PROBLEMS AND EXERCISES

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1. Monodromy

(1) Show that if $X \xrightarrow{\pi} Y$ is a d-to-1 branched cover of irreducible varieties, then the monodromy group of π is transitive. Show the converse: that if the monodromy group of π is transitive, then X is irreducible.

(2) Let $X \xrightarrow{\pi} Y$ be a d-to-1 cover of irreducible varieties. Show that the monodromy group of π is 2-transitive if and only if the variety

$$\{(x_1, x_2, y) \mid x_1, x_2 \in \pi^{-1}(y), y \in Y\}$$

has two maximal dimensional irreducible components

(3) Suppose $F_c(x)$ is a zero-dimensional parametrized polynomial system with variables $x_1, \ldots, x_n$ and parameters $c_1, \ldots, c_k$ defined over the real numbers. i.e. we have a branched cover

$$\{(x, c) \mid F_c(x) = 0\} \to \mathbb{C}^k_c$$
$$(x, c) \mapsto c$$

Is it possible for a real path to produce a nontrivial monodromy permutation? Under which conditions can this happen?

2. Witness Sets

(1) Prove the trace test for plane curves.

Hint: for one direction, it is useful to know that the monodromy group

$$\{(x, L) \mid x \in X \cap L\} \xrightarrow{\pi} \mathrm{Gr}(2,3)$$
$$(x, L) \mapsto L$$

is the full symmetric group

(2) Compute a witness set for

$$\mathrm{SO}(5) = \{M \in \mathrm{Mat}_{\mathbb{C}}(5, 5) \mid MM^T = \mathrm{id}, \det(M) = 1\}$$

(3) How many maximal dimensional irreducible components does

$$\mathrm{HSO}(4) = \{M \in \mathrm{Mat}_{\mathbb{C}}(4, 4) \mid MM^T = \mathrm{id}, \det(M) = 1, M_{i,i} = 0 \text{ for all } i\}$$

have? What are their degrees? How do they intersect?

(4) The Lüroth hypersurface $\mathfrak{L}$ is the hypersurface in the space of plane quartics parameterized by

$$(\mathbb{C}^3)^5 \to \mathbb{P}^{15}$$
$$(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \mapsto \sum_{i=1}^{5} \prod_{j \neq i} \ell_j \text{ where } \ell_i = a_i x + b_i y + c_i$$

Compute a witness set for $\mathfrak{L}$. What is its degree?

3. Other

(1) A conic is the zero set of a quadratic polynomial

$$c(x,y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$$

with $a_i \in \mathbb{C}$.

[Emiris and Tzoumas](http://www.win.tue.nl/EWCG2005/Proceedings/38.pdf) write that there are 184 complex circles that are tangent to 3 general conics C_1, C_2 and C_3. This means, that there are 184 complex solutions (a_1, a_2, r) such that there exists some $(x,y) \in \mathbb{C}^2$ with

- $(x-a_1)^2 + (y-a_2)^2 = r$,
- $(x,y) \in C_i$ for $1 \le i \le 3$ and
- $(x-a_1, y-a_2)$ spans the normal space of C_i at (x,y) for $1 \le i \le 3$.

(a) Setup the polynomial system for 3 general conics and verify that this system has indeed 184 solutions

(b) Consider the three conics

$$C_1 = \{y = -x^2 + 2x + 5\}, C_2 = \{y = 2x^2 + 5x - 8\}$$
$$\text{and } C_3 = \{y = 8x^2 - 3x - 2\}.$$

How many circles are tangent to these 3 conics? How many of them are real?

(c) Find a configuration of 3 conics with as many real solutions as possible. It is possible to find 184 real solutions?

(2) A real algebraic variety is the common zero set of polynomials $f_1, \dots, f_m \in \mathbb{R}[x_1, \dots, x_n]$ denoted by $X = V(f_1, \dots, f_m)$.

A bottleneck of X is defined to be a pair of distinct points $x, y \in X$ such that $x - y$ is orthogonal to the tangent space T_xX and to T_yX.

In [DEW18](https://arxiv.org/abs/1904.04502) it is shown that a generic plane curve of degree d has $d^4 5d^2 + 4d$ bottleneck pairs. This is called the bottleneck degree of the curve.

Consider the curve $X = V(f)$ defined by $f = (x^4 + y^4 - 1)(x^2 + y^2 - 2) + x^5y$.

(a) Write down definining equations for computing all bottlenecks.

(b) What is the Bottleneck degree of X? How many real bottlenecks does it have?

(c) What are the coordinates smallest bottleneck pair?

(d) What effect do different start systems have on the number of paths necessary to track?

(e) Visualize all bottlenecks for your favorite plane curve

(3) Consider a general quartic surface $f \in \mathbb{C}[x,y,z]$.

This is defined by a random polynomial $f \in \mathbb{C}[x,y,z]$ of degree 4.

We want to count the number of planes in three-space which are tangent to f in at least 3 points.

(a) Set up polynomial systems to compute all tritangent planes of a general quartic surface. (Hint you should obtain a polynomial system in 11 variables).

(b) What is the Bezout bound of the system in (a)?

(c) Use the monodromy method to solve the system from (a).