{C}[z_0, z_1]_d \setminus \Sigma$ is path-connected. This causes polynomials in $\mathbb{C}[z_0, z_1]_d \setminus \Sigma$ to admit the generic behavior of having d distinct zeros in $\mathbb{C}\mathrm{P}^1$, because we continuously deform the zero set of any $f_1 \notin \Sigma$ to the zero set of any other $f_2 \notin \Sigma$.
2. The zero set $Z_{\mathbb{C}}(f) \subset \mathbb{C}\mathrm{P}^2$ of a generic $f \in \mathbb{C}[z_0, z_1, z_2]_{(d)}$ of degree d is homeomorphic to a surface of genus $g = \frac{(d-1)(d-2)}{2}$. In this case what happens is that there exists a polynomial $\mathrm{disc} : \mathbb{C}[z_0, z_1, z_2]_{(d)} \to \mathbb{C}$, which vanishes exactly at polynomials whose corresponding zero set in the projective plane is singular. Again, we call $\Sigma = Z_{\mathbb{C}}(\mathrm{disc})$ the discriminant. Outside of the discriminant the topology of $Z(f)$ all look the same: the reason is again that $\mathbb{C}[z_0, z_1, z_2]_{(d)} \setminus \Sigma$ is path-connected by Lemma 1.5.
3. Let $\mathbb{C}[z_0, z_1]_{(3)}$ be the space of homogeneous complex polynomials of degree 3. Inside this space there is the cone $X^{\mathbb{C}}$ of polynomials which are powers of linear forms: $X^{\mathbb{C}} = \{f \in \mathbb{C}[z_0, z_1]_{(3)} \mid \exists \ell \in \mathbb{C}[z_0, z_1]_{(1)} : f = \ell^3\}$. The linear span of $X^{\mathbb{C}}$ is the whole $\mathbb{C}[z_0, z_1]_{(3)}$, therefore for every $f \in \mathbb{C}[z_0, z_1]_{(3)}$ there exist $\ell_1, \dots, \ell_s \in \mathbb{C}[z_0, z_1]_{(1)}$ and $\alpha_1, \dots, \alpha_s \in \mathbb{C}$ such that $f = \sum_{i=1}^{s} \alpha_i \ell_i^3$. For the generic $f \in \mathbb{C}[z_0, z_1]_{(3)}$ the minimal s for having this is $s = 2$. This means that there is a discriminant $\Sigma \subsetneq \mathbb{C}[z_0, z_1]_{(3)}$, which is a proper algebraic subset, such that this property holds outside Σ.
4. The zero set $Z_{\mathbb{C}}(f) \subset \mathbb{C}\mathrm{P}^3$ of a generic cubic $f \in \mathbb{C}[z_0, z_1, z_2, z_3]_{(3)}$ contains 27 complex lines. We will discuss in details this type of problems later, but still let us now try to see what is happening, at least in an informal way. The set of lines in $\mathbb{C}\mathrm{P}^3$ is itself a manifold, which is called the Grassmanian of (projective) lines and denoted by $\mathbb{G}(1, 3)$ (1-dimensional projective subspaces of 3-dimensional projective space). There is a rank-4 complex vector bundle $E \to \mathbb{G}(1, 3)$ whose fiber over a line $\ell \in \mathbb{C}\mathrm{P}^3$ consists of homogeneous polynomials of degree 3 over this line. Every polynomial $f \in \mathbb{C}[z_0, z_1, z_2, z_3]_{(3)}$ defines naturally a section $\sigma_f : \mathbb{G}(1, 3) \to E$ by $\sigma_f(\ell) = f|_\ell$ and a line ℓ is contained in $Z_{\mathbb{C}}(f)$ if and only if $\sigma_f(\ell) = 0$.

The discriminant $\Sigma \subset \mathbb{C}[z_0, z_1, z_2, z_3]_{(3)}$ consists of those polynomials whose section σ_f is not transversal to the zero section

In most cases the properties we will be interested in are described by a list of numbers associated to elements of some parameter space S. Let us re-interpret the statement from Example 1.3 using this language. If $S = P(\mathbb{C}[z_0, z_1]_{(d)} = \mathbb{CP}^d$ is the projective space of complex polynomials of degree d, we might be interested in the number of zeroes of these polynomials. We can interpret this number as a map $\beta : \mathbb{CP}^d \to \mathbb{C}$ given by

$$\beta : f \mapsto \#Z(f).$$

This β is a constant map outside $\Sigma = \{f \mid \mathrm{Res}(f, f') = 0\}$.

The next definition gives a rigorous definition for genericity in our setting.

Definition 1.4 (Generic Properties)**.** Let S be a semialgebraic set[1]. We say that a property β is *generic* for the elements of S if there exists a semialgebraic set $\Sigma \subset S$ of codimension at least one in S such that the property β is true for all elements in $S \backslash \Sigma$. We call the largest (by inclusion) such Σ the *discriminant* of the property β.

When working over the complex numbers most properties are generic in the sense that the discriminant is a proper *complex* algebraic set. Since proper complex algebraic sets in $\mathbb{CP}^N$ do not disconnect the whole space, these properties are constant on an open dense set. This is a simple observation that we record in the next lemma.

Lemma 1.5. *Let $\Sigma \subsetneq \mathbb{CP}^N$ be a proper algebraic subset. Then, $\mathbb{CP}^N \backslash \Sigma$ is path-connected.*

Proof. Let $z_1, z_2 \in \mathbb{CP}^N \backslash \Sigma$. Choose a complex linear space $L \subset \mathbb{CP}^N$ of dimension one, such that $z_1, z_2 \in L$. Then, $L \cap \Sigma$ is a subvariety of L. Since L is irreducible, if $\dim(L \cap \Sigma) = 1$, we must have $L \subset \Sigma$, but this contradicts $z_1, z_2 \notin \Sigma$. Thus, we have $\dim(L \cap \Sigma) = 0$, which means that L intersects Σ in finitely many points. Since L is of complex dimension one, it is of real dimenson two, and thus $L \backslash \Sigma$ is path-connected. We find a real path from z_1 to z_2 that does not intersect Σ. □

Very often the properties that we will be interested in are values of some semi-algebraic functions $\beta : S \to \mathbb{C}^n$, as in the second point from Example 1.3. To see this, let $S = P(\mathbb{C}[z_0, \ldots, z_n]_{(d)})$ be the projective space of polynomials and consider the "property" $\beta : S \to \mathbb{C}^{2n+1}$ given by $\beta(f) = (b_0(Z_{\mathbb{C}}(f)), \ldots, b_{2n}(Z_{\mathbb{C}}(f)))$

[1] A semialgebraic set $S \subset \mathbb{R}^n$ is a finite union and intersections of sets of the form $\{f \leq 0\}$ or $\{f < 0\}$, with $f \in \mathbb{R}[x_0, \ldots, x_n]$.

(i.e. $\beta(f)$ is the list of the Betti numbers of the zero set of f in $\mathbb{C}\mathrm{P}^n$; this number does not depend on the representative of f that we pick, as a nonzero multiple of a polynomial has the same zero set as the original polynomial). The property β in this case takes a constant value on the complement of a complex discriminant $\Sigma \subset S$. In other words, there exists $\beta_0 \in \mathbb{C}^{2n+1}$ such that for all $f \in S\backslash\Sigma$ we have $\beta(Z_{\mathbb{C}}(f)) = \beta_0$. In the case $n = 2$, because the genus is $\frac{(d-1)(d-2)}{2}$, we have that $\beta_0 = (1, (d-1)(d-2), 1)$. A similar argument can be done for the third point in Example 1.3: the property "number of lines on the zero set of f" is constant outside a complex discriminant $\Sigma \subset \mathbb{C}[z_0, \ldots, z_3]_{(3)}$.

As already briefly discussed in the beginning of this section, the topological reason for the existence of such strong generic properties over the complex numbers ultimately is Lemma 1.5. The additional technical ingredient that one needs to deduce that topological properties are stable under nondegenerate deformations goes under the name of *Thom's Isotopy Lemma* and we will prove it and discuss its implications later.

1.2 Real discriminants

Moving to the real world, let us copy the notation from the preceding section to the real numbers.

Definition 1.6 (Real projective space)**.** The real projective space $\mathbb{R}\mathrm{P}^n$ of dimension n is defined to be the set of lines through the origin in $\mathbb{R}^{n+1}$. That is, $\mathbb{R}\mathrm{P}^n := (\mathbb{R}^{n+1}\backslash\{0\})/\sim$, where $y \sim z$, if and only of there exists some $\lambda \in \mathbb{R}\backslash\{0\}$ with $y = \lambda z$. For a point $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ we denote by $[x_0 : x_1 : \ldots : x_n]$ its equivalence class in $\mathbb{R}\mathrm{P}^n$.

Similar to before, we define the projection

$$P : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}\mathrm{P}^n,\ (x_0, x_1, \ldots, x_n) \mapsto [x_0 : x_1 : \ldots : x_n]. \tag{1.1}$$

The space of real homogeneous polynomials is

$$\mathbb{R}[x_0, \ldots, x_n]_{(d)} := \left\{ \sum_{|\alpha|=d} f_\alpha\, x^\alpha \;\middle|\; (f_\alpha) \in \mathbb{R}^N \right\},\ \text{where } N = \binom{n+d}{d}.$$

The projective space of real polynomials is $P(\mathbb{R}[x_0, \ldots, x_n]_{(d)})$. The real projective zero set of k polynomials $f = (f_1, \ldots, f_k)$ is

$$Z(f) = \{[x] \in \mathbb{R}\mathrm{P}^n : f_1(x) = 0, \ldots, f_k(x) = 0\}.$$

Over the Reals we do not have in general an analogue of Lemma 1.5: a proper real

algebraic set can in general disconnect the ambient space. To see this, let us look again at the problems discussed in example 1.3, but from the real point of view.

Example 1.7. Let us start by noticing that the complex properties studied in Example 1.3 are still generic over the reals, in the sense that for the generic *real* polynomial the structure of the *complex* zero set has a constant generic behavior; the structure of the *real* zero set is instead highly dependent on the coefficients of f and there is no "generic" behaviour.

1. A generic univariate polynomial $f \in \mathbb{R}[x]_d$ of degree d has at most d distinct zeros in $\mathbb{R}$, but this number can range anywhere between $\frac{1+(-1)^{d+1}}{2}$ and d. In particular there is no generic number of real zeroes.

   A property which is generic is having real *distinct* zeroes. In this case, however, the real discriminant is not algebraic, but rather just semialgebraic. Unless $d = 2$ it not coincide with the real part of $\{\mathrm{Res}(f, f') = 0\}$: the equation $\mathrm{Res}(f, f') = 0$, which is real for real f, tells us whether f has a double root, but this root can also be complex. The subset of the real part of $\{\mathrm{Res}(f, f') = 0\}$ which corresponds to polynomials with a double real root is only a piece of this discriminant and this piece is selected by imposing some extra inequalities on the coefficients of the polynomial.

2. The zero set $Z(f) \subset \mathbb{R}\mathrm{P}^2$ of a generic $f \in \mathbb{R}[x_0, x_1, x_2]_{(d)}$ is a smooth curve (being smooth is a generic property) but the topology of this curve depends on the coefficients of the polynomial – Harnack's inequality tells that

$$b_0(Z(f)) \leq \frac{(d-1)(d-2)}{2} + 1. \tag{1.2}$$

   For instance $\{x_0^2+x_1^2+x_2^2 = 0\} \subset \mathbb{R}\mathrm{P}^2$ is empty and $\{x_0^2-x_1^2-x_2^2 = 0\} \subset \mathbb{R}\mathrm{P}^2$ is homeomorphic to a circle (they are both smooth).

3. Let $\mathbb{R}[x_0, x_1]_{(3)}$ be the space of homogeneous real polynomials of degree 3. Inside this space there is the cone X of polynomials which are powers of real linear forms: $X = \{f \in \mathbb{R}[x_0, x_1]_{(3)} \mid \exists \ell \in \mathbb{R}[x_0, x_1]_{(1)} : f = \ell^3\}$. The linear span of X is the whole $\mathbb{R}[z_0, z_1]_{(3)}$, as in the complex case. Therefore, for every polynomial $f \in \mathbb{R}[z_0, z_1]_{(3)}$ there exist $\ell_1, \ldots, \ell_s \in \mathbb{R}[x_0, x_1]_{(1)}$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ such that $f = \sum_{i=1}^{s} \alpha_i \ell_i^3$. However now, differently than from the complex case, there is no generic minimal value that the number s can take. In fact, denoting by $\mathrm{rk}_{\mathbb{R}}(f)$ the minimum such s we have that $\mathrm{rk}_{\mathbb{R}}(f) = 2$ whenever a polynomial has one real zero and $\mathrm{rk}_{\mathbb{R}}(f) = 3$ whenever it has 3 real zeroes.

4. The zero set $Z \subset \mathbb{R}\mathrm{P}^3$ of a generic cubic $f \in \mathbb{R}[x_0, x_1, x_2, x_3]_{(3)}$ is smooth and it can contain either $27, 15, 7$ or 3 real lines.

Remark. There exists a *generic* way of counting the lines on $Z(f)$: it is possible to canonically associate a sign $s(\ell)$ to each line $\ell \subset Z(f)$ and the number $\sum_{\ell \subset Z(f)} s(\ell)$ (a signed count) is generically equal to 3.

1.3 Reasonable probability distributions

In the quote of Ginibre it says "one assumes a reasonable probability distribution". He was probably thinking of physically meaningful distributions. But for us this means the following: suppose that $\mathcal{F}$ is a space of geometric objects endowed with a probability distribution, and that $X : \mathcal{F} \to \mathbb{R}^m$ is a random variable on $\mathcal{F}$. If X has symmetries, by which we mean that there is a group G acting on $\mathcal{F}$, such that $X(g \cdot f) = X(f)$ for all g, then the probability distribution is reasonable, if it is invariant under G; that is $g \cdot f \sim f$. This interpretation follows the *Erlangen program* by Felix Klein. In "A comparative review of recent researches in geometry" [12] Klein lays out a perspective on geometry based on a group of symmetries:

> *"Geometric properties are characterized by their remaining invariant under the transformations of the principal group."*

He writes that geometry should be seen as the following comprehensive problem.

> *"Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group."*

Therefore, reasonable probability distributions are distributions which respect geometry in Klein's sense. A reasonable probability distribution should not prefer one instance over another if they share the same geometry.

To illustrate this line of thought, we recall Booth's example from the beginning of this section. The space of geometric objects $\mathcal{F}$ is the space of univariate polynomials of degree 8 with coefficients in $\{-1, 1\}$. The random variable $X(f)$ is the number of real zeros of the polynomial $f \in \mathcal{F}$. The group $G = \{-1, 1\}$ acts on $\mathcal{F}$ as $g.f(x) = 1 + \epsilon'_1 x + \epsilon'_2 x^2 + \epsilon'_3 x^3 + \epsilon'_4 x^4 + \epsilon'_5 x^5 + \epsilon'_6 x^6 + \epsilon'_7 x^7 + \epsilon'_8 x^8$, where $\epsilon'_i = \epsilon_i g^i$. Since for all i we have $\epsilon_i g^i \in \{-\epsilon_i, \epsilon_i\}$ and since $\epsilon_i \sim -\epsilon_i$, we see that $gf \sim f$. In this sense, the distribution proposed by Booth is reasonable. In many cases the space $\mathcal{F}$ comes with the structure of a smooth manifold (e.g. a vector space, a Lie group or a homogeneous space) and in this case a "reasonable" probability distribution should be absolutely continuous with respect to Lebesgue measure (notice that the notion of sets of measure zero is well defined on a smooth manifold and independent of the possible choice of an actual measure).

In these lectures, when $\mathcal{F}$ is a linear space (e.g. the space of polynomials) we will mostly consider a special class of distributions called *gaussian*. The reason for this is that the set of gaussian distributions is rich enough to describe interesting phenomena and simple enough to be able to put our hands on it.

Definition 1.8 (Nondegenerate gaussian distribution)**.** A probability distribution on $\mathbb{R}^N$ is said to be *nondegenerate gaussian* if there exist a positive definite symmetric matrix $Q \in \mathrm{Sym}(N, \mathbb{R})$ and a vector $\mu \in \mathbb{R}^N$ such that for all $U \subseteq \mathbb{R}^N$ measurable subset we have:

$$\mathbb{P}(U) = \frac{1}{((2\pi)^N \det(Q))^{1/2}} \int_U e^{-\frac{(y-\mu)^T Q^{-1}(y-\mu)}{2}} dy. \tag{1.3}$$

Whenever $\mu = 0$ the distribution is called *centered*. The *standard gaussian distribution* corresponds to the choice $Q = \mathbf{1}$ and $\mu = 0$. For a random variables ξ on the real line distributed as a standard gaussian we will write $\xi \sim N(0, 1)$ and sometimes also call it a *standard normal*. More generally, if $X \in \mathbb{R}^N$ has probability density (1.3), we will say that X is a multivariate nondegenerate gaussian variable with mean μ and covariance matrix $\Sigma = Q^{-1}$, and we will write $X \sim N(\mu, \Sigma)$.

From now on we will always assume that Gaussian distributions are nondegenerate and centered.

