Metric and Random Algebraic Geometry

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Contents

1 How many zeros of a polynomial are real?

How many zeros has a polynomial? The answer to this question is taught in a basic algebra course: it is equal to the degree of the polynomial. This is known as the fundamental theorem of algebra.

However, this assumes that the question was stated as *How many complex zeros does a polynomial have?*. Yet, if the person, who asked that question, had in mind a real polynomial and real zeros, the answer is less clear. For instance, the polynomial $x^2 + ax + b$ has two real zeros, if $a^2 - 4b > 0$, it has one real zero, if $a^2 - 4b = 0$, and in the case $a^2 - 4b < 0$ it has no real zeros. The situation is depicted in Figure 1.1. This simple yet important example shows already that we can not give an answer to the above question in terms of the degree of the polynomial. Instead, we have to use a list of algebraic equalities and inequalities. While for polynomials of degree 2 this was simple enough for us to understand, more complicated counting problems pose uncomparably harder challenges. Think of the number of eigenvalues of an $n \times n$ matrix. The number of complex eigenvalues is always n (counted with multiplicity). But the algebraic constraints for the number of real solutions are already so complicated, that it is very difficult just to compute this number without computing all eigenvalues in the first place.

In this book we want to lay out an alternative perspective on counting problems like the ones above. Instead of computing a deterministic real picture, we want to understand its *statistical properties*. This thinking is not new: already in the 1930s and 1940s Littlewood, Offord [26] and Kac [19, 20] considered real zeros of random polynomials. Later, in 1973, Montgomery [31] introduced randomness to number therory. In the 1950s, Wigner [40], Dyson [7] and others proposed using probability for understanding models in theoretical physics. Ginibre [14] summarizes their motivation as follows.

> *"In the absence of any precise knowledge [...], one assumes a reasonable probability distribution [...], from which one deduces statistical properties [...]. Apart from the intrinsic interest of the problem, one may hope that the methods and results will provide further insight in the cases of physical interest or suggest as yet lacking applications."*

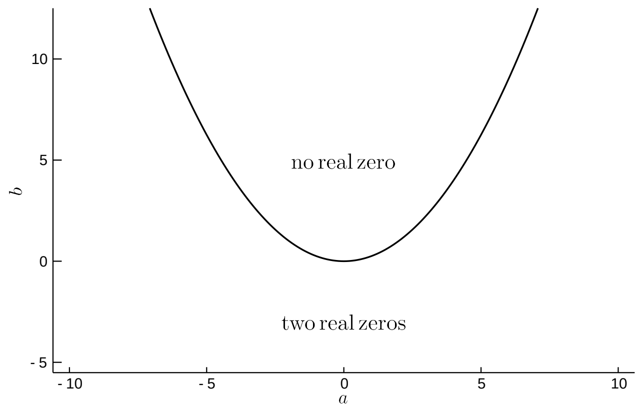

Figure 1.1: The configuration space for real zeros of the polynomial $f = x^2 + ax + b$. The blue curve $a^2 - 4b = 0$ is called the discriminant. If (a, b) is below the discriminant, then f has two real zeros. If it is above, it has no real zero. Polynomials on the discriminant have one real zero.

Although written in the context of statistical physics, Ginibre's words perfectly outline the ideas we wish to present with this book: we want to use tools from probability theory to understand the nature of algebraic–geometric objects.

Edelman and Kostlan [8] condense the probabilistic approach in the title of their seminal paper "How many zeros of a random polynomial are real?" (the answer is in Example 1.14 below). We chose the title of this introductory section as a homage of their work. Starting from their results, we explore in this book algebraic geometry from a probabilistic point of view. Our name for this new field of research is *Random Algebraic Geometry*.

Here is an illustrative example of what we have in mind: consider the degree 8 polynomial $f_\epsilon(x) = 1 + \epsilon_1 x + \epsilon_2 x^2 + \epsilon_3 x^3 + \epsilon_4 x^4 + \epsilon_5 x^5 + \epsilon_6 x^6 + \epsilon_7 x^7 + \epsilon_8 x^8$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_8) \in \{-1, 1\}^8$. This polynomial can have $0, 2, 4, 6$ or 8 zeros, because complex zeros come in conjugate pairs. Instead of attempting to understand the equations separating the regions with a certain number of real solutions, we endow the coefficients of f_ϵ with a probability distribution. We assume that $\epsilon_1, \ldots, \epsilon_8$ are independent random variables with $\mathbb{P}\{\epsilon_i = 1\} = \frac{1}{2}$ for $1 \le i \le 8$, and we denote by $n(\epsilon)$ the random variable "number of real zeros of f_ϵ". Booth [4] showed that

$$\mathbb{P}\{n(\epsilon) = 0\} = \frac{58}{2^8}, \quad \mathbb{P}\{n(\epsilon) = 2\} = \frac{190}{2^8}, \quad \mathbb{P}\{n(\epsilon) = 4\} = \frac{8}{2^8}, \quad \text{and}$$
$$\mathbb{P}\{n(\epsilon) = 6\} = \mathbb{P}\{n(\epsilon) = 8\} = 0,$$

which shows that f_ϵ has at most 4 zeros, and having more than 2 zeros is unlikely.

In Booth's example we have access to the full probability law. However, during this book we will encounter many situations in which computing the probability law is too ambitious. Instead, it is often feasible to compute or estimate the expected value of a random geometric property. For instance, in Booth's example the expected value of the number of roots is $\mathbb{E}\, n(\epsilon) = 1.609375$. Just based on this information we can conclude that having a large number of zeros is unlikely.

Interestingly, many of the expected values we will meet later in this book obey what is called the "square-root law": the expected number of real solutions is roughly the square-root of the number of complex solutions. If this law holds, it immediately implies that instances, for which the number of real solutions equal the number of complex solutions, are *rarae aves*. This phenomenon, which is specific of a particular, but natural, probability distribution that we will work with, has several manifestation: from geometry (expectation of volumes of real algebraic sets) to topology (expectation of Betti numbers).

1.1 Discriminants

Let us have a closer look at the picture in Figure 1.1. We can see that the *discriminant* $\Sigma_{\mathbb{R}} := \{(a,b) \in \mathbb{R}^2 \mid a^2 - 4b = 0\}$ divides the real (a,b)-plane into two components – one, where the number of real zeros is two, and one, where there are no real zeros. This is because the discriminant is a curve of *real* codimension 1. The complex picture is different: here, the complex curve $\Sigma_{\mathbb{C}} = \{(a,b) \in \mathbb{C}^2 \mid a^2 - 4b = 0\}$ is of *complex* codimension one. In particular, it is of real codimesion two, and $\mathbb{C}^2 \backslash \Sigma_{\mathbb{C}}$ is path-connected! We show this in Lemma 1.5, but it can also be seen in Figure 1.2. This is the reason for why each polynomial of degree 2 outside $\Sigma_{\mathbb{C}}$ has two complex zeros: a function which is locally constant on a connected space is constant. We say that having two complex zeros is a *generic property*. We will give a more precise definition of this later in Definition 1.4.

In algebraic geometry, it is more appropriate to work with zeros of polynomials in *projective space* rather than with zeros in $\mathbb{C}^n$. The definition of projective space comes next.

Definition 1.1 (Complex projective Space)**.** The complex projective space $\mathbb{C}\mathrm{P}^n$ of dimension n is defined to be the set of lines through the origin in $\mathbb{C}^{n+1}$. That is, $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1}\backslash\{0\})/\sim$, where $y \sim z$, if and only of there exists some $\lambda \in \mathbb{C}\backslash\{0\}$ with $y = \lambda z$. For a point $(z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ we denote by $[z_0, z_1, \ldots, z_n]$ its equivalence class in $\mathbb{C}\mathrm{P}^n$.

For completing the terminology, and distinguishing it from projective space, we say that $\mathbb{C}^n$ is an n-dimensional *affine complex space.*

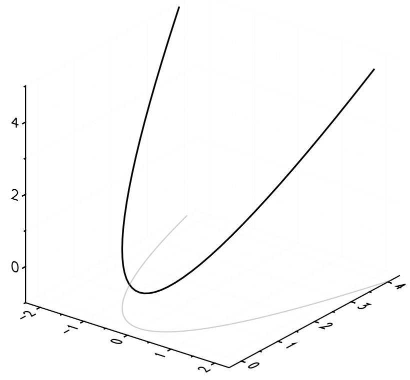

Figure 1.2: The picture shows the part of the complex discriminant $(a_1 + ia_2)^2 - 4(b_1 + ib_2) = 0$, where $a_1 = 2a_2$ As can be seen from the picture, the discriminant is of real codimension two. Because one can "go around" the discriminant without crossing it, a generic complex polynomial of degree 2 has two complex zeros.

The map

$$P : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{C}\mathrm{P}^n \ , (z_0, z_1, \dots, z_n) \mapsto [z_0 : \dots : z_n]$$

projects $(n+1)$-dimensional affine space onto n-dimensional projective space. On the other hand, the map $\psi : \mathbb{C}^n \to \mathbb{C}\mathrm{P}^n$, $(z_1, \dots, z_n) \mapsto [1, z_1 : \dots : z_n]$ embeds n-dimensional affine space into n-dimensional projective space. Using this embedding we can define the zero sets in Example 1.3 to be in $\mathbb{C}\mathrm{P}^n$.

Projective zero sets are defined by *homogeneous polynomials*. It is common to use the notation $f = \sum_{|\alpha|=d} f_\alpha \, z^\alpha$ for complex homogeneous polynomials of degree d in $n+1$ variables, where $\alpha = (\alpha_0, \dots, \alpha_n) \in \mathbb{N}^{n+1}$, $z^\alpha = \prod_{i=0}^n z_i^{\alpha_i}$ and $|\alpha| = \alpha_0 + \cdots + \alpha_n$. The space of homogeneous polynomials of degree d in $n+1$ many variables is

$$\mathbb{C}[x_0, \dots, x_n]_{(d)} := \left\{ \sum_{|\alpha|=d} f_\alpha \, z^\alpha \;\middle|\; (f_\alpha) \in \mathbb{C}^N \right\}, \text{ where } N = \binom{n+d}{d},$$

and the projective space of polynomials is thus $\mathbb{C}\mathrm{P}^{N-1}$. The complex projective zero set of k polynomials $f = (f_1, \dots, f_k)$, where the i-th polynomial is $f_i \in \mathbb{C}[z_0, \dots, z_n]_{(d_i)}$, is

$$Z_\mathbb{C}(f) = \{[z] \in \mathbb{C}\mathrm{P}^n : f_1(z) = 0, \dots, f_k(z) = 0\}.$$

For a simplified notation we also denote by $Z_\mathbb{C}(f)$ the zero set of f in $\mathbb{C}^{n+1}$.

Remark 1.2. A polynomial $f \in \mathbb{C}[z_0, \ldots, z_n]_{(d)}$ is not a function on the complex projective space $\mathbb{C}\mathrm{P}^n$, but its zero set is still well defined.

Example 1.3. Here are a few more examples of generic properties. The first generalizes our introductory example to higher degrees.

(1) A generic homogeneous polynomial $f \in \mathbb{C}[z_0, z_1]_{(d)}$ of degree d has d distinct zeros in $\mathbb{C}\mathrm{P}^1$ unless $\mathrm{Res}(f, f') = 0$ (i.e. the resultant of f and f' is zero). We define the polynomial map $\mathrm{disc}(f) := \mathrm{Res}(f, f')$ that associates to a polynomial f the resultant $\mathrm{Res}(f, f')$. Then, the zero set $\Sigma = Z_{\mathbb{C}}(\mathrm{disc})$ of this polynomial is a proper algebraic set in $\mathbb{C}[z_0, z_1]_d$, which we again call the discriminant. By Lemma 1.5 below, $\mathbb{C}[z_0, z_1]_d \setminus \Sigma$ is path-connected. This causes polynomials in $\mathbb{C}[z_0, z_1]_d \setminus \Sigma$ to admit the generic behavior of having d distinct zeros in $\mathbb{C}\mathrm{P}^1$, because we continuously deform the zero set of any $f_1 \notin \Sigma$ to the zero set of any other $f_2 \notin \Sigma$.

(2) The zero set $Z_{\mathbb{C}}(f) \subset \mathbb{C}\mathrm{P}^2$ of a generic $f \in \mathbb{C}[z_0, z_1, z_2]_{(d)}$ of degree d is homeomorphic to a surface of genus $g = \frac{(d-1)(d-2)}{2}$. In this case what happens is that there exists a polynomial $\mathrm{disc} : \mathbb{C}[z_0, z_1, z_2]_{(d)} \to \mathbb{C}$, which vanishes exactly at polynomials whose corresponding zero set in the projective plane is singular. Again, we call $\Sigma = Z_{\mathbb{C}}(\mathrm{disc})$ the discriminant. Outside of the discriminant the topology of $Z(f)$ all look the same: the reason is again that $\mathbb{C}[z_0, z_1, z_2]_{(d)} \setminus \Sigma$ is path-connected by Lemma 1.5.

(3) Let $\mathbb{C}[z_0, z_1]_{(3)}$ be the space of homogeneous complex polynomials of degree 3. Inside this space there is the cone $X^{\mathbb{C}}$ of polynomials which are powers of linear forms: $X^{\mathbb{C}} = \{f \in \mathbb{C}[z_0, z_1]_{(3)} \mid \exists \ell \in \mathbb{C}[z_0, z_1]_{(1)} : f = \ell^3\}$. The linear span of $X^{\mathbb{C}}$ is the whole $\mathbb{C}[z_0, z_1]_{(3)}$, therefore for every $f \in \mathbb{C}[z_0, z_1]_{(3)}$ there exist $\ell_1, \ldots, \ell_s \in \mathbb{C}[z_0, z_1]_{(1)}$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{C}$ such that $f = \sum_{i=1}^{s} \alpha_i \ell_i^3$. For the generic $f \in \mathbb{C}[z_0, z_1]_{(3)}$ the minimal s for having this is $s = 2$. This means that there is a discriminant $\Sigma \subsetneq \mathbb{C}[z_0, z_1]_{(3)}$, which is a proper algebraic subset, such that this property holds outside Σ.

(4) The zero set $Z_{\mathbb{C}}(f) \subset \mathbb{C}\mathrm{P}^3$ of a generic cubic $f \in \mathbb{C}[z_0, z_1, z_2, z_3]_{(3)}$ contains 27 complex lines. We will discuss in details this type of problems later, but still let us now try to see what is happening, at least in an informal way. The set of lines in $\mathbb{C}\mathrm{P}^3$ is itself a manifold, which is called the Grassmanian of (projective) lines and denoted by $\mathbb{G}(1, 3)$ (1-dimensional projective subspaces of 3-dimensional projective space). There is a rank-4 complex vector bundle $E \to \mathbb{G}(1, 3)$ whose fiber over a line $\ell \in \mathbb{C}\mathrm{P}^3$ consists of homogeneous polynomials of degree 3 over this line. Every polynomial $f \in \mathbb{C}[z_0, z_1, z_2, z_3]_{(3)}$ defines naturally a section $\sigma_f : \mathbb{G}(1, 3) \to E$ by $\sigma_f(\ell) = f|_\ell$ and a line ℓ is contained in $Z_{\mathbb{C}}(f)$ if and only if $\sigma_f(\ell) = 0$.

The discriminant $\Sigma \subset \mathbb{C}[z_0, z_1, z_2, z_3]_{(3)}$ consists of those polynomials whose section σ_f is not transversal to the zero section

In most cases the properties we will be interested in are described by a list of numbers associated to elements of some parameter space S. Let us re-interpret the statement from Example 1.3 using this language. If $S = P(\mathbb{C}[z_0, z_1]_{(d)} = \mathbb{C}\mathrm{P}^d$ is the projective space of complex polynomials of degree d, we might be interested in the number of zeroes of these polynomials. We can interpret this number as a map $\beta : \mathbb{C}\mathrm{P}^d \to \mathbb{C}$ given by

$$\beta : f \mapsto \#Z(f).$$

This β is a constant map outside $\Sigma = \{f \mid \mathrm{Res}(f, f') = 0\}$.

The next definition gives a rigorous definition for genericity in our setting.

Definition 1.4 (Generic Properties)**.** Let S be a semialgebraic set[1]. We say that a property β is *generic* for the elements of S if there exists a semialgebraic set $\Sigma \subset S$ of codimension at least one in S such that the property β is true for all elements in $S \backslash \Sigma$. We call the largest (by inclusion) such Σ the *discriminant* of the property β.

When working over the complex numbers most properties are generic in the sense that the discriminant is a proper *complex* algebraic set. Since proper complex algebraic sets in $\mathbb{C}\mathrm{P}^N$ do not disconnect the whole space, these properties are constant on an open dense set. This is a simple observation that we record in the next lemma.

Lemma 1.5. *Let $\Sigma \subsetneq \mathbb{C}\mathrm{P}^N$ be a proper algebraic subset. Then, $\mathbb{C}\mathrm{P}^N \backslash \Sigma$ is path-connected.*

Proof. Let $z_1, z_2 \in \mathbb{C}\mathrm{P}^N \backslash \Sigma$. Choose a complex linear space $L \subset \mathbb{C}\mathrm{P}^N$ of dimension one, such that $z_1, z_2 \in L$. Then, $L \cap \Sigma$ is a subvariety of L. Since L is irreducible, if $\dim(L \cap \Sigma) = 1$, we must have $L \subset \Sigma$, but this contradicts $z_1, z_2 \notin \Sigma$. Thus, we have $\dim(L \cap \Sigma) = 0$, which means that L intersects Σ in finitely many points. Since L is of complex dimension one, it is of real dimenson two, and thus $L \backslash \Sigma$ is path-connected. We find a real path from z_1 to z_2 that does not intersect Σ. $\square$

Very often the properties that we will be interested in are values of some semi-algebraic functions $\beta : S \to \mathbb{C}^n$, as in the second point from Example 1.3. To see this, let $S = P(\mathbb{C}[z_0, \ldots, z_n]_{(d)})$ be the projective space of polynomials and consider the "property" $\beta : S \to \mathbb{C}^{2n+1}$ given by $\beta(f) = (b_0(Z_{\mathbb{C}}(f)), \ldots, b_{2n}(Z_{\mathbb{C}}(f)))$

[1]A semialgebraic set $S \subset \mathbb{R}^n$ is a finite union and intersections of sets of the form $\{f \leq 0\}$ or $\{f < 0\}$, with $f \in \mathbb{R}[x_0, \ldots, x_n]$.

(i.e. $\beta(f)$ is the list of the Betti numbers of the zero set of f in $\mathbb{C}\mathrm{P}^n$; this number does not depend on the representative of f that we pick, as a nonzero multiple of a polynomial has the same zero set as the original polynomial). The property β in this case takes a constant value on the complement of a complex discriminant $\Sigma \subset S$. In other words, there exists $\beta_0 \in \mathbb{C}^{2n+1}$ such that for all $f \in S \backslash \Sigma$ we have $\beta(Z_{\mathbb{C}}(f)) = \beta_0$. In the case $n = 2$, because the genus is $\frac{(d-1)(d-2)}{2}$, we have that $\beta_0 = (1, (d-1)(d-2), 1)$. A similar argument can be done for the third point in Example 1.3: the property "number of lines on the zero set of f" is constant outside a complex discriminant $\Sigma \subset \mathbb{C}[z_0, \ldots, z_3]_{(3)}$.

As already briefly discussed in the beginning of this section, the topological reason for the existence of such strong generic properties over the complex numbers ultimately is Lemma 1.5. The additional technical ingredient that one needs to deduce that topological properties are stable under nondegenerate deformations goes under the name of *Thom's Isotopy Lemma* and we will prove it and discuss its implications later.

1.2 Real discriminants

Moving to the real world, let us copy the notation from the preceding section to the real numbers.

Definition 1.6 (Real projective space)**.** The real projective space $\mathbb{R}\mathrm{P}^n$ of dimension n is defined to be the set of lines through the origin in $\mathbb{R}^{n+1}$. That is, $\mathbb{R}\mathrm{P}^n := (\mathbb{R}^{n+1} \backslash \{0\}) / \sim$, where $y \sim z$, if and only of there exists some $\lambda \in \mathbb{R}\backslash\{0\}$ with $y = \lambda z$. For a point $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ we denote by $[x_0 : x_1 : \ldots : x_n]$ its equivalence class in $\mathbb{R}\mathrm{P}^n$.

Similar to before, we define the projection

$$P : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}\mathrm{P}^n, \ (x_0, x_1, \ldots, x_n) \mapsto [x_0 : x_1 : \ldots : x_n].$$

The space of real homogeneous polynomials is

$$\mathbb{R}[x_0, \ldots, x_n]_{(d)} := \left\{ \sum_{|\alpha|=d} f_\alpha \, x^\alpha \;\middle|\; (f_\alpha) \in \mathbb{R}^N \right\}, \text{ where } N = \binom{n+d}{d}.$$

The projective space of real polynomials is $P(\mathbb{R}[x_0, \ldots, x_n]_{(d)})$. The real projective zero set of k polynomials $f = (f_1, \ldots, f_k)$ is

$$Z(f) = \{[x] \in \mathbb{R}\mathrm{P}^n : f_1(x) = 0, \ldots, f_k(x) = 0\}.$$

Over the Reals we do not have in general an analogue of Lemma 1.5: a proper real algebraic set can in general disconnect the ambient space. To see this, let us look again at the problems discussed in example 1.3, but from the real point of view.

Example 1.7. Let us start by noticing that the complex properties studied in Example 1.3 are still generic over the reals, in the sense that for the generic *real* polynomial the structure of the *complex* zero set has a constant generic behavior; the structure of the *real* zero set is instead highly dependent on the coefficients of f and there is no "generic" behaviour.

(1) A generic univariate polynomial $f \in \mathbb{R}[x]_d$ of degree d has at most d distinct zeros in $\mathbb{R}$, but this number can range anywhere between $\frac{1+(-1)^{d+1}}{2}$ and d. In particular there is no generic number of real zeroes.

    A property which is generic is having real *distinct* zeroes. In this case, however, the real discriminant is not algebraic, but rather just semialgebraic. Unless $d = 2$ it not coincide with the real part of $\{\mathrm{Res}(f, f') = 0\}$: the equation $\mathrm{Res}(f, f') = 0$, which is real for real f, tells us whether f has a double root, but this root can also be complex. The subset of the real part of $\{\mathrm{Res}(f, f') = 0\}$ which corresponds to polynomials with a double real root is only a piece of this discriminant and this piece is selected by imposing some extra inequalities on the coefficients of the polynomial.

(2) The zero set $Z(f) \subset \mathbb{R}\mathrm{P}^2$ of a generic $f \in \mathbb{R}[x_0, x_1, x_2]_{(d)}$ is a smooth curve (being smooth is a generic property) but the topology of this curve depends on the coefficients of the polynomial – Harnack's inequality tells that

$$b_0(Z(f)) \leq \frac{(d-1)(d-2)}{2} + 1.$$

    For instance $\{x_0^2+x_1^2+x_2^2 = 0\} \subset \mathbb{R}\mathrm{P}^2$ is empty and $\{x_0^2-x_1^2-x_2^2 = 0\} \subset \mathbb{R}\mathrm{P}^2$ is homeomorphic to a circle (they are both smooth).

(3) Let $\mathbb{R}[x_0, x_1]_{(3)}$ be the space of homogeneous real polynomials of degree 3. Inside this space there is the cone X of polynomials which are powers of real linear forms: $X = \{f \in \mathbb{R}[x_0, x_1]_{(3)} \mid \exists \ell \in \mathbb{R}[x_0, x_1]_{(1)} : f = \ell^3\}$. The linear span of X is the whole $\mathbb{R}[z_0, z_1]_{(3)}$, as in the complex case. Therefore, for every polynomial $f \in \mathbb{R}[z_0, z_1]_{(3)}$ there exist $\ell_1, \ldots, \ell_s \in \mathbb{R}[x_0, x_1]_{(1)}$ and $\alpha_1, \ldots, \alpha_s \in \mathbb{R}$ such that $f = \sum_{i=1}^{s} \alpha_i \ell_i^3$. However now, differently than from the complex case, there is no generic minimal value that the number s can take. In fact, denoting by $\mathrm{rk}_{\mathbb{R}}(f)$ the minimum such s we have that $\mathrm{rk}_{\mathbb{R}}(f) = 2$ whenever a polynomial has one real zero and $\mathrm{rk}_{\mathbb{R}}(f) = 3$ whenever it has 3 real zeroes.

(4) The zero set $Z \subset \mathbb{R}\mathrm{P}^3$ of a generic cubic $f \in \mathbb{R}[x_0, x_1, x_2, x_3]_{(3)}$ is smooth and it can contain either $27, 15, 7$ or 3 real lines.

Remark 1.8. There exists a *generic* way of counting the lines on $Z(f)$: it is possible to canonically associate a sign $s(\ell)$ to each line $\ell \subset Z(f)$ and the number $\sum_{\ell \subset Z(f)} s(\ell)$ (a signed count) is generically equal to 3.

1.3 Reasonable probability distributions

In the quote of Ginibre it says "one assumes a reasonable probability distribution". He was probably thinking of physically meaningful distributions. But for us this means the following: suppose that $\mathcal{F}$ is a space of geometric objects endowed with a probability distribution, and that $X : \mathcal{F} \to \mathbb{R}^m$ is a random variable on $\mathcal{F}$. If X has symmetries, by which we mean that there is a group G acting on $\mathcal{F}$, such that $X(g \cdot f) = X(f)$ for all g, then the probability distribution is reasonable, if it is invariant under G; that is $g \cdot f \sim f$. This interpretation follows the *Erlangen program* by Felix Klein. In "A comparative review of recent researches in geometry" [21] Klein lays out a perspective on geometry based on a group of symmetries:

> *"Geometric properties are characterized by their remaining invariant under the transformations of the principal group."*

He writes that geometry should be seen as the following comprehensive problem.

> *"Given a manifoldness and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group."*

Therefore, reasonable probability distributions are distributions which respect geometry in Klein's sense. A reasonable probability distribution should not prefer one instance over another if they share the same geometry.

To illustrate this line of thought, we recall Booth's example from the beginning of this section. The space of geometric objects $\mathcal{F}$ is the space of univariate polynomials of degree 8 with coefficients in $\{-1, 1\}$. The random variable $X(f)$ is the number of real zeros of the polynomial $f \in \mathcal{F}$. The group $G = \{-1, 1\}$ acts on $\mathcal{F}$ as $g.f(x) = 1 + \epsilon_1' x + \epsilon_2' x^2 + \epsilon_3' x^3 + \epsilon_4' x^4 + \epsilon_5' x^5 + \epsilon_6' x^6 + \epsilon_7' x^7 + \epsilon_8' x^8$, where $\epsilon_i' = \epsilon_i g^i$. Since for all i we have $\epsilon_i g^i \in \{-\epsilon_i, \epsilon_i\}$ and since $\epsilon_i \sim -\epsilon_i$, we see that $gf \sim f$. In this sense, the distribution proposed by Booth is reasonable. In many cases the space $\mathcal{F}$ comes with the structure of a smooth manifold (e.g. a vector space, a Lie group or a homogeneous space) and in this case a "reasonable" probability distribution should be absolutely continuous with respect to Lebesgue measure (notice that the notion of sets of measure zero is well defined on a smooth manifold and independent of the possible choice of an actual measure).

Let us introduce the Gaussian distribution.

Definition 1.9 (Nondegenerate Gaussian distribution)**.** A probability distribution on $\mathbb{R}^N$ is said to be *nondegenerate Gaussian* if there exist a positive definite symmetric matrix $\Sigma \in \mathrm{Sym}(N, \mathbb{R})$ and a vector $\mu \in \mathbb{R}^N$ such that for all $U \subseteq \mathbb{R}^N$ measurable subset we have:

$$\mathbb{P}(U) = \frac{1}{((2\pi)^N \det(\Sigma))^{1/2}} \int_U e^{-\frac{(y-\mu)^T \Sigma^{-1}(y-\mu)}{2}} dy.$$

Whenever $\mu = 0$ the distribution is called *centered*. The *standard Gaussian distribution* corresponds to the choice $Q = \mathbf{1}$ and $\mu = 0$. For a random variables ξ on the real line distributed as a standard Gaussian we will write $\xi \sim N(0, 1)$ and sometimes also call it a *standard normal*. More generally, if $X \in \mathbb{R}^N$ has a Gaussian density, we will say that X is a multivariate nondegenerate Gaussian variable with mean μ and covariance matrix Σ, and we will write $X \sim N(\mu, \Sigma)$.

From now on we will always assume that Gaussian distributions are nondegenerate and centered.

In these lectures, when $\mathcal{F}$ is a linear space (e.g. the space of polynomials) we will mostly consider a special class of distributions called *Gaussian*. The reason for this is that we are interested in zeros of polynomials, and they are invariant under scaling of polynomials. Therefore, a reasonable probability distribution should be induced by a distribution in projective space $P(\mathbb{R}[x_0, \ldots, x_n]_{(d)})$. What we mean by this is the following. The space of real polynomials $\mathbb{R}[x_0, \ldots, x_n]_{(d)}$ is a real vector space of dimension $N = \binom{n+d}{d}$ and therefore it is isomorphic to $\mathbb{R}^N$. We fix an isomorphism

$$\varphi : \mathbb{R}^N \to \mathbb{R}[x_0, \ldots, x_n]_{(d)}$$

between these two spaces (for example the isomorphism could be given by the coefficients list of the polynomial in some basis). Then, we fix on $\mathbb{R}^N$ a nondegenerate Gaussian distribution $N(\mu, \Sigma)$ in the sense of Definition 1.9. Then, a Gaussian distribution on $\mathbb{R}[x_0, \ldots, x_n]_{(d)}$ is defined as follows:

$$\mathbb{P}(f \in A) = \frac{1}{((2\pi)^N \det(\Sigma))^{1/2}} \int_{\varphi^{-1}(A)} e^{-\frac{(y-\mu)^T \Sigma^{-1}(y-\mu)}{2}} dy.$$

That is, if $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^N$ and $b_i := \varphi(e_i)$, then

$$f = \xi_1 b_1 + \cdots + \xi_N b_N,$$

where $\xi_1, \dots, \xi_N$ is a family of i.i.d. standard normal random variables.

A random variable $f \in \mathbb{R}[x_0, \dots, x_n]_{(d)}$ with a reasonable probability distribution should then be given by $f = \varphi(X)$, where $X \in \mathbb{R}^N$ has a density, has independent entries, and is invariant under transformations by the orthogonal group $O(N)$. The last point reflects the fact that we do not want any preferred direction in $\mathbb{R}[x_0, \dots, x_n]_{(d)}$, because the zero set of a polynomial only depend on its class in projective space $P(\mathbb{R}[x_0, \dots, x_n]_{(d)})$. The Gaussian distribution satisfies these requirements and the next lemma shows that it is the only probability distribution with this property.

Lemma 1.10. *Let $X = (X_1, \dots, X_N)$ be a random vector with a density $\phi(X)$ such that*

(1) the X_i are independent;

(2) for all $U \in O(N)$ we have $UX \sim X$.

Then, $X \sim N(0, \sigma^2 \mathbf{1}_N)$ for some $\sigma^2 > 0$. Here, $\mathbf{1}_N$ denotes the identity matrix.

Proof. Since the X_i are independent, we have $\phi(x) = \phi_1(x_1) \cdots \phi_N(x_N)$. Since permutation matrices are orthogonal, we have $X_i \sim X_j$ for every pair i, j. Moreover, $X_i \sim -X_i$ so that ϕ_i only depends on X_i^2. We get for every $1 \leq i \leq N$ that $\phi_i(x_i) = \lambda(x_i)$ for some function λ. We have then $\phi(x) = \lambda(x_1) \cdots \lambda(x_N)$. Next, we use that $X \sim (X_1^2 + X_2^2, 0, X_3, \dots, X_N)$ to deduce that

$$\lambda(x_1^2 + x_2^2)\lambda(0) = \lambda(x_1)\lambda(x_2).$$

This shows first that $\lambda(0) \neq 0$, and second by setting $\theta(u) := \lambda(u)/\lambda(0)$ we get $\theta(x_1^2 + x_2^2) = \theta(x_1^2)\theta(x_2^2)$. This forces θ to be the exponential map. There exists $a, b \in \mathbb{R}$ with $\theta(u) = ae^{\frac{b}{2}u}$, so that

$$\phi(x) = \lambda(x_1^2) \cdots \lambda(x_N^2) = a^{2N} e^{\frac{b}{2} x^T x}.$$

Hence, X must be a Gaussian random variable with covariance matrix $\sigma^2 \mathbf{1}_N$ for some $\sigma^2 > 0$. □

Let us discuss one important property of centered Gaussian distributions. The matrix $\Sigma > 0$ in Section 1.3 is positive definite and it therefore defines a scalar product on $\mathbb{R}^N$ by the rule $\langle y_1, y_2 \rangle_\Sigma := y_1^T \Sigma^{-1} y_2$. If we choose an orthonormal basis $\mathcal{B}_\Sigma = \{e_j\}_{j=1,\dots,N}$ for the scalar product $\langle \cdot, \cdot \rangle_\Sigma$, then a random element X from the Gaussian distribution Section 1.3 can be written as: $X = \sum_{j=1}^N \xi_j e_j$, where $\xi_1, \dots, \xi_N$ is a family of i.i.d. standard normal random variables. Conversely, any scalar product $\langle , \langle$ in $\mathbb{R}^N$ implies a centered Gaussian distribution with density $(2\pi)^N(\det(\Sigma))^{\frac{1}{2}}e^{-\frac{\langle y,y\rangle}{2}}$, where Σ is the covariance matrix defined by $\Sigma_{i,j} = \mathrm{Cov}(y_i, y_j)$. This shows that there is a one-to-one correspondence between centered Gaussian distributions and inner products in $\mathbb{R}^N$. This observation will play a crucial practical role later in the book, when dealing with space of random Gaussian functions, for which we will need the presentation as a Gaussian combination of some basis elements.

We want to introduce now a reasonable probability distribution on the space $\mathbb{R}[x_0, \ldots, x_n]_{(d)}$ and, in line with the previous discussion, we require that such distribution satisfies some invariance suggested by the geometry of the objects we are considering.

(1) We want it to be Gaussian for the reasons discussed above.
(2) We want to get a model of randomness for which there are no preferred points or directions in the projective space $\mathbb{R}\mathrm{P}^n$. Using the language of group invariance, there is a representation $\rho : O(n+1) \to \mathrm{GL}(\mathbb{R}[x_0, \ldots, x_n]_{(d)})$ given by change of variables and we require our distribution to satisfy the property of being invariant under all elements of $\rho(O(n+1))$.

It turns out that the two conditions above do not identify uniquely a probability distribution, and in fact, as we will see later in these lectures, there is a whole family of such distributions. We will call them *invariant distributions*.

1.4 The Kostlan distribution

The Kostlan distribution is a special case of an invariant distribution which has some additional special features that make it good for comparisons with complex algebraic geometry. In order to define it, it is helpful to use the following notation:

$$\binom{d}{\alpha} := \frac{d!}{\alpha_0! \cdots \alpha_n!}.$$

Choose the isomorphism $\varphi_{\mathrm{Kostlan}} : \mathbb{R}^N \to \mathbb{R}[x_0, \ldots, x_n]_{(d)}$ defined by

$$\varphi_{\mathrm{Kostlan}}((f_\alpha)_\alpha) = \sum_{|\alpha|=d} f_\alpha \cdot \sqrt{\binom{d}{\alpha}} x_0^{\alpha_0} \cdots x_n^{\alpha_n}.$$

Then, for a measurable $A \subseteq \mathbb{R}[x_0, \ldots, x_n]_{(d)}$ its probability with respect to the Kostlan distribution is defined to be:

$$\mathbb{P}(f \in A) = \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\varphi^{-1}_{\mathrm{Kostlan}}(A)} e^{-\frac{\|y\|^2}{2}} dy.$$

A simple way to write down a Kostlan polynomial is by taking a combination of standard Gaussians as follows:

$$f(x) = \sum_{|\alpha|=d} \xi_\alpha \cdot \sqrt{\binom{d}{\alpha}}\, x_0^{\alpha_0} \cdots x_n^{\alpha_n},$$

where $\{\xi_\alpha\}_{|\alpha|=d}$ is a family of standard, independent Gaussian variables on $\mathbb{R}$. Similarly, a complex Kostlan polynomial is Section 1.4 where $\{\xi_\alpha\}_{|\alpha|=d}$ is a family of standard, independent Gaussian variables on $\mathbb{C}$.

Kostlan polynomials are invariant as recorded in the next lemma.

Lemma 1.11. *The Kostlan distribution is invariant under orthogonal change of variables. The complex Kostlan distribution is invariant under unitary change of variables.*

The Kostlan distribution, among the invariant ones, is the unique (up to multiples) for which a random polynomial can be written as a combination of independent Gaussians in front of the standard monomial basis. The next proposition follows from a result, which we state in Theorem 8.4.

Proposition 1.12. *Among the invariant distributions, the Kostlan one is the unique (up to multiples) such that a random polynomial can be written as a linear combination of the standard monomial basis with coefficients independent Gaussians.*

1.5 Expected properties

As we have seen, if the discriminant is a complex algebraic set, we have strong genericity over the complex numbers: the reason for this is Lemma 1.5, which says that the complex discriminant does not disconnect $\mathbb{C}\mathrm{P}^N$. However, if the discriminant is a real hypersurface, in general it might disconnect $\mathbb{R}\mathrm{P}^N$, this is why in Figure 1.1 there are two regions with different properties. Therefore, over the real numbers we might not have a notion of strong genericity, and we adopt a random point of view. The next definition is the probabilistic analogue of Definition 1.4.

Definition 1.13 (Expected Properties)**.** Let S be a semialgebraic set. A measurable property is a measurable function $\beta : S \to \mathbb{C}^m$. If we have a (reasonable) probability distribution on S, we call $\mathbb{E}_{s\in S}\, \beta(s)$ the expected property.

In fact, Definition 1.4 is a special case of Definition 1.13. Let us revisit Example 1.3 from a probabilistic point of view.

Example 1.14. Let us endow the space of real polynomials with the Kostlan distribution. Then we can ask for the expectation of the real version of the properties that we have discussed in Example 1.3.

(1) Let $f \in \mathbb{R}[x_0, x_1]_{(d)}$ be a Kostlan polynomial of degree d in 2 variables. Then, for the generic element $f \in \mathbb{R}[x_0, x_1]_{(d)}$ the number of complex zeroes is d, but the expected number of real zeros of f is $\sqrt{d}$.

(2) Let $f \in \mathbb{R}[x_0, x_1, x_2]_{(d)}$ be a Kostlan polynomial of degree d in 3 variables. There exist constants $c, C > 0$ such that the expected value of the zero-th Betti number $b_0(f)$ of $Z(f)$ satisfies $cd \leq \mathbb{E}\, b_0(f) \leq Cd$.

(3) Let $f \in \mathbb{R}[x_0, x_1]_{(3)}$ be a Kostlan polynomial, then the expectation of its real rank $\mathrm{rk}_{\mathbb{R}}(f)$ is $\frac{9-\sqrt{3}}{2}$.

(4) Let $f \in \mathbb{R}[x_0, x_1, x_2, x_3]_{(3)}$ be a Kostlan polynomial of degree 3 in 3 variables. Then, the expected number of real lines on $Z(f)$ is $6\sqrt{2} - 3$.

The first example was proven in [8], the second in [13], and the third is actually a consequence of the first example, but we will also prove them in the remainder of these lectures. The fourth example was proved in [36]. We want to emphasize that the first two of those examples obey a square-root law – the expected value of the real property has the order of the square root of the generic value of the complex property.

In closing of this introductory lecture we want to explain why generic properties are, in fact, random properties in disguise. The essence of this is a simple observation: suppose $z \in \mathbb{C}\mathrm{P}^N$ is a random variable that is supported on some full-dimensional subset of $\mathbb{C}\mathrm{P}^N$. In particular, this implies that, if β is a property with discriminant Σ, and if $\Sigma \subsetneq \mathbb{C}\mathrm{P}^N$ is an algebraic variety, then $\mathbb{P}\{z \in \Sigma\} = 0$, and so $\mathbb{P}\{\beta(z) \text{ has the generic value}\} = 1$. Therefore

$$\mathbb{E}\, \beta(z) = \text{ generic value of } \beta(z).$$

It is interesting to approach the problem of computing generic properties from a probablistic point of view.

This strategy becomes more effective as counting problems over the complex numbers becomes more complicated. Consider $f = \sum_{i=0}^d c_i x_0^i x_1^{d-i} \in \mathbb{C}[x_0, x_i]_{(d)}$, where the real and imaginary parts of the c_i are independent Gaussian random variables such that $\Re(c_i) \sim N(0, \frac{1}{2}\binom{d}{i})$ and $\Im(c_i) \sim N(0, \frac{1}{2}\binom{d}{i})$ (the factor $\frac{1}{2}$ is for normalizing the variance to $\mathbb{E}\,|c_i|^2 = 1$). Such a polynomial is called a *complex Kostlan polynomial.* The distribution we have put on the coefficients is absolutely continuous with respect to Lebesgue measure on the space of coefficients, and in fact the distribution of $P(f)$ is supported on the whole $\mathbb{C}\mathrm{P}^d$. Therefore, we know that with probability one we have that $\#Z_{\mathbb{C}}(f)$ equals some constant (we know this constant is d, but let's pretend for a second that we did not know this). Then, if we can find a way (and there is such a way) to compute by elementary means the expectation of $\#Z_{\mathbb{C}}(f)$, we have found its generic value.

2 Riemannian manifolds and probability

In this chapter we will recall some basic concepts from differential geometry, leading to the notion of *volume* of a Riemannian manifold. We will use this definition for computing the volumes of several basic objects in algebraic geometry: spheres, projective spaces, orthogonal groups, and the Grassmannian, and then interpret these in simple probabilistic terms. For more details on smooth manifolds and Riemannian geometry we refer the reader to [25].

2.1 Basics from differential geometry

We assume the reader is familiar with the notions of smooth manifolds, smooth maps and differential forms. Here we recall just some basic definitions, with the purpose of setting our notation and introducing some examples of special interest for what follows.

2.1.1 Basic notions and examples

Definition 2.1 (Smooth Manifold)**.** A smooth manifold M of dimension m is a Hausdorff and second countable topological space together with a family of homeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subset \mathbb{R}^m$, $\alpha \in A$, where $U_\alpha \subset M$, $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^m$ are open sets, such that

(1) $M = \bigcup_{\alpha\in A} U_\alpha$;

(2) The change of coordinates $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is smooth for all $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$.

Each pair $(U_\alpha, \varphi_\alpha)$ is called a *chart*. The family $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ is called an *atlas* for M. If we replace $\mathbb{R}^m$ by $\mathbb{C}^m$ and if we require that each change of coordinates is a holomorphic map, we call M a *complex manifold*.

We will assume that the atlas we work with is maximal, so that we will have all possible charts available.

Example 2.2. We consider the unit circle: $M = S^1 = \{x \in \mathbb{R}^2 \mid x^T x = 1\}$. We can cover S^1 with two charts $S^1 = U_1 \cup U_2$, where $U_1 = M \setminus \{(0,1)\}$ and $U_2 = M \setminus \{(0,-1)\}$. The homeomorphisms are the two *stereographic projections* $\varphi_1(x,y) = \frac{x}{1-y}$ and $\varphi_2(x,y) = \frac{x}{1+y}$. The change of coordinates is the smooth map $(\varphi_2 \circ \varphi_1^{-1})(t) = t^{-1}$, so that S^1 is indeed a smooth manifold.

Example 2.3 (Real projective spaces)**.** The real projective space $\mathbb{R}\mathrm{P}^n$ can be endowed with the structure of smooth manifold as follows. Recall that $\mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1}\backslash\{0\})/\sim$, where $p_1 \sim p_2$ if and only if there exists $\lambda \neq 0$ such that $p_1 = \lambda p_2$. We denote by $[x_0, \ldots, x_n]$ the equivalence class of $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}\backslash\{0\}$ (the x_j are called *homogeneous coordinates*). For every $j = 0, \ldots, n$ consider the open set $U_j \subset \mathbb{R}\mathrm{P}^n$ defined by:

$$U_j = \{[x_0, \ldots, x_n] \text{ such that } x_j \neq 0\},$$

together with the homeomorphism $\psi_j : U_j \to \mathbb{R}^n$ given by:

$$\psi_j([x_0, \ldots, x_n]) = \left(\frac{x_0}{x_j}, \ldots, \frac{\widehat{x_j}}{x_j}, \ldots, \frac{x_n}{x_j}\right)$$

(here the "hat" symbol denotes that this element has been removed from the list). The inverse $\psi_j^{-1} : \mathbb{R}^n \to \mathbb{R}\mathrm{P}^n$ is given by:

$$\psi_j^{-1}(y_0, \ldots, \widehat{y}_j, \ldots, y_n) = [y_0, \ldots, 1, \ldots y_n],$$

where the "1" is in position j. As a consequence, for every $i \neq j$ we have:

$$\psi_i \circ \psi_j^{-1}(y_0, \ldots, \widehat{y}_j, \ldots, y_n) = \left(\frac{y_0}{y_i}, \ldots, \frac{\widehat{y}_i}{y_i}, \ldots, \frac{1}{y_i}, \ldots \frac{y_n}{y_j}\right),$$

which is a diffeomorphism of $\mathbb{R}^n\backslash\{0\}$ to itself.

Generalizing the previous example we obtain *real Grassmannians.*

Example 2.4 (real Grassmannians)**.** The real Grassmannian $G(k,n)$ consists of the set of all k-dimensional vector subspaces of $\mathbb{R}^n$, endowed with the quotient topology of the map:

$$q : \{M \in \mathbb{R}^{n\times k} \text{ such that } \mathrm{rk}(M) = k\} \to G(k,n),$$

where $q(M) = \mathrm{span}\{\text{columns of } M\}$ In other words, $G(k,n)$ (as a topological space) can be considered as the quotient of the set of $n \times k$ real matrices of rank k

(viewed as a subset of $\mathbb{R}^{n\times k}$) under the equivalence relation:

$$M_1 \sim M_2 \iff \text{there exists } L \in \mathrm{GL}(\mathbb{R}^k) \text{ such that } M_1 = M_2 L.$$

Observe that $G(1,n) = \mathbb{R}\mathrm{P}^{n-1}$ and that the above definition mimics the equivalence relation $v_1 \sim v_2$ if and only if there exists $\lambda \in \mathrm{GL}(\mathbb{R}) = \mathbb{R}\backslash\{0\}$ such that $v_1 = \lambda v_2$.

We want to endow $G(k,n)$ with the structure of a smooth manifold. For every multi-index $J = (j_1, \ldots, j_k) \in \left\{ {n \atop k} \right\}$ we denote by $M|_J$ the $k \times k$ submatrix of M obtained by selecting the rows $j_1, \ldots, j_k$ (in this way $M|_{J^c}$ denotes the $(n-k)\times k$ submatrix of M obtained by selecting the complementary rows). For every such multi-index J we define the open set:

$$U_J = \{[M] \in G(k,n) \text{ such that } \det(M|_J) \neq 0\}.$$

(Note that this set is well defined.) Mimicking again the definition for projective spaces, we define the manifold charts $\psi_J : U_J \to \mathbb{R}^{(n-k)\times k}$ by:

$$\psi_J([M]) = (MM_J^{-1})|_{J^c}.$$

The expression of the inverse of a matrix in terms of its determinant and its cofactor sshows that for every pair of indices $J_1, J_2 \in \left\{ {n \atop k} \right\}$ the map $\psi_{J_2} \circ \psi_{J_1}^{-1}$ is smooth. In this way $\{(U_J, \psi_J)\}_{J \in \left\{ {n \atop k} \right\}}$ is a smooth atlas for the $k(n-k)$-dimensional manifold $G(k,n)$.

Exercise 2.1. Fill in all the details in the previous definition of the smooth structure on the Grassmannian.

Example 2.5 (The Complex projective line)**.** Recall that the complex line $\mathbb{C}\mathrm{P}^1$ is defined as the quotient space $(\mathbb{C}^2\backslash\{0\})/\sim$ where $(z_0, z_1) \sim \lambda(z_0, z_1)$ for every $\lambda \in \mathbb{C}\backslash\{0\}$. As we did for real projective spaces, we denote by $[z_0, z_1]$ the homogeneous coordinates of a point on $\mathbb{C}\mathrm{P}^1$. Consider the two open sets $U_0 = \{z_0 \neq 0\}$ and $U_1 = \{z_1 \neq 0\}$ together with the charts $\psi_j : U_j \to \mathbb{C} \simeq \mathbb{R}^2$ for $j = 0, 1$ which are given by:

$$\psi_1([z_0, z_1]) = \frac{z_0}{z_1} \quad \text{and} \quad \psi_0([z_0, z_1]) = \frac{z_1}{z_0}.$$

We have that $\psi_0 \circ \psi_1^{-1}(z) = \frac{1}{z}$, which is a holomorphic map $\mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ and consequently, using the identification $\mathbb{C} \simeq \mathbb{R}^2$, a smooth map $\mathbb{R}^2\backslash\{0\} \to \mathbb{R}^2\backslash\{0\}$. If we wanted, we could also work with ψ_j as a real map, as follows (however, as the reader will see, using the field structure of $\mathbb{C} \simeq \mathbb{R}^2$ simplifies a lot the computations). Given $(x_0, y_0, x_1, y_1) \in \mathbb{R}^4$, let $(x_0 + iy_0, x_1 + iy_1) = (z_0, z_1) \in \mathbb{C}^2$.

We can write $\psi_1 : U_1 \to \mathbb{C} \simeq \mathbb{R}^2$ as

$$\psi_1([x_0 + iy_0, x_1 + iy_1]) = \frac{x_0 + iy_0}{x_1 + iy_1} = \frac{x_0x_1 + y_0y_1}{x_1^2 + y_1^2} + i \cdot \frac{y_0x_1 - x_0y_1}{x_1^2 + y_1^2}$$

which means that the real map $\psi_1 : U_1 \to \mathbb{R}^2$ is given by:

$$\psi_1([x_0 + iy_0, x_1 + iy_1]) = \left(\frac{x_0x_1 + y_0y_1}{x_1^2 + y_1^2}, \frac{y_0x_1 - x_0y_1}{x_1^2 + y_1^2}\right),$$

with inverse $\psi_1^{-1} : \mathbb{R}^2 \to \mathbb{C}\mathrm{P}^1$ given by:

$$\psi_1^{-1}(x, y) = [1, x + iy].$$

In particular $\psi_0 \circ \psi_1|_{U_0\cap U_1}^{-1}$ is given by $(x, y) \mapsto \frac{1}{x^2+y^2}(x, -y)$, which is indeed a smooth map $\mathbb{R}^2\backslash\{0\} \to \mathbb{R}^2\backslash\{0\}$.
The complex projective line $\mathbb{C}\mathrm{P}^1$ is homeomorphic to S^2 and in fact, as smooth manifolds, they are indistinguishable.

Exercise 2.2. Generalize the previous example and prove that $\mathbb{C}\mathrm{P}^n$ can be endowed with the structure of a smooth $2n$-dimensional manifold. Moreover, prove that the atlas for $\mathbb{C}\mathrm{P}^n$ can be chosen in such a way that the change of coordinates are holomorphic maps between open subsets of $\mathbb{C}^n$, so that $\mathbb{C}\mathrm{P}^n$ is a complex manifold.

It will be useful for the sequel also to recall some notation on tangent spaces and differential of a smooth map. To this end, recall that a *derivation* of M at x is a linear function $\mathrm{D} : \mathscr{C}^\infty(M, \mathbb{R}) \to \mathbb{R}$ such that $\mathrm{D}(fg) = \mathrm{D}(f)g(x) + f(x)\mathrm{D}(g)$. The (abstract) *tangent space* of M at a point $x \in M$ is then

$$T_xM := \{\mathrm{D} : \mathscr{C}^\infty(M, \mathbb{R}) \to \mathbb{R} \mid \mathrm{D} \text{ is a derivation of } M \text{ at } x\},$$

and we have $\dim T_xM = \dim M$ for all $x \in M$; see [25, Proposition 3.10].

As an example, let us consider the tangent space to $\mathbb{R}^m$. At a point $a \in \mathbb{R}^m$ the tangent space $T_a\mathbb{R}^m$ consists of all directional derivatives at a (see [25, Proposition 3.2]):

$$T_a\mathbb{R}^m := \mathrm{span}\left\{\frac{\partial}{\partial x_i}\Big|_a \;\Big|\; i = 1, \dots, m\right\} \cong \mathbb{R}^m.$$

Let now (φ, U) be a chart of M, $x \in M$ and $a := \varphi(x)$. We denote by $(\varphi^{-1})_*(\frac{\partial}{\partial x_i}\big|_a)$ the derivation that acts as

$$(\varphi^{-1})_*\Big(\frac{\partial}{\partial x_i}\Big|_a\Big)(f) := \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(a), \text{ for } f \in \mathscr{C}^\infty(M, \mathbb{R}).$$

The map $(\varphi^{-1})_*$ is also called a *push-forward* for derivations, and it is a linear isomorphism of n-dimensional vector spaces. This, together with Subsection 2.1.1 implies

$$T_xM = \operatorname{span}\Big\{(\varphi^{-1})_*\Big(\frac{\partial}{\partial x_i}\Big|_a\Big) \;\Big|\; i=1,\dots,m\Big\}, \quad \text{where } a=\varphi(x).$$

Example 2.6. As an example we consider the $(n-1)$-dimensional unit sphere $S^{n-1}=\{x\in\mathbb{R}^n \mid x^Tx=1\}$. Let $x\in S^{n-1}$. Then, we can identify

$$T_xS^{n-1}\cong x^\perp=\{y\in\mathbb{R}^n \mid x^Ty=0\}.$$

Every point $y\in x^\perp$ is identified with the derivation that acts as the directional derivative in direction y; i.e., $y(f)=\sum_{i=1}^{n+1} y_i\frac{\partial}{\partial x_i}F(x)$ for $F\in\mathscr{C}^\infty(S^n,\mathbb{R})$.

In the previous example we identified T_xS^n with a linear subspace of $\mathbb{R}^{n+1}$. In fact, for every submanifold $M\hookrightarrow\mathbb{R}^n$ we have such an identification. Let $M\hookrightarrow\mathbb{R}^n$ and $x\in M$. For a curve $\gamma:(-\epsilon,\epsilon)\to M$ with $\gamma(0)=x$ we define the *geometric tangent vector* $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)|_{t=0}$. The linear span of all such tangent vectors is called the *geometric tangent space* of M at x.

Lemma 2.7. *The geometric tangent space of a submanifold $M\hookrightarrow\mathbb{R}^n$ at x is isomorphic to T_xM.*

Proof. Let $\varphi:U\to\mathbb{R}^m$ be a chart of M with $x\in M$ and $a=\varphi(x)$. Then, writing the curve γ in coordinates, we get a curve $(\varphi\circ\gamma)(t)$ in $\mathbb{R}^m$ through a. It's derivative $\frac{\mathrm{d}}{\mathrm{d}t}(\varphi\circ\gamma)|_{t=0}$ is a vector v in $\mathbb{R}^m$, which we can identify with the derivation $\sum_{i=1}^n v_i\frac{\partial}{\partial x_i}|_a\in T_a\mathbb{R}^m$. Using Subsection 2.1.1, we get an identification of the geometric tangent space with M. □

Let now $F:M\to N$ be a smooth map. We denote by $\mathrm{D}_xF:T_xM\to T_{F(x)}N$ its differential. Let us consider D_xF in coordinates: suppose $x\in U$ and $F(x)\in V$. We check how D_xF acts on a basis for T_xM For this, we denote $a:=\varphi(x)$ and $b=\psi(F(x))$. There exists $c_{i,j}\in\mathbb{R}$ such that

$$\mathrm{D}_xF((\varphi^{-1})_*(\tfrac{\partial}{\partial x_i}|_a))(f)=\sum_{i=1}^n c_{i,j}\frac{\partial}{\partial x_i}(f\circ\psi^{-1})(b) \quad \text{for } f\in\mathscr{C}^\infty(N,\mathbb{R}),$$

because the derivations $(\psi^{-1})_*(\frac{\partial}{\partial x_i}|_b)$ form a basis for $T_{F(x)}N$, by Subsection 2.1.1. On the other hand, $\mathrm{D}_xF((\varphi^{-1})_*(\frac{\partial}{\partial x_i}|_a))(f)=\frac{\partial}{\partial x_i}(f\circ F\circ\varphi^{-1})(a)$, by definition of

$\mathrm{D}_x F$. Therefore,

$$C = (c_{i,j}) \in \mathbb{R}^{n\times n}, \quad \text{where} \quad c_{i,j} = \frac{\partial}{\partial x_i}(x_j \circ \psi \circ F \circ \varphi^{-1})(a),$$

and where x_j is the j-th coordinate functions, represents $\mathrm{D}_x F$ relative to the bases $((\varphi^{-1})_*(\frac{\partial}{\partial x_i}|_a))$ and $((\psi^{-1})_*(\frac{\partial}{\partial x_i}|_b))$.

A point $x \in M$ is called a *regular point* of $F : M \to N$ if $\mathrm{D}_x F : T_x M \to T_{F(x)} N$ is surjective. Note that a necessary condition for regular points to exist is $\dim M \geq \dim N$. We call a point $y \in N$ a *regular value* of F, if every $x \in F^{-1}(y)$ is a regular point of F. For a proof of the following see, e.g., [5, Theorem A.9].

Proposition 2.8. *The fiber $F^{-1}(y)$ of a regular value $y \in N$ is a smooth submanifold of dimension $\dim M - \dim N$. Its tangent space at x is $T_x F^{-1}(y) = \ker \mathrm{D}_x F$.*

A useful consequence of this proposition is the following result.

Corollary 2.9. *Let $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n]$ be k polynomials in $n \geq k$ variables and let $M = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_k(x) = 0\}$ be the real algebraic set defined as the zero set of the f_i. We assume that for each $x \in M$, the Jacobian matrix $J(x) = (\frac{\partial}{\partial x_j} f_i(x))_{i=1,\ldots,k, j=1,\ldots,n} \in \mathbb{R}^{k\times n}$ has full rank. Then:*

(1) $M \hookrightarrow \mathbb{R}^m$ is a smooth manifold of dimension $n - k$;

(2) the geometric tangent space is $T_x M \cong \ker J(x)$.

Proof. The first item follows from Proposition 2.8 by noticing that 0 is a regular value of the smooth map $\mathbb{R}^m \to \mathbb{R}^k, x \mapsto (f_1(x), \ldots, f_k(x))$. The second item also follows from Proposition 2.8 and the fact that $J(x)$ is the derivative of this map in coordinates. □

Remark 2.10. In fact, the statement of Corollary 2.9 is still true, if we replace polynomials by smooth maps.

A smooth map $F : M \to N$ whose differential is a submersion at every point enjoys, of course, some special properties. For instance, if F is surjective and proper (i.e. the preimage under F of a compact set in N is compact in M) the map is a *locally trivial fibration*: there exists a smooth manifold L such that for every point $y \in N$ there exists a neighborhood U of y and a diffeomorphism $\varphi : F^{-1}(U) \to U \times L$ such that $p_1 \circ \varphi = F$, where $p_1 : U \times L \to U$ denotes the projection on the first factor. This result is called *Ehresmann's fibration lemma.*

We can use Proposition 2.8 also to produce submanifolds of real projective spaces, as follows.

