Max-Planck-Institut für
Mathematik
in den Naturwissenschaften

Computing and decomposing tensors

Download `Tensorlab` at `https://www.tensorlab.net`. Download `MATLAB` code for computing the geometric condition number and the tensor rank decomposition and for the GEVD algorithm at `http://personal-homepages.mis.mpg.de/breiding/tensors_cond_and_pba.zip`. Uncompress the `.m` files. Add the files to the current `MATLAB` session by typing `addpath(p)`, where `p` is a `char` of the directory where you saved the `.m` files to.

Exercise 1

Generate a random real $3 \times 3 \times 2$ tensor `T` of rank 3 by sampling factor matrices $A \in \mathbb{R}^{3\times 3}, B \in \mathbb{R}^{3\times 3}, C \in \mathbb{R}^{2\times 3}$. In `Tensorlab` the tensor with factor matrices A, B, C is generated with `T = cpdgen({A,B,C})`. Compute the rank-3-CPD of `T` by using `cpd(T,3)`. What is the output format of `cpd`? Compare the output to `T`. Compute the Frobenius norm of the difference by using `frob`.

Exercise 2

Generate 20 random real 2×2 matrices. Compute their ranks. Generate 20 random real $2\times 2\times 2$ tensors and compute their ranks. Compare to the matrix case. Compute the Kruskal rank of the tensor decompositions. Is the decomposition unique?

Exercise 3

Generate a (random) tensor `T` $\in \mathbb{R}^{4\times 3\times 3}$ and matrices $M_1 \in \mathbb{R}^{2\times 4}, M_2 \in \mathbb{R}^{2\times 3}$ and $M_3 \in \mathbb{R}^{2\times 3}$. Implement the multilinear multiplication $(M_1, M_2, M_3) \cdot$ `T`. `Tensorlab`'s `tens2mat` function is helpful. What is the format of the output?

Exercise 4

Implement an algorithm computing the Higher Order Singular Value Decomposition (HOSVD) algorithm (you may use the `svd` command in `MATLAB`). Return the multilinear rank as output. Compare with `Tensorlab`'s `HOSVD` command. What is the multilinear rank of a random real $3 \times 3 \times 2$ tensor? What is the multilinear rank of a random real $3 \times 3 \times 2$ tensor of rank 3?

Exercise 5

Compute the standard HOSVD of a $3 \times 3 \times 9$ tensor of multilinear rank $(2, 2, 2)$. Then, compute the same HOSVD of the reshaped tensor of format $9 \times 3 \times 3$. Compare both computations with an ST-HOSVD. Compare the speed of the computations. What are the respective computational complexities?

Exercise 6

Generate the $4 \times 3 \times 3$ tensor with factor matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This tensor has rank 3. Compute its CPD in `Tensorlab`. Investigate the output. What are the norms of the rank-1 summands? Consider the condition number of the output: `cpdgeomcond(output)`.

Exercise 7

Sample a random orthogonal matrix $Q \in \mathbb{R}^{5\times 2}$. For instance, this can be done by sampling $A \in \mathbb{R}^{5\times 2}$ and passing it to `MATLAB`'s `orth` function. Sample a random tensor $\mathtt{T} \in \mathbb{R}^{6\times 5\times 5}$ of rank 3. Compute the CPD of $\mathtt{T}$ with GEVD (generalized eigenvalue decomposition) using the matrix $\mathtt{Q}$ for the projection: `cpd_pba(T, 3,` Q`)`.

Now, let $U \in \mathbb{R}^{5\times 5}$ be an orthogonal matrix whose first two colums are Q. Sample orthogonal matrices $A \in \mathbb{R}^{5\times 3}, B \in \mathbb{R}^{4\times 3}$ and put

$$C := 0.4 \cdot U \begin{bmatrix} 1.5 & 1 & 1 \\ -1 & -1.5 & 1 \\ -1 & 1 & -1.5 \\ -1 & 1 & 1 \\ -0.99 & 1 & 1 \end{bmatrix},$$

Let $\mathtt{S}$ be the tensor with factor matrices A, B, C. Run `cpd_gevd(S, 3,` Q`)` and compare the result with the output of `cpd`.

Finally, compute the CPD of $\mathtt{T}$, $(I_6, I_5, Q) \cdot \mathtt{T}$, $\mathtt{S}$ and $(I_6, I_5, Q) \cdot \mathtt{S}$ and compare the condition numbers of the decompositions you get (I_n is the $n \times n$ identity matrix).

Exercise 8

Generate random factor matrices $A \in \mathbb{R}^{3\times 2}, B \in \mathbb{R}^{3\times 2}, C \in \mathbb{R}^{3\times 2}$. Let $\mathtt{T}$ be the corresponding tensor. Furthermore, sample a $3 \times 3 \times 3$ tensor $\mathtt{X}$ (the rank of $\mathtt{X}$ is not important) and factor matrices $A' \in \mathbb{R}^{3\times 2}, B' \in \mathbb{R}^{3\times 2}, C' \in \mathbb{R}^{3\times 2}$. Denote $\Delta A = 10^{-6}\frac{A'}{\|A'\|_F}$, $\Delta B = 10^{-6}\frac{B'}{\|B'\|_F}$, $\Delta C = 10^{-6}\frac{C'}{\|C'\|_F}$. For $k \in \{0, 10^{-14}, 10^{-12}\}$ let $\mathtt{T}_k = \mathtt{T} + k\frac{\mathtt{X}}{\|\mathtt{X}\|_F}$ and compute the CPD of $\mathtt{T}_k$ for a maximal number of iterations $1 \le m \le 5$ and with initial guess $(A + \Delta A, B + \Delta B, C + \Delta C)$ and plot the residuals $\|\mathtt{T}_k - \mathtt{O}_m\|_F$ against m (here, $\mathtt{O}_m$ is the output from CPD).

Exercise 9

compute CPD of some actual data tensor (e.g. the nice amino acid data set): first compute an ST-HOSVD compression to rank (10,10,10), what is approximation error? then compute CPD of the compressed tensor. Compare with computing the CPD directly from input tensor (one needs to disable the automatic compression in TL). Which is faster? Is there a difference in the approximation error? Have a look at factor matrices, compute kruskal ranks (is it unique?), compute condition number (is it stable?), etc. This could maybe be the final wrap-up exercise?