Let us discuss one important property of gaussian distributions. The matrix $Q > 0$ in (1.4) is positive definite and it therefore defines a scalar product on $\mathbb{R}^N$ by the rule $\langle y_1, y_2 \rangle_Q := y_1^T Q y_2$. If we choose an orthonormal basis $\mathcal{B}_Q = \{e_j\}_{j=1,\dots,N}$ for the scalar product $\langle \cdot, \cdot, \rangle_Q$, then a random element X from the gaussian distribution (1.4) can be written as: $X = \sum_{j=1}^N \xi_j \cdot e_j$ (i.e. X is a linear combination of the basis elements with independent standard gaussian coefficients). This observation will play a crucial practical role later in the book, when dealing with space of random gaussian functions, for which we will need the presentation as a gaussian combination of some basis elements.

We want to introduce now a reasonable probability distribution on the space $\mathbb{R}[x_0, \dots, x_n]_{(d)}$ and, in line with the previous discussion, we require that such distribution satisfies some invariance suggested by the geometry of the objects we are considering.

1. We want it to be "simple", and that is why we require it to be Gaussian, in the following sense. The space of real polynomials $\mathbb{R}[x_0, \dots, x_n]_{(d)}$ is a real vector space of dimension $N = \binom{n+d}{d}$ and therefore it is isomorphic to $\mathbb{R}^N$. We fix a linear isomorphism

$$\varphi : \mathbb{R}^N \to \mathbb{R}[x_0, \dots, x_n]_{(d)}$$

between these two spaces (for example the isomorphism could be given by the coefficients list of the polynomial in some basis). Then, we fix on $\mathbb{R}^N$ a nondegenerate Gaussian distribution $N(Q, \mu)$ in the sense of Definition 1.8. Then, a Gaussian distribution on $\mathbb{R}[x_0, \ldots, x_n]_{(d)}$ is defined as follows:

$$\mathbb{P}(f \in A) = \frac{1}{((2\pi)^N \det(Q))^{1/2}} \int_{\varphi^{-1}(A)} e^{-\frac{(y-\mu)^T Q^{-1}(y-\mu)}{2}} dy. \tag{1.4}$$

2. A second requirement reflects the fact that the zero set of f and $-f$ are the same. Thinking in terms of group actions, the group $\mathbb{Z}_2 = \{\pm 1\}$ acts on the space of polynomials by $f \mapsto -f$ and we want our distribution to be invariant under this action. This forces $\mu = 0$ in (1.4).
3. Third – maybe the most important requirement, is that we want to get a model of randomness for which there are no preferred points or directions in the projective space $\mathbb{R}\mathrm{P}^n$. Using the language of group invariance, there is a representation $\rho : O(n+1) \to \mathrm{GL}(\mathbb{R}[x_0, \ldots, x_n]_{(d)})$ given by change of variables and we require our distribution to satisfy the property of being invariant under all elements of $\rho(O(n+1))$.

It turns out that the three conditions above do not identify uniquely a probability distribution, and in fact, as we will see later in these lectures, there is a whole family of such distributions. We will call them *invariant distributions*.

1.3.1 The Kostlan distribution

The Kostlan distribution is a special case of an invariant distribution which has some additional special features that make it good for comparisons with complex algebraic geometry. In order to define it, it is helpful to use the following notation:

$$\binom{d}{\alpha} := \frac{d!}{\alpha_0! \cdots \alpha_n!}.$$

Choose the linear isomorphism $\varphi_{\mathrm{Kostlan}} : \mathbb{R}^N \to \mathbb{R}[x_0, \ldots, x_n]_{(d)}$ defined by

$$\varphi_{\mathrm{Kostlan}}((f_\alpha)_\alpha) = \sum_{|\alpha|=d} f_\alpha \cdot \sqrt{\binom{d}{\alpha}} x_0^{\alpha_0} \cdots x_n^{\alpha_n}. \tag{1.5}$$

Then, for a measurable $A \subseteq \mathbb{R}[x_0, \ldots, x_n]_{(d)}$ its probability with respect to the Kostlan distribution is defined to be:

$$\mathbb{P}(f \in A) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\varphi^{-1}_{\text{Kostlan}}(A)} e^{-\frac{\|y\|^2}{2}} dy. \tag{1.6}$$

Kostlan polynomials are invariant as recorded in the next lemma.

Lemma 1.9. *The Kostlan distribution is an invariant distribution.*

We postpone the proof of this Lemma until we give a thorough discussion of probability distributions which are invariant under group actions.

A simple way to write down a Kostlan polynomial is by taking a combination of standard gaussians as follows:

$$f(x) = \sum_{|\alpha|=d} \xi_\alpha \cdot \sqrt{\binom{d}{\alpha}}\, x_0^{\alpha_0} \cdots x_n^{\alpha_n}, \tag{1.7}$$

where $\{\xi_\alpha\}_{|\alpha|=d}$ is a family of standard, independent gaussian variables on $\mathbb{R}$. The Kostlan distribution, among the invariant ones, is the unique (up to multiples) for which a random polynomial can be written as a combination of independent gaussians in front of the standard monomial basis.

Proposition 1.10. *Among the invariant distributions, the Kostlan one is the unique (up to multiples) such that a random polynomial can be written as a linear combination of the standard monomial basis with coefficients independent gaussians.*

1.4 Expected properties

As we have seen, if the discriminant is a complex algebraic set, we have strong genericity over the complex numbers: the reason for this is Lemma 1.5, which says that the complex discriminant does not disconnect $\mathbb{C}\mathrm{P}^N$. However, if the discriminant is a real hypersurface, in general it might disconnect $\mathbb{R}\mathrm{P}^N$, this is why in Figure 1.1 there are two regions with different properties. Therefore, over the real numbers we might not have a notion of strong genericity, and we adopt a random point of view. The next definition is the probabilistic analogue of Definition 1.4.

Definition 1.11 (Expected Properties)**.** Let S be a semialgebraic set. A measurable property is a measurable function $\beta : S \to \mathbb{C}^m$. If we have a (reasonable) probability distribution on S, we call $\mathbb{E}_{s \in S}\, \beta(s)$ the expected property.

In fact, Definition 1.4 is a special case of Definition 1.11. We will discuss this in Subsection 1.4.1 below. First, let us revisit Example 1.3 from a probabilistic point of view.

Example 1.12. Let us endow the space of real polynomials with the Kostlan distribution. Then we can ask for the expectation of the real version of the properties that we have discussed in Example 1.3.

1. Let $f \in \mathbb{R}[x_0, x_1]_{(d)}$ be a Kostlan polynomial of degree d in 2 variables. Then, for the generic element $f \in \mathbb{R}[x_0, x_1]_{(d)}$ the number of complex zeroes is d, but the expected number of real zeros of f is $\sqrt{d}$.
2. Let $f \in \mathbb{R}[x_0, x_1, x_2]_{(d)}$ be a Kostlan polynomial of degree d in 3 variables. There exist constants $c, C > 0$ such that the expected value of the zero-th Betti number $b_0(f)$ of $Z(f)$ satisfies $cd \leq \mathbb{E}\, b_0(f) \leq Cd$.
3. Let $f \in \mathbb{R}[x_0, x_1]_{(3)}$ be a Kostlan polynomial, then the expectation of its real rank $\mathrm{rk}_{\mathbb{R}}(f)$ is $\frac{9-\sqrt{3}}{2}$.
4. Let $f \in \mathbb{R}[x_0, x_1, x_2, x_3]_{(3)}$ be a Kostlan polynomial of degree 3 in 3 variables. Then, the expected number of real lines on $Z(f)$ is $6\sqrt{2} - 3$.

The first example was proven in [6], the second in [8], and the third is actually a consequence of the first example, but we will also prove them in the remainder of these lectures. The fourth example was proved in [17]. We want to emphasize that the first two of those examples obey a square-root law – the expected value of the real property has the order of the square root of the generic value of the complex property.

1.4.1 Generic properties are expected properties

In closing of this introductory lecture we want to explain why generic properties are, in fact, random properties in disguise. The essence of this is a simple observation: suppose $z \in \mathbb{C}\mathrm{P}^N$ is a random variable that is supported on some full-dimensional subset of $\mathbb{C}\mathrm{P}^N$. In particular, this implies that, if β is a property with discriminant Σ, and if $\Sigma \subsetneq \mathbb{C}\mathrm{P}^N$ is an algebraic variety, then $\mathbb{P}\{z \in \Sigma\} = 0$, and so $\mathbb{P}\{\beta(z) \text{ has the generic value}\} = 1$. Therefore

$$\mathbb{E}\, \beta(z) = \text{ generic value of } \beta(z).$$

It is interesting to approach the problem of computing generic properties from a probablistic point of view.

This strategy becomes more effective as counting problems over the complex numbers becomes more complicated. Consider $f = \sum_{i=0}^{d} c_i x_0^i x_1^{d-i} \in \mathbb{C}[x_0, x_i]_{(d)}$,

where the real and imaginary parts of the c_i are independent Gaussian random variables such that $\Re(c_i) \sim N(0, \frac{1}{2}\binom{d}{i})$ and $\Im(c_i) \sim N(0, \frac{1}{2}\binom{d}{i})$ (the factor $\frac{1}{2}$ is for normalizing the variance to $\mathbb{E}\,|c_i|^2 = 1$). Such a polynomial is called a *complex Kostlan polynomial*. The distribution we have put on the coefficients is absolutely continuous with respect to Lebesgue measure on the space of coefficients, and in fact the distribution of $P(f)$ is supported on the whole $\mathbb{C}\mathrm{P}^d$. Therefore, we know that with probability one we have that $\#Z_{\mathbb{C}}(f)$ equals some constant (we know this constant is d, but let's pretend for a second that we did not know this). Then, if we can find a way (and there is such a way) to compute by elementary means the expectation of $\#Z_{\mathbb{C}}(f)$, we have found its generic value.

2 Volumes

In this chapter we will consider the definition of the integral of a measure on a Riemannian manifold M. This will lead to a natural definition of the *volume* of M. We will use this definition for computing the volumes of several basic objects in algebraic geometry: spheres, projective spaces, orthogonal groups, and the Grassmannian. We will start with some basics from Riemannian differential geometry. For a general introduction into this topic we refer to [13].

2.1 Differential geometry basics

The first basic definition is that of a smooth manifold M.

Definition 2.1 (Smooth Manifold)**.** A smooth manifold M is a Hausdorff and second countable topological space together with a family of homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^m$, $\alpha \in A$, where $U_\alpha \subset M$ is open, such that

1. $M \subset \bigcup_{\alpha\in A} U_\alpha$;
2. The change of coordinates $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth for all $\alpha, \beta \in A$.

Each pair $(U_\alpha, \varphi_\alpha)$ is called a *chart*. The family $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ is called an *atlas* for M. The *dimension* if M is m.

If we replace $\mathbb{R}^m$ by $\mathbb{C}^m$ and if we require that each change of coordinates is a holomorphic map, we call M a *complex manifold*.

Example 2.2. We consider the unit circle: $M = S^1 = \{x \in \mathbb{R}^2 \mid x^T x = 1\}$. We can cover S^1 with two charts $S^1 = U_1 \cup U_2$, where $U_1 = M \setminus \{(0,1)\}$ and $U_2 = M \setminus \{(0,-1)\}$. The homeomorphisms are the two *stereographic projections* $\varphi_1(x,y) = \frac{x}{1-y}$ and $\varphi_2(x,y) = \frac{x}{1+y}$. The change of coordinates is the smooth map $(\varphi_2 \circ \varphi_1^{-1})(t) = t^{-1}$, so that S^1 is indeed a smooth manifold.

In the following, M and N will be smooth manifolds of dimensions m and n, respectively.

Definition 2.3. We say that $F : M \to N$ is smooth, if $\psi \circ F \circ \varphi^{-1}$ is smooth for every pair of charts (U, φ) of M and (V, ψ) of N, such that $F(U) \subset V$. We denote by $C^\infty(M, N)$ the space of all smooth functions from M to N. We say that a smooth map $F : M \to N$ is a *diffeomorphism*, if it is invertible with smooth inverse $F^{-1} : M \to N$.

Notice that the definition of smooth map is compatible with the smooth structures of M in the following sense. If we restrict to the intersection of two charts $F|_{U_\alpha \cap U_\beta} : U_\alpha \cap U_\beta \to V$, then $\psi \circ F \circ \varphi_\alpha^{-1}$ is smooth if and only if the composition $\psi \circ F \circ \varphi_\beta^{-1} = (\psi \circ F \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})$ is smooth. A similar argument shows the compatibility of Definition 2.3 with the smooth structure of N.

A *derivation* of M at x is a linear function $D : C^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that $D(fg) = D(f)g(x) + f(x)D(g)$. The (abstract) *tangent space* of M at a point $x \in M$ is then

$$T_x M := \{D : C^\infty(M, \mathbb{R}) \to \mathbb{R} \mid D \text{ is a derivation of } M \text{ at } x\},$$

and we have $\dim T_x M = \dim M$ for all $x \in M$; see [13, Proposition 3.10]. As an example, let us consider the tangent space of $\mathbb{R}^m$. at a point $a \in \mathbb{R}^m$ the tangent space $T_a\mathbb{R}^m$ consists of all directional derivatives at a (see [13, Proposition 3.2]):

$$T_a\mathbb{R}^m := \operatorname{span}\left\{ \frac{\partial}{\partial x_i}\Big|_a \;\middle|\; i = 1, \dots, m \right\} \cong \mathbb{R}^m. \tag{2.1}$$

Let now (φ, U) be a chart of M, $x \in M$ and $a := \varphi(x)$. We denote by $(\varphi^{-1})_*(\frac{\partial}{\partial x_i}\big|_a)$ the derivation that acts as

$$(\varphi^{-1})_*\Big(\frac{\partial}{\partial x_i}\Big|_a\Big)(f) := \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(a), \text{ for } f \in C^\infty(M, \mathbb{R}). \tag{2.2}$$

The map $(\varphi^{-1})_*$ is also called a *push-forward* for derivations, and it is a linear isomorphism of n-dimensional vector spaces. This together with (2.1) implies

$$T_x M = \operatorname{span}\left\{ (\varphi^{-1})_*\Big(\frac{\partial}{\partial x_i}\Big|_a\Big) \;\middle|\; i = 1, \dots, m \right\}, \quad \text{where } a = \varphi(x). \tag{2.3}$$

Example 2.4. As an example we consider the $(n-1)$-dimensional unit sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid x^T x = 1\}$. Let $x \in S^{n-1}$. Then, we can identify

$$T_x S^{n-1} \cong x^\perp = \{y \in \mathbb{R}^n \mid x^T y = 0\}. \tag{2.4}$$

Every point $y \in x^\perp$ is identified with the derivation that acts as the directional derivative in direction y; i.e., $y(f) = \sum_{i=1}^{n+1} y_i \frac{\partial}{\partial x_i} F(x)$ for $F \in C^\infty(S^n, \mathbb{R})$.

In the previous example we identified $T_x S^n$ with a linear subspace of $\mathbb{R}^{n+1}$. In fact, for every manifold $M \hookrightarrow \mathbb{R}^n$ embedded in $\mathbb{R}^n$ we have such an identification: let $M \hookrightarrow \mathbb{R}^n$ and $x \in M$. For a curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x$ we define the *geometric tangent vector* $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)|_{t=0}$. The linear span of all such tangent vectors is called the *geometric tangent space* of M at x.

Lemma 2.5. *The geometric tangent space of M at x is isomorphic to $T_x M$.*

Proof. Let $\varphi : U \to \mathbb{R}^m$ be a chart of M with $x \in M$ and $a = \varphi(x)$. Then, writing the curve γ in coordinates, we get a curve $(\varphi \circ \gamma)(t)$ in $\mathbb{R}^m$ through a. It's derivative $\frac{\mathrm{d}}{\mathrm{d}t}(\varphi\circ\gamma)|_{t=0}$ is a vector v in $\mathbb{R}^m$, which we can identify with the derivation $\sum_{i=1}^n v_i \frac{\partial}{\partial x_i}|_a \in T_a\mathbb{R}^m$. Using (2.3), we get an identification of the geometric tangent space with M. □

Next, we introduce smooth maps between manifolds.

Definition 2.6. Let $F : M \to N$ be smooth and $x \in M$. The *derivative* of F at x is the linear map $\mathrm{D}_x F : T_x M \to T_{F(x)} N$, such that $\mathrm{D}_x F(v)(f) := v(f \circ F)$ for all $f \in C^\infty(N, \mathbb{R})$. We also write, $F_* := \mathrm{D}_x F$. We say that F is a *submersion*, if $\mathrm{D}_x F$ is surjective for all $x \in M$. We say that is is an *immersion*, if $\mathrm{D}_x F$ is injective for all $x \in M$.

Let us consider $\mathrm{D}_x F$ in coordinates: suppose $x \in U$ and $F(x) \in V$. We check how $\mathrm{D}_x F$ acts on a basis for $T_x M$ For this, we denote $a := \varphi(x)$ and $b = \psi(F(x))$. There exists $c_{i,j} \in \mathbb{R}$ such that

$$\mathrm{D}_x F((\varphi^{-1})_*(\tfrac{\partial}{\partial x_i}|_a))(f) = \sum_{i=1}^n c_{i,j} \frac{\partial}{\partial x_i}(f \circ \psi^{-1})(b) \quad \text{for } f \in C^\infty(N, \mathbb{R}),$$

because the derivations $(\psi^{-1})_*(\frac{\partial}{\partial x_i}|_b)$ form a basis for $T_{F(x)}N$, by (2.3). On the other hand, $\mathrm{D}_x F((\varphi^{-1})_*(\frac{\partial}{\partial x_i}|_a))(f) = \frac{\partial}{\partial x_i}(f \circ F \circ \varphi^{-1})(a)$, by definition of $\mathrm{D}_x F$. Therefore,

$$C = (c_{i,j}) \in \mathbb{R}^{n\times n}, \quad \text{where } \; c_{i,j} = \frac{\partial}{\partial x_i}(x_j \circ \psi \circ F \circ \varphi^{-1})(a), \tag{2.5}$$

and where x_j is the j-th coordinate functions, represents $\mathrm{D}_x F$ relative to the bases $((\varphi^{-1})_*(\frac{\partial}{\partial x_i}|_a))$ and $((\psi^{-1})_*(\frac{\partial}{\partial x_i}|_b))$.