Example 2.11 (Smooth projective hypersurfaces)**.** Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ be a homogeneous polynomial of degree d. Since F is homogeneous, the following set $Z(F) \subset \mathbb{R}\mathrm{P}^n$ is well defined:

$$Z(F) = \{[x_0, \ldots, x_n] \in \mathbb{R}\mathrm{P}^n \text{ such that } F(x_0, \ldots, x_n) = 0\}.$$

If the vector

$$\nabla F := \begin{pmatrix} \frac{\partial F}{\partial x_0} \\ \vdots \\ \frac{\partial F}{\partial x_n} \end{pmatrix}$$

(the gradient of F) is nonzero at every non-zero point of $\{F = 0\} \subset \mathbb{R}^{n+1}$ (a non-degeneracy condition), then $Z(F)$ is a smooth submanifold of $\mathbb{R}\mathrm{P}^n$. To see this we use the fact that being a submanifold is a local property, i.e. in order to prove that $Z(F)$ is a submanifold, it is enough to cover $\mathbb{R}\mathrm{P}^n$ with the open sets $\mathbb{R}\mathrm{P}^n = \bigcup_{j=}^n U_j$ defined in Example 2.3 and prove that $Z(F) \cap U_j$ is a submanifold of $\mathbb{R}\mathrm{P}^n$ for every $j = 0, \ldots, n$. Moreover, since being a submanifold is invariant under diffeomorphisms, it is enough to prove that $\psi_j(Z(F) \cap U_j)$ is a submanifold of $\mathbb{R}^n$ for every $j = 0, \ldots, n$ (the $\psi_j : U_j \to \mathbb{R}^n$ are the manifold charts for $\mathbb{R}\mathrm{P}^n$). Now, if we define the function $f_j : \mathbb{R}^n \to \mathbb{R}$ by $f_j(y_0, \ldots, \widehat{y}_j, \ldots, y_n) = F(y_0, \ldots, 1, \ldots, y_n)$, the set $\psi_j(Z(F) \cap U_j)$ is given by the equation:

$$\psi_j(Z(F) \cap U_j) = \{f_j = 0\}.$$

The condition $\nabla F|_{\{F=0\}\setminus\{0\}} \neq 0$ tells that the equation $\{f_j = 0\}$ is regular (i.e. 0 is a regular value of f_j). In fact if for some $y \in \{f_j = 0\}$ we had $\nabla f_j(y) = 0$, then:

$$\begin{aligned} \nabla F(y_0, \ldots, 1, \ldots, y_n)^T &= \left(\frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_j}, \ldots, \frac{\partial F}{\partial x_n}\right)\Bigg|_{(y_0,\ldots,1,\ldots,y_n)} \\ &= \left(0, \ldots, 0, \frac{\partial F}{\partial x_j}\bigg|_{(y_0,\ldots,1,\ldots,y_n)}, 0, \ldots, 0\right) \\ &= (0, \ldots, 0, d \cdot F(y_0, \ldots, 1, \ldots, y_n), 0, \ldots, 0) \\ &= 0 \end{aligned}$$

contradicting the non-degeneracy condition (in the last line we have used Euler's identity for homogeneous functions $d \cdot F(x) = \sum_{j=0}^n x_j \frac{\partial F}{\partial x_j}(x)$.) We will review this example in Example 2.19 below.

Exercise 2.3. Let $F : \mathbb{R}^{n+1} \to \mathbb{R}$ as in the previous example be a homogeneous polynomial of degree d and define the function $f = F|_{S^n}$. Prove that if for every $x \in \mathbb{R}^{n+1}\setminus\{0\}$ such that $F(x) = 0$ we have $\nabla F(x) \neq 0$, then the equation $\{f = 0\}$

is a regular equation on S^n (i.e. $\{f = 0\}$ is a smooth submanifold of S^n, of dimension $n-1$ if nonempty). Prove that the covering map $q : S^n \to \mathbb{R}\mathrm{P}^n$ restricts to a smooth covering map $p|_{\{f=0\}} : \{f = 0\} \to Z(F) \subset \mathbb{R}\mathrm{P}^n$.

2.1.2 Vector bundles

Definition 2.12 (Vector bundle)**.** A *vector bundle* of rank k is a triple (π, E, M) where E and M are smooth manifolds and

$$\pi : E \longrightarrow M$$

is a smooth map such that:

(1) there is an open cover $M = \bigcup_{\alpha\in A} V_\alpha$ and diffeomorphisms $\psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^k$ such that the following diagram is commutative:

$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr, "\pi"'] & & V_\alpha \times \mathbb{R}^k \arrow[dl, "p_1"] \\ & V_\alpha & \end{tikzcd}$$

(p_1 is the projection on the first factor). The family $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ is called a *vector bundle atlas*, and each ψ_α is called a *trivialization*.

(2) whenever $V_\alpha \cap V_\beta \neq \emptyset$, there is a continuous map $g_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathrm{GL}(\mathbb{R}^k)$ such that the map $\psi_\alpha \circ \psi_\beta^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{R}^k \to (V_\alpha \cap V_\beta) \times \mathbb{R}^k$ is given by:

$$(x, v) \mapsto (x, g_{\alpha\beta}(x) \cdot v).$$

The *fiber* over $x \in M$ is the vector space $E_x = \pi^{-1}(x)$. Given $X \subset M$ the *restriction* of E is the vector bundle $(\pi|_{\pi^{-1}(X)}, \pi^{-1}(X), X)$ (the vector space structure is well defined by condition (2) above). The family $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ is called the *cocycle* of the bundle and satisfies the interesting properties:

$$g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x) \quad \forall x \in V_\alpha \cap V_\beta \cap V_\gamma, \quad g_{\alpha\alpha}(x) = \mathbb{1}.$$

A *sub-bundle* of E is a vector bundle $\pi_0 : E_0 \to M$ such that $E_0 \subseteq E$, $\pi_0 = \pi|_{E_0}$ and such that there exists a vector bundle atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ for E with the following property: for every trivialization $\psi_\alpha : E|_{V_\alpha} \to V_\alpha \times \mathbb{R}^k$ we have $\psi_\alpha(E|_{V_\alpha} \cap E_0) = V_\alpha \times \mathbb{R}^r$ and the pair $(\psi_\alpha|_{\pi^{-1}(V_\alpha)\cap E_0}, V_\alpha)$ belong to the vector bundle atlas of E_0 (here r is the rank of E_0).

Example 2.13 (The trivial bundle). The trivial rank-k bundle on M is simply $E = M \times \mathbb{R}^k$ with $\pi : M \times \mathbb{R}^k \to M$ given by the projection on the first factor (in other words there is a single bundle trivialization: $V_\alpha = M$ and $\psi_\alpha = \mathrm{id}_M$).

Example 2.14 (How to construct a vector bundle). Let us consider an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of a smooth manifold M and for every $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq 0$ assume we are given a smooth function $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}^k)$ such that the family $\{g_{\alpha\beta}\}$ satisfies properties (2.1.2). Then we can construct a vector bundle E with cocylce $\{g_{\alpha\beta}\}$ by setting:

$$E = \left(\coprod_{\alpha \in A} U_\alpha \times \mathbb{R}^k \right) / \sim,$$

where $(x_1, v_1)_{\alpha_1} \sim (x_2, v_2)_{\alpha_2}$ if and only if $x_1 = x_2$ and $v_1 = g_{\alpha_1\alpha_2}(x)v_2$. We endow E with the quotient topology from the defining equivalence relation "$\sim$" and the projection map $\pi : E \to M$ is simply given by $[(x, v)] \mapsto x$.

Example 2.15 (Tangent bundle). Let M be a smooth manifold of dimension m and $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ be a smooth atlas for M. Using Example 2.14, we see that the tangent bundle TM of M is the vector bundle with cocycle

$$g_{\alpha\beta}(x) := J(\psi_\alpha \circ \psi_\beta^{-1})(\psi_\beta(x)).$$

Example 2.16 (The tautological bundle). The tautological bundle $\tau_{k,n}$ is a vector bundle on the Grassmnannian $G(k, n)$ which, as a topological space, is defined by:

$$\tau_{k,n} = \{(W, x) \in G(k, n) \times \mathbb{R}^n \mid x \in W \simeq \mathbb{R}^k\}.$$

The projection $\pi : \tau_{k,n} \to G(k, n)$ equals the restriction to $\tau_{k,n} \subset G(k, n) \times \mathbb{R}^n$ of the projection on the first factor $G(k, n) \times \mathbb{R}^n \to G(k, n)$.

Definition 2.17 (Section of a vector bundle). Let $E \xrightarrow{\pi} M$ be a vector bundle. A *section* of E is a smooth map $s : M \to E$ such that $\pi(s(x)) = x$ for every $x \in M$, i.e. such that the following diagram is commutative:

$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "s"'] \end{tikzcd}$$

The *zero* section is the section that associates to every point $x \in M$ the zero vector in E_x. A section of the tangent bundle TM is called a *vector field*.

Remark 2.18 (How to build a section of a vector bundle). Let $E \xrightarrow{\pi} M$ be a rank-k vector bundle and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of M such that $\psi_\alpha : E|_{U_\alpha} \simeq U_\alpha \times \mathbb{R}^k$ for every $\alpha \in A$. Let also $\{g_{\alpha\beta}\}$ be the corresponding cocyle. Given a section $s : M \to E$, for every $\alpha \in A$ we can consider the map $\psi_\alpha \circ s : U_\alpha \to U_\alpha \times \mathbb{R}^k$ (the section in the trivialization), which has the following form:

$$\psi_\alpha(s(x)) = (x, \sigma_\alpha(x)),$$

for some smooth function $\sigma_\alpha : U_\alpha \to \mathbb{R}^k$. Using the definition of the cocyle, we see that the family of maps $\{\sigma_\alpha\}$ satisfies the relation:

$$\sigma_\alpha(x) = g_{\alpha\beta}(x)\sigma_\beta(x) \quad \forall x \in U_\alpha \cap U_\beta.$$

Conversely, given a family of smooth functions $\{\sigma_\alpha : U_\alpha \to \mathbb{R}^k\}$ such that (2.18) is satisfied, this family defines a section of E (i.e. the unique section which in in the trivializaztions has the expression (2.18)).

Example 2.19 (Polynomials and vector bundles over projective spaces)**.** Let $\mathcal{U} = \{U_j\}_{j=0}^n$ be the open cover $U_j = \{x_j \neq 0\}$ of $\mathbb{RP}^n$. For every $d \in \mathbb{Z}$ consider the cocycle:

$$g_{ij}([x_0, \ldots, x_n]) = \left(\frac{x_j}{x_i}\right)^d, \quad x_i, x_j \neq 0.$$

We denote by $O_{\mathbb{RP}^n}(d)$ the vector bundle constructed with the procedure of Example 2.14 corresponding to this cocyle. An interesting fact is that, when $d \geq 0$, every homogeneous polynomial of degree d defines in a natural way a section of $O_{\mathbb{RP}^n}(d)$, as follows. Given $p \in \mathbb{R}[x_0, \ldots, x_n]_{(d)}$ define the family of functions $\{(\sigma_p)_j : U_j \to \mathbb{R}\}_{j=0}^n$ by

$$(\sigma_p)_j([x_0, \ldots, x_n]) = p\left(\frac{x_0}{x_j}, \ldots, \frac{x_j}{x_j}, \ldots, \frac{x_n}{x_j}\right).$$

Observe now that for every i, j and $[x] \in U_i \cap U_j$ we have:

$$\begin{aligned} g_{ij}([x])(\sigma_p)_j([x]) &= \left(\frac{x_j}{x_i}\right)^d p\left(\frac{x_0}{x_j}, \ldots, \frac{x_j}{x_j}, \ldots, \frac{x_n}{x_j}\right) \\ &= p\left(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right) = (\sigma_p)_i([x]) \end{aligned}$$

(where in the second equality we have used the fact that p is homogeneous of degree d). In particular, following the discussion of Remark 2.18, we see that the family $\{(\sigma_p)_j\}_{j=0}^n$ defines a section σ_p of $O_{\mathbb{RP}^n}(d)$. (Note also that the map

$p \mapsto \sigma_p$ is linear.) The procedure of de–homogenization of a polynomial with respect to a variable can be geometrically described as the procedure of looking at the polynomial as a section in the corresponding trivialization.

2.2 The Riemannian volume

In this section we introduce the notion of Riemannian volume: this is a measure defined on the sigma algebra of Borel sets of a smooth manifold, which can be constructed as soon as one has a Riemannian structure on such manifold. We start recalling some basic definitions.

2.2.1 Riemannian manifolds and integrals

Definition 2.20. A *Riemannian manifold* is a pair (M, g), where M is a smooth manifold and g assigns to each tangent space T_xM a positive definite bilinear form $\mathrm{g}(x) : T_xM \times T_xM \to \mathbb{R}$ such that for every pair of vector fields $v, w : M \to TM$ the function $x \mapsto \mathrm{g}(x)(u(x), v(x))$ is smooth. We call g the *Riemannian metric* on M.

Similarly, a *Hermitian manifold* is a pair (M, h), where M is a complex manifold and h assigns to each tangent space T_xM a positive definite Hermitian form, in a smooth way as above. We call h the *Hermitian metric* on M.

Every Hermitian manifold is also a Riemannian manifold by taking the Riemannian metric to be the real part of the Hermitian metric: $\mathrm{g} = \frac{1}{2}(\mathrm{h} + \overline{\mathrm{h}})$.

Given an atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ for M and a point $x \in U_\alpha$ for a fixed α we can represent $\mathrm{g}(x)$ in coordinates by the $n \times n$ matrix $\mathrm{g}_\alpha(x)$ with coordinates

$$\mathrm{g}_\alpha(x)_{i,j} := \mathrm{g}(x)((\varphi_\alpha^{-1})_*(\tfrac{\partial}{\partial x_i} \mid_a), (\varphi_\alpha^{-1})_*(\tfrac{\partial}{\partial x_j} \mid_a)), \quad \text{where } a = \varphi_\alpha(x).$$

Let us see how g_α behaves under coordinate changes.

Lemma 2.21. *If* $x \in U_\alpha \cap U_\beta$, *then* $\det(\mathrm{g}_\alpha(x)) = \det(\mathrm{g}_\beta(x)) \det(J_{\alpha,\beta}(x))^2$, *where*

$$J_{\alpha,\beta}(x) = \left[\frac{\partial(\varphi_\beta \circ \varphi_\alpha^{-1})}{\partial x_j}(\varphi_\alpha(x))\right]_{j=1,\dots,m} \in \mathbb{R}^{m \times m}.$$

Proof. Let $a := \varphi_\alpha(x)$ and $b := \varphi_\beta(x)$, and denote by $v_i := (\varphi_\alpha^{-1})_*(\frac{\partial}{\partial x_i} \mid_a)$ and $w_i := (\varphi_\beta^{-1})_*(\frac{\partial}{\partial x_j} \mid_b)$ tangent vectors. In Subsection 2.1.1 we take F to be the identify to see that $v_i = \sum_{j=1}^{n} c_{i,j}\, w_i$ for $J_{\alpha,\beta}(x) = (c_{i,j})^T$. This implies

$$\mathrm{g}(x)(v_i, v_j) = \sum_{k=1}^{m}\sum_{\ell=1}^{m} c_{i,k}c_{j,\ell}\, \mathrm{g}(x)(w_k, w_\ell);$$

i.e., $\mathrm{g}_\alpha(x) = J_{\alpha,\beta}(x)^T \mathrm{g}_\beta(x) J_{\alpha,\beta}(x)$. Taking determinants concludes the proof. □

We first discuss how to integrate functions on a Riemannian manifold.

Definition 2.22 (Integral of a function on a Riemannian manifold)**.** Let (M, g) be a Riemannian manifold with atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$. Let $\{\rho_\alpha : U_\alpha \to \mathbb{R}\}_{\alpha\in A}$ be a partition of unity[1] subordinated to the open cover $\{U_\alpha\}_{\alpha\in A}$. The integral of a measurable function $f : M \to \mathbb{R}$ is defined by:

$$\int_M f(x)\,\mathrm{vol}_{\mathrm{g}}(\mathrm{d}x) := \sum_{\alpha\in A}\int_{\varphi_\alpha(U_\alpha)} \left((f\cdot\rho_\alpha)\circ\varphi_\alpha^{-1}\right)(x)\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))}\,\mathrm{d}x_1\cdots\mathrm{d}x_n,$$

where on the right hand side of this equation we have standard Lebesgue integrals, and g_α is the matrix from Subsection 2.2.1. We will say that a measurable function $f : M \to \mathbb{R}$ is *integrable* if $\int_M |f(x)|\mathrm{vol}_{\mathrm{g}}(\mathrm{d}x)$ is finite.

In the following, whenever it will be clear to which Riemannian metric we refer, we will simply denote by

$$\int_M f(x)\,\mathrm{d}x := \int_M f(x)\,\mathrm{vol}_{\mathrm{g}}(\mathrm{d}x)$$

the integral of a measurable function.

Definition 2.22 is based on the choice of a partition of unity. Next we show, that the definition is actually independent of this choice.

Lemma 2.23. *The definition of the integral in Definition 2.22 is independent of the choice of partition of unity.*

Proof. Let $\{p_\alpha \mid \alpha \in A\}$ and $\{q_\alpha \mid \alpha \in A\}$ be two partitions of unity subordinated

[1] A *partition of unity* for M subordinated to the open cover $\{U_\alpha\}_{\alpha\in A}$ is a family of smooth functions $\{\rho_\alpha : U_\alpha \to \mathbb{R}\}_{\alpha\in A}$, such that: (1) $0 \le \rho_\alpha(x) \le 1$ for all $x \in M$ and $\alpha \in A$; (2) for every $x \in M$ we have $\rho_\alpha(x) > 0$ for only finitely many $\alpha \in A$; and (3) $\sum_{\alpha\in A}\rho_\alpha(x) = 1$ for all $x \in M$. Such a family always exists; see, e.g., [25, Theorem 2.23].

to the open cover $\{U_\alpha\}_{\alpha\in A}$. Then, we have

$$\begin{aligned}
&\sum_{\alpha\in A}\int_{\varphi_\alpha(U_\alpha)} \left((f\cdot p_\alpha)\circ\varphi_\alpha^{-1}\right)(x)\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=&\sum_{\alpha\in A}\int_{\varphi_\alpha(U_\alpha)} \Big((f\cdot p_\alpha\cdot\sum_{\beta\in A} q_\beta)\circ\varphi_\alpha^{-1}\Big)(x)\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\\
=&\sum_{\alpha\in A}\sum_{\beta\in A}\int_{\varphi_\alpha(U_\alpha\cap U_\beta)} \left((f\cdot p_\alpha\cdot q_\beta)\circ\varphi_\alpha^{-1}\right)(x)\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))}\,\mathrm{d}x_1\cdots\mathrm{d}x_n,
\end{aligned}$$

the second equality, because $\sum_{\beta\in A} q_\beta = 1$, and the last equality, because $p_\alpha\cdot q_\beta$ is zero outside of $U_\alpha\cap U_\beta$. For the last term we can use a change of variables $\varphi_\alpha(U_\alpha\cap U_\beta)\to\varphi_\beta(U_\alpha\cap U_\beta)$, $x\mapsto y=(\varphi_\beta\circ\varphi_\alpha^{-1})(x)$. By Lemma Lemma 2.21 we have $\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))} = |\det J_{\alpha,\beta}|\sqrt{\det \mathrm{g}_\beta(\varphi_\beta^{-1}(y))}$, where $J_{\alpha,\beta}$ is the Jacobian matrix of $\varphi_\beta\circ\varphi_\alpha^{-1}$ at x. This cancels with $|\det J_{\alpha,\beta}|^{-1}$, which we get from the change of variables formula for the Lebesgue integral. We can now go the chain of equalities backwards and interchange the roles of p_α and q_β. □

We are now ready to introduce the notion of Riemannian volume. In this context, this means that we are introducing a special measure on the Borel sigma algebra of M, through the help of the Riemannian metric. This measure is called the *Riemannian volume*.

Definition 2.24 (The Riemannian volume)**.** Let (M,g) be a smooth Riemannian manifold of dimension m and $U\subset M$ be a Borel subset. The *Riemannian volume* is the measure vol_g defined on Borel subsets $U\subseteq M$ by

$$\mathrm{vol}_g(U) := \int_M \chi_U\,\mathrm{vol}_\mathrm{g}(\mathrm{d}x),$$

where χ_U denotes the characteristic function of U, the measurable function equal to 1 on U and to zero everywhere else. If the metric g is clear from the context, we also write $\mathrm{vol}(U) := \mathrm{vol}_\mathrm{g}(U)$. If we wish to emphasize the dimension of M we write $\mathrm{vol}_m(U) := \mathrm{vol}(U)$.

If (M,g) is a Riemannian manifold, and $X\hookrightarrow M$ is a submanifold we define its volume as the volume of X with respect to the Riemannian metric $\mathrm{g}|_X$. In other words, we first restrict the Riemannian metric from the ambient space, and then consider the corresponding Riemannian measure. For instance, the volume of a curve is its length, and the volume of a surface is its area.

Computing the volume of a manifold directly using Definition 2.24 is often difficult. In most cases, we can simplify the calculation by using the coarea formula from the next section. See for instance Example 2.31, where we compute the volume of the unit circle.

In many cases we can define the volume of X even when it is not smooth, for instance in the semialgebraic case (see Chapter 3). More precisely, assume that M is an m-dimensional semialgebraic and smooth manifold, endowed with a Riemannian metric g, and $X \subseteq M$ is a semialgebraic set of dimension $s \leq m$. Then, by Theorem 3.14 below, X can be partitioned into finitely many smooth and semialgebraic subsets $X = \coprod_{j=1}^{N} X_j$. Let $U \subseteq X$ be a Borel subset; we define its volume (with respect to the volume induced by g on X) by:

$$\mathrm{vol}_{\mathrm{g}|_X}(U) = \sum_{\dim(X_j)=s} \mathrm{vol}_{\mathrm{g}|_{X_j}}(U \cap X_j).$$

For instance, when X is an algebraic subset of M, this definition coincides with *declaring* the set of singular points of X to be of Riemannian measure zero, and then considering the volume measure induced by $\mathrm{g}|_{\mathrm{smooth}(X)}$ on the set of smooth points of X.

2.2.2 Measure theoretic considerations

Our definition of the Riemannian measure views it as a linear functional on the set $\mathscr{C}^0_c(M, \mathbb{R})$ of compactly supported, continuous functions. More precisely, recall that a *(Radon) measure* on a topological space X is a measure μ defined on the sigma algebra of Borel subsets of X such that: (1) for every Borel set $A \subset X$ we have $\mu(A) = \sup_{K \subset A} \mu(K)$, where the sup is over all compact sets $K \subset X$ contained in A (i.e. μ is *inner regular*) and (2) for every $x \in X$ there exists a neighborhood U_x such that $\mu(U_x) < \infty$ (i.e. μ is *locally finite*).

When X is locally compact and Hausdorff, Radon measures on X can be characterized as continuous positive linear functionals on $\mathscr{C}^0_c(X, \mathbb{R})$, as follows. First, for every compact set $K \subset X$ we denote by $C^0_K(X, \mathbb{R})$ the set of compactly supported, continuous functions $f : X \to \mathbb{R}$ with support contained in K. The vector space $C^0_K(X, \mathbb{R})$ is a topological space with the topology induced by the supremum norm metric. Then we endow $\mathscr{C}^0_c(X, \mathbb{R})$ with the direct limit topology: a set $U \subset \mathscr{C}^0_c(X, \mathbb{R})$ is open if and only if $U \cap C^0_K(X, \mathbb{R})$ is open for every compact set $K \subset X$. Let now $\eta : \mathscr{C}^0_c(X, \mathbb{R}) \to \mathbb{R}$ be a continuous, positive linear functional, i.e. $\eta(f) \geq 0$ whenever $f \geq 0$ (note: the positivity of η actually implies its continuity). By Riesz Representation Theorem [35, Theorem 2.14], there exists a unique Radon measure μ such that for every $f \in \mathscr{C}^0_c(X, \mathbb{R})$:

$$\eta(f) = \int_X f(x)\mu(dx).$$

In this way the set of Radon measures on X can be identified with the set of positive, continuous, linear functionals on $\mathscr{C}_c^0(X,\mathbb{R})$.

If M is a smooth, orientable manifold and α is a nowhere vanishing top form on M (i.e. a volume form), the map

$$f \mapsto \int_M f\alpha, \quad f \in \mathscr{C}_c^0(M,\mathbb{R}).$$

is a positive linear functional. Hence, by the Riesz Representation Theorem, there exists a unique Radon measure on M, denoted by $|\alpha|$, such that for $f \in \mathscr{C}_c^0(M,\mathbb{R})$:

$$\int_M f\alpha = \int_M f(x)|\alpha|(dx).$$

The measure $|\alpha|$ is called the measure associated to α (notice that in the above formula, the left hand side denotes the integral on M, as a differentiable manifold, of the differential top form $f\alpha$, and the right hand side denotes the integral on M, as a measurable space, of the function f with respect to the measure $|\alpha|$). If now (M,g) is an orientable Riemannian manifold, the Riemannian volume form $\omega_g \in \Omega^m(M)$ is defined as follows. Let $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in A}$ be an oriented atlas for M and $\{\rho_\alpha\}_{\alpha\in A}$ be a partition of unity subordinated to it. Then:

$$\omega_g := \sum_{\alpha\in A} \rho_\alpha\varphi_\alpha^* \left(|\det g_\alpha(y)|^{\frac{1}{2}}\, dy_{\alpha,1}\wedge\cdots\wedge dy_{\alpha,m}\right).$$

In this case we have

$$|\omega_g| = \mathrm{vol}_g.$$

One can also introduce the notion of “sets of measure zero” for smooth manifolds (not necessarily Riemannian), sayiing that $W \subset M$ *has measure zero*, if for every chart (U,φ) for M the set $\varphi(W\cap U)\subset\mathbb{R}^m$ is a set of measure zero. The notion of sets of measure zero only depends on the differentiable structure (see [25, Proposition 6.5]). Note that, if a set U has measure zero in M, then for every Riemannian metric g on M we have $\mathrm{vol}_{\mathrm{g}}(U)=0$.

2.2.3 The coarea formula

The coarea formula is a key tool for our purposes. Basically, this formula shows how integrals transform under smooth maps. A well known special case is integration by substitution. The coarea formula generalizes this from integrals defined on the real line to integrals defined on Riemannian manifolds.

Let M, N be Riemannian manifolds and $F : M \to N$ be a smooth map. Recall that $x \in M$ is a regular point of F, if $\mathrm{D}_x F$ is surjective. For any $x \in M$ the Riemannian metric on M defines orthogonality on $T_x M$; i.e., $v, w \in T_x M$ are orthogonal, if and only if $\mathrm{g}(x)(v, w) = 0$. For a regular point x of F this implies that the restriction of $\mathrm{D}_x F$ to the orthogonal complement of its kernel is a linear isomorphism. The absolute value of the determinant of that isomorphism, computed with respect to orthonormal bases in $T_x M$ and $T_{F(x)} N$, respectively, is the *normal Jacobian of* F *at* x. Let us summarize this in a definition.

Definition 2.25. Let $F : M \to N$ be a smooth map between Riemannian manifolds and $x \in M$ be a regular point of F. Let $(\,\cdot\,)^\perp$ denote the operation of taking orthogonal complement with respect to the corresponding Riemannian metric. The normal Jacobian of F at x is defined as

$$\mathrm{NJ}(F, x) := \left|\det\left(\mathrm{D}_x F|_{(\ker \mathrm{D}_x F)^\perp}\right)\right|,$$

where the determinant is computed with respect to orthonormal bases in the source and in $T_x M$ and $T_{F(x)} N$ (this definition does not depend on the choice of the bases). If $x \in M$ is not a regular point of F, we set $\mathrm{NJ}(F, v) = 0$.

We are now equipped with all we need to state the coarea formula.

Theorem 2.26 (The coarea formula)**.** *Suppose that* M, N *are Riemannian manifolds with* $\dim M \geq \dim N$*, and let* $F : M \to N$ *be a surjective smooth map. Then we have for any integrable function* $h : M \to \mathbb{R}$ *that*

$$\int_M h(x)\, \mathrm{d}x = \int_{y \in N} \left(\int_{x \in F^{-1}(y)} \frac{h(x)}{\mathrm{NJ}(F, x)}\, \mathrm{d}x \right) \mathrm{d}y.$$

Notice that by Sard's lemma the set of point in N, which are not regular values, is a measure zero set, and can therefore be ignored in the integration, so that the normal Jacobian is always positive.

In the case when $F : M \to N$ is a diffeomorphism, $F^{-1}(y)$ contains a single element for all $y \in N$. Therefore, we get the following simplification of the coarea formula in this case.

Corollary 2.27 (The change of variables formula for manifolds)**.** *Suppose that* M, N *are Riemannian manifolds, and let* $F : M \to N$ *be a diffeomorphism and* $h : M \to \mathbb{R}$ *be integrable. Then:* $\int_M \mathrm{NJ}(F, x)\, h(v)\, \mathrm{d}x = \int_N h(F^{-1}(y))\, \mathrm{d}y$.

If M and N are complex Hermitian manifolds, and $F : M \to N$ is a complex differentiable map (i.e., F is complex differentiable for every choice of coordinates using charts), then the normal Jacobian of F can be written in a simpler form. Recall that a given a Hermitian metric h on a complex manifold M, then we can construct a Riemannian metric g on M by taking the real part $\mathrm{g} = \frac{1}{2}(\mathrm{h} + \overline{\mathrm{h}})$ of h. In this context, the coarea formula for a complex map takes the following form.

Lemma 2.28. *Let $F : M \to N$ be a complex differentiable map between complex hermitian manifolds and $x \in M$. Then:*

$$\mathrm{NJ}(F,x) := \left| \det(\mathrm{D}_x^{\mathbb{C}} F\big|_{(\ker \mathrm{D}_x F)^\perp}) \right|^2,$$

where $\mathrm{D}_x^{\mathbb{C}} F$ denotes the differential of F viewed as a complex map and the determinant is with respect to hermitian-orthonormal bases, made of real vectors, for $(\ker D_x F)^\perp$ and $T_{F(x)}N$ viewed as complex vector spaces.

Proof. Choose hermitian-orthonormal bases, made of real vectors, for $(\ker \mathrm{D}_x F)^\perp$ and $T_{F(x)}N$ viewed as a m-dimensional complex vector spaces. Denoting these bases by $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_m\}$ respectively, we can construct the new bases $\{e_1, \ldots, e_m, \sqrt{-1} \cdot e_1, \ldots, \sqrt{-1} \cdot e_m\}$ and $\{f_1, \ldots, f_m, \sqrt{-1} \cdot f_1, \ldots, \sqrt{-1} \cdot f_m\}$, which are orthonormal bases for $(\ker D_x F)^\perp$ and $T_{F(x)}N$ as real vector spaces with the Riemannian metric $\mathrm{g} = \frac{1}{2}(\mathrm{h} + \overline{\mathrm{h}})$ induced by the Hermitian metric h. Let the complex Jacobian with respect to these bases be $J^{\mathbb{C}} = A + iB$, where $A, B \in \mathbb{R}^{m \times m}$ are real matrices. Then, the real Jacobian has the shape $J = \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)$. Therefore, applying the formula for the normal Jacobian in the Riemannian case we get: $\mathrm{NJ}(F,x) = \left|\det \left(\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right)\right| = |\det(A + iB)|^2$, which shows the assertion. □

2.2.4 Isometries and Riemannian submersions

An important class of smooth functions between manifolds in the context of integrals are *isometries.*

Definition 2.29. Let (M, g) and $(N, \tilde{\mathrm{g}})$ be Riemannian manifolds and let $F : M \to N$ a smooth map. The map F is an isometry, if it is a diffeomorphism and for all $x \in M$ and $v, w \in T_x M$ we have

$$\mathrm{g}(x)(v,w) = \tilde{\mathrm{g}}(F(x))(\mathrm{D}_x F(v), \mathrm{D}_x F(w));$$

i.e., at every point $x \in M$ the derivative $\mathrm{D}_x F$ is an isometry of Euclidean spaces. If there is an isometry $F : M \to N$, we say that M is isometric to N.

We have the following consequence of this definition.

Lemma 2.30. *Let* (M, g) *and* $(N, \tilde{\mathrm{g}})$ *be Riemannian manifolds and* $F : M \to N$ *be an isometry. Then:* $\mathrm{vol}(M) = \mathrm{vol}(N)$.

Proof. Since F is an isometry, we have $\mathrm{NJ}(F, x) = 1$ for all $x \in M$, because $\mathrm{D}_x F$ maps an orthonormal basis of $T_x M$ to an orthonormal basis of $T_{F(x)} N$. Furthermore, we have $\mathrm{vol}(F^{-1}(x)) = 1$ for all $w \in N$, because F is invertible. Then the coarea formula from Theorem 2.26 implies that $\mathrm{vol}(M) = \mathrm{vol}(N)$. $\square$

Example 2.31. Consider again the unit circle S^1. We want to compute the volume of S^1 relative to the Riemannian metric that is given by defining the bilinear form on $T_x S^1 \cong x^\perp$ (see Example 2.6) to be the standard Euclidean inner product in $\mathbb{R}^2$ restricted to $x^\perp$. For this, we define $F : (0, 2\pi) \to S^1, t \mapsto (\cos(t), \sin(t))$, which is smooth. Then, the derivative of F at t is $\mathrm{D}_t F(s) = (-\sin(s), \cos(s))$. Since $(-\sin(s))^2 + \cos(s) = 1$, we see that F is an isometry of $(0, 2\pi)$ and $S^1 \setminus \{(1, 0)\}$. Since $\{(1, 0)\}$ is of measure zero, we have $\mathrm{vol}(S^1) = \mathrm{vol}(S^1 \setminus \{(1, 0)\})$ and, since F is an isometry, this implies $\mathrm{vol}(S^1) = \mathrm{vol}((0, 2\pi)) = 2\pi$.

A weaker property than being an isometry is being a Riemannian submersion.

Definition 2.32. Let (M, g) and $(N, \tilde{\mathrm{g}})$ be Riemannian manifolds. A smooth map $F : M \to N$ is called a Riemannian submersion, if for all $x \in M$ the differential $\mathrm{D}_x F$ is surjective and if we have $\mathrm{g}(x)(v, w) = \tilde{\mathrm{g}}(F(x))(\mathrm{D}_x F(v), \mathrm{D}_x F(w))$ for all $v, w \in (\ker \mathrm{D}_x F)^\perp$; i.e., at every point $x \in M$ the derivative $\mathrm{D}_x F$ when restricted to the orthogonal complement of its kernel is an isometry of Euclidean spaces.

Using essentially the same arguments as for the proof of Lemma 2.30 we get the following

Lemma 2.33. *Let* (M, g) *and* $(N, \tilde{\mathrm{g}})$ *be Riemannian manifolds and* $F : M \to N$ *an Riemannian submersion. Then,* $\mathrm{vol}(M) = \int \mathrm{vol}(F^{-1}(y))\, \mathrm{d}y$. *In particular, if the volume of the fibers is constant, that is,* $\mathrm{vol}(F^{-1}(y)) = \mathrm{vol}(F^{-1}(y_0))$ *for all* $y \in N$ *and a fixed* $y_0 \in N$, *then* $\mathrm{vol}(M) = \mathrm{vol}(N) \cdot \mathrm{vol}(F^{-1}(y_0))$.

2.2.5 Volume of the sphere and projective space

Probably the most important example of a manifold is the n-dimensional unit sphere $S^n \hookrightarrow \mathbb{R}^{n+1}$. For us the sphere will be endowed with the Riemannian metric, which in turn is the restriction of the Euclidean structure on $\mathbb{R}^{n+1}$.

In Example 2.31 we computed its volume in the case $n = 1$ using the parametrization by polar coordinates, which turned out to be an isometry. For $n \geq 2$ we can again use polar coordinates.

Proposition 2.34. $\mathrm{vol}(S^n) = \dfrac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$.

For instance, we have

$$\mathrm{vol}(S^1) = 2\pi, \quad \mathrm{vol}(S^2) = 4\pi, \quad \mathrm{vol}(S^3) = 2\pi^2,$$

where we have used that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof of Proposition 2.34. Consider $\phi : S^n \times \mathbb{R}_{>0} \to \mathbb{R}^{n+1}\backslash\{0\}, (s,r) \mapsto rs$. Its derivative is $\mathrm{D}\phi(s,r)(\dot{s},\dot{r}) = r\dot{s} + \dot{r}s$. Let $\dot{s}_1, \ldots, \dot{s}_n$ be a basis of the tangent space $T_sS^n \cong s^\perp$ (see Example 2.6). Then, $\det \mathrm{D}\phi(s,r) = \det \begin{bmatrix} r\dot{s}_1 & \cdots & r\dot{s}_n & s \end{bmatrix} = r^n$.

Consider now the integral $\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x$ (since $\frac{1}{(\sqrt{2\pi})^{n+1}} e^{-\frac{1}{2}\|x\|^2}$ is the density of a standard Gaussian random variable in $\mathbb{R}^{n+1}$, the value of this integral is $\sqrt{2\pi}^{n+1}$, but let us prove it directly). The coarea formula (Theorem 2.26) implies

$$\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = \int_{S^n\times\mathbb{R}_{>0}} e^{-\frac{1}{2}r^2}\, \det(\mathrm{D}\phi(s,r))\,\mathrm{d}s\mathrm{d}r = \int_{S^n\times\mathbb{R}_{>0}} e^{-\frac{1}{2}r^2}\, r^n\,\mathrm{d}s\mathrm{d}r.$$

By Tonelli's theorem, this is equal to

$$\begin{aligned} \int_{S^n} \mathrm{d}s \int_{\mathbb{R}_{>0}} e^{-\frac{1}{2}r^2}\, r^n\,\mathrm{d}r &= \mathrm{vol}(S^n) \int_{\mathbb{R}_{>0}} e^{-t}\,(2t)^{\frac{n-1}{2}}\,\mathrm{d}t, t = \frac{r^2}{2} \\ &= \sqrt{2}^{n-1}\,\mathrm{vol}(S^n)\,\Gamma(\tfrac{n+1}{2}). \end{aligned}$$

Now, for $n = 1$ we know from Example 2.31 that $\mathrm{vol}(S^1) = 2\pi$. Moreover, we have $\Gamma(1) = 1$, so that $\int_{\mathbb{R}^2} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = 2\pi$. Since the exponential map is a group homomorphism from $(\mathbb{R}, +) \to (\mathbb{R}_{>0}, \cdot)$, we have $\int_{\mathbb{R}^k} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = (\int_{\mathbb{R}} e^{-\frac{1}{2}x^2}\,\mathrm{d}x)^k$, for every k, which implies that

$$\int_{\mathbb{R}^{n+1}} e^{-\frac{1}{2}\|x\|^2}\,\mathrm{d}x = \sqrt{2\pi}^{n+1}.$$

We use this in the equation above to obtain the asserted formula. $\square$

We will now introduce a Riemannian metric and consequently a Riemannian volume on the real projective space $\mathbb{R}\mathrm{P}^n$. In order to do this, observe first that the antipodal map $x \mapsto -x$ is an isometry for the Euclidean structure on the sphere. Therefore, the Riemannian metric g_{S^n} on S^n descend to a Riemannian metric $\mathrm{g}_{\mathbb{R}\mathrm{P}^n}$ on $\mathbb{R}\mathrm{P}^n$, meaning that we declare a metric on $\mathbb{R}\mathrm{P}^n$ that makes P a Riemannian submersion. With this metric we have that (S^n, g_{S^n}) and $(\mathbb{R}\mathrm{P}^n, \mathrm{g}_{\mathbb{R}\mathrm{P}^n})$ are locally isometric. For $x \in S^n$ we have $T_x S^n = T_{-x} S^n = x^\perp$, and we can identify

$$T_x \mathbb{R}\mathrm{P}^n \cong x^\perp.$$

The Riemannian metric on $\mathbb{R}\mathrm{P}^n$ is then given by

$$\mathrm{g}_{\mathbb{R}\mathrm{P}^n}(x)(v, w) := v^T w,$$

where $x \in \mathbb{R}\mathrm{P}^n$ and $v, w \in x^\perp$.

Let $P : S^n \to \mathbb{R}\mathrm{P}^n$ the projection that identifies antipodal points. Because the volume of the preimage $P^{-1}(x)$ is 2 for all $x \in \mathbb{R}\mathrm{P}^n$, Lemma 2.33 implies that the volume of a submanifold $X \hookrightarrow \mathbb{R}\mathrm{P}^n$ is given as $\mathrm{vol}(X) := \frac{1}{2}\mathrm{vol}(P^{-1}(X))$. In particular, the volume of the projective space with this metric is half the volume of the sphere:

$$\mathrm{vol}(\mathbb{R}\mathrm{P}^n) = \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

We make a similar construction for the complex projective space $\mathbb{C}\mathrm{P}^n$. The main difference being that the projection $P_{\mathbb{C}} : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ has positive dimensional fibers. Namely, $P_{\mathbb{C}}^{-1}(x) = \{\xi x_0 \mid |\xi| = 1\}$, where $P(x_0) = x$. The tangent space at $x_0 \in S^{2n+1}$ is $T_{x_0} S^{2n+1} \cong x_0^\perp = \{y \in \mathbb{C}^{n+1} \mid \Re(x^T \overline{y}) = 0\}$. Note that $\Re(x^T \overline{y}) = 0$ if and only if $x^T \overline{y} \in i\mathbb{R}$. Then, we have $\Re(\xi x^T \overline{y}) = 0$ for all ξ with $|\xi| = 1$, if and only if $x^T \overline{y} = 0$. This yields

$$T_x \mathbb{C}\mathrm{P}^n \cong x^{\perp_{\mathbb{C}}} := \{y \in \mathbb{C}^{n+1} \mid x^T \overline{y} = 0\}.$$

The Hermitian metric on $\mathbb{C}\mathrm{P}^n$ is given by $\mathrm{h}(x)(v, w) = v^T \overline{w}$, where $x \in \mathbb{C}\mathrm{P}^n$ and $v, w \in x^{\perp_{\mathbb{C}}}$, and, consequently, the Riemannian metric on $\mathbb{C}\mathrm{P}^n$ is

$$\mathrm{g}(x)(v, w) = \Re(v^T \overline{w}).$$

Since the preimage $P_{\mathbb{C}}^{-1}(x)$ is a isometric to a circle, we have $\mathrm{vol}(P_{\mathbb{C}}^{-1}(x)) = 2\pi$ for all x. Together with Lemma 2.33 this implies that the volume of a submanifold $X \hookrightarrow \mathbb{C}\mathrm{P}^n$ is $\mathrm{vol}(X) := \frac{1}{2\pi}\mathrm{vol}(P_{\mathbb{C}}^{-1}(X))$. In particular, the volume of the projective space with this metric is:

$$\mathrm{vol}(\mathbb{C}\mathrm{P}^n) = \frac{\pi^n}{n!},$$

where we have used that $\Gamma(n+1) = n!$.

2.2.6 Volume of the Orthogonal and Unitary group

The *orthogonal group* $O(n)$ is the group of matrices $Q \in \mathbb{R}^{n\times n}$ such that $Q^TQ = \mathbf{1}$. This is a system of $\frac{n(n+1)}{2}$ polynomials in n^2 many variables. Let $Q \in O(n)$. The kernel of the Jacobian matrix $J(Q)$ of this system of polynomial equations is $\ker J(Q) = \{R \in \mathbb{R}^{n\times n} \mid Q^TR + R^TQ = 0\}$, so $\dim\ker J(Q) = \frac{n(n-1)}{2}$. We can therefore apply Corollary 2.9 to deduce that $O(n)$ is a smooth manifold of dimension

$$\dim O(n) = \frac{n(n-1)}{2}.$$

and that the geometric tangent is given by

$$T_QO(n) = \{R \in \mathbb{R}^{n\times n} \mid Q^TR + R^TQ = 0\} = Q \cdot T_{\mathbf{1}}O(n).$$

We consider the orthogonal group as a Riemannian manifold endowed with the metric that is the restriction of the Euclidean structure of $\mathbb{R}^{n\times n}$:

$$\mathrm{g}(Q)(R_1, R_2) = \tfrac{1}{2}\mathrm{tr}(R_1^TR_2).$$

The next proposition gives the volume of $O(n)$ with respect to this metric structure.

Proposition 2.35. $\mathrm{vol}(O(n)) = \displaystyle\prod_{k=0}^{n-1} \mathrm{vol}(S^k).$

Proof. Consider the smooth surjective map $F : O(n) \to S^{n-1}$ that maps $Q \in O(n)$ to its first column Qe_1, where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. We compute the normal Jacobian of F. The derivative of F is $\mathrm{D}_QF(R) = Re_1$ for $R \in T_QO(n)$. Let $E_{i,j} \in \mathbb{R}^{n\times n}$ be the matrix that has a 1 as the (i, j)-th entry, a -1 in the (j, i)-th entry and zeros elsewhere. Then, $\{QE_{i,j} \mid 1 \le i < j \le n\}$ is an orthonormal basis for $T_QO(n)$. The inner product between $\mathrm{D}_QF(QE_{i,j})$ and $\mathrm{D}_QF(QE_{k,\ell})$ is then

$$(QE_{i,j}e_1)^T(QE_{k,\ell}e_1) = (E_{i,j}e_1)^TE_{k,\ell}e_1 = \begin{cases} 1, & \text{if } k = i = 1 \text{ and } j = \ell \\ 0, & \text{otherwise} \end{cases}$$

Hence, D_QF maps an orthonormal basis of $(\ker \mathrm{D}_QF)^\perp$ to an orthonormal basis of $(Qe_1)^\perp$, which shows that $\mathrm{NJ}(F, Q) = 1$; i.e., F is a Riemannian submersion. Moreover, $F^{-1}(Qe_1) = QF^{-1}(e_1) = \{Q\left[\begin{smallmatrix} 1 & 0 \\ 0 & R \end{smallmatrix}\right] \mid R \in O(n-1)\}$, so that $\mathrm{vol}(F^{-1}(q)) = \mathrm{vol}(O(n-1))$ for all $q \in S^{n-1}$. We can now use Lemma 2.33 to deduce that

$$\mathrm{vol}(O(n)) = \mathrm{vol}(S^{n-1})\mathrm{vol}(O(n-1)).$$

Induction on n proves the assertion. $\square$

The unitary group consists of matrices $Q \in \mathbb{C}^{n\times n}$ such that $Q\overline{Q}^T = \mathbf{1}$; the Riemannian metric on $U(n)$ is obtained by restricting to it the metric on $\mathbb{C}^{n\times n}$ given by the Euclidean structure $\mathrm{g}(Q)(A_1, A_2) = \mathfrak{Re}(\frac{1}{2}\mathrm{tr}(A_1^T\overline{A}_2))$. Arguing as for the orthogonal group, we can show that the complex dimension of $U(n)$ is

$$\dim_{\mathbb{C}} U(n) = \frac{n(n-1)}{2},$$

and that its volume is

$$\mathrm{vol}(U(n)) = \prod_{k=0}^{n-1} \mathrm{vol}(S^{2k+1}).$$

(We will come back to these formulas when we discuss volumes of Riemannian homogenous spaces.)

3 Semialgebraic geometry

In this section we introduce *semialgebraic sets*. They describe a large class of sets and are rigid enough to allow being able to practically work with them.

3.1 Semialgebraic sets and functions

We start with the definition of semialgebraic sets and functions.

Definition 3.1 (Semialgebraic sets)**.** A *basic semialgebraic set* is a subset $S \subseteq \mathbb{R}^n$ defined by a system of algebraic equations and inequalities:

$$S = \{x \in \mathbb{R}^n \,|\, p_1(x) = 0,\, \ldots,\, p_a(x) = 0,\, q_1(x) > 0,\, \ldots,\, q_b(x) > 0\},$$

where $p_1, \ldots, p_a, q_1, \ldots, q_b \in \mathbb{R}[x_1, \ldots, x_n]$ are real polynomials. A *semialgebraic set* is a finite union of basic semialgebraic sets.

Definition 3.2 (Semialgebraic functions)**.** Let A and B be semialgebraic sets. A function $f : A \to B$ is called *semialgebraic* if its graph $\mathrm{Graph}(f) \subset A \times B$ is semialgebraic.

Next comes a list of examples and simple properties.

(1) Finite unions, intersections, complements and cartesian products of semialgebraic sets are semialgebraic.

(2) The composition, the sum and the product of semialgebraic functions is a semialgebraic function.

(3) If $S \subset \mathbb{R}$, then S is a finite union of points and open intervals. Similarly, if $S \subset \mathbb{R}^n$ and $L \subset \mathbb{R}^n$ is an affine line, then $L \cap S$ is a finite union of points and open intervals.

(4) If $F : \mathbb{R}^n \to \mathbb{R}^m$ is semialgebraic and $S \in \mathbb{R}^m$ is semialgebraic, then $F^{-1}(S)$ is semialgebraic.

(5) Identifying $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ via $(x_1 + iy_1, \ldots, x_n + iy_n) \mapsto (x_1, y_1, \ldots, x_n, y_n)$, we see that the complex zero set of complex polynomials is semialgebraic as a subset of $\mathbb{R}^{2n}$.

It will be practical for us to be able to define also semialgebraic subsets of projective spaces (and Grassmannians). This is done as follows. First, for every $0 \leq k \leq n$ consider

$$\Gamma_{k,n} := \{A \in \mathbb{R}^{n\times n} \,|\, A^T = A,\ A^2 = A,\ \mathrm{tr}(A) = k\}.$$

This is the algebraic set of matrices which are orthogonal projections onto vector subspaces of dimension k. It follows from Corollary 2.9 that $\Gamma_{k,n}$ is also a smooth submanifold of $\mathbb{R}^{n\times n}$. Let now $G(k,n)$ denote the Grassmannian of real vector subspaces of dimension k in $\mathbb{R}^n$. We define the map

$$\psi_{k,n} : G(k,n) \to \Gamma_{k,n}$$

by sending each space $W \subseteq \mathbb{R}^n$ to the matrix representing the orthogonal projection onto it.

Definition 3.3. We will say that $S \subset G(k,n)$ is semialgebraic if $S = \psi_{k,n}^{-1}(R)$ for some $R \in \mathcal{S}_{n\times n}$.

With this definition, the projective zero set $Z_{\mathbb{R}\mathrm{P}^n}(p) := \{[x] \in \mathbb{R}\mathrm{P}^n \,|\, p(x) = 0\}$ of a homogeneous polynomial $p \in \mathbb{R}[x_0,\ldots,x_n]_{(d)}$ is semialgebraic. For instance, writing a matrix as $A = (a_{ij})$, we see that for every $i = 1,\ldots,n$ the hyperplane $Z_{\mathbb{R}\mathrm{P}^n}(x_i)$ is semialgebraic, since $\{x_i = 0\} = \psi_{1,n}^{-1}(\{a_{ii} = 0\})$. Semialgebraic subsets of $\mathbb{R}\mathrm{P}^1$ are finite union of points and intervals.

An important result in the context of semialgebraic sets is that this class is closed under projections, as stated in the next theorem.

Theorem 3.4 (Tarski–Seidenberg)**.** *Let $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be the projection on the last n coordinates. If $S \subset \mathbb{R}^{n+k}$ is semialgebraic, then $\pi(S)$ is semialgebraic.*

For a proof see, e.g., [3, Section 2.2].

Here is a small example illustrating the Tarski-Seidenberg theorem.

Example 3.5. Consider the algebraic set $S = \{y_1x^2 + y_2x + y_3 = 0\} \subset \mathbb{R}^4$ and let $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ be the projection on the y-coordinates. Then $y \in \pi(S)$, if and only if

$$((y_1 \neq 0) \wedge (y_1^2 - 4y_2y_3 \geq 0)) \vee ((y_1 = 0) \wedge (y_2 \neq 0)) \vee (y_1 = y_2 = y_3 = 0).$$

In other words, $y_1x^2 + y_2x + y_3 = 0$ has a solution x over the real numbers, if and only if y satisfies Example 3.5.

We conclude this section with some corollaries of Theorem 3.4. The standard proof of this theorem is based on the *Tarski–Seidenberg algorithm.* This is an algorithm that can be used to compute formulas like in Example 3.5. Consider a semialgebraic set $S \subset \mathbb{R}^{n+k}$ defined by inequalities in $(x, y) \in \mathbb{R}^{n+k}$ with $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^k$, and let $\pi : S \to \mathbb{R}^k$ be the projection onto the y-coordinate. The algorithm produces finitely many systems of inequalities in y, such that at least one of these systems has a solution, if and only if there is a point $(x, y) \in \pi^{-1}(y)$. In this sense, the Tarski–Seidenberg algorithm gives a constructive proof for the Tarski–Seidenberg theorem.

As a direct corollary of Theorem 3.4, we get the following.

Corollary 3.6. *Let $f : A \to B$ be a semialgebraic function and $S \subseteq A$ be a semialgebraic set. Then $f(A)$ is semialgebraic.*

Proof. Let $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^n$. Denote by $\Gamma \subset A \times B$ the graph of f and by $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ the projection on the last coordinates. Then $\Gamma \cap (S \times B) \subset \mathbb{R}^{n+k}$ is semialgebraic and $f(S) = \pi(\Gamma \cap (S \times B))$ is semialgebraic by Theorem 3.4. $\square$

To appreciate more the power of Theorem 3.4, we introduce the following notion.

Definition 3.7 (First order formulas)**.** The set of first order formulas is inductively defined by the following rules:

(1) For every $n \in \mathbb{N}$ and for every $p \in \mathbb{R}[x_1, \ldots, x_n]$, both $(p(x_1, \ldots, x_n) = 0)$ and $(p(x_1, \ldots, x_n) > 0)$ are first order formulas.

(2) If Φ and Ψ are first order formulas, then $\Phi \wedge \Psi$, $\Phi \vee \Psi$ and $\neg\Psi$ are first order formulas.

(3) If Φ is a first order formula and $x \in \mathbb{R}$, then both $(\exists x \Phi)$ and $(\forall x \Phi)$ are first order formulas.

First order formulas obtained only using the rules (1) and (2) above are called *quantfier free formulas.* Notice that, $S \subset \mathbb{R}^n$ is a semialgebraic set if and only if the following is true: there is a quantifier free formula Φ such that $y \in S$ if and only if $\Phi(y)$. In this case we will say that "Φ defines S", or "S is defined by Φ". With this notation, since the formula $\forall x \Phi$ is equivalent to $\neg\exists x \neg\Phi$, Theorem 3.4 can be rephrased as follows.

Theorem 3.8. *Every set defined by a first order formula is a semialgebraic set.*

This is particularly practical, as shown by the next proposition. We denote by $\mathrm{clos}(S)$ and $\mathrm{int}(S)$ the Euclidean closure and interior, respectively.

Proposition 3.9. *Let $S \subseteq \mathbb{R}^n$ be semialgebraic set. Then,* $\operatorname{clos}(S)$ *and* $\operatorname{int}(S)$ *are also semialgebraic.*

Proof. Let Φ be a first order formula defining S and define the first order formula $\Psi(x) := (\forall \epsilon > 0 \, \exists y \in \mathbb{R}^n \, \Phi(y) \wedge (\|x-y\|^2 < \epsilon))$. Then $\operatorname{clos}(S)$ is defined by Ψ, so that Theorem 3.8 implies it is semialgebraic. A similar argument proves that the interior is semialgebraic. □

Example 3.10. Let $A \subseteq \mathbb{R}^n$ be a semialgebraic set. We denote by $\operatorname{dist}_A : \mathbb{R}^n \to \mathbb{R}$ the Euclidean distance function from A, i.e. $\operatorname{dist}_A(x) = \inf_{y \in A} \|x-y\|$. The distance function from A is a continuous, nonnegative semialgebraic function, vanishing exactly on $\operatorname{clos}(A)$.

In the world of complex algebraic geometry, the quantifier elimination theorem is called *Chevalley's Theorem*. In order to state this result, we need a preliminary definition. A *basic constructible set* is a set $C \subseteq \mathbb{C}^n$ of the form

$$C = \{z \in \mathbb{C}^n \mid p_1(z) = \cdots = p_a(z) = 0, q_1(z) \neq 0, \ldots, q_b(z) \neq 0\},$$

where $p_1, \ldots, p_a, q_1, \ldots q_b \in \mathbb{C}[z_1, \ldots, z_n]$. A *constructible set* is a finite union of basic constructible sets.

Theorem 3.11 (Chevalley)**.** *Let $C \subseteq \mathbb{C}^{n+k}$ be constructible and $\pi : \mathbb{C}^{n+k} \to \mathbb{C}^n$ be the projection on the last coordinates. Then $\pi(C)$ is constructible.*

Proof. See, e.g., [27, Theorem 4.19]. □

3.2 Decomposition of semialgebraic sets and their stratification

In this section we discuss the decomposition of semialgebraic sets into simple objects. We start with the following definition.

Definition 3.12 (Nash submanifold)**.** A semialgebraic set $S \subseteq \mathbb{R}^n$ is called a *Nash submanifold*, if for every point $x \in S$ there is an open semialgebraic neighborhood U_x of x in $\mathbb{R}^n$ and a smooth semialgebraic diffeomorphism $\varphi : U_x \to \mathbb{R}^n$ such that $S \cap U_x = \varphi^{-1}(\{0\} \times \mathbb{R}^k)$. The dimension of S is k. A *Nash map* between Nash submanifolds is a smooth semialgebraic map.

We will prove in Theorem 3.14 that every semialgebraic set can be decomposed into a disjoint union of Nash manifolds, each diffeomorphic to $(0,1)^k$ for some k, called *cell*. Before proving the theorem, we will need an auxiliary lemma and some notation.

Given $p_1, \ldots, p_s \in \mathbb{R}[t]$, we denote by $t_1 < \cdots < t_r$ the set of all the possible roots of those polynomial from the previous list which are nonzero. We also set $t_0 := -\infty$ and $t_{r+1} := +\infty$. We denote by $I_k := (t_k, t_{k+1})$ and observe that the sign of each polynomial $p_1, \ldots, p_s$ is constant on each of these intervals. We denote by $\sigma(p_1, \ldots, p_2) \in \{-1, 0, 1\}^{s\times(2r+1)}$ the following matrix:

$$\sigma(p_1, \ldots, p_2) := \begin{pmatrix} \operatorname{sign}(p_1|_{I_0}) & \operatorname{sign}(p_1(t_1)) & \cdots & \operatorname{sign}(p_1(t_r)) & \operatorname{sign}(p_1|_{I_r}) \\ \vdots & \vdots & & \vdots & \vdots \\ \operatorname{sign}(p_s|_{I_0}) & \operatorname{sign}(p_s(t_1)) & \cdots & \operatorname{sign}(p_s(t_r)) & \operatorname{sign}(p_s|_{I_r}) \end{pmatrix}$$

Letting $d := \max\{\deg(p_i)\}$, we denote by $\Omega_{s,d}$ the set of all $s \times (2rs+1)$ matrices with entries in $\{-1, 0, 1\}$, for $r = 0, \ldots, d$.

Lemma 3.13. *Given polynomials $p_1, \ldots, p_s \in \mathbb{R}[x, t]$, where $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, there exists a finite partition*

$$\mathbb{R}^n = \bigsqcup_{i=1}^{a} A_i$$

with the property that for every $i = 1, \ldots, a$ the set A_i is semialgebraic and there exists $r_i > 0$ and semialgebraic continuous functions:

$$\xi_{i,j} : A_i \to \mathbb{R},\ 1 \le j \le r_i,$$

such that $\xi_{i,1} < \cdots < \xi_{i,r_i}$ and for every $x \in A_i$ the set $\{\xi_{i,1}(x), \ldots, \xi_{i,r_i}(x)\}$ consists of all the roots of the nonzero polynomials $\{p_1(x, \cdot), \ldots, p_s(x, \cdot)\} \subset \mathbb{R}[t]$.

Proof. We assume that the set $\{p_1, \ldots, p_s\}$ is closed under taking derivatives with respect to the t variable (if we construct the functions $\{\xi_{i,j}\}$ for a larger set of polynomials, the desired family of functions will be a subset of them).

Denote by $d = \max_i(\deg(p_i))$. Using the above notation, for every $\sigma \in \Omega_{s,d}$, denote by $A_\sigma \subseteq \mathbb{R}^n$ the semialgebraic set:

$$A_\sigma := \{x \in \mathbb{R}^n \,|\, \sigma(p_1(x, \cdot), \ldots, p_s(x, \cdot)) = \sigma\}.$$

(This set is semialgebraic by Theorem 3.8.) Observe that we can write:

$$\mathbb{R}^n = \bigsqcup_{\sigma\in\Omega_{s,d}} A_\sigma = \bigsqcup_{i=1}^{a} A_i,$$

where for each A_i equals one of the nonempty A_σ.

For every $i = 1, \dots, a$, by construction, there exists $r_i > 0$ such that for every $x \in A_i$ the nonzero polynomials among the $p_1(x, \cdot), \dots, p_s(x, \cdot)$ all together have r_i real roots. We denote these roots by

$$\xi_{i,1}(x) < \cdots < \xi_{i,r_i}(x).$$

This defines the desired functions. We prove now that they are semialgebraic. The graph of $\xi_{i,j}$ is given by

$$\mathrm{Graph}(\xi_{i,j}) = \Big\{(x,t) \in A_i \times \mathbb{R} \,|\, \exists (t_1, \dots, t_{r_i}) \in \mathbb{R}^{r_i}, t_1 < \cdots < t_{r_i},\, t = t_j, \\ \prod_{p_k \neq 0} p_k(x, t_1) = \cdots = \prod_{p_k \neq 0} p_k(x, t_{r_i}) = 0\Big\}.$$

Therefore by Theorem 3.8, each $\xi_{i,j}$ is semialgebraic.

We prove now that these functions are continuous. Pick $x \in A_i$. Since $\{p_1, \dots, p_s\}$ is closed under differentiation with respect to the t variable, for every $j = 1, \dots, r_i$, the number $t_j = \xi_{i,j}(x)$ is a simple root for at least one $p_{i_j}(x, \cdot)$. In particular

$$\frac{\partial}{\partial t} p_{i_j}(x, t_j) \neq 0,$$

and by the Implicit Function Theorem there exists a neighborhood $W_{x,j} = U_{x,j} \times I_j$ of (x, t_j) in $\mathbb{R}^n \times \mathbb{R}$ and a smooth function $\gamma_{i,j} : U_{x,j} \to I_j$ such that

$$\{(x,t) \,|\, p_{i_j}(x,t) = 0\} \cap W_{x,j} = \mathrm{Graph}(\gamma_{i,j}).$$

We can repeat this argument for every $j = 1, \dots, r_i$ and, setting $U_x := \bigcap_j U_{x,j}$, we get the existence of functions $\gamma_{i,1}, \dots, \gamma_{i,r_i} : U_x \to \mathbb{R}$ such that $\gamma_{i,j}(x) = \xi_{i,j}(x)$. Moreover, by possibly shrinking U_x, since the $\gamma_{i,j}$ are smooth, we see that

$$\gamma_{i,1} < \cdots < \gamma_{i,r_i}.$$

Consequently, $\gamma_{i,j}|_{S_i} = \xi_{i,j}$, which proves the continuity. $\square$

We are now ready to state and prove the main theorem of this section.

Theorem 3.14. *Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set. Then there exists a finite partition*

$$S = \bigsqcup_{i=1}^{\ell} S_i,$$

such that each S_i is a Nash submanifold of $\mathbb{R}^n$ that is Nash diffeomorphic to a cell.

Proof. The proof is by induction on n. The case $n = 1$ is trivially true, since semialgebraic sets in $\mathbb{R}$ are just finite union of points and intervals.

For the inductive step, let $S \subseteq \mathbb{R}^{n+1}$ be a semialgebraic set defined by a formula which involves the polynomials $p_1, \ldots, p_s \in \mathbb{R}[x, t]$ (here $(x, t) \in \mathbb{R}^n \times \mathbb{R}$). Using Lemma 3.13, we get a partition

$$\mathbb{R}^n = \bigsqcup_{i=1}^{a} A_i.$$

Using the inductive step, we can partition each A_i into finitely many semialgebraic sets, each Nash diffeomorphic to a cell. That is, for every $i = 1, \ldots, a$ we have $A_i = \bigsqcup_{k=1}^{k_i} B_{i,k}$, where $B_{i,k} \simeq (0, 1)^{\dim(B_{i,k})}$. After relabeling, we can write:

$$\mathbb{R}^n = \bigsqcup_{k=1}^{b} B_k,$$

where $B_k \simeq (0, 1)^{\dim(B_k)}$. Moreover, again by Lemma 3.13, restricting the functions $\{\xi_{i,j} : A_i \to \mathbb{R}\}$ to each each $B_{i,k}$ and relabeling, for every $k = 1, \ldots, b$, there are continuous semialgebraic functions $\{\xi_{k,j} : B_k \to \mathbb{R}\}_{j=1}^{r_k}$ such that for every $k = 1, \ldots, b$ and every $x \in B_k$ the set $\{\xi_{k,1}(x), \ldots, \xi_{k,r_k}(x)\}$ consists of the set of all zeroes of the nonzero polynomials in $\{p_1(x, \cdot), \ldots, p_s(x, \cdot)\} \in \mathbb{R}[t]$ and $\sigma(p_1(x, \cdot), \ldots, p_s(x, \cdot))$ is constant on B_k. Notice also that, from the proof of Lemma 3.13 it follows that the $\xi_{k,j}$ are restrictions to B_k (which now is a smooth manifold) of smooth functions, and therefore they are themselves smooth.

For every $k = 1, \ldots, b$ and every $j = 1, \ldots, r_k$ we denote by

$$G_{j,k} := \mathrm{Graph}(\xi_{j,k}).$$

Each $G_{k,j}$, being a graph of a smooth semialgebraic function, is Nash diffeomorphic to its domain $B_k \simeq (0, 1)^{\dim(B_k)}$.

Moreover, for every $k = 1, \ldots, b$ we define $\xi_{k,0} := -\infty$ and $\xi_{k,r_k+1} := +\infty$ and for every $j = 0, \ldots, r_k$ we denote

$$T_{k,j} := \big\{(x, t) \,|\, x \in B_k,\ \xi_{k,j}(x) < t < \xi_{k,j+1}(x)\big\}.$$

Each $T_{k,j}$ is Nash diffeomorphic to $B_k \times (0, 1) \simeq (0, 1)^{\dim(B_k)+1}$.

With this notation, we get a partition of $\mathbb{R}^{n+1}$ into semialgebraic sets:

$$\mathbb{R}^{n+1} = \left(\bigsqcup_{k=1}^{b} \bigsqcup_{j=1}^{r_k} G_{k,j} \right) \sqcup \left(\bigsqcup_{k=1}^{b} \bigsqcup_{j=0}^{r_k} T_{k,j} \right).$$

Since S is the union of some of these sets, this concludes the proof. □

As a corollary we get the following interesting result.

Corollary 3.15. *Let $S \subseteq \mathbb{R}^n$ be semialgebraic. Then S has finitely many connected components, each of which is semialgebraic.*

Proof. Decompose $S = \sqcup S_i$, as in Theorem 3.14. On the set of cells $\{S_1, \dots, S_\ell\}$ we impose the equivalence relation generated by

$$S_i \sim S_j \iff S_i \cap \operatorname{clos}(S_j) \neq \emptyset.$$

Denote by $\{C_1, \dots, C_b\}$ the set where each C_i is the union of cells in the same equivalence class. Then $S = \sqcup_i C_i$, with each C_i which is semialgebraic and closed in S. Since cells are connected, these are the components of S. □

Exercise 3.1. Inspect the proof of Theorem 3.14 and deduce a quantitative bound on the number of connected components of a semialgebraic set $S \subseteq \mathbb{R}^n$ defined by a family of s polynomials of degree bounded by d.

Exercise 3.2. Is there a direct way to prove that semialgebraic sets have finitely many connected components, just using Definition 3.1 (and maybe Theorem 3.4)?

The decomposition of a semialgebraic set as in Theorem 3.14 can be made stronger, by requiring that cells "meet" in a nice way. To make this rigorous we introduce the following definition.

Definition 3.16 (Nash stratification)**.** A *Nash stratification* of a semialgebraic set $S \subseteq \mathbb{R}^n$ is a decomposition

$$S = \bigsqcup_{i=1}^{\ell} S_i$$

into finitely many connected Nash submanifolds (called *strata*) such that

$$S_i \cap \operatorname{clos}(S_j) \neq \emptyset \implies S_i \subset \operatorname{clos}(S_j).$$

for every $1 \leq i, j \leq \ell$ with $i \neq j$.

The following result is a refinement of Theorem 3.14.

Theorem 3.17. *Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set and $\{F_j\}_{j=1}^a$ be a family of semialgebraic subsets of S. There exists a Nash stratification of S such that each F_j is a union of strata.*

Proof. See [3, Proposition 9.18]. □

We can use Theorem 3.14 to give the following definition.

Definition 3.18. Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set and

$$S = \bigsqcup_{i=1}^{\ell} S_i$$

be a decomposition as in Definition 3.12. The *dimension* of S is

$$\dim(S) := \max_{1 \le i \le \ell} \dim(S_i),$$

where $\dim(S_i)$ is as in Definition 3.12. Alternatively, using Theorem 3.17, it is the maximum of the dimensions of its strata.

We list now a sequence of simple properties and facts related to notion of dimension.

(1) If $S = \bigcup_{i=1}^a A_i$, with each A_i semialgebraic, then $\dim(S) = \max_i \dim(A_i)$.

(2) For A, B semialgebraic, $\dim(A \times B) = \dim(A) + \dim(B)$ (with the convention that the dimension of the emptyset is $\dim(\emptyset) := -\infty$).

(3) If $S \subseteq \mathbb{R}^n$ and $\dim(S) \le n - 1$, then S is contained in an algebraic hypersurface.

Let us conclude this section with a comment on Sard's Lemma in the semialgebraic world. Recall that, given a smooth map $f : M \to N$ between smooth manifolds, a point $x \in M$ is a critical point for f if the differential $D_x f : T_x M \to T_{f(x)} N$ is not surjective. Denoting by $\mathrm{crit}(f) \subseteq M$ the set of critical points of f, the set of its critical values is defined to be $f(\mathrm{crit}(f))$. Sard's Lemma [30] states now that $f(\mathrm{crit}(f))$ has measure zero in N. If we strenghten the hypothesis on f, namely we assume it to be semialgebraic, the conclusions are also stronger.

Theorem 3.19 (Semialgebraic Sard's Lemma)**.** *Let $f : M \to N$ be a Nash map between Nash manifolds. The set of critical values of f is a semialgebraic susbet of N of codimension at least one.*

Proof. The set of critical points of f is semialgebraic in M and, consequently, the set of critical values is semialgebraic in N by Corollary 3.6. Sard's Lemma states now that $f(\text{crit}(f))$ has measure zero in N and, using the fact that it is semialgebraic, it follows that $\dim(f(\text{crit}(f))) < \dim(N)$, because otherwise it would not have measure zero. $\square$

Exercise 3.3. Prove the following semialgebraic version of the constant rank theorem. Let $f : A \to B$ be a Nash map between open semialgebraic sets $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$. Assume that for every $x \in A$ the rank of $\mathrm{D}_x f : T_x S_1 \to T_{f(x)} S_2$ is r. Prove that for every $a \in A$ there exist semialgebraic neighborhood U of a and V of $f(a)$ with $f(U) \subset V$ and Nash diffeomorphisms $\varphi : U \to (0,1)^m$ and $\psi : V \to (0,1)^n$ such that $\psi \circ f \circ \varphi^{-1}$ is the projection on the first r coordinates composed with the inclusion (i.e.; $\psi \circ f \circ \varphi^{-1}(x_1, \ldots, x_m) = (x_1, \ldots, x_r, 0, \ldots, 0)$ for every $x \in (0,1)^m$).

Exercise 3.4. Try to give a proof of the Theorem 3.19, only using the results we have presented so far (including the semialgebraic constant rank theorem from exercise 3.3), but without using the classical Sard's Lemma.

3.3 Cohomology of semialgebraic sets

The main goal of this section is to prove that semilalgebraic sets have finite cohomology. More precisely, for a semialgebraic set S (or, more general, a topological space) and $k \in \mathbb{N}$, let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ be the field with two elements, and denote by

$$b_k(S) := \dim(H^k(S; \mathbb{Z}_2))$$

the k-th Betti number for $\mathbb{Z}_2$-coefficients and the total Betti number by

$$b(S) := \sum_{k \geq 0} b_k(S).$$

We have the following.

Theorem 3.20. *Let S be a semialgebraic set. Then $b(S) < \infty$.*

We prove this theorem towards the end of the section. The proof relies on the fact that semialgebraic sets can be triangulated; see Theorem 3.23. A technical result that we will need for Theorem 3.20 is Lemma 3.25.

There is a precise reason for choosing the coefficients of the cohomology to be $\mathbb{Z}_2$. This allows to compare the sum of the Betti numbers of the real part of an algebraic set with those of its complex part. More precisely, let $\tau : X \to X$ be an involution of a triangulable topological space X and denote by $\mathrm{fix}(\tau) \subset X$ the set of its fixed points. The basic example we have in mind is the case $X = \mathbb{C}Z \subset \mathbb{C}\mathrm{P}^n$ is an algebraic set defined by real equations and $\tau : \mathbb{C}Z \to \mathbb{C}Z$ is the complex conjugation, so that $\mathrm{fix}(\tau) = \mathbb{R}Z$, the real part of $\mathbb{C}Z$. Smith's theory on involution implies that $\sum_{k=0}^{\infty} \dim(H^k(\mathrm{fix}(\tau), \mathbb{Z}_2)) \leq \sum_{k=0}^{\infty} \dim(H^k(X, \mathbb{Z}_2))$. This is not true if we work with different coefficients.

We recall the notion of simplicial complex.

Definition 3.21 (Finite simplicial complex in $\mathbb{R}^n$)**.** A *k–simplex* $\sigma = [a_0, \ldots, a_k]$ in $\mathbb{R}^n$ is the convex hull of $k+1$ affinely independent points $\{a_0, \ldots, a_k\} \subset \mathbb{R}^n$:

$$\sigma = \{\lambda_0 a_0 + \cdots + \lambda_k a_k \mid \lambda_0 + \cdots + \lambda_k = 1, \lambda_0 \geq 0, \ldots, \lambda_k \geq 0\}.$$

The ℓ–simplex $[a_{i_0}, \ldots, a_{i_\ell}]$, $\ell \leq k$, is said to be an ℓ–face of $[a_0, \ldots, a_k]$.

A *finite simplicial complex in* $\mathbb{R}^n$ is a finite collection $K = \{\sigma_j\}_{j \in J}$ of simplices in $\mathbb{R}^n$ such that (1) for every $j \in J$ all the faces of σ_j belong to K and (2) if $\sigma_i, \sigma_j \in K$, then, either $\sigma_i \cap \sigma_j = \emptyset$ or $\sigma_i \cap \sigma_j$ is a face of both σ_i and σ_j. A *subcomplex* L of K is a subset $L \subseteq K$ which is itself a simplicial complex.

If $K = \{\sigma_j\}_{j \in J}$ is a finite simplicial complex in $\mathbb{R}^n$, we denote

$$|K| := \bigcup_{j \in J} \sigma_j \subset \mathbb{R}^n.$$

While K provides a combinatorial description the complex, $|K|$ is a topological object with the following structure.

Lemma 3.22. *Let K be a simplicial complex in $\mathbb{R}^n$. Then, $|K|$ is a semialgebraic subset of $\mathbb{R}^n$.*

Proof. This follows from the fact that simplices are semialgebraic. □

The following theorem shows that compact algebraic susbets of $\mathbb{R}^n$ can be triangulated using simplicial complexes.

Theorem 3.23 (Triangulation of compact semialgebraic sets)**.** *Let $S \subset \mathbb{R}^n$ be a compact semialgebraic set and $\{S_1, \ldots, S_a\}$ be a finite family of semialgebraic subsets of S. There exist $\ell \geq 0$, a finite simplicial complex $K = \{\sigma_j\}_{j \in J}$ in $\mathbb{R}^\ell$ and a semialgebraic homeomorphism*

$$\varphi : |K| \to S$$

such that for every $i = 1, \dots, a$ there exists $J_i \subseteq J$ such that

$$S_i = \bigcup_{j \in J_i} \varphi(\mathrm{int}(\sigma_j)).$$

Proof. See [3, Theorem 9.2.1]. □

In the context of the previous theorem we also say that each S_i is a *union of open simplices.*

Remark 3.24. If $S \subseteq \mathbb{R}^n$ is not compact, we can still triangulate it as follows. We first embed $\mathbb{R}^n \hookrightarrow S^n$ and view S as a semialgebraic subset of S^n. Then $\mathrm{clos}(S)$ is compact and, using Theorem 3.23, we can triangulate it in such a way that S is a union of open simplices.

Unions of open simplices are slightly more complicated than just subcomplexes. However, the following observation allows to study them, up to homotopy, as if they were subcomplexes.

Lemma 3.25. *Let K be a finite simplicial complex and $S \subset |K|$ be a union of open simplices. Let also K' denote the first barycentric subdivision of K. There exists a subcomplex $L \subset K'$ such that $|L|$ is homotopy equivalent to S.*

Proof. Denote by $\{v_0, \dots, v_r\}$ the vertices (i.e. 0–simplices) of K'. We define a function $g : |K'| \to [0, 1]$ as follows. First, we set for every $j = 0, \dots, r$,

$$g(v_j) := \begin{cases} 0 & \text{if } v_j \in S \\ 1 & \text{if } v_j \notin S \end{cases}.$$

Then we extend g to all $|K'|$ by linearity: whenever $[v_j]_{j \in J}$ is a face of K', we set

$$g\Big(\sum_{j \in J} \lambda_j v_j\Big) := \sum_{j \in J} \lambda_j g(v_j).$$

Recall from Lemma 3.22 that that $|K'|$ is semialgebraic. Denote by M the simplicial complex given by the union of all the cells in $g^{-1}(1) \subset |K'|$ and by L complex given as the union of all the cells in $g^{-1}(0) \subset |K'|$. Notice that both L and M are subcomplexes. They are closed, since g is continuous. Moreover if a set of vertices $\{v_{j_0}, \dots, v_{j_k}\} \subset \{v_0, \dots, v_r\}$ is entirely contained in L (respectively M), then the simplex $[v_{j_0}, \dots, v_{j_k}] \in L$ (respectively $[v_{j_0}, \dots, v_{j_k}] \in M$). Observe that:

$$|L| \subseteq S \subseteq |K| \setminus |M|$$

by the definition of g. We will define now a deformation retraction of S onto $|L|$.

For every simplex $\sigma = [a_0, \dots, a_k] \in K'$, up to relabeling, assume that the first ℓ vertices are $a_0, \dots, a_\ell \in L$ and the rest is $a_{\ell+1}, \dots, a_k \in M$. For $y = \sum_{i=0}^k \lambda_i a_i \in \sigma$, we define the continuous function

$$\alpha(y) := \sum_{i=0}^{\ell} \lambda_i$$

Let us denote $\tau := \sigma \setminus |M|$. We have $\alpha(y) > 0$ for $y \in \tau$. This implies that the restriction $\alpha|_\tau : \tau \to (0, \infty)$ gives a continuous function $\sigma \setminus (\sigma \cap |M|) \to (0, \infty)$. We then get a continuous map $\varphi_\sigma : \tau \times [0, 1] \to \tau$ defined by

$$\varphi_\sigma(y, t) = \frac{(1-t)\alpha(y) + t}{\alpha(y)} \sum_{i=0}^{\ell} \lambda_i a_i + (1-t) \sum_{i=\ell+1}^{k} \lambda_i a_i$$

for $y = \sum_{i=0}^k \lambda_i a_i$. To see that $\varphi_\sigma(y, t) \in \tau$ we check that

$$\frac{(1-t)\alpha(y) + t}{\alpha(y)} \sum_{i=0}^{\ell} \lambda_i + (1-t) \sum_{i=\ell+1}^{k} \lambda_i = (1-t)\alpha(y) + t + (1-t) \sum_{i=\ell+1}^{k} \lambda_i = 1.$$

Notice that for every $t \in [0, 1]$ and $y \in \sigma \cap S$ we have $\varphi_\sigma(y, t) \in \sigma \cap S$, because $\varphi_\sigma(y, t)$ does not increase the coefficients λ_i for $i \geq \ell + 1$ (which belong to K). This can also be verified from Figure 3.1.

Now, given a face σ' of σ, we have that

$$\varphi_{\sigma'} = \varphi_\sigma|_{\sigma' \times [0,1]}.$$

Therefore, we can glue together the functions φ_τ and find a continuous function

$$\varphi : S \times [0, 1] \to S$$

We have that $\varphi(y, t) = y$ for every $y \in |L|$ and $t \in [0, 1]$. Furthermore, for every $t \in [0, 1]$ and $y \in S$ we have $\varphi(y, t) \in S$ and $\varphi(y, 1) \in |L|$. Therefore, φ defines a retraction from S to $|L|$. □

We can now prove Theorem 3.20.

Proof of Theorem 3.20. Realize first S as a finite union of open simplices of a finite simplicial complex K, using Remark 3.24. Then, by Lemma 3.25, S is homotopy equivalent to the total space $|L|$ of a subcomplex $L \subseteq K$. Therefore $b(S)$ is bounded by the number of cells of L, which is finite. □

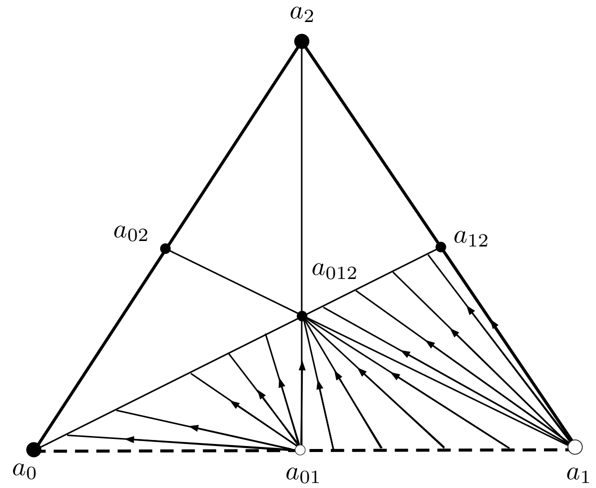

Figure 3.1: A visual proof of Lemma 3.25. In this picture $|K|$ is the simplex $[a_0, a_1, a_2]$, and $S = |K| \setminus (\mathrm{int}([a_0, a_1]) \cup [a_1])$ and $|L| = [a_0, a_{12}, a_2]$.

Exercise 3.5. Use the Universal Coefficients Theorem [16, Theorem 3.2] to prove that, for a topological space X with finite total Betti number $b(X) < \infty$ we have $\sum_{k=0}^{\infty} \mathrm{rank}(H^k(X, \mathbb{Z})) \leq b(X)$. In particular, whatever bound we produce for the sum of the Betti numbers with coefficients in $\mathbb{Z}_2$, this bound will also work for Betti numbers with coefficients in $\mathbb{Z}$.

3.4 The mapping cyclinder of semialgebraic functions

In this section we discuss the mapping cyclinder of continuous semialgebraic functions. We will use this notion to show that the zero sets of such functions are homotopy equivalent to tubular neighborhoods; see Corollary 3.30. The main result of this section is Theorem 3.29.

We first recall the following result extending Theorem 3.23.

Theorem 3.26. *Let $S \subset \mathbb{R}^n$ be a compact semialgebraic set, $\{S_1, \ldots, S_a\}$ be a finite family of semialgebraic subsets of S and $f : S \to \mathbb{R}$ be a semialgebraic, continuous function. There exists a finite simplicial complex $K = \{\sigma_j\}_{j \in J}$ in $\mathbb{R}^{n+1}$ and a semialgebraic homeomorphism $\varphi : |K| \to S$ such that*

(1) each S_i is the union of open simplices; and

(2) the map

$$f \circ \varphi : |K| \to \mathbb{R}$$

is affine on every simplex.

Proof. See [3, Theorem 9.4.1]. □

As a corollary we can derive some useful results on neighborhoods of semialgebraic sets. We start by recalling the definition of the *Mapping cylinder*.

Definition 3.27 (Mapping cylinder)**.** Let $g : B \to Z$ be a continuous map between topological spaces. We define the *mapping cylinder* M_g to be the topological space

$$M_g := \big((B \times [0,1]) \sqcup Z\big)/ \sim,$$

where the equivalence relation "$\sim$" is defined by $(b, 0) \sim g(b)$ for all $b \in B$.

Definition 3.28 (Mapping cylinder neighborhood)**.** Let S be a topological space and $Z \subseteq S$ be a subspace. We call a closed neighborhood W of Z a *mapping cylinder neighborhood*, if there exists a subspace $B \subset W$ such that:

(1) $W \setminus B$ is an open neighborhood of Z;

(2) there exist a continuous map $g : B \to Z$ and a homeomorphism $h : M_g \to W$ such that
$$h|_{Z \cup B} = \mathrm{id}_{Z \cup B}.$$

The subspace B is also called the *boundary* of W.

Let us recall some useful properties of mapping cylinders.

(1) The mapping cylinder M_g deformation retracts to Z.

(2) If $Z \subset S$ has a mapping cylinder neighborhood, then the pair (S, Z) has the homotopy extension property: given $f_0 : S \to Y$ and a homotopy $\widetilde{f_t} : Z \to Y$ such that $\widetilde{f_0} = f_0|_Z$, there is a homotopy $f_t : S \to Y$ such that $f_t|_Z = \widetilde{f_t}$.

(3) Mapping cylinders satisfy a universal property: for every space Y and continuous maps $f_1 : B \times [0,1] \to Y$ and $f_2 : Z \to Y$ such that $f_1(b, 0) = f_2(g(b))$ for every $b \in B$, there exists a unique continuous map $h : M_g \to Y$ which

makes the following diagram commute:

$$
\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, hook, "i_0"'] & Z \arrow[d, hook] \arrow[ddr, bend left, "f_2"] & \\
B \times [0,1] \arrow[r, "q"] \arrow[drr, bend right, "f_1"'] & M_g \arrow[dr, dashed, "\exists ! h"'] & \\
& & Y
\end{tikzcd},
$$

where $q : B \times [0,1] \to M_g$ is the quotient map and $i_0 : B \to B \times [0,1]$ the inclusion $b \mapsto (b, 0)$.

The next result shows that compact semialgebraic sets have semialgebraic mapping cylinder neighborhoods.

Theorem 3.29. *Let S be a compact semialgebraic set and $f : S \to [0, \infty)$ be a continuous semialgebraic function. Denote by $Z := f^{-1}(0)$ the zero set of f. There exists $\epsilon_f > 0$ such that for every $0 < \epsilon < \epsilon_f$ the set $\{f \le \epsilon\}$ is a mapping cylinder neighborhood of Z in S. In particular, the inclusions $Z \hookrightarrow \{f < \epsilon\} \hookrightarrow \{f \le \epsilon\}$ are homotopy equivalences.*

Proof. Let $\varphi : |K| \to S$ be a triangulation of S such that $\widetilde{f} := f \circ \varphi : |K| \to [0, \infty)$ is affine on each simplex. Such a φ exists by Theorem 3.26. Since $\widetilde{f}$ is affine on each simplex, if $\widetilde{f}^{-1}(0)$ contains a simplex σ, it also contains all the faces of σ. Therefore it is a subcomplex, which we denote by

$$|L| := \widetilde{f}^{-1}(0).$$

In particular, we have

$$Z = \varphi(|L|).$$

We denote by $V(K)$ the set of 0–simplices of the complex K and define

$$\epsilon_f := \min_{\substack{v \in V(K) \\ v \notin L}} \widetilde{f}(v);$$

the minimum exists, because $\widetilde{f}$ is continuous.

We will first construct, for every $0 < \epsilon < \epsilon_f$, a continuous semialgebraic map

$$T : f^{-1}(\epsilon) \times [0, \epsilon] \to \{f \le \epsilon\} \subset S$$

such that $f(T(x,t)) = t$ for every $(x,t) \in f^{-1}(\epsilon) \times [0, \epsilon]$ and such that $T|_{f^{-1}(\epsilon) \times (0, \epsilon]}$ is a homeomorphism onto $\{f \le \epsilon\} \setminus Z$.

To this end, let $x \in f^{-1}(\epsilon) \in S$ and denote by $y = \varphi^{-1}(x) \in |K|$. Then y belongs to some simplex $\sigma = [a_0, \dots, a_m]$ of K. Up to reordering, we can assume that $a_0, \dots, a_k \in |L|$ and $a_{k+1}, \dots, a_m \notin |L|$. We can write:

$$y = \sum_{i=0}^{k} \lambda_i a_i + \sum_{j=k+1}^{m} \lambda_j a_j,$$

with $\lambda_0 + \cdots + \lambda_m = 1$ and $\lambda_0, \dots, \lambda_m \geq 0$. As in Section 3.3 we denote $\alpha(y) = \sum_{i=0}^{k} \lambda_i$. We claim that $0 < \alpha(y) < 1$. In fact, if $\alpha(y) = 0$, then the only the last $m - k$ coefficients of y are nonzero and so $\widetilde{f}(y) = \sum_{j=k+1}^{m} \lambda_j \widetilde{f}(a_j) > \epsilon_f$, which contradicts $\widetilde{f}(y) = \epsilon < \epsilon_f$. On the other hand, if $\alpha(y) = 1$, then only the first k coefficients of y are nonzero and we have $\widetilde{f}(y) = \sum_{j=0}^{k} \lambda_j \widetilde{f}(a_j) = 0$, which contradicts $\widetilde{f}(y) = \epsilon > 0$.

The map T is defined by

$$T(x,t) := \varphi\left(\sum_{i=0}^{k} \frac{t\alpha(y) + \epsilon - t}{\epsilon\alpha(y)} \lambda_i a_i + \sum_{j=k+1}^{m} \frac{t}{\epsilon} \lambda_j a_j \right),$$

where $y = \varphi^{-1}(x) = \sum_{i=0}^{m} \lambda_i a_i$. Observe also that the map T is also surjective.

Let us now use T to construct the mapping cylinder neighborhood. We set $W := \{f \leq \epsilon\} \subset S$, $B := f^{-1}(\epsilon) \subset S$ and define $g : B \to Z$ by

$$g(b) := T(b, 0) \quad \text{for } b \in B.$$

Following Definition 3.27 this defines the mapping cylinder M_g. By item (3) we then have a commutative diagram of maps

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, hook, "i_0"'] & Z \arrow[d, hook] \arrow[ddr, bend left, "i_Z"] & \\
B \times [0,\epsilon] \arrow[r, "q"] \arrow[drr, bend right, "T"'] & M_g \arrow[dr, dashed, "\exists! h"'] & \\
 & & W
\end{tikzcd},$$

where $i_Z : Z \to W$ denotes the inclusion, and $i_0(b) = (b, 0)$, and q is the quotient map. By the universal property of mapping cylinders (see item (3)) we get the existence of a map $h : M_g \to W$. Since this map is injective, M_g is compact and W is Hausdorff, it is also a homeomorphism onto its image. Since T is surjective, and q is surjective, h is surjective as well and $h : M_g \to W$ is a homeomorphism.

Moreover, $W \setminus B = \{f < \epsilon\}$ is an open neighborhood of Z. Finally, for every $z \in Z$ and $b \in B = f^{-1}(\epsilon)$ we have

$$h(z) = i_Z(z) = z \quad \text{and} \quad h(b, \epsilon) = T(b, \epsilon) = b.$$

This shows $h|_{(B\times\{\epsilon\})\cup Z} = \mathrm{id}_{(B\times\{\epsilon\})\cup Z}$, which concludes the proof. □

From the previous result we get the following corollary.

Corollary 3.30. *Let $Z \subset \mathbb{R}^n$ be a compact semialgebraic set. There exists $\epsilon_Z > 0$ such that for all $0 < \epsilon < \epsilon_Z$ the inclusions*

$$Z \hookrightarrow \{x \in \mathbb{R}^n \,|\, \mathrm{dist}_Z(x) < \epsilon\} \hookrightarrow \{x \in \mathbb{R}^n \,|\, \mathrm{dist}_Z(x) \leq \epsilon\}$$

are homotopy equivalences.

Proof. The function $\mathrm{dist}_Z : \mathbb{R}^n \to [0, \infty)$ is proper, continuous, semialgebraic and it vanishes on Z. The result follows by applying Theorem 3.29 in the case $S = \{\mathrm{dist}_Z \leq 1\}$ (which is compact and semialgebraic) and $f = \mathrm{dist}_Z|_S$. □

We discuss now an interesting corollary, which allows to "regularize" smooth semialgebraic equalities and inequalities.

Corollary 3.31 (Regularization of smooth equalities and inequalities)**.** *Let S be a compact Nash manifold and $f : S \to \mathbb{R}$ be a smooth semialgebraic function.*

(1) There exists $\epsilon_1 > 0$ such that for all $0 < \epsilon < \epsilon_1$ the inclusion

$$\{f \leq 0\} \hookrightarrow \{f \leq \epsilon\}$$

is a homotopy equivalence and ϵ is a regular value of f.

(2) There exists $\epsilon_2 > 0$ such that for all $0 < \epsilon < \epsilon_2$ the inclusion

$$\{f = 0\} \hookrightarrow \{|f| \leq \epsilon\}$$

is a homotopy equivalence and both ϵ and $-\epsilon$ are regular values of f.

Proof. Let us deal first with the case (1). Let us define $f_1 := f \cdot \chi_{\{f \geq 0\}} : S \to [0, \infty)$. Notice that f_1 is continuous and semialgebraic; moreover, for every $\epsilon > 0$:

$$\{f \leq 0\} = \{f_1 = 0\} \quad \text{and} \quad \{f \leq \epsilon\} = \{f_1 \leq \epsilon\}.$$

By Theorem 3.29, there exists $\epsilon' > 0$ such that for all $0 < \epsilon < \epsilon'$ the inclusion $\{f \leq 0\} \hookrightarrow \{f \leq \epsilon\}$ is a homotopy equivalence. By Theorem 3.19, $f(\mathrm{crit}(f)) \subset \mathbb{R}$

is semialgebraic of dimension zero and, consequently, $f(\text{crit}(f)) = \{c_1, \ldots, c_k\}$. In particular there exists $\epsilon'' > 0$ such that the interval $(0, \epsilon'')$ contains no critical value of f. Defining $\epsilon_1 =: \min\{\epsilon', \epsilon''\}$, the conclusion follows.

As for point (2), we consider the function $f_2 := f^2$. Notice that f_2 is continuous and semialgebraic; moreover, for every $\epsilon > 0$:

$$\{f = 0\} = \{f_2 = 0\} \quad \text{and} \quad \{|f| \le \epsilon\} = \{f_2 \le \epsilon^2\}.$$

The proof follows now the same strategy as for point (1). $\square$

3.5 Semialgebraic triviality

We discuss now a fundamental tool, which guarantees that continuous semialgebraic maps can be "partitioned" into fibrations.

Theorem 3.32. *Let $f : A \to B$ be a semialgebraic continuous map of semialgebraic sets. There exists a finite partition into semialgebraic sets*

$$B = \bigsqcup_{j=1}^{\ell} B_j,$$

such that for every $j = 1, \ldots, \ell$ there exist a semialgebraic set $F_j \subset A$, called fiber, *and a semialgebraic homeomorphism $\theta_j : f^{-1}(B_j) \to B_j \times F_j$, called* trivialization, *such that the following diagram commutes:*

$$\begin{tikzcd} f^{-1}(B_j) \arrow[rr, "\theta_j"] \arrow[dr, "f"'] & & B_j \times F_j \arrow[dl, "p_1"] \\ & B_j & \end{tikzcd}$$

Here, $p_1 : B_j \times F_j \to B_j$ denotes the projection on the first factor.

Proof. See [3, Theorem 9.3.2]. $\square$

Remark 3.33. If the function $f : A \to B$ from Theorem 3.32 is not continuous, the conclusion of the statement is false. For instance, let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^{-1}$ for $x \neq 0$ and $f(0, y) = y$. For this function we cannot partition the target space into finitely many pieces over which obtaining a semialgebraic trivialization.

An immediate corollary is the following result, that says that gives a bound on the dimension of a continuous image of a semialgebraic set.

Corollary 3.34. *Let $f : A \to B$ a continuous semialgebraic map of semialgebraic sets. Then*
$$\dim(f(A)) \leq \dim(A).$$

Proof. We use Theorem 3.32 and write $B = \sqcup B_j$ with $f^{-1}(B_j) \simeq B_j \times F_j$. Then we can write $f(A) = \bigsqcup_{F_j \neq \emptyset} B_j$ and, consequently,

$$\begin{aligned}\dim(f(A)) = \sup_{F_j \neq \emptyset} \dim(B_j) &\leq \sup_{F_j \neq \emptyset} (\dim(B_j) + \dim(F_j)) \\ &= \sup_{F_j \neq \emptyset} \dim(f^{-1}(B_j)) \leq \dim A.\end{aligned}$$

□

The next corollary uses Theorem 3.32 to show that there are only finitely many homeomorphism types among the common zero set of polynomials of degree d in n variables.

Corollary 3.35. *Let $n, d \in \mathbb{N}$ and consider the collection of zero sets of polynomials of degree in $\mathbb{R}^n$:*

$$\mathcal{Z}(n,d) := \{Z \subseteq \mathbb{R}^n \,|\, Z = Z(p_1, \ldots, p_s),\ p_1, \ldots, p_s \in \mathbb{R}[x_1, \ldots, x_n]_d, s \in \mathbb{N}\}.$$

There exist real algebraic sets $Z_1, \ldots, Z_\ell \subseteq \mathbb{R}^n$ such that every $Z \in \mathcal{Z}(n,d)$ is semialgebraically homeomorphic to one Z_j.

Proof. Observe first that, given $p_1, \ldots, p_s \in \mathbb{R}[x_1, \ldots, x_n]_d$, we have

$$Z(p_1, \ldots, p_s) = Z(p_1^2 + \cdots + p_s^2).$$

In particular

$$\mathcal{Z}(n,d) \subseteq \{Z \subseteq \mathbb{R}^n \,|\, Z = Z(p),\ p \in \mathbb{R}[x_1, \ldots, x_n]_{2d}\}.$$

Denote by
$$A := \{(x,p) \in \mathbb{R}^n \times \mathbb{R}[x_1, \ldots, x_n]_{2d} \,|\, p(x) = 0\}$$
and observe that A is an algebraic subset of $\mathbb{R}^n \times \mathbb{R}[x_1, \ldots, x_n]_{2d}$. Denote

$$B := \mathbb{R}[x_1, \ldots, x_n]_{2d}$$

and $f : A \to B$ be the restriction of the projection on the second factor. Notice also that, given $p \in B$, the fiber $f^{-1}(p) \subset \mathbb{R}^n \times \{p\}$ can be identified $Z(p) \subset \mathbb{R}^n$. By Theorem 3.32 we can write

$$B = \bigsqcup_{j=1}^{\ell} B_j$$

and there are $Z_1, \ldots, Z_\ell \subset \mathbb{R}^n$ (which are real algebraic, since the fibers of f are real algebraic) such that for every $j = 1, \ldots, \ell$ and for every $p \in B_j$, the fiber $f^{-1}(p)$ is semialgebraically homeomorphic to Z_j. Therefore, because of Section 3.5, every $Z \in \mathcal{Z}(n, d)$ is semialgebraically homeomorphic to some Z_j. □

Exercise 3.6. State and prove a projective version of Corollary 3.35, as well as a version for the family of complex zero sets of complex polynomials in $\mathbb{C}^n$.

For $r > 0$ we denote the closed ball with center $x \in \mathbb{R}^n$ and of radius r by

$$D(x, r) := \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}.$$

The next two results use Theorem 3.32 to give information on the local structure of semialgebraic sets and their structure at infinity.

Proposition 3.36 (Local conic structure)**.** *Let $S \in \mathbb{R}^n$ be a semialgebraic set and $x \in S$ be a nonisolated point. For every $\epsilon > 0$ consider the intersection $S \cap \partial D(x, \epsilon)$ and define*

$$\mathrm{cone}(S \cap \partial D(x, \epsilon)) := \{x + t(v - x) \mid v \in S \cap \partial D(x, \epsilon), t \in [0, 1]\}.$$

If $\epsilon > 0$ is small enough there is a semialgebraic homeomorphism

$$\varphi : D(x, \epsilon) \cap S \to \mathrm{cone}(S \cap \partial D(x, \epsilon))$$

such that $\|x - y\| = \|x - \varphi(y)\|$ for every $y \in D(x, \epsilon) \cap S$.

Proof. Let $f : S \to [0, \infty)$ be the continuous semialgebraic map $f(y) := \|x - y\|$. By Theorem 3.32 there exists $\epsilon' > 0$, a semialgebraic set F homeomorphic to $f^{-1}(\epsilon')$ and a semialgebraic homeomorphism

$$\theta : \{y \in S \mid 0 < \|x - y\| < \epsilon'\} \to (0, \epsilon') \times F$$

such that the first component of $\theta(y)$ is $\|x - y\|$.

We can assume $\epsilon < \epsilon'$, so that $f^{-1}(\epsilon')$ is homeomorphic to $f^{-1}(\epsilon)$. Notice that $f^{-1}(\epsilon) = S \cap \partial D(x, \epsilon)$. So, replacing F by $f^{-1}(\epsilon)$ we get a semialgebraic homeomorphism $\{y \in S \mid 0 < \|x-y\| < \epsilon'\} \to (0,\epsilon) \times (S \cap \partial D(x,\epsilon))$. Restricting the left side of this to $D(x,\epsilon) \cap S$ we get a homeomorphism

$$\phi : D(x,\epsilon) \cap S \to (0,\epsilon) \times (S \cap \partial D(x,\epsilon)) .$$

Denoting by (ϕ_1, ϕ_2) the components of ϕ, we have $\phi_1(y) = \theta_1(y) = \|x-y\|$. The map φ is now defined as follows:

$$\varphi(y) := \begin{cases} x & \text{if } y = x \\ x + \phi_1(y) \frac{x-\phi_2(y)}{\|x-\phi_2(y)\|} & \text{otherwise} \end{cases} .$$

We have $\|x - \varphi(y)\| = \phi_1(y) = \|x-y\|$, and φ is invertible and its inverse is a continuous semialgebraic. □

Corollary 3.37 (Conic structure at infinity)**.** *Let $S \subseteq \mathbb{R}^n$ be semialgebraic. There exists $R > 0$ such that S deformation retracts on $S \cap D(x,r)$ for every $r > R$.*

Proof. If S is bounded, then $S \cap D(x,r) = S$ for $r > 0$ large enough. In the case S is not bounded, consider the semialgebraic homeomorphism $\eta : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by $x \mapsto x\|x\|^{-2}$. We define the semialgebraic set $\widetilde{S} := \eta(S \setminus \{0\}) \cup \{0\}$. Observe that

$$\eta\left(S \cap \{\|x\| \geq r\}\right) = \widetilde{S} \cap \left\{\|y\| \leq r^{-1}\right\} \setminus \{0\}$$

(and the two sets are semialgebraically homeomorphic, the homeomorphism being given by the restriction of η). Applying now Proposition 3.36 to the semialgebraic set $\widetilde{S}$, we get the existence of $\epsilon > 0$ small enough such that for $\frac{1}{r} < \epsilon$ we have a semialgebraic homeomorphism

$$\theta : \widetilde{S} \cap \left\{\|y\| \leq r^{-1}\right\} \to \operatorname{cone}(\widetilde{S} \cap \partial D(0, r^{-1})).$$

Removing the origin from $\widetilde{S}$, and using the cone structure, we get a deformation retraction of $\widetilde{S} \cap \{\|y\| \leq r^{-1}\}$ onto $\widetilde{S} \cap \partial D(0, r^{-1})$.

Since $\eta : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is a semialgebraic homeomorphism, the deformation retraction above can be seen as a deformation retraction of $S \cap \{\|x\| \geq r\}$ onto $S \cap \partial D(x,r)$, and this glues to a deformation retraction of S onto $S \cap D(x,r)$. □

4 Topology of algebraic sets

In this section we introduce the notion of isotopy and prove *Thom's Isotopy Lemma*, which is a fundamental tool in differential topology, and which will play a crucial role also in real algebraic geometry.

4.1 Thom's Isotopy Lemma

We start with the definition of isotopy.

Definition 4.1 (Isotopy)**.** Let M, N be smooth manifolds. An *isotopy* is a smooth map $\phi : N \times I \to M$ such that for every $t \in I$ the map $\phi_t := \phi(\cdot, t) : N \to M$ is an embedding. The *track* of the isotopy is the map $\widehat{\phi} : N \times I \to M \times I$ defined by $\widehat{\phi}(x,t) := (\phi(x,t), t)$. When $N = M$, each ϕ_t is a diffeomorphism and $\phi_0 = \mathrm{id}_M$, the isotopy is called an *ambient isotopy*.

The notion of isotopy is strongly related to the notion of time–dependent vector field, as we will prove in Lemma 4.3.

Definition 4.2 (Time–dependent vector field)**.** A *time–dependent vector field* on a smooth manifold M is a map $v : M \times I \to TM$ such that $v(x,t) \in T_xM$ for all $(x,t) \in M \times I$. (Given a time–dependent vector field v, we still denote by $v : M \times \mathbb{R} \to TM$ a smooth extension of it on the whole $M \times \mathbb{R}$.) We denote by $\widehat{v}$ the vector field on $M \times \mathbb{R}$ defined by

$$\widehat{v}(x,t) := (v(x,t), 1) \in T_{(x,t)}(M \times I) \simeq T_xM \times \mathbb{R}.$$

Lemma 4.3. *Let M be a compact manifold and $v : M \times I \to TM$ be a time–dependent vector field. Denote by $\widehat{v}$ the vector field on $M \times \mathbb{R}$ defined as in Definition 4.2, and by $\widehat{\phi} : (M \times \mathbb{R}) \times \mathbb{R} \to M \times \mathbb{R}$ its flow. Let also $p_1 : M \times I \to M$ the projection on the first factor. The map $\phi : M \times I \to M$ defined by*

$$\phi(x,t) := p_1(\widehat{\phi}(x,0,t))$$

is the unique ambient isotopy such that

$$\frac{\partial\phi}{\partial t}(x,t) = v(\phi(x,t),t).$$

Proof. Since M is compact, the vector field $\widehat{v} = (v,1)$ is bounded and therefore its flow $\widehat{\phi}$ on $M \times \mathbb{R}$ is complete and is characterized by

$$\frac{\partial\widehat{\phi}}{\partial t}(x,s,t) = \widehat{v}(\widehat{\phi}(x,s,t)).$$

Notice that $\widehat{\phi}(x,s,t) = (\widehat{\phi}^{(1)}(x,s,t), s+t)$. In particular $\widehat{\phi}_t : M \times \mathbb{R} \to M \times \mathbb{R}$ is a diffeomorphism with $\widehat{\phi}_0 = \mathrm{id}_{M\times\mathbb{R}}$ and such that $\widehat{\phi}_t(x,0) \in M \times \{t\} \simeq M$. This implies that ϕ_t is an isotopy. Moreover, from (4.1) we have

$$\frac{\partial\widehat{\phi}}{\partial t}(x,0,t) = \widehat{v}(\widehat{\phi}(x,0,t)) = (v(\phi(x,t),t),1),$$

and (4.3) follows by just looking at the first component of the previous identity. □

We are now in the position of stating Thom's Isotopy Lemma. There are many versions of this statement, all of them relying on the basic version that we prove here.

Theorem 4.4 (Thom's Isotopy Lemma)**.** *Let M be a smooth compact manifold and $F : M \times I \to \mathbb{R}$ be a smooth map such that, for every $t \in I$, zero is a regular value of the function $f_t := F(\cdot,t) : M \to \mathbb{R}$. There exists an ambient isotopy $\phi_t : M \to M$ such that for all $t \in I$*

$$\phi_t(\{f_0 = 0\}) = \{f_t = 0\} \quad \textit{and} \quad \phi_t(\{f_0 \le 0\}) = \{f_t \le 0\}.$$

In particular, for every $t \in I$ the level sets $\{f_t = 0\}$ and $\{f_0 = 0\}$ are diffeomorphic, and the same is true for the Lebesgue sets $\{f_t \le 0\}$ and $\{f_0 \le 0\}$.

Proof. Denote by $\widehat{Z} := Z(F) \subset M \times I$ and by $Z_t := Z(f_t) \subset M$.

We claim that, in order to prove the statement, it is enough to find a time–dependent vector field $v : M \times I \to TM$ such that the corresponding vector field $\widehat{v}$ on $M \times I$, defined by Definition 4.2, is tangent to $\widehat{Z}$. The isotopy $\phi : M \times I \to M$ will then be defined by restricting the flow $\widehat{\phi}$ of $\widehat{v}$ to $M \times \{0\}$, as in Lemma 4.3. In fact, if $\widehat{v}$ is tangent to $\widehat{Z}$, its flow will preserve $\widehat{Z}$. In particular, if $x \in Z_0$, then $(x,0) \in \widehat{Z}$ and $\widehat{\phi}_t(x,0) = (\phi(x,t),t) \in M \times \{t\}$ will also belong to $\widehat{Z}$. This means $f_t(\phi_t(x)) = F(\phi(x,t),t) = 0$, i.e. $\phi_t(Z_0) \subseteq Z_t$. In order to see the other inclusion,

$\phi_t(Z_0) \supseteq Z_t$, we argue similarly: we consider the flow $\widehat{\psi}$ of the vector field $-\widehat{v}$ on $M \times I$ and observe hat $\widehat{\psi}_t = \widehat{\phi}_t^{-1}$ and that $\widehat{\psi}$ also preserves $\widehat{Z}$. We set

$$\psi_t := \widehat{\psi}|_{M\times\{t\}} : M \simeq M \times \{t\} \to M \times \{0\} \simeq M.$$

Arguing as above, we see that $\psi_t(Z_t) \subseteq Z_0$ and, since $\psi_t = \phi_t^{-1}$, applying ϕ_t on both sides of this inclusion gives $Z_t \subseteq \phi_t(Z_0)$. This proves Theorem 4.4. (Notice also that, since $\phi_t : M \to M$ is a diffeomorphism, then $\phi_t|_{Z_0} : Z_0 \to Z_t$ is also a diffeomorphism.)

Similarly, since $\widehat{\phi}_t$ preserves the zero set of F, by continuity, it also preserves the Lebesgue sets $\{F < 0\}$ and $\{F > 0\}$, which implies the right hand side of (4.4).

It remains to prove the existence of the time–dependent vector field v. Notice that the condition that the flow of $\widehat{v}$ preserves $\widehat{Z}$ is equivalent to the condition that $\widehat{v}$ is tangent to $\widehat{Z}$, which in turn is equivalent to:

$$F(x,t) = 0 \implies \frac{\partial F}{\partial x}(x,t)v(x,t) + \frac{\partial F}{\partial t}(x,t) = 0$$

(this is simply obtained by differentiating F along the trajectories of $\widehat{v}$).

In order to construct v we proceed as follows. Pick first $z_0 = (x_0, t_0) \in \widehat{Z}$. We claim that there exists a neighborhood U_{z_0} of z_0 in $M\times I$ and a map $v_{z_0} : U_{z_0} \to TM$ such that for every $(x,t) \in U_{z_0}$ we have $v_{z_0}(x,t) \in T_xM$ and

$$\frac{\partial F}{\partial x}(x,t)v_{z_0}(x,t) + \frac{\partial F}{\partial t}(x,t) = 0.$$

Since $z_0 \in \widehat{F}$, then $f_{t_0}(x_0) = 0$ and the linear map $\frac{\partial F}{\partial x}(x_0,t_0) = d_{x_0}f_{t_0}$ is surjective by assumption. In local coordinates near (x_0,t_0), we can write

$$\frac{\partial F}{\partial x}(x,t) = \ell_1(x,t)dx_1 + \cdots + \ell_n(x,t)dx_n,$$

and, up to relabeling, we can assume $\ell_1(x_0,t_0) \neq 0$. Therefore the function $\ell_1(t,x)$ is separated from zero in a neighborhood U_{z_0} of (x_0,t_0) and on this neighborhood we can define:

$$v_{z_0}(x,t) := -\frac{\partial F}{\partial t}(x,t)\ell_1(x,t)^{-1}\frac{\partial}{\partial x_1}.$$

The map $v_{z_0} : U_{z_0} \to TM$ is smooth, and for every $(x,t) \in U_{z_0}$ satisfies $v_{z_0}(x,t) \in T_xM$ and solves Section 4.1.

Let now $\widehat{U} := \{F \neq 0\}$ and $\mathcal{U}$ be the open cover $\mathcal{U} := \{\widehat{U}\} \cup \{U_{z_0}\}_{z_0\in\widehat{Z}}$ of $M\times I$.

Denote by $\{\widehat{\rho}\} \cup \{\rho_{z_0}\}_{z_0 \in \widehat{Z}}$ a partition of unity subordinated to $\mathcal{U}$ and define:

$$v(x,t) := \sum_{z_0 \in \widehat{Z}} \rho_{z_0}(x,t) v_{z_0}(x,t).$$

Notice that this is a well–defined time–dependent vector field on M.

Moreover, if $(x,t) \in \widehat{Z}$:

$$\begin{aligned}
\frac{\partial F}{\partial x}(x,t)v(x,t) &= \frac{\partial F}{\partial x}(x,t)\left(\sum_{z_0 \in \widehat{Z}} \rho_{z_0}(x,t) v_{z_0}(x,t)\right) \\
&= \sum_{z_0 \in \widehat{Z}} \frac{\partial F}{\partial x}(x,t)\rho_{z_0}(x,t) v_{z_0}(x,t) \\
&= \sum_{z_0 \in \widehat{Z}} \rho_{z_0}(x,t)\left(-\frac{\partial F}{\partial t}(x,t)\right) \\
&= -\frac{\partial F}{\partial t}(x,t),
\end{aligned}$$

where in the last step we used the fact that $\widehat{\rho}|_{\widehat{Z}} \equiv 0$, in particular $\sum_{z_0 \in \widehat{Z}} \rho_{z_0}(x,t) = 1$ if $(x,t) \in \widehat{Z}$. This proves Section 4.1 and concludes the proof. $\square$

Motivated by the statement of Theorem 4.4, we introduce the following notation.

Definition 4.5. Let M be a compact manifold and $Z_0, Z_1 \subseteq M$ be submanifolds. We will say that Z_0 and Z_1 are ambient isotopic, and write $(M, Z_0) \sim (M, Z_1)$, if there exists an ambient isotopy $\phi_t : M \to M$ such that $\phi_1(Z_0) = Z_1$.

The following is a stronger version of Thom's Isotopy Lemma.

Theorem 4.6. *Let M be a compact manifold, N be a smooth manifold and $A \hookrightarrow N$ be a smooth, closed submanifold. Let $F : M \times I \to N$ be a smooth map such that $f_t := F(\cdot, t) : M \to N$ is transversal to A for every $t \in I$. Then $(M, f_0^{-1}(A)) \sim (M, f_1^{-1}(A))$.*

Proof. The proof is essentially the same as the proof of Theorem 4.4 and we only sketch it, leaving the details to the reader.

Denote by $\widehat{Z} := F^{-1}(A) \subset M \times I$ and by $Z_t := f_t^{-1}(A) \subset M$. As for Theorem 4.4, in order to prove the statement, it is enough to find a time–dependent vector field $v : M \times I \to TM$ such that the corresponding vector field $\widehat{v}$ on $M \times I$, defined by Definition 4.2, is tangent to $\widehat{Z}$. The isotopy $\phi : M \times I \to M$ will then again be defined by restricting the flow $\widehat{\phi}$ of $\widehat{v}$ to $M \times \{0\}$, as in Lemma 4.3.

Now the condition that the flow of $\widehat{v}$ preserves $\widehat{Z}$ is equivalent to the condition that $\widehat{v}$ is tangent to $\widehat{Z}$, which in turn is equivalent to:

$$F(x,t) \in A \implies \frac{\partial F}{\partial x}(x,t)v(x,t) + \frac{\partial F}{\partial t}(x,t) \in T_{F(x,t)}A$$

(this is simply obtained by differentiating F along the trajectories of $\widehat{v}$).

Using the fact that A is a submanifold, for every point $y \in A$ we can find a chart $\psi : U_y \to \mathbb{R}^n$ on a neighborhood of y in N such that $A \cap U_y = \psi^{-1}(\{0\} \times \mathbb{R}^{n-k})$, where $k = n - \dim A$. On this neighborhood we define $g_y : U_y \to \mathbb{R}^k$ by composing ψ with the projection on $\mathbb{R}^k$, so that $A \cap U_y = g_y^{-1}(0)$, with $0 \in \mathbb{R}^k$ a regular value of g_y. In this way we get an open cover $\mathcal{W} = \{U_y\}_{y \in A} \cup \{A^c\}$ of N and, taking its preimage under F an open cover $F^{-1}\mathcal{W}$ of M. Since M is compact, we can assume that $F^{-1}\mathcal{W} = \{W_j\}_{j=0}^r$, where $W_0 = F^{-1}(A^c)$ and, for $j = 1, \dots, r$, $W_j = F^{-1}(U_{y_j})$, with $y_j \in A$. On each $W_j = F^{-1}(U_{y_j})$ we have a smooth map $f_j := g_{y_j} \circ F : W_j \to \mathbb{R}^k$ such that $\widehat{Z} \cap W_j = f_j^{-1}(0)$. Moreover the condition Section 4.1, which tells that $\widehat{v}$ is tangent to $\widehat{Z}$, can be reformulated as follows: for every $j = 1, \dots, r$

$$(x,t) \in \widehat{Z} \cap W_j \implies \frac{\partial f_j}{\partial x}(x,t)v(x,t) + \frac{\partial f_j}{\partial t}(x,t) = 0 \in \mathbb{R}^k.$$

Notice that if $(x,t) \in \widehat{Z} \cap W_j \cap W_i$, then

$$\frac{\partial f_j}{\partial x}(x,t)v(x,t) + \frac{\partial f_j}{\partial t}(x,t) = 0 \iff \frac{\partial f_i}{\partial x}(x,t)v(x,t) + \frac{\partial f_i}{\partial t}(x,t) = 0.$$

In particular, for every $(x,t) \in \widehat{Z}$, enough for v to fulfill the condition Section 4.1 only for one $j = 1, \dots, r$ such that $(x,t) \in W_j$.

In order to construct v we proceed as follows. Pick first $z_0 = (x_0, t_0) \in \widehat{Z}$ and assume $z_0 \in W_j$. We claim that there exists a neighborhood U_{z_0} of z_0 in $M \times I$ and a map $v_{z_0} : U_{z_0} \to TM$ such that for every $(x,t) \in U_{z_0}$ we have $v_{z_0}(x,t) \in T_xM$ and

$$\frac{\partial f_j}{\partial x}(x,t)v_{z_0}(x,t) + \frac{\partial f_j}{\partial t}(x,t) = 0.$$

Since $z_0 \in \widehat{F}$, then $f_{t_0}(x_0) \in A$ and the linear map $\frac{\partial f_i}{\partial x}(x_0, t_0)$ is surjective by assumption. In local coordinates near (x_0, t_0), we can write

$$\frac{\partial f_i}{\partial x}(x,t) = (\ell_{1,1}(x,t)dx_1 + \cdots + \ell_{1,n}(x,t)dx_n, \ldots, \ell_{k,1}(x,t)dx_1 + \cdots + \ell_{k,n}(x,t)dx_n)$$

and, up to relabeling, we can assume that the matrix $L(x,t) := (\ell_{i,j}(x,t))_{1\leq i,j\leq k}$ is invertible at (x_0, t_0). Therefore the $k \times k$ matrix $L(x,t)$ is invertible in a neighborhood U_{z_0} of (x_0, t_0) and on this neighborhood, setting

$$L(x,t)^{-1}\frac{\partial f_j}{\partial t}(x,t) = (\nu_1(x,t), \ldots, \nu_k(x,t)) \in \mathbb{R}^k$$

we define:

$$v_{z_0}(x,t) := -\sum_{i=1}^{k} \nu_i(x,t)\frac{\partial}{\partial x_i}.$$

The map $v_{z_0} : U_{z_0} \to TM$ is smooth, and for every $(x,t) \in U_{z_0}$ satisfies $v_{z_0}(x,t) \in T_xM$ and solves Section 4.1.

The last step of the proof consists in gluing the local fields $\{v_{z_0}\}_{z_0\in\widehat{Z}}$ with the help of a partition of unity, as in the proof of Theorem 4.4. The global solution v obtained in this way solves the problem, since each local solution does it and since Section 4.1 is linear (this can be checked as in the end of the proof of Theorem 4.4). □

Exercise 4.1. Fill in the details of the proof of Theorem 4.6.

Definition 4.7 (First jet bundle and the $\mathscr{C}^1$–topology)**.** Let M be a smooth compact manifold T^*M its cotangent bundle and $\varepsilon_1 = M \times \mathbb{R}$ the trivial bundle. We define the first jet bundle $\pi : J^1(M,\mathbb{R}) \to M$ as:

$$J^1(M,\mathbb{R}) := \varepsilon_1 \oplus T^*M.$$

This is a vector bundle of rank $r = \dim M + 1$. Given $f \in \mathscr{C}^1(M,\mathbb{R})$, we define its first jet extension $j^1 f \in \mathscr{C}^0(M, J^1(M,\mathbb{R}))$ by

$$j^1 f(x) := (f(x), d_x f).$$

Let also $\|\cdot\| : J^1(M,\mathbb{R}) \to \mathbb{R}$ be a continuous function such that for every $x \in M$ its restriction to $J^1(M,\mathbb{R})|_x$ is a norm. Given $f \in \mathscr{C}^1(M,\mathbb{R})$ we define its $\mathscr{C}^1$–norm by:

$$\|f\|_{\mathscr{C}^1(M,\mathbb{R})} := \max_{x\in M} \|j^1 f(x)\|$$

With this notation, $(\mathscr{C}^1(M,\mathbb{R}), \|\cdot\|_{\mathscr{C}^1(M,\mathbb{R})})$ is a Banach space. The topology on this space does not depend on the choice of $\|\cdot\| : T^*M \to \mathbb{R}$ and is called the $\mathscr{C}^1$–topology.