A point $x \in M$ is called a *regular point* of $F : M \to N$ if $\mathrm{D}_x F : T_x M \to T_{F(x)} N$ is surjective. Note that a necessary condition for regular points to exist is $\dim M \geq \dim N$. We call a point $y \in N$ a *regular value* of F, if every $x \in F^{-1}(y)$ is a regular point of F. For a proof of the following see, e.g., [4, Theorem A.9].

Proposition 2.7. *The fiber $F^{-1}(y)$ of a regular value $y \in N$ is a smooth manifold of dimension* $\dim M - \dim N$. *Its tangent space at x is $T_x F^{-1}(y) = \ker \mathrm{D}_x F$.*

A useful consequence of this proposition is the following result.

Corollary 2.8. *Let $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n]$ be k polynomials in $n \geq k$ variables and let $M = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_k(x) = 0\}$ be the real algebraic variety defined as the zero set of the f_i. We assume that for each $x \in M$, the Jacobian matrix $J(x) = (\frac{\partial}{\partial x_j} f_i(x))_{i=1,\ldots,k, j=1,\ldots,n} \in \mathbb{R}^{k \times n}$ has full rank. Then:*

1. *$M \hookrightarrow \mathbb{R}^m$ is a smooth manifold of dimension $n - k$;*
2. *the geometric tangent space is $T_x M \cong \ker J(x)$.*

Proof. The first item follows from Proposition 2.7 by noticing that 0 is a regular value of the smooth map $\mathbb{R}^m \to \mathbb{R}^k, x \mapsto (f_1(x), \ldots, f_k(x))$. The second item also follows from Proposition 2.7 and the fact that $J(x)$ is the derivative of this map in coordinates. □

Remark 2.9. In fact, the statement of Corollary 2.8 is still true, if we replace by polynomials by smooth maps.

2.2 The integral of a function on a Riemannian manifold

In the spirit of Definition 2.3 we say that a function $f : M \to \mathbb{R}$ is *measurable*, if $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}$ is measurable for every of chart $(U, \varphi : U \to \mathbb{R}^m)$ of M. In this section we discuss how to integrate measurable function on on a manifold M. For this, M must be a *Riemannian manifold*.

Definition 2.10. A *Riemannian manifold* is a pair (M, g), where M is a smooth manifold and g assigns to each tangent space $T_x M$ a positive definite bilinear form $\mathrm{g}(x) : T_x M \times T_x M \to \mathbb{R}$. We also call g the *Riemannian metric* on M.

A *Hermitian manifold* is a pair (M, h), where M is a complex manifold and h assigns to each tangent space $T_x M$ a positive definite Hermitian form. We call h the *Hermitian metric* on M.

Every Hermitian manifold is also a Riemannian manifold by taking the Riemannian metric to be the real part of the Hermitian metric: $\mathrm{g} = \frac{1}{2}(\mathrm{h} + \overline{\mathrm{h}})$.

Given an atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ for M and a point $x \in U_\alpha$ for a fixed α we can represent $\mathrm{g}(x)$ in coordinates by the $n \times n$ matrix $g_\alpha(x)$ with coordinates

$$g_\alpha(x)_{i,j} := \mathrm{g}(x)((\varphi_\alpha^{-1})_*(\tfrac{\partial}{\partial x_i} \mid_a), (\varphi_\alpha^{-1})_*(\tfrac{\partial}{\partial x_j} \mid_a)), \quad \text{where } a = \varphi_\alpha(x). \tag{2.6}$$

Let us see how g_α behaves under coordinate changes.

Lemma 2.11. *If $x \in U_\alpha \cap U_\beta$, then*

$$\det(g_\alpha(x)) = \det(g_\beta(x)) \det(J_{\alpha,\beta}(x))^2,$$

where

$$J_{\alpha,\beta}(x) = \left[\frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})}{\partial x_j}(\varphi_\alpha(x)) \right]_{j=1,\dots,m} \in \mathbb{R}^{m\times m}.$$

Proof. Let $a := \varphi_\alpha(x)$ and $b := \varphi_\beta(x)$, and denote by $v_i := (\varphi_\alpha^{-1})_*(\frac{\partial}{\partial x_i}\,|_a)$ and $w_i := (\varphi_\beta^{-1})_*(\frac{\partial}{\partial x_j}\,|_b)$ tangent vectors. In (2.5) we take F to be the identify to see that $v_i = \sum_{j=1}^n c_{i,j}\, w_i$ for $J_{\alpha,\beta}(x) = (c_{i,j})^T$. This implies

$$\mathrm{g}(x)(v_i, v_j) = \sum_{k=1}^m \sum_{\ell=1}^m c_{i,k} c_{j,\ell}\, \mathrm{g}(x)(w_k, w_\ell);$$

i.e., $g_\alpha(x) = J_{\alpha,\beta}(x)^T g_\beta(x) J_{\alpha,\beta}(x)$. Taking determinants finishes the proof. □

A *partition of unity* for M subordinated to the open cover $\{U_\alpha\}_{\alpha\in A}$ is a family of continuous functions $\{p_\alpha : U_\alpha \to \mathbb{R}\}_{\alpha\in A}$, such that: (1) $0 \le p_\alpha(x) \le 1$ for all $x \in M$ and $\alpha \in A$; (2) $p_\alpha(x) > 0$ for only finitely many α; and (3) $\sum_{\alpha\in A} p_\alpha(x) = 1$ for all $x \in M$. Such a family always exists; see, e.g., [13, Theorem 2.23]. We need it to define the integral of a measurable function.

Definition 2.12 (Integral of a function on a Riemannian manifold)**.** Let (M, g) be a Riemannian manifold with atlas $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$. Let $\{p_\alpha : U_\alpha \to \mathbb{R}\}_{\alpha\in A}$ be a partition of unity subordinated to the open cover $\{U_\alpha\}_{\alpha\in A}$. The integral of a measurable function $f : M \to \mathbb{R}$ is (if it exists):

$$\int_M f(x)\, \mathrm{dvol}_\mathrm{g}(x) := \sum_{\alpha\in A} \int_{\varphi_\alpha(U_\alpha)} \left((f \cdot p_\alpha) \circ \varphi_\alpha^{-1}\right)(x) \sqrt{\det g_\alpha(\varphi_\alpha^{-1}(x))}\, \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where on the right hand side of this equation we have standard Lebesgue integrals, and g_α is the matrix from (2.6). We will say that a measurable function $f : M \to \mathbb{R}$ is *integrable* if $\int_M |f(x)| \mathrm{dvol}_\mathrm{g}(x)$ is finite.

In the following, whenever it will be clear to which Riemannian metric we refer, we will simply denote by

$$\int_M f(x)\, \mathrm{d}x := \int_M f(x)\, \mathrm{dvol}_\mathrm{g}(x)$$

the integral of a measurable function.

Definition 2.12 is based on the choice of a partition of unity. Next we show, that the definition is actually independent of this choice.

Lemma 2.13. *The definition of the integral in Definition 2.12 is independent of the choice of partition of unity.*

Proof. Let $\{p_\alpha \mid \alpha \in A\}$ and $\{q_\alpha \mid \alpha \in A\}$ be two partitions of unity subordinated to the open cover $\{U_\alpha\}_{\alpha\in A}$. Then, we have

$$\begin{aligned}
&\sum_{\alpha\in A}\int_{\varphi_\alpha(U_\alpha)} \left((f\cdot p_\alpha)\circ\varphi_\alpha^{-1}\right)(x)\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=&\sum_{\alpha\in A}\int_{\varphi_\alpha(U_\alpha)} \left(\left(f\cdot p_\alpha\cdot\sum_{\beta\in A} q_\beta\right)\circ\varphi_\alpha^{-1}\right)(x)\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=&\sum_{\alpha\in A}\sum_{\beta\in A}\int_{\varphi_\alpha(U_\alpha\cap U_\beta)} \left((f\cdot p_\alpha\cdot q_\beta)\circ\varphi_\alpha^{-1}\right)(x)\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))}\,\mathrm{d}x_1\cdots\mathrm{d}x_n,
\end{aligned}$$

the second equality, because $\sum_{\beta\in A} q_\beta = 1$, and the last equality, because $p_\alpha\cdot q_\beta$ is zero outside of $U_\alpha\cap U_\beta$. For the last term we can use a change of variables $\varphi_\alpha(U_\alpha\cap U_\beta)\to\varphi_\beta(U_\alpha\cap U_\beta)$, $x\mapsto y=(\varphi_\beta\circ\varphi_\alpha^{-1})(x)$. By Lemma Lemma 2.11 we have $\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))} = |\det J_{\alpha,\beta}|\sqrt{\det \mathrm{g}_\beta(\varphi_\beta^{-1}(y))}$, where $J_{\alpha,\beta}$ is the Jacobian matrix of $\varphi_\beta\circ\varphi_\alpha^{-1}$ at x. This cancels with $|\det J_{\alpha,\beta}|^{-1}$, which we get from the change of variables formula for the Lebesgue integral. We can now go the chain of equalities backwards and interchange the roles of p_α and q_β. □

2.2.1 The Riemannian volume

We are now ready to introduce the notion of Riemannian volume. In this context, this means that we are introducing a special measure on the Borel sigma algebra of M, through the help of the Riemannian metric. This measure is called the *Riemannian volume*.

Definition 2.14 (The Riemannian volume)**.** Let (M,g) be a smooth Riemannian manifold of dimension m and $U\subset M$ be a Borel subset. We define the volume of U and denote it by $\mathrm{vol}_\mathrm{g}(U)$ by

$$\mathrm{vol}_g(U) = \int_U \mathrm{dvol}_\mathrm{g}(x).$$

In other word we are integrating the measurable function $f\equiv 1$ on the set U. If the metric g is clear from the context, we also write $\mathrm{vol}(U) := \mathrm{vol}_\mathrm{g}(U)$. If we wish to emphasize the dimension of M we write $\mathrm{vol}_m(U) := \mathrm{vol}(U)$.

Definition 2.14 induces a measure on M, which we call the *Lebesgue measure*.

Computing the volume of a manifold directly using Definition 2.14 is often difficult. In most cases, we can simplify the calculation by using the coarea formula from the next section. See for instance Example 2.23, where we compute the volume of the unit circle.

Given a submanifold $X \hookrightarrow M$ we define its volume as the volume of X with respect to the volume density $\mathrm{vol}_{\mathrm{g}|_X}$,:

$$\mathrm{vol}(X) = \int_X \mathrm{dvol}_{\mathrm{g}|_X}.$$

i.e. we first restrict the Riemannian metric on X and obtain the metric $\mathrm{g}|_X$ and *then* we consider its volume measure on X. For instance, the volume of a curve is its length, and the volume of a surface is its area.

In many cases we can define the volume of X even when it is not smooth, for instance in the semialgebraic case. More precisely, assume that M is an m-dimensional semialgebraic and smooth manifold, endowed with a Riemannian metric g, and $X \subseteq M$ is a semialgebraic set of dimension $s \leq m$. Then, following [2, Proposition 9.1.8], X can be partitioned into finitely many smooth and semialgebraic subsets $X = \coprod_{j=1}^{N} X_j$. Let $U \subseteq X$ be a Borel subset; we define its volume (with respect to the volume induced by g on X) by:

$$\mathrm{vol}_{\mathrm{g}|_X}(U) = \sum_{\dim(X_j)=s} \mathrm{vol}_{\mathrm{g}|_{X_j}}(U \cap X_j).$$

For instance, when X is an algebraic subset of M, this definition coincides with *declaring* the set of singular points of X to be of Riemannian measure zero, and then considering the volume measure induced by $\mathrm{g}|_{\mathrm{smooth}(X)}$ on the set of smooth points of X.

Remark 2.15 (Sets of measure zero). One can also introduce the notion of "sets of measure zero" for smooth manifolds (not necessarily Riemannian). The definition is as follows. Let $D = [a_1, b_1] \times \cdots \times [a_m, b_m]$ be a cube in $\mathbb{R}^m$. We will set $\mu(D) = \prod_{i=1}^{m} |b_i - a_i|$, and we will say that $W \subset \mathbb{R}^m$ *has measure zero* if for every $\epsilon > 0$ there exists a countable family of cubes $\{D_k\}_{k \in J}$ (countable means that the cardinality of the index set J is either finite or countable) such that $W \subset \bigcup_{k \in J} D_k$ and $\sum_{k \in J} \mu(D_k) \leq \epsilon$. If M is a smooth manifold, we will say that $W \subset M$ *has measure zero*, if for every chart (U, φ) for M the set $\varphi(W \cap U) \subset \mathbb{R}^m$ is a set of measure zero.

A measure zero set in M cannot contain any open set: if V is open inside some measure zero set, then for some chart (U, φ) we have that $\varphi(U \cap V) \subset \mathbb{R}^m$ is open

and nonempty. In particular $\psi(U \cap V)$ contains a cube D with $\mu(D) = \delta > 0$. If we now try to cover $\varphi(U \cap V)$ with a countable collection of cubes $\{D_k\}_{k\in J}$, this collection must cover D and $\sum_{k\in J} \mu(D_k) \geq \delta$. In particular if $W \subset M$ has measure zero, its complement is dense: let $V \subset M$ be any open set, then $V \cap W^c \neq \emptyset$, otherwise V would be contained in W. The reason why the notion of sets of measure zero only depends on the differentiable structure is the following result.

Proposition 2.16. *Let $U \subset \mathbb{R}^m$ be an open set and $W \subset U$ be of measure zero. If $F : U \to \mathbb{R}^n$ is a smooth map, then $F(W)$ has measure zero.*

Proof. See [13, Proposition 6.5]. □

Note that, if a set U has measure zero in M, then for every Riemannian metric g on M we have $\mathrm{vol}_g(U) = 0$.

2.3 The coarea formula

The coarea formula is a key tool in these lectures. Basically, this formula shows how integrals transform under smooth maps. A well known special case is integration by substitution. The coarea formula generalizes this from integrals defined on the real line to integrals defined on Riemannian manifolds.

Let M, N be Riemannian manifolds and $F : M \to N$ be a smooth map. Recall that $x \in M$ is a regular point of F, if $\mathrm{D}_x F$ is surjective. For any $x \in M$ the Riemannian metric on M defines orthogonality on T_xM; i.e., $v, w \in T_xM$ are orthogonal, if and only if $\mathrm{g}(x)(v, w) = 0$. For a regular point x of F this implies that the restriction of $\mathrm{D}_x F$ to the orthogonal complement of its kernel is a linear isomorphism. The absolute value of the determinant of that isomorphism, computed with respect to orthonormal bases in T_xM and $T_{F(x)}N$, respectively, is the *normal Jacobian of F at x*. Let us summarize this in a definition.

Definition 2.17. Let $F : M \to N$ be a smooth map between Riemannian manifolds and $x \in M$ be a regular point of F. Let $(\,\cdot\,)^\perp$ denote the operation of taking orthogonal complement with respect to the corresponding Riemannian metric. The normal Jacobian of F at x is defined as

$$\mathrm{NJ}(F, x) := \left|\det\left(\mathrm{D}_x F|_{(\ker \mathrm{D}_x F)^\perp}\right)\right|,$$

where the determinant is computed with respect to orthonormal bases in the source and in T_xM and $T_{F(x)}N$ (this definition does not depend on the choice of the bases). If $x \in M$ is not a regular point of F, we set $\mathrm{NJ}(F, v) = 0$.

We are now equipped with all we need to state the coarea formula.

Theorem 2.18 (The coarea formula)**.** *Suppose that M, N are Riemannian manifolds with $\dim M \geq \dim N$, and let $F : M \to N$ be a surjective smooth map. Then we have for any integrable function $h : M \to \mathbb{R}$ that*

$$\int_M h(x)\,\mathrm{d}x = \int_{y\in N}\left(\int_{x\in F^{-1}(y)} \frac{h(x)}{\mathrm{NJ}(F,x)}\,\mathrm{d}x\right)\mathrm{d}y.$$

Notice that by Sard's lemma the set of point in N, which are not regular values, is a measure zero set, and can therefore be ignored in the integration, so that the normal Jacobian is always positive.

In the case when $F : M \to N$ is a diffeomorphism, $F^{-1}(y)$ contains a single element for all $y \in N$. Therefore, we get the following simplification of the coarea formula in this case.

Corollary 2.19 (The change of variables formula for manifolds)**.** *Suppose that M, N are Riemannian manifolds, and let $F : M \to N$ be a diffeomorphism and $h : M \to \mathbb{R}$ be integrable. Then: $\int_M \mathrm{NJ}(F,x)\,h(v)\,\mathrm{d}x = \int_N h(F^{-1}(y))\,\mathrm{d}y$.*

If M and N are complex Hermitian manifolds, and $F : M \to N$ is a complex differentiable map (i.e., F is complex differentiable for every choice of coordinates using charts), then the coarea formula still applies, and the normal Jacobian can be written in a simpler form. Recall that a given a Hermitian metric h on a complex manifold M, then we can construct a Riemannian metric g on M by taking the real part $\mathrm{g} = \frac{1}{2}(\mathrm{h} + \overline{\mathrm{h}})$ of h. In this context, the coarea formula for a complex map takes the following form.