Lemma 4.8. *Let M be a compact manifold and $f : M \to \mathbb{R}$ with zero a regular value. There exists $\delta(f) > 0$ such that*

$$\|f-g\|_{\mathscr{C}^1(M,\mathbb{R})} < \delta(f) \implies (M, Z(f)) \sim (M, Z(g)) \quad \textit{and} \quad (M, \{f \le 0\}) \sim (M, \{g \le 0\}).$$

Proof. Let us denote by $E := J^1(M, \mathbb{R})$, equipped with a norm $\|\cdot\| : E \to \mathbb{R}$ as in Definition 4.7. Given a function $h \in \mathscr{C}^1(M, \mathbb{R})$, we introduce the quantity:

$$\delta(h) := \min_{x \in M} \|j^1 f(x)\|.$$

Observe that $\delta(h) > 0$ if and only if zero is a regular value of h.

By assumption $\delta(f) > 0$. Denote by $f_t := f + t(g - f) \in \mathscr{C}^1(M, \mathbb{R})$. We will show that, if $\|g - f\|_{\mathscr{C}^1(M,\mathbb{R})} < \delta(f)$, then, for every $t \in I$, zero is a regular value of f_t, and the conclusion will follow from Theorem 4.4.

Let $j^1 f_t(x) := (f_t(x), d_x f_t) \in \mathbb{R} \times T^*_x M \simeq \mathbb{R} \times \mathbb{R}^n$. The assumption $\delta(f) > 0$ implies that for every $x \in M$ the vector $j^1 f_0(x)$ lies outside the closed ball $B_{E_x}(0, \delta(f))$. On the other hand, the fact that $\|g - f\|_{\mathscr{C}^1(M,\mathbb{R})} < \delta(f)$ implies that $j^1 f_0(x) - j^1 f_1(x)$ lies inside the interior of the ball $B_{E_x}(0, \delta(f))$. Since for every $x \in M$

$$j^1 f_t(x) = j^1 f_0(x) + t(j^1 f_0(x) - j^1 f_1(x)),$$

and since $|t| \le 1$, then $j^1 f_t(x)$ cannot be zero. In particular,

$$\delta(f_t) = \min_{x \in M} \|j^1 f_t(x)\| > 0,$$

and zero is a regular value of f_t for every $t \in I$, as claimed. $\square$

4.2 A bound on the Betti numbers of real algebraic sets

In this section we prove a result that estimates the sum of the Betti numbers of a real algebraic set Z in $\mathbb{RP}^n$ defined by polynomial equations of degree at most d, i.e. $b(Z) \le (2d)^n$ (see Theorem 4.13). This bound is usually refererred to as *Thom–Milnor's bound*, since a bound of this type was independently proved by Thom in [38] and Milnor in [29]. The proofs are similar, and are based on Morse theory. Our proof also uses Morse theory, but it is slightly different from the classical ones.

From Morse theory we recall only the main ingredient, Theorem 4.10; the interested reader is referred to [28]. Notice that, if one is only interested in estimating the number of connected components of the *complement* of an algebraic set, Morse theory is not needed, see Theorem 4.15.

Definition 4.9 (Morse function)**.** Let M be a smooth manifold and $f \in \mathscr{C}^2(M, \mathbb{R})$. We will say that f is a *Morse* function if the map $j^1 f : M \to J^1(M, \mathbb{R})$ defined in Definition 4.7 is transversal to the zero section.

Exercise 4.2. Prove that a function $f : M \to \mathbb{R}$ is Morse if and only if for every $y \in \mathrm{crit}(f)$ and every chart $\varphi : U \to \mathbb{R}^n$ on a neighborhood of y the Hessian matrix $\mathrm{He}(f \circ \varphi^{-1})(\varphi(y))$ is nondegenerate.

A Morse function on a compact manifold has only finitely many critical points (the nondegeneracy condition implies that they are isolated).

The importance of Morse functions is given by the following result, which shows that if $f : M \to \mathbb{R}$ is Morse, the number of its critical points gives an upper bound for the sum of the Betti numbers of M.

Theorem 4.10 (Morse inequalities)**.** *Let M be a compact manifold and $f : M \to \mathbb{R}$ be a Morse function. Then, for every $c \in \mathbb{R}$ we have*

$$b\left(\{f \leq c\}\right) \leq \#\left(\mathrm{crit}(f) \cap \{f \leq c\}\right).$$

Proof. See [28, Theorem 5.2]. □

We will now interpret a polynomial of even degree as a function on the projective space, and give a bound for the number of its critical points when it is a Morse function, as follows. For every $p \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ we define the function $f_p : \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ by

$$f_p([x]) := \frac{p(x)}{\|x\|^{2d}}.$$

The function f_p is well defined, since for every $\lambda \neq 0$ we have

$$f_p([\lambda x]) = \frac{p(\lambda x)}{\|\lambda x\|^{2d}} = \frac{\lambda^{2d} p(x)}{\lambda^{2d}\|x\|^{2d}} = f_p([x]).$$

(This would not work for polynomials of *odd* degree.) Notice that the correspondence $p \mapsto f_p$ gives a linear embedding $\mathbb{R}[x_0, \ldots, x_n]_{(2d)} \hookrightarrow \mathscr{C}^1(\mathbb{R}\mathrm{P}^n, \mathbb{R})$.

Lemma 4.11. *For every $n, d \in \mathbb{N}$, let $\mathcal{M}_{n,2d} \subset \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ be the set defined by*

$$\mathcal{M}_{n,2d} := \{p \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)} \mid f_p : \mathbb{R}\mathrm{P}^n \to \mathbb{R} \text{ is Morse}\}.$$

The following properties are true:

(1) $\mathcal{M}_{n,2d}$ *is a dense semialgebraic subset of* $\mathbb{R}[x_0, \dots, x_n]_{(2d)}$.

(2) For every $p \in \mathcal{M}_{n,2d}$ *we have*

$$\#\mathrm{crit}(f_p) \leq (2d)^n.$$

Proof. Let us prove point (1). Consider the standard atlas $\{(U_i, \varphi_i)\}_{i=0}^n$ for $\mathbb{RP}^n$, i.e. $U_i = \{x_i \neq 0\}$ and $\varphi_i([x]) = x_i^{-1}(x_0, \dots, \widehat{x_i}, \dots, x_n)$. For every $p \in \mathbb{R}[x_0, \dots, x_n]_{(2d)}$ consider the function

$$f_{p,i} := f_p \circ \varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}.$$

Notice that f_p is Morse if and only if $f_{p,i}$ is Morse for every $i = 0, \dots n$ (this is because the notion of nondegenerate critical point is invariant under composition with a diffeomorphism). We will prove that for every $i = 0, \dots, n$ the set

$$\mathcal{M}_{n,2d,i} = \{p \,|\, f_{p,i} : \mathbb{R}^n \to \mathbb{R} \text{ is Morse}\} \subseteq \mathbb{R}[x_0, \dots, x_n]_{(2d)}$$

is dense and semialgebraic. The conclusion will follow from the fact that

$$\mathcal{M}_{n,2d} = \bigcap_{i=0}^n \mathcal{M}_{n,2d,i}.$$

In order to prove that each $\mathcal{M}_{n,2d,i}$ is dense and semialgebraic, we assume, without loss of generality, that $i = 0$. Observe that $f_{p,0}$ is Morse if and only if zero is a regular value of its gradient $y \mapsto \nabla f_{p,0}(y)$. Consider therefore the smooth map $G_0 : \mathbb{R}[x_0, \dots, x_n]_{(2d)} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$G_0(p, y) := \nabla f_{p,0}(y).$$

We will show that G_0 is a submersion, and the Parametric Transversality Theorem[1] will imply that the set $\mathcal{M}_{n,2d,0}$ of p such that $y \mapsto \nabla f_{p,0}(y)$ is transversal to zero has full measure in $\mathbb{R}[x_0, \dots, x_n]_{(2d)}$. Since this set is clearly semialgebraic, the conclusion follows.

In order to prove that G_0 is a submersion, observe first that G_0 is linear in the variable p. Hence, denoting by $y = (y_1, \dots, y_n)$ the variable in $\mathbb{R}^n$, for every

[1] The theorem states the following: let $G : P \times M \to N$ be a smooth map between smooth manifolds. We think of this as a "family" of smooth maps $\{g_p : M \to N\}_{p \in P}$ depending on the parameter $p \in P$, where $g_p(x) := G(p, x)$. Let $A \hookrightarrow N$ be a smooth submanifold and assume that G is transversal to A. Then, the set of $p \in P$ such that g_p *is not* transversal to A has zero measure in P – in particular the set of $p \in P$ such that g_p is transversal to A is dense in P. See [17, Chapter 3, Theorem 2.7].

$p, q \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$:

$$\frac{\partial}{\partial t} G_0(p + tq, y)\Big|_{t=0} = \nabla f_{q,0}(y) = G_0(q, y).$$

This means that, in order to prove that G_0 is a submersion, it is enough to find, for every $y, w \in \mathbb{R}^n$ a polynomial $p_{y,w}$ such that $G_0(p_{y,w}, y) = w$.

Given $p \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$, observe that

$$f_{p,0}(y) = \frac{p(1, y)}{(1 + \|y\|^2)^d}.$$

The gradient of $f_{p,0}$ can be easily computed as

$$\nabla f_{p,0}(y) = \frac{1}{(1+\|y\|^2)^d}\bigg(\frac{\partial p}{\partial x_1}(1, y) + 2dy_1(1 + \|y\|^2)^{-1}p(1, y), \ldots \\ \ldots, \frac{\partial p}{\partial x_n}(1, y) + 2dy_n(1 + \|y\|^2)^{-1}p(1, y)\bigg).$$

Given $y, w \in \mathbb{R}^n$ consider the polynomial $p_{y,w}$ defined by

$$p_{y,w}(x_0, \ldots, x_n) := (1 + \|y\|^2)^d x_0^{d-1} \sum_{i=1}^{n} w_i(x_i - y_i x_0).$$

An immediate computation shows that

$$(D_{(p,y)}G_0)p_{y,w} = \frac{\partial}{\partial t} G_0(p + tp_{y,w}, y)\Big|_{t=0} = G(p_{y,w}, y) = w.$$

This means that for every $(p, y) \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ the differential $D_{(p,y)}G_0$ is surjective and concludes the proof of point (1).

In order to prove point (2), let $p \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ and consider the following commutative diagram:

$$\begin{tikzcd} S^n \arrow[d, "\pi"'] \arrow[dr, "p|_{S^n}"] & \\ \mathbb{RP}^n \arrow[r, "f_p"'] & \mathbb{R} \end{tikzcd},$$

where π is the covering map. Since π is a local diffeomorphism, if $p \in \mathcal{M}_{n,2d}$,

$$\#\mathrm{crit}(f_p) = \tfrac{1}{2}\#\mathrm{crit}(p|_{S^n}).$$

A point $x \in S^n$ is critical for $p|_{S^n}$ if and only if there is $\lambda \neq 0$ such that

$$\nabla p(x) = \lambda x.$$

Up to composing p with a rotation, we may assume that $(1, 0, \dots, 0)$ *is not* critical for $p|_{S^n}$. Therefore the condition for $x \in \mathbb{R}^{n+1}$ being critical, in this case, reduces to

$$\|x\|^2 = 1 \quad \text{and} \quad x_0 \frac{\partial p}{\partial x_i}(x) - x_i \frac{\partial p}{\partial x_0}(x) = 0 \quad \forall i = 1, \dots, n.$$

Since $p \in \mathcal{M}_{n,2d}$, this is a nondegenerate system of $(n+1)$ equations in $\mathbb{R}^{n+1}$ of degrees $(2, 2d, \dots, 2d)$. By Bézout's theorem, the number of solutions of this system is bounded by $2(2d)^n$. This, together with Section 4.2, implies the statement. $\square$

Remark 4.12. Interpreting a homogeneous polynomial as a symmetric tensor, it is proved in [6] that for the generic $p \in \mathbb{R}[x_0, \dots, x_n]_{(d)}$ one has $\#\text{crit}(p|_{S^n}) \leq 2((d-1)^{n+1} - 1)/(d-2)$, which refines the bound from the previous proof. No better bound can be found: in [24] it is proved that there exists $p \in \mathbb{R}[x_0, \dots, x_n]_{(d)}$ attaining such bound.

We are now ready to state the main result of this section.

Theorem 4.13 (Thom–Milnor's bound)**.** *Let $Z \subseteq \mathbb{R}\mathrm{P}^n$ be an algebraic set defined by polynomials of degree at most d. Then*

$$b(Z) \leq (2d)^n.$$

Proof. Let us write $Z = Z(p_1, \dots, p_s) \subseteq \mathbb{R}\mathrm{P}^n$, with $p_i \in \mathbb{R}[x_0, \dots, x_n]_{(d_i)}$ and $d_i \leq d$. Define the polynomial $q \in \mathbb{R}[x_0, \dots, x_n]_{(2d)}$ by

$$q(x) := \|x\|^{2d-2d_1} p_1(x)^2 + \cdots + \|x\|^{2d-2d_s} p_s(x)^2.$$

Notice that zero set of the function $f_q : \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ (defined by Section 4.2) coincides with Z:

$$Z(f_q) = Z(q) = Z(p_1, \dots, p_s).$$

Moreover, since $f_q \geq 0$, then $Z(f_q) = \{f_q \leq 0\}$ and it follows from Corollary 3.31 that there exists $\epsilon(q) > 0$ such that for every $0 < \epsilon < \epsilon(q)$ the inclusion

$$Z = \{f_q \leq 0\} \hookrightarrow \{f_q \leq \epsilon\}$$

is a homotopy equivalence and ϵ is a regular value of f_q.

Let $q_\epsilon := q - \epsilon \| \cdot \|^{2d} \in \mathbb{R}[x_0, \dots, x_n]_{(2d)}$, so that

$$\{f_q \leq \epsilon\} = \{f_{q_\epsilon} \leq 0\}.$$

The condition that ϵ is a regular value of f_q is equivalent to the fact that zero is a regular value of f_{q_ϵ}. Therefore, let $\delta(f_{q_\epsilon}) > 0$ be given by Lemma 4.8. Let now $r \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ such that:

$$r \in \mathcal{M}_{n,2d} \quad \text{and} \quad \|f_r - f_{q_\epsilon}\|_{\mathscr{C}^1(\mathbb{R}\mathrm{P}^n,\mathbb{R})} < \delta(f_{q_\epsilon}).$$

Such a polynomial r exists because $\mathcal{M}_{n,2d} \subseteq \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ is dense (Lemma 4.11) and the set $\{r \,|\, \|f_r - f_{q_\epsilon}\|_{\mathscr{C}^1(\mathbb{R}\mathrm{P}^n,\mathbb{R})} < \delta(f_{q_\epsilon})\} \subset \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ is open and nonempty (all norms on a finite dimensional vector space are equivalent).

Lemma 4.8 implies now that $\{f_{q_\epsilon} \leq 0\}$ and $\{f_r \leq 0\}$ are diffeomorphic. Moreover, since f_r is Morse, we can apply Theorem 4.10 and get

$$b(Z) = b(\{f_{q_\epsilon} \leq 0\}) = b(\{f_r \leq 0\}) \leq \#\mathrm{crit}(f_r).$$

The conclusion of the theorem follows now from item (2). □

As a corollary of the proof[2] of the previous theorem we get the following.

Corollary 4.14. *Denote by $\pi : S^n \to \mathbb{R}\mathrm{P}^n$ the double cover map, and let $Z \subset \mathbb{R}\mathrm{P}^n$ be an algebraic set defined by polynomials of degree at most d. Then $b(\pi^{-1}(Z)) \leq 2(2d)^n$.*

Proof. The proof runs exactly the same as the proof of Theorem 4.13, where now we use directly the function $q|_{S^n} : S^n \to \mathbb{R}$ instead of f_q. In the last step pf the proof we use the fact that if $r \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ is such that $f_r : \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ is Morse, then also $r|_{S^n}$ is Morse and $\#\mathrm{crit}(r|_{S^n}) = 2\#\mathrm{crit}(f_r)$, which is bounded by item (2). □

Exercise 4.3. Using a Mayer–Vietoris argument, prove that if an algebraic set $Z \subset \mathbb{R}^n$ is defined by equations of degree at most d, then[3] $b(Z) \leq O(d^n)$.

The next theorem estimates the number of connected components of the complement of an algebraic set. This will be useful when the algebraic set is a "discriminant" and, combined with Theorem 4.6, it gives a way to givea bound on the number of isotopy classes of nondegenerate objects in a family (see Example 4.16 below). The proof of Theorem 4.15 does not use Morse theory.

Theorem 4.15. *If $Z \subseteq \mathbb{R}\mathrm{P}^n$ is an algebraic set defined by polynomials of degree at most d, then*

$$b_0(\mathbb{R}\mathrm{P}^n \setminus Z) \leq (2d)^n.$$

[2] This also follow from the statement of Theorem 4.13, using the fact that for a double cover $\pi : X \to Y$ of triangulable sets we have $b(Y) \leq 2b(X)$.

[3] Actually, as it is proved in [29], $b(Z) \leq d(2d-1)^{n-1}$.

Proof. The proof is similar to the proof of Theorem 4.13. We write $Z = Z(p_1, \ldots, p_s) \subseteq \mathbb{R}\mathrm{P}^n$, with $p_i \in \mathbb{R}[x_0, \ldots, x_n]_{(d_i)}$ and $d_i \leq d$, and define the polynomial $q \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ by

$$q(x) := \|x\|^{2d-2d_1} p_1(x)^2 + \cdots + \|x\|^{2d-2d_s} p_s(x)^2.$$

Now we have $\mathbb{R}\mathrm{P}^n \setminus Z = \{f_q > 0\}$, where $f_q : \mathbb{R}\mathrm{P}^n \to \mathbb{R}$ is defined by Section 4.2.

By Theorem 3.29, there exists $\epsilon_{f_q} > 0$ such that for every $0 < \epsilon < \epsilon_{f_q}$ the set $\{f \leq \epsilon\}$ is a mapping cylinder neighborhood of Z in $\mathbb{R}\mathrm{P}^n$. In particular, for such $0 < \epsilon < \epsilon_{f_q}$ the mapping cylinder neighborhood property implies that $\{f_q \leq \epsilon\} \setminus Z$ deformation retracts to $\{f_q = \epsilon\}$ and this in turn yields a deformation retraction of $\mathbb{R}\mathrm{P}^n \setminus Z$ onto $\{f_q \geq \epsilon\}$. This implies:

$$b_0(\mathbb{R}\mathrm{P}^n \setminus Z) = b_0(\{f_q \geq \epsilon\}).$$

By the semialgebraic Sard's lemma we can also assume that $(0, \epsilon_{f_q})$ contains no critical values of f_q and ϵ is a regular value of f_q. Let $q_\epsilon := q - \epsilon \| \cdot \|^{2d} \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$, so that

$$\{f_q \geq \epsilon\} = \{-f_q + \epsilon \leq 0\} = \{f_{-q_\epsilon} \leq 0\}.$$

The condition that ϵ is a regular value of f_q is equivalent to the fact that zero is a regular value of f_{-q_ϵ}. Therefore, let $\delta(f_{-q_\epsilon}) > 0$ be given by Lemma 4.8. Let now $r \in \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ such that:

$$r \in \mathcal{M}_{n,2d} \quad \text{and} \quad \|f_r - f_{-q_\epsilon}\|_{\mathscr{C}^1(\mathbb{R}\mathrm{P}^n, \mathbb{R})} < \delta(f_{-q_\epsilon}).$$

As above, such a polynomial r exists because $\mathcal{M}_{n,2d} \subseteq \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ is dense (Lemma 4.11) and the set $\{r \mid \|f_r - f_{q_\epsilon}\|_{\mathscr{C}^1(\mathbb{R}\mathrm{P}^n, \mathbb{R})} < \delta(f_{q_\epsilon})\} \subset \mathbb{R}[x_0, \ldots, x_n]_{(2d)}$ is open and nonempty.

Together with Section 4.2, Lemma 4.8 implies that

$$b_0(\mathbb{R}\mathrm{P}^n \setminus Z) = b_0(\{f_{-q_\epsilon} \leq 0\}) = b_0(\{f_r \leq 0\}).$$

Each connected component of $\{f_r \leq 0\}$ is a smooth compact manifold with boundary a component of $\{r = 0\}$. In the interior of each component the function f_r takes strictly negative values and, by compactness, it attains a minimum. Therefore the number of connected components of $\{f_r \leq 0\}$ is bounded by the number of its local minima, which in turn is bounded by the number of its critical points. Since f_r is Morse, the conclusion follows from the bound item (2) in Lemma 4.11. $\square$

Example 4.16. Let $N = \binom{n+d}{d} - 1$ and identify $\mathbb{RP}^N$ with the projectivization of $\mathbb{R}[x_0, \ldots, x_n]_{(d)}$. Let $\Sigma \subset \mathbb{RP}^N$ be the set of polynomials p (up to multiples) whose real zero set is singular. We will see later that this is contained in an algebraic set $Z \subset \mathbb{RP}^N$ of degree $\delta := (n+1)(d-1)^n$. In particular, for any two polynomials p_0, p_1 in the same connected component of $\mathbb{RP}^N \setminus Z$, by Theorem 4.6, we have $(\mathbb{RP}^n, Z(p_0)) \sim (\mathbb{RP}^n, Z(p_1))$. The number of classes $[(\mathbb{RP}^n, Z(p))]$ with $Z(p)$ nonsingular is bounded by $b_0(\mathbb{RP}^N \setminus \Sigma) \leq b_0(\mathbb{RP}^N \setminus Z) \leq (2\delta)^N$ (see Corollary 9.16).

Exercise 4.4. Using the ideas from the proof of Theorem 4.13, try to prove the following statement *without* using Morse Theory: there exists $c_n > 0$ such that for every algebraic set $Z \subseteq \mathbb{RP}^n$ defined by polynomial equations of degree bounded by d, we have $b_0(Z) \leq c_n d^n$.

4.3 The fundamental class of a real algebraic set

The goal of this section is to prove the following theorem. As above, recall that, for a triangulable space Z and $j \in \mathbb{N}$, we denote by $H_j(Z)$ the j–th homology group with coefficients in $\mathbb{Z}_2$.

Theorem 4.17. *Let $Z \subset \mathbb{R}^n$ be a compact real algebraic set of dimension k and $\varphi : |K| \to Z$ be a semialgebraic triangulation. The sum of all the k–dimensional simplices in K is a $\mathbb{Z}_2$–cycle which represents a* nonzero *element in $H_k(Z)$; this element is independent of the choice of the triangulation.*

The nonzero element from the previous statement will be denoted by $[Z] \in H_k(Z)$ and called *the fundamental class of Z*.

In order to establish the result we will need to prove first some lemmas, essentially with the purpose of showing that the statement is true when Z is a curve. The proof of the general case will then be a consequence of the result for curves. We begin with the following.

Lemma 4.18. *Let $Z \subset \mathbb{R}^n$ be an algebraic set of dimension k. For every $v \in S^{n-1}$ denote by $\pi_v : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the orthogonal projection onto $\mathbb{R}^{n-1} \simeq \{v\}^\perp$. The set*

$$\Sigma := \{v \in S^{n-1} \,|\, \pi_v(x) = \pi_v(y) \text{ for infinitely many couples } (x,y) \in (Z \times Z) \setminus \Delta\}$$

is a semialgebraic set of dimension $\dim(\Sigma) < 2k$.

Proof. Consider the map $\Psi : (Z \times Z) \setminus \Delta \to S^{n-1}$ defined by

$$(x, y) \mapsto \frac{x - y}{\|x - y\|}.$$

The map Ψ is semialgebraic and continuous. Moreover, by construction

$$\Sigma = \{v \in S^{n-1} \mid \dim(\Psi^{-1}(v)) \geq 1\}.$$

Using Theorem 3.32, we can write:

$$S^{n-1} = \bigsqcup_{j=1}^{\ell} S_j,$$

with each S_j semialgebraic and such that there exists F_j semialgebraic with $\Psi^{-1}(S_j) \simeq S_j \times F_j$. For every $j = 1, \dots, \ell$

$$\dim(S_j) + \dim(F_j) \leq \dim\left((Z \times Z) \setminus \Delta\right) = 2k.$$

In particular, if $\dim(F_j) \geq 1$, then $\dim(S_j) \leq 2k - 1$. By Section 4.3 we have

$$\Sigma = \bigsqcup_{\dim(F_j) \geq 1} S_j,$$

and it follows that $\dim(\Sigma) \leq 2k - 1$. □

Lemma 4.19. *Let $\Gamma \subset \mathbb{R}^n$ be a real algebraic curve, $n \geq 3$ and $z \in \Gamma$. There exist a vector $v \in S^{n-1}$, a semialgebraic neighborhood U of z in Γ, a semialgebraic neighborhood W of $\pi_v(z)$ in $\pi_v(\Gamma)$, and a finite set $\Gamma_0 \subset \Gamma$ such that $\pi_v|_U : U \to W$ is a homeomorphism and $\pi_v|_{\Gamma \setminus \Gamma_0}$ is injective.*

Proof. If $n \geq 3$, by Lemma 4.18, we have $\dim(\Sigma) < 2 \leq n - 1$ and therefore there exists $v \in S^{n-1} \setminus \Sigma$. Then there exists $\Gamma_0 := \{z_1, \dots, z_\ell\} \subset \Gamma$ such that $\pi_v|_{\Gamma \setminus \Gamma_0}$ is injective. Moreover, by taking $X := D(z, \epsilon) \cap \Gamma$, for $\epsilon > 0$ small enough, we may also assume that there is no other point in X other than z mapping to $\pi_v(z)$. Since X is compact and $\pi_v|_X$ is injective, X is mapped homeomorphically onto its image. We set $W := B(\pi_v(z), \delta)$, for $\delta > 0$ small enough such that $W \subset \pi_v(X)$, and $U := \pi_v^{-1}(W)$. This concludes the proof. □

We will need also the following fact from real algebraic geometry. The proof is not difficult, but it requires some more advanced notions, so we omit it.

Proposition 4.20. *Let $Z \subset \mathbb{R}^n$ be an irreducible real algebraic set and $f : Z \to \mathbb{R}^m$ be a regular map (e.g. a projection onto a linear subspace). Assume that there exists a proper algebraic set $Z_0 \subsetneq Z$ such that $f|_{Z \setminus Z_0}$ is injective. Then*

$$\dim\left(\mathrm{clos}^{\mathscr{Z}}(f(Z)) \setminus f(Z)\right) < \dim(Z) = \dim\left(\mathrm{clos}^{\mathscr{Z}}(f(Z))\right).$$

Proof. See [3, Lemma 11.4.3]. □

Lemma 4.21. *Let $\Gamma \subset \mathbb{R}^n$ be a real algebraic curve, $n \geq 3$, and $z \in \Gamma$. There exists a linear projection $\pi : \mathbb{R}^n \to \mathbb{R}^2$, a real algebraic curve $\widetilde{\Gamma} \subset \mathbb{R}^2$, a neighborhood U of z in Γ and a neighborhood W of $\pi(z)$ in $\widetilde{\Gamma}$ such that $\pi|_U : U \to W$ is a homeomorphism.*

Proof. This follows by iteratively applying Lemma 4.19 together with Proposition 4.20, setting $\widetilde{\Gamma} := \mathrm{clos}^{\mathscr{Z}}(\pi_v(\Gamma))$ and possibly shrinking at each step the obtained neighborhoods. □

Proof of Theorem 4.17. For every $\tau \in K$ of dimension $k-1$, we denote by $\nu(\tau)$ the number of k–simplices having τ as a face. Then, for every σ of dimension k,

$$\partial\sigma = \sum_{\dim(\tau)=k-1} \nu(\tau)\tau.$$

We will prove that $\nu(\tau)$ is even and, since we are working with coefficients in the field $\mathbb{Z}_2$, the first part of the statement will follow. Notice that for every *smooth* point $z \in Z$, the element $[Z]$ restricts to a generator of $H_k(Z, Z \setminus \{z\}) \simeq \mathbb{Z}_2$, which proves that $[Z]$ is independent of the triangulation.

For every $(k-1)$–dimensional simplex τ pick a point $z \in \varphi(\mathrm{int}(\tau))$ and stratify Z into Nash submanifolds such that all the images of the simplices of the triangulation are union of strata. Let $A \simeq \mathbb{R}^{n-k+1}$ be an affine space through z transversal to all the strata of this stratification. Such affine space exists by the Parametric Transversality Theorem, see footnote 1. Then A misses all the strata of Z of dimension smaller than $k-1$ and, it if intersects a k–dimensional stratum, the intersection is transversal. This implies that $\Gamma := A \cap Z$ is a an algebraic set of dimension at most one in $\mathbb{R}^{n-k+1}$. Let $\psi : |L| \to \Gamma$ be a semialgebraic triangulation of this set in such a way that z is a vertex. There is a one–to–one correspondence between the k–simplices of K having τ as a face and the 1–simplices of L having z as a vertex. If Γ has dimension zero, then there is no 1–simplex in the triangulation and $\nu(\tau) = 0$. Otherwise, Γ has dimension one, and we are therefore reduced to prove the statement in the case of a curve.

In the case of a curve $Z \subset \mathbb{R}^{n-k+1}$, for a point $z \in Z$ the quantity $\nu(z)$ only depends on the local behavior of the curve near z, up to semialgebraic homeomorphisms. Therefore, applying Lemma 4.21, we can assume that $Z \subset \mathbb{R}^2$ is a plane curve.

If $z \in Z$ is a smooth point, then $\nu(z) = 2$. Otherwise, let $z = (a, b)$ be a singular point of Z. Let $p \in \mathbb{R}[x, y]$ be a polynomial defining Z. Up to a linear change of coordinates, we can assume that the p is monic with respect to y (i.e.

$p(x,y) = y^d + p_1(x)y^{d-1} + \cdots + p_d(x))$, that $z = (0,0)$ and that this is the only singular point on the line $\{x = 0\}$. Notice that, this last condition implies that for $\epsilon > 0$ small enough there are no other singular points of Z in the strip $(-\epsilon, \epsilon) \times \mathbb{R}$ (since the number of singular points is finite).

Let $\alpha : \mathbb{R} \to \mathbb{R}$ be the discriminant of p with respect to the y–variable, i.e.

$$\alpha(x) := \mathrm{disc}_y(p(x,y)).$$

The function α is itself a nonzero polynomial with $\alpha(0) = 0$. Since α is not identically zero, it follows that for (a possibly smaller) $\epsilon > 0$ all the polynomials $p(x,\cdot) \in \mathbb{R}[y]$ with $|x| < \epsilon$ and $x \neq 0$ have nondegenerate zeroes. Using Lemma 3.13, possibly shrinking $\epsilon > 0$ even further, we get open sets $A_1 = (-\epsilon, 0)$ and $A_2 = (0, \epsilon)$ and semialgebraic continuous functions $\xi_{1,1} < \cdots < \xi_{1,r_1} : A_1 \to \mathbb{R}$ and $\xi_{2,1} < \cdots < \gamma_{2,r_2} : A_2 \to \mathbb{R}$ such that for $i = 1,2$ and for all $t \in A_i$ the set $\{\xi_{i,1}(x), \dots, \xi_{i,r_i}(x)\}$ consists of all the roots of $p(x,\cdot) \in \mathbb{R}[y]$. Since p is a real polynomial, and since the zeroes of $p(x,\cdot)$ are nondegnerate, $r_1 \equiv d \bmod 2$ and $r_2 \equiv d \bmod 2$, which implies $r_1 + r_2$ is even.

Since p is monic with respect to y, for every $j = 1, \dots, r_1$, the function $\xi_{1,j}$ has a finite limit as $x \to 0$. If this limit is different from zero, then $(0, \lim_{x\to 0} \xi_{1,j}(x))$ is a smooth point of Z and there exists a unique $\xi_{2,i(j)}$ such that $\lim_{x\to 0} \xi_{2,i(j)}(x) = \lim_{x\to 0} \xi_{1,j}(x) \neq 0$. Therefore

$$\begin{aligned} r + s &= \#\{\xi_{i,j} \mid \lim_{x\to 0} \xi_{i,j}(x) = 0\} + 2\#\{\xi_{1,j} \mid \lim_{x\to 0} \xi_{1,j}(x) \neq 0\} \\ &= \nu(z) + 2\#\{\xi_{1,j} \mid \lim_{x\to 0} \xi_{1,j}(x) \neq 0\}. \end{aligned}$$

Together with the fact that $r + s$ is even, this implies that $\nu(z)$ is also even. □

As a corollary of Theorem 4.17 we see that a real algebraic hypersurface Z in $\mathbb{R}^n$ *must* separate the space into several connected components. (This is not true for a real algebraic hypersurface in $\mathbb{RP}^n$: take for instance $Z = \mathbb{RP}^{n-1}$, then $\mathbb{RP}^n \setminus Z$ is connected.)

Theorem 4.22. *Let $Z \subset \mathbb{R}^n$ be an algebraic set of dimension $n-1$ (possibly singular, possibly non compact). Then $\mathbb{R}^n \setminus Z$ has at least two connected components.*

Proof. Let us first prove the result in the case Z is compact. Including $\mathbb{R}^n \hookrightarrow S^n$, we can think of Z as a subset of S^n. The number of connected components of $\mathbb{R}^n \setminus Z$ is the same as $S^n \setminus Z$. By Alexander duality[4], working with $\mathbb{Z}_2$ coefficients,

[4] Alexander duality states the following. Let $Z \subset S^n$ be a compact, nonempty, locally contractible, proper subspace of the sphere, then, for all $j \in \mathbb{N}$, there is an isomorphism

for every $j \in \mathbb{N}$ we have
$$\widetilde{b}_j(S^n \setminus Z) = \widetilde{b}_{n-j-1}(Z).$$
In particular:
$$b_{n-1}(Z) = \widetilde{b}_{n-1}(Z) = \widetilde{b}_0(S^n \setminus Z) = b_0(S^n \setminus Z) - 1.$$
Theorem 4.17 implies now that $b_{n-1}(Z) \geq 1$, which in turn gives $b_0(S^n \setminus Z) \geq 2$. Since $b_0(S^n \setminus Z) = b_0(S^n \setminus Z; \mathbb{Z})$ (there are no torsion elements in the zero–th homology), this implies the claim.

If Z is not compact we argue as follows. First, up to a translation, we can assume that Z does not contain the origin. Let $p \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial such that $Z = Z(p)$ and write
$$p(x) = \sum_{|\alpha| \leq d} c_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}.$$
Then we can define the new polynomial $q \in \mathbb{R}[y_1, \ldots, y_n]$ by
$$q(y) := \sum_{|\alpha| \leq d} c_\alpha \|y\|^{2d+2-|\alpha|} y_1^{\alpha_1} \cdots y_n^{\alpha_n}.$$
Denoting by $\eta : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be the inversion $\eta(x) := x\|x\|^{-2}$, we see that the zero set $\widetilde{Z} := Z(q) \subset \mathbb{R}^n$ is compact and that $\eta(Z) = \widetilde{Z} \setminus \{0\}$. In particular $b_0(\mathbb{R}^n \setminus Z) = b_0(\mathbb{R}^n \setminus \widetilde{Z})$ and the result follows now from the previous part. □

The previous result has a complex counterpart.

Theorem 4.23. *Let $Z \subset \mathbb{C}^m$ be a complex algebraic set of complex dimension $m-1$. Then $\mathbb{C}^m \setminus Z$ is connected and its first fundamental group is nonzero.*

Proof. We argue as in the proof of Theorem 4.22. Assuming that $0 \notin Z$, we see that there exists a *real* algebraic set $\widetilde{Z} \subset \mathbb{C}^m \simeq \mathbb{R}^{2m}$ such that $(\mathbb{C}^m \setminus \{0\}, Z) \simeq (\mathbb{C}^m \setminus$

$\widetilde{H}_j(S^n \setminus Z; \mathbb{Z}) \simeq \widetilde{H}^{n-j-1}(Z; \mathbb{Z})$. This is a consequence of a more general duality theorem, [16, Theorem 3.44], which states that if a compact n–dimensional manifold M is orientable and if $Z \subset M$ is a compact, locally contractible, proper subspace, then for every $j \in \mathbb{N}$ there is an isomorphism $H_j(M, M \setminus Z; \mathbb{Z}) \simeq H^{n-j}(Z; \mathbb{Z})$. This last result is valid for every coefficient ring R, as soon as M is R–orientable. Since every compact n–dimensional manifold M is $\mathbb{Z}_2$–orientable, the same proof as [16, Theorem 3.44] gives:
$$H_j(M, M \setminus Z; \mathbb{Z}_2) \simeq H^{n-j}(Z; \mathbb{Z}_2) \quad \forall j \in \mathbb{N}.$$
In the case of the sphere, combined with the long exact sequence of the pair $(S^n, S^n \setminus Z)$, this gives Section 4.3.

$\{0\}, \widetilde{Z} \setminus \{0\})$. Denote by $\widehat{\mathbb{C}^m} \simeq S^{2m}$ the one point compactification of $\mathbb{C}^m$. Adding back the origin to the pair $(\mathbb{C}^m \setminus \{0\}, Z)$, since $0 \notin Z$, we get a homeomorphism $\mathbb{C}^m \setminus Z \simeq (\widehat{\mathbb{C}^m} \setminus \{0\}) \setminus (\widetilde{Z} \setminus \{0\})$. This in turn gives

$$\mathbb{C}^m \setminus Z \simeq \widehat{\mathbb{C}^m} \setminus \widetilde{Z} \simeq S^{2m} \setminus \widetilde{Z}.$$

Applying now Alexander duality, we see that

$$\widetilde{b}_0(\mathbb{C}^m \setminus Z) = \widetilde{b}_{2m-1}(\widetilde{Z}) \quad \text{and} \quad \widetilde{b}_1(\mathbb{C}^m \setminus Z) = \widetilde{b}_{2m-2}(\widetilde{Z}).$$

(These are identities of Betti numbers with coefficient ring $\mathbb{Z}_2$.)

Since $\widetilde{Z}$ has real dimension $2m-2$, its homology in dimension $2m-1$ vanishes and, consequently, from the left hand side of Section 4.3 we deduce:

$$b_0(\mathbb{C}^m \setminus Z) = \widetilde{b}_0(\mathbb{C}^m \setminus Z) + 1 = 1.$$

The first part of the statement follows now again from the fact that there is no torsion in the zero–th homology and $b_0(\mathbb{C}^m \setminus Z)$ equals the number of connected components of $\mathbb{C}^m \setminus Z$.

For the second part of the statement, we use the fact that, for a connected topological space X, the group $H_1(X;\mathbb{Z})$ is the abelianization of the first fundamental group. In particular, if $H_1(X;\mathbb{Z}) \neq 0$, then also $\pi_1(X) \neq 0$. Assume by contradiction that $H_1(\mathbb{C}^m \setminus Z;\mathbb{Z}) = 0$. Then, by the Universal Coefficients theorem,

$$H_1(\mathbb{C}^m \setminus Z;\mathbb{Z}_2) \simeq (H_1(\mathbb{C}^m \setminus Z;\mathbb{Z}) \otimes \mathbb{Z}_2) \oplus \mathrm{Tor}(H_0(\mathbb{C}^m \setminus Z;\mathbb{Z}), \mathbb{Z}_2).$$

Since $H_0(\mathbb{C}^m \setminus Z;\mathbb{Z})$ has not torsion, $\mathrm{Tor}(H_0(\mathbb{C}^m \setminus Z;\mathbb{Z}), \mathbb{Z}_2) = 0$. Moreover, assuming that $H_1(\mathbb{C}^m \setminus Z;\mathbb{Z}) = 0$, then $H_1(\mathbb{C}^m \setminus Z;\mathbb{Z}) \otimes \mathbb{Z}_2 = 0$. Now Section 4.3 would imply that $H_1(\mathbb{C}^m \setminus Z;\mathbb{Z}_2) = 0$, which is in contradiction with the right hand side of (4.3). This concludes the proof. □

Taken together, Theorem 4.22 and Theorem 4.23 tell that real affine algebraic hypersurface *must* separate the space whereas complex affine algebraic hypersurfaces *cannot* separate (but their complements *must* have monodromy).

5 The Kac-Rice formula

In this chapter we address the basic problem of counting the number of solutions of a random equation – many interesting questions from geometry to topology can be reduced to a problem where we have to count points.

To be more specific, suppose we are given a *random map*

$$f : \mathbb{R}^m \to \mathbb{R}^k$$

with $m \geq k$. This means we have a random vector $f(x) \in \mathbb{R}^k$ for every $x \in \mathbb{R}^m$. In the literature, f is also called a *random field*. In the case $m = k$ we will be interested in computing $\mathbb{E}\,\#\,(\{f = 0\} \cap U)$, where $U \subseteq \mathbb{R}^k$ is a measurable subset. More generally, we compute the $(m-k)$-dimensional volume $\mathbb{E}\,\mathrm{vol}\,(\{f = 0\} \cap U)$. For an example of a random map we can take $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sum_{i=0}^{d} \xi_i x^i$, where $\{\xi_i\}_{i=0,\ldots,d}$ is a family of independent standard gaussians. This defines a random polynomial called a *Kac polynomial* (see also Definition 5.7 below). Another random map is given by taking the dehomogenization of a Kostlan polynomial $\sum_{i=0}^{d} \xi_i \binom{d}{i}^{\frac{1}{2}} x^i$ from Section 1.4. We can also consider the case where f is given by a complex random polynomial, using the identification of real vector spaces $\mathbb{R}^2 \cong \mathbb{C}$.

As before, $C^\infty(\mathbb{R}^m, \mathbb{R}^k)$ denotes the space of smooth function $\mathbb{R}^m \to \mathbb{R}^k$. Following the discussion in Section 1.3, we define a random map to be an element of some finite-dimensional Gaussian space of smooth functions. The motivates the next definition.

Definition 5.1 (Random Gaussian maps)**.** Let $\mathcal{F} = \{f_0, f_1, \ldots, f_\ell\} \subset C^\infty(\mathbb{R}^m, \mathbb{R}^k)$ be finite with $\ell, m \geq k$. The random Gaussian map induced by $\mathcal{F}$ is

$$f(x) = f_0(x) + \xi_1 f_1(x) + \cdots + \xi_\ell f_\ell(x),$$

where $\{\xi_i\}_{i=1,\ldots,\ell}$ is a family of i.i.d. $N(0,1)$ random variables.

Let $V := \mathrm{span}(\{f_1, \ldots, f_\ell\})$ be the vector space spanned by the last ℓ elements in $\mathcal{F}$. If $f_1, \ldots, f_\ell$ are linearly independent, we get an isomorphism

$$\varphi_{\mathcal{F}} : \mathbb{R}^\ell \to f_0 + V,\ y \mapsto f_0(x) + \sum_{i=1}^{\ell} y_i f_i(x).$$

Following Section 1.3, we get a Gaussian distribution on V in the following way:

$$\mathbb{P}(f \in U) = \frac{1}{\sqrt{2\pi}^\ell} \int_{\varphi_{\mathcal{F}}^{-1}(U)} e^{-\frac{\|y\|^2}{2}} \,\mathrm{d}y, \text{ where } U \subset V \text{ is measurable.}$$

For example, in the case of Kac polynomials we have $\mathcal{F} = \{0, 1, x, \ldots, x^d\}$. For defining a complex Kac polynomials we would choose $\mathcal{F} = \{0, a_0, \ldots, a_d, b_0, \ldots, b_d\}$ where $a_k(x, y) = \frac{1}{2}\left(\begin{smallmatrix} \Re(x+iy)^k \\ 0 \end{smallmatrix}\right)$ and $b_k(x, y) = \frac{1}{2}\left(\begin{smallmatrix} 0 \\ \Im(x+iy)^k \end{smallmatrix}\right)$.

5.1 The Kac-Rice formula in Euclidean Space

We will now establish the framework, in which we can compute the expected volume of zeros of random maps.

For this, we assume that $f : \mathbb{R}^m \to \mathbb{R}^k$ is a random map induced by a set of smooth functions $\mathcal{F} = \{f_0, f_1, \ldots, f_\ell\}$, in the sense of Definition 5.1. Here we assume that $f(x)$ has a nondegenerate distribution, i.e. that for almost all $x \in \mathbb{R}^m$ the covariance matrix is positive definite:

$$\mathbb{E}\,(f(x) - f_0(x))(f(x) - f_0(x))^T \succ 0$$

Furthermore, we let $Jf(x) \in \mathbb{R}^{k\times m}$ be the jacobian matrix of f at x. The normal Jacobian is $\mathrm{NJ}(f, x) = \sqrt{\det(Jf(x)Jf(x)^T)}$. The Kac-Rice density for $\mathcal{F}$ is given in terms of the expected value of the normal Jacobian.

Definition 5.2 (The Kac-Rice density)**.** The Kac-Rice density of $\mathcal{F}$ at x is

$$\rho(x) = \mathbb{E}\left[\ \mathrm{NJ}(f, x)\ \middle|\ f(x) = 0\ \right] \cdot \phi_{f(x)}(0),$$

where $\phi_{f(x)}(0)$ is the density of the random vector $f(x)$ evaluated at zero.

For every $x \in \mathbb{R}^m$ the vector-matrix pair $(f(x), Jf(x)) \in \mathbb{R}^m \times \mathbb{R}^{k\times m}$ is Gaussian. In the homogeneous case where $f_0 = 0$, the Gaussian regression formula from [2, Proposition 1.2] gives way for computing the conditional expectation in Definition 5.2.

Proposition 5.3. *Let $(\xi, \eta) \in \mathbb{R}^r \times \mathbb{R}^s$ be a centered Gaussian vector such that the covariance matrix of ξ is nondegenerate: $A = \mathbb{E}\,\xi\xi^T \succ 0$. Let us also define the matrices: $B = \mathbb{E}\,\eta\eta^T$ and $C = \mathbb{E}\,\eta\xi^T$. Then for every measurable function $h : \mathbb{R}^s \to \mathbb{R}$ we have:*

$$\mathbb{E}\left\{h(\eta) \,\middle|\, \xi = 0\right\} = \mathbb{E}\,h(\zeta),$$

where $\zeta \in \mathbb{R}^s$ is a Gaussian vector with mean zero and covariance $B - CA^{-1}C^T$.

In the case when the pair $(f(x), Jf(x))$ itself has a density we can give an alternative formulation for the Kac-Rice density. Let us denote by $p : \mathbb{R}^k \times \mathbb{R}^{k\times m} \times \mathbb{R}^m \to \mathbb{R}$ the joint density of the pair $(f(x), Jf(x))$. This means that p is the function defined by the requirement that for every measurable subset $A \subseteq \mathbb{R}^k \times \mathbb{R}^{k\times m}$ we have for a fixed $x \in \mathbb{R}^m$

$$\mathbb{P}\left((f(x), Jf(x)) \in A\right) = \int_A p(v, J, x)\,\mathrm{d}v\mathrm{d}J.$$

The Kac-Rice density for $\mathcal{F}$ in this case is given by

$$\rho(x) = \int_{\mathbb{R}^{k\times m}} \sqrt{\det(JJ^T)}\, p(0, J, x)\,\mathrm{d}J.$$

Now comes the Kac-Rice formula. We prove a special case of the more general formula in [2, Theorem 6.2].

Theorem 5.4 (Kac-Rice formula)**.** *Let $\ell, m \geq k$ and let $f : \mathbb{R}^m \to \mathbb{R}^k$ be the random map induced by $\mathcal{F} \subset C^\infty(\mathbb{R}^m, \mathbb{R}^k)$, $\#\mathcal{F} = \ell + 1$. Assume that for almost all $x \in \mathbb{R}^m$ we have*

(1) $f(x)$ has a nondegenerate distribution;

(2) the probability of $\det(Jf(x)Jf(x)^T) = 0$ conditioned on the event $f(x) = 0$ is equal to zero; i.e., $\mathbb{P}\{\det(Jf(x)Jf(x)^T) = 0 \mid f(x) = 0\} = 0$.

Then, almost surely the zero set of f is an $(m-k)$-dimensional smooth manifold. The volume of the zeros of f in a measurable set $U \subseteq \mathbb{R}^m$ is given by the formula:

$$\mathbb{E}\ \mathrm{vol}_{m-k}(\{x \in U \mid f(x) = 0\}) = \int_U \rho(x)\,\mathrm{d}x.$$

In particular, if $m = k$, we have $\mathbb{E}\ \#\{x \in U \mid f(x) = 0\} = \int_U \rho(x)\,\mathrm{d}x$.

The theorem shows the role of the Kac-Rice density $\rho(x)$: it is the density of the zeros of f. We thus call $\rho(x)$ also *root density*. This theorem is actually a special case of a more general result, which we state next.

Theorem 5.5. *Let $h : \mathbb{R}^m \to \mathbb{R}$ be a measurable function. Under the assumptions from Theorem 5.4 the zero set $Z(f) = \{x \in \mathbb{R}^m \mid f(x) = 0\}$ is almost surely an $(m-k)$-dimensional smooth submanifold of $\mathbb{R}^m$ and we have*

$$\mathbb{E} \int_{Z(f)\cap U} h(x)\,\mathrm{d}x = \int_U h(x)\rho(x)\,\mathrm{d}x$$

for any measurable set $U \subset \mathbb{R}^m$.

Before we prove Theorem 5.5, let us recall the next result on multivariate Gaussians; for a proof see, e.g., [32, Theorem 1.2.6].

Lemma 5.6. *Let $X = (X_1, \ldots, X_n)$ be Gaussian with mean μ and covariance matrix Σ, and let $A \in \mathbb{R}^{m\times n}$. Then, $AX \sim N(A\mu, A\Sigma A^T)$.*

Now, we prove Theorem 5.4.

Proof of Theorem 5.5. Let $Jf(x)$ be the Jacobian matrix of f at x. We use the following shorthand notations:

$$J := Jf(x) \quad \text{ and } \quad M = \begin{bmatrix} f_1(x) & \cdots & f_\ell(x)\end{bmatrix} \in \mathbb{R}^{k\times \ell}.$$

By Proposition 2.8, the zero set $Z(f) = f^{-1}(0)$ is a smooth manifold of dimension $m-k$, if $\det(JJ^T) \neq 0$ on $f^{-1}(0)$. This holds almost surely due to our assumption that $\mathbb{P}\{\det(JJ^T) = 0 \mid f(x) = 0\} = 0$.

Let us write $f = \varphi_{\mathcal{F}}(y) = f_0 + My$ as in Chapter 5. Then, if $\xi \in \mathbb{R}^\ell$ is a vector of i.i.d. Gaussians we have

$$\mathbb{E}(f(x) - f_0(x))(f(x) - f_0(x))^T = \mathbb{E}\, M\xi\xi^T M^T = MM^T.$$

Let $W \subset \mathbb{R}^\ell \times \mathbb{R}^m$ be the subset of points (y, x) such that $\det(MM^T) \neq 0$. By assumption, W is an open dense subset. We define the incidence correspondence

$$Z := \{(y, x) \in W \mid f(x) = 0, \text{ where } f = \varphi_{\mathcal{F}}(y)\}.$$

If Z is empty, then $f(x) = 0$ has no solution in $\mathbb{R}^m$. In this case, the Kac-Rice formula is trivially true. In the following, we assume that $Z \neq \emptyset$. We show that Z is a manifold. To see this, we define $g : W \to \mathbb{R}^k, (y, x) \mapsto f(x)$. This map has Jacobian matrix $Jg(y, x) = \left[\begin{smallmatrix} M & J\end{smallmatrix}\right] \in \mathbb{R}^{k\times(\ell+m)}$. Since M has rank k, this shows that 0 is a regular value of g. Proposition 2.8 implies that $Z = g^{-1}(0)$ is a smooth manifold of dimension $\ell + m - k$ with tangent space

$$T_{(y,x)}Z = \left\{(\dot{y}, \dot{x}) \in \mathbb{R}^\ell \times \mathbb{R}^m \mid M\dot{y} = -J\dot{x}\right\}.$$

Let us define the coordinate projections $\pi_1 : Z \to \mathbb{R}^\ell$ and $\pi_2 : Z \to \mathbb{R}^k$. Since M has rank k, π_2 is a submersion. By [25, Proposition 4.28], submersions are open maps, hence $\pi_2(Z)$ is open in $\mathbb{R}^k$. If moreover JJ^T is invertible at (y,x), the zero set $Z(f) = (\pi_2 \circ \pi_1^{-1})(y)$ is a manifold as we have discussed above. In this case, we can isometrically identify $Z(f) \cong \pi_1^{-1}(y)$. We apply the coarea formula (Theorem 2.26) first to π_1 to get

$$\int_{Z(f)\cap U} h(x)\,\mathrm{d}x = \int_{\mathbb{R}^\ell} \Big(\int_{\pi_1^{-1}(y)} \chi_U(x)\, h(x)\,\mathrm{d}x \Big) \frac{1}{\sqrt{2\pi}^{\ell}} e^{-\frac{\|y\|^2}{2}}\,\mathrm{d}y,$$

where χ_U is the indicator function of U. Next, we apply the coarea formula to π_2 to Section 5.1 and get

$$\int_{Z(f)\cap U} h(x)\,\mathrm{d}x = \int_{\mathbb{R}^m} \chi_U(x)\, h(x)\, \tilde{\rho}(x)\,\mathrm{d}x,$$

where

$$\tilde{\rho}(x) = \int_{\pi_2^{-1}(x)} \frac{\mathrm{NJ}(\pi_1,(y,x))}{\mathrm{NJ}(\pi_2,(y,x))} \frac{1}{\sqrt{2\pi}^{\ell}} e^{-\frac{\|y\|^2}{2}}\,\mathrm{d}(y,x).$$

We have to show that $\rho(x) = \tilde{\rho}(x)$. First, we compute the ratio of normal jacobians. For this, we fix $(y,x) \in Z$ such that JJ^T is invertible (recall that this holds for almost everywhere). We define $K := T_{(y,x)}Z \cap ((\ker M)^\perp \times (\ker J)^\perp)$ and denote the coordinate projections by $p_1 : K \to \mathbb{R}^\ell$ and $p_2 : K \to \mathbb{R}^m$. We have an orthogonal decomposition

$$T_{(y,x)}Z = \ker M \oplus \ker J \oplus K.$$

If $(u,v,w) \in \ker M \oplus \ker J \oplus K$ is a decomposition of a point in $T_{(y,x)}Z$ relative to this decomposition, its images under the derivatives of π_1 and π_2 are $\mathrm{D}_{(y,x)}\pi_1(u,v,w) = (u, p_1(w))$ and $\mathrm{D}_{(y,x)}\pi_2(u,v,w) = (v, p_2(w))$. Furthermore, for $w = (w_1,w_2) \in K$ we have that $w_1 = -(M|_{(\ker M)^\perp})^{-1}(J|_{(\ker J)^\perp})(w_2)$. This implies that both p_1 and p_2 are invertible and $\det(p_1 \circ p_2^{-1}) = \det((M|_{(\ker M)^\perp})^{-1}(J|_{(\ker J)^\perp}))$. We get

$$\begin{aligned}\frac{\mathrm{NJ}(\pi_1,(y,x))}{\mathrm{NJ}(\pi_2,(y,x))} = \frac{|\det(p_1)|}{|\det(p_2)|} = |\det(p_1 \circ p_2^{-1})| &= \det((M|_{(\ker M)^\perp})^{-1}(J|_{(\ker J)^\perp}))\\ &= \frac{\sqrt{\det(JJ^T)}}{\sqrt{\det(MM^T)}}.\end{aligned}$$

For a fixed $x \in \mathbb{R}^m$ we have $\pi_2^{-1}(x) = (y_0 + \ker M) \times \{x\}$, where $y_0 \in (\ker M)^\perp$ satisfies $f_0(x) + My_0 = 0$. This shows

$$\tilde{\rho}(x) = \frac{1}{\sqrt{\det(MM^T)}} \int_{\ker M} \sqrt{\det(JJ^T)} \cdot \frac{1}{\sqrt{(2\pi)^\ell}} \cdot e^{-\frac{\|y-y_0\|^2}{2}}\,\mathrm{d}y.$$

Recall that $\rho(x) = \mathbb{E}[\, \mathrm{NJ}(f,x) \mid f(x) = 0\,] \cdot \phi_{f(x)}(0)$. Using $\sqrt{\det(JJ^T)} = \mathrm{NJ}(f,x)$ we have

$$\rho(x) = \frac{\int_{\ker M} \sqrt{\det(JJ^T)} \cdot \frac{1}{\sqrt{(2\pi)^\ell}} \cdot e^{-\frac{\|y-y_0\|^2}{2}} \,\mathrm{d}y}{\int_{\ker M} \frac{1}{\sqrt{(2\pi)^\ell}} \cdot e^{-\frac{\|y-y_0\|^2}{2}} \,\mathrm{d}y} \cdot \phi_{f(x)}(0).$$

Because y_0 is chosen to be orthogonal to $\ker M$ we have $\|y - y_0\|^2 = \|y\|^2 + \|y_0\|^2$. The dimension of $\ker M$ is $\ell - k$. Let $U \in O(\ell)$ be an orthogonal matrix so that $MU^T = [\, M' \; 0\,]$, where $M' \in \mathbb{R}^{k\times k}$ is invertible. We have $y \in \ker M$ if and only if $Uy = (0, z) \in \{0\} \times \mathbb{R}^{\ell - k}$. The denominator in Section 5.1 therefore is

$$\int_{\ker M} \frac{1}{\sqrt{(2\pi)^\ell}} \cdot e^{-\frac{\|y-y_0\|^2}{2}} \,\mathrm{d}y = \frac{1}{\sqrt{(2\pi)^k}} \cdot e^{-\frac{\|y_0\|^2}{2}}.$$

Finally, as $f(x) = f_0(x) + My = M(y - y_0)$ we have by Lemma 5.6 that

$$\phi_{f(x)}(0) = \frac{1}{\sqrt{(2\pi)^k \det(MM^T)}} \cdot e^{-\frac{\|y_0\|}{2}}.$$

If we plug both these identities into Section 5.1 we see that $\rho(x) = \tilde{\rho}(x)$. □

5.2 Root density of Kac polynomials

In the first chapter, specifically in Section 1.4, we discussed that Kostlan polynomials are a reasonable class of random maps. Another class are Kac polynomials as defined at the beginning of this chapter. Let us recall their definition.

Definition 5.7 (Kac polynomials)**.** Let $f = \sum_{i=0}^d \xi_i\, x^i$ be a univariate polnomial of degree d. We call f a Kac polynomial, if the ξ_i are i.i.d. $N(0,1)$ random variables.

However, Kac polynomials are not entirely reasonable in the sense of Section 1.3. We observe that Kac polynomials are only invariant under the symmetry $f \mapsto -f$, and in fact, as one can see from the plot of the root density of these polynomials in Figure 5.1, the zeroes are more likely to be near the points $\pm 1 \in \mathbb{R}$ (hence there are privileged points for this model of randomness). In particular, for $U = [a, b]$ with either $a, b \gg 0$ or $a, b \ll 0$, the expected number of zeros $\mathbb{E}\ \#(\{f = 0\} \cap U)$ is almost zero. Yet, Kac polynomials are still interesting, at least because they have historically been the first examples to be studied. For this reason and to illustrate the use of the Kac-Rice formula we study them in this section.

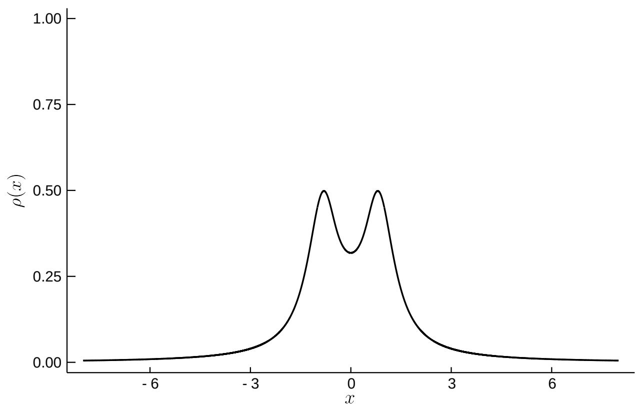

Figure 5.1: The root density $\rho(x)$ of Kac polynomials of degree $d = 4$. The two peaks are at $x = -1$ and $x = 1$. This means that a root of a Kac polynomial is most likely close to -1 or 1.