Lemma 2.20. *Let $F : M \to N$ be a complex differentiable map between complex hermitian manifolds and $x \in M$. Then:*

$$\mathrm{NJ}(F,x) := \left|\det(\mathrm{D}^{\mathbb{C}}_x F\big|_{(\ker \mathrm{D}_x F)^\perp})\right|^2,$$

where $\mathrm{D}^{\mathbb{C}}_x F$ denotes the differential of F viewed as a complex map and the determinant is with respect to hermitian-orthonormal bases, made of real vectors, for $(\ker D_x F)^\perp$ and $T_{F(x)}N$ viewed as complex vector spaces.

Proof. Choose hermitian-orthonormal bases, made of real vectors, for $(\ker \mathrm{D}_x F)^\perp$ and $T_{F(x)}N$ viewed as a m-dimensional complex vector spaces. Denoting these bases by $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_m\}$ respectively, we can construct the new bases $\{e_1, \ldots, e_m, \sqrt{-1}\cdot e_1, \ldots, \sqrt{-1}\cdot e_m\}$ and $\{f_1, \ldots, f_m, \sqrt{-1}\cdot f_1, \ldots, \sqrt{-1}\cdot f_m\}$, which are orthonormal bases for $(\ker D_x F)^\perp$ and $T_{F(x)}N$ as real vector spaces with

the Riemannian metric $\mathrm{g} = \frac{1}{2}(\mathrm{h} + \overline{\mathrm{h}})$ induced by the Hermitian metric h. Let the complex Jacobian with respect to these bases be $J^{\mathbb{C}} = A + iB$, where $A, B \in \mathbb{R}^{m \times m}$ are real matrices. Then, the real Jacobian has the shape $J = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$. Therefore, applying the formula for the normal Jacobian in the Riemannian case we get: $\mathrm{NJ}(F, x) = \left|\det \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)\right| = |\det(A + iB)|^2$, which shows the assertion. □

2.3.1 Isometries and Riemannian submersions

An important class of smooth functions between manifolds in the context of integrals are *isometries*.

Definition 2.21. Let (M, g) and $(N, \tilde{\mathrm{g}})$ be Riemannian manifolds and let $F : M \to N$ a smooth map. The map F is an isometry, if it is a diffeomorphism and for all $x \in M$ and $v, w \in T_x M$ we have

$$\mathrm{g}(x)(v, w) = \tilde{\mathrm{g}}(F(x))(\mathrm{D}_x F(v), \mathrm{D}_x F(w));$$

i.e., at every point $x \in M$ the derivative $\mathrm{D}_x F$ is an isometry of Hilbert spaces. If there is an isometry $F : M \to N$, we say that M is isometric to N.

We have the following consequence of this definition.

Lemma 2.22. *Let (M, g) and $(N, \tilde{\mathrm{g}})$ be Riemannian manifolds and $F : M \to N$ be an isometry. Then:* $\mathrm{vol}(M) = \mathrm{vol}(N)$.

Proof. Since F is an isometry, we have $\mathrm{NJ}(F, x) = 1$ for all $x \in M$, because $\mathrm{D}_x F$ maps an orthonormal basis of $T_x M$ to an orthonormal basis of $T_{F(x)} N$. Furthermore, we have $\mathrm{vol}(F^{-1}(x)) = 1$ for all $w \in N$, because F is invertible. Then the coarea formula from Theorem 2.18 implies that $\mathrm{vol}(M) = \mathrm{vol}(N)$. □

Example 2.23. Consider again the unit circle S^1. We want to compute the volume of S^1 relative to the Riemannian metric that is given by defining the bilinear form on $T_x S^1 \cong x^\perp$ (see Example 2.4) to be the standard Euclidean inner product in $\mathbb{R}^2$ restricted to $x^\perp$. For this, we define $F : (0, 2\pi) \to S^1, t \mapsto (\cos(t), \sin(t))$, which is smooth. Then, the derivative of F at t is $\mathrm{D}_t F(s) = (-\sin(s), \cos(s))$. Since $(-\sin(s))^2 + \cos(s) = 1$, we see that F is an isometry of $(0, 2\pi)$ and $S^1 \setminus \{(1, 0)\}$. Since $\{(1, 0)\}$ is of measure zero, we have $\mathrm{vol}(S^1) = \mathrm{vol}(S^1 \setminus \{(1, 0)\})$ and, since F is an isometry, this implies $\mathrm{vol}(S^1) = \mathrm{vol}((0, 2\pi)) = 2\pi$.

A weaker property than being an isometry is being a Riemannian submersion.

Definition 2.24. Let (M, g) and $(N, \tilde{\mathrm{g}})$ be Riemannian manifolds. A smooth map $F : M \to N$ is called a Riemannian submersion, if for all $x \in M$ the differential $\mathrm{D}_x F$ is surjective and if we have $\mathrm{g}(x)(v, w) = \tilde{\mathrm{g}}(F(x))(\mathrm{D}_x F(v), \mathrm{D}_x F(w))$ for all $v, w \in (\ker \mathrm{D}_x F)^\perp$; i.e., at every point $x \in M$ the derivative $\mathrm{D}_x F$ when restricted to the orthogonal complement of its kernel is an isometry of Hilbert spaces.

Using essentially the same arguments as for the proof of Lemma 2.22 we get the following

Lemma 2.25. *Let (M, g) and $(N, \tilde{\mathrm{g}})$ be Riemannian manifolds and $F : M \to N$ an Riemannian submersion. Then, $\mathrm{vol}(M) = \int \mathrm{vol}(F^{-1}(y))\,\mathrm{d}y$. In particular, if the volume of the fibers is constant, that is, $\mathrm{vol}(F^{-1}(y)) = \mathrm{vol}(F^{-1}(y_0))$ for all $y \in N$ and a fixed $y_0 \in N$, then $\mathrm{vol}(M) = \mathrm{vol}(N) \cdot \mathrm{vol}(F^{-1}(y_0))$.*

2.4 Volume of the sphere and projective space

Probably the most important example of a manifold is the n-dimensional unit sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$. For us the sphere will be endowed with the Riemannian metric, which in turn is the restriction of the Euclidean structure on $\mathbb{R}^{n+1}$.

In Example 2.23 we computed its volume in the case $n = 1$ using the parametrization by polar coordinates, which turned out to be an isometry. For $n \geq 2$ we could still use polar coordinates, but the computations gets more involved. Instead, we will use another argument using the coarea formula.

Proposition 2.26. $\mathrm{vol}(S^n) = \dfrac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$.

Proof. The map $\phi : S^n \times \mathbb{R}_{>0} \to \mathbb{R}^{n+1} \backslash \{0\}, (s, r) \mapsto rs$ is smooth. Its derivative is $\mathrm{D}\phi(s, r)(\dot{s}, \dot{r}) = r\dot{s} + \dot{r}s$. Let $\dot{s}_1, \ldots, \dot{s}_n$ be a basis of the tangent space $T_s S^n \cong s^\perp$ (see (2.4)). Then, $\det \mathrm{D}\phi(s, r) = \det \begin{bmatrix} r\dot{s}_1 & \cdots & r\dot{s}_n & s \end{bmatrix} = r^n$.

Consider now the integral $\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x$ (since $\frac{1}{(\sqrt{2\pi})^{n+1}} e^{-\frac{1}{2}\|x\|^2}$ is the density of a standard Gaussian random variable in $\mathbb{R}^{n+1}$, the value of this integral is $\sqrt{2\pi}^{n+1}$, but let us prove it directly). The coarea formula (Theorem 2.18) implies

$$\begin{aligned} \int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x &= \int_{S^n \times \mathbb{R}_{>0}} e^{-\frac{1}{2}r^2} \det(\mathrm{D}\phi(s, r))\,\mathrm{d}s\mathrm{d}r \\ &= \int_{S^n \times \mathbb{R}_{>0}} e^{-\frac{1}{2}r^2} r^n\,\mathrm{d}s\mathrm{d}r \end{aligned}$$

By Tonelli's theorem, this is equal to

$$\int_{S^n} \mathrm{d}s \int_{\mathbb{R}_{>0}} e^{-\frac{1}{2}r^2}\, r^n \,\mathrm{d}r = \mathrm{vol}(S^n) \int_{\mathbb{R}_{>0}} e^{-t}\,(2t)^{\frac{n-1}{2}}\,\mathrm{d}t, t = \frac{r^2}{2}$$
$$= \sqrt{2}^{n-1}\,\mathrm{vol}(S^n)\,\Gamma(\tfrac{n+1}{2}).$$

Now, for $n = 1$ we know from Example 2.23 that $\mathrm{vol}(S^1) = 2\pi$. Moreover, we have $\Gamma(1) = 1$, so that $\int_{\mathbb{R}^2} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = 2\pi$. Since the exponential map is a group homomorphism from $(\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$, we have $\int_{\mathbb{R}^k} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = (\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,\mathrm{d}x)^k$, for every k, which implies that

$$\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = \sqrt{2\pi}^{n+1}.$$

We use this in the equation above to obtain the asserted formula. □

We will now introduce a Riemannian metric and consequently a Riemannian volume on the real projective space $\mathbb{R}\mathrm{P}^n$. In order to do this, observe first that the antipodal map $x \mapsto -x$ is an isometry for the Euclidean structure on the sphere. Therefore, the Riemannian metric g_{S^n} on S^n descend to a Riemannian metric $\mathrm{g}_{\mathbb{R}\mathrm{P}^n}$ on $\mathbb{R}\mathrm{P}^n$, meaning that we declare a metric on $\mathbb{R}\mathrm{P}^n$ that makes P a Riemannian submersion. With this metric we have that (S^n, g_{S^n}) and $(\mathbb{R}\mathrm{P}^n, \mathrm{g}_{\mathbb{R}\mathrm{P}^n})$ are locally isometric. For $x \in S^n$ we have $T_xS^n = T_{-x}S^n = x^\perp$, and we can identify

$$T_x\mathbb{R}\mathrm{P}^n \cong x^\perp.$$

The Riemannian metric on $\mathbb{R}\mathrm{P}^n$ is then given by

$$\mathrm{g}_{\mathbb{R}\mathrm{P}^n}(x)(v, w) := v^T w, \tag{2.7}$$

where $x \in \mathbb{R}\mathrm{P}^n$ and $v, w \in x^\perp$.

Let $P : S^n \to \mathbb{R}\mathrm{P}^n$ the projection that identifies antipodal points. Because the volume of the preimage $P^{-1}(x)$ is 2 for all $x \in \mathbb{R}\mathrm{P}^n$, Lemma 2.25 implies that the volume of a submanifold $X \hookrightarrow \mathbb{R}\mathrm{P}^n$ is given as $\mathrm{vol}(X) := \frac{1}{2}\mathrm{vol}(P^{-1}(X))$. In particular, the volume of the projective space with this metric is half the volume of the sphere:

$$\mathrm{vol}(\mathbb{R}\mathrm{P}^n) = \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}. \tag{2.8}$$

We make a similar construction for the complex projective space $\mathbb{C}\mathrm{P}^n$. The main difference being that the projection $P_{\mathbb{C}} : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ has positive dimensional fibers. Namely, $P_{\mathbb{C}}^{-1}(x) = \{\xi x_0 \mid |\xi| = 1\}$, where $P(x_0) = x$ The tangent space

at $x_0 \in S^{2n+1}$ is $T_{x_0}S^{2n+1} \cong x_0^\perp = \{y \in \mathbb{C}^{n+1} \mid \Re(x^T\overline{y}) = 0\}$. Note that $\Re(x^T\overline{y}) = 0$ if and only if $x^T\overline{y} \in i\mathbb{R}$. Then, we have $\Re(\xi x^T\overline{y}) = 0$ for all ξ with $|\xi| = 1$, if and only if $x^T\overline{y} = 0$. This yields

$$T_x\mathbb{C}\mathrm{P}^n \cong x^{\perp_{\mathbb{C}}} := \{y \in \mathbb{C}^{n+1} \mid x^T\overline{y} = 0\}.$$

The Hermitian metric on $\mathbb{C}\mathrm{P}^n$ is given by $\mathrm{h}(x)(v,w) = v^T\overline{w}$, where $x \in \mathbb{C}\mathrm{P}^n$ and $v, w \in x^{\perp_{\mathbb{C}}}$, and, consequently, the Riemannian metric on $\mathbb{C}\mathrm{P}^n$ is

$$\mathrm{g}(x)(v,w) = \Re(v^T\overline{w}). \tag{2.9}$$

Since the preimage $P_{\mathbb{C}}^{-1}(x)$ is a isometric to a circle, we have $\mathrm{vol}(P_{\mathbb{C}}^{-1}(x)) = 2\pi$ for all x. Together with Lemma 2.25 this implies that the volume of a submanifold $X \hookrightarrow \mathbb{C}\mathrm{P}^n$ is $\mathrm{vol}(X) := \frac{1}{2\pi}\mathrm{vol}(P_{\mathbb{C}}^{-1}(X))$. In particular, the volume of the projective space with this metric is:

$$\mathrm{vol}(\mathbb{C}\mathrm{P}^n) = \frac{\pi^n}{n!}, \tag{2.10}$$

where we have used that $\Gamma(n+1) = n!$.

2.5 Volume of the orthogonal group

The *orthogonal group* $O(n)$ is the group of matrices $Q \in \mathbb{R}^{n\times n}$ such that $Q^TQ = \mathbf{1}$. This is a system of $\frac{n(n+1)}{2}$ polynomials in n^2 many variables. Let $Q \in O(n)$. The kernel of the Jacobian matrix $J(Q)$ of this system of equations is

$$\ker J(Q) = \{R \in \mathbb{R}^{n\times n} \mid Q^TR + R^TQ = 0\}, \tag{2.11}$$

so $\dim\ker J(Q) = \frac{n(n-1)}{2}$. We can therefore apply Corollary 2.8 to deduce that $O(n)$ is a smooth manifold of dimension

$$\dim O(n) = \frac{n(n-1)}{2}.$$

We consider the orthogonal group as a Riemannian manifold endowed with the Euclidean structure $\mathrm{g}(Q)(R_1,R_2) = \frac{1}{2}\mathrm{tr}(R_1^TR_2)$. This is called the *Frobenius* scalar product. The next proposition gives the volume of $O(n)$.

Proposition 2.27. $\mathrm{vol}(O(n)) = \prod_{k=0}^{n-1}\mathrm{vol}(S^k)$.

Proof. Consider the smooth surjective map $F : O(n) \to S^{n-1}$ that maps $Q \in O(n)$ to its first column Qe_1, where $e_1 = (1,0,\ldots,0) \in \mathbb{R}^n$. We compute the normal

Jacobian of F. Using Corollary 2.8 and (2.11) we see that the geometric tangent space of the orthogonal group at $Q \in O(n)$ is given by

$$T_Q O(n) \cong \{R \in \mathbb{R}^{n\times n} \mid Q^T R + R^T Q = 0\}.$$

The derivative of F is then $\mathrm{D}_Q F(R) = Re_1$ for $R \in T_Q O(n)$. Let $E_{i,j} \in \mathbb{R}^{n\times n}$ be the matrix that has a 1 as the (i,j)-th entry, a -1 in the (j,i)-th entry and zeros elsewhere. Then, $\{QE_{i,j} \mid 1 \le i < j \le n\}$ is an orthonormal basis for $T_Q O(n)$. The inner product between $\mathrm{D}_Q F(QE_{i,j})$ and $\mathrm{D}_Q F(QE_{k,\ell})$ is then

$$(QE_{i,j}e_1)^T(QE_{k,\ell}e_1) = (E_{i,j}e_1)^T E_{k,\ell}e_1 = \begin{cases} 1, & \text{if } k = i = 1 \text{ and } j = \ell \\ 0, & \text{otherwise} \end{cases}$$

Hence, $\mathrm{D}_Q F$ maps an orthonormal basis of $(\ker \mathrm{D}_Q F)^\perp$ to an orthonormal basis of $(Qe_1)^\perp$, which shows that $\mathrm{NJ}(F,Q) = 1$; i.e., F is a Riemannian submersion. Moreover, $F^{-1}(Qe_1) = QF^{-1}(e_1) = \{Q\left[\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right] \mid R \in O(n-1)\}$, so that $\mathrm{vol}(F^{-1}(q)) = \mathrm{vol}(O(n-1))$ for all $q \in S^{n-1}$. We can now use Lemma 2.25 to deduce that

$$\mathrm{vol}(O(n)) = \mathrm{vol}(S^{n-1})\mathrm{vol}(O(n-1)).$$

Induction on n proves the assertion. □

The unitary group consists of matrices $Q \in \mathbb{C}^{n\times n}$ such that $Q\overline{Q}^T = \mathbf{1}$; the Riemannian metric on $U(n)$ is obtained by restricting to it the metric on $\mathbb{C}^{n\times n}$ given by the Euclidean structure $\mathrm{g}(Q)(A_1, A_2) = \mathfrak{Re}(\frac{1}{2}\mathrm{tr}(A_1^T \overline{A}_2))$. Arguing as for the orthogonal group, we can show that the complex dimension of $U(n)$ is

$$\dim_{\mathbb{C}} U(n) = \frac{n(n-1)}{2}, \tag{2.12}$$

and that its volume is

$$\mathrm{vol}(U(n)) = \prod_{k=0}^{n-1} \mathrm{vol}(S^{2k+1}). \tag{2.13}$$

We will come back to these formulas in next section, when we discuss volumes of Riemannian homogenous spaces.