In this subsection we want to illustrate Theorem 5.4 in the case of Kac polynomials. The following theorem appeared in [19].

Theorem 5.8. *The root density of a Kac polynomial is*

$$\rho(x) = \frac{1}{\pi}\,\frac{\sqrt{1 - h(x)^2}}{1 - x^2}, \qquad \textit{where} \qquad h(x) = \frac{(d+1)x^d(1-x^2)}{1 - x^{2(d+1)}}.$$

It should be mentioned that one has not been able to derive a closed formula for $\int_{\mathbb{R}} \rho(x)\,\mathrm{d}x$. In Kac's original paper [19] he considered instead the *asymptotic* behavior of this integral. This is a recurring theme in this book: often one can not compute closed expressions of expected properties, but one can estimate the asymptotics when one (or several) parameters go to infinity. In the case of Kac polynomials we have the following.

Theorem 5.9. *The number of zeros of Kac polynomials satisfies for $d \to \infty$:*

$$\mathbb{E}\ \#Z(f) \sim \frac{2}{\pi}\log(d).$$

We prove Theorem 5.8 and Theorem 5.9.

Proof of Theorem 5.8 and Theorem 5.9. The following proof appeared in [19]. For a Kac polynomial f we have $f(x) = \sum_{i=0}^d \xi_i x^i$, and $Jf(x) = \sum_{i=0}^d i\xi_i x^{i-1}$. Consider the *Kac Rational normal curve*

$$\vartheta(x) = (1, x, x^2, \ldots, x^d)^T.$$

Let us also write $\xi = (\xi_0, \ldots, \xi_d)^T$. Then, $f(x) = \xi^T\vartheta(x)$ and $Jf(x) = \xi^T\vartheta'(x)$, where $\vartheta'(x)$ denotes the derivative of $\vartheta(x)$. By Lemma 5.6, $(f(x), Jf(x))$ is a vector of Gaussian random variables with covariance matrix that is invertible almost everywhere:

$$\Sigma = \begin{bmatrix} \vartheta^T\vartheta & \vartheta^T\vartheta' \\ (\vartheta')^T\vartheta & (\vartheta')^T\vartheta' \end{bmatrix}.$$

The joint density of $(u, v) = (f(x), Jf(x)) \in \mathbb{R}^2$ therefore is

$$p(u, v, x) = \frac{1}{2\pi\sqrt{\det \Sigma}}\, e^{-\frac{1}{2}\,(u,v)\Sigma^{-1}(u,v)^T}.$$

This implies $p(0, v, x) = (2\pi\sqrt{\det \Sigma})^{-1}\, e^{-\frac{v^2}{2}\,\alpha}$, where $\alpha = (\det \Sigma)^{-1}\, \vartheta^T\vartheta$. We can use the fomula for the Kac-Rice density in Section 5.1 to get

$$\rho(x) = \int_{\mathbb{R}} |v|\, p(0, v, x)\, \mathrm{d}v = \frac{1}{2\pi\sqrt{\det \Sigma}} \int_{\mathbb{R}} |v|\; e^{-\frac{v^2}{2}\,\alpha}\, \mathrm{d}v = \frac{1}{\pi}\, \frac{\sqrt{\det \Sigma}}{\vartheta^T\vartheta}.$$

Let us simplify this expression a little further. We have

$$\frac{\sqrt{\det \Sigma}}{\vartheta^T\vartheta} = \sqrt{\frac{\vartheta^T\vartheta \cdot (\vartheta')^T\vartheta' - (\vartheta^T\vartheta')^2}{\vartheta^T\vartheta}} = \sqrt{\frac{\partial^2}{\partial x \partial y}\Big|_{x=y} \log \vartheta^T\vartheta(y)}.$$

We have $\vartheta(x)^T\vartheta(y) = \sum_{i=0}^{d}(xy)^i = (1 - (xy)^{d+1})/(1 - xy)$ and so

$$\frac{\partial^2}{\partial x \partial y}\Big|_{x=y} \log \vartheta(x)^T\vartheta(y) = -\frac{(d+1)^2 x^{2d}}{(1 - x^{2(d+1)})^2} + \frac{1}{(1 - x^2)^2}.$$

Thus, we have $\rho(x) = \frac{1}{\pi}\, \frac{\sqrt{1-h(x)^2}}{1-x^2}$, where $h(x) = \frac{(d+1)x^d(1-x^2)}{1-x^{2(d+1)}}$. This finishes the proof of Theorem 5.8.

For proving Theorem 5.8 we observe that $\rho(x)$ is symmetric around 0, so that

$$\mathbb{E}\ \#\{f = 0\} = 2\int_0^\infty \rho(x)\ \mathrm{d}x = 2\left(\int_0^1 \rho(x)\ \mathrm{d}x + \int_1^\infty \rho(x)\ \mathrm{d}x\right).$$

In the right interval, we make a change of variables $y = \frac{1}{x}$, which reveals that the two integrals in the sum are equal. Thus, we have $\mathbb{E}\ \#\{f = 0\} = 4\int_0^1 \rho(y)\ \mathrm{d}y$. Using the formula for the geometric sum, and the fact that $0 \le y \le 1$, we have

$$h(y) = \frac{(d+1)y^d(1-y^2)}{(1-y)(1+y+\cdots+y^{2d+1})} \ge \frac{y^d(1+y)}{2},$$

which implies

$$1 - h(y)^2 \leq (1 - \tfrac{1}{2}y^d(1+y))(1 + \tfrac{1}{2}y^d(1+y)) \leq 2 - y^d(1+y).$$

By the mean value theorem, for a fixed y there exists some $y < \theta < 1$ such that

$$\frac{(2 - 1^d(1+1)) - (2 - y^d(1+y))}{1-y} = \frac{y^d(1+y) - 2}{1-y} = -d\theta^{d-1}(1+\theta) - \theta^d.$$

Since $\theta < 1$, this implies $1 - h(y)^2 \leq 2 - y^d(1+y) < (1-y)(2d+1)$. Moreover, for $0 \leq y \leq 1$ we have $1 - y^2 \geq 1 - y$, so that

$$\frac{\sqrt{1-h(y)^2}}{1-y^2} \leq \frac{\sqrt{1-h(y)^2}}{1-y} < \sqrt{\frac{2d+1}{1-y}}.$$

On the other hand, Section 5.2 implies that $h(y) \geq 0$, and so

$$\frac{\sqrt{1-h(y)^2}}{1-y^2} \leq \frac{1}{1-y^2}.$$

A combination of Section 5.2 and Section 5.2 yields

$$\begin{aligned}
\mathbb{E}\,\#\{f=0\} &\leq \frac{4}{\pi}\left(\int_0^{1-\frac{1}{d}} \frac{1}{1-y^2}\,\mathrm{d}y + \int_{1-\frac{1}{d}}^1 \sqrt{\frac{2d+1}{1-y}}\,\mathrm{d}y\right) \\
&= \frac{4}{\pi}\left(\frac{\log(2-\frac{1}{d})}{2} + \frac{\log(d)}{2} + 2\sqrt{\frac{2d+1}{d}}\right) \\
&\leq \frac{4\log(2)}{\pi} + \frac{2\log(d)}{\pi} + \frac{8\sqrt{3}}{\pi} \leq \frac{2\log(d)}{\pi} + 6,
\end{aligned}$$

where in the penultimate step we have used $d \geq 1$.

To finish the proof of Theorem 5.9 we also need a lower bound for $\mathbb{E}\,\#\{f=0\}$. For this we consider a number $0 \leq \delta < 1$. Then, since $h(y) \leq (d+1)y^d$, we have $h(y) \leq (d+1)(1-d^{\delta-1})^d$ for $0 \leq y \leq 1 - d^{\delta-1}$. This implies

$$\begin{aligned}
\mathbb{E}\,\#\{f=0\} &= \frac{4}{\pi}\int_0^1 \rho(y)\,\mathrm{d}y \\
&\geq \frac{4}{\pi}\int_0^{1-d^{\delta-1}} \frac{\sqrt{1-(d+1)^2(1-d^{\delta-1})^{2d}}}{1-y^2}\,\mathrm{d}y \\
&= \frac{2}{\pi}\sqrt{1-(d+1)^2(1-d^{\delta-1})^{2d}}\,\big(\log(2-d^{\delta-1}) + (1-\delta)\log(d)\big)
\end{aligned}$$

Let us write $d' = d^{1-\delta}$. Since $1-\delta > 0$ we have

$$(d+1)(1-d^{\delta-1})^d = (d+1)\Big(\Big(1-\frac{1}{d'}\Big)^{d'}\Big)^{d^\delta} \overset{d\to\infty}{\sim} (d+1)e^{-d^\delta} \overset{d\to\infty}{\to} 0,$$

which in combination with Section 5.2 and Section 5.2 shows that

$$\mathbb{E}\ \#\{f=0\} \sim \frac{2}{\pi}\log(d) \text{ for } d\to\infty.$$

This finishes the proof of Theorem 5.9. □

5.3 The Kac-Rice formula for random maps on manifolds

In this section we extend the Kac-Rice formula to random maps defined on a Riemannian manifold (M, g). Let $m = \dim M$. Similar to Definition 5.1 we say that a finite collection of smooth functions $\mathcal{F} = \{f_0, f_1, \ldots, f_\ell\} \subset C^\infty(M, \mathbb{R}^k)$ induces the random Gaussian map

$$f(x) = f_0(x) + \xi_1 f_1(x) + \cdots + \xi_\ell f_\ell(x),$$

where $\{\xi_i\}_{i=1,\ldots,\ell}$ is a family of i.i.d. $N(0,1)$ random variables. Assuming that the density of $f(x)$ is nondegenerate, the analogue of Definition 5.2 for manifolds is the following.

Definition 5.10 (The Kac-Rice density for random maps on manifolds)**.** The Kac-Rice density of $\mathcal{F}$ at x is

$$\rho(x) = \mathbb{E}\left[\mathrm{NJ}(f,x)\,\middle|\, f(x)=0\right]\cdot \phi_{f(x)}(0),$$

where $\phi_{f(x)}(0)$ is the density of the random vector $f(x)$ evaluated at zero, and the determinant is defined to be the determinant of $\mathrm{D}_x f$ in coordinates with respect to orthonormal bases in T_xM and $T_{f(x)}\mathbb{R}^k$; see Subsection 2.1.1.

Here is the Kac-Rice formula for manifolds

Theorem 5.11 (Kac-Rice formula for random maps on manifolds)**.** *Let $\ell, m \geq k$. Let (M,g) be a Riemannian manifold of dimension m and $f: M\to\mathbb{R}^k$ be the random map induced by $\mathcal{F}\subset C^\infty(M,\mathbb{R}^k)$, $\#\mathcal{F} = \ell+1$. Assume that for almost all $x\in M$ we have*

(1) $f(x)$ is nondegenerate (i.e., $\mathbb{E}(f(x)-f_0(x))^T(f(x)-f_0(x)) \succ 0$);

(2) $\mathbb{P}\{\mathrm{NJ}(f,x) = 0 \mid f(x)=0\} = 0$.

Then, almost surely the zero set of f is an $(m-k)$-dimensional smooth manifold. The volume of the zeros of f in a measurable set $U \subseteq M$ is given by the formula:

$$\mathbb{E}\ \mathrm{vol}_{m-k}(\{x \in U \mid f(x) = 0\}) = \int_U \rho(x)\, \mathrm{dvol}_{\mathrm{g}}(x).$$

In particular, if $m = k$, we have $\mathbb{E}\ \#\{x \in U \mid f(x) = 0\} = \int_U \rho(x)\, \mathrm{dvol}_{\mathrm{g}}(x)$.

As in the case of random maps between Euclidean cases the Kac-Rice formula is a special case of the following more general formula.

Theorem 5.12. *Let $h : \mathbb{M} \to \mathbb{R}$ be a measurable function. Under the assumptions from Theorem 5.11 the zero set $Z(f) = \{x \in \mathbb{R}^m \mid f(x) = 0\}$ is almost surely an $(m-k)$-dimensional smooth submanifold of M and we have*

$$\mathbb{E} \int_{Z(f)\cap U} h(x)\, \mathrm{d}x = \int_U h(x)\rho(x)\, \mathrm{d}x$$

for any measurable subset $U \subset M$.

Proof. If $\mathrm{NJ}(f, x) \neq 0$ on $f^{-1}(0)$, by Proposition 2.8 the zero set $Z(f) = f^{-1}(0)$ is a smooth manifold of dimension $m - k$. This holds almost surely due to our assumption $\mathbb{P}\{\det(JJ^T) = 0 \mid f(x) = 0\} = 0$. Let χ_U be the indicator function of U. Let $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ be an atlas for M and $\{p_\alpha : U_\alpha \to \mathbb{R}\}_{\alpha\in A}$ be a partition of unity subordinated to $\{U_\alpha\}_{\alpha\in A}$. For a fixed $x \in \varphi_\alpha(U_\alpha)$ let us abbreviate $\mathrm{g}_\alpha := \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))$. Let $v_1, \ldots, v_m$ denote an orthonormal basis of $T_{\varphi_\alpha(x)}M$ so that $v_1, \ldots, v_{m-k}$ span $T_x Z(f)$. Similarly, let $w_1, \ldots, w_m$ denote an orthonormal basis of $\mathbb{R}^m$ such that $T_x Z(f \circ \varphi_\alpha^{-1})$ is spanned by $w_1, \ldots, w_{m-k}$. Let us denote by $C = (c_{i,j}) \in \mathbb{R}^{m\times m}$ the matrix such that $(\varphi_\alpha^{-1})_*(w_i) = \sum_{i=1}^m c_{i,j} v_j$. By construction, $c_{i,j} = 0$ when $i \leq m-k$ and $j \geq k$, so

$$C = \begin{bmatrix} C_1 & 0 \\ * & C_2 \end{bmatrix}$$

with matrices $C_1 \in \mathbb{R}^{(m-k)\times(m-k)}$ and $C_2 \in \mathbb{R}^{k\times k}$. We have $g_\alpha = CC^T$, so that

$$\det \mathrm{g}_\alpha = \det(C_1 C_1^T)\ \det(C_2 C_2^T).$$

By Proposition 2.8 we have $T_x Z(f) = \ker \mathrm{D}_x f$, so that $(\rho \circ \varphi_\alpha^{-1})(x)\sqrt{\det(C_2 C_2^T)}$ is the Kac-Rice density at x of the random map $f \circ \varphi_\alpha^{-1}$. Setting $a(x) := \chi_U(x)\, h(x)$

and using Theorem 5.5 we get

$$\begin{aligned}\int_M a(x)\rho(x)\,\mathrm{dvol}_{\mathrm{g}}(x) &= \sum_{\alpha\in A}\int_{\varphi_\alpha(U_\alpha)} \left((a\cdot\rho\cdot p_\alpha)\circ\varphi_\alpha^{-1}\right)(x)\sqrt{\det \mathrm{g}_\alpha(\varphi_\alpha^{-1}(x))}\,\mathrm{d}x\\ &= \mathbb{E}\sum_{\alpha\in A}\int_{Z(f\circ\varphi_\alpha^{-1})} \left((a\cdot p_\alpha)\circ\varphi_\alpha^{-1}\right)(x)\sqrt{\det(C_1C_1^T)}\,\mathrm{d}x.\end{aligned}$$

The matrix $C_1C_1^T$ describes the metric g restricted to $Z(f)$, so that

$$\int_M a(x)\rho(x)\,\mathrm{dvol}_{\mathrm{g}}(x) = \mathbb{E}\sum_{\alpha\in A}\int_{Z(f)\cap U_\alpha} \chi_U(x)\,p(x)\,h(x)\,\mathrm{d}x = \mathbb{E}\int_{Z(f)} \chi_U(x)\,h(x)\,\mathrm{d}x.$$

This finishes the proof. □

5.4 Root density of systems of Kostlan polynomials

As an illustrative example, we compute the Kac-Rice density (Definition 5.10) of a system of $k \le n$ independent Kostlan polynomials of degrees $(d_1,\dots,d_k)$ defined on the sphere S^n. Recall from Section 1.4 that a Kostlan polynomial is a random homogeneous polynomial defined by $\sum_{|\alpha|=d}\xi_\alpha\binom{d}{\alpha}^{1/2}x^\alpha$, where $x=(x_0,\dots,x_n)$ and $\{\xi_\alpha\}$ is a family of i.i.d. $N(0,1)$ random variables. We denote a system of k independent Kostlan polynomials by

$$f(x) = \begin{bmatrix}\sum_{|\alpha|=d_1}\xi_\alpha^{(1)}\binom{d_1}{\alpha}^{1/2}x^\alpha\\ \vdots\\ \sum_{|\alpha|=d_k}\xi_\alpha^{(k)}\binom{d_k}{\alpha}^{1/2}x^\alpha\end{bmatrix}.$$

We also apply the Kac-Rice formula to $f(x)$ to compute the expected volume of zeros on S^n. This also computes the volume of zeros in $\mathbb{R}\mathrm{P}^n$, since

$$\mathbb{E}\,\mathrm{vol}\{x\in\mathbb{R}\mathrm{P}^n \mid f(x)=0\} = \frac{1}{2}\,\mathbb{E}\,\mathrm{vol}\{x\in S^n \mid f(x)=0\};$$

see the discussion in Subsection 2.2.5. Edelman and Kostlan showed in their seminal paper [8] that in the case of univariate Kostlan polynomials ($k=n=1$) of degree d the expected number of zeros of f in $\mathbb{R}\mathrm{P}^1$ is $\sqrt{d}$. This is a special case of the general situation in the next theorem.

Theorem 5.13. *Let $f(x)$ be a system of $k\le n$ independent Kostlan polynomials of degrees $(d_1,\dots,d_k)$ in $n+1$ variables. Then, almost surely the zero set of f is an $(n-k)$-dimensional submanifold of S^n, the Kac-Rice density of the real zeros* *of f is constant and equal to*

$$\rho(x) = \frac{\mathrm{vol}(\mathbb{R}\mathrm{P}^{n-k})}{\mathrm{vol}(\mathbb{R}\mathrm{P}^{n})}\sqrt{d_1\cdots d_k}.$$

The expected volume of the zero set in $\mathbb{R}\mathrm{P}^n$ is

$$\mathbb{E}\,\mathrm{vol}_{n-k}\{x\in\mathbb{R}\mathrm{P}^n \mid f(x)=0\} = \mathrm{vol}(\mathbb{R}\mathrm{P}^{n-k})\sqrt{d_1\cdots d_k}.$$

In particular, if $k=n$ we expect $\sqrt{d_1\cdots d_n}$ many zeros.

While in this section we will use the Kac-Rice formula for polynomials defined on the sphere to prove Theorem 5.13, in the next section we will prove Theorem 5.13 by treating polynomials as section of an appropriate line bundle over $\mathbb{R}\mathrm{P}^1$. Going from the first to the second proof we will make a conceptual jump when introducing the notion of a random section of a vector bundle.

Now, we prove Theorem 5.13.

Proof of Theorem 5.13. By Section 5.4, the volume of zeros of f in $\mathbb{R}\mathrm{P}^n$ is half the volume of zeros of f in S^n. We compute the latter. The system of Kostlan polynomials in Section 5.4 defines a random map on S^n, which satisfies the assumptions of Theorem 5.11 so that the zero set is almost surely $(n-k)$-dimensional an

$$\mathbb{E}\,\mathrm{vol}_{n-k}\{x\in S^n \mid f=0\} = \int_{S^n}\rho(x)\,\mathrm{d}x.$$

Let us compute the Kac-Rice density $\rho(x)$. By Lemma 1.11 we have $f\sim f\circ U$ for every $U\in O(n+1)$, and so $\rho(Ux)=\rho(x)$. Moreover, $O(n+1)$ acts transitively on S^n which shows that ρ is in fact constant on S^n. This implies

$$\int_{S^n}\rho(x)\,\mathrm{d}x = \rho(e_0)\,\mathrm{vol}(S^n),$$

where $e_0=(1,0,\ldots,0)\in S^n$. For computing the root density at e_0, we recall from that the tangent space of S^n at e_0 is $T_{e_0}S^n = e_0^\perp$ We choose $\{e_1,\ldots,e_n\}$ both as orthonormal basis for $e_0^\perp$ and for $\mathbb{R}^n$. The matrix of $\mathrm{D}_{e_0}f$ with respect to these bases is

$$J = \begin{bmatrix}\frac{\partial f}{\partial x_1}(e_0) & \cdots & \frac{\partial f}{\partial x_n}(e_0)\end{bmatrix} = \begin{bmatrix}\sqrt{d_1}\xi_{(d-1,1,0,\ldots,0)} & \cdots & \sqrt{d_1}\xi_{(d-1,0,0,\ldots,1)}\\ \vdots & \ddots & \vdots\\ \sqrt{d_k}\xi_{(d-1,1,0,\ldots,0)} & \cdots & \sqrt{d_k}\xi_{(d-1,0,0,\ldots,1)}\end{bmatrix}\in\mathbb{R}^{k\times n}.$$

Furthermore, $f(e_0) = (\xi_{(d_1,0,\ldots,0)}, \ldots, \xi_{(d_k,0,\ldots,0)})^T$ so that it's density at 0 is

$$\phi_{f(x)}(0) = \sqrt{2\pi}^{-k}.$$

We see that J is independent of $f(e_0)$. Using $\mathrm{NJ}(f,x) = \sqrt{\det(JJ^T)}$ we get

$$\mathbb{E}\left[\sqrt{\det(JJ^T)}\,|f(e_0) = 0\right] = \mathbb{E}\sqrt{\det(JJ^T)} = \sqrt{d_1\cdots d_k}\,\mathbb{E}\sqrt{\det(XX^T)},$$

where $X \in \mathbb{R}^{k\times n}$ is a matrix whose entries are i.i.d. $N(0,1)$ random variables.

All this implies $\rho(e_0) = (2\pi)^{-\frac{k}{2}}\sqrt{d_1\cdots d_k}\;\mathbb{E}\sqrt{\det(XX^T)}$, and so

$$\mathbb{E}\,\mathrm{vol}_{n-k}\{x \in S^n \mid f = 0\} = \frac{\mathrm{vol}(S^n)}{\sqrt{2\pi}^k}\sqrt{d_1\cdots d_k}\;\mathbb{E}\sqrt{\det(XX^T)}.$$

We consider the case $d_1 = \cdots = d_k = 1$, where the system of Kostlan polynomials consists of k linear equations with i.i.d. $N(0,1)$ coefficients. With probability one the k equations define k hyperplanes in $\mathbb{R}^{n+1}$ in general position so that their intersection is a linear space of dimension $n-k+1$. Its intersection with S^n is a sphere of dimension $(n-k)$. The expected volume of zeros on the sphere in this case is $\mathrm{vol}(S^{n-k})$. Plugging this into Section 5.4 we get

$$\mathrm{vol}(S^{n-k}) = \mathrm{vol}(S^n)\,(2\pi)^{-\frac{n}{2}}\;\mathbb{E}\sqrt{\det(XX^T)},$$

so that $\rho(e_0) = (\mathrm{vol}(S^{n-k})/\mathrm{vol}(S^n))\sqrt{d_1\cdots d_k} = (\mathrm{vol}(\mathbb{RP}^{n-k})/\mathrm{vol}(\mathbb{RP}^n))\sqrt{d_1\cdots d_k}$ and, consequently, $\mathbb{E}\,\mathrm{vol}\{x \in S^n \mid f = 0\} = 2\mathrm{vol}(\mathbb{RP}^{n-k})\sqrt{d_1\cdots d_k}$. □

An important corollary from the proof of Theorem 5.13 is the following result.

Corollary 5.14. *Let $X \in \mathbb{R}^{k\times n}$, $k \le n$, be a matrix whose entries are i.i.d. $N(0,1)$ random variables. Then,*

$$\mathbb{E}\sqrt{\det(XX^T)} = \sqrt{2\pi}^n\,\frac{\mathrm{vol}(\mathbb{RP}^{n-k})}{\mathrm{vol}(\mathbb{RP}^n)} = \sqrt{2^n\pi^{n-k}}\,\frac{\Gamma\left(\frac{n-k+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Proof. We have $\mathrm{vol}(S^{n-k}) = \mathrm{vol}(S^n)\,(2\pi)^{-\frac{n}{2}}\;\mathbb{E}\sqrt{\det(XX^T)}$ by Section 5.4. This yields $\mathbb{E}\sqrt{\det(XX^T)} = (2\pi)^{\frac{n}{2}}\,\mathrm{vol}(\mathbb{RP}^{n-k})/\mathrm{vol}(\mathbb{RP}^n)$. Using from Subsection 2.2.5 that the volume of projective space is $\mathrm{vol}(\mathbb{RP}^n) = \pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right)$ gives the second equality. □

In the important special case $k = n$ the corollary gives the expected absolute determinant of a matrix $X \in \mathbb{R}^{n\times n}$ filled with i.i.d. standard normal random variables: $\mathbb{E}\,|\det(X)| = \sqrt{2^n\pi^{-1}}\,\Gamma\left(\frac{n+1}{2}\right)^{-1}$.

We can use the same strategy to compute the volume of complex zeros in $\mathbb{C}\mathrm{P}^n$ of complex Kostlan polynomials. Let $k \leq n$ and

$$f(x) = \begin{bmatrix} \sum_{|\alpha|=d_1} \zeta_\alpha^{(1)} \binom{d}{\alpha}^{1/2} x^\alpha \\ \vdots \\ \sum_{|\alpha|=d_k} \zeta_\alpha^{(k)} \binom{d}{\alpha}^{1/2} x^\alpha \end{bmatrix},$$

where the $\zeta_\alpha^{(i)}$ are i.i.d. complex Gaussian random variables (a complex Gaussian random variable is of the form $\zeta = \xi_1 + \sqrt{-1}\xi_2$, where ξ_1 and ξ_2 are independent and both $N(0,1)$). Using the Kac-Rice formula we can show that the zero set of $f(x) \in \mathbb{C}\mathrm{P}^n$ is almost surely an $(n-k)$-dimensional submanifold of expected volume $\mathbb{E}\,\mathrm{vol}_{n-k}\{x \in \mathbb{C}\mathrm{P}^n \mid f(x) = 0\} = d_1 \cdots d_k$. The reason why we don't get the square root as in Theorem 5.13 is Lemma 2.28: if $J = \left[\, \frac{\partial f}{\partial x_1}(e_0) \cdots \frac{\partial f}{\partial x_n}(e_0) \,\right]$ is the matrix of *complex derivatives* of f at e_0, the expected value in Section 5.4 becomes $\mathbb{E}\det(JJ^T) = d_1 \cdots d_k\, \mathbb{E}\,|\det(XX^T)|$, where $X \in \mathbb{C}^{k\times n}$ is a complex Gaussian matrix. In fact, the volume of the zero set is generic, because it is locally constant on the complement of the discriminant consisting of polynomial systems f whose zero set has a singular point, which by Lemma 1.5 is connected. Therefore, we have

$$\mathrm{vol}_{n-k}\{x \in \mathbb{C}\mathrm{P}^n \mid f(x) = 0\} = d_1 \cdots d_k \quad \text{almost surely.}$$

In the case $n = k$ this number is $d_1 \cdots d_n$, which is the *generic* number of complex zeros of $f(x)$. Indeed, the number $d_1 \cdots d_n$ is implied by Bézout's theorem. But Bézout's theorem gives in fact more: it asserts that, if the number of zeros of $f(x)$ is finite, then it has $d_1 \cdots d_n$ zeros counted with multiplicity. The result in Section 5.4 is weaker in this regard: it only shows that the the number of zeros of a *generic* system is $d_1 \cdots d_n$. On the other hand, Section 5.4 is also stronger than the Bézout theorem, because it also measures the number of solutions in the case $k < n$.

5.5 Random sections of vector bundles

In the final section of this chapter we develop the most general version of the Kac-Rice formula. While Theorem 5.11 generalizes the Kac-Rice formula from random maps on Euclidean spaces to random maps on manifolds, here we take the next conceptual step: we measure the zero set of *random sections* of vector bundles. Let us first recall the definition and basic properties of vector bundles.

Definition 5.15 (Vector bundles)**.** A *vector bundle* of rank k is a triple (π, E, M)

where E and M are smooth manifolds and

$$\pi : E \to M$$

is a submersion such that there is an open cover $M = \bigcup_{\alpha\in A} V_\alpha$ and diffeomorphisms $\psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{R}^k$ with the following two properties:

(1) The following diagram is commutative, p_1 is the projection on the first factor:

$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[rr, "\psi_\alpha"] \arrow[dr, "\pi"'] & & V_\alpha \times \mathbb{R} \arrow[dl, "p_1"] \\ & V_\alpha & \end{tikzcd}$$

(2) Whenever $V_\alpha \cap V_\beta \neq \emptyset$, there is a continuous map $g_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathrm{GL}(\mathbb{R}^k)$ such that the map $\psi_\alpha \circ \psi_\beta^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{R}^k \to (V_\alpha \cap V_\beta) \times \mathbb{R}^k$ is given by $(x, v) \mapsto (x, g_{\alpha\beta}(x) \cdot v)$.

The family $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in A}$ is called a *vector bundle atlas*, and each ψ_α is called a *trivialization*.

Let (π, E, M) be a vector bundle. We call M the *base space* of the bundle. The *fiber* over $x \in M$ is the vector space

$$E_x := \pi^{-1}(x).$$

The family $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ is called the *cocycle* of the bundle and satisfies $g_{\alpha\alpha}(x) = \mathbf{1}$ and $g_{\alpha\beta}(x)g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$.

Definition 5.16 (Sections)**.** Let $\pi : E \to M$ be a vector bundle. A *section* of E is a smooth map $s : M \to E$ such that $\pi(s(x)) = x$ for every $x \in M$. The *zero* section is the section that associates to every point $x \in M$ the zero vector in E_x.

Similar to the definition of random maps we now introduce the notion of a *random section*: a finite set $\mathcal{F} = \mathrm{span}\,\{\sigma_0, \sigma_1, \ldots, \sigma_\ell\}$ of sections defines the random section

$$\sigma = \sigma_0 + \xi_1\sigma_1 + \cdots + \xi_\ell\sigma_\ell,$$

with $\{\xi_k\}_{k=1,\ldots,\ell}$ a family of i.i.d. standard real gaussian variables. In particular, if we have the trivial bundle $M \times \mathbb{R}^k$ this reduces to a random map on M. For the vector bundle analogue of the Kac-Rice density, we work with the next definition.

Definition 5.17 (Riemannian vector bundle)**.** A *Riemannian vector bundle* of rank k is a vector bundle (π, E, M) of rank k, where $(E, \tilde{\mathrm{g}})$ and (M, g) are Riemannian manifolds such that $\pi : E \to M$ is a Riemannian submersion.

If (π, E, M) is a Riemannian vector bundle of rank k, we have the orthogonal decomposition $T_zE \cong T_xM \oplus \mathbb{R}^k$, where $x = \pi(z)$. Let $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in A}$ be a vector bundle atlas with vector bundle trivializations $\psi_\alpha : \pi^{-1}(V_\alpha) \to V_\alpha \times V_\alpha \times \mathbb{R}^k$. For every α we define the local section

$$\sigma_\alpha := \psi_\alpha \circ \sigma|_{V_\alpha} : V_\alpha \to \mathbb{R}^k,$$

First, we define the local Kac-Rice density for random sections.

Definition 5.18. Let $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in A}$ be a vector bundle atlas for E. The Kac-Rice density at V_α of the collections of sections $\mathcal{F}$ at $x \in V_\alpha$ is

$$\rho_\alpha(x) = \mathbb{E}\left[\mathrm{NJ}(\sigma_\alpha, x)\,\middle|\sigma_\alpha(x) = 0\right] \cdot \phi_{\sigma_\alpha(x)}(0),$$

where $\phi_{\sigma_\alpha(x)}(x)$ is the density of the random vector $\sigma_\alpha(x)$ evaluated at zero.

We need the local definition together with the following proposition to get a global notion for the Kac-Rice density.

Proposition 5.19. *For every $\alpha, \beta \in A$ such that $V_\alpha \cap V_\beta \neq \emptyset$ we have for almost all $x \in V_\alpha \cap V_\beta$ that $\rho_\alpha(x) = \rho_\beta(x)$.*

Proof. For every α the map $p_2 \circ \sigma_\alpha$ satisfies the assumptions of Theorem 5.11. Moreover, $\sigma_\alpha = \mathrm{id}_{V_\alpha} \times (p_2 \circ \sigma_\alpha)$ so that $\mathrm{NJ}(\sigma_\alpha, x) = \mathrm{NJ}(p_2 \circ \sigma_\alpha, x)$. We therefore have $\mathbb{E}\,\mathrm{vol}\{x \in U \mid \sigma_\alpha(x) = 0\} = \int_U \rho_\alpha(x)\,\mathrm{dvol}_{\mathrm{g}}(x)$ for every measurable subset $U \subset V_\alpha$. For every pair α, β we $\sigma_\alpha(x) = 0$ if and only if $\sigma_\beta(x) = 0$. This shows that for $U \subset V_\alpha \cap V_\beta$ we have $\int_U \rho_\alpha(x)\,\mathrm{dvol}_{\mathrm{g}}(x) = \int_U \rho_\beta(x)\,\mathrm{dvol}_{\mathrm{g}}(x)$. Since this equality holds for all measurable subsets we see that $\rho_\alpha(x) = \rho_\beta(x)$ for almost all $x \in V_\alpha \cap V_\beta$. $\square$

The proposition shows that the following definition does not depend on the partition of unity up to sets of measure zero.

Definition 5.20 (The Kac-Rice density for random sections of vector bundles)**.** Let $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in A}$ be a vector bundle atlas for E and $\{q_\alpha : V_\alpha \to \mathbb{R}\}_{\alpha\in A}$ be a partition of unity subordinated to the open cover $M = \bigcup_{\alpha\in A} V_\alpha$. Then, the Kac-Rice density of the collections of sections $\mathcal{F}$ at $x \in M$ is

$$\rho(x) = \sum_{\alpha\in A} q_\alpha(x)\rho_\alpha(x).$$

Now comes the Kac-Rice formula for vector bundles.

Theorem 5.21 (Kac-Rice formulas for random sections of vector bundles)**.** *Let $\ell, m \geq k$. Let (π, E, M) be a Riemannian vector bundle of rank-k. Let $m := \dim M$ and $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in A}$ be a vector bundle atlas for the bundle. Let $\sigma : M \to E$ be a random section of E defined by the family of sections $\mathcal{F}$, $\#\mathcal{F} = \ell + 1$. Assume that for almost all $x \in M$:*

(1) σ is nondegenerate (i.e., $\mathbb{E}(\sigma(x) - \sigma_0(x))^T(\sigma(x) - \sigma_0(x)) \succ 0$);

(2) $\mathbb{P}\{\mathrm{NJ}(\sigma_\alpha, x) = 0 \mid \sigma_\alpha(x) = 0\} = 0$ for every $\alpha \in A$.

Then, almost surely the zero set of σ is a $(m - k)$-dimensional submanifold of M and we have for every measurable set $U \subset M$:

$$\mathbb{E}\,\mathrm{vol}\{x \in U \mid \sigma(x) = 0\} = \int_U \rho(x)\,\mathrm{dvol_g}(x).$$

Proof. Let $\{q_\alpha : V_\alpha \to \mathbb{R}\}_{\alpha\in A}$ be a partition of unity subordinated to the open cover $M = \bigcup_{\alpha\in A} V_\alpha$. For every α we have $\mathrm{NJ}(\sigma_\alpha, x) \neq 0$ on $f^{-1}(0)$ almost surely, so that Proposition 2.8 implies that the zero set $Z(\sigma_\alpha) = f^{-1}(0)$ is a smooth manifold of dimension $m - k$. We have $Z(\sigma_\alpha) = Z(\sigma) \cap V_\alpha$, so that we can use the atlas of each $Z(\sigma_\alpha)$ to define an atlas for $Z(\sigma)$. This shows that the zero set of σ is also a smooth manifold of dimension $m - k$. Furthermore, σ_α satisfies the assumptions of Theorem 5.12 so that

$$\int_{U\cap V_\alpha} q_\alpha(x)\rho_\alpha(x)\,\mathrm{dvol_g}(x) = \mathbb{E}\int_{Z(\sigma_\alpha)\cap U\cap V_\alpha} q_\alpha(x)\,\mathrm{d}x,$$

Then,

$$\begin{aligned}\int_U \rho(x)\,\mathrm{dvol_g}(x) &= \sum_{\alpha\in A}\int_{U\cap V_\alpha} q_\alpha(x)\rho_\alpha(x)\,\mathrm{dvol_g}(x)\\ &= \sum_{\alpha\in A}\mathbb{E}\int_{\int_{Z(\sigma_\alpha)\cap U\cap V_\alpha}} q_\alpha(x)\,\mathrm{d}x\\ &= \sum_{\alpha\in A} q_\alpha(x)\;\mathbb{E}\,\mathrm{vol}\{x \in U\cap V_\alpha \mid \sigma(x) = 0\}\\ &= \mathbb{E}\,\mathrm{vol}\{x \in U \mid \sigma(x) = 0\}.\end{aligned}$$

This finishes the proof. □

To illustrate the theorem let us discuss how to prove Theorem 5.13 using vector bundles. To simplify, let us restrict to the case when we have one Kostlan polynomial f in $n + 1$ many variables.

For this we introduce the real line bundle

$$\pi : \mathcal{O}_{\mathbb{RP}^n}(d) \to \mathbb{RP}^n$$

defined as follows: we cover $\mathbb{RP}^n$ with the open sets $V_i = \{[x_0 : \ldots : x_n] \mid x_i \neq 0\}$ and define the cocycle $h_{ij} : VU_i \cap V_j \to \mathrm{GL}(\mathbb{R}),\ [x_0 : \cdots : x_n] \mapsto (\frac{x_i}{x_j})^d$. As a topological space $\mathcal{O}_{\mathbb{RP}^1}(d)$ is defined to be the quotient topological space:

$$\mathcal{O}_{\mathbb{RP}^n}(d) = \Big(\bigsqcup_{i=0}^{n} V_i \times \mathbb{R}\Big)/\sim,$$

where we identify $(x, v) \in V_i \times \mathbb{R}$ with $(x, w) \in V_j \times \mathbb{R}$ if and only if $h_{i,j}(x)v = w$. The projection map in Section 5.5 is $\pi(x, v) = x$. The Kostlan polynomial f defines a local random section $\sigma_i : V_i \to V_i \times \mathbb{R}$ by $\sigma_i(x) = f(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i})$. If ψ_i, ψ_j denote the trivializations on U_i and U_j, we have

$$\psi_1\psi_0^{-1}(\sigma_i(x)) = (x, (\tfrac{x_i}{x_j})^d f(\tfrac{x_0}{x_i}, \ldots, \tfrac{x_n}{x_i})) = (x, f(\tfrac{x_0}{x_j}, \ldots, \tfrac{x_n}{x_j})) = \sigma_j(x).$$

which implies that the two local sections agree on their overlap. We can therefore the global section σ_f which takes the value $(\psi_i^{-1} \circ \sigma_i)(x)$ on V_i. By definition, the local sections as defined in Section 5.5 are $\sigma_i = \psi_i \circ \sigma$. We can therefore apply Theorem 5.21 to conclude that $\mathbb{E}\,\mathrm{vol}\{x \in \mathbb{RP}^n \mid \sigma_f(x) = 0\} = \int_U \rho(x)\,\mathrm{d}x$. Lemma 1.11 implies that the Kac-Rice density is constant on $\mathbb{RP}^n$ so that we have $\mathbb{E}\,\mathrm{vol}\{x \in \mathbb{RP}^n \mid \sigma_f(x) = 0\} = \mathrm{vol}(\mathbb{RP}^n)\,\rho(e_0)$. The value of the Kac-Rice density is computed as in the proof of Theorem 5.13.

5.6 There are $6\sqrt{2} - 3$ lines on a real cubic surface

We use the Kac-Rice formula for vector bundles to show [36, Theorem 5], which computes the expected number of lines contained in the cubic surface $Z(f) \subset \mathbb{RP}^3$, where f is a Kostlan polynomial of degree 3 in 4 variables. We have discussed in Example 1.7 that, generically, a cubic surface in $\mathbb{RP}^3$ can contain either 27, 15, 7 or 3 real lines. We show the following for Kostlan polynomials.

Theorem 5.22. *The expected number of lines on a cubic surface in $\mathbb{RP}^n$ defined by a Kostlan polynomial of degree 3 in 4 variables is $6\sqrt{2} - 3$.*

Let us put the problem of counting real lines on hypersurfaces in the above framework of measuring zero sets of sections on vector bundles. Let $\mathbb{G}(1, 3)$ be the Grassmannian of lines in $\mathbb{RP}^3$. Every line $L \in \mathbb{G}(1, 3)$ corresponds to a two-dimensional linear space $\hat{L}$. This way we can identify $\mathbb{G}(1, 3) \cong G(2, 4)$, where the latter is the Grassmanian of two-dimensional linear spaces in $\mathbb{R}^4$ (see Section 6.4). We we put the manifold structure quotient manifold structure on $\mathbb{G}(1,3) \cong G(2,4)$ (see Section 6.4) so the Grassmanian admits the structure of a homogenous $O(4)$-space with a left- and right-invariant Riemannian metric.

Let us consider the vector bundle

$$\pi : E \to \mathbb{G}(1,3),$$

where at $L \in \mathbb{G}(1,3)$ the fiber E_L is the set of homogeneous cubic polynomials in four variables restricted to L. The vector bundle structure is defined as follows. Consider the open cover $\mathbb{G}(1,3) \cong G(2,4) = \bigcup_{1\leq i<j\leq 4} V_{i,j}$ defined by charts

$$\varphi_{i,j}^{-1} : \mathbb{R}^{2\times 2} \to V_{i,j},\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \operatorname{rowspan}\Big(\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} P_{i,j} \Big),$$

where $P_{i,j}$ is the matrix for the permutation interchanging 1 with i and 2 with j. We have the trivializations $\psi_{i,j} : \pi^{-1}(V_{i,j}) \to V_{i,j} \times \mathbb{R}[s,t]_{(3)}$ defined for $\varphi_{i,j}(L) = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ by evaluating a polynomial h at these variables:

$$h(x_0, x_1, x_2, x_3)|_L \mapsto h\Big(\begin{bmatrix} s & t \end{bmatrix} \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} P_{i,j} \Big).$$

The cocycles of E are $g_{(i,j),(\ell,k)}(L,f) = (L, f \circ A_{(i,j),(k,\ell)}^T)$, where $A_{(i,j),(k,\ell)} \in \mathbb{R}^{2\times 2}$ is the matrix such that

$$A_{(i,j)(k,\ell)} \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} P_{i,j} = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix} P_{k,\ell}.$$

We need the following lemma.

Lemma 5.23. *For every $1 \leq i < j \leq 4$ the derivative of $\varphi_{i,j}^{-1}$ at $0 \in \mathbb{R}^{2\times 2}$ is an orthogonal linear map.*

Proof. It is enough to prove this for $i = 1, j = 2$, because the matrices $P_{i,j}$ are orthogonal matrices and the orthogonal group acts on $G(k,n)$ by isometries. Let $\varphi := \varphi_{1,2}$ and

$$L_0 := \varphi^{-1}(0) = \operatorname{span}\{(1,0,0,0)^T, (0,1,0,0)^T\}.$$

If $\varphi : O(4) \to G(2,4)$ is the quotient map, we have $\pi(\mathbf{1}_4) = L_0$, where $\mathbf{1}_4$ is the 4×4 identity matrix.

Let us consider the following four curves in $\mathbb{R}^{2\times 2}$.

$$\gamma_1(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & \sin(t) \end{bmatrix}, \; \gamma_2(t) = \begin{bmatrix} \sin(t) & 0 \\ 0 & -\sin(t) \end{bmatrix},$$
$$\gamma_3(t) = \begin{bmatrix} 0 & \sin(t) \\ \sin(t) & 0 \end{bmatrix}, \; \gamma_4(t) = \begin{bmatrix} 0 & \sin(t) \\ -\sin(t) & 0 \end{bmatrix}.$$

Their derivatives at $t = 0$ span $\mathbb{R}^{2\times 2}$ and they define the curves

$$\theta_i(t) := \cos(t)\mathbf{1}_4 + \begin{bmatrix} 0 & -\gamma_i(t) \\ \gamma_i(t) & 0 \end{bmatrix} \in O(4).$$

We have $\theta_i(0) = \mathbf{1}_4$ for all i and $(\varphi^{-1} \circ \gamma_i)(t) = (\pi \circ \theta_i)(t)$, so that

$$\mathrm{D}_0\varphi^{-1}(\gamma_i'(0)) = \mathrm{D}_{\mathbf{1}_4}\pi(\theta_i'(0)).$$

The derivatives $\theta_i'(0) \in T_{\mathbf{1}}O(n)$ satisfy

$$\theta_i'(0) = \begin{bmatrix} 0 & -\gamma_i'(0) \\ \gamma_i'(0) & 0 \end{bmatrix}.$$

Following Section 6.4, we see that $\mathrm{D}_{\mathbf{1}_4}\pi(\theta_i'(0)) = \mathrm{D}_0\varphi^{-1}(\gamma_i'(0)) = \gamma_i'(0)$, which shows that $\mathrm{D}_0\varphi^{-1}$ is the identity with respect to a choice of orthonormal basis of $\mathbb{R}^{2\times 2}$, hence an orthogonal linear map. $\square$

Let us now pick a random Kostlan polynomial

$$f(x) = \sum_{|\alpha|=3} \xi_\alpha \cdot \left(\frac{3!}{\alpha_0!\alpha_1!\alpha_2!\alpha_3!}\right)^{1/2} x_0^{\alpha_0}x_1^{\alpha_1}x_2^{\alpha_0}x_3^{\alpha_3}$$

where $\{\xi_\alpha\}_{|\alpha|=3}$ is a family of i.i.d. standard real gaussian variables. The zero set $Z(f) \subset \mathbb{R}\mathrm{P}^3$ is a random real cubic surface. The random polynomial f defines a random section $\sigma_f : \mathbb{G}(1,3) \to E$ by restriction $\sigma_f(L) = (L, f|_L)$ and we have

$$\{\text{real lines on } Z(f)\} = \{\text{zeros of } \sigma_f\}.$$

Proof of Theorem 5.22. We can use Theorem 5.21 for computing the expectation of the number of lines as $\mathbb{E}\,\#\{\text{real lines on } Z(f)\} = \int_{\mathbb{G}(1,3)} \rho(L)\,\mathrm{d}L$. Since $f \sim f \circ U$ for every $U \in O(4)$ by Lemma 1.11, the Kac-Rice density is constant and we have

$$\mathbb{E}\,\#\{\text{real lines on } Z(f)\} = \mathrm{vol}(\mathbb{G}(1,3))\,\rho(L_0),$$

where $L_0 = \mathrm{rowspan}(\left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right])$. Let us compute $\rho(L_0)$. Locally around L_0 we have the chart $\varphi := \varphi_{(1,2)}$ as defined in Section 5.6: $\varphi^{-1}([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}]) = \operatorname{rowspan}([\begin{smallmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{smallmatrix}])$. The trivialization $\psi := \psi_{(1,2)}$ is given by

$$\psi(L, h(x_0, x_1, x_2, x_3)|_L) := (L, h(s, t, sa + tc, sb + td)), \text{ where } \varphi(L) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

As basis for the vector space of cubics we take $\mathcal{B} = \{s^i t^{3-i} \mid 0 \le i \le 3\}$. We have $L_0 = \varphi^{-1}(0)$ and we have

$$\sigma_f(L) = \big(L, \sum_{|\alpha|=3} \xi_\alpha \cdot \tbinom{3}{\alpha}^{1/2} s^{\alpha_0} t^{\alpha_1} (sa + tc)^{\alpha_2} (sb + td)^{\alpha_3}\big).$$

Taking derivatives of the second component with respect to the variables a, b, c, d, evaluating at 0 and representing them as vectors with respect to the basis $\mathcal{B}$ gives

$$M := \sqrt{3} \begin{bmatrix} 0 & \xi_{(0,2,1,0)} & 0 & \xi_{(0,2,0,1)} \\ \xi_{(0,2,1,0)} & \sqrt{2}\,\xi_{(1,1,1,0)} & \xi_{(0,2,0,1)} & \sqrt{2}\,\xi_{(1,1,0,1)} \\ \sqrt{2}\,\xi_{(1,1,1,0)} & \xi_{(2,0,1,0)} & \sqrt{2}\,\xi_{(1,1,0,1)} & \xi_{(2,0,0,1)} \\ \xi_{(2,0,1,0)} & 0 & \xi_{(2,0,0,1)} & 0 \end{bmatrix}.$$

This matrix represents $\mathrm{D}_0(\psi \circ \sigma_f \circ \varphi^{-1})$ with respect to the chosen bases. Moreover,

$$\mathrm{NJ}(\psi \circ \sigma_f, L_0) = |\det M|,$$

because $\mathrm{D}_{L_0}\varphi$ is orthogonal as we have shown in Lemma 5.23. Moreover, we observe the second component of $\sigma_f(L_0)$ is $\sum_{i=0}^{3} \xi_{(i,3-i,0,0)} \binom{3}{\alpha}^{1/2} s^i t^{3-i}$, which is independent of the random matrix above. The density of this random vector with respect to the basis $\mathcal{B}$ is the density of $(\xi_{(0,3,0,0)}, \sqrt{3}\,\xi_{(1,2,0,0)}, \sqrt{3}\,\xi_{(2,1,0,0)}, \xi_{(3,0,0,0)})$. The density of this random vector evaluated at zero is $(2\pi)^{-1}(6\pi)^{-1}$. Therefore,

$$\rho(L_0) = \mathbb{E}[\,|\det(M)| \;\big|\; \sigma_f(L_0) = 0\,] \cdot \frac{1}{12\pi^2} = \mathbb{E}\,|\det(M)| \cdot \frac{1}{12\pi^2}.$$

In [36, Section 3] it is shown that $\mathbb{E}\,|\det(M)| = 3^2(4\sqrt{2} - 2)$, so that

$$\rho(L_0) = \frac{3^2(4\sqrt{2} - 2)}{12\pi^2} = \frac{6\sqrt{2} - 3}{2\pi^2}.$$

Using from Section 6.4 that $\operatorname{vol}(G(2,4)) = 2\pi^2$ and Section 5.6 we get $6\sqrt{2} - 3$ as the expected number of lines. $\square$

6 Homogeneous spaces and integral geometry

A situation that we will often encounter in these lectures is when a manifold is a *homogeneous space*. In the first two sections of this chapter we recall the basic definitions and properties of Lie groups and homogeneous spaces. For a more detailed treatment of this subject we refer to [25, Section 7 & Section 21]. We start with the definition of Lie groups.

6.1 Lie groups

A *Lie group* G is a smooth manifold that is also a group, such that the multiplication $\mathrm{mul} : G \times G \to G, (g,h) \mapsto gh$ and the inversion $i : G \to G, g \mapsto g^{-1}$ are smooth maps. Let $g \in G$. We define the left- and right-translation of g to be the maps

$$L_g : G \to G, h \mapsto gh, \quad \text{and} \quad R_g : G \to G, h \mapsto hg.$$

As $L_g = \mathrm{mul} \circ (h \mapsto (g,h))$ is a composition of smooth maps it is smooth. Furthermore, L_g has the smooth inverse $L_{g^{-1}}$, so that L_g is in fact a diffeomorphism. Similarly, R_g is an isomorphism. For $g, h \in G$ and $v \in T_hG$ we write $\mathrm{D}_h L_g(v) =: gv$. Similarly, $\mathrm{D}_g R_g(v) =: vg$.

Example 6.1. Examples of Lie groups are $\mathbb{R}^n$ with the smooth Euclidean structure and vector addition as group operation, the general linear group $\mathrm{GL}(n,\mathbb{R})$ with the smooth structure inherited from $\mathbb{R}^{n\times n}$ with matrix multiplication as group operation, and the orthogonal group $O(n)$ as a submanifold of $\mathrm{GL}(n,\mathbb{R})$. Similarly, $\mathrm{GL}(n,\mathbb{C})$ and the unitary group $U(n)$ are Lie groups.

Let G be a Lie group of dimension m, and let $e \in G$ be the identity element. For every $g \in G$, left-translation is a diffeomorphism, which implies that the derivative $\mathrm{D}_e L_g$ is invertible. Consequently, we have $T_gG = \mathrm{D}_e L_g(T_eG) = g\, T_eG$. This implies

$$T_gG = (gh^{-1})\, T_hG, \quad \text{for } g, h \in G.$$

6.2 The Haar measure

We discuss how to define a left-invariant Riemannian structure on a Lie group G. Let $m := \dim(G)$. We choose any basis $\{u_1, \ldots, u_n\}$ of T_eG. We define an inner product on T_eG by declaring this basis to be orthonormal:

$$\mathrm{g}(e)\Big(\sum_{i=1}^m \lambda_i u_i, \sum_{i=1}^m \mu_i u_i\Big) := \sum_{i=1}^m \lambda_i \mu_i.$$

This defines a Riemannian metric g on G in the following way: for $g \in G$ we set

$$\mathrm{g}(g)(v, w) := \mathrm{g}(e)(g^{-1}v, g^{-1}w), \text{ for } v, w \in T_gG,$$

which is well-defined by Section 6.1. By construction, the Riemannian metric is *left-invariant* and with respect to this metric any left-translation $L_g : G \to G$ is an isometry. The left-invariant metric defines a left-invariant measure on G by Definition 2.24, called the *Haar measure* on G.

The next result states that, despite making a choice of basis above, the resulting measure is unique up to scaling.

Theorem 6.2. *There is a unique left-invariant measure on G up to scaling.*

Proof. The existence of such a measure is given by the construction above. For uniqueness let μ and ν be two left-invariant measures on G. We have for any measurable set $U \subset G$ that $\mu(U) + \nu(U) = 0$ implies $\mu(U) = 0$. Hence, μ is absolutely continuous with respect to $\mu + \nu$. The Radon-Nikodym theorem (see, e.g., [11, Section 23]) implies that there exists a measurable function $\phi : G \to \mathbb{R}$ with $\mu = \phi\,(\mu + \nu)$. We show that ϕ is constant $(\mu + \nu)$–almost everywhere: for any measurable subset $W \subset G$ and every $g \in G$ we have

$$\mu(L_g(W)) = \int_{L_g(W)} \phi \,\mathrm{d}(\mu + \nu) = \int_W (\phi \circ L_g)\,\mathrm{d}(\mu + \nu),$$

by left-invariance of $\mu + \nu$. On the other hand, by left-invariance of μ, we have $\mu(L_g(W)) = \mu(W) = \int_W \phi\,\mathrm{d}(\mu + \nu)$, and so $\int_W \big((\phi \circ L_g) - \phi\big)\,\mathrm{d}(\mu + \nu) = 0$, which implies that $(\phi \circ L_g) - \phi = 0$ almost everywhere. $\square$

A consequence of this theorem is that there is a *unique* left-invariant probability measure on G (if it exists).

Corollary 6.3. *Let G be a Lie group. Then, there is a unique left-invariant probability measure on G (if it exists).*

Proof. Let G be a Lie group and $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be two left-invariant probability measures on G. By Theorem 6.2, they are multiples of each other, so there exists $c \neq 0$ with $\mathbb{P} = c \cdot \tilde{\mathbb{P}}$. This implies $1 = \mathbb{P}(G) = c \cdot \tilde{\mathbb{P}}(G) = c$, so $\mathbb{P} = \tilde{\mathbb{P}}$. □

In the following, we denote by vol the left-invariant measure on G that we have constructed above, after declaring some basis in $\mathfrak{g}$ to be orthonormal. Then, for every $g \in G$ the measure $\mu(A) := \mathrm{vol}(R_g(A)) = \mathrm{vol}(Ag)$ is also left-invariant. Since G is locally compact, there exists a measurable subset $A \subseteq G$ with the property that $0 < \mathrm{vol}(A) < \infty$. Hence, by Theorem 6.2 there exists a real number $m(g)$ such that $\mu = m(g)\,\mathrm{vol}$. This defines the so-called *modular function*.

Definition 6.4. The modular function of G is defined by

$$m(g) := \frac{\mathrm{vol}(Ag)}{\mathrm{vol}(A)}, \quad A \subseteq G,\ 0 < \mathrm{vol}(A) < \infty.$$

Here are some basic properties of the modular function.

Lemma 6.5. *Let G be a Lie group.*

(1) The modular function is a group homomorphism $m : G \to \mathbb{R}_{>0}$.

(2) If $\mathrm{vol}(G) < \infty$, then m is constant and equal to one.

Proof. We have $m : G \to \mathbb{R}_{\geq 0}$ since volumes are nonnegative. Let $A \subseteq G$ be measurable with $0 < \mathrm{vol}(A) < \infty$. For every $g \in G$ right-translation $R_g : G \to G$ is a diffemorphism, which implies $0 < \mathrm{vol}(Ag) < \infty$. Then, for $g, h \in G$ we have

$$m(gh) = \frac{\mathrm{vol}(Agh)}{\mathrm{vol}(A)} = \frac{\mathrm{vol}(Agh)}{\mathrm{vol}(Ag)}\,\frac{\mathrm{vol}(Ag)}{\mathrm{vol}(A)} = m(g)m(h).$$

If $m(g) = 0$ for some $g \in G$, then $1 = m(e) = m(g)m(g^{-1}) = 0$, which is a contradiction. Hence $m(g) > 0$ for every g. This settles the first part of the lemma. For the second part we use $A = G$ to get $m(g) = \mathrm{vol}(Gg)/\mathrm{vol}(G) = 1$. □

In particular, this shows that compact Lie groups are unimodular.

Exercise 6.1. Prove that a discrete group is unimodular.

6.3 Volumes of homogeneous spaces

Let G be a Lie group and M be a smooth manifold. We say that G *acts* on M, if we have a group action $G \times M \to M$, $(g, x) \mapsto g \cdot x$ that is smooth and transitive, meaning that for all $x, y \in M$ we can find $g \in G$ such that $g \cdot x = y$. In this case, we say that M is a *homogeneous space*. We denote $\mathbf{0} := H = \pi(e) \in M$. As for Lie groups we write $L_g : M \to M, x \mapsto g \cdot x$, and we write $\mathrm{D}_x L_g(v) =: gv$, $v \in T_x M$.

If M is a Riemannian manifold and G is endowed with a left-invariant metric, and for every $g \in G$ the map L_g is an isometry, we say that G acts on M isometrically and we call M a *Riemannian homogeneous space.*

Here are some examples of homogeneous spaces.

Example 6.6. The orthogonal group $O(n)$ acts isometrically on the sphere S^{n-1} and on projective space $\mathbb{R}\mathrm{P}^{n-1}$. The unitary group $U(n)$ acts isometrically on the sphere S^{2n-1} and on complex projective space $\mathbb{C}\mathrm{P}^{n-1}$. In general, every Lie group acts isometrically on itself.

The next two results imply that quotients of Lie groups completely classify homogeneous spaces. The first is [25, Theorem 21.17]

Theorem 6.7. *Let G be a Lie group and let $H \subset G$ be a closed subgroup. The left coset space G/H is a topological manifold of dimension equal to $\dim(G) - \dim(H)$, and it has a unique smooth structure such that the quotient map $\pi : G \mapsto G/H$ is a smooth submersion. The quotien G/H us a homogeneous space under the left action of G on G/H.*

In the following we will always assume that the quotient space is endowed with the unique smooth structure from Theorem 6.7. In fact, all homogeneous spaces arise in this way, as next Theorem says. To state the theorem we need to define for $p \in M$ the *isotropy group*

$$G_p := \{g \in G \mid g \cdot p = p\}.$$

For a proof of the following result see [25, Theorem 21.18].

Theorem 6.8. *Let G be a Lie group, M be a homogeneous G-space and $p \in M$. The isotropy group G_p is a closed subgroup of G, and*

$$\phi_p : G/G_p \to M, \quad gG_p \mapsto g \cdot p$$

is an equivariant diffeomorphism.

A natural way to build *Riemannian* homogeneous G-spaces is to start with a Lie group G endowed with a left invariant Riemannian metric, which is also *right invariant* under the action of a compact subgroup H.

Proposition 6.9. *Let G be a Lie group with a left-invariant Riemannian metric which is right invariant for a closed subgroup H. This metric induces a unique Riemannian metric on the homogeneous space G/H that makes the quotient map $\pi : G \to G/H$ a Riemannian submersion.*

Proof. The required metric is built as follows. Given an element $p \in G/H$, chose an element $g \in G$ such that $\pi(g) = p$. Then $D_g\pi|_{(\ker D_g\pi)^\perp} : (\ker D_g\pi)^\perp \to T_p(G/H)$ is a linear isomorphism and there is a unique way to declare it to be a Euclidean isometry. We show that the induced metric on $T_p(G/H)$ does not depend on the choice of g such that $\pi(g) = p$, so that we get a well-defined Riemannian submersion.

The invariance of the metric on G under the action of H induces an isometry by right-translation $R_h : G \to G$ for every $h \in H$. For $g, g' \in G$ such that $g' = gh$ we then have an Euclidean isometry

$$\mathrm{D}_g R_h : T_g G \to T_{g'}G, v \mapsto vh$$

Let $v \in \ker D_g\pi$. We show that $vh \in \ker D_{g'}\pi$. This would imply that $\mathrm{D}_g R_h$ maps $(\ker D_g\pi)^\perp$ isometrically to $(\ker D_{g'}\pi)^\perp$, so the induced metric on T_pG/H does not depend on the choice of g. To see this, we take $f \in C^\infty(G/H, \mathbb{R})$. Then:

$$D_{g'}\pi(vh)(f) = v(f \circ \pi \circ R_h).$$

By construction, we have $f \circ \pi \circ R_h = f \circ \pi$. Moreover, $v(f \circ \pi) = D_g\pi(v)(f) = 0$, because $v \in \ker D_g\pi$. This shows that $vh \in \ker D_{g'}\pi$. $\square$

We call the metric induced on G/H as in Proposition 6.9 the *quotient metric*. Observe that G/H with the quotient metric is a Riemannian homogeneous space. If G is compact, it is unimodular by Lemma 6.5, and so the metric on G is right-invariant for any closed subgroup H of G. We have the following useful result.

Theorem 6.10. *Let G be a compact Riemannian Lie group endowed with a left-invariant Riemannian metric. Let $H \subset G$ be a closed subgroup. Endow G/H with the quotient metric. Then:*

$$\mathrm{vol}(G/H) = \frac{\mathrm{vol}(G)}{\mathrm{vol}(H)}.$$

Here, the volume of H is the one induced by restricting the Riemannian metric to H and then taking the corresponding Riemannian measure.

Proof. The quotient map $\pi : G \mapsto G/H$ is a Riemannian submersion. Lemma 2.33 implies that $\mathrm{vol}(G) = \int_{w \in G/H} \mathrm{vol}(\pi^{-1}(w))\, \mathrm{d}w$, where $\mathrm{d}w$ denotes the integration with respect to the Riemannian measure of G/H. Because G acts on itself by isometries, for all $g \in G$ we have that the map $h \mapsto gh$ from H to gH is an isometry of submanifolds of G with the induced Riemannian metric and consequently, if $\pi(g) = w$: $\mathrm{vol}(\pi^{-1}(w)) = \mathrm{vol}(gH) = \mathrm{vol}(H)$. This finishes the proof. $\square$

Since the orthogonal $O(n)$ and the unitary group $U(n)$ are compact, we can use Theorem 6.10 to compute the volumes of $O(n)$ and $U(n)$. The orthogonal group acts on S^n by rotations. The isotropy group $O(n)_x$ of $x \in S^{n-1}$ is the orthogonal group that rotates the orthogonal complement $e_1^\perp$, so it is isometric to $O(n-1)$. Theorem 6.8 implies that S^{n-1} is diffeomorphic to $O(n)/O(n-1)$. Thus, if we endow S^{n-1} with the quotient structure, then

$$\mathrm{vol}(S^{n-1}) = \frac{\mathrm{vol}(O(n))}{\mathrm{vol}(O(n-1))},$$

by Theorem 6.10. In fact, we have shown in the proof of Proposition 2.35 that the quotient structure agrees with the Euclidean metric on S^{n-1} inherited from $\mathbb{R}^n$. This is why we get the same formula. The orthogonal group also acts on $\mathbb{R}\mathrm{P}^{n-1}$, but with isotropy group $O(1) \times O(n-1)$, so that

$$\mathrm{vol}(\mathbb{R}\mathrm{P}^{n-1}) = \frac{\mathrm{vol}(O(n))}{\mathrm{vol}(O(1)) \cdot \mathrm{vol}(O(n-1))} = \tfrac{1}{2}\mathrm{vol}(S^{n-1}),$$

which agrees with Subsection 2.2.5. With the same argumentation we have that

$$\mathrm{vol}(S^{2n-1}) = \frac{\mathrm{vol}(U(n))}{\mathrm{vol}(U(n-1)},$$

because $U(n)$ acts on S^{2n-1} seen as the complex sphere of points $x \in \mathbb{C}^n$ with $x^*x = 1$ and has isotropy group $U(n-1)$. This shows Subsection 2.2.6. Furthermore, $U(n)$ acts on complex projective space $\mathbb{C}\mathrm{P}^{2n-1}$ with isotropy group $U(1)\times U(n-1)$, so that Theorem 6.10 implies

$$\mathrm{vol}(\mathbb{C}\mathrm{P}^{n-1}) = \frac{\mathrm{vol}(U(n))}{\mathrm{vol}(U(1)) \cdot \mathrm{vol}(U(n-1))} = \frac{1}{2\pi}\mathrm{vol}(S^{2n-1}),$$

which agrees with Subsection 2.2.5.

6.4 Grassmannians

We compute the volume of the *Grassmannian.* This is the space of all k-dimensional linear spaces in $\mathbb{R}^n$:

$$G(k,n) := \{L \subset \mathbb{R}^n \mid L \text{ is a linear space of dimension } k\}$$

Similarly, we denote the complex Grassmannian by

$$G_{\mathbb{C}}(k,n) := \{L \subset \mathbb{C}^n \mid L \text{ is a complex linear space of dimension } k\}.$$

Both $G(k,n)$ and $G_{\mathbb{C}}(k,n)$ are homogeneous spaces in the following way. The orthogonal group acts on the Grassmanian $G(k,n)$ by $Q \cdot L := \{Q\ell \mid \ell \in L\}$ for $Q \in O(n)$ and $L \in G(k,n)$. The isotropy group of $L_0 = \operatorname{span}\{e_1,\dots,e_k\}$, where e_i is the i-th standard basis vector in $\mathbb{R}^n$ is $G_{L_0} = O(k) \times O(n-k)$; so that we have a bijective map $O(n)/(O(k)\times O(n-k)) \to G(k,n)$. We *define* a Riemannian manifold structure on $G(k,n)$ by declaring this map to be an isometry. By Proposition 6.9, the quotient map $\pi : O(n) \to G(k,n)$ is a Riemannian submersion, which implies that for $L \in G(k,n)$ we have an isometry $(\ker \mathrm{D}_Q\pi)^\perp \cong T_LG(k,n)$, where $\pi(Q) = L$.

Let us consider the tangent space $T_{L_0}G(k,n)$. We have $\pi(\mathbf{1}_n) = L_0$, where $\mathbf{1}_n$ denotes the $n \times n$ identity matrix. The tangent space of $G(k,n)$ at L_0 can be isometrically identified with the orthogonal complement of $\ker \mathrm{D}_{\mathbf{1}_n}\pi$ and we have $\ker \mathrm{D}_{\mathbf{1}_n}\pi = T_{\mathbf{1}_n}G_{L_0} = T_{\mathbf{1}_k}O(k) \times T_{\mathbf{1}_{n-k}}O(n-k)$. We have computed the tangent space of the orthogonal group in Subsection 2.2.6:

$$T_{\mathbf{1}_n}O(n) = \left\{ \begin{bmatrix} A & -C \\ C & B \end{bmatrix} \mid A \in T_{\mathbf{1}_k}O(k), B \in T_{\mathbf{1}_{n-k}}O(n-k), C \in \mathbb{R}^{(n-k)\times k} \right\}.$$

Therefore, we have

$$T_{L_0}G(k,n) = \left\{ \begin{bmatrix} 0 & -C \\ C^T & 0 \end{bmatrix} \mid C \in \mathbb{R}^{(n-k)\times k} \right\} \cong \operatorname{Hom}(L_0, L_0^\perp)$$

as $\left[\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right]$ represents elements in $\operatorname{Hom}(L_0, L_0^\perp)$ in coordinates. Since for $L \in G(k,n)$ and $Q \in O(n)$ with $\pi(Q) = L$ we have $T_QO(n) = Q \cdot T_{\mathbf{1}_n}O(n)$ by Subsection 2.2.6 and $G_L = Q \cdot G_{L_0} \cdot Q^T$, the tangent space of the Grassmanian at $L \in G(k,n)$ is

$$T_LG(k,n) = \operatorname{Hom}(L, L^\perp)$$

Moreover, by Subsection 2.2.6 the metric on $T_{L_0}G(k,n)$ is

$$g_L(C_1, C_2) = \operatorname{tr}(C_1^T C_2), \text{ for } C_1, C_2 \in \operatorname{Hom}(L, L^\perp).$$

We proceed similarly for the complex Grassmannian. We now have the following theorem.

Theorem 6.11. *On the Grassmannian we put the quotient metric from the $O(n)$ action on $G(k,n)$. Similarly, on $G_{\mathbb{C}}(k,n)$ we put the quotient metric from the $U(n)$ action. With these structures:*

$$\operatorname{vol}(G(k,n)) = \frac{\operatorname{vol}(O(n))}{\operatorname{vol}(O(k)) \cdot \operatorname{vol}(O(n-k))}, \quad \textit{and}$$
$$\operatorname{vol}(G_{\mathbb{C}}(k,n)) = \frac{\operatorname{vol}(U(n))}{\operatorname{vol}(U(k)) \cdot \operatorname{vol}(U(n-k))}.$$

Proof. This follows immediately from Theorem 6.10. □

For example, $\operatorname{vol}(G(2,4)) = \operatorname{vol}(O(4))/\operatorname{vol}(O(2))^2$, which by Proposition 2.35 is equal to $\operatorname{vol}(S^3) \cdot \operatorname{vol}(S^2)/(\operatorname{vol}(S^1) \cdot \operatorname{vol}(S^0))$. U sing Subsection 2.2.5 we get

$$\operatorname{vol}(G(2,4)) = 2\pi^2.$$

The volumes of other Grassmannians can be computed similarly.

6.5 Integral geometry

Let G be a unimodular Lie group with a left-invariant Riemannian metric, and let $H \subseteq G$ be a compact subgroup. Then, by Proposition 6.9 we know that the homogeneous space $M \cong G/H$ has a left invariant metric. Let us denote the corresponding measure on M by vol.

Definition 6.12. A submanifold $X \hookrightarrow M$ is called *cohomogeneous*, if G acts transitively on tangent spaces of X; i.e., for all $x, y \in X$ there exists $g \in G$ with $gx = y$ and $g\, T_x X = T_y X$.

In this section we will prove the *integral geometry formula in Riemannian homogeneous space*, that computes the average volume of $X \cap g \cdot Y$ for cohomogeneous submanifolds X and Y. For this, we need to introduce a definition of *angle* between subspaces. Let $V, W \subset T_{\mathbf{0}}M$, $\mathbf{0} := \pi(e)$, and let $v_1, \ldots, v_m$ be an orthonormal basis for V and $w_1, \ldots, w_n$ be an orthonormal basis for W. We define

$$\sigma(V,W) := \sqrt{\det(1_n - A^T A)}, \quad A := \begin{bmatrix} \langle v_1, w_1\rangle & \cdots & \langle v_1, w_n\rangle \\ & \ddots & \\ \langle v_m, w_1\rangle & \cdots & \langle v_m, w_n\rangle \end{bmatrix}.$$

Since $H = \pi(e)$ is the stabilizer group of $\pi(e)$, we have $h\,W \subset T_{\pi(e)}M$ for every $h \in G$ and $W \subset T_eG$. The *average angle* between $V, W \subset T_eG$ is

$$\sigma_{\mathrm{av}}(V,W) := \int_H \sigma(V, hW)\,\mathrm{d}h.$$

For two cohomogeneous submanifolds $X, Y \hookrightarrow M$ we then set

$$\sigma(X,Y) := \sigma_{\mathrm{av}}(N_{\mathbf{0}}\, a^{-1}X,\ N_{\mathbf{0}}\, b^{-1}X),$$

where $a, b \in G$ are such that $\pi(a) \in X$ and $\pi(b) \in Y$. Due to cohomogeneity Section 6.5 is independent of the choices of a and b.

We will prove the following theorem in Section 6.7 below.

Theorem 6.13. *Let G be a unimodular Lie group with a left-invariant metric, and H be a compact subgroup. Let $M := G/H$ and denote $N := \dim(M)$. Let $X \hookrightarrow M$ and $Y \hookrightarrow M$ be cohomogeneous submanifolds of codimensions m and n, respectively. We assume that $X \cap g \cdot Y \neq \emptyset$ for a full-dimensional subset of G. Then, for almost all $g \in G$ we have that $X \cap g \cdot Y$ is either empty or a submanifold of codimension $m + n$, and*

$$\int_G \mathrm{vol}_{N-m-n}(X \cap g \cdot Y)\,\mathrm{d}g = \sigma(X,Y)\,\mathrm{vol}_{N-m}(X)\,\mathrm{vol}_{N-n}(Y).$$

Howard [18] proved a more general formula for any submanifolds and allowing G to be not unimodular, but here in this book we focus on the special case from the above theorem. Before we prove Theorem 6.13, let us first discuss its consequences in the case when M is projective space.

6.6 Probabilistic intersection theory in projective space

We want to apply Theorem 6.13 to real projective space $M = \mathbb{RP}^N$. Recall that $\mathbb{RP}^N \cong G/H$ for $G = O(N+1)$ and $H := O(1) \times O(N)$. The orthogonal group is compact, hence H is compact, and by Lemma 6.5 $O(N+1)$ is unimodular. Therefore, real projective space falls into the setting of this section. The next lemma shows that all submanifolds of projective space are cohomogeneous.

Lemma 6.14. *Let $X \hookrightarrow \mathbb{R}\mathrm{P}^N$ be a submanifold. Then X is cohomogeneous.*

Proof. Let $x, y \in X$ and let $g \in G$ with $L_g(x) = y$. Put $V := g\,T_xX$. Without restriction we can assume that $y = e_0 = [1 : 0 : \ldots : 0]$. The stabilizer of e_0 is $H = O(1) \times O(N)$, and $T_{e_0}\mathbb{R}\mathrm{P}^N = e_0^\perp$. Take $h = (h_1, h_2) \in O(1) \times O(N)$. Then, for $v \in e_0^\perp$ we have $hv = h_2 v$. Therefore, H acts as the orthogonal group $O(n)$ on $e_0^\perp \cong \mathbb{R}^N$. The orthogonal group acts transitively on the Grassmanian of $(\dim X)$-dimensional planes in $\mathbb{R}^N$. This shows that we can find $h \in H$ such that $h\,V = T_{e_0}X$. Setting $g' := hg$ we have $L_{g'}(x) = g'x = y$ and $g'\,T_xX = T_yX$. $\square$

In the proof of Lemma 6.14 we used the fact that H acts transitively on every Grassmanian in $T_{e_0}\mathbb{R}\mathrm{P}^N$. This implies in fact more: it implies that $\sigma(X, Y)$ does not depend on X and Y but only on their codimensions. In this case, let us denote $\sigma(m, n) := \sigma(X, Y)$. We have the following result.

Lemma 6.15. $\sigma(m,n) = \dfrac{\mathrm{vol}(\mathbb{R}\mathrm{P}^{N-m-n})}{\mathrm{vol}(\mathbb{R}\mathrm{P}^{N-m}) \cdot \mathrm{vol}(\mathbb{R}\mathrm{P}^{N-n})} \cdot \mathrm{vol}(O(N+1)).$

Proof. We take $X = \mathbb{R}\mathrm{P}^{N-m} \times \{0\}^m$ and $X = \mathbb{R}\mathrm{P}^{N-n} \times \{0\}^n$. Then, X and $g \cdot Y$ intersect transversally for almost all $g \in O(N+1)$, so that $X \cap g \cdot Y$ is a linear space of dimension $N - m - n$. Hence, $\int_{O(N+1)} \mathrm{vol}_{N-m-n}(X \cap g \cdot Y)\, \mathrm{d}g = \mathrm{vol}(\mathbb{R}\mathrm{P}^{N-m-n})$. Plugging this into Theorem 6.13 gives the asserted formula. $\square$

Combining this lemma with the integral geometry formula (Theorem 6.13) implies the following important result.

Corollary 6.16. *Let $X, Y \hookrightarrow \mathbb{R}\mathrm{P}^N$ be submanifolds of codimensions m and n, respectively. Then, we have that $X \cap g \cdot Y$ is either empty or a submanifold of codimension $m+n$ for almost all $g \in G$*

$$\mathop{\mathbb{E}}_{g \sim \mathrm{Unif}(O(N+1))} \mathrm{vol}(X \cap g \cdot Y) = \mathrm{vol}(\mathbb{R}\mathrm{P}^{N-m-n}) \cdot \frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{R}\mathrm{P}^{N-m})} \cdot \frac{\mathrm{vol}(Y)}{\mathrm{vol}(\mathbb{R}\mathrm{P}^{N-n})}.$$

We can argue similar for submanifolds of complex projective space. But here, we have the special situation that the volume of $X \cap g \cdot Y$ is a generic value. Hence, when translatin Corollary 6.16 to complex projective space we are taking the expected value over a function, which is constant almost everywhere. This is why we don't have an expected value in the following corollary.

Corollary 6.17. *Let $X, Y \hookrightarrow \mathbb{C}\mathrm{P}^N$ be submanifolds of codimensions m and n, respectively. Then, for almost all $g \in G$ we have that $X \cap g \cdot Y$ is a submanifold of codimension $m+n$, and*

$$\mathrm{vol}(X \cap g \cdot Y) = \mathrm{vol}(\mathbb{C}\mathrm{P}^{N-m-n}) \cdot \frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{C}\mathrm{P}^{N-m})} \cdot \frac{\mathrm{vol}(Y)}{\mathrm{vol}(\mathbb{C}\mathrm{P}^{N-n})}.$$

We use Corollary 6.17 to prove a probabilistic version of intersection theory in complex projective space. Let $X \subseteq \mathbb{C}\mathrm{P}^N$ be a projective variety of codimension m. Recall that the degree of a projective variety $X \subseteq \mathbb{C}\mathrm{P}^N$ denoted $\deg(X)$, is the number of points in the intersection $X \cap L$, where L is a generic linear space of dimension m.

Recall from Bézout's theorem, that if Y is another subvariety of $\mathbb{C}\mathrm{P}^N$, that intersects X transversally, then the intersection $X \cap Y$ is a subvariety of degree $\deg(X) \cdot \deg(Y)$. This follows from the fact that the *Chow-ring* of $\mathbb{C}\mathrm{P}^N$ is generated by hyperplane classes (see [9, Theorem 2.1]).

Let $L := \mathbb{C}\mathrm{P}^m \times \{0\}^{N-m}$. It follows from Theorem 6.13 that X and $g \cdot L$ intersect transversally for almost all $g \in U(N+1)$. Furthermore, recall that we defined the volume of X to be the volume of the smooth locus of X, which is an embedded submanifold of $\mathbb{C}\mathrm{P}^N$. We may therefore apply Corollary 6.17 to the smooth locus of X. This yields the following formula.

Theorem 6.18. *The degree of a complex projective variety equals its normalized volume:*

$$\deg(X) = \frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{C}\mathrm{P}^{N-m})}.$$

Now, for subvarieties of X and Y of codimensions m and n, respectively, combining Theorem 6.18 with Corollary 6.17 implies that for almost all g:

$$\deg(X) \cdot \deg(Y) = \deg(X \cap g \cdot Y).$$

Therefore, we have shown that $X \cap g \cdot Y$ is a variety of degree $d \cdot d'$.

For subvarieties of real projective space there is no concept of degree, because there is no general intersection number. Nevertheless, in analogy with Theorem 6.18 we can make the following definition.

Definition 6.19. Let X be a subvariety of $\mathbb{R}\mathrm{P}^N$ of codimension m. We define the *average degree* of X to be

$$\mathrm{adeg}(X) := \mathop{\mathbb{E}}_{g \sim \mathrm{Unif}(O(N+1))} \#(X \cap g \cdot L),$$

where $L = \mathbb{R}\mathrm{P}^m \times \{0\}^{N-m}$ is a linear space of dimension m.

It follows from Corollary 6.16 that the average degree of a real variety X equals its normalized volume; i.e., $\operatorname{adeg}(X) = \operatorname{vol}(X)/\operatorname{vol}(\mathbb{R}\mathrm{P}^{N-m})$, $m = \operatorname{codim}(X)$. Therefore, for subvarieties X and Y we have by Corollary 6.16 that

$$\operatorname{adeg}(X) \operatorname{adeg}(X) = \mathop{\mathbb{E}}_{g \in O(N+1)} \operatorname{adeg}(X \cap g \cdot Y),$$

which is the real analogue of Theorem 6.18.

6.7 Proof of the integral geometry formula

In this section we prove Theorem 6.13. We follow the monograph by Howard [18].

The first step is the following lemma.

Lemma 6.20. *Let $\iota : G \to G, g \mapsto g^{-1}$ be the inversion map. Its derivative at $g \in G$ is given by $\mathrm{D}_g\iota(\dot{g}) = -g^{-1}\dot{g}g^{-1}$, $\dot{g} \in T_gG$.*

Proof. Let $\alpha(t)$ be a curve in G with $g = \alpha(0)$. We write $\dot{g} := \frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)|_{t=0}$. Differentiating $\alpha(t)\alpha(t)^{-1} = e$ gives $0 = \dot{g}\, g^{-1} + g\,(\mathrm{D}_g\iota(\dot{g}))$. □

Next, let $\pi : G \to M := G/H$ be the quotient map. We denote

$$\hat{X} := \pi^{-1}(X) \quad \text{and} \quad \hat{Y} := \pi^{-1}(Y).$$

As π is a fibration $\hat{X}$ and $\hat{Y}$ are smooth manifolds of G of codimensions $m = \operatorname{codim}(X)$ and $n = \operatorname{codim}(Y)$, respectively. A central role in the proof is played by the following map:

$$F : \hat{X} \times \hat{Y} \to G,\ (a, b) \mapsto ab^{-1}.$$

For every $g \in G$ in the image of F we have

$$F^{-1}(g) = \{(a, b) \in \hat{X} \times \hat{Y} \mid a = gb\} = \operatorname{Graph}(L_{g^{-1}}|_{\hat{X} \cap g\hat{Y}}).$$

Therefore, $X \cap g \cdot Y \neq \emptyset$ if and only if $F^{-1}(g) \neq \emptyset$.

The next lemma computes the derivative of F.

Lemma 6.21. *Let $(a, b) \in \hat{X} \times \hat{Y}$. For $(\dot{a}, \dot{b}) \in T_a\hat{X} \times T_b\hat{Y}$ we have*

$$\mathrm{D}_{(a,b)}F(\dot{a}, \dot{b}) = (\dot{a} - g\dot{b})b^{-1},$$

where $g = F(a, b) = ab^{-1}$.

Proof. Let $\iota(g) = g^{-1}$ be the inversion map. Then, $F(a,b) = a\,\iota(b)$, and so by Lemma 6.20 $\mathrm{D}_{(a,b)}(\dot a, \dot b) = \dot a\, b^{-1} + a\,\mathrm{D}_b\iota(\dot b) = \dot a b^{-1} - ab^{-1}\dot b b^{-1}$. □

We have the following proposition, which proves the first part of Theorem 6.13.

Proposition 6.22. *For almost all $g \in G$ either $\hat X \cap g\hat Y$ is empty or a submanifold of G of codimension $m+n$.*

Proof. Let us assume that $\hat X \cap g\hat Y \neq \emptyset$. By Section 6.7, the preimage of g under F is $F^{-1}(g) = \mathrm{Graph}(L_{g^{-1}}|_{\hat X \cap g\hat Y})$. Therefore, we have $F^{-1}(g) \neq \emptyset$. Let $(a,b) \in F^{-1}(g)$ and $(\dot a, \dot b) \in T_a\hat X \times T_b\hat Y$. By Lemma 6.21, we have $\mathrm{D}_{(a,b)}(\dot a, \dot b) = (\dot a - g\dot b)b^{-1}$. It follows that the image of $\mathrm{D}_{(a,b)}F$ is

$$\mathrm{Im}(\mathrm{D}_{(a,b)}F) = (T_a\hat X + g\,T_b\hat Y)b^{-1} = (T_a\hat X + T_a(g\hat Y))b^{-1}.$$

By Sard's lemma, the nonregular values of F are a subset of G of measure zero. Since the set of all $g \in G$ such that $\hat X \cap g\hat Y \neq \emptyset$ is a full-dimensional subset of G (because otherwise $\int_G \mathrm{vol}(X \cap gY)\,\mathrm{d}g = 0$), almost all g in the image of F are regular values. For such g the characterization of $\mathrm{Im}(\mathrm{D}_{(a,b)}F)$ in Section 6.7 implies that $T_a\hat X + T_a(g\hat Y) = T_gG$ for all $a \in \hat X \cap g\hat Y$. Hence, $\hat X$ and $g\hat Y$ intersect transversally, and so by [25, Theorem 6.30] $\hat X \cap g\hat Y \hookrightarrow G$ is a submanifold of codimension $m+n$. □

We need another basic lemma from linear algebra. For this, recall the angle $\sigma(V,W)$ between linear spaces V, W from Section 6.5.

Lemma 6.23. *Let $m < \ell$ and $V, W \in O(\ell)$ be orthogonal matrices. Let us write $V = [V_1\, V_2]$ and $W = [W_1\, W_2]$ with $V_1, W_1 \in \mathbb{R}^{\ell\times m}$. We identify the matrices with their columnspans. Then, $\sigma(V_1, W_2) = \sigma(V_2, W_1)$.*

Proof. Recall that $\sigma(V_1, W_2) = \sqrt{\det(\mathbf{1}_m - V_1^TW_2W_2^TV_1)}$. Without restriction we can assume that $W = \mathbf{1}_\ell$. Since $V_1^TV_1 = \mathbf{1}_m$ and $\mathbf{1}_\ell = W_1W_1^T + W_2W_2^T$, this shows $\mathbf{1}_m = V_1^T(W_1W_1^T + W_2W_2^T)V_1$. From this we get $\mathbf{1}_m - V_1^TW_2W_2^TV_1 = V_1^TW_1W_1^TV_1$. The assertion follows now from the symmetry of this argument. □

Next, we compute the normal Jacobian of F at a point.

Proposition 6.24. *For almost all $(a,b) \in \hat X \times \hat Y$ the normal Jacobian of F is*

$$\mathrm{NJ}(F,(a,b)) = \sqrt{2}^k\,\sigma(N_a\hat X, N_ag\hat Y),$$

where $k = \dim(\hat X \cap g\hat Y)$.

Proof. Recall from Lemma 6.21 that $\mathrm{D}_{(a,b)}F(\dot{a},\dot{b}) = (\dot{a} - g\dot{b})b^{-1}$, where $g = ab^{-1}$. Since left-multiplication by g and right-multiplication by b are isometries, we have $\mathrm{NJ}(F,(a,b)) = \sqrt{\det(J^T J)}$, where J is the matrix with respect to orthonormal bases of the linear map

$$\phi : T_a\hat{X} \times T_a g\hat{Y} \to T_a G,\ (\dot{a},\dot{b}) \mapsto \dot{a} - \dot{b}.$$

Let us write $V := T_a\hat{X}$ and $W := T_a g\hat{Y}$ and $U := V \cap W$. Then, we have $\ker(\phi) = \{(\dot{a},\dot{a}) \mid \dot{a} \in U\}$ and so

$$\ker(\phi)^\perp = (V \cap U^\perp \times \{0\}) \oplus (\{0\} \times W \cap U^\perp) \oplus \{(\dot{a}, -\dot{a}) \mid \dot{a} \in U\},$$

which is an orthogonal decomposition. By Proposition 6.22, U has codimension $m+n$ in T_aG. Therefore, $\dim(V\cap U^\perp) = n$ and $\dim(V\cap U^\perp) = m$. Let $u_1, \ldots, u_k$ be an orthonormal basis for U, and $v_1, \ldots, v_n$ be an orthonormal basis for $V\cap U^\perp$, and $w_1, \ldots, w_m$ be an orthonormal basis for $W\cap U^\perp$. Then, $\phi(v_i, 0) = v_i$, $\phi(0, w_j) = w_j$ and $\phi(\frac{1}{\sqrt{2}}(u_\ell, -u_\ell)) = \sqrt{2}u_\ell$, which shows that

$$J^T J = \begin{bmatrix} \mathbf{1}_n & B & 0 \\ B^T & \mathbf{1}_m & 0 \\ 0 & 0 & 2\,\mathbf{1}_k \end{bmatrix}, \quad \text{where} \quad B = \begin{bmatrix} \langle v_1, w_1\rangle & \cdots & \langle v_1, w_m\rangle \\ & \ddots & \\ \langle v_n, w_1\rangle & \cdots & \langle v_n, w_m\rangle \end{bmatrix}.$$

Hence,

$$\begin{aligned} \mathrm{NJ}(F,(a,b)) = \sqrt{\det(J^T J)} = \sqrt{2}^k \sqrt{\det \begin{bmatrix} \mathbf{1}_n & B \\ B^T & \mathbf{1}_m \end{bmatrix}} &= \sqrt{2}^k \sqrt{\det(\mathbf{1}_n - BB^T)} \\ &= \sqrt{2}^k\, \sigma(V\cap U^\perp, W \cap U^\perp). \end{aligned}$$

From Lemma 6.23 and the fact that the orthogonal complement of $V \cap U^\perp$ in $U^\perp$ is $V^\perp$, and that the orthogonal complement of $W\cap U^\perp$ in $U^\perp$ is $W^\perp$, we get $\sigma(V\cap U^\perp, W\cap U^\perp) = \sigma(V^\perp, W^\perp)$. □

We can now prove the integral geometry formula.

Proof of Theorem 6.13. The fact that $X \cap gY$ is either empty or a submanifold of codimension $m+n$ for almost all $g \in G$ follows from Proposition 6.22. Fix $g \in G$ such that $X \cap gY \neq \emptyset$. Then, $\hat{X} \cap g\hat{Y} \neq \emptyset$. Since π is a Riemannian submersion,

$$\mathrm{vol}(\hat{X} \cap g\hat{Y}) = \mathrm{vol}(X \cap gY)\,\mathrm{vol}(H),$$

(the volume of H is finite due to compactness). Recall $F^{-1}(g) = \mathrm{Graph}(L_{g^{-1}}|_{\hat{X}\cap g\hat{Y}})$

from Section 6.7. We have a smooth map $\phi_g : \hat{X} \cap g\hat{Y} \to F^{-1}(g), a \mapsto (a, g^{-1}a)$. Since left-translation is an isometry, $\mathrm{NJ}(\phi_g, a) = \sqrt{2}^k$, $k = \dim(\hat{X} \cap g\hat{Y})$. This implies, using the coarea formula (Theorem 2.26): $\sqrt{2}^k \operatorname{vol}(\hat{X} \cap g\hat{Y}) = \operatorname{vol}(F^{-1}(g))$. On the other hand, again using the coarea formula and Proposition 6.24, we have

$$\int_G \operatorname{vol}(F^{-1}(g))\, \mathrm{d}g = \sqrt{2}^k \int_{\hat{X}\times\hat{Y}} \sigma(N_a\hat{X}, N_a g\hat{Y})\, \mathrm{d}(a,b), \quad g = ab^{-1}.$$

Altogether, this implies

$$\operatorname{vol}(\hat{X} \cap g\hat{Y}) \operatorname{vol}(H) = \int_{\hat{X}\times\hat{Y}} \sigma(N_a\hat{X}, N_a g\hat{Y})\, \mathrm{d}(a,b).$$

We proceed with the right-hand side. First, we use that left-translation by a^{-1} is an isometry to see that $\sigma(N_a\hat{X}, N_a g\hat{Y}) = \sigma(N_e\, a^{-1}\hat{X}, N_e\, b^{-1}\hat{Y})$, where $e \in G$ is the identity. Using that $\pi : G \to G/H$ is a Riemannian submersion we get

$$\int_{\hat{X}\times\hat{Y}} \sigma(N_a\hat{X}, N_a g\hat{Y})\, \mathrm{d}(a,b) = \int_{X\times Y} \Big(\int_{aH\times bH} \sigma(N_e\, \eta^{-1}\hat{X}, N_e\, \theta^{-1}\hat{Y})\, \mathrm{d}(\eta,\theta) \Big) \mathrm{d}(x,y),$$

where $\pi(a) = x$ and $\pi(b) = y$. Next, a change of variables $\eta = ag$ and $\theta = bh$, $g, h \in H$, and using the fact that H stabilizes T_eG, turns this integral into

$$\int_{X\times Y} \Big(\int_{H\times H} \sigma(g^{-1}\, \hat{V}, h^{-1}\, \hat{W})\, \mathrm{d}(g,h) \Big) \mathrm{d}(x,y),$$
$$\text{where} \quad \hat{V} = N_e\, a^{-1}\hat{X},\ \hat{W} = N_e\, b^{-1}\hat{Y}.$$

We have $H \subset a^{-1}\hat{X}$ and so $T_eH = \ker \mathrm{D}_e\pi \subset T_e a^{-1}\hat{X}$, which implies that $\mathrm{D}_e\pi$ restricts to an isometry $N_e\, a^{-1}\hat{X} \to N_{\mathbf{0}}\, a^{-1}X$, $\mathbf{0} = \pi(e)$. This shows that the last integral is equal to

$$\int_{X\times Y} \Big(\int_{H\times H} \sigma(g^{-1}\, V, h^{-1}\, W)\, \mathrm{d}(g,h) \Big) \mathrm{d}(x,y),$$

where $V = N_{\mathbf{0}}\, a^{-1}X$, $W = N_{\mathbf{0}}\, b^{-1}Y$. Since G acts transitively on tangent spaces of X and Y, we have that the inner integral is independent of a and b, so that

$$\int_{H\times H} \sigma(g^{-1}\, V, h^{-1}\, W)\, \mathrm{d}(g,h) = \int_{H\times H} \sigma(V, (hg)^{-1}\, W)\, \mathrm{d}(g,h) = \sigma(X,Y)\operatorname{vol}(H).$$

Plugging this into Section 6.7 yields $\operatorname{vol}(\hat{X} \cap g\hat{Y}) \operatorname{vol}(H) = \int_{X\times Y} \sigma(X,Y)\operatorname{vol}(H)\, \mathrm{d}(x,y)$, which gives $\operatorname{vol}(X \cap gY) = \sigma(X,Y)\operatorname{vol}(X)\operatorname{vol}(Y)$. $\square$

7 Representation theory

In this section we start adopting the metric point of view and endow a complex or real vector space V with in inner product. Clearly, there are multiple choices of inner products which we can put. We want to select among them those which have a geometric meaning, which are *reasonable* in the sense of Chapter 1. We assume that there is a group G acting on V and we look at inner products on V, which are invariant under the action by G. Furthermore, an invariant inner product induces an invariant norm, but not all norms comes from inner products. Therefore, towards the end of this chapter we also consider invariant norms on V.

The setting in this chapter is general, but the scenario we have in mind is when V is the space of polynomials and G is the orthogonal or unitary group acting by change of variables. This special case will be worked out in Chapter 8.

We start by recalling some basics facts from *representation theory*. For more details we refer to the textbooks [10, 12].

In the following, we denote by $\mathrm{GL}(V)$ the group of invertible linear maps from V to V. That is, real linear maps, if V is a real vector space, and complex linear, if V is complex.

Definition 7.1. Let G be a topological group.

(1) A representation of the group G is a (real or complex) vector space together with a continuous group homomorphism $\rho : G \to \mathrm{GL}(V)$.

(2) If a linear subspace $W \subset V$ satisfies $\rho(g)W = W$ we call it G-invariant.

(3) The representation is reducible, if there exists a G-invariant linear subsspace $0 \neq W \subsetneq V$. Otherwise, it is called irreducible.

(4) The representation is called continuous, if ρ is continuous

If $W \subset V$ is a G-invariant subspace, any $\rho(g)$ can be regarded as an element in $\mathrm{GL}(W)$. Therefore, W is also a representation for G. We call W a *sub-representation* of V.

In the following, we will always consider continuous representations.

Definition 7.2. Let $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$ be two representations of a group G. We say that a linear map $\phi : V \to W$ is a G-homomorphism, if for all $g \in G$ and $v \in V$ we have $\phi(\rho_V(g)v) = \rho_W(g)\phi(v)$.

From another perspective, a linear map $\phi : V \to W$ is a G-homomorphism, if and only if the following diagram commutes.

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\rho_1(g)"'] & W \arrow[d, "\rho_2(g)"] \\ V \arrow[r, "L"] & W \end{tikzcd}$$

Definition 7.3. Let $\rho_V : G \to \mathrm{GL}(V)$ and $\rho_W : G \to \mathrm{GL}(W)$ be two representations of a group G. We call the representations *isomorphic*, if there is a linear isomorphism $\phi : V \to W$, which is also a G-homomorphism.

The map from the previous definitions is also called an *intertwining map*.

Lemma 7.4. *Let $\phi : V \to W$ be a G-homomorphism between two representations of a group G. Then,* $\ker\phi$ *is a sub-representation of V and the image of ϕ is sub-representation of W.*

Proof. For all $g \in G$ and $v \in \ker\phi$ we have $\phi(\rho_V(g)v) = \rho_W(g)\phi(v) = 0$, so $\ker\phi$ is G-invariant and hence a sub-representation of V. On the other hand, for any $w = \phi(v)$ in the image of ϕ we have $\rho_W(g)w = \phi(\rho_V(g)v)$. This shows that the image of ϕ is G-invariant, and hence a sub-representation of W. $\square$

The next lemma, called *Schur's lemma*, is a central yet powerful observation about irreducible representations.

Lemma 7.5 (Schur's lemma)**.** *Let $\phi : V \to W$ be a G-homomorphism between irreducible representations of a group G. Then, either $\phi = 0$ or ϕ is an isomorphism.*

Proof. By Lemma 7.4, $\ker\phi$ is a sub-representation of V. Since V is irreducible, this implies $\ker\phi = 0$ or $\ker\phi = V$; i.e., $\phi = 0$ or ϕ is injective. Furthermore, again by Lemma 7.4, the image of ϕ is a sub-representation of W. Since W is irreducible, this shows $\phi(V) = 0$ or $\phi(V) = W$; i.e., $\phi = 0$ or ϕ is surjective. $\square$

Proposition 7.6. *Let $\rho : G \to \mathrm{GL}(V)$ be a continuous representation of a compact group G. There exists on V a scalar product which is ρ–invariant.*

Proof. Let $H := \mathrm{im}(\rho)$. First observe that a scalar product $\langle \cdot, \cdot \rangle$ on V is ρ–invariant if it is H–invariant, i.e.

$$\langle hv_1, hv_2 \rangle = \langle v_1, v_2 \rangle \quad \forall v_1, v_2 \in V, \forall h \in H.$$

It will therefore be enough to construct a H–invariant scalar product.

Observe now that H is compact (since ρ is continuous and G is compact), and it is a subgroup of $\mathrm{GL}(V)$, which is a Lie group. In particular H is a closed subgroup of $\mathrm{GL}(V)$, and it is therefore a Lie group. By Theorem 6.2 there exists a (unique) left–invariant Haar measure μ on H. Let $\langle \cdot, \cdot \rangle$ be any scalar prodcut on V and define, for $v_1, v_2 \in V$,

$$\langle v_1, v_2 \rangle_\rho := \int_H \langle hv_1, hv_2 \rangle \mu(dh).$$

Clearly Chapter 7 defines a scalar product on V. In order to see that it is H–invariant, we observe that for every $\widetilde{h} \in H$

$$\langle \widetilde{h}v_1, \widetilde{h}v_2 \rangle_\rho = \int_H \langle h\widetilde{h}v_1, h\widetilde{h}v_2 \rangle \mu(dh) = \int_H \langle hv_1, hv_2 \rangle \mu(dh) = \langle v_1, v_2 \rangle_\rho,$$

where in the second step we have used the fact that the measure μ is right–invariant. □

Note that, because of Proposition 7.6, if G is compact we can always assume that the representation takes values in the orthogonal group of V.

We now work towards understanding the structure of the space of invariant inner products. In the following $\rho : V \to \mathrm{GL}(V)$ will be a fixed representation of a compact Lie group G. Given an inner product we define the orthogonal complement of a subspace W by

$$W^\perp := \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

Lemma 7.7. *Let $\langle\ ,\ \rangle$ be a G-invariant inner product on V. Let $W \subset V$ be a sub-representation. Then, the orthogonal projection $P : V \to W$ from V onto W is a G-homomorphism.*

Proof. Pick $v \in V = W \otimes W^\perp$ and decompose it as $v = w + w_\perp$ with $w \in W$ and $w_\perp \in W^\perp$. Let $g \in G$. Then, by definition of the orthogonal projection we have $P(v) = w$. We have $\rho(g)v = \rho(g)w + \rho(g)w^\perp$, and $\rho(g)w \in W$, because W is G-invariant. Furthermore, we have $\langle \rho(g)w_\perp, u \rangle = \langle w_\perp, \rho(g^{-1})u \rangle = 0$ for all $u \in W$, because $\rho(g^{-1})u \in W$. This shows that $\rho(g)w_\perp \in W^\perp$ and so we have $P(\rho(g)v) = \rho(g)w = \rho(g)P(v)$; i.e., P is a G-homomorphism. □

An immediate corollary of this lemma is the following result.

Corollary 7.8. *Let $\langle\ ,\ \rangle$ be a G-invariant inner product on V. Let $W \subset V$ be a sub-representation. Then, also the orthogonal complement $W^{\perp}$ is a sub-representation.*

Proof. Let $P : V \to W$ be the orthogonal projection. We have

$$\ker W^{\perp} = \ker P.$$

By Lemma 7.7, P is a G-homomorphism, and so by Lemma 7.4 $\ker P$ is a sub-representation of V. □

The next result tells us that representations can be orthogonally decomposed into subsrepresentations

Corollary 7.9. *Let $\rho : G \to \mathrm{GL}(V)$ be a continuous representation of a compact group and let $\langle\cdot,\cdot\rangle$ be a ρ–invariant scalar product on V (which exists by Proposition 7.6). Then V can be decomposed as an* orthogonal *direct sum*

$$V = \bigoplus_{k=1}^{r} V_k,$$

such that each V_k is ρ–invariant and the restriction $\rho|_{V_k} : G \to \mathrm{GL}(V_k)$ is an irreducible representation.

Proof. The proof is by induction on the dimension of V, the case $\dim(V) = 1$ being trivial: $V = V_1$ is irreducible. Let therefore $\dim(V) > 1$. If ρ is irreducible, there is nothing to prove. Otherwise, let $W \subsetneq V$ be an invariant subspace. By Corollary 7.8, $W^{\perp} \subsetneq V$ is also a proper invariant subspace, and the result follows by induction, since $\dim(W), \dim(W^{\perp}) < \dim(V)$. □

We can rephrase the decomposition in Corollary 7.9 by grouping isomorphic terms, so that we get a decomposition

$$V \simeq \bigoplus_{i=1}^{\nu} V_i^{m_i},$$

where $V_i^{m_i} = V_i^{(1)} \oplus \cdots \oplus V_i^{(m_i)}$, and V_i and V_j are non-isomorphic for $i \neq j$. In fact, two subrepresentations that are non-isomorphic will be orthogonal with respect to *any* invariant inner product. We show this next.

Lemma 7.10. *Let $W, Z \subset V$ be two irreducible sub-representations of V and $\langle\ ,\ \rangle$ be a G-invariant inner product on V.*

(1) If $W \neq Z$, then $W \cap Z = 0$.

(2) If W and Z are not isomorphic as representations of G, we have $W \perp Z$; i.e., $\langle w, z\rangle = 0$ for all $w \in W$ and $z \in Z$.

Proof. For the first item let us assume that $0 \neq x \in W \cap Z$. Consider the orthogonal projection $P : V \to W$. By Lemma 7.7, P is a G-homomorphism. Since Z is G-invariant, also the restriction $P|_Z : Z \to W$ is a G-homomorphism. We have $P|_Z(x) = x$, so that $P|_Z - \mathrm{id}_Z$ is not an isomorphism. By Schur's lemma (Lemma 7.5), this implies $P|_Z - \mathrm{id}_Z = 0$, hence $W = Z$. For the second item, we use that $P|_Z$ can't be an isomorphism, because W and Z are non-isomorphic. Again by Schur's lemma (Lemma 7.5), $P|_Z = 0$, which implies that $Z \subset \ker P = W^\perp$. □

We now come to the main theorem of this first section describing the structure of invariant inner products.

Theorem 7.11. *Let $\rho : G \to \mathrm{GL}(V)$ be a continuous representation of a compact group, with $\dim(V) = n$. Let $\mathrm{Sym}_n^+(\mathbb{R})$ denote the cone of positive definite symmetric matrices and $\mathrm{Her}_n^+(\mathbb{C})$ the cone of positive definite Hermitian matrices.*

(1) If V is a real vector space, the set of ρ–invariant scalar products on V can be identified with the linear section

$$\{P \in \mathrm{Sym}_n^+(\mathbb{R}) \mid \rho(g)P = P\rho(g) \text{ for all } g \in G\}.$$

(2) If V is a complex vector space, the set of ρ–invariant scalar products on V can be identified with

$$\{P \in \mathrm{Her}_n^+(\mathbb{C}) \mid \rho(g)P = P\rho(g) \text{ for all } g \in G\}.$$

(3) In both cases, if ρ is irreducible, this cone is one–dimensional and there is on V only one ρ–invariant scalar product, up to multiples.

Proof. We work out the real case. The complex case is one-to-one.

Let $\langle\cdot,\cdot\rangle$ be a ρ–invariant scalar product (which exists by Proposition 7.6) and pick an orthonormal basis $\{e_1, \dots, e_n\}$ for V with respect to $\langle\cdot,\cdot\rangle$. Using this basis we identify V with $\mathbb{R}^n$ (through the list of coordinates of a vector) and we observe that $\rho(g) \in O(n)$ for all $g \in G$. Let now $\langle\cdot,\cdot\rangle_2$ be another ρ–invariant scalar product. There exists $P \in \mathrm{Sym}_n(\mathbb{R})$ such that

$$\langle v_1, v_2\rangle_2 = v_1^T P v_2 \quad \text{for } v_1, v_2 \in V.$$

Moreover, by invariance of $\langle \cdot, \cdot \rangle_2$ we have $\rho(g)^T P \rho(g) = P$ for all $g \in G$, i.e. $P\rho(g) = \rho(g)P, \quad \forall g \in G$. This shows that the set of ρ–invariant scalar products has the asserted structure.

Assume now that ρ is irreducible. Since P is symmetric, it has a real eigenvalue λ. Therefore, $P - \lambda \mathbf{1}_n$ has a kernel, and so by Schur's lemma Lemma 7.5, it is zero. This implies $P = \lambda \mathbf{1}_n$, so that in this case the cone is one–dimensional. □

7.1 Invariant Hermitian structures

In this section and the next section we will now refine Theorem 7.11 in the complex and the real case separately. We start with the complex case.

In this section V is a *complex* vector space of complex dimension n. First, we give a stronger version of Schur's lemma in the complex setting.

Lemma 7.12 (Schur's Lemma, complex version)**.** *Let $\phi : V \to W$ be a G-homomorphism between irreducible of a group G. Then, ϕ is either zero or an isomorphism. Moreover, if $V = W$, there exists $\lambda \in \mathbb{C}$ such that $\phi = \lambda \mathbf{1}$.*

Proof. The first part follows from Lemma 7.5. For the second part of the proof, pick an eigenvalue $\lambda \in \mathbb{C}$ of ϕ. Then $\phi - \lambda \mathbf{1}$ is a G-homomorphism, which is not an isomorphism, so it is zero. This shows $\phi = \lambda \mathbf{1}$, □

We can use this result and the decomposition in Chapter 7 to classify all ρ–invariant Hermitian products.

Theorem 7.13. *Let V be a complex vector space and $\rho : G \to \mathrm{GL}(V)$ be a continuous representation of a compact group. Let $\nu, m_1, \ldots, m_\nu$ be given by Chapter 7 and, for every $i = 1, \ldots, \nu$ denote by $d_i := \dim(V_i)$. The set $\mathcal{H}^\rho$ of ρ–invariant Hermitian structures on V is isomorphic to the semialgebraic cone*

$$\mathcal{H}^\rho \simeq \bigoplus_{i=1}^{\nu} \mathrm{Her}^+_{m_i}.$$

In fact, the proof of Theorem 7.13 shows that the inner product associated to $(Q_1, \ldots, Q_\nu) \in \bigoplus_{i=1}^{\nu} \mathrm{Her}^+_{m_i}$ is represented by the Hermitian matrix

$$H = \begin{bmatrix} \mathbf{1}_{d_1} \otimes Q_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathbf{1}_{d_\nu} \otimes Q_\nu \end{bmatrix},$$

where $\otimes$ denotes the Kronecker product.

Proof of Theorem 7.13. Let h_0 be a ρ–invariant Hermitian product, and let

$$V \simeq \bigoplus_{i=1}^{\nu} V_i^{m_i}$$

be the decomposition from Chapter 7, so that each V_i is irreducible.

For every i, j let us pick an orthonormal basis $\{e_{i,k}^j\}_{k=1,\dots,d_i}$ for the j-th factor in $V_i^{m_i}$. Collating these basis, we get an orthonormal basis $\{e_1, \dots, e_n\}$ for V. In this basis the Hermitian product can be written as

$$h_0(v, w) = w^* v.$$

Let now h be another ρ–invariant Hermitian product and let H be the Hermitian matrix that relative to the basis above gives

$$h(v, w) = w^* H v.$$

Recall that V_i and V_j are non-isomorphic, if $i \neq j$. Therefore, by Lemma 7.10, we have $h(v, w) = 0$ for all $v \in V_i$ and $w \in V_j$. This implies that H is block-diagonal:

$$H = \begin{bmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_\nu \end{bmatrix}$$

with $B_i \in \mathrm{Her}^+_{d_i m_i}$. The third item in Theorem 7.11 implies that the restriction of h to each V_i is a multiple of h_0. Therefore, B_i consists of $m_i \times m_i$ blocks which are multiples of the identity matrix $\mathbf{1}_{d_i}$; i.e.,

$$B_i = \begin{bmatrix} \lambda_{1,1}^{(i)} \mathbf{1}_{d_i} & \cdots & \lambda_{1,m_i}^{(i)} \mathbf{1}_{d_i} \\ \vdots & \ddots & \vdots \\ \overline{\lambda}_{1,m_i}^{(i)} \mathbf{1}_{d_i} & \cdots & \lambda_{m_i,m_i}^{(i)} \mathbf{1}_{d_i} \end{bmatrix} = \mathbf{1}_{d_i} \otimes Q_i,$$

where $\lambda_{k,\ell}^{(i)} \in \mathbb{C}$ with $1 \leq k \leq \ell \leq m_i$, and

$$Q_i := \begin{bmatrix} \lambda_{1,1}^{(i)} & \cdots & \lambda_{1,m_i}^{(i)} \\ \vdots & \ddots & \vdots \\ \overline{\lambda}_{1,m_i}^{(i)} & \cdots & \lambda_{m_i,m_i}^{(i)} \end{bmatrix}$$

is positive definite Hermitian. This proves that an invariant Hermitian product is uniquely determined by a block–diagonal matrix with blocks $B_i \in \mathbf{1}_{d_i} \otimes \mathrm{Her}^+_{m_i}$.

Conversely, every such matrix gives an invariant Hermitian product. $\square$

7.2 Classification of real invariant inner products

Let us introduce some notation first, following [1].

Given a real vector space V, we denote by $cV := \mathbb{C} \otimes_{\mathbb{R}} V$ its complexification and, if $\rho : G \to \mathrm{GL}(V)$ is a representation, $c\rho : G \to \mathrm{GL}(cV)$ is the representation given by $c\rho(g) := 1 \otimes \rho(g)$.

If W is a complex vector space, we denote by rW the underlying real vector space and, if $\rho : G \to \mathrm{GL}(W)$ is a complex representation, we denote the representation induced by the inclusion $\mathrm{GL}(W) \hookrightarrow \mathrm{GL}(rW)$ by $r\rho : G \to \mathrm{GL}(rW)$.

Let now V be a fixed real vector space, and denote

$$W := cV.$$

The vector space W comes with a conjugation

$$\tau : W \to W, \quad \tau(z \otimes v) := \overline{z} \otimes v.$$

Such a map squares to the identity, and it induces a linear map, which we also denote $\tau : rW \to rW$. The real vector space rW can be decomposed into its eigenspaces:

$$rW = (rW)_+ \oplus (rW)_-, \quad \tau|_{(rW)_\pm} = \pm \mathbf{1}_{(rW)_\pm}.$$

The map $v \mapsto 1 \otimes v$ gives an isomorphism $V \simeq (rW)_+$; similarly, the map $v \mapsto i \otimes v$ gives an isomorphism $V \simeq (rW)_-$. In coordinates we have $rW \simeq V \oplus V$ and these isomorphisms are simply $v \mapsto (v, 0) \in V \oplus V$ and $v \mapsto (0, v) \in V \oplus V$.

In the previous section we relied on a specialization of Schur's lemma (Lemma 7.5) to the complex setting. Here, we need a version for real vector spaces. We use the setup above to prove the following proposition.

Proposition 7.14. *Let $\rho : G \to \mathrm{GL}(V)$ be a real irreducible representation of a compact group G. The ρ–invariant endomorphisms of V*

$$\mathrm{End}(V)^\rho := \{L \in \mathrm{End}(V) \mid L\rho(g) = \rho(g)L \text{ for all } g \in G\}$$

form a division algebra. In particular, $\mathrm{End}(V)^\rho$ is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, according to the following cases:

(1) If $c\rho$ is irreducible, then $\mathrm{End}(V)^\rho = \{\lambda \mathbf{1} \mid \lambda \in \mathbb{R}\} \simeq \mathbb{R}$.

(2) *If $c\rho$ is not irreducible and $cV \simeq W_1 \oplus W_2$ with W_1 not isomorphic to W_2 (as complex representations), then V carries a complex structure $I : V \to V$, and $\mathrm{End}(V)^\rho = \{a_1\mathbf{1} + a_2 I \mid a_1, a_2 \in \mathbb{R}\} \simeq \mathbb{C}$.*

(3) *If $c\rho$ is not irreducible and $cV \simeq W \oplus W$ (as complex representations), then V carries a quaternionic structure $I, J, K : V \to V$, and we have that $\mathrm{End}(V)^\rho = \{a_1\mathbf{1} + a_2 I + b_1 J + b_2 K \mid a_1, a_2, b_1, b_2 \in \mathbb{R}\} \simeq \mathbb{H}$.*

Let h be an invariant scalar product for the represetation $\rho : G \to \mathrm{GL}(V)$. Using Proposition 7.14 we can refine the partition inChapter 7 by grouping into isomorphism classes of real representations, according to the set of ρ–invariant endomorphism for each summand. We can write:

$$V \simeq \bigoplus_{\beta=1,2,4} \bigoplus_{i=1}^{\nu_\beta} V_{\beta,i}^{m_{\beta,i}},$$

where $V_{\beta,i}^{m_{\beta,i}} = V_{\beta,i}^{(1)} \oplus \cdots \oplus V_{\beta,i}^{(m_{\beta,i})}$ and such that for every $\beta = 1, 2, 4$ and for every $i = 1, \ldots, \nu_\beta$ and $j = 1, \ldots, m_{\beta,i}$, the subspace $V_{\beta,i}^{(j)}$ is an irreducible subrepresentation with

$$\mathrm{End}(V_{\beta,i}^{(j)})^{\rho_{\beta,i}{}^{(j)}} \simeq \begin{cases} \mathbb{R} & \beta = 1 \\ \mathbb{C} & \beta = 2 \\ \mathbb{H} & \beta = 4 \end{cases}.$$

(Notice that in this case $\dim V_{\beta,i}^{(j)}$ is divisible by β.). Moreover,

$$V_{\beta,i}^{(j_1)} \simeq V_{\beta,i}^{(j_2)}$$

as representations and, if $(\beta_1, i_1) \neq (\beta_2, i_2)$,

$$V_{\beta_1,i_1}^{(j_1)} \perp V_{\beta_2,i_2}^{(j_2)}$$

with respect to h.

In order to finally state the result on classification of real invariant scalar product we introduce the following notation. For $\beta = 1, 2, 4$ we denote by $\mathrm{Her}_{\beta,n}$ the set of symmetric matrices ($\beta = 1$), Hermitian matrices ($\beta = 2$), quaternionic hemitian matrices ($\beta = 4$). The latter correspond to the set of Hermitian matrices on $\mathbb{H}^n \simeq \mathbb{C}^n$ which commutes with the quaternionic structure. Similarly we define the intersections of these sets with the positive definite cone by $\mathrm{Her}_{\beta,n}^+$.

Theorem 7.15. *Let $\rho : G \to \mathrm{GL}(V)$ be a continuous representation of a compact group. Let $\{\nu_\beta, m_{\beta,i}\}_{\beta,i}$ be given by Section 7.2 and, for every $\beta = 1, 2, 4$, and*

$i = 1, \dots, \nu$ write $\dim(V_{\beta,i}) = d_{\beta,i}$. The set $\mathcal{H}^\rho$ of ρ–invariant scalar prodcut on V is isomorphic to the semialgebraic cone

$$\mathcal{H}^\rho \simeq \bigoplus_{\beta=1,2,4} \bigoplus_{i=1}^{\nu_\beta} \mathrm{Her}^+_{\beta,m_{\beta,i}}.$$

Proof. The proof proceeds as the proof of Theorem 7.13. Let h_0 be the ρ–invariant scalar product giving the decomposition Section 7.2. For every β, i, j let us pick an orthonormal basis $\{e^{(j)}_{\beta,i,k}\}_{k=1,\dots,\beta d_{\beta,i}}$ for $V^{(j)}_{\beta,i}$. Collating these bases, we get an orthonormal basis $\{e_1, \dots, e_n\}$ for V. In this basis the scalar product can be written as

$$h(v, w) = w^T v.$$

The set of ρ–invariant scalar products is isomorphic to

$$\begin{aligned} \mathcal{H}^\rho &\simeq \{H \in \mathrm{Her}^+_{1,n} \,|\, \rho(g)^T H \rho(g) = H \quad \forall g \in G\} \\ &= \{H \in \mathrm{Her}^+_{1,n} \,|\, H\rho(g) = \rho(g) H \quad \forall g \in G\}. \end{aligned}$$

The restriction of $h \in \mathcal{H}^\rho$ to each $V^{(j)}_{\beta,i}$ is a multiple of h_0 by Theorem 7.11 (since $\rho^{(j)}_i$ is irreducible). Therefore the corresponding orthogonal matrix H, defined by $h(v, w) = w^T H v$ for all $v, w \in V$, is a block-matrix (we call this blocks "first order blocks"), with each block which is itself a block matrix (we call these blocks "second order blocks") and whose diagonal blocks are multiples of the identity. Off–diagonal blocks correspond to either endomorphisms between isomorphic or between non–isomorphic spaces.

First order off–diagonal blocks are zero, since the corresponding representations are non–isomorphic. Second–order diagonal blocks on diagonal first–order blocks are multiple of the identity and second–order off–diagonal blocks on diagonal first–order blocks are endomorphisms belonging to $\mathrm{End}(V^{(j)}_{\beta,i})^{\rho^{(j)}_{\beta,i}} \simeq \mathbb{R}, \mathbb{C}, \mathbb{H}$. Using Proposition 7.14 we see that each diagonal first order block is of the form $\mathbf{1}_{d_{\beta,i}} \otimes Q_{\beta,i}$ with $Q_{\beta,i} \in \mathrm{Her}_{\beta,m_i}$. Since the eigenvalues of $\mathbf{1}_{d_{\beta,i}} \otimes Q_{\beta,i}$ are the same as the eigenvalues of $Q_{\beta,i}$ (with multiplicities), $\mathbf{1}_{d_{\beta,i}} \otimes Q_{\beta,i}$ is positive definite if and only if $Q_{\beta,i}$ is positive definite.