3 Riemannian homogeneous spaces

A situation that we will often encounter in these lectures is when a manifold is a *homogeneous space*. In the first two sections of this chapter we recall the basic definitions and properties of Lie groups and homogeneous spaces. For a more detailed treatment of this subject we refer to [13, Section 7 & Section 21]. We start with the definition of Lie groups.

3.1 Lie groups

A *Lie group* G is a smooth manifold that is also a group, such that the multiplication $\mathrm{mul} : G \times G \to G, (g, h) \mapsto gh$ and the inversion $i : G \to G, g \mapsto g^{-1}$ are smooth maps. Let $g \in G$. We define the left- and right-translation of g to be the maps

$$L_g : G \to G, h \mapsto gh, \quad \text{and} \quad R_g : G \to G, h \mapsto hg.$$

As $L_g = \mathrm{mul} \circ (h \mapsto (g, h))$ is a composition of smooth maps it is smooth. Furthermore, L_g has the smooth inverse $L_{g^{-1}}$, so that L_g is in fact a diffeomorphism. Similarly, R_g is an isomorphism.

Example 3.1. Examples of Lie groups are $\mathbb{R}^n$ with the smooth Euclidean structure and vector addition as group operation, the general linear group $\mathrm{GL}(n, \mathbb{R})$ with the smooth structure inherited from $\mathbb{R}^{n \times n}$ with matrix multiplication as group operation, and the orthogonal group $O(n)$ as a submanifold of $\mathrm{GL}(n, \mathbb{R})$. Similarly, $\mathrm{GL}(n, \mathbb{C})$ and the unitary group $U(n)$ are Lie groups.

Let G be a Lie group of dimension m, and let $e \in G$ be the identity element. We denote the tangent space of G at e by

$$\mathfrak{g} := T_e G.$$

For every $g \in G$, left-translation is a diffeomorphism, which implies that the derivative $\mathrm{D}_e L_g$ is invertible. As a consequence we have $T_g G = \mathrm{D}_e L_g(\mathfrak{g}) = (L_g)_*(\mathfrak{g})$ (using the push-forward notation $(L_g)_*$ for the derivative; see Definition 2.6). This implies

$$T_g G = (L_{gh^{-1}})_*(T_h G), \quad \text{for } g, h \in G. \tag{3.1}$$

Remark 3.2. The vector space $\mathfrak{g}$ is called the *Lie-Algebra* of G. It is an algebra with respect to the Lie bracket coming from G. But we will not need the algebra structure here. For more details the reader is referred to [13, Section 7].

Next, we discuss how to define a Riemannian structure on a Lie group G of dimension m. For this, we choose any basis $\{u_1, \dots, u_n\}$ of $\mathfrak{g}$. We define an inner product on $\mathfrak{g}$ by desclaring this basis to be orthonormal:

$$\mathrm{g}(e)\Big(\sum_{i=1}^{m} \lambda_i u_i, \sum_{i=1}^{m} \mu_i u_i\Big) := \sum_{i=1}^{m} \lambda_i \mu_i.$$

This defines a Riemannian metric g on G in the following way: for $g \in G$ we set

$$\mathrm{g}(g)(v, w) := \mathrm{g}(e)\big((L_{g^{-1}})_*(v), (L_{g^{-1}})_*(w)\big), \text{ for } v, w \in T_g G,$$

which is well-defined by (3.1). By construction, the Riemannian is *left-invariant*, so it defines a *left-invariant measure* on G by Definition 2.14. With respect to this metric any left-translation $L_g : G \to G$ is an isometry.

The next result states that, despite making a choice of basis above, the resulting measure is unique up to scaling.

Theorem 3.3. *There is a unique left-invariant measure on G up to scaling.*

Proof. The existence of such a measure is given by the construction above. For uniqueness[1] let μ and ν be two left-invariant measures on G. We have for any measurable set $U \subset G$ that $\mu(U) + \nu(U) = 0$ implies $\mu(U) = 0$. Hence, μ is absolutely continuous with respect to $\mu + \nu$. The Radon-Nikodym theorem (see, e.g., [7, Section 23]) implies that there exists a measurable function $\phi : G \to \mathbb{R}$ with $\mu = \phi\,(\mu+\nu)$. We show that ϕ is constant $(\mu+\nu)$–almost everywhere: for any measurable subset $W \subset G$ and every $g \in G$ we have $\mu(L_g(W)) = \int_{L_g(W)} \phi \,\mathrm{d}(\mu + \nu) = \int_W (\phi \circ L_g)\,\mathrm{d}(\mu + \nu)$, by left-invariance of $\mu + \nu$. On the other hand, by left-invariance of μ, we have $\mu(L_g(W)) = \mu(W) = \int_W \phi\,\mathrm{d}(\mu + \nu)$, and so

$$\int_W \big((\phi \circ L_g) - \phi\big)\,\mathrm{d}(\mu + \nu) = 0,$$

which implies that $(\phi \circ L_g) - \phi = 0$ almost everywhere. □

A consequence of this theorem is that there is a *unique* left-invariant probability measure on G (if it exists).

[1] We use a proof idea by `StackOverflow` user *YCor*.

Corollary 3.4. *Let G be a Lie group. Then, there is a unique left-invariant probability measure on G (if it exists).*

Proof. Let G be a Lie group and $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be two left-invariant probability measures on G. By Theorem 3.3, they are multiples of each other, so there exists $c \neq 0$ with $\mathbb{P} = c \cdot \tilde{\mathbb{P}}$. This implies $1 = \mathbb{P}(G) = c \cdot \tilde{\mathbb{P}}(G) = c$, so $\mathbb{P} = \tilde{\mathbb{P}}$. □

3.2 Volumes of homogeneous spaces

Let G be a Lie group and M be a smooth manifold. We say that G *acts* on M, if we have a group action $G \times M \to M$, $(g, x) \mapsto g \cdot x$ that is smooth and transitive, meaning that for all $x, y \in M$ we can find $g \in G$ such that $g \cdot x = y$. In this case, we say that M is a *homogeneous space*.

If M is a Riemannian manifold and G is endowed with a left-invariant metric, and for every $g \in G$ the map $x \mapsto g.x$ is an isometry, we say that G acts on M isometrically and we call M a *Riemannian homogeneous space*.

Here are some examples of homogeneous spaces.

Example 3.5. The orthogonal group $O(n)$ acts isometrically on the sphere S^{n-1} and on projective space $\mathbb{R}\mathrm{P}^{n-1}$. The unitary group $U(n)$ acts isometrically on the sphere S^{2n-1} and on complex projective space $\mathbb{C}\mathrm{P}^{n-1}$. In general, every Lie group acts isometrically on itself.

The next two results imply that quotients of Lie groups completely classify homogeneous spaces. The first is [13, Theorem 21.17]

Theorem 3.6. *Let G be a Lie group and let $H \subset G$ be a closed subgroup. The left coset space G/H is a topological manifold of dimension equal to* $\dim(G) - \dim(H)$*, and it has a unique smooth structure such that the quotient map $\pi : G \mapsto G/H$ is a smooth submersion. The quotien G/H us a homogeneous space under the left action of G on G/H.*

In the following we will always assume that the quotient space is endowed with the unique smooth structure from Theorem 3.6. In fact, all homogeneous spaces arise in this way, as next Theorem says. To state the theorem we need to define for $p \in M$ the *isotropy group*

$$G_p := \{g \in G \mid g \cdot p = p\}.$$

For a proof of the following result see [13, Theorem 21.18].

Theorem 3.7. *Let G be a Lie group, M be a homogeneous G-space and $p \in M$. The isotropy group G_p is a closed subgroup of G, and*

$$\phi_p : G/G_p \to M, \quad gG_p \mapsto g \cdot p$$

is an equivariant diffeomorphism.

A natural way to build *Riemannian* homogeneous G-spaces is to start with a Lie group G endowed with a left invariant Riemannian metric, which is also *right invariant* under the action of a compact subgroup H.

Proposition 3.8. *Let G be a Lie group with a left-invariant Riemannian metric which is right invariant for a compact a subgroup H. This metric induces a unique Riemannian metric on the homogeneous space G/H that makes the quotient map $\pi : G \to G/H$ a Riemannian submersion.*

Proof. The required metric is built as follows. Given an element $p \in G/H$, chose an element $g \in G$ such that $\pi(g) = p$. Then $D_g\pi|_{(\ker D_g\pi)^\perp} : (\ker D_g\pi)^\perp \to T_p(G/H)$ is a linear isomorphism and there is a unique way to declare it to be a Euclidean isometry. We show that the induced metric on $T_p(G/H)$ does not depend on the choice of g such that $\pi(g) = p$, so that we get a well-defined Riemannian submersion.

The invariance of the metric on G under the action of H induces an isometry by right-translation $R_h : G \to G$ for every $h \in H$. For $g, g' \in G$ such that $g' = gh$ we then have an Euclidean isometry

$$(R_h)_* : T_gG \to T_{g'}G.$$

Let $v \in \ker D_g\pi$. We show that $(R_h)_*(v) \in \ker D_{g'}\pi$. This would imply that $(R_h)_*$ maps $(\ker D_g\pi)^\perp$ isometrically to $(\ker D_{g'}\pi)^\perp$, so that the induced metric on $T_p(G/H)$ does not depend on the choice of g. To see this, we take $f \in C^\infty(G/H, \mathbb{R})$. Then:

$$D_{g'}\pi((R_h)_*(v))(f) = v(f \circ \pi \circ R_h).$$

By construction, we have $f \circ \pi \circ R_h = f \circ \pi$. Moreover, $v(f \circ \pi) = D_g\pi(v)(f) = 0$, because $v \in \ker D_g\pi$. This shows that $(R_h)_*(v) \in \ker D_{g'}\pi$. □

Definition 3.9. We call the metric induced on G/H as in Proposition 3.8 the *quotient metric*.

Observe that, in this situation G/H with the quotient metric is a Riemannian homogeneous space and we have the following useful result.

Theorem 3.10. *Let G be a Riemannian Lie group, endowed with a left-invariant metric which is also right-invariant under the compact subgroup H. Endow G/H with the quotient metric. Then:*

$$\operatorname{vol}(G/H) = \frac{\operatorname{vol}(G)}{\operatorname{vol}(H)}.$$

Here, the volume of H is the one induced by restricting the Riemannian metric to H and then taking the corresponding Riemannian measure.

Proof. The quotient map $\pi : G \mapsto G/H$ is a Riemannian submersion. Lemma 2.25 implies that $\operatorname{vol}(G) = \int_{w \in G/H} \operatorname{vol}(\pi^{-1}(w))\, \mathrm{d}w$, where $\mathrm{d}w$ denotes the integration with respect to the Riemannian measure of G/H. Because G acts on itself by isometries, for all $g \in G$ we have that the map $h \mapsto gh$ from H to gH is an isometry of submanifolds of G with the induced Riemannian metric and consequently, if $\pi(g) = w$: $\operatorname{vol}(\pi^{-1}(w)) = \operatorname{vol}(gH) = \operatorname{vol}(H)$. This finishes the proof. □

We can use Theorem 3.10 to compute the volumes of the orthogonal $O(n)$ and the unitary group $U(n)$ (recall that we computed the volume of $O(n)$ already in Section 2.5). The orthogonal group acts on S^n by rotations. The isotropy group $O(n)_x$ of $x \in S^{n-1}$ is the orthogonal group that rotates the orthogonal complement $e_1^\perp$, so it is isometric to $O(n-1)$. Theorem 3.7 implies that S^{n-1} is diffeomorphic to $O(n)/O(n-1)$. Thus, if we endow S^{n-1} with the quotient structure, then

$$\operatorname{vol}(S^{n-1}) = \frac{\operatorname{vol}(O(n))}{\operatorname{vol}(O(n-1))},$$

by Theorem 3.10. In fact, we have shown in the proof of Proposition 2.27 that the quotient structure agrees with the Euclidean metric on S^{n-1} inherited from $\mathbb{R}^n$. This is why we get the same formula. The orthogonal group also acts on $\mathbb{R}\mathrm{P}^{n-1}$, but with isotropy group $\mathbb{Z}_2 \times O(n-1)$, so that

$$\operatorname{vol}(\mathbb{R}\mathrm{P}^{n-1}) = \frac{\operatorname{vol}(O(n))}{\operatorname{vol}(\mathbb{Z}_2) \cdot \operatorname{vol}(O(n-1))} = \tfrac{1}{2}\operatorname{vol}(S^{n-1}),$$

which agrees with (2.8). With the same argumentation we have that

$$\operatorname{vol}(S^{2n-1}) = \frac{\operatorname{vol}(U(n))}{\operatorname{vol}(U(n-1)},$$

because $U(n)$ acts on S^{2n-1} seen as the complex sphere of points $x \in \mathbb{C}^n$ with $x^*x = 1$ and has isotropy group $U(n-1)$. This shows (2.13). Furthermore, $U(n)$

acts on complex projective space $\mathbb{C}\mathrm{P}^{2n-1}$ with isotropy group $S^1 \times U(n-1)$, so that Theorem 3.10 implies

$$\mathrm{vol}(\mathbb{C}\mathrm{P}^{n-1}) = \frac{\mathrm{vol}(U(n))}{\mathrm{vol}(S^1)\cdot\mathrm{vol}(U(n-1))} = \frac{1}{2\pi}\mathrm{vol}(S^{2n-1}),$$

which agrees with (2.10).

3.2.1 The Grassmannian

We compute the volume of the *Grassmannian*. This is the space

$$G(k,n) := \{L \subset \mathbb{R}^n \mid L \text{ is a linear space of dimension } k\}$$

of all k-dimensional linear spaces in $\mathbb{R}^n$. Similarly,

$$G_{\mathbb{C}}(k,n) := \{L \subset \mathbb{C}^n \mid L \text{ is a complex linear space of dimension } k\}.$$

Both $G(k,n)$ and $G_{\mathbb{C}}(k,n)$ are homogeneous spaces in the following way. The orthogonal group acts on $G(k,n)$ by $Q \cdot L := \{Q\ell \mid \ell \in L\}$ for $Q \in O(n)$ and $L \in G(k,n)$. The isotropy group of $L_0 = \mathrm{span}\{e_1,\ldots,e_k\}$, where e_i is the i-th standard basis vector in $\mathbb{R}^n$ is $G_{L_0} = O(k)\times O(n-k)$; so that we have a bijective map $O(n)/(O(k)\times O(n-k)) \to G(k,n)$. We *define* a Riemannian manifold structure on $G(k,n)$ by declaring this map to be an isometry. We proceed similarly for the complex Grassmannian. With these identifications:

$$G(k,n) \cong O(n)/(O(k)\times O(n-k)), \quad G_{\mathbb{C}}(k,n) \cong U(n)/(U(k)\times U(n-k)) \tag{3.2}$$

We now have the following theorem.

Theorem 3.11. *On the Grassmannian we put the metrics from the identifcations* (3.2)*. With these structures:*

$$\mathrm{vol}(G(k,n)) = \frac{\mathrm{vol}(O(n))}{\mathrm{vol}(O(k))\cdot\mathrm{vol}(O(n-k))}, \quad \textit{and}$$

$$\mathrm{vol}(G_{\mathbb{C}}(k,n)) = \frac{\mathrm{vol}(U(n))}{\mathrm{vol}(U(k))\cdot\mathrm{vol}(U(n-k))}.$$

Proof. This follows immediately from Theorem 3.10. □

4 The Kac-Rice formula

In this chapter we address the basic problem of counting the number of solutions of a random equation – many interesting questions from geometry to topology can be reduced to a problem where we have to count points.

To be more specific, suppose we are given a *random map*

$$f : \mathbb{R}^m \to \mathbb{R}^k$$

with $m \geq k$. This means we have a random vector $f(x) \in \mathbb{R}^k$ for every $x \in \mathbb{R}^m$. In the literature, f is also called a *random field*. In the case $m = k$ we will be interested in computing $\mathbb{E}\,\#\left(\{f = 0\} \cap A\right)$, where $A \subseteq \mathbb{R}^k$ is a measurable subset. More generally, we compute the $(m-k)$-dimensional volume $\mathbb{E}\,\mathrm{vol}\left(\{f = 0\} \cap A\right)$. For an example of a random map we can take $f : \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{i=0}^{d} \xi_i x^i$, where $\{\xi_i\}_{i=0,\dots,d}$ is a family of independent standard gaussians. This defines a random polynomial called a *Kac polynomial* (see also Definition 4.7 below). Another random map is given by taking the dehomogenization of a Kostlan polynomial $\sum_{i=0}^{d} \xi_i \binom{d}{i}^{\frac{1}{2}} x^i$ from (1.7). We can also consider the case where f is given by a complex random polynomial, using the identification of real vector spaces $\mathbb{R}^2 \cong \mathbb{C}$.

As before, $C^\infty(\mathbb{R}^m, \mathbb{R}^k)$ denotes the space of smooth function $\mathbb{R}^m \to \mathbb{R}^k$. Following the discussion in Section 1.3, we define a random map to be an element of some finite-dimensional Gaussian space of smooth functions. The motivates the next definition.