This proves that the matrix H representing a ρ–invariant scalar product is block–diagonal, with each block in $\mathbf{1}_{d_{\beta,i}} \otimes \mathrm{Her}^+_{\beta,m_i}$, which proves the statement. □

Corollary 7.16. *Let $\rho : G \to \mathrm{GL}(V)$ be a continuous representation of a compact group G, and write*

$$V = \bigoplus_{k=1}^{r} V_k,$$

as in Corollary 7.9, with each V_k *invariant and* $\rho_k := \rho|_{V_k}$ *irreducible. If the representations* (V_k, ρ_k) *are pairwise non–isomorphic, the set of* ρ*–invariant scalar products is isomorphic to a polyhedral cone* $\mathcal{S}^\rho \simeq (0, \infty)^r$*. Moreover, if the above decomposition into irreducible subspaces is orthogonal with respect to some* ρ*–invariant scalar product, then it is orthogonal with respect to any* ρ*–invariant scalar product.*

Proof. The first part of the statement follows from Proposition 7.14. In this case we have $m_{\beta,i} \equiv 1$ and

$$\mathbf{1}_{d_{\beta,i}} \otimes \mathrm{Her}^+_{\beta,m_{\beta,i}} \simeq (0, \infty).$$

Given an invariant scalar product $\langle \cdot, \cdot \rangle$ for which the above decomposition is orthogonal, any other scalar product h can be written as

$$h = \sum_{k=1}^{r} \lambda_k \langle \cdot, \cdot \rangle|_{V_k},$$

for some $(\lambda_1, \dots, \lambda_r) \in (0, \infty)^r$. In particular the decomposition is orthogonal also for h. $\square$

Proof of Proposition 7.14. If $\rho : G \to \mathrm{GL}(V)$ is a representation, then both $(rW)_+$ and $(rW)_-$ are invariant subspaces for $rc\rho$ and the above isomorphism $V \simeq (rW)_+$ is an isomorphism of representations.

Observe first that the set $\mathrm{End}(V)^\rho$ is a division algebra: every nonzero endomorphism $L \in \mathrm{End}(V)^\rho$ is invertible by Lemma 7.5, and $L_1, L_2 \in \mathrm{End}(V)^\rho$ implies $L_1 L_2 \in \mathrm{End}(V)^\rho$. Therefore $\mathrm{End}(V)^\rho$, as a ring, can be either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

To start with, let us assume that $c\rho$ is irreducible. Then $\mathrm{End}^{\mathbb{C}}(cV)^{c\rho} \simeq \mathbb{C}$ by Lemma 7.12, and $\mathrm{End}(V)^\rho$ consists of the real multiples of the identity by Lemma 7.12.

Let us assume now that $c\rho$ is not irreducible. Since G is compact, we get a decomposition into irreducible complex subspaces:

$$cV \simeq W_1 \oplus \cdots \oplus W_\ell.$$

Using Section 7.2, we see that $rcV \simeq (rcV)_+ \oplus (rcV)_- \simeq V \oplus V$. Applying this to Section 7.2, gives

$$V \oplus V \simeq rcV \simeq rW_1 \oplus \cdots \oplus rW_\ell.$$

In particular $\ell = 2$ and $rW_1 \simeq rW_2 \simeq V$. This proves also that V carries a complex structure $I : V \to V$, being isomorphic to rW_1. It will be convenient to set $W := W_1$ and $\overline{W} := \tau W$. Since $c\rho$ comes from a real representation, $\overline{W}$ is invariant and $W \cap \overline{W} = \{0\}$. In particular:

$$cV \simeq W \oplus \overline{W}.$$

The complex conjugation $\tau : cV \to cV$, in this decomposition, acts as $\tau(w_1, w_2) = (\overline{w_2}, \overline{w_1})$. Two possibilities now can happen: either W and $\overline{W}$ are not isomorphic (as complex representations) or they are isomorphic.

If W is not isomorphic to $\overline{W}$ then, by Lemma 7.12 we have $\mathrm{End}(cV)^{c\rho} \simeq \mathbb{C} \oplus \mathbb{C}$. Writing $cV \simeq W \oplus \overline{W}$, we see that an endomorphism $L : cV \to cV$ commutes with $c\rho$ and is real if and only if it has the form

$$L = \begin{pmatrix} \lambda \mathbf{1} & 0 \\ 0 & \overline{\lambda}\mathbf{1} \end{pmatrix}, \quad \lambda \in \mathbb{C}.$$

Writing $\lambda = a_1 + ia_2$, such an endomorphism acts on

$$V \simeq (rcV)_+ = \{(w, \overline{w}) \,|\, w \in W\}$$

as $L = a_1\mathbf{1} + a_2 I$, as claimed.

If instead $W \simeq \overline{W}$ (as complex representations), let h be a Hermitian invariant form on W. Then the correspondence $w \mapsto h(\cdot, w)$ gives an isomorphism of representations

$$W \simeq W^*.$$

Such isomorphism gives an element $B \in \mathrm{Hom}^{\mathbb{C}}(W, W^*)^{c\rho}$ and we define a real isomorphism $A : rW \to rW$ by

$$h(Aw_1, w_2) = \mathrm{Re}(B(w_1, w_2)) \quad \forall w_1, w_2 \in W.$$

Notice that the space $\mathrm{Hom}^{\mathbb{C}}(W, W^*)$ decomposes into the invariant subspaces of symmetric and skewsymmetric homomorphisms; by Schur's Lemma $\mathrm{Hom}^{\mathbb{C}}(W, W^*)^{c\rho}$ is one dimensional, thus either $B^* = B$ or $B^* = -B$.

Using the polar decomposition, we can write $A = PJ$ with P a positive definite real endomorphism and J an orthogonal real endomorphism. We observe the following facts: (1) A is anti–linear (since $h(Aiw_1, w_2) = \mathrm{Re}(B(iw_1, w_2)) = \mathrm{Re}(B(w_1, iw_2)) = h(iAw_1, -w_2) = -h(iAw_1, w_2)$); (2) $A^T = \pm A$ (since $h(Aw_1, w_2) = \mathrm{Re}(B(w_1, w_2)) = \pm\mathrm{Re}(B(w_1, w_2)) = \pm h(w_1, A^T w_2)$, where we have used the fact that B is either symmetric or skew–symmetric; (3) P is $\mathbb{C}$–linear (since $P = (AA^T)^{\frac{1}{2}}$, it is a power series in AA^T, which is $\mathbb{C}$–linear); (4) J is anti–linear (since A is anti–linear and P is $\mathbb{C}$–linear). These facts together imply that $J^2 = \pm\mathbf{1}$: in fact $A^T = \pm A = \pm J^T P^T = \pm J^{-1} P = J^T P^T = J^T P$, which implies $J^T = \pm J^{-1}$. Moreover, since rW is irreducible, $J^2 = -\mathbf{1}$, otherwise its (-1)–eigenspace would be a proper ($J \neq \mathbf{1}$ since it is anti–linear) invariant subspace.

Summarizing: we have proved that there exists a real endomorphism $J : rW \to rW$ which is anti–linear and such that $J^2 = -\mathbf{1}$. This gives a quaternionic structure (I, J, K) on $V \simeq rW$, where I is the multiplication by i, and $K := IJ$.

Let us denote now by $\phi : W \oplus W \to W \oplus \overline{W}$ the isomorphism

$$\phi(w_1, w_2) := (w_1, \overline{Jw_2}).$$

By Lemma 7.12, we have $\mathrm{End}^{\mathbb{C}}(cV)^{c\rho} \simeq \mathrm{GL}_2(\mathbb{C})$ and, if $L \in \mathrm{End}^{\mathbb{C}}(cV)^{c\rho}$, we can write

$$\phi^{-1} L \phi = \begin{pmatrix} a_{1,1}\mathbf{1} & a_{1,2}\mathbf{1} \\ a_{2,1}\mathbf{1} & a_{2,2}\mathbf{1} \end{pmatrix}, \quad a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2} \in \mathbb{C}.$$

Observe also that for every $(w_1, w_2) \in W \oplus W$,

$$\phi^{-1} \tau \phi(w_1, w_2) = (Jw_2, -Jw_1).$$

The set of real endomoprhisms commuting with ρ can be identified with the set of complex endomoprhisms L commuting with $c\rho$ and with τ. In the coordinates given by ϕ, this set can be identified with the set of matrices $\phi^{-1}L\phi$, as in the right hand side of Section 7.2, such that

$$\phi^{-1} L \phi \phi^{-1} \tau \phi^{-1} = \phi^{-1} \tau \phi^{-1} \phi^{-1} L \phi.$$

An explicit computation gives:

$$\phi^{-1} L \phi \phi^{-1} \tau \phi^{-1}(w_1, w_2) = (a_{1,1} Jw_2 - a_{1,2} Jw_1, a_{2,1} Jw_2 - a_{2,2} Jw_1),$$

and, using the fact that J is antilinear,

$$\phi^{-1} \tau \phi^{-1} \phi^{-1} L \phi(w_1, w_2) = (\overline{a_{2,1}} Jw_1 + \overline{a_{2,2}} Jw_2, -\overline{a_{1,1}} Jw_1 - \overline{a_{1,2}} Jw_2).$$

In order for $\phi^{-1}L\phi$ to represent an element in $\mathrm{End}(V)^\rho$ we must therefore have

$$a_{2,1} = -\overline{a_{1,2}} \quad \text{and} \quad a_{2,2} = \overline{a_{1,1}}.$$

This means that $\phi^{-1}L\phi$ must be of the form

$$\phi^{-1} L \phi = A \otimes \mathbf{1} \quad \text{with} \quad A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}.$$

The set of matrices $A \in \mathrm{GL}_2(\mathbb{C})$ as above is isomorphic, as a ring, to the quaternions and therefore $\mathrm{End}(V)^\rho \simeq \mathbb{H}$.

Let us finally identify the set of quaternions as precise elements in $\mathrm{End}(V)^\rho$. Given a quaternion A, writing $\alpha = a_1 + ia_2$ and $\beta = b_1 + ib_2$ we can decompose

$$A = a_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + a_2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + b_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

The endomorphism $L = A \otimes \mathbf{1}$ acts on $V \simeq (rcV)_+ = \{(w, \overline{w}) \mid w \in W\}$. as $L = a_1\mathbf{1} + a_2 I + b_1 J + b_2 K$, as claimed. $\square$

8 Invariant inner products on the space of polynomials

The goal of this lecture is to classify all Gaussian distributions on the space of homogeneous polynomials od degree d in $n+1$ many variables that are invariant under change of variables by the orthogonal group $O(n+1)$ or the unitary group $U(n+1)$.

Definition 8.1. Let $f \in \mathbb{R}[x_0,\ldots,x_n]_{(d)}$ be a random homogeneous polnomial of degre d in $n+1$ variables. We call the distribution of f an invariant distribution, if $f \circ U = f$ for all $U \in O(n+1)$. Similarly, we say that $f \in \mathbb{C}[x_0,\ldots,x_n]_{(d)}$ has an invariant distribution, if $f \circ U = f$ for all $U \in U(n+1)$.

We saw already in Lemma 1.11 that the Kostlan distribution is an invariant distribution. We will see that over the complex numbers the Kostlan distribution is the only invariant distribution, but that over the Reals there is a whole family of invariant distribution.

Recall from Section 1.3 that there is a one-to-one correspondence between centered Gaussian distributions and inner products in a real vector space. Therefore, for classifying invariant distributions in $\mathbb{R}[x_0,\ldots,x_n]_{(d)}$ it is enough to classify all invariant inner products.

8.1 Complex invariant distributions

In this section let us write

$$\mathcal{P}_{n,d}^{\mathbb{C}} := \mathbb{C}[x_0,\ldots,x_n]_{(d)}.$$

Theorem 7.13 implies that for classifying the invariant inner products on $\mathcal{P}_{n,d}^{\mathbb{C}}$, which are invariant under unitary change of variables, we have to study the representation $\rho^{\mathbb{C}} : U(n+1) \to \mathrm{GL}(\mathcal{P}_{n,d}^{\mathbb{C}})$ defined by

$$\rho^{\mathbb{C}}(U)f = f \circ U^T \text{ for } U \in U(n+1).$$

The reason why we put the transpose here is that we want $\rho^{\mathbb{C}}$ to be a group homomorphism; i.e., $\rho^{\mathbb{C}}(U_1U_2) = \rho^{\mathbb{C}}(U_1)\rho^{\mathbb{C}}(U_2)$.

The main theorem of this section is as follows.

Theorem 8.2. *The representation* $\rho^{\mathbb{C}} : U(n+1) \to \mathrm{GL}(\mathcal{P}^{\mathbb{C}}_{n,d})$ *is irreducible.*

Together with Theorem 7.13 this theorem implies that on $\mathcal{P}^{\mathbb{C}}_{n,d}$ there is (up to scaling) a unique inner product which is invariant by unitary change of variables. Consequently, there is a unique $U(n+1)$-invariant Gaussian distribution on $\mathcal{P}^{\mathbb{C}}_{n,d}$ up to scaling. Since the complex Kostlan distribution is $U(n+1)$-invariant, it must be this one. In other words, the only inner products on $\mathcal{P}^{\mathbb{C}}_{n,d}$, which are $U(n+1)$-invariant, are multiples of

$$\langle \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha}^{\frac{1}{2}} x^\alpha, \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha}^{\frac{1}{2}} x^\alpha \rangle_{\mathrm{BW}} = \sum_{|\alpha|=d} f_\alpha \overline{g_\alpha},$$

called the *Bombieri–Weyl* hermitian product.

For the proof of Theorem 8.2 we need an auxiliary lemma. The intersection of $U(n+1)$ with the space of diagonal matrices is called the *torus*, denoted

$$T(n+1) = \{\mathrm{diag}(\lambda) \mid \lambda = (\lambda_0, \ldots, \lambda_n) \in \mathbb{C}^{n+1}, |\lambda_i| = 1\} \subset U(n+1).$$

Lemma 8.3. *For every* $\beta \in \mathbb{N}^{n+1}$ *there exists* $\mathrm{diag}(\lambda) \in T(n+1)$ *such that* $\lambda^\alpha \neq \lambda^\beta$ *for all* $\alpha \in \mathbb{N}^{n+1}$ *with* $\alpha \neq \beta$.

Proof. Let us enumerate the exponent vectors $\alpha \in \mathbb{N}^{n+1}$ with $\alpha \neq \beta$ by $\alpha^{(i)}$, $1 \leq i \leq \binom{n+d}{d} - 1$. Notice that $\lambda^\alpha = \lambda^\beta$, if and only if $\lambda^{\alpha-\beta} = 1$. We prove the lemma by induction on the supscripts of the $\alpha^{(i)}$. Choose any λ and let $t_i := \lambda^{\alpha^{(i)}-\beta}$. For the induction step we can assume that $t_1, \ldots, t_{k-1} \neq 1$. If $t_k \neq 1$, we are done. Otherwise, without restriction, we can assume that $\alpha_1^{(k)} - \beta_1 \neq 0$. Then, there are only finitely many $\mu \in \mathbb{C}$ such that $\mu^{\alpha_1^{(i)}-\beta_1} = t_i^{-1}$ for $1 \leq i \leq k$. Choosing any other $\mu \in \mathbb{C}$, $|\mu| = 1$, and setting $\lambda' = \mathrm{diag}(\mu, 1, \ldots, 1)\lambda$ we have that $(\lambda')^{\alpha^{(i)}-\beta} \neq 1$ for $1 \leq i \leq k$. $\square$

Now, we prove Theorem 8.2. The idea of the next proof is taken from [39].

Proof of Theorem 8.2. Let $0 \neq W \subset \mathcal{P}^{\mathbb{C}}_{n,d}$ be a $U(n+1)$-invariant subspace. The proof consists of two steps. First, we show that if

$$f = \sum_{|\alpha|=d} c_\alpha \binom{d}{\alpha}^{\frac{1}{2}} x^\alpha \in W,$$

then $x^\alpha \in W$ for every α with $c_\alpha \neq 0$.

To see this, let $f \in W$. If $f = c_\alpha x^\alpha$, then $x^\alpha \in W$. Otherwise, there is another nonzero coefficient in f, say $\beta \neq \alpha$ with $c_\beta \neq 0$. We act on f by diagonal matrices. Let $D = \operatorname{diag}(\lambda_0, \dots, \lambda_n) \in T(n+1)$ be such that $\lambda^\alpha - \lambda^\beta \neq 0$ for all $\alpha \neq \beta$. Such a matrix exists by Lemma 8.3. Let us define

$$g := \rho(D)f - \lambda^\beta f \in W.$$

We have

$$g = \sum_{\alpha \neq \beta} c_\alpha (\lambda^\alpha - \lambda^\beta) x^\alpha,$$

so the number of nonzero coefficients of g is exactly one less than the number of coefficients of f. This shows that we can reduce the number of terms in f by one. Repeating this argument, we see that we can get any monomial in f with nonzero coefficient as an element in W.

Now comes the second step. The argument above shows that there is an α with $x^\alpha \in W$. Let now

$$U = \begin{bmatrix} \frac{1}{\sqrt{n+1}} & \cdots & \frac{1}{\sqrt{n+1}} \\ & H & \end{bmatrix} \in U(n+1)$$

with $H \in \mathbb{C}^{(n-1)\times n}$. Then, we have

$$\rho^{\mathbb{C}}(U)x^\alpha = (U^T x)^\alpha = \prod_{i=0}^{n} \left(\tfrac{1}{\sqrt{n+1}}\, x_0 + h_{i,1}x_1 + \cdots + h_{i,n}x_n\right) = \frac{1}{\sqrt{n+1}^{\,d}}\, x_0^d + g(x),$$

such that x_0^d does not appear in $g(x)$. Therefore, the monomial x_0^d has nonzero coefficient in $\rho^{\mathbb{C}}(U)x^\alpha \in W$, which implies that $x_0^d \in W$. But then

$$\rho(U)x_0^d = \tfrac{1}{\sqrt{n+1}^{\,d}}(x_0 + \cdots + x_n) \in W,$$

which has nonzero coefficient for all monomials. Hence, all monomials are in W and so $W = \mathcal{P}^{\mathbb{C}}_{n,d}$. Hence, $\mathcal{P}^{\mathbb{C}}_{n,d}$ is irreducible. $\square$

8.2 Real invariant distributions

In the last section we have seen that the representation of the unitary group in complex homogeneous polynomial of degree d is irreducible. Over the real numbers the situation is different. Let us denote

$$\mathcal{P}_{n,d} := \mathbb{R}[x_0, \ldots, x_n]_{(d)}.$$

The representation $\rho : O(n+1) \to \mathrm{GL}(\mathcal{P}_{n,d})$ defined by orthogonal changes of variables $\rho(U)f = f \circ U^T$ is not irreducible for $d \geq 2$! To see this, consider the subspace $W_{n,d} := \|x\|^2 \mathcal{P}_{n,d-2}$, where $\|x\|^2 = x_0^2 + \cdots + x_n^2$ denotes the polynomial given by sum-of-squares of the variables. For all $f \in \mathcal{P}_{n,d-2}$ and $U \in O(n+1)$ we have $\rho(U)(\|x\|^2 \cdot f) = \|x\|^2 \cdot (f \circ U^T) \in W_{n,d}$. Therefore, $W_{n,d}$ is an invariant subspace and so $\mathcal{P}_{n,d}$ can't be irreducible. Over the complex numbers this argument fails, because there is no $U(n+1)$-invariant complex polynomial – the squared hermitian norm $\overline{x}^T x$ is not a polynomial, because complex conjugation is not an algebraic map.

Hence, by Theorem 7.15 there is not a unique invariant inner product, and, consequently, no unique invariant Gaussian probability distribution on $\mathcal{P}_{n,d}$. One goal of this section is to recall the work by Kostlan [8, 22, 23] who classified all such invariant inner products. For this, we compute a decomposition of $\mathcal{P}_{n,d}$ into a direct sum of irreducible representations. We will do this in the next section. Before, however, we will prove the following theorem.

Theorem 8.4. *There is a unique $O(n+1)$-invariant inner product on $\mathcal{P}_{n,d}$ up to scaling for which monomials are orthogonal.*

Recall that the Kostlan inner product is

$$\langle \sum_{|\alpha|=d} f_\alpha \binom{d}{\alpha}^{\frac{1}{2}} x^\alpha, \sum_{|\alpha|=d} g_\alpha \binom{d}{\alpha}^{\frac{1}{2}} x^\alpha \rangle = \sum_{|\alpha|=d} f_\alpha g_\alpha.$$

Monomials are orthogonal with respect to this product, so it must be the one from Theorem 8.4.

For the proof of Theorem 8.4 we need to make some general observations. Let $0 \neq V \subset \mathcal{P}_{n,d}$ be a subrepresentation and $\langle\, , \,\rangle$ be an invariant inner product on V. This induces an isomorphism of dual spaces $V \to V^*$, $g \mapsto (f \mapsto \langle f, g\rangle)$. For a fixed $z \in S^n$ consider the evaluation map

$$\mathrm{eval}_z : V \to \mathbb{R}, f \mapsto f(z).$$

This is a linear map, hence an element in V^*. Let us denote its dual by τ_z, so that

$$\langle f, \tau_z \rangle = f(z) \text{ for all } f \in V.$$

Then, we have $\langle f, \rho(U)\tau_z \rangle = \langle \rho(U^T)f, \tau_z\rangle = f(Uz) = \langle f, \tau_{Uz}\rangle$ for all $U \in O(n+1)$ and $f \in V$. Therefore,

$$\rho(U)\tau_z = \tau_{Uz}.$$

We remark that $\tau_z \neq 0$, because otherwise $\tau_{Uz} = \rho(U)\tau_z = 0$ for every U, so that $V = 0$, which is a contradiction to our assumption $V \neq 0$.. Furthermore, the linear span of $\{\tau_x \mid x \in S^n\}$ is V. To see this, suppose that linear span is $W \subsetneq V$. Then, for $0 \neq f \in W^\perp$ we have $0 = \langle f, \tau_x\rangle = f(x)$ for all $x \in S^n$. This contradicts the assumption $f \neq 0$.

We use this construction to prove Theorem 8.4.

Proof of Theorem 8.4. For the Kostlan inner product monomials are orthogonal. This shows existence. For uniqueness we let $\langle\ ,\ \rangle$ be an $O(n+1)$-invariant inner product on $\mathcal{P}_{n,d}$ such that monomials are orthogonal. As in 8.2 for $z \in S^n$ let τ_z be the dual of the evaluation map at z relative to $\langle\ ,\ \rangle$. Since the τ_z span $\mathcal{P}_{n,d}$, the inner product is completely determined by the values of $\langle \tau_x, \tau_y\rangle$ for $x, y \in S^n$. Let $e_0 = (1, 0, \ldots, 0) \in S^n$. Then, for every multiindex α we have

$$\langle x^\alpha, \tau_{e_0}\rangle = (e_0)^\alpha = \begin{cases} 1, & \alpha = (d, 0, \ldots, 0) \\ 0, & \text{otherwise} \end{cases}.$$

Since monomials are orthogonal this shows that $\tau_{e_0} = \lambda x_0^d$ for some $\lambda \neq 0$. This implies that for every $u \in S^n$ we have $\langle \tau_{e_0}, \tau_u\rangle = \lambda\langle e_0, u\rangle^d$. Let $U \in O(n+1)$ be such that $Ux = e_0$ and $Uy = u$. Then, by invariance, we have

$$\langle \tau_x, \tau_y\rangle = \langle \rho(U)\tau_x, \rho(U)\tau_y\rangle = \langle \tau_{Ux}, \tau_{Uy}\rangle = \lambda\langle e_0, u\rangle^d = \lambda\langle x, y\rangle^d,$$

where for the second equality we have used Section 8.2. This shows that $\langle\ ,\ \rangle$ is unique up to scaling. □

8.2.1 Spherical harmonics

Consider the differential operator

$$\nabla^2 = \frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

We interpret ∇^2 as a linear map $\mathcal{P}_{n,d} \to \mathcal{P}_{n,d-2}$ defined by $\nabla^2 f = \sum_{i=0}^n \frac{\partial}{\partial x_0} f$. Solutions to $\nabla^2 f = 0$ are called *spherical harmonic polynomial.* We denote the vector space of all spherical harmonic polynomials of a fixed degree by

$$H_{n,d} := \{f \in \mathcal{P}_{n,d} \mid \nabla^2 f = 0\}.$$

The main theorem of this section is the following result.

Theorem 8.5. *For all n, d:*

1. *(1) $H_{n,d}$ is an irreducible representation of $O(n+1)$.*
2. *(2) We have the following decomposition of $\mathcal{P}_{n,d}$ into irreducible representations:*

$$\mathcal{P}_{n,d} = \bigoplus_{k=0}^{\lfloor \frac{d}{2} \rfloor} \|x\|^{2k} H_{n,d-2k}.$$

3. *(3) The decomposition above is orthogonal for any inner product on $\mathcal{P}_{n,d}$, which is invariant by the $O(n+1)$ action.*

Before we prove this theorem, let us discuss its implications. Let $\langle\ ,\ \rangle$ an invariant inner product on $\mathcal{P}_{n,d}$ and let

$$f = f_0 + \cdots + f_{\lfloor \frac{d}{2} \rfloor} \text{ and } g = g_0 + \cdots + g_{\lfloor \frac{d}{2} \rfloor}$$

with $f_k, g_k \in \|x\|^{2k} H_{n,d-2k}$. Then, the third item in Theorem 8.5 implies that the inner product between f and g is

$$\langle f, g \rangle = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \langle f_k, g_k \rangle.$$

However, on every component $\|x\|^{2k} H_{n,d-2k}$ there is a unique invariant inner product up to scaling by Theorem 7.15. After normalization, let this unique inner product be $\langle\ ,\ \rangle_k$. There exists positive numbers

$$\lambda_0, \ldots, \lambda_{\lfloor \frac{d}{2} \rfloor} > 0,$$

such that

$$\langle f, g \rangle = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \lambda_k \, \langle f_k, g_k \rangle_k.$$

Hence, there is a $(\lfloor \frac{d}{2} \rfloor + 1)$-dimensional family of invariant inner products on $\mathcal{P}_{n,d}$ and they are completely determined by the choice of the numbers in Subsection 8.2.1.

For the proof of Theorem 8.5 we need a few auxiliary results.

Lemma 8.6. $\nabla^2 : \mathcal{P}_{n,d} \to \mathcal{P}_{n,d-2}$ *is a* $O(n+1)$*-homomorphism.*

Proof. Linearity of ∇^2 follows from the linearity of derivatives. Let $f \in \mathcal{P}_{n,d}$. The *Hessian* of f is the matrix $H(f)(x) = [\frac{\partial^2}{\partial x_i \partial x_j} f]_{0\leq i,j\leq n}$. For every $U \in O(n+1)$ we have $H(f \circ U^T)(x) = U\, H(f)(U^T x)\, U^T$. Since $\nabla^2 f(x) = \mathrm{Trace}(H(f)(x))$, we have

$$\begin{aligned}\nabla^2(\rho(U)f) = \nabla^2(f \circ U^T) &= \mathrm{Trace}(U\, H(f)(U^T x)\, U^T)\\ &= \mathrm{Trace}(H(f)(U^T x))\\ &= (\nabla^2 f) \circ U^T\\ &= \rho(U)(\nabla^2 f).\end{aligned}$$

which shows that ∇^2 commutes with the $O(n+1)$-action. □

This lemma in combination with Lemma 7.4 already shows that $H_{n,d} = \ker \nabla^2$ is a representation, which gives part of the first item in Theorem 8.5. for showing that $H_{n,d}$ is irreducible we have to do more work. The next lemma classifies $H_{n,d}$ as eigenspaces of a particular operator.

Lemma 8.7. *For every* $f \in \|x\|^{2k} H_{n,d-2k}$ *we have*

$$\|x\|^2 \nabla^2 f = 2k(n+2d-2k-1)f;$$

i.e., the $\|x\|^2 H_{n,d-2k}$ *are eigenspaces of the operator* $\|x\|^2\nabla^2 : \mathcal{P}_{n,d} \to \mathcal{P}_{n,d}$ *for different eigenvalues.*

Proof. Let $g \in H_{n,d-2k}$. Then, $\nabla^2 g = 0$. This implies for $f = \|x\|^{2k} \cdot g$ that

$$\nabla^2 f = (\nabla^2 \|x\|^{2k}) \cdot g + 2(\nabla \|x\|^{2k})^T \nabla g,$$

where ∇ denotes the gradient. The gradient of $\|x\|^{2k}$ is $\nabla\|x\|^{2k} = 2k\|x\|^{2k-2}x$. The polynomial g is homogeneous of degree $d-2k$, which implies that $x^T g = (d-2k)g$. We get

$$(\nabla\|x\|^{2k})^T\, \nabla g = 2k(d-2k)\|x\|^{2k-2} \cdot g.$$

Furthermore, we have

$$\nabla^2\|x\|^{2k} = 2k(n+2k-1)\|x\|^{2k-2}.$$

Altogether:

$$\|x\|^{2k}\nabla^2 f = 2k(n+2k-1)f + 4k(d-2k)f = 2k(n+2d-2k-1)f.$$

This finishes the proof. □

We need one last result for the proof of Theorem 8.5. The idea for the proof the next lemma is from [37].

Lemma 8.8. *Let $z \in S^n$. Up to scaling, there is a unique $0 \neq f \in H_{n,d}$ such that $\rho(U)f = f$ for all $U \in O(n+1)_z$.*

Proof. We chose an invariant inner product on $H_{n,d}$. As in Section 8.2 this defines the polynomial τ_z, which is the dual of the evaluation at map z. By Section 8.2 we have for every $U \in O(n+1)_z$ that

$$\rho(U)\tau_z = \tau_{Uz} = \tau_z.$$

This shows existence. Next, we show uniqueness. Without restriction we can assume that $z = e_0 = (1, 0, \dots, 0)$. Suppose that $f \in H_{n,d}$ satisfies $\rho(U)f = f$ for all $U \in O(n+1)_z$. Let us write

$$f(x) = \sum_{i=0}^{d} f_i(\hat{x})\, x_0^{d-i},$$

where f_i is a homogeneous polnymomial in $\hat{x} = (x_1, \dots, x_n)$ of degree i. We have

$$O(n+1)_z = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} \mid U \in O(n). \right\}.$$

Let us act on f by an element of $O(n+1)_z$:

$$f(x) = (f \circ \begin{bmatrix} 1 & 0 \\ 0 & U^T \end{bmatrix})(x) = \sum_{i=0}^{d} f_i(U^T\hat{x})\, x_0^{d-i}.$$

This implies that $f_i(\hat{x}) = f_i(U^T\hat{x})$ for all $0 \leq i \leq d$ and all $U \in O(n)$. Consequently, there exists coefficients c_k such that

$$f_i = \begin{cases} c_k\, \|\hat{x}\|^{2k}, & \text{if } i = 2k \text{ is even} \\ 0, & \text{otherwise} \end{cases}.$$

We get $f(x) = \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} c_k \, \|\hat{x}\|^{2k} \, x_0^{d-2k}$. Furthermore,

$$\begin{aligned} 0 = \nabla^2 f &= \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} c_k \, \nabla^2 \|\hat{x}\|^{2k} \, x_0^{d-2k} \\ &= \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} c_k \, \hat{\nabla}^2 (\|\hat{x}\|^{2k}) \, x_0^{d-2k} + c_k \, \|\hat{x}\|^{2k} (d-2k) x_0^{d-2k-2}, \end{aligned}$$

where $\hat{\nabla}^2 := \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. This implies $c_k \, 2k(n+2k-3) + c_{k-1}(d-2k+2) = 0$. where we have used from Subsection 8.2.1 that $\nabla^2 \|x\|^{2k-2} = 2k(n+2k-3)\|x\|^{2k-2}$. This shows that the c_k satisfy a recurrence relation and so they are uniquely determined up to scaling. Therefore, f is unique up to scaling. □

Now, we can prove Theorem 8.5.

Proof of Theorem 8.5. First, ∇^2 is a $O(n+1)$-homomorphism by Lemma 8.6. Lemma 7.4 implies that $H_{n,d} = \ker \Delta^2 f$ is a representation for the orthogonal group. Consequently, $\|x\|^{2k} H_{n,d-2k}$ is a representation for every k. Moreover, $\|x\|^{2k} H_{n,d-2k}$ is irreducible, if and only if $H_{n,d-2k}$ is irreducible.

We show that $H_{n,d}$ is irreducible. We can decompose $H_{n,d} = V_1 \oplus \cdots \oplus V_p$ into a direct sum of irreducible subrepresentations. Suppose that $p \geq 2$, and let $z \in S^n$. On each V_i we choose an invariant inner product, and let $\tau_z^{(i)} \in V_i$ be the dual of the evaluation map, as in Section 8.2. By Section 8.2 we have for every $U \in O(n+1)_z$ that $\rho(U)\tau_z^{(i)} = \tau_{Uz}^{(i)} = \tau_z^{(i)}$. Hence, for every $1 \leq i \leq p$ there is a nonzero element that is invariant under the action by $O(n+1)_z$. By Lemma 7.10 we have $V_i \cap V_j \neq 0$ for $i \neq j$. This implies that we can find at least two linearly independent $O(n+1)_z$-invariant elements in $H_{d,k}$. This contradicts Lemma 8.8. Therefore, $p = 1$ and $H_{n,d}$ is irreducible. This proves the first item in Theorem 8.5.

For the second item, we observe that, since $H_{n,d}$ is the kernel of the linear map $\nabla^2 : \mathcal{P}_{n,d} \to \mathcal{P}_{n,d-2}$ we have $\dim H_{n,d} \geq \dim \mathcal{P}_{n,d} - \dim \mathcal{P}_{n,d-2}$. Furthermore, the spaces $\|x\|^{2k} H_{n,d-2k}$ are eigenspaces of the operator $\|x\|^2 \nabla^2$ for different eigenvalues, so that $(\|x\|^{2k} H_{n,d-2k}) \cap (\|x\|^{2\ell} H_{n,d-2\ell}) = 0$ for $k \neq \ell$. Moreover,

$\dim \|x\|^{2k} H_{n,d-2k} = \dim H_{n,d-2k}$. All this implies

$$\begin{aligned}\dim \mathcal{P}_{n,d} \geq \dim \bigoplus_{k=0}^{\lfloor \frac{d}{2} \rfloor} \|x\|^{2k} H_{n,d-2k} &= \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \dim \|x\|^{2k} H_{n,d-2k} \\ &\geq \sum_{k=0}^{\lfloor \frac{d}{2} \rfloor} \dim \mathcal{P}_{n,d} - \dim \mathcal{P}_{n,d-2} \\ &= \dim \mathcal{P}_{n,d}.\end{aligned}$$

This shows $\mathcal{P}_{n,d} = \bigoplus_{k=0}^{\lfloor \frac{d}{2} \rfloor} \|x\|^{2k} H_{n,d-2k}$, and, in particular,

$$\dim H_{n,d} = \dim \mathcal{P}_{n,d} - \dim \mathcal{P}_{n,d-2}.$$

For the third item we define $d_{n,d} := \dim \mathcal{P}_{n,d}$. Then, $d_{n,d} = \sum_{k=0}^{d} d_{n-1,l}$, because $\mathcal{P}_{n,d}$ is isomorphic to the vector space of polynomials in n variables of degree at most d by setting $x_0 = 1$. This shows

$$\dim H_{n,d} = d_{n-1,d} + d_{n-1,d-1} > d_{n-1,d-2} + d_{n-1,d-3} = \dim H_{n,d-2}.$$

So the spaces $\|x\|^{2k} H_{n,d-2k}$ are all of pairwise different dimension, hence pairwise non-isomorphic. Lemma Lemma 7.10 implies that they are pairwise orthogonal for ony $O(n+1)$-invariant inner product on $\mathcal{P}_{n,d}$. □

9 Discriminants

In this section we start focusing on discriminants, real and complex. A discriminant, loosely speaking, is a subsets of a space of geometric objects (parametrized by a smooth manifold), consisting of those objects with singularities of some chosen type. Examples of discriminant, in this sense, are: the set of matrices with zero or multiple eigenvalues, the set of functions whose critical points are degenerate, the set of degenerate geometric complexes, the set of singular hypersurfaces of given degree (in real or complex projective space). When the set of objects under consideration is finite dimensional, discriminants are usually singular algebraic varieties, whose stratifications correspond to the classification of degenerations of the corresponding objects. In the case the parameter space P for the geometric objects is a finite dimensional vector space and the corresponding discriminant D is an algebraic set (real or complex), Alexander duality relates the topology of $P \setminus D$ with the topology of the one–point compactification of D.

9.1 The main theorem of elimination theory

We begin with the following fundamental result, which is again related to quantifier elimination. Recall from Theorem 3.11 that the image under a projection $\pi : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ of an algebraic set $Z \subseteq \mathbb{C}^{n+m}$ is constructible, but in general it is not algebraic. The next result says that, in order to have a closed map, in the Zariski topology, it is enough to take the projective closure of the $\mathbb{C}^n$–factor.

In the following we denote by

$$\mathscr{P}^{\mathbb{C}}_{n,d} := \mathbb{C}[z_0, \ldots, z_n]_{(d)},$$

the space of complex homogeneous polynomials of degree d, and by $N+1 := \binom{n+d}{d}$ its complex dimension.

Theorem 9.1. *The projection on the second factor $\pi_2 : \mathbb{C}\mathrm{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$ is closed in the Zariski topology.*

Proof. Denote by $z = (z_0, \ldots, z_n)$ and by $y = (y_1, \ldots, y_m)$ and let $Z = Z(p_1, \ldots, p_s) \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}^m$ be an algebraic set defined by polynomials $p_1, \ldots, p_s \in \mathbb{C}[z, y]$, with each p_i homogeneous of degree d_i in the z–variables. We need to prove that $\pi_2(Z)$ is Zariski closed or, equivalently, that $\mathbb{C}^m \setminus \pi_2(Z)$ is Zariski open. To this end, we define

$$I(y) := (p_1(z,y), \dots, p_s(z,y)) \subset \mathbb{C}[z],$$

the ideal generated by the defining polynomials with the y–variables "frozen".

The point $y \in \mathbb{C}^m$ *does not* belong to $\pi_2(Z)$ if and only if the polynomials $p_1(z,y), \dots, p_s(z,y)$ do not have a common zero in $\mathbb{C}\mathrm{P}^n$. By Hilbert's Nullstellensatz, this is equivalent to

$$\sqrt{I(y)} := \{q \in \mathbb{C}[z] \mid \exists d \geq 1 \text{ such that } q^d \in I(y)\} \supseteq (z_0, \dots, z_n).$$

For $d \geq 1$ let us also consider the set

$$A_d := \{y \in \mathbb{C}^m \mid (z_0, \dots, z_n)^d \subseteq I(y)\}.$$

Then $y \notin \pi_2(Z)$ if and only if there exists $d \geq 1$ such that $y \in A_d$ and, consequently:

$$\mathbb{C}^m \setminus \pi_2(Z) = \bigcup_{d \geq 1} A_d.$$

We will prove that each A_d is open in the Zariski topology, from which the conclusion follows.

Observe now that $(z_0, \dots, z_n)^d \subseteq I(y)$ if and only if for every $k \geq d$ we have an inclusion of the homogeneous parts $(z_0, \dots, z_n)^d \cap \mathscr{P}^{\mathbb{C}}_{n,k} \subseteq I(y) \cap \mathscr{P}^{\mathbb{C}}_{n,k}$, which in turn is verified if and only if

$$(z_0, \dots, z_n)^d \cap \mathscr{P}^{\mathbb{C}}_{n,d} \subseteq I(y) \cap \mathscr{P}^{\mathbb{C}}_{n,d}.$$

Since $(z_0, \dots, z_n)^d \cap \mathscr{P}^{\mathbb{C}}_{n,d} = \mathscr{P}^{\mathbb{C}}_{n,d}$, we have

$$A_d = \{y \in \mathbb{C}^m \mid I(y) \cap \mathscr{P}^{\mathbb{C}}_{n,d} = \mathscr{P}^{\mathbb{C}}_{n,d}\}.$$

Denote now by $L(y) : \mathscr{P}^{\mathbb{C}}_{n,d-d_1} \oplus \cdots \oplus \mathscr{P}^{\mathbb{C}}_{n,d-d_s} \to \mathscr{P}^{\mathbb{C}}_{n,d}$ the linear map:

$$(q_1(z), \dots, q_s(z)) \mapsto \sum_{j=1}^{s} q_j(z) p_j(z,y).$$

Then $I(y) \cap \mathscr{P}^{\mathbb{C}}_{n,d} = \mathscr{P}^{\mathbb{C}}_{n,d}$ if and only if the linear map $L(y)$ is surjective. Denoting by $M(y)$ the matrix representing $L(y)$ with respect to some fixed bases, we see that the entries of $M(y)$ are polynomials in y. The linear map $L(y)$ is surjective if and only if $M(y)$ has a $(N+1) \times (N+1)$ nonzero minor (recall that $N+1 = \dim_{\mathbb{C}} \mathscr{P}^{\mathbb{C}}_{n,d}$), and the set of $y \in \mathbb{C}^m$ such that this is verified is therefore Zariski open. This proves the claim. $\square$

As a corollary, one can immediately prove the following.

Corollary 9.2. *The projection on the second factor $\pi_2 : \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^m$ is closed in the Zariski topology.*

Proof. This follows immediately from Theorem 9.1. In fact, consider, for $j = 0, \dots, n$, the open set $U_j := \{[y] \in \mathbb{C}\mathrm{P}^m \,|\, y_j \neq 0\}$. If $Z \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n$ is closed, $\pi_2(Z)$ is closed if and only if for every $j = 0, \dots, n$ the set $\pi_2(Z) \cap U_j$ is closed. Since $Z \cap (\mathbb{C}\mathrm{P}^n \times U_j)$ is closed in $\mathbb{C}\mathrm{P}^n \times U_j \simeq \mathbb{C}\mathrm{P}^n \times \mathbb{C}^m$, the conclusion follows from Theorem 9.1. $\square$

The next lemma will be especially useful in the context of real algebraic geometry.

Lemma 9.3. *Let $\pi_2 : \mathbb{C}\mathrm{P}^n \times \mathbb{C}^m \to \mathbb{C}^m$ be the projection on the second factor and $Z \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}^m$ be an algebraic set defined by polynomial equations with rational coefficients. Then $\pi_2(Z)$ is also defined by equations with rational coefficients. The same result holds true for the projection on the second factor of an algebraic set $Z \subset \mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^m$.*

Proof. The result follows from inspecting the proof of Theorem 9.1. In fact, recall from that proof that $\mathbb{C}^m \setminus \pi_2(Z) = \cup_d A_d$, where

$$A_d = \{y \in \mathbb{C}^m \,|\, I(y) \cap \mathscr{P}^{\mathbb{C}}_{n,d} = \mathscr{P}^{\mathbb{C}}_{n,d}\}.$$

If we prove that the complement of each A_d is defined by equations with rational coefficients, then so is $\pi_2(Z) = \cap_d(\mathbb{C}^m \setminus A_d)$. In the proof of Theorem 9.1 we have introduced the linear map $L(y) : \mathscr{P}^{\mathbb{C}}_{n,d-d_1} \oplus \cdots \oplus \mathscr{P}^{\mathbb{C}}_{n,d-d_s} \to \mathscr{P}^{\mathbb{C}}_{n,d}$, which was defined by

$$(q_1(z), \dots, q_s(z)) \mapsto \sum_{j=1}^{s} q_j(z) p_j(z, y),$$

and showed that $I(y) \cap \mathscr{P}^{\mathbb{C}}_{n,d} = \mathscr{P}^{\mathbb{C}}_{n,d}$ if and only if the linear map $L(y)$ is surjective. Since now each $p_j(z, y)$ is with rational coefficients, choosing rational basis for the space of polynomials, we see that the matrix $M(y)$, representing $L(y)$ with respect to these bases, has entries which are polynomials in y with rational coefficients. The linear map $L(y)$ is not surjective if and only if all $(N+1) \times (N+1)$ minors of $M(y)$ vanish. This is now a polynomial condition on y with rational coefficients.

The second part of the statement is similar to Corollary 9.2 and is left to the reader. $\square$

9.2 The discriminant in the space of complex polynomials

For the study of discriminants, we will focus mainly on the case of singular hypersurfaces in projective space, but the techniques we will discuss are quite general.

Definition 9.4 (Complex discriminant)**.** We denote by $\mathscr{D}^{\mathbb{C}}_{n,d} \subset \mathscr{P}^{\mathbb{C}}_{n,d}$ the set of polynomials whose projective zero set is singular, i.e.

$$\mathscr{D}^{\mathbb{C}}_{n,d} := \left\{ p \in \mathscr{P}^{\mathbb{C}}_{n,d} \,\middle|\, \exists z \neq 0,\, p(z) = \frac{\partial p}{\partial z_0}(z) = \cdots = \frac{\partial p}{\partial z_n}(z) = 0 \right\}.$$

Since a nonzero polynomial $p \in \mathscr{D}^{\mathbb{C}}_{n,d}$ if and only if $\lambda p \in \mathscr{D}^{\mathbb{C}}_{n,d}$ for every $\lambda \in \mathbb{C}^*$, we can define the projectivization of the discriminant

$$\mathrm{D}^{\mathbb{C}}_{n,d} := \mathrm{P}\left(\mathscr{D}^{\mathbb{C}}_{n,d}\right) \subset \mathrm{P}\left(\mathscr{P}^{\mathbb{C}}_{n,d}\right) \simeq \mathbb{C}\mathrm{P}^N.$$

It follows immediately from Theorem 9.1 that both $\mathscr{D}^{\mathbb{C}}_{n,d}$ and $\mathrm{D}^{\mathbb{C}}_{n,d}$ are complex algebraic sets. In fact, consider the algebraic set

$$\mathscr{W}^{\mathbb{C}}_{n,d} := \left\{ ([z], p) \in \mathbb{C}\mathrm{P}^n \times \mathscr{P}^{\mathbb{C}}_{n,d} \,\middle|\, \frac{\partial p}{\partial z_0}(z) = \cdots = \frac{\partial p}{\partial z_n}(z) = 0 \right\}.$$

Notice that, by Euler's identity, the conditions $\frac{\partial p}{\partial z_0}(z) = \cdots = \frac{\partial p}{\partial z_n}(z) = 0$ imply $p(z) = 0$ (since the polynomial p is homogeneous). In particular, denoting by $\pi_2 : \mathbb{C}\mathrm{P}^n \times \mathscr{P}^{\mathbb{C}}_{n,d} \to \mathscr{P}^{\mathbb{C}}_{n,d}$ the projection on the second factor, we have $\pi_2(\mathscr{W}^{\mathbb{C}}_{n,d}) = \mathscr{D}^{\mathbb{C}}_{n,d}$, which is therefore Zariski closed by Theorem 9.1.

Similarly, one can consider the algebraic set

$$W^{\mathbb{C}}_{n,d} := \left\{ ([z], [p]) \in \mathbb{C}\mathrm{P}^n \times \mathrm{P}\left(\mathscr{P}^{\mathbb{C}}_{n,d}\right) \,\middle|\, \frac{\partial p}{\partial z_0}(z) = \cdots = \frac{\partial p}{\partial z_n}(z) = 0 \right\},$$

and $\mathrm{D}^{\mathbb{C}}_{n,d} = \pi_2(W^{\mathbb{C}}_{n,d})$ is Zariski closed by Corollary 9.2. Because of the projective nature of the set of singular polynomials (which is a cone), it is somehow more natural to work with this second set. The set $W^{\mathbb{C}}_{n,d}$ is called the "universal singularity".

Proposition 9.5. *The discriminant $\mathrm{D}^{\mathbb{C}}_{n,d}$ is a complex algebraic set of real dimension $2(N-1)$.*

Proof. Since $\mathrm{D}^{\mathbb{C}}_{n,d}$ is a proper complex algebraic subset, its real dimension is at most $2(N-1)$ (the tangent space at a smooth point to a complex algebraic set is a complex space). It is enough to prove that there exists a polynomial $\widetilde{p} \in \mathscr{D}_{n,d}^{\mathbb{C}}$ and a neighborhood $U_{\widetilde{p}}$ of $[\widetilde{p}] \in \mathbb{C}\mathrm{P}^N$ in $\mathrm{D}_{n,d}^{\mathbb{C}}$ such that $\dim(U_{\widetilde{p}}) = 2(N-1)$.

Consider the algebraic set $W = W_{n,d}^{\mathbb{C}}$ defined in Section 9.2, together with the projection on the first factor $\pi_1 : W \to \mathbb{C}\mathrm{P}^n$. We claim that for every $[z] \in \mathbb{C}\mathrm{P}^n$ we have $\pi_1^{-1}([z]) \simeq \mathbb{C}\mathrm{P}^{N-n-1}$. Let us consider first the special case $[z] = [1, 0, \ldots, 0]$. Writing a polynomial $p \in \mathscr{P}_{n,d}^{\mathbb{C}}$ as

$$p(z_0, \ldots, z_n) = c_0 z_0^d + z_0^{d-1}(c_1 z_1 + \ldots + c_n z_n) + \sum_{k=2}^{d} z_0^{d-k} p_k(z_1, \ldots, z_n),$$

where p_k is a homogeneous polynomial of degree k, we see that $[p] \in \pi_1^{-1}([1, 0, \ldots, 0])$ (i.e. $Z(p)$ has a singularity at $[1, 0, \ldots, 0]$) if and only if

$$c_0 = c_1 = \cdots = c_n = 0.$$

This proves that $\pi_1^{-1}([1, 0, \ldots, 0]) \simeq \mathbb{C}\mathrm{P}^{N-n-1}$ (we are imposing $n+1$ linear, independent complex conditions on a complex projective space of dimension N).

Let now $R \in \mathrm{GL}(\mathbb{C}^{n+1})$ be a linear map such that $R \cdot (1, 0, \ldots, 0) = (z_0, \ldots, z_n)$. Then the map $\rho(R) : p \mapsto p \circ R^{-1}$ defines a linear map $\rho(R) \in \mathrm{GL}(\mathbb{C}^{N+1})$ (this is the action of $\mathrm{GL}(\mathbb{C}^{n+1})$ on polynomials by change of variables). The map $\rho(R)$, being linear and invertible, induces a linear map from $\mathbb{C}\mathrm{P}^N$ to itself, and

$$\rho(R)(\pi_1^{-1}([1, 0, \ldots, 0]) = \pi_1^{-1}([z_0, \ldots, z_n]).$$

This shows that

$$\pi_1^{-1}([z]) \simeq \mathbb{C}\mathrm{P}^{N-n-1}, \quad \forall [z] \in \mathbb{C}\mathrm{P}^n.$$

Pick now $q \in \mathscr{P}_{n-1,2}^{\mathbb{C}}$ such that $Z(q) \subset \mathbb{C}\mathrm{P}^{n-1}$ is nonsingular and consider the polynomial

$$\widetilde{p}(z_0, \ldots, z_n) := z_0^{d-2} q(z_1, \ldots, z_n).$$

Since for this polynomial we have Section 9.2, then $[\widetilde{p}] \in \pi_1^{-1}([1, 0, \ldots, 0]) \subset \mathrm{D}_{n,d}^{\mathbb{C}}$. Consider now the smooth map

$$\psi : \mathrm{GL}(\mathbb{C}^{n+1}) \times \pi_1^{-1}([1, 0, \ldots, 0]) \to \mathbb{C}\mathrm{P}^N$$

defined by

$$\psi(R, [p]) := [p \circ R]\,.$$

The image of this map is contained in $\mathrm{D}_{n,d}^{\mathbb{C}}$. Moreover, it is immediate to verify that

$$\mathrm{rank}_{\mathbb{C}} \left(D_{(\mathbf{1}, [\widetilde{p}])} \psi \right) = N - 1.$$

In particular, the real rank of $D_{(R,[p])}\psi$ is $2(N-1)$ on a neighborhood B of $(\mathbf{1},[\widetilde{p}])$ and, by the Rank Theorem, possibly shrinking the neighborhood, $U_{\widetilde{p}} := \psi(B) \hookrightarrow \mathbb{C}\mathrm{P}^N$ is a smooth submanifold of real dimension $2(N-1)$, contained in $\mathrm{D}^{\mathbb{C}}_{n,d}$. This proves the statement. □

The discriminant admits a nice characterization as a dual variety. Let us recall first the general definition.

Definition 9.6 (Dual variety)**.** Let $X \subset \mathbb{C}\mathrm{P}^m$ be a complex algebraic variety and denote by $(\mathbb{C}\mathrm{P}^m)^\vee$ the dual projective space, i.e. the space of projective hyperplanes in $\mathbb{C}\mathrm{P}^m$. We denote by X^{sm} the set of smooth points of X and, for every $x \in X^{\mathrm{sm}}$ we say that $H \in (\mathbb{C}\mathrm{P}^m)^\vee$ is tangent to X at x if: $x \in H$ and $T_xX \subseteq T_xH$. We denote by $X^\vee \subset (\mathbb{C}\mathrm{P}^m)^\vee$ the algebraic set

$$X^\vee := \mathrm{clos}^{\mathscr{Z}}\left(\{H \in (\mathbb{C}\mathrm{P}^m)^\vee \mid H \text{ is tangent to } X \text{ at a smooth point}\}\right),$$

and call it the variety dual to X.

The discriminant in the space of polynomials is related to the *Veronese variety* $V^{\mathbb{C}}_{n,d} \subset \mathbb{C}\mathrm{P}^N$, which is the image of the *Veronese embedding* $\nu^{\mathbb{C}}_{n,d} : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^N$, defined by

$$\nu^{\mathbb{C}}_{n,d}([z_0,\dots,z_n]) := \left[(z_0^{\alpha_0}\cdots z_n^{\alpha_n})_{|\alpha|=d}\right].$$

For example, $\nu^{\mathbb{C}}_{1,d} : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^d$ is the map

$$\nu^{\mathbb{C}}_{1,d}([z_0,z_1]) := [z_0^d, z_0^{d-1}z_1,\dots,z_1^d],$$

and the Veronese variety $V^{\mathbb{C}}_{1,d} \subset \mathbb{C}\mathrm{P}^d$ is the rational normal curve of degree d; the next result implies that the set $\mathrm{D}^{\mathbb{C}}_{1,d}$ of polynomials with multiple zeroes, projectively, is the dual to the rational normal curve in $\mathbb{C}\mathrm{P}^d$.

Lemma 9.7. *The projectivization of the discriminant is dual to the Veronese variety:*

$$\mathrm{D}^{\mathbb{C}}_{n,d} = (V^{\mathbb{C}}_{n,d})^\vee.$$

Proof. Denote by $[(\mu_\alpha)_{|\alpha|=d}]$ homogeneous coordinates on the codomain of the Veronese map $\nu^{\mathbb{C}}_{n,d} : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^N$. Given a nonzero polynomial $p \in \mathscr{P}^{\mathbb{C}}_{n,d}$, we can write $p = \sum_{|\alpha|=d} c_\alpha x_0^{\alpha_0}\cdots x_n^{\alpha_n}$ and associate to it a hyperplane in $\mathbb{C}\mathrm{P}^N$ by

$$H_p := \left\{[(\mu_\alpha)_{|\alpha|=d}] \in \mathbb{C}\mathrm{P}^N \;\middle|\; \sum_{|\alpha|=d} c_\alpha \mu_\alpha = 0\right\}.$$

In this way we can identify $\mathrm{P}(\mathscr{P}_{n,d}^{\mathbb{C}})$ with $(\mathbb{C}\mathrm{P}^N)^\vee$. Notice that, for every $p \in \mathscr{P}_{n,d}^{\mathbb{C}}$ we have

$$\nu_{n,d}(Z(p)) = V_{n,d}^{\mathbb{C}} \cap H_p.$$

Since $\nu_{n,d}$ is an embedding, $Z(p) \subset \mathbb{C}\mathrm{P}^n$ is smooth if and only if $\nu_{n,d}(Z(p)) \subset \mathbb{C}\mathrm{P}^N$ is smooth. In particular, $Z(p)$ is smooth if and only if the hyperplane H_p is transversal to $V_{n,d}^{\mathbb{C}}$. The set of projective hyperplanes which *are not* transversal to $V_{n,d}^{\mathbb{C}}$, i.e. the dual variety $(V_{n,d}^{\mathbb{C}})^\vee$, can therefore be identified with the projectivization of the set of polynomials defining a singular hypersurface, i.e. with $\mathrm{D}_{n,d}^{\mathbb{C}}$. □

Notice that, so far, we have only proved that the discriminant $\mathrm{D}_{n,d}^{\mathbb{C}}$ is a semialgebraic set of dimension $2(N-1)$ (Proposition 9.5), which is also a complex algebraic set. The fact that it is the dual variety to the Veronese variety, which is an irreducible algebraic set, implies that $\mathrm{D}_{n,d}^{\mathbb{C}}$ is also irreducible (as a complex algebraic set), and that it is a hypersurface (i.e. it is defined by a single polynomial equation in $\mathbb{C}\mathrm{P}^N$).

Corollary 9.8. *The discriminant is an irreducible, complex algebraic hypersurface in $\mathbb{C}\mathrm{P}^N$.*

Proof. Let $A \subset \mathbb{C}\mathrm{P}^N \times (\mathbb{C}\mathrm{P}^N)^\vee$ be the algebraic set:

$$A := \{([x], H) \mid H \text{ is tangent to } V_{n,d}^{\mathbb{C}} \text{ at } x\}.$$

Denote by $\pi_1 : A \to \mathbb{C}\mathrm{P}^N$ and $\pi_2 : A \to (\mathbb{C}\mathrm{P}^N)^\vee$ the two projections. The fibers of $\pi_1 : A \to V_{n,d}^{\mathbb{C}}$ are projective spaces and, since $V_{n,d}^{\mathbb{C}}$ is irreducible (being smooth), A is also irreducible. Since $\mathrm{D}_{n,d}^{\mathbb{C}}$ equals $\pi_2(A)$, by Lemma 9.7, then it is irreducible.[1]

By Proposition 9.5, the dimension of $\mathrm{D}_{n,d}^{\mathbb{C}}$, as a complex algebraic set is $N-1$. Let now $p \in \mathbb{C}[w_0, \ldots, w_N]$ be a homogeneous polynomial vanishing on $\mathrm{D}_{n,d}^{\mathbb{C}}$. We can write $p = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where each p_i is irreducible. Since $\mathrm{D}_{n,d}^{\mathbb{C}}$ is irreducible, there exists p_i such that $Z(p_i) \supseteq \mathrm{D}_{n,d}^{\mathbb{C}}$ and, since $\dim_{\mathbb{C}}(\mathrm{D}_{n,d}^{\mathbb{C}}) = N-1$, it follows that $Z(p_i) = \mathrm{D}_{n,d}^{\mathbb{C}}$. □

The description of the discriminant as a dual variety is also useful for proving the following property, which says that the generic singular hypersurface has only one singular point. In order to state the result we define the following set:

$$\widetilde{\mathrm{D}}_{n,d}^{\mathbb{C}} = \left\{[p] \in \mathbb{C}\mathrm{P}^N \mid Z_{\mathbb{C}\mathrm{P}^n}(p) \text{ has only one singular point}\right\}.$$

[1]Note that we have actually proved that if X is smooth, $X^\vee$ is irreducible.

Proposition 9.9. *The set* $\mathrm{D}_{n,d}^{\mathbb{C}} \setminus \widetilde{\mathrm{D}}_{n,d}^{\mathbb{C}}$ *is semialgebraic of real codimension at least* 4 *in* $\mathbb{C}\mathrm{P}^N$*. In particular, the zero set of a generic point* $[p] \in \mathrm{D}_{n,d}^{\mathbb{C}}$ *has only one singular point.*

Proof. Denote by $V_{n,d}^{\mathbb{C}}$ the Veronese variety in $\mathbb{C}\mathrm{P}^N$. By Lemma 9.7,

$$\mathrm{D}_{n,d}^{\mathbb{C}} = (V_{n,d}^{\mathbb{C}})^{\vee}.$$

Notice that the set $\mathrm{D}_{n,d}^{\mathbb{C}} \setminus \widetilde{\mathrm{D}}_{n,d}^{\mathbb{C}}$ we are interested in coincides with the set of hyperplanes in $\mathbb{C}\mathrm{P}^N$ which are tangent to $V_{n,d}^{\mathbb{C}}$ in at least two points. We claim that $\mathrm{D}_{n,d}^{\mathbb{C}} \setminus \widetilde{\mathrm{D}}_{n,d}^{\mathbb{C}}$ is the continuous semialgebraic image of a semialgebraic set of dimension $2(N-2)$, and in particular, it has dimension smaller than or equal to $2(N-2)$. From this the result is immediate.