Definition 4.1 (Random Gaussian maps)**.** Let $\mathcal{F} = \{f_1, \dots, f_\ell\} \subset C^\infty(\mathbb{R}^m, \mathbb{R}^k)$ be finite with $\ell, m \geq k$. The random Gaussian map induced by $\mathcal{F}$ is

$$f(x) = \xi_1 f_1(x) + \cdots + \xi_\ell f_\ell(x),$$

where $\{\xi_i\}_{i=1,\dots,\ell}$ is a family of i.i.d. $N(0,1)$ random variables.

Let $V := \mathrm{span}(\{f_1, \dots, f_\ell\})$ be the vector space spanned by the elements in $\mathcal{F}$. If $f_1, \dots, f_\ell$ are linearly independent, we get a linear isomorphism

$$\varphi_{\mathcal{F}} : \mathbb{R}^\ell \to V,\ y \mapsto \sum_{i=1}^{\ell} y_i f_i(x). \tag{4.1}$$

Following Definition 1.8, we get a Gaussian distribution on V in the following way:

$$\mathbb{P}(f \in U) = \frac{1}{\sqrt{2\pi}^{\ell}} \int_{\varphi_{\mathcal{F}}^{-1}(U)} e^{-\frac{\|y\|^2}{2}} \, \mathrm{d}y, \text{ where } U \subset V \text{ is measurable.} \tag{4.2}$$

For example, in the case of Kac polynomials we have $\mathcal{F} = \{1, x, \ldots, x^d\}$. For defining a complex Kac polynomials we would choose $\mathcal{F} = \{a_0, \ldots, a_d, b_0, \ldots, b_d\}$ where $a_k(x,y) = \frac{1}{2}\begin{pmatrix} \Re(x+iy)^k \\ 0 \end{pmatrix}$ and $b_k(x,y) = \frac{1}{2}\begin{pmatrix} 0 \\ \Im(x+iy)^k \end{pmatrix}$.

4.1 The Kac-Rice formula

We will now establish the framework, in which we can compute the expected volume of zeros of random maps.

For this, we assume that $f : \mathbb{R}^m \to \mathbb{R}^k$ is a random map induced by a set of smooth functions $\mathcal{F} = \{f_1, \ldots, f_\ell\}$, in the sense of Definition 4.1. Here we assume that $f(x)$ has a nondegenerate distribution, i.e. that for almost all $x \in \mathbb{R}^m$ the covariance matrix is positive definite:

$$\mathbb{E}\, f(x) f(x)^T \succ 0 \tag{4.3}$$

Furthermore, we let $Jf(x) \in \mathbb{R}^{k \times m}$ be the jacobian matrix of f at x. The normal Jacobian is $\mathrm{NJ}(f,x) = \sqrt{\det(Jf(x)Jf(x)^T)}$. The Kac-Rice density for $\mathcal{F}$ is given in terms of the expected value of the normal Jacobian.

Definition 4.2 (The Kac-Rice density)**.** The Kac-Rice density of $\mathcal{F}$ at x is

$$\rho(x) = \mathbb{E}\left[\mathrm{NJ}(f,x) \;\middle|\; f(x) = 0 \right] \cdot \phi_{f(x)}(0),$$

where $\phi_{f(x)}(0)$ is the density of the random vector $f(x)$ evaluated at zero.

For every $x \in \mathbb{R}^m$ the vector-matrix pair $(f(x), Jf(x)) \in \mathbb{R}^m \times \mathbb{R}^{k \times m}$ is Gaussian. The Gaussian regression formula from [1, Proposition 1.2] gives way for computing the conditional expectation in Definition 4.2.

Proposition 4.3. *Let $(\xi, \eta) \in \mathbb{R}^r \times \mathbb{R}^s$ be a centered Gaussian vector such that the covariance matrix of ξ is nondegenerate: $A = \mathbb{E}\,\xi\xi^T \succ 0$. Let us also define the matrices: $B = \mathbb{E}\,\eta\eta^T$ and $C = \mathbb{E}\,\eta\xi^T$. Then for every measurable function $h : \mathbb{R}^s \to \mathbb{R}$ we have:*

$$\mathbb{E}\left\{ h(\eta) \,\middle|\, \xi = 0 \right\} = \mathbb{E}\, h(\zeta),$$

where $\zeta \in \mathbb{R}^s$ is a Gaussian vector with mean zero and covariance $B - CA^{-1}C^T$.

In the case when the pair $(f(x), Jf(x))$ itself has a nondegenerate distribution we can give an alternative formulation for the Kac-Rice density. Let us denote by $p : \mathbb{R}^k \times \mathbb{R}^{k\times m} \times \mathbb{R}^m \to \mathbb{R}$ the joint density of the pair $(f(x), Jf(x))$. This means that p is the function defined by the requirement that for every measurable subset $A \subseteq \mathbb{R}^k \times \mathbb{R}^{k\times m}$ we have for a fixed $x \in \mathbb{R}^m$

$$\mathbb{P}\left((f(x), Jf(x)) \in A\right) = \int_A p(v, M, x)\,\mathrm{d}v\mathrm{d}M.$$

The Kac-Rice density for $\mathcal{F}$ in this case is given by

$$\rho(x) = \int_{\mathbb{R}^{k\times m}} \sqrt{\det(MM^T)}\, p(0, M, x)\,\mathrm{d}M. \tag{4.4}$$

Now comes the Kac-Rice formula. We prove a special case of the more general formula in [1, Theorem 6.2].

Theorem 4.4 (Kac-Rice formula)**.** *Let $\ell, m \geq k$ and let $f : \mathbb{R}^m \to \mathbb{R}^k$ be the random map induced by $\mathcal{F} \subset C^\infty(\mathbb{R}^m, \mathbb{R}^k)$, $\#\mathcal{F} = \ell$. Assume that for almost all $x \in \mathbb{R}^m$ we have*

1. *$f(x)$ has a nondegenerate distribution;*
2. *the probability of $\det(Jf(x)Jf(x)^T) = 0$ conditioned on the event $f(x) = 0$ is equal to zero; i.e., $\mathbb{P}\{\det(Jf(x)Jf(x)^T) = 0 \mid f(x) = 0\} = 0$.*

Then, almost surely the zero set of f is an $(m-k)$-dimensional smooth manifold. The volume of the zeros of f in a measurable set $A \subseteq \mathbb{R}^m$ is given by the formula:

$$\mathbb{E}\ \mathrm{vol}_{m-k}(\{x \in A \mid f(x) = 0\}) = \int_A \rho(x)\,\mathrm{d}x.$$

In particular, if $m = k$, we have $\mathbb{E}\ \#\{x \in A \mid f(x) = 0\} = \int_A \rho(x)\,\mathrm{d}x$.

The theorem shows the role of the Kac-Rice density $\rho(x)$: it is the density of the zeros of f. We thus call $\rho(x)$ also *root density*. This theorem is actually a special case of a more general result, which we state next.

Theorem 4.5. *Let $h : \mathbb{R}^m \to \mathbb{R}$ be a measurable function. Under the assumptions from Theorem 4.4 the zero set $Z(f) = \{x \in \mathbb{R}^m \mid f(x) = 0\}$ is almost surely an $(m-k)$-dimensional smooth submanifold of $\mathbb{R}^m$ and we have*

$$\mathbb{E} \int_{Z(f)\cap A} h(x)\,\mathrm{d}x = \int_A h(x)\rho(x)\,\mathrm{d}x.$$

Before we prove Theorem 4.5, let us recall the next result on multivariate Gaussians, which shows how they behave under linear transformations; for a proof see, e.g., [16, Theorem 1.2.6].

Lemma 4.6. *Let $X = (X_1, \dots, X_n)$ be Gaussian with mean μ and covariance matrix Σ, and let $A \in \mathbb{R}^{m\times n}$. Then, $AX \sim N(A\mu, A\Sigma A^T)$.*

Now, we prove Theorem 4.4.

Proof of Theorem 4.5. Let $W \subset \mathbb{R}^\ell \times \mathbb{R}^m$ be the subset of points (y, x) such that $\det(\mathbb{E}\, f(x)f(x)^T) \neq 0$, where $f = \varphi_{\mathcal{F}}(y)$ is as in (4.1). By assumption, W is an open dense subset. We define the incidence correspondence

$$Z := \{(y, x) \in W \mid f(x) = 0, \text{ where } f = \varphi_{\mathcal{F}}(y)\}.$$

If Z is empty, then $f(x) = 0$ has no solution in $\mathbb{R}^m$. In this case, the Kac-Rice formula is trivially true. In the following, we assume that $Z \neq \emptyset$. Let $Jf(x)$ be the Jacobian matrix of f at x. We use the following shorthand notations:

$$J := Jf(x) \quad \text{and} \quad M = \begin{bmatrix} f_1(x) & \cdots & f_\ell(x) \end{bmatrix} \in \mathbb{R}^{k\times \ell}.$$

We show that Z is a manifold. To see this, we define $g : W \to \mathbb{R}^n, (y, x) \mapsto f(x)$. This map has Jacobian matrix $Jg(y, x) = \begin{bmatrix} M & J \end{bmatrix} \in \mathbb{R}^{m\times(\ell+m)}$. We observe that $MM^T = \mathbb{E}\, Myy^TM = \mathbb{E}\, f(x)f(x)^T \succ 0$, so M has rank k, which shows that 0 is a regular value of g. Proposition 2.7 implies that $Z = g^{-1}(0)$ is a smooth manifold of dimension $\ell + m - k$ with tangent space

$$T_{(y,x)}Z = \left\{(\dot y, \dot x) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid M\dot y = -J\dot x\right\}.$$

Let us define the coordinate projections $\pi_1 : Z \to \mathbb{R}^\ell$ and $\pi_2 : Z \to \mathbb{R}^k$. Since M has rank k, π_2 is a submersion. By [13, Proposition 4.28], submersions are open maps, hence $\pi_2(Z)$ is open in $\mathbb{R}^k$. If moreover JJ^T is invertible at (y, x), the derivative of π_1 at (y, x) is of rank ℓ, so that the preimage $\pi_1^{-1}(y)$ is a smooth manifold of dimension $m - k$ by Proposition 2.7. This holds almost surely due to our assumption $\mathbb{P}\{\det(JJ^T) = 0 \mid f(x) = 0\} = 0$. We can apply the coarea formula (Theorem 2.18) first to π_1 and then to π_2 we get

$$\begin{aligned}
&\mathbb{E} \int_{Z(f)\cap A} h(x)\, \mathrm{d}x \\
&= \int_{\mathbb{R}^\ell} \Big(\int_{\pi_1^{-1}(y)} \chi_A(x)\, h(x)\, \mathrm{d}x \Big) \frac{1}{\sqrt{2\pi}^\ell} e^{-\frac{\|y\|^2}{2}}\, \mathrm{d}y && (4.5)\\
&= \int_{\mathbb{R}^m} \chi_A(x)\, h(x) \Big(\int_{\pi_2^{-1}(x)} \frac{\mathrm{NJ}(\pi_1, (y, x))}{\mathrm{NJ}(\pi_2, (y, x))} \frac{1}{\sqrt{2\pi}^\ell} e^{-\frac{\|y\|^2}{2}}\, \mathrm{d}(y, x) \Big)\, \mathrm{d}x, && (4.6)
\end{aligned}$$

where χ_A is the indicator function of A. We have to show that the inner integral equals $\rho(x)$. First, we compute the ratio of normal jacobians. For this, we fix $(y, x) \in Z$ such that JJ^T is invertible (recall that this holds for almost everywhere). We define the subspace $K := T_{(y,x)}Z \cap ((\ker M)^\perp \times (\ker J)^\perp)$ and denote the coordinate projections by $p_1 : K \to \mathbb{R}^\ell$ and $p_2 : K \to \mathbb{R}^m$. We have an orthogonal decomposition

$$T_{(y,x)}Z = \ker M \oplus \ker J \oplus K.$$

If $(u, v, w) \in \ker M \oplus \ker J \oplus K$ is a decomposition of a point in $T_{(y,x)}Z$ relative to this decomposition, its images under the derivatives of π_1 and π_2 are $\mathrm{D}_{(y,x)}\pi_1(u, v, w) = (u, p_1(w))$ and $\mathrm{D}_{(y,x)}\pi_2(u, v, w) = (v, p_2(w))$. Furthermore, for $w = (w_1, w_2) \in K$ we have that $w_1 = -(M|_{(\ker M)^\perp})^{-1}(J|_{(\ker J)^\perp})(w_2)$. This implies that both p_1 and p_2 are invertible and $\det(p_1 \circ p_2^{-1}) = \det((M|_{(\ker M)^\perp})^{-1}(J|_{(\ker J)^\perp}))$. We get

$$\begin{aligned}\frac{\mathrm{NJ}(\pi_1, (y, x))}{\mathrm{NJ}(\pi_2, (y, x))} = \frac{|\det(p_1)|}{|\det(p_2)|} &= |\det(p_1 \circ p_2^{-1})| \\ &= \det((M|_{(\ker M)^\perp})^{-1}(J|_{(\ker J)^\perp})) \\ &= \frac{\sqrt{\det(JJ^T)}}{\sqrt{\det(MM^T)}}.\end{aligned}$$

Next, we observe that for a fixed $x \in \mathbb{R}^m$ we have $\pi^{-1}(x) = \ker M \times \{x\}$. Since M is of full-rank, the dimension of $\ker M$ is $\ell - k$. By rotational invariance of the Gaussian distribution, we can assume that $\ker M = \mathrm{span}\{e_{k+1}, \ldots, e_\ell\}$, so that $M = [\, M' \; 0\,]$, where $M' \in \mathbb{R}^{k\times k}$ is invertible. In this case, $y = (y_1, \ldots, y_\ell) \in \ker M$ if and only if $y_1 = \cdots = y_k = 0$. Let us write $z := (y_{k+1}, \ldots, y_\ell)$. The inner integral in (4.6) becomes

$$\frac{1}{\sqrt{(2\pi)^k \det(MM^T)}} \int_{\mathbb{R}^{\ell-k}} \sqrt{\det(JJ^T)} \cdot \frac{1}{\sqrt{(2\pi)^{\ell-k}}} e^{-\frac{\|z\|^2}{2}} \, \mathrm{d}z.$$

The integral in this equation is $\mathbb{E}[\, \sqrt{\det(JJ^T)} \,|\, f(x) = 0 \,]$, and we have that $\sqrt{\det(JJ^T)} = \mathrm{NJ}(f, x)$. Moreover, $\phi_{f(x)}(0) = ((2\pi)^k \det(MM^T))^{-\frac{1}{2}}$ by Lemma 4.6 and the fact that $f(x) = My$. This shows the inner integral in (4.6) is $\rho(x)$. □

4.2 Root density of Kac polynomials

In the first chapter, specifically in Subsection 1.3.1, we discussed that Kostlan polynomials are a reasonable class of random maps. Another class are Kac polynomials as defined at the beginning of this chapter. Let us recall their definition.

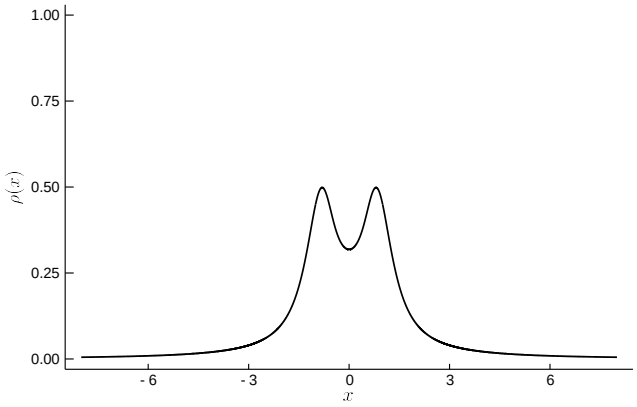


Figure 4.1: The root density $\rho(x)$ of Kac polynomials of degree $d = 4$. The two peaks are at $x = -1$ and $x = 1$. This means that a root of a Kac polynomial is most likely close to -1 or 1.

Definition 4.7 (Kac polynomials)**.** Let $f = \sum_{i=0}^{d} \xi_i \, x^i$ be a univariate polnomial of degree d. We call f a Kac polynomial, if the ξ_i are i.i.d. $N(0, 1)$ random variables.

However, Kac polynomials are not entirely reasonable in the sense of Section 1.3. We observe that Kac polynomials are only invariant under the symmetry $f \mapsto -f$, and in fact, as one can see from the plot of the root density of these polynomials in Figure 4.1, the zeroes are more likely to be near the points $\pm 1 \in \mathbb{R}$ (hence there are privileged points for this model of randomness). In particular, for $U = [a, b]$ with either $a, b \gg 0$ or $a, b \ll 0$, the expected number of zeros $\mathbb{E} \, \#(\{f = 0\} \cap U)$ is almost zero. Yet, Kac polynomials are still interesting, at least because they have historically been the first examples to be studied. For this reason and to illustrate the use of the Kac-Rice formula we study them in this section.

In this subsection we want to illustrate Theorem 4.4 in the case of Kac polynomials. The following theorem appeared in [10].

Theorem 4.8. *The root density of a Kac polynomial is*

$$\rho(x) = \frac{1}{\pi} \frac{\sqrt{1 - h(x)^2}}{1 - x^2}, \qquad \textit{where} \qquad h(x) = \frac{(d+1)x^d(1 - x^2)}{1 - x^{2(d+1)}}.$$

It should be mentioned that one has not been able to derive a closed formula for $\int_{\mathbb{R}} \rho(x) \, \mathrm{d}x$. In Kac's original paper [10] he considered instead the *asymptotic* behavior of this integral. This is a recurring theme in this book: often one can not compute closed expressions of expected properties, but one can estimate the asymptotics when one (or several) parameters go to infinity. In the case of Kac polynomials we have the following.

Theorem 4.9. *The number of zeros of Kac polynomials satisfies for $d \to \infty$:*

$$\mathbb{E}\ \#Z(f) \sim \frac{2}{\pi}\ \log(d).$$

We prove Theorem 4.8 and Theorem 4.9.

Proof of Theorem 4.8 and Theorem 4.9. The following proof appeared in [10]. For a Kac polynomial f we have $f(x) = \sum_{i=0}^{d} \xi_i x^i$, and $Jf(x) = \sum_{i=0}^{d} i\xi_i x^{i-1}$. Consider the *Kac Rational normal curve*

$$\vartheta(x) = (1, x, x^2, \dots, x^d)^T.$$

Let us also write $\xi = (\xi_0, \dots, \xi_d)^T$. Then, $f(x) = \xi^T \vartheta(x)$ and $Jf(x) = \xi^T \vartheta'(x)$, where $\vartheta'(x)$ denotes the derivative of $\vartheta(x)$. By Lemma 4.6, $(f(x), Jf(x))$ is a vector of Gaussian random variables with covariance matrix that is invertible almost everywhere:

$$\Sigma = \begin{bmatrix} \vartheta^T\vartheta & \vartheta^T\vartheta' \\ (\vartheta')^T\vartheta & (\vartheta')^T\vartheta' \end{bmatrix}.$$

The joint density of $(u, v) = (f(x), Jf(x)) \in \mathbb{R}^2$ therefore is

$$p(u, v, x) = \frac{1}{2\pi\sqrt{\det \Sigma}}\, e^{-\frac{1}{2}\,(u,v)\Sigma^{-1}(u,v)^T}.$$

This implies $p(0, v, x) = (2\pi\sqrt{\det \Sigma})^{-1}\, e^{-\frac{v^2}{2}\,\alpha}$, where $\alpha = (\det \Sigma)^{-1}\, \vartheta^T\vartheta$. We can use the fomula for the Kac-Rice density in (4.4) to get

$$\rho(x) = \int_{\mathbb{R}} |v|\, p(0, v, x)\, \mathrm{d}v = \frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\mathbb{R}} |v|\ e^{-\frac{v^2}{2}\,\alpha}\, \mathrm{d}v = \frac{1}{\pi}\, \frac{\sqrt{\det \Sigma}}{\vartheta^T\vartheta}.$$

Let us simplify this expression a little further. We have

$$\frac{\sqrt{\det \Sigma}}{\vartheta^T\vartheta} = \sqrt{\frac{\vartheta^T\vartheta \cdot (\vartheta')^T\vartheta' - (\vartheta^T\vartheta')^2}{\vartheta^T\vartheta}} = \sqrt{\frac{\partial^2}{\partial x \partial y}\ \Big|_{x=y}\ \log \vartheta^T\vartheta(y)}.$$

We have $\vartheta(x)^T\vartheta(y) = \sum_{i=0}^{d}(xy)^i = (1 - (xy)^{d+1})/(1 - xy)$ and so

$$\frac{\partial^2}{\partial x \partial y}\ \Big|_{x=y}\ \log \vartheta(x)^T\vartheta(y) = -\frac{(d+1)^2 x^{2d}}{(1 - x^{2(d+1)})^2} + \frac{1}{(1 - x^2)^2}.$$

Thus, we have $\rho(x) = \frac{1}{\pi}\, \frac{\sqrt{1-h(x)^2}}{1-x^2}$, where $h(x) = \frac{(d+1)x^d(1-x^2)}{1-x^{2(d+1)}}$. This finishes the proof of Theorem 4.8.

For proving Theorem 4.8 we observe that $\rho(x)$ is symmetric around 0, so that

$$\mathbb{E}\ \#\{f=0\} = 2\int_0^\infty \rho(x)\ \mathrm{d}x = 2\left(\int_0^1 \rho(x)\ \mathrm{d}x + \int_1^\infty \rho(x)\ \mathrm{d}x\right).$$

In the right interval, we make a change of variables $y = \frac{1}{x}$, which reveals that the two integrals in the sum are equal. Thus, we have $\mathbb{E}\ \#\{f=0\} = 4\int_0^1 \rho(y)\ \mathrm{d}y$. Using the formula for the geometric sum, and the fact that $0 \le y \le 1$, we have

$$h(y) = \frac{(d+1)y^d(1-y^2)}{(1-y)(1+y+\cdots+y^{2d+1})} \ge \frac{y^d(1+y)}{2}, \tag{4.7}$$

which implies

$$1-h(y)^2 \le (1-\tfrac{1}{2}y^d(1+y))(1+\tfrac{1}{2}y^d(1+y)) \le 2-y^d(1+y).$$

By the mean value theorem, for a fixed y there exists some $y<\theta<1$ such that

$$\frac{(2-1^d(1+1))-(2-y^d(1+y))}{1-y} = \frac{y^d(1+y)-2}{1-y} = -d\theta^{d-1}(1+\theta)-\theta^d.$$

Since $\theta<1$, this implies $1-h(y)^2 \le 2-y^d(1+y) < (1-y)(2d+1)$. Moreover, for $0\le y\le 1$ we have $1-y^2 \ge 1-y$, so that

$$\frac{\sqrt{1-h(y)^2}}{1-y^2} \le \frac{\sqrt{1-h(y)^2}}{1-y} < \sqrt{\frac{2d+1}{1-y}}. \tag{4.8}$$

On the other hand, (4.7) implies that $h(y)\ge 0$, and so

$$\frac{\sqrt{1-h(y)^2}}{1-y^2} \le \frac{1}{1-y^2}. \tag{4.9}$$

A combination of (4.8) and (4.9) yields

$$\begin{aligned}\mathbb{E}\ \#\{f=0\} &\le \frac{4}{\pi}\left(\int_0^{1-\frac{1}{d}} \frac{1}{1-y^2}\ \mathrm{d}y + \int_{1-\frac{1}{d}}^1 \sqrt{\frac{2d+1}{1-y}}\ \mathrm{d}y\right) \\ &= \frac{4}{\pi}\left(\frac{\log(2-\frac{1}{d})}{2} + \frac{\log(d)}{2} + 2\sqrt{\frac{2d+1}{d}}\right) \\ &\le \frac{4\log(2)}{\pi} + \frac{2\log(d)}{\pi} + \frac{8\sqrt{3}}{\pi} \le \frac{2\log(d)}{\pi} + 6,\end{aligned} \tag{4.10}$$

where in the penultimate step we have used $d\ge 1$.

To finish the proof of Theorem 4.9 we also need a lower bound for $\mathbb{E}\ \#\{f=0\}$. For this we consider a number $0 \le \delta < 1$. Then, since $h(y) \le (d+1)y^d$, we have

$$h(y) \le (d+1)(1-d^{\delta-1})^d \quad \text{for} \quad 0 \le y \le 1-d^{\delta-1}.$$

This implies

$$\begin{aligned}\mathbb{E}\ \#\{f=0\} &= \frac{4}{\pi}\int_0^1 \rho(y)\,\mathrm{d}y \qquad (4.11)\\ &\ge \frac{4}{\pi}\int_0^{1-d^{\delta-1}} \frac{\sqrt{1-(d+1)^2(1-d^{\delta-1})^{2d}}}{1-y^2}\,\mathrm{d}y\\ &= \frac{2}{\pi}\sqrt{1-(d+1)^2(1-d^{\delta-1})^{2d}}\,\big(\log(2-d^{\delta-1})+(1-\delta)\log(d)\big)\end{aligned}$$

Let us write $d' = d^{1-\delta}$. Since $1-\delta > 0$ we have

$$(d+1)(1-d^{\delta-1})^d = (d+1)\Big(\Big(1-\frac{1}{d'}\Big)^{d'}\Big)^{d^\delta} \overset{d\to\infty}{\sim} (d+1)e^{-d^\delta} \overset{d\to\infty}{\to} 0,$$

which in combination with (4.11) and (4.10) shows that

$$\mathbb{E}\ \#\{f=0\} \sim \frac{2}{\pi}\log(d) \text{ for } d\to\infty.$$

This finishes the proof of Theorem 4.9. □

4.3 The Kac-Rice formula for random maps on manifolds

In this section we extend the Kac-Rice formula to random maps defined on a Riemannian manifold (M, g). Let $m = \dim M$. Similar to Definition 4.1 we say that a finite collection of smooth functions $\mathcal{F} = \{f_1, \dots, f_\ell\} \subset C^\infty(M, \mathbb{R}^k)$ induces the random Gaussian map $f(x) = \xi_1 f_1(x) + \cdots + \xi_\ell f_\ell(x)$, where $\{\xi_i\}_{i=1,\dots,\ell}$ is a family of i.i.d. $N(0,1)$ random variables. Assuming that the density of $f(x)$ is nondegenerate, the analogue of Definition 4.2 for manifolds is the following.

Definition 4.10 (The Kac-Rice density for random maps on manifolds)**.** The Kac-Rice density of $\mathcal{F}$ at x is

$$\rho(x) = \mathbb{E}\left[\mathrm{NJ}(f,x)\,\middle|\,f(x)=0\right]\cdot\phi_{f(x)}(0),$$

where $\phi_{f(x)}(0)$ is the density of the random vector $f(x)$ evaluated at zero, and the determinant is defined to be the determinant of $\mathrm{D}_x f$ in coordinates with respect to orthonormal bases in T_xM and $T_{f(x)}\mathbb{R}^k$; see (2.5).

Here is the Kac-Rice formula for manifolds

Theorem 4.11 (Kac-Rice formula for random maps on manifolds)**.** *Let $\ell, m \geq k$. Let (M, g) be a Riemannian manifold of dimension m and $f : M \to \mathbb{R}^k$ be the random map induced by $\mathcal{F} \subset C^\infty(M, \mathbb{R}^k)$, $\#\mathcal{F} = \ell$. Assume that for almost all $x \in M$ we have*

1. *$f(x)$ has a nondegenerate distribution; i.e., $\mathbb{E}\, f(x)f(x)^T \succ 0$, and*
2. *$\mathbb{P}\{\mathrm{NJ}(f, x) = 0 \mid f(x) = 0\} = 0$.*

Then, almost surely the zero set of f is an $(m-k)$-dimensional smooth manifold. The volume of the zeros of f in a measurable set $A \subseteq M$ is given by the formula:

$$\mathbb{E}\ \mathrm{vol}_{m-k}(\{x \in A \mid f(x) = 0\}) = \int_A \rho(x)\, \mathrm{dvol}_{\mathrm{g}}(x).$$

In particular, if $m = k$, we have $\mathbb{E}\ \#\{x \in A \mid f(x) = 0\} = \int_A \rho(x)\, \mathrm{dvol}_{\mathrm{g}}(x)$.

As in the case of random maps between Euclidean cases the Kac-Rice formula is a special case of the following more general formula.

Theorem 4.12. *Let $h : \mathbb{M} \to \mathbb{R}$ be a measurable function. Under the assumptions from Theorem 4.11 the zero set $Z(f) = \{x \in \mathbb{R}^m \mid f(x) = 0\}$ is almost surely an $(m-k)$-dimensional smooth submanifold of M and we have*

$$\mathbb{E} \int_{Z(f)\cap A} h(x)\, \mathrm{d}x = \int_A h(x)\rho(x)\, \mathrm{d}x.$$

Proof. The proof is almost one-to-one to the proof of Theorem 4.5: we define $W \subset M \times \mathbb{R}^k$ be the subset of points (y, x) such that $\det(\mathbb{E}\, f(x)f(x)^T) \neq 0$, and consider $Z := \{(y, x) \in W \mid f(x) = 0$, where $f(x) = \sum_{i=1}^{\ell} y_i f_i(x)\}$. As in the proof of Theorem 4.4 we can apply the coarea formula to the coordinate projections $\pi_1 : Z \to \mathbb{R}^\ell$ and $\pi_2 : Z \to M$ to conclude. □

4.4 The number of zeros of Kostlan polynomials

As an illustrative example, we compute the Kac-Rice density (Definition 4.17) of a system of $k \leq n$ independent Kostlan polynomials of degrees $(d_1, \ldots, d_k)$ defined on the sphere S^n. Recall from (1.7) that a Kostlan polynomial is a random homogeneous polynomial defined by $\sum_{|\alpha|=d} \xi_\alpha \binom{d}{\alpha}^{1/2} x^\alpha$, where $x = (x_0, \ldots, x_n)$

and $\{\xi_\alpha\}$ is a family of i.i.d. $N(0,1)$ random variables. We denote a system of k independent Kostlan polynomials by

$$f(x) = \begin{bmatrix} \sum_{|\alpha|=d_1} \xi_\alpha^{(1)} \binom{d_1}{\alpha}^{1/2} x^\alpha \\ \vdots \\ \sum_{|\alpha|=d_k} \xi_\alpha^{(k)} \binom{d_k}{\alpha}^{1/2} x^\alpha \end{bmatrix}. \tag{4.12}$$

We also apply the Kac-Rice formula to $f(x)$ to compute the expected volume of zeros on S^n. This also computes the volume of zeros in $\mathbb{R}\mathrm{P}^n$, since

$$\mathbb{E}\,\mathrm{vol}\{x \in \mathbb{R}\mathrm{P}^n \mid f(x) = 0\} = \frac{1}{2}\,\mathbb{E}\,\mathrm{vol}\{x \in S^n \mid f(x) = 0\}; \tag{4.13}$$

see the discussion in Section 2.4. Edelman and Kostlan showed in their seminal paper [6] that in the case of univariate Kostlan polynomials ($k = n = 1$) of degree d the expected number of zeros of f in $\mathbb{R}\mathrm{P}^1$ is $\sqrt{d}$. This is a special case of the general situation in the next theorem.

Theorem 4.13. *Let $f(x)$ be a system of $k \leq n$ independent Kostlan polynomials of degrees $(d_1, \ldots, d_k)$ in $n+1$ variables. Then, almost surely the zero set of f is an $(n-k)$-dimensional submanifold of S^n, the Kac-Rice density of the real zeros of f is constant and equal to*

$$\rho(x) = \frac{\mathrm{vol}(\mathbb{R}\mathrm{P}^{n-k})}{\mathrm{vol}(\mathbb{R}\mathrm{P}^n)} \sqrt{d_1 \cdots d_k}.$$

The expected volume of the zero set in $\mathbb{R}\mathrm{P}^n$ is

$$\mathbb{E}\,\mathrm{vol}_{n-k}\{x \in \mathbb{R}\mathrm{P}^n \mid f(x) = 0\} = \mathrm{vol}(\mathbb{R}\mathrm{P}^{n-k})\sqrt{d_1 \cdots d_k}.$$

In particular, if $k = n$ we expect $\sqrt{d_1 \cdots d_n}$ many zeros.

While in this section we will use the Kac-Rice formula for polynomials defined on the sphere to prove Theorem 4.13, in the next section we will prove Theorem 4.13 by treating polynomials as section of an appropriate line bundle over $\mathbb{R}\mathrm{P}^1$. Going from the first to the second proof we will make a conceptual jump when introducing the notion of a random section of a vector bundle.

An crucial part in the proof of Theorem 4.13 is the invariance of Kostlan polynomials under orthogonal changes of variables that we already stated in Lemma 1.9. For us this means, that Kostlan polynomials are reasonable in the sense of Section 1.3, because they have no preferred locations for their projective zeros – unlike Kac polynomials whose projective zeros are most likely to be close to $\pm 1 \in \mathbb{R}$.

Later, we will discuss invariant distributions of polynomials in general, but let us now give a short proof for Kostlan polynomials.

Proof of Lemma 1.9. From Lemma 4.6 it follows that for every $\alpha \in \mathbb{N}^{n+1}$ with $|\alpha| = d$ the random variable $\xi_\alpha \binom{d}{\alpha}^{1/2}$ has the same distribution as the sum of $\binom{d}{\alpha}$ independent $N(0,1)$ random variables. Notice that there are exactly $\binom{d}{\alpha}$ multiindices $(i_1, \dots, i_d)$ such that $\#\{j \mid i_j = k\} = \alpha_k$ for all $k = 0, \dots, n$. This implies that $g(x) = \sum_{0 \le i_1, \dots, i_d \le n} \xi_{(i_1,\dots,i_d)}\, x_{i_1} \cdots x_{i_d}$, where $\xi_{(i_1,\dots,i_d)}$ are i.i.d. $N(0,1)$-random variables, is a Kostlan polynomial. We regard this polynomial as a *multilinear map* $h(z_1, \dots, z_d) = \sum_{0 \le i_1, \dots, i_d \le n} \xi_{(i_1,\dots,i_d)}\, (z_1)_{i_1} \cdots (z_d)_{i_d}$, evaluated at $(z_1, \dots, z_d) = (x, \dots, x)$. Let $X \in \mathbb{R}^{(n+1)\times(n+1)^{d-1}}$ be the Gaussian matrix with entries $X_{k,(i_2,\dots,i_d)} = \xi_{(k,i_2,\dots,i_d)}$. We can write

$$h(z_1, \dots, z_d) = z_1^T X y, \quad \text{where} \quad y = ((z_2)_{i_1} \cdots (z_d)_{i_d})_{0 \le i_2, \dots, i_d \le n}.$$

Let now $U \in O(n+1)$. Then, $(Uz_1)^T M y = z_1 (U^T M) y$ and $U^T M$ is also a Gaussian matrix with independent $N(0,1)$ entries by Lemma 4.6. By symmetry, we see that $h(Uz_1, \dots, Uz_d) \sim h(z_1, \dots, z_d)$ and hence $f(Ux) \sim f(x)$. □

Now, we prove Theorem 4.13.