For every $x \in V_{n,d}^{\mathbb{C}}$, denote by $\mathrm{D}_x^{\mathbb{C}} \subset (\mathbb{C}\mathrm{P}^N)^{\vee}$ the set of hyperplanes which are tangent to $V_{n,d}^{\mathbb{C}}$ at x. Then $\mathrm{D}_x^{\mathbb{C}} \simeq \mathbb{C}\mathrm{P}^{N-n-1}$ and, for $x_1 \neq x_2$,

$$\mathrm{D}_{x_1}^{\mathbb{C}} \cap \mathrm{D}_{x_2}^{\mathbb{C}} \simeq \mathbb{C}\mathrm{P}^{N-2n-2}.$$

(This can be proved picking the two special points $x_1 = [1, 0, \ldots, 0]$ and $x_2 = [0, 1, 0, \ldots, 0]$ and then using the linear action of $\mathrm{GL}(\mathbb{C}^{n+1})$.)

Let now $Y \subset \big((V_{n,d}^{\mathbb{C}} \times V_{n,d}^{\mathbb{C}}) \setminus \Delta\big) \times \mathbb{C}\mathrm{P}^N$ be the semialgebraic set:

$$Y = \left\{ ((x_1, x_2), [p]) \in \big((V_{n,d}^{\mathbb{C}} \times V_{n,d}^{\mathbb{C}}) \setminus \Delta\big) \times \mathbb{C}\mathrm{P}^N \,\middle|\, [p] \in \mathrm{D}_{x_1}^{\mathbb{C}} \cap \mathrm{D}_{x_2}^{\mathbb{C}} \right\}.$$

By Section 9.2, the fibers of the projection

$$\pi_1 : Y \to \big((V_{n,d}^{\mathbb{C}} \times V_{n,d}^{\mathbb{C}}) \setminus \Delta\big)$$

are all of the same dimension $2(N-2n-2)$ and, since $\dim\big((V_{n,d}^{\mathbb{C}} \times V_{n,d}^{\mathbb{C}}) \setminus \Delta\big) = 4n$, we have $\dim(Y) = 2(N-2)$.

On the other hand, denoting by $\pi_2 : Y \to \mathbb{C}\mathrm{P}^N$ the projection on the second factor, we see that

$$\pi_2(Y) = \mathrm{D}_{n,d}^{\mathbb{C}} \setminus \widetilde{\mathrm{D}}_{n,d}^{\mathbb{C}},$$

and the conclusion follows now from Corollary 3.34.

□

We can use Proposition 9.9 to compute the degree of an irreducible polynomial vanishing on $\mathrm{D}_{n,d}^{\mathbb{C}}$. Let us first recall the following definition.

Definition 9.10 (Degree of a complex hypersurface)**.** Let $Z \subset \mathbb{C}\mathrm{P}^m$ be a complex hypersurface, i.e. the zero locus of a single homogeneous polynomial p. Assume that p has no repeated prime factors. The degree of p is called the degree of Z.

The degree of a complex hypersurface has the following geometric interpretation (later we will also give a measure–theoretic interpretation).

Lemma 9.11. *Let $Z \subset \mathbb{C}\mathrm{P}^m$ be a complex hypersurface of degree d. Then, for the generic complex line $\ell \subset \mathbb{C}\mathrm{P}^m$, the intersection $\ell \cap Z$ is transversal and consists of d points.*

Proof. Let $p = p_1 \cdots p_k$ be a homogeneous polynomial with each p_i irreducible, such that $Z(p) = Z$. Then $Z(p_1), \ldots, Z(p_k)$ are the irreducible components of Z. The degree of Z is, by definition, $\deg(Z) = \deg(p_1) + \cdots + \deg(p_k)$. Let $Z = \sqcup_j N_j$ with each N_j a Nash submanifold, and denote by $\widetilde{Z}$ the union of all the strata of real dimension $2m - 2$. A generic complex line ℓ (which has dimension two) meets all the strata N_j transversally and therefore it only meets $\widetilde{Z}$, since $Z \setminus \widetilde{Z}$ has real codimension more than 2 in $\mathbb{C}\mathrm{P}^m$. Each $Z(p_i)$ is a union of strata and we define $\widetilde{Z}_i = \widetilde{Z} \cap Z(p_i)$. Then, for the generic line ℓ, we have:

$$\#\ell \cap Z = \sum_{i=1}^{k} \#\ell \cap \widetilde{Z}_i.$$

For each $i = 1, \ldots, k$, and for the generic ℓ, the set $\ell \cap Z(p_i) = \ell \cap \widetilde{Z}_i$ consists of the zeroes of $p_i|_\ell$. Since the intersection, for the generic ℓ, is transversal, it follows that

$$\#\ell \cap Z(p_i) = \#\ell \cap \widetilde{Z}_i = \deg(p_i),$$

and the conclusion follows from Section 9.2. $\square$

Finally we compute the degree of $\mathrm{D}^{\mathbb{C}}_{n,d}$.

Theorem 9.12. *The discriminant $\mathrm{D}^{\mathbb{C}}_{n,d}$ is defined by a single irreducible polynomial equation, with integer coefficients and of degree $(n+1)(d-1)^n$.*

Proof. The first part of the statement, being defined by a single irreducible polynomial equation, follows from Corollary 9.8. The fact that this polynomial can be taken to be with integer coefficients follows from Lemma 9.3 using the fact that Section 9.2 is defined by polynomials with integer coefficients.

In order to compute the degree, we consider again the map $\pi_2 : W^{\mathbb{C}}_{n,d} \to \mathrm{D}^{\mathbb{C}}_{n,d}$. The main idea of the proof is to show that for the generic complex line ℓ we have:

$$\#\ell \cap \mathrm{D}^{\mathbb{C}}_{n,d} = \#\pi_2^{-1}(\ell) \cap W^{\mathbb{C}}_{n,d},$$

with transversal intersection both on the left and on the right. Leaving the proof of this aside for the moment, let us see how this property implies the statement.

The advantage of Section 9.2 is that we have explicit equations of the objects we are intersecting on the right hand side. More precisely, $\pi_2^{-1}(\ell) = \mathbb{C}\mathrm{P}^n \times \{\ell\}$ is defined by $(N-1)$ complex polynomial equations of degree 0 in the z–variables and of degree 1 in the p–variables. Similarly, $W^{\mathbb{C}}_{n,d}$ is defined by $n+1$ complex polynomial equations of degree $(d-1)$ in the z–variables and of degree 1 in the p–variables. Since the intersections in Section 9.2 are transversal, the multi-homogeneous Bézout theorem (Theorem 9.13) implies now that the number of points of intersection is the coefficients of $t_1^n t_2^N$ in the polynomial

$$t_2^{N-1}((d-1)t_1 + t_2)^{n+1} = (n+1)(d-1)^n t_1^n t_2^N + O(t_2^{N-1}).$$

Putting all this together, Lemma 9.11 implies:

$$\deg(\mathrm{D}^{\mathbb{C}}_{n,d}) = \#\ell \cap \mathrm{D}^{\mathbb{C}}_{n,d} = \#\pi_2^{-1}(\ell) \cap W^{\mathbb{C}}_{n,d} = (n+1)(d-1)^n.$$

It remains to prove that for a generic complex line ℓ we have Section 9.2 with transversal intersections. To this end, recall the definition of the set $\widetilde{\mathrm{D}}^{\mathbb{C}}_{n,d}$ from Section 9.2:

$$\widetilde{\mathrm{D}}^{\mathbb{C}}_{n,d} = \{[p] \mid Z_{\mathbb{C}\mathrm{P}^n}(p) \text{ has only one singular point}\}.$$

Recall that by Proposition 9.9 the complement of $\widetilde{\mathrm{D}}^{\mathbb{C}}_{n,d}$ has real codimension 2 in $\mathrm{D}^{\mathbb{C}}_{n,d}$.

We will find a semialgebraic subset $\widetilde{B} \subset \widetilde{\mathrm{D}}^{\mathbb{C}}_{n,d}$ with complement in $\mathrm{D}^{\mathbb{C}}_{n,d}$ of real codimension at least 2, and therefore at least 4 in $\mathbb{C}\mathrm{P}^N$, and such that a line $\ell \subset \mathbb{C}\mathrm{P}^N$ is transversal to $\widetilde{B}$ if and only if $\pi_2^{-1}(\ell)$ is transversal to $W^{\mathbb{C}}_{n,d}$. For the generic complex line ℓ, transversal to $\widetilde{B}$, we will therefore have

$$\#\ell \cap \mathrm{D}^{\mathbb{C}}_{n,d} = \#\ell \cap \widetilde{B}.$$

Moreover, since π_2 is one–to–one over $\widetilde{B} \subseteq \widetilde{\mathrm{D}^{\mathbb{C}}}_{n,d}$ (Proposition 9.9), we will also have:

$$\#\ell \cap \widetilde{B} = \#\pi_2^{-1}(\ell) \cap W,$$

with the intersection $\pi_2^{-1}(\ell) \cap W$ transversal.

The set $\widetilde{B}$ is constructed as follows. We first stratify $\widetilde{\mathrm{D}}_{n,d}^{\mathbb{C}} = \sqcup_j D_j$ into Nash submanifolds and we set B to be the union of the strata of dimension

$$2(N-1) = \dim(\mathrm{D}_{n,d}^{\mathbb{C}}).$$

This set B is a smooth, complex submanifold. Then we set $A := \pi_2^{-1}(B)$. We define $\widetilde{A} \subseteq A$ to be the set of points $w \in A$ such that w is a smooth point and $\mathrm{rk}_{\mathbb{C}}(D_w\pi_2) = N-1$. We claim that the complement of $\widetilde{A}$ is of dimension strictly less than $2(N-1)$. First, the complement of the set of smooth points has dimension less than $2(N-1)$ by Theorem 3.17. Second, on the set of smooth points, the set where $\mathrm{rk}_{\mathbb{C}}(D_w\pi_2) \neq N-1$ has dimension less than $2(N-1)$. In fact, the set where $\mathrm{rk}_{\mathbb{C}}(D_w\pi_2) < N-1$ is a complex algebraic subset and, if this subset is not proper, there would be a semialgebraic subset $A' \subset A$ of dimension $2(N-1)$ such that $\mathrm{rank}_{\mathbb{C}}(D\pi_2) < N$ on A'. Therefore $\pi_2(A')$ would be contained in a subset of $\pi_2(A)$ of relative measure zero, which is in contradiction with $\pi_2|_A$ being one–to–one and B being smooth and of dimension $2(N-1)$.

We define $\widetilde{B} := \pi_2(\widetilde{A})$. Since $\mathrm{rank}_{\mathbb{C}}(D_w\pi_2) = N-1$ for all $w \in \widetilde{A}$, it follows that $\pi_2 : \widetilde{A} \to \widetilde{B}$ is a local biholomorphism. In particular, if ℓ is transversal to $\widetilde{B}$ at $[p] \in \widetilde{B}$, then the tangent space $T_{\pi_2^{-1}([p])}W$ is transversal to $\pi_2^{-1}(\ell)$, as claimed. Therefore the set $\widetilde{B}$ has the desired properties and this concludes the proof □

In the proof of Theorem 9.12 we have used the multi–homogeneous Bézout theorem, which we now prove.

Theorem 9.13 (Multi–homogeneous Bézout)**.** *Let $Z_1, \ldots, Z_m$ be complex algebraic hypersurfaces in $\mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_k}$, where $m = n_1 + \cdots + n_k$ and each Z_i is defined by a polynomial multi–homogeneous of degree $(d_{i,1}, \ldots, d_{i,k})$. Then, if the intersection $Z_1 \cap \cdots \cap Z_k$ is transversal, the number of points of intersection is given by the coefficient of $t_1^{n_1} \cdots t_k^{n_k}$ in the polynomial*

$$\prod_{j=1}^{m} (d_{i,1}t_1 + \cdots + d_{i,k}t_k).$$

Proof. Denote by $X := \mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_k}$. By Künneth theorem we have:

$$H^*(X;\mathbb{Z}) = \bigotimes_{j=1}^{k} \mathbb{Z}[t_j]/t_j^{n_j+1} \simeq \mathbb{Z}[t_1, \ldots, t_k]/(t_1^{n_1}, \ldots, t_k^{n_k}),$$

where, denoting by $\pi_i : X \to \mathbb{C}\mathrm{P}^{n_i}$ the projection on the i–th factor and by $[\omega_i] \in H^2(\mathbb{C}\mathrm{P}^{n_i};\mathbb{Z})$ a 2–form generating the cohomology, we have $t_i = \pi_i^*[\omega_i]$. In particular

$$H^{2m}(X;\mathbb{Z}) = \langle t_1^{n_1} \cdots t_k^{n_k} \rangle.$$

For a $\mathbb{Z}$–orientable manifold Y, denote by $P : H_*(Y;\mathbb{Z}) \to H^{\dim(Y)-*}(Y;\mathbb{Z})$ the Poincaré duality. For every $j = 1,\dots,k$, the class $[\omega_j]$ is the Poincaré dual of the homology class of a hyperplane $[H_j] \in H^{2n_j-2}(\mathbb{C}\mathrm{P}^{n_j};\mathbb{Z})$ and

$$t_j = P[\mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_{j-1}} \times H_j \times \mathbb{C}\mathrm{P}^{n_{j+1}} \cdots \times \mathbb{C}\mathrm{P}^{n_k}].$$

Since each Z_i is a complex hypersurface in X, it follows that its fundamental class[2] $[Z_i]$ represents an element in $H^{2m-2}(X;\mathbb{Z})$ and, since the intersection $Z_1 \cap \cdots \cap Z_m$ is transversal, we have

$$[Z_1] \cdots [Z_m] = (\# Z_1 \cap \cdots \cap Z_m) \cdot 1 \in H_0(X;\mathbb{Z}),$$

where the product on the left hand side in Section 9.2 denotes the intersection product.

For every $i = 1,\dots,m$, denote by $z_i(t) = c_{i,1}t_1 + \cdots + c_{i,k}t_k \in H^2(X;\mathbb{Z})$ the Poincaré dual of $[Z_i]$. Since the intersection product in homology is Poincaré dual to the cup product (see [15, pag. 59]), and since $P[1] = t_1^{n_1} \cdots t_k^{n_k}$ it follows that

$$\prod_{i=1}^{m} z_i(t) = (\# Z_1 \cap \cdots \cap Z_m) t_1^{n_1} \cdots t_k^{n_k}.$$

To conclude the proof it remains to prove that, for every $i = 1,\dots,m$, we have

$$(c_{i,1},\dots,c_{i,k}) = (d_{i,1},\dots,d_{i,k}).$$

In order to see this, for every $j = 1,\dots,k$, denote by $Y_j \subset X$ the hypersurface

$$Y_j := \mathbb{C}\mathrm{P}^{n_1} \times \cdots \times \mathbb{C}\mathrm{P}^{n_{j-1}} \times H_j \times \mathbb{C}\mathrm{P}^{n_{j+1}} \cdots \times \mathbb{C}\mathrm{P}^{n_k},$$

and, letting $\ell_j \subset \mathbb{C}\mathrm{P}^{n_j}$ be a line and $p_l \in \mathbb{C}\mathrm{P}^{n_l}, l \neq j$, be generic points, denote by $L_j \subset X$ the complex curve

$$L_j := \{p_1\} \times \cdots \times \{p_{j-1}\} \times \ell_j \times \{p_{j+1}\} \times \cdots \{p_k\}.$$

[2] Complex algebraic sets in a complex projective variety carry a well defined fundamental class with coefficients in $\mathbb{Z}$, see [15, pag. 61].

Then, for every $1 \leq i_1, i_2 \leq m$ we have:

$$[L_{i_1}] \cdot [Y_{i_2}] = \delta_{i_1,i_2} \cdot 1 \in H_0(X;\mathbb{Z}).$$

In particular, letting for every $j = 1, \dots, m$,

$$[Z_i] = c_{i,1}[Y_1] + \cdots + c_{i,k}[Y_k],$$

we see that, if L_j is transversal to Z_i, then $c_{i,j} = [Z_i] \cdot [L_j] = \#Z_i \cap L_j$. The cardinality $\#Z_i \cap L_j$ is now given by the number of (nondegenerate) points of intersections between the hypersurface $Z_i \cap \mathbb{C}\mathrm{P}^{n_j}$ (defined by a polynomial of degree $d_{i,j}$ in $\mathbb{C}\mathrm{P}^{n_j}$) and the line ℓ_j; this number equals $d_{i,j}$ by the classical Bézout theorem. Therefore $c_{i,j} = d_{i,j}$ and this concludes the proof. □

9.3 The discriminant in the space of real polynomials

In this section we start looking at the real version of the previous picture. We denote the space of real homogeneous polynomials of degree d by:

$$\mathscr{P}_{n,d} := \mathbb{R}[x_0, \dots, x_n]_{(d)}.$$

Inside the space of real polynomials sits the *real part* of the complex discriminant:

$$\mathscr{D}_{n,d} := \mathscr{D}^{\mathbb{C}}_{n,d} \cap \mathscr{P}_{n,d} = \left\{ p \in \mathscr{P}_{n,d} \,\middle|\, \exists [z] \in \mathbb{C}\mathrm{P}^n,\, p(z) = \frac{\partial p}{\partial x_0}(z) = \cdots = \frac{\partial p}{\partial x_n}(z) = 0 \right\}.$$

This set is, in general, different from the set of real polynomials whose real zero set is singular:

$$\mathscr{S}_{n,d} := \left\{ p \in \mathscr{P}_{n,d} \,\middle|\, \exists [x] \in \mathbb{R}\mathrm{P}^n,\, p(x) = \frac{\partial p}{\partial x_0}(z) = \cdots = \frac{\partial p}{\partial x_n}(x) = 0 \right\}.$$

Projectively, we have inclusions

$$\Sigma_{n,d} := \mathrm{P}\left(\mathscr{S}_{n,d}\right) \subset \mathrm{D}_{n,d} := \mathrm{P}\left(\mathscr{D}_{n,d}\right) \subset \mathbb{R}\mathrm{P}^N = \mathrm{P}\left(\mathscr{P}_{n,d}\right).$$

We denote by $V_{n,d} \subset (\mathbb{R}\mathrm{P}^N)^\vee$ the image of the real Veronese map $\nu_{n,d} : \mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^N$, defined by restricting the Veronese map Section 9.2.

Proposition 9.14. *The following properties are true:*

(1) The set $\mathrm{D}_{n,d}$ is a real algebraic set of codimension one in $\mathbb{R}\mathrm{P}^N$, defined by a homogeneous polynomial of degree $(n+1)(d-1)^n$. This algebraic set equals the real part of the dual variety $(V_{n,d}^{\mathbb{C}})^\vee$.

(2) The set $\Sigma_{n,d}$ is a semialgebraic set of codimension one in $\mathbb{R}\mathrm{P}^N$. It can be identified with the set of real hyperplanes $H \subset (\mathbb{R}\mathrm{P}^N)^\vee$ which are tangent to the real variety $V_{n,d}$.

Proof. Since $\mathrm{D}_{n,d}^{\mathbb{C}}$ is defined by a homogeneous polynomial with integer coefficients (Theorem 9.12), by restricting this polynomial we get a defining polynomial for $\mathrm{D}_{n,d}$, with degree $(n+1)(d-1)^n$. To see that $\dim(\mathrm{D}_{n,d}) = N-1$, consider the set

$$\mathrm{D}_{[1,0,\ldots,0]}^{\mathbb{R}} := \{[p] \in \mathbb{R}\mathrm{P}^N \mid [1,\ldots,0] \text{ is a singular point of } Z_{\mathbb{C}\mathrm{P}^n}(p)\} \simeq \mathbb{R}\mathrm{P}^{N-n-1}.$$

Restricting the map ψ from Section 9.2, we get a map

$$\psi : \mathrm{GL}(\mathbb{R}^{n+1}) \times \mathrm{D}_{[1,0,\ldots,0]}^{\mathbb{R}} \to \mathbb{R}\mathrm{P}^N.$$

Pick now $q \in \mathscr{P}_{n-1,2}$ such that $Z(q) \subset \mathbb{C}\mathrm{P}^{n-1}$ is nonsingular and consider the polynomial

$$\widetilde{p}(x_0,\ldots,x_n) := x_0^{d-2} q(x_1,\ldots,x_n).$$

Then

$$\operatorname{rank}\left(D_{(\mathbf{1},[\widetilde{p}])}\psi\right) = N-1.$$

In particular, on a neighborhood B of $(\mathbf{1},[\widetilde{p}])$ and, by the Rank Theorem, possibly shrinking the neighborhood, $U_{\widetilde{p}} := \psi(B) \hookrightarrow \mathbb{R}\mathrm{P}^N$ is a smooth submanifold of real dimension $(N-1)$, contained in $\mathrm{D}_{n,d}$. This proves the first part of the statement.

Since $\Sigma_{n,d} \subseteq \mathrm{D}_{n,d}$ and the image of the map Section 9.3 is contained in $\Sigma_{n,d}$, the second part of the statement follows from the previous point. (The descriptions as dual objects are clear from the geometry of the Veronese map.) □

We are now interested in the description of the generic point on $\mathrm{D}_{n,d}$ and $\Sigma_{n,d}$. To this end we introduce the following set:

$$\widetilde{\Sigma}_{n,d} := \{[p] \in \mathbb{R}\mathrm{P}^N \mid Z_{\mathbb{R}\mathrm{P}^n}(p) \text{ has only one singular point}\}.$$

The following description gives an analogue to Proposition 9.9.

Proposition 9.15. *The sets $\mathrm{D}_{n,d} \setminus \widetilde{\Sigma}_{n,d}$ and $\Sigma_{n,d} \setminus \widetilde{\Sigma}_{n,d}$ are semialgebraic of codimension at least 2 in $\mathbb{R}\mathrm{P}^N$. In particular, the zero set of a generic point $[p] \in \mathrm{D}_{n,d}$ or $[p] \in \Sigma_{n,d}$ has only one singular point, and this singular point is real.*

Proof. Let us prove first that $\Sigma_{n,d} \setminus \widetilde{\Sigma}_{n,d}$ has dimension at most $N-2$. We use the dual description of $\Sigma_{n,d}$ as the set of real hyperplanes $H \subset (\mathbb{R}\mathrm{P}^N)^\vee$ which are tangent to the real Veronese variety $V_{n,d}$. For $x \in V_{n,d}$, denote by

$$\Sigma_x := \{H \subset (\mathbb{R}\mathrm{P}^N)^\vee \,|\, H \text{ is tangent to } x \text{ at } V_{n,d}\} \simeq \mathbb{R}\mathrm{P}^{N-n-1}.$$

Arguing as in the proof of Proposition 9.9, one can easily see that for $x_1 \neq x_2 \in V_{n,d}$, we have

$$\Sigma_{x_1} \cap \Sigma_{x_2} \simeq \mathbb{R}\mathrm{P}^{N-2n-2} = \Sigma_{[1,0,\ldots,0]} \cap \Sigma_{[0,1,0,\ldots,0]}.$$

Let now $S \subset ((V_{n,d} \times V_{n,d}) \setminus \Delta) \times \mathbb{R}\mathrm{P}^N$ be the set:

$$S = \left\{ ((x_1,x_2),[p]) \in ((V_{n,d} \times V_{n,d}) \setminus \Delta) \times \mathbb{R}\mathrm{P}^N \,\middle|\, [p] \in \Sigma_{x_1} \cap \Sigma_{x_2} \right\}.$$

By Section 9.3, the fibers of the projection

$$\pi_1 : S \to (V_{n,d} \times V_{n,d}) \setminus \Delta$$

are all of the same real dimension $(N-2n-2)$ and, since $\dim((V_{n,d} \times V_{n,d}) \setminus \Delta)) = 2n$, we have $\dim(S) = (N-2)$.

On the other hand, denoting by $\pi_2 : S \to \mathbb{R}\mathrm{P}^N$ the projection on the second factor, we see that

$$\pi_2(S) = \Sigma_{n,d} \setminus \widetilde{\Sigma}_{n,d},$$

and the conclusion follows now from Corollary 3.34.

Let us now estimate the dimension of $\mathrm{D}_{n,d} \setminus \widetilde{\Sigma}_{n,d}$. Notice that if a point $[p]$ is in $\mathrm{D}_{n,d}$ but not in $\widetilde{\Sigma}_{n,d}$, then two possibilities can occurr:

(1) $Z_{\mathbb{R}\mathrm{P}^n}(p)$ has more than one real singularity, which implies $[p] \in \Sigma_{n,d} \setminus \widetilde{\Sigma}_{n,d}$.

(2) $Z_{\mathbb{R}\mathrm{P}^n}(p)$ is smooth, but $Z_{\mathbb{C}\mathrm{P}^n}(p)$ has at least one singular point $[w] \in \mathbb{C}\mathrm{P}^n \setminus \mathbb{R}\mathrm{P}^n$. In this case, since $[p]$ is real, the complex conjugation map

$$\sigma : \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^n, \quad [z_0,\ldots,z_n] \mapsto [\overline{z}_0,\ldots,\overline{z}_n],$$

preserves the zero set $\sigma(Z_{\mathbb{C}\mathrm{P}^n}(p)) = Z_{\mathbb{C}\mathrm{P}^n}(p)$ and $\sigma([w]) \neq [w]$ is also a singular point. In particular $Z_{\mathbb{C}\mathrm{P}^n}(p)$ has at least two singular points, which are complex conjugate.

Therefore

$$\mathrm{D}_{n,d} \setminus \widetilde{\Sigma}_{n,d} = \left(\Sigma_{n,d} \setminus \widetilde{\Sigma}_{n,d}\right) \cup \mathrm{D}',$$

where D' consists of real polynomials with at least two nonreal singular points which are complex conjugate. Since we have already proved that $\dim(\Sigma_{n,d} \setminus \widetilde{\Sigma}_{n,d}) \leq$

$N-2$, it remains to prove that also $\dim(\mathrm{D}') \leq N-2$.

Observe now that, using the notation of Section 9.2, given $[w] \in \mathbb{C}\mathrm{P}^n \setminus \mathbb{R}\mathrm{P}^n$, the projective space $\mathrm{D}^{\mathbb{C}}_{[w]} \cap \mathrm{D}^{\mathbb{C}}_{\sigma([w])} \simeq \mathbb{C}\mathrm{P}^{N-2n-2}$ is real and, by Section 9.2,

$$\left(\mathrm{D}^{\mathbb{C}}_{[w]} \cap \mathrm{D}^{\mathbb{C}}_{\sigma([w])}\right) \cap \mathbb{R}\mathrm{P}^N \simeq \mathbb{R}\mathrm{P}^{N-2n-2}.$$

Let now $T \subset (\mathbb{C}\mathrm{P}^n \setminus \mathbb{R}\mathrm{P}^n) \times \mathbb{R}\mathrm{P}^N$ be the set:

$$T := \left\{ ([w],[p]) \in (\mathbb{C}\mathrm{P}^n \setminus \mathbb{R}\mathrm{P}^n) \times \mathbb{R}\mathrm{P}^N \;\middle|\; [p] \in \left(\mathrm{D}^{\mathbb{C}}_{[w]} \cap \mathrm{D}^{\mathbb{C}}_{\sigma([w])}\right) \cap \mathbb{R}\mathrm{P}^N \right\}.$$

By Section 9.3, the fibers of the projection

$$\pi_1 : T \to \mathbb{C}\mathrm{P}^n \setminus \mathbb{R}\mathrm{P}^n$$

are all of the same real dimension $(N-2n-2)$ and, since $\dim(\mathbb{C}\mathrm{P}^n \setminus \mathbb{R}\mathrm{P}^n) = 2n$, we have $\dim(T) = (N-2)$.

On the other hand, denoting by $\pi_2 : T \to \mathbb{R}\mathrm{P}^N$ the projection on the second factor, we see that

$$\pi_2(S) \supseteq \mathrm{D}',$$

and the conclusion follows now from Corollary 3.34.

□

Corollary 9.16. *The number of isotopy classes of real smooth hypersurfaces of degree d in $\mathbb{R}\mathrm{P}^n$ is bounded by:*

$$b_0(\mathbb{R}\mathrm{P}^N \setminus \Sigma_{n,d}) = b_0(\mathbb{R}\mathrm{P}^N \setminus \mathrm{D}_{n,d}) \leq (2(n+1)(d-1)^n)^{\binom{n+d}{d}-1}.$$

Proof. The number of isotopy classes of real smooth hypersurfaces of degree d, by Thom's Isotopy Lemma, is bounded by $b_0(\mathbb{R}\mathrm{P}^N \setminus \Sigma_{n,d})$. The first equality follows now from Proposition 9.14, and the inequality follows from Proposition 9.14 and Theorem 4.15. □

Remark 9.17 (Hilbert's Sixteenth Problem). At the Paris ICM in 1900, D. Hilbert asked for the study of "the number, shape and relative position" of the components of a smooth hypersurface of degree d in $\mathbb{R}\mathrm{P}^n$. This is known as (the first part of) Hilbert's Sixteenth Problem. Hilbert's question essentially requires to understand the components of $\mathbb{R}\mathrm{P}^N \setminus \Sigma_{n,d}$ and the topological invariants associated to the zero set of a generic element of each component. To give a feeling of the "untractability" of this problem in general, let us define the two quantities:

$$\theta_{n,d} := \#\{\text{isotopy classes of pairs } (\mathbb{R}\mathrm{P}^n, Z(p)) \text{ with } p \in \mathscr{P}_{n,d}\},$$

$$\beta_{n,d} := b_0(\mathbb{RP}^N \setminus \Sigma_{n,d}).$$

Then we have the inequalities:

$$\theta_{n,d} \le \beta_{n,d} \le (2(n+1)(d-1)^n)^{\binom{n+d}{d}-1},$$

where the first inequality comes from Thom's Isotopy Lemma and the second one from Corollary 9.16. Already in the case of curves, Orevkov and Kharlamov [33] have proved that there exists $c_1, c_2 > 0$ such that

$$c_1 d^2 \le \log \theta_{2,d} \le \log \beta_{2,d} \le c_2 d^2 \log d.$$

A case–by–case enumeration of the components of $\mathbb{RP}^N \setminus \Sigma_{n,d}$ is not possible and not understood already in the case of plane curves of degree 9. This motivates a probabilistic approach to Hilbert's problem: what are "the number, shape and relative position" of the components of a *random* smooth hypersurface of degree d in $\mathbb{RP}^n$? In other words, what is the structure of $(\mathbb{RP}^n, Z(p))$ for a *typical* $p \in \mathscr{P}_{n,d}$? Notice the switch from the world "generic", from complex algebraic geometry, to "random", more suitable for real algebraic geometry. Of course, this requires introducing measure–theoretic tools in the picture, and this will be done in the next chapters.

Example 9.18. Let us look at the complement of the real discriminant in the space of *monic* univariate polynomials of degree d:

$$\{p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \,|\, p \text{ has no real multiple root}\} = (\mathscr{P}_{1,d} \setminus \mathscr{S}_{1,d}) \cap \{a_d = 1\}.$$

We claim that

$$(\mathscr{P}_{1,d} \setminus \mathscr{S}_{1,d}) \cap \{a_d = 1\} = \bigsqcup_{d-k \text{ even}} V_k,$$

where each V_k is a contractible component. We define V_k to be the set of real monic polynomials with k nondegenerate real zeroes. The number of real nondegenerate zeroes of a real polynomial of degree d without degenerate real zeroes, is congruent to d modulo 2, which implies the decomposition as in (9.18). For $d-k$ even, consider the polynomial:

$$p_k(x) := (x^2+1)^{\frac{d-k}{2}} \prod_{j=1}^{k} (x-j).$$

We claim that V_k deformation retracts to $\{p_k\}$. Given a polynomial $p \in V_k$, let us write:

$$p(x) = \left(\prod_{j=1}^{k}(x-\lambda_j(p))\right)\left(\prod_{\ell=1}^{\frac{d-k}{2}}(x-\alpha_\ell(p))(x-\overline{\alpha}_\ell(p))\right).$$

If we require that $\lambda_1(p) < \cdots < \lambda_k(p)$ and that $\mathrm{Im}(\alpha_\ell(p)) > 0$, for $j = 1, \dots, k$ we get well defined and continuous functions:

$$\lambda_j : V_k \to \mathbb{R}, \quad \text{and} \quad \left\{\alpha_1, \dots, \alpha_{\frac{d-k}{2}}\right\} : V_k \to \mathrm{SP}^{\frac{d-k}{2}}(H),$$

where $\mathrm{SP}^m(X) := X^m/S_m$ denotes the m–fold symmetric product of a topological space[3] and $H = \{\mathrm{Im}(z) > 0\}$ is the upper half plane.

The desired deformation retraction is defined by:

$$g_t(p) := \left(\prod_{j=1}^{k}(x - tj - (1-t)\lambda_j(p))\right)\left(\prod_{\ell=1}^{\frac{d-k}{2}}(x - ti - (1-t)\alpha_\ell(p))(x + t_i - (1-t)\overline{\alpha}_\ell(p))\right).$$

Then, $g_0 = \mathrm{id}$ and $g_1 \equiv p_k$. (Notice that $\mathrm{im}(g_t) \subset \mathscr{P}_{1,d} \setminus \mathscr{S}_{1,d}$, but it intersects $\mathscr{D}_{1,d}$.)

9.4 The discriminant in the space of real quadrics

In this section we discuss the special case of the real discriminant in the space of quadratic forms. Recall first that, there is a linear isomorphism

$$\mathscr{P}_{n,2} \simeq \mathrm{Sym}_{n+1}(\mathbb{R}),$$

given by associating to every quadratic form $q \in \mathscr{P}_{n,2}$ the unique real symmetric matrix $Q \in \mathrm{Sym}_{n+1}(\mathbb{R})$ defined by

$$q(x) = \langle x, Qx \rangle.$$

The following result shows that, under this linear isomorphim, the discriminant is the set of matrices with zero determinant.

Lemma 9.19. *A quadratic form q belongs to $\mathscr{D}_{n,2}$ if and only if the corresponding matrix Q has zero determinant. Moreover, $\mathscr{D}_{n,2} = \mathscr{S}_{n,2}$.*

[3]For instance, $\mathrm{SP}^d(\mathbb{CP}^1) \simeq \mathbb{CP}^d$, by the Fundamental Theorem of Algebra.

Proof. For every $z \in \mathbb{C}^{n+1}$, denote by $\nabla q(z) \in \mathbb{C}^{n+1}$ the gradient of q at z:

$$\nabla q(z) := \left(\frac{\partial q}{\partial x_0}(z), \ldots, \frac{\partial q}{\partial x_n}(z) \right).$$

Then $q \in \mathscr{D}_{n,2}$ if and only if there exists $z \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $\nabla q(z) = 0$.

Using the isomorphim Section 9.4, one immediately sees that $\nabla q(z) = 2Qz$. In particular $q \in \mathscr{D}_{n,2}$ if and only if there exists $z \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $Qz = 0$, i.e. if and only if $\det(Q) = 0$.

Since Q is a real symmetric matrix, its eigenspaces, and in particular its kernel, are real. Therefore there exists $z \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $Qz = 0$ if and only if there exists $x \in \mathbb{R}^{n+1} \setminus \{0\}$ such that $Qx = 0$. This proves that $\mathscr{D}_{n,2} = \mathscr{S}_{n,2}$. □

Given $q \in \mathscr{P}_{n,2}$ (respectively $Q \in \mathrm{Sym}_{n+1}(\mathbb{R})$), we denote by $(\mathrm{i}^+(q), \mathrm{i}^-(q))$ (respectively $(\mathrm{i}^+(Q), \mathrm{i}^-(Q))$) its signature, where $\mathrm{i}^+(q) = \mathrm{i}^+(Q)$ denote the number of positive, and $\mathrm{i}^-(q) = \mathrm{i}^-(Q)$ the number of negative, eigenvalues of Q.

Proposition 9.20. *The components of $\mathscr{P}_{n,2} \setminus \mathscr{D}_{n,2}$ are labeled by integers $0 \le k \le n+1$:*

$$\mathscr{P}_{n,2} \setminus \mathscr{D}_{n,2} = \bigsqcup_{k=0}^{n+1} U_k,$$

where U_k denotes the component consisting of quadratic forms with signature $(k, n+1-k)$. Given $q \in U_k$, its zero set $Z_{\mathbb{RP}^n}(q)$ is diffeomorphic to $\left(S^{k-1} \times S^{n-k}\right) / \sim$, where the quotient relation is given by $(x, y) \sim (-x, -y)$.

Proof. The decomposition of Proposition 9.20 is clear from the description of U_k. The fact that each U_k is connected follows from the fact that the special orthogonal group is connected. In fact U_k is the continuous image of the connected space $SO(n+1) \times (\mathbb{R}^k_+ \times \mathbb{R}^{n+1-k}_-)$ under the map:

$$(R, a, b) \mapsto R^T \mathrm{diag}(a, b) R,$$

where $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_{n+1-k})$.

By Thom's Isotopy Lemma, the zero sets of any $q_1, q_2 \in U_k$ are diffeomorphic. Therefore it is enough to look at the special case of $Z_{\mathbb{RP}^n}(\|x\|^2 - \|y\|^2)$, where $(x, y) \in \mathbb{R}^k \times \mathbb{R}^{n+1-k}$. The zero set of $\|x\|^2 - \|y\|^2$ in the sphere $S^n = \{\|x\|^2 + \|y\|^2 = 1\}$ is diffeomorphic to $S^{k-1} \times S^{n-k}$. The zero set in $\mathbb{RP}^n$ is the quotient of the spherical one under the antipodal map action, and the result follows. □

From the previous result we deduce in particular that:

$$b_0(\mathscr{P}_{n,2} \setminus \mathscr{D}_{n,2}) = n+2 \quad \text{and} \quad b_0(\mathbb{R}\mathrm{P}^N \setminus \mathrm{D}_{n,2}) = \left\lfloor \frac{n+2}{2} \right\rfloor + 1.$$

The structure of $\mathscr{P}_{n,2} \setminus \mathscr{D}_{n,2}$ is actually much richer, as we will shortly see.

For every real symmetric matrix Q we denote by $\lambda_1(Q) \leq \cdots \leq \lambda_{n+1}(Q)$ its eigenevalues, and by $V_\lambda(Q) := \{v \in \mathbb{R}^{n+1} \,|\, Qv = \lambda v.\}$ We define also the "positive eigenspace" by:

$$V_+(Q) := \bigoplus_{\{\lambda=\lambda(Q)>0\}} V_\lambda.$$

Denoting by U_k the set of real symmetric matrices with nonzerodeterminant and k positive eigenvalues, if $Q \in U_k$ then $V_+(Q) \simeq \mathbb{R}^k$ and we have a vector bundle:

$$E_k := \{(Q, v) \in \mathrm{Sym}_{n+1}(\mathbb{R}) \times \mathbb{R}^{n+1} \,|\, v \in V_+(Q)\} \subseteq U_k \times \mathbb{R}^{n+1}.$$

The projection $p : E_k \to U_k$ is the restriction of the projection on the first factor, and the fiber over a point Q is $E_k|_Q = V_+(Q)$.

In order to state the next result, we introduce also the notation $\pi : \tau_{k,n+1} \to G(k, n+1)$ for the tautological vector bundle on $G(k, n+1)$.

Theorem 9.21. *The map $\varphi : U_k \to G(k, n+1)$ defined by $\varphi(Q) := V_+(Q)$ is a homotopy equivalence. Moreover $\varphi^*\tau_{k,n+1} = E_k$.*

Proof. For $W \in G(k, n+1)$, let us denote by $P_W \in \mathrm{Sym}_{n+1}(\mathbb{R})$ the orthogonal projection onto W. Notice that, if $\{w_1, \ldots, w_k\}$ is an orthonormal basis of W,

$$P_W = (w_1, \ldots, w_k)(w_1, \ldots, w_k)^T.$$

This is well defined since, if $\{v_1, \ldots, v_k\}$ is another orthonormal basis for W, there exists $H \in O(k)$ such that $(v_1, \ldots, v_k) = (w_1, \ldots, w_k)H$ and

$$(v_1, \ldots, v_k)(v_1, \ldots, v_k)^T = (w_1, \ldots, w_k)HH^T(w_1, \ldots, w_k)^T = (w_1, \ldots, w_k)(w_1, \ldots, w_k)^T.$$

We define the map $\phi : G(k, n+1) \to U_k$ (the homotopy inverse of φ) by

$$\phi(W) := P_W - P_{W^\perp}.$$

First observe that $\varphi \circ \phi = \mathrm{id}_{G(k,n+1)}$. To prove that φ is a homotopy equivalence, it is therefore enough to prove that $g_0 := \phi \circ \varphi$ is homototopic to id_{U_k}. Notice that

$g_0(Q) = P_{V_+(Q)} - P_{V^+(Q)^\perp}$, and we can define the linear homotopy:

$$g_t(Q) := tQ + (1-t)\left(P_{V_+(Q)} - P_{V^+(Q)^\perp}\right).$$

Clearly $g_t : U_k \to \mathrm{Sym}_{n+1}(\mathbb{R})$, $g_0 = \phi \circ \varphi$ and $g_1 = \mathrm{id}_{U_k}$. We need to prove that $g_t(U_k) \subseteq U_k$ for all $0 < t < 1$. Given a symmetric matrix Q, let $R \in O(n+1)$ such that

$$R^T Q R = \mathrm{diag}(a_1, \ldots, a_k, b_1, \ldots, b_{n+1-k}),$$

where $b_1 \leq \cdots \leq b_{n+1-k} < 0 < a_1 \leq \cdots \leq a_k$ are the eigenvalues of Q. Then the first k columns of R form an orthonormal basis for $V_+(Q)$ and the last columns an orthonormal basis for $V_+(Q)^\perp$. In particular, using the fact that $RR^T = \mathbf{1}_{n+1}$,

$$R\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_{n+1-k} \end{pmatrix} R^T = P_{V_+(Q)} - P_{V_+(Q)^\perp}.$$

Therefore:

$$\begin{aligned} g_t(Q) &= tQ + (1-t)\left(P_{V_+(Q)} - P_{V^+(Q)^\perp}\right) \\ &= tR\mathrm{diag}(a_1, \ldots, a_k, b_1, \ldots, b_{n+1-k})R^T + (1-t)R\begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_{n+1-k} \end{pmatrix} R^T \\ &= R\mathrm{diag}(ta_1 + (1-t), \ldots, ta_k + (1-t), tb_1 + (t-1), \ldots, tb_{n+1-k}(t-1))R^T, \end{aligned}$$

which shows that $(\mathrm{i}^+(g_t(Q)), \mathrm{i}^-(g_t(Q))) = (k, n+1-k)$, i.e. $g_t(Q) \in U_k$.

Finally, since $\varphi(Q) = V_+(Q) = E_k|_Q \subseteq \mathbb{R}^{n+1}$, the fact that $\varphi^*\tau_{k,n+1} = E_k$ is clear. □

Corollary 9.22. *There is a homotopy equivalence*

$$\mathscr{P}_{n,2} \setminus \mathscr{D}_{n,2} \sim \coprod_{k=0}^{n+1} G(k, n+1).$$

In particular, $b(\mathscr{P}_{n,2} \setminus \mathscr{D}_{n,2}) = 2^{n+1}$.

Proof. The homotopy equivalence Corollary 9.22 follows immediately from Theorem 9.21 and Proposition 9.20. The second part of the statement follows from the fact that $b(G(k, n+1)) = \binom{n+1}{k}$. □

Exercise 9.1. Denote by Her_n the set of $n \times n$ hermitian matrices:

$$\mathrm{Her}_n := \{Q \in \mathbb{C}^{n\times n} \mid Q = Q^*\}.$$

Following the lines of Theorem 9.21, prove that the set $\mathrm{Her}_n \setminus \{\det = 0\}$ is homotopy equivalent to a disjoint union of Grassmannians: $\mathrm{Her}_n \setminus \{\det = 0\} \sim \bigsqcup_{k=0}^n G(k,n)$. Deduce that the inclusion $\mathrm{Sym}_n(\mathbb{R}) \setminus \{\det = 0\} \hookrightarrow \mathrm{Her}_n \setminus \{\det = 0\}$ is a homotopy equivalence.

9.5 The distance to the discriminant

In this section we give a formula for the distance, in the Bombieri–Weyl norm, between a polynomial and the discriminant. We formulate the result using the inclusion $\mathscr{P}_{n,d} \hookrightarrow C^\infty(S^n, \mathbb{R})$, i.e. identifying a polynomial P with its restriction $p := P|_{S^n}$. Recalling Section 9.3, we see that the discriminant $\mathscr{S}_{n,d} \subset \mathscr{P}_{n,d}$ is given by

$$\begin{aligned}\mathscr{S}_{n,d} &= \left\{ p \in \mathscr{P}_{n,d} \,\middle|\, \exists x \in S^n,\, p(x) = \frac{\partial p}{\partial x_0}(x) = \cdots = \frac{\partial p}{\partial x_n}(x) = 0 \right\} \\ &= \left\{ p \in \mathscr{P}_{n,d} \,\middle|\, \exists x \in S^n,\, p(x) = 0, \nabla p(x) = 0 \right\},\end{aligned}$$

where $\nabla p(x) \in T_x S^n$ denotes the gradient of $p : S^n \to \mathbb{R}$ (in this case it is just the orthogonal projection on the tangent space to the sphere of the gradient of the homogeneous polynomial). The following result was proved by C. Raffalli in [34].

Theorem 9.23. *The distance between a polynomial $p \in \mathscr{P}_{n,d}$ and the real discriminant $\mathscr{S}_{n,d}$, in the metric induced by the Bombieri–Weyl norm, is given by*

$$\mathrm{dist}_{\mathrm{BW}}(P, \mathscr{S}_{n,d}) = \min_{u \in S^n} \left(|p(u)|^2 + \frac{\|\nabla p(u)\|^2}{d} \right)^{\frac{1}{2}}.$$

Moreover, if $u_0 \in S^n$ is a point where the minimum on the right hand side of Theorem 9.23 is attained, then a polynomial $s \in \mathscr{S}_{n,d}$ such that $\|s - p\|_{\mathrm{BW}} = \mathrm{dist}_{\mathrm{BW}}(p, \mathscr{S}_{n,d})$ is given by

$$s(x) := p(x) - p(u_0)\langle u_0, x\rangle^d - \langle x, \nabla p(u_0)\rangle \langle x, u_0\rangle^{d-1}.$$

Proof. In order to simplify notations, denote by $\mathscr{S} = \mathscr{S}_{n,d}$ and, for $u \in S^n$, by $\mathscr{S}_u$ the set

$$\mathscr{S}_u = \left\{ p \in \mathscr{P}_{n,d} \,\middle|\, p(u) = 0, \nabla p(u) = 0 \right\}.$$

Notice that, for every $u \in S^n$, the set $\mathscr{S}_u$ is a linear space and that $\mathscr{S}_{n,d} = \bigcup_u \mathscr{S}_u$. In particular

$$\mathrm{dist}_{\mathrm{BW}}(p, \mathscr{S}_{n,d}) = \min_{u \in S^n} \mathrm{dist}_{\mathrm{BW}}(p, \mathscr{S}_u).$$

Given $u \in S^n$, let now $g \in O(n+1)$ such that $gu = e_0 = (1, 0, \dots, 0)$. Since the Bombieri–Weyl metric is orthogonally invariant, we have

$$\mathrm{dist}_{\mathrm{BW}}(p, \mathscr{S}_u) = \mathrm{dist}_{\mathrm{BW}}(\rho_{n,d}(g)p, \rho_{n,d}(g)\mathscr{S}_u) = \mathrm{dist}_{\mathrm{BW}}(p \circ g^{-1}, \mathscr{S}_{e_0}).$$

Denote by $\widetilde{p} = p \circ g^{-1}$. Then

$$\widetilde{p}(x) = a_d x_0^d + \sum_{k=1}^{n} a_{d-1,k} x_0^{d-1} x_k + x_0^{d-2} P_2(x),$$

with $P_2 \in \mathscr{P}_{n,d-2}$. Observe that $x_0^{d-2} P_2(x) \in \mathscr{S}_{e_0}$. By definition of the Bombieri–Weyl scalar product, for which the monomials are orthogonal, the three summands above belong to orthogonal subspaces. Consequently, using again the fact that monomials are orthogonal, and that $\|x_0^d\|_{\mathrm{BW}} = 1$ and $\|x_0^{d-1} x_k\|_{\mathrm{BW}} = \sqrt{d}$, we have:

$$\mathrm{dist}_{\mathrm{BW}}(p, \mathscr{S}_{e_0}) = \left\| a_d x_0^d + \sum_{k=1}^{n} a_{d-1,k} x_0^{d-1} x_k \right\|_{\mathrm{BW}} = \left(a_d^2 + \frac{1}{d} \sum_{k=1}^{n} a_{d-1,k}^2 \right)^{\frac{1}{2}}.$$

Observe also that $a_d = \widetilde{p}(e_0)$ and that $(0, a_{d-1,1}, \dots, a_{d-1,n}) = \nabla \widetilde{p}(e_0)$, which means

$$\mathrm{dist}_{\mathrm{BW}}(\widetilde{p}, \mathscr{S}_{e_0}) = \left(\widetilde{p}(e_0)^2 + \frac{1}{d} \|\nabla \widetilde{p}(e_0)\|^2 \right)^{\frac{1}{2}}.$$

Since $\widetilde{p} = p \circ g^{-1}$, we have $\widetilde{p}(e_0) = p(u)$. Moreover, for every $v \in T_{e_0} S^n$

$$\begin{aligned} \langle \nabla \widetilde{p}(e_0), v \rangle &= (d_{e_0} \widetilde{p}) v = d_{e_0}(p \circ g^{-1}) v \\ &= d_{g^{-1} e_0} p \circ g^{-1} v = \langle \nabla p(u), g^{-1} v \rangle \\ &= \langle g \nabla p(u), v \rangle, \end{aligned}$$

which shows that $\nabla \widetilde{p}(e_0) = g \nabla p(u)$. In this way, using Section 9.5, we can rewrite Section 9.5 as

$$\mathrm{dist}_{\mathrm{BW}}(p, \mathscr{S}_u) = \mathrm{dist}_{\mathrm{BW}}(p, \mathscr{S}_{e_0}) = \left(|p(u)|^2 + \frac{\|\nabla p(u)\|^2}{d} \right)^{\frac{1}{2}}.$$

Together with Section 9.5, this implies Theorem 9.23.

For the second part of the statement, let u_0 be such that

$$\mathrm{dist}(p, \mathscr{S}) = \left(|p(u_0)|^2 + \frac{\|\nabla p(u_0)\|^2}{d} \right)^{\frac{1}{2}} = \mathrm{dist}(p, \mathscr{S}_{u_0})$$

and pick $g \in O(n+1)$ such that $gu_0 = e_0$. Writing as before $\widetilde{p} := p \circ g^{-1}$, we know from Section 9.5 that the point of closest distance to $\widetilde{p}$ from $\mathscr{S}_{e_0}$ is the polynomial $x_0^{d-2}R_2(x)$ appearing in Section 9.5. This means that

$$s \circ g^{-1} = \widetilde{p}(x) - a_d x_0^d - \sum_{k=1}^{n} a_{d-1,k} x_0^{d-1} x_k.$$

We have already observed that $a_d = p(u_0)$. Notice also that $x_0^k = \langle x, e_0 \rangle^k$ and that

$$\sum_{k=1}^{n} a_{d-1,k} x_k = \langle x, \nabla \widetilde{p}(e_0) \rangle.$$

This means that

$$(s \circ g^{-1})(x) = \widetilde{p}(x) - p(u_0)\langle x, e_0 \rangle^d - \langle \nabla \widetilde{p}(e_0), x \rangle \langle x, e_0 \rangle^{d-1}.$$

In particular

$$\begin{aligned} s(x) &= p(x) - p(u_0)\langle gx, e_0 \rangle^d - \langle \nabla \widetilde{p}(e_0), gx \rangle \langle gx, e_0 \rangle^{d-1} \\ &= p(x) - p(u_0)\langle x, g^{-1}e_0 \rangle^d - \langle g^{-1}\nabla \widetilde{p}(e_0), x \rangle \langle x, g^{-1}e_0 \rangle^{d-1} \\ &= p(x) - p(u_0)\langle x, u_0 \rangle^d - \langle \nabla p(u_0), x \rangle \langle x, u_0 \rangle^{d-1}. \end{aligned}$$

This concludes the proof. □

In the special case of polynomials of degree two (quadrics) the previous result can be considered as a special case of the Eckart–Young Theorem for symmetric matrices, which we now discuss. To start with, we introduce the following definition.

Definition 9.24 (Frobenius Scalar Product)**.** The Frobenius scalar product is the scalar product defined on the space of matrices $\mathbb{R}^{n\times n}$ by

$$\langle A, B \rangle_F := \operatorname{tr}(AB^T).$$

Given a matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma_1(A) \leq \cdots \leq \sigma_n(A)$ its *singular values*, i.e. the square roots of the eigenvalues, arranged in increasing order, of AA^T (a positive semidefinite matrix). We recall the Eckart–Young theorem.

Theorem 9.25. *The distance, in the Frobenius norm, between a matrix $A \in \mathbb{R}^{n\times n}$ and the set $S \subset \mathbb{R}^{n\times n}$ of matrices with zero determinant, is given by:*

$$\operatorname{dist}_F(A, S) = \sigma_1(A).$$

Proof. If the matrix A is already with zero determinant, $\sigma_1(A) = 0 = \mathrm{dist}_F(A, S)$. Otherwise assume that A is invertible, in which case

$$\sigma_1(A) = \|A^{-1}\|_{\mathrm{op}}^{-1},$$

where $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm. To see this identity, simply use the fact that, for every matrix B, there exists R_1, R_2 orthogonal matrices such that $R_1BR_2 = \mathrm{diag}(\sigma_1(B), \ldots, \sigma_n(B))$, which implies $\|B\|_{\mathrm{op}} = \sigma_n(B)$. In the case $B = A^{-1}$, we have $\sigma_i(B) = \sigma_{n-i+1}(A)^{-1}$ and

$$\|A^{-1}\|_{\mathrm{op}} = \|B\|_{\mathrm{op}} = \sigma_n(B) = \sigma_1(A)^{-1},$$

which is equivalent to Section 9.5.

We prove now the inequality $\mathrm{dist}_F(A, S) \leq \|A^{-1}\|_{\mathrm{op}}^{-1}$. Let $v \in \mathbb{R}^n$ of norm one such that $\|A^{-1}v\| = \|A^{-1}\|_{\mathrm{op}}$ and consider $u := \|A^{-1}\|_{\mathrm{op}}^{-1}A^{-1}v$. Then u has norm one and we can find $R \in O(n)$ such that $Re_1 = u$, so that Au is the first column of AR. In particular, using the fact that, if R is orthogonal, $X \mapsto XR$ is an orthogonal transformation in the Frobenius norm and that $\det(X) = \det(XR)$,

$$\begin{aligned}\mathrm{dist}_F(A, S) &= \mathrm{dist}_F(AR, S) \\ &\leq \mathrm{dist}_F(AB - (Au, 0, \ldots, 0), S) + \|(Au, 0, \ldots, 0)\|_F = \|A^{-1}\|_{\mathrm{op}}^{-1}.\end{aligned}$$

To see the opposite inequality, let $Z \in S$ such that $\|A - Z\|_F = \mathrm{dist}(A, S)$. Pick a norm one element v in the kernel of Z (which is nonzero) and consider $R \in O(n)$ such that $Re_1 = v$. The first column of MR is zero and $\|ARe_1\| \leq \|(A - Z)R\|_F = \mathrm{dist}_F(A, S)$. In particular

$$\|A^{-1}\|_{\mathrm{op}} \geq \frac{\|Re_1\|}{\|ARe_1\|} \geq \frac{1}{\mathrm{dist}_F(A, S)}.$$

This proves the inequality $\mathrm{dist}_F(A, S) \geq \|A^{-1}\|_{\mathrm{op}}^{-1}$ and concludes the proof. □

To get back to the case of symmetric matrices, the orthogonal group $O(n)$ acts on $\mathrm{Sym}_n(\mathbb{R})$ by congruence, i.e. there is a representation $\rho_{\mathrm{Sym}_n} : O(n) \to \mathrm{GL}(\mathrm{Sym}_n(\mathbb{R}))$ given by $\rho_{\mathrm{Sym}_n}(R)(Q) := R^TQR$. Observe that for every $R \in O(n)$ the linear map $\rho_{\mathrm{Sym}_n}(R) : \mathrm{Sym}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$ is orthogonal with respect to the Frobenius scalar product. Recall also from Section 9.4 that there is a linear isomorphism

$$\phi : \mathrm{Sym}_n(\mathbb{R}) \to \mathscr{P}_{n-1,2}$$

given by associating to every $Q \in \mathrm{Sym}_n(\mathbb{R})$ the quadratic form $q \in \mathscr{P}_{n-1,2}$ defined

by
$$q(x) := \langle x, Qx\rangle.$$

Proposition 9.26. *The isomorphism Section 9.5 is an isometry of Euclidean spaces:*
$$\phi : (\mathrm{Sym}_n(\mathbb{R}), \langle\cdot,\cdot\rangle_F) \simeq (\mathscr{P}_{n-1,2}, \langle\cdot,\cdot\rangle_{BW}).$$
Moreover, ϕ is an isomorphism of representations: $\rho_{n-1,2}\phi = \phi\rho_{\mathrm{Sym}_n}$.

Proof. It suffices to show that ϕ sends a Frobenius orthonormal basis to a Bombieri–Weyl orthonormal one. As is can be easily verified, the following is an orthonormal basis for $\mathrm{Sym}_n(\mathbb{R})$ for the Frobenius scalar product
$$\{E_{ii}, i = 1, \dots, n\} \cup \left\{\frac{1}{\sqrt{2}}(E_{ij} + E_{ji}), i < j\right\}.$$
By construction we have
$$\phi(E_{ii}) = x_i^2 \quad \text{and} \quad \phi\left(\frac{1}{\sqrt{2}}(E_{ij} + E_{ji})\right) = \sqrt{2}x_ix_j.$$
By definition of the Bombieri–Weyl scalar product (Section 8.1),
$$\left\{x_i^2, i = 1, \dots, n\right\} \cup \left\{\sqrt{2}x_ix_j, i < j\right\}$$
is a an orthonormal basis for $(\mathscr{P}_{n-1,2}, \langle\cdot,\cdot\rangle_{\mathrm{BW}})$. Together with Section 9.5 this proves the first part of the statement.

The second part is immediate from the definition of ϕ: given $R \in O(n)$ and $Q \in \mathrm{Sym}_n(\mathbb{R})$, we have
$$\begin{aligned}\phi(\rho_{\mathrm{Sym}_n}(R))(Q)(x) = \langle x, R^TQRx\rangle = \langle Rx, QRx\rangle &= \phi(Q)(Rx)\\ &= \rho_{n-1,2}(\phi(Q))(x).\end{aligned}$$
□

Exercise 9.2. Let $Q \in \mathrm{Sym}_n(\mathbb{R})$ and denote by $q \in \mathscr{P}_{n-1,2}$ the quadratic form defined by $q(x) = \langle x, Qx\rangle$. Prove that
$$\sigma_1(Q) = \min_{u\in S^n}\left(|q(u)|^2 + \frac{\|\nabla q(u)\|^2}{2}\right)^{\frac{1}{2}}.$$

Notice that exercise 9.2 together with Theorem 9.25 do not imply Theorem 9.23 for the space of quadrics. In fact, since $S \cap \mathrm{Sym}_n(\mathbb{R}) \subsetneq S$, a priori the distance from Q to $S \cap \mathrm{Sym}_n(\mathbb{R})$ could be larger than $\mathrm{dist}(Q, S)$ (which is computed in the whole space $\mathbb{R}^{n \times n}$).

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