Proof of Theorem 4.13. By (4.13), the volume of zeros of f in $\mathbb{R}\mathrm{P}^n$ is half the volume of zeros of f in S^n. We compute the latter. The system of Kostlan polynomials in (4.12) defines a random map on S^n, which satisfies the assumptions of Theorem 4.11 so that the zero set is almost surely $(n-k)$-dimensional an

$$\mathbb{E}\, \mathrm{vol}_{n-k}\{x \in S^n \mid f = 0\} = \int_{S^n} \rho(x)\, \mathrm{d}x.$$

Let us compute the Kac-Rice density $\rho(x)$. By Lemma 1.9 we have $f \sim f \circ U$ for every $U \in O(n+1)$, and so $\rho(Ux) = \rho(x)$. Moreover, $O(n+1)$ acts transitively on S^n which shows that ρ is in fact constant on S^n. This implies

$$\int_{S^n} \rho(x)\, \mathrm{d}x = \rho(e_0)\, \mathrm{vol}(S^n),$$

where $e_0 = (1, 0, \dots, 0) \in S^n$. For computing the root density at e_0, we recall from that the tangent space of S^n at e_0 is $T_{e_0} S^n = e_0^\perp$ We choose $\{e_1, \dots, e_n\}$ both as orthonormal basis for $e_0^\perp$ and for $\mathbb{R}^n$. The matrix of $\mathrm{D}_{e_0} f$ with respect to these bases is

$$J = \begin{bmatrix} \frac{\partial f}{\partial x_1}(e_0) & \cdots & \frac{\partial f}{\partial x_n}(e_0) \end{bmatrix} = \begin{bmatrix} \sqrt{d_1}\xi_{(d-1,1,0,\dots,0)} & \cdots & \sqrt{d_1}\xi_{(d-1,0,0,\dots,1)} \\ \vdots & \ddots & \vdots \\ \sqrt{d_k}\xi_{(d-1,1,0,\dots,0)} & \cdots & \sqrt{d_k}\xi_{(d-1,0,0,\dots,1)} \end{bmatrix} \in \mathbb{R}^{k \times n}.$$

Furthermore, $f(e_0) = (\xi_{(d_1,0,\dots,0)}, \dots, \xi_{(d_k,0,\dots,0)})^T$ so that it's density at 0 is

$$\phi_{f(x)}(0) = \frac{1}{\sqrt{2\pi}^k}.$$

We see that J is independent of $f(e_0)$. Using $\mathrm{NJ}(f,x) = \sqrt{\det(JJ^T)}$ we get

$$\mathbb{E}\left[\sqrt{\det(JJ^T)}\,|f(e_0)=0\right] = \mathbb{E}\sqrt{\det(JJ^T)} = \sqrt{d_1\cdots d_k}\,\mathbb{E}\sqrt{\det(XX^T)}, \quad (4.14)$$

where $X \in \mathbb{R}^{k\times n}$ is a matrix whose entries are i.i.d. $N(0,1)$ random variables.

All this implies $\rho(e_0) = (2\pi)^{-\frac{k}{2}}\sqrt{d_1\cdots d_k}\;\mathbb{E}\sqrt{\det(XX^T)}$, and so

$$\mathbb{E}\,\mathrm{vol}_{n-k}\{x\in S^n \mid f=0\} = \frac{\mathrm{vol}(S^n)}{\sqrt{2\pi}^k}\sqrt{d_1\cdots d_k}\;\mathbb{E}\sqrt{\det(XX^T)}. \quad (4.15)$$

We consider the case $d_1 = \cdots = d_k = 1$, where the system of Kostlan polynomials consists of k linear equations with i.i.d. $N(0,1)$ coefficients. With probability one the k equations define k hyperplanes in $\mathbb{R}^{n+1}$ in general position so that their intersection is a linear space of dimension $n-k+1$. Its intersection with S^n is a $(n-k)$-sphere. The expected volume of zeros on the sphere in this case is $\mathrm{vol}(S^{n-k})$. Plugging this into (4.15) we get $\mathrm{vol}(S^{n-k}) = \mathrm{vol}(S^n)\,(2\pi)^{-\frac{n}{2}}\;\mathbb{E}\sqrt{\det(XX^T)}$, so that $\rho(e_0) = (\mathrm{vol}(S^{n-k})/\mathrm{vol}(S^n))\sqrt{d_1\cdots d_k} = (\mathrm{vol}(\mathbb{R}\mathrm{P}^{n-k})/\mathrm{vol}(\mathbb{R}\mathrm{P}^n))\sqrt{d_1\cdots d_k}$ and, consequently, $\mathbb{E}\,\mathrm{vol}\{x\in S^n \mid f=0\} = 2\mathrm{vol}(\mathbb{R}\mathrm{P}^{n-k})\sqrt{d_1\cdots d_k}$. □

An important corollary from the proof of Theorem 4.13 is the following result.

Corollary 4.14. *Let $X \in \mathbb{R}^{k\times n}$, $k \le n$, be a matrix whose entries are i.i.d. $N(0,1)$ random variables. Then,*

$$\mathbb{E}\sqrt{\det(XX^T)} = \sqrt{2\pi}^n\,\frac{\mathrm{vol}(\mathbb{R}\mathrm{P}^{n-k})}{\mathrm{vol}(\mathbb{R}\mathrm{P}^n)} = \sqrt{2^n\pi^{n-k}}\,\frac{\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Proof. We have shown $\mathrm{vol}(S^{n-k}) = \mathrm{vol}(S^n)\,(2\pi)^{-\frac{n}{2}}\;\mathbb{E}\sqrt{\det(XX^T)}$ at the end of the proof of Theorem 4.13. This yields $\mathbb{E}\sqrt{\det(XX^T)} = (2\pi)^{\frac{n}{2}}\,\mathrm{vol}(\mathbb{R}\mathrm{P}^{n-k})/\mathrm{vol}(\mathbb{R}\mathrm{P}^n)$. Using from (2.8) that the volume of projective space is $\mathrm{vol}(\mathbb{R}\mathrm{P}^n) = \pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ gives the second equality. □

In the important special case $k=n$ the corollary gives the expected absolute determinant of a matrix $X \in \mathbb{R}^{n\times n}$ filled with i.i.d. standard normal random variables: $\mathbb{E}\,|\det(X)| = \sqrt{2^n\pi^{-1}}\,\Gamma\left(\frac{n+1}{2}\right)^{-1}$.

We can use the same strategy to compute the volume of complex zeros in $\mathbb{C}\mathrm{P}^n$ of complex Kostlan polynomials. Let $k \leq n$ and

$$f(x) = \begin{bmatrix} \sum_{|\alpha|=d_1} \zeta_\alpha^{(1)} \binom{d}{\alpha}^{1/2} x^\alpha \\ \vdots \\ \sum_{|\alpha|=d_k} \zeta_\alpha^{(k)} \binom{d}{\alpha}^{1/2} x^\alpha \end{bmatrix}, \tag{4.16}$$

where the $\zeta_\alpha^{(i)}$ are i.i.d. complex Gaussian random variables (a complex Gaussian random variable is of the form $\zeta = \xi_1 + \sqrt{-1}\xi_2$, where ξ_1 and ξ_2 are independent and both $N(0,1)$). Using the Kac-Rice formula we can show that the zero set of $f(x) \in \mathbb{C}\mathrm{P}^n$ is almost surely an $(n-k)$-dimensional submanifold of expected volume

$$\mathbb{E}\,\mathrm{vol}_{n-k}\{x \in \mathbb{C}\mathrm{P}^n \mid f(x) = 0\} = d_1 \cdots d_k.$$

The reason why we don't get the square root as in Theorem 4.13 is Lemma 2.20: if $J = [\, \frac{\partial f}{\partial x_1}(e_0) \;\cdots\; \frac{\partial f}{\partial x_n}(e_0) \,]$ is the matrix of *complex derivatives* of f at e_0, the expected value in (4.14) becomes $\mathbb{E}\det(JJ^T) = d_1 \cdots d_k\, \mathbb{E}\,|\det(XX^T)|$, where now $X \in \mathbb{C}^{k\times n}$ is a complex Gaussian matrix. In fact, the volume of the zero set is generic, because it is locally constant on the complement of the discriminant consisting of polynomial systems f whose zero set has a singular point, which by Lemma 1.5 is connected; see the discussion in Subsection 1.4.1. Therefore, we have

$$\mathrm{vol}_{n-k}\{x \in \mathbb{C}\mathrm{P}^n \mid f(x) = 0\} = d_1 \cdots d_k \;\text{ almost surely.} \tag{4.17}$$

In the case $n = k$ this number is $d_1 \cdots d_n$, which is the *generic* number of complex zeros of $f(x)$. Indeed, the number $d_1 \cdots d_n$ is implied by Bézout's theorem. But Bézout's theorem gives in fact more: it asserts that, if the number of zeros of $f(x)$ is finite, then it has $d_1 \cdots d_n$ zeros counted with multiplicity. The result in (4.17) is weaker in this regard: it only shows that the the number of zeros of a *generic* system is $d_1 \cdots d_n$. On the other hand, (4.17) is also stronger than the Bézout theorem, because it also measures the number of solutions in the case $k < n$.

4.5 Random sections of vector bundles

In the final section of this chapter we develop the most general version of the Kac-Rice formula. While Theorem 4.11 generalizes the Kac-Rice formula from random maps on Euclidean spaces to random maps on manifolds, here we make another conceptual step: we measure the zero set of *random sections* of vector bundles.

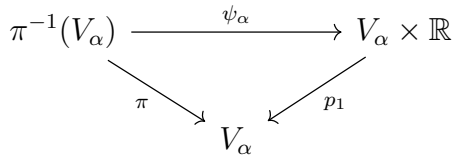
Let us first recall the definition and basic properties of vector bundles.

Definition 4.15. A *vector bundle* of rank k is a triple (π, E, M) where E and M are smooth manifolds and

$$\pi : E \to M$$

is a smooth map such that there is an open cover $M = \bigcup_{\alpha \in A} V_\alpha$ and diffeomorphisms $\psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^k$ with the following two properties:

1. The following diagram is commutative:



   (p_1 is the projection on the first factor).

2. Whenever $V_\alpha \cap V_\beta \neq \emptyset$, there is a continuous map $g_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathrm{GL}(\mathbb{R}^k)$ such that the map

$$\psi_\alpha \circ \psi_\beta^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{R}^k \to (V_\alpha \cap V_\beta) \times \mathbb{R}^k$$

   is given by $(x, v) \mapsto (x, g_{\alpha\beta}(x) \cdot v)$.

The family $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in A}$ is called a *vector bundle atlas*, and each ψ_α is called a *trivialization*.

Let (π, E, M) be a vector bundle. We call M the *base space* of the bundle. The *fiber* over $x \in M$ is the vector space

$$E_x := \pi^{-1}(x).$$

The family $\{g_{\alpha\beta}\}_{\alpha,\beta \in A}$ is called the *cocycle* of the bundle and satisfies $g_{\alpha\alpha}(x) = \mathbf{1}$ and $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$.

Definition 4.16. Let $\pi : E \to M$ be a vector bundle. A *section* of E is a smooth map $s : M \to E$ such that $\pi(s(x)) = x$ for every $x \in M$. The *zero* section is the section that associates to every point $x \in M$ the zero vector in E_x.

Similar to the definition of random maps we now introduce the notion of a *random section*: a finite set $\mathcal{F} = \mathrm{span}\{\sigma_1, \ldots, \sigma_\ell\}$ of sections defines the random section

$$\sigma = \xi_1 \sigma_1 + \cdots + \xi_\ell \sigma_\ell,$$

with $\{\xi_k\}_{k=1,\ldots,\ell}$ a family of i.i.d. standard real gaussian variables. In particular, if we have the trivial bundle $M \times \mathbb{R}^k$ this reduces to a random map on M.

For the vector bundle analogue of the Kac-Rice density, we consider a Riemannian manifold (M, g) of dimension m and rank-k vector bundle $\pi : E \to M$. Let $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be a vector bundle atlas with vector bundle trivializations $\psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^k$, we can identify. For every α we define the local function

$$\sigma_\alpha := p_2 \circ \psi_\alpha \circ \sigma|_{V_\alpha} : V_\alpha \to \mathbb{R}^k,$$

where $p_2 : V_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$ is the projection onto the second coordinate. First, we define the local Kac-Rice density for random sections.

Definition 4.17. Let $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be a vector bundle atlas for E. The Kac-Rice density at V_α of the collections of sections $\mathcal{F}$ at $x \in M$ is

$$\rho_\alpha(x) = \mathbb{E}\left[\mathrm{NJ}(\sigma_\alpha, x) \,\middle|\, \sigma_\alpha(x) = 0\right] \cdot \phi_{\sigma_\alpha(x)}(0),$$

where $\phi_{\sigma_\alpha(x)}(0)$ is the density of the random vector $\sigma_\alpha(x)$ evaluated at zero.

We need the local definition together with the following proposition to get a global notion for the Kac-Rice density.

Proposition 4.18. *For every $\alpha, \beta \in A$ such that $V_\alpha \cap V_\beta \neq \emptyset$ we have for almost all $x \in V_\alpha \cap V_\beta$ that $\rho_\alpha(x) = \rho_\beta(x)$.*

Proof. For every α the map σ_α satisfies the assumptions of Theorem 4.11 so that we have $\mathbb{E}\,\mathrm{vol}\{x \in U \mid \sigma_\alpha(x) = 0\} = \int_U \rho_\alpha(x)\,\mathrm{dvol_g}(x)$ for every measurable subset $U \subset V_\alpha$. For every pair α, β we $\sigma_\alpha(x) = 0$ if and only if $\sigma_\beta(x) = 0$. This shows that for $U \subset V_\alpha \cap V_\beta$ we have $\int_U \rho_\alpha(x)\,\mathrm{dvol_g}(x) = \int_U \rho_\beta(x)\,\mathrm{dvol_g}(x)$. Since this equality holds for all measurable subsets we see that $\rho_\alpha(x) = \rho_\beta(x)$ for almost all $x \in V_\alpha \cap V_\beta$. □

The proposition shows that the following definition does not depend on the partition of unity up to sets of measure zero.

Definition 4.19 (The Kac-Rice density for random sections of vector bundles)**.** Let $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be a vector bundle atlas for E and $\{q_\alpha : V_\alpha \to \mathbb{R}\}_{\alpha \in A}$ be a partition of unity subordinated to the open cover $M = \bigcup_{\alpha \in A} V_\alpha$. Then, the Kac-Rice density of the collections of sections $\mathcal{F}$ at $x \in M$ is

$$\rho(x) = \sum_{\alpha \in A} q_\alpha(x) \rho_\alpha(x).$$

Now comes the Kac-Rice formula for vector bundles.

Theorem 4.20 (Kac-Rice formulas for random sections of vector bundles)**.** *Let $\ell, m \geq k$. Let (M, g) be a Riemannian manifold of dimension m and $\pi : E \to M$ be a rank-k vector bundle with vector bundle atlas $\{(V_\alpha, \psi_\alpha)\}_{\alpha \in A}$. Let $\sigma : M \to E$ be a random section of E. Assume that for almost all $x \in M$ and every $\alpha \in A$:*

1. *$\sigma_\alpha(x)$ has a nondegenerate distribution; i.e., $\mathbb{E}\, \sigma_\alpha(x)\sigma_\alpha(x)^T \succ 0$, and*
2. *$\mathbb{P}\{\mathrm{NJ}(\sigma_\alpha, x) = 0 \mid \sigma(x) = 0\} = 0$.*

Then, almost surely the zero set of σ is a $(m-k)$-dimensional submanifold of M and we have for every measurable set $U \subset M$:

$$\mathbb{E}\, \mathrm{vol}\{x \in U \mid \sigma(x) = 0\} = \int_U \rho(x)\, \mathrm{dvol}_{\mathrm{g}}(x).$$

Proof. Let $\{q_\alpha : V_\alpha \to \mathbb{R}\}_{\alpha \in A}$ be a partition of unity subordinated to the open cover $M = \bigcup_{\alpha \in A} V_\alpha$. For every α the map σ_α satisfies the assumptions of Theorem 4.12 so that

$$\int_{U \cap V_\alpha} q_\alpha(x)\rho_\alpha(x)\, \mathrm{dvol}_{\mathrm{g}}(x) = \mathbb{E} \int_{Z(\sigma_\alpha) \cap U \cap V_\alpha} q_\alpha(x)\, \mathrm{d}x,$$

where $Z(\sigma_\alpha) = \{x \in M \mid \sigma_\alpha(x) = 0\}$. Then,

$$\begin{aligned}
\int_U \rho(x)\, \mathrm{dvol}_{\mathrm{g}}(x) &= \sum_{\alpha \in A} \int_{U \cap V_\alpha} q_\alpha(x)\rho_\alpha(x)\, \mathrm{dvol}_{\mathrm{g}}(x) \\
&= \sum_{\alpha \in A} \mathbb{E} \int_{\int_{Z(\sigma_\alpha) \cap U \cap V_\alpha}} q_\alpha(x)\, \mathrm{d}x \\
&= \mathbb{E}\, \mathrm{vol}\{x \in U \mid \sigma_\alpha(x) = 0\}.
\end{aligned}$$

This finishes the proof. □

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