

# PROBLEMS AND EXERCISES

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## 1. CODING EXERCISES

- (1) A conic is the zero set of a quadratic polynomial

$$c(x, y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$$

with  $a_i \in \mathbb{C}$ .

Emiris and Tzoumas showed that there are 184 complex circles that are tangent to 3 general conics  $C_1$ ,  $C_2$  and  $C_3$ . This means, that there are 184 complex solutions  $(a_1, a_2, r)$  such that there exists some  $(x, y) \in \mathbb{C}^2$  with

$$(x - a_1)^2 + (y - a_2)^2 = r$$

$$(x, y) \in C_i \text{ for } 1 \leq i \leq 3$$

$$(x - a_1, y - a_2) \text{ spans the normal space of } C_i \text{ at } (x, y) \text{ for } 1 \leq i \leq 3.$$

- (a) Define the polynomial system for 3 general conics and verify that this system has indeed 184 solutions. Use certification.
- (b) Consider the three conics

$$C_1 = \{y = -x^2 + 2x + 5\},$$

$$C_2 = \{y = 2x^2 + 5x - 8\},$$

$$C_3 = \{y = 8x^2 - 3x - 2\}.$$

How many circles are tangent to these 3 conics? How many of them are real?

- (c) Find a configuration of 3 conics with as many real solutions as possible. It is possible to find 184 real solutions?
- (2) A real algebraic variety is the common zero set of polynomials  $f_1, \dots, f_m \in \mathbb{R}[x_1, \dots, x_n]$  denoted by  $X = V(f_1, \dots, f_m)$ . A bottleneck of  $X$  is defined to be a pair of distinct points  $x, y \in X$  such that  $x - y$  is orthogonal to the tangent space  $T_x X$  and to  $T_y X$ .

It was recently shown that a generic plane curve of degree  $d$  has  $d^4 - 5d^2 + 4d$  bottleneck pairs. This is called the bottleneck degree of the curve.

Consider the curve  $X = V(f)$  defined by

$$f = (x^4 + y^4 - 1)(x^2 + y^2 - 2) + x^5y.$$

- (a) Write down defining equations for computing all bottlenecks.
  - (b) What is the Bottleneck degree of  $X$ ? How many real bottlenecks does it have?
  - (c) What are the coordinates smallest bottleneck pair?
  - (d) What effect do different start systems have on the number of paths necessary to track?
- (3) Consider a general quartic surface  $X \in \mathbb{C}^3$ . This is defined by a random polynomial  $f \in \mathbb{C}[x, y, z]$  of degree 4. `HC.jl` provides functions to sample random polynomials. We want to count the number of planes in three-space which are tangent to  $f = 0$  in at least 3 points.
- (a) Set up polynomial systems to compute all tritangent planes of a general quartic surface. (*Hint: you should obtain a polynomial system in 11 variables*).
  - (b) Use monodromy to solve the system from (a).
- (4) Extend the triangulation example from two to three (or more) cameras.
- (5) Verify that the configurations of 5 conics at this link has 3264 real conics, which are simultaneously tangent to all 5 of them. Use certification methods to obtain a proof!
- (6) A quintic threefold is the zero set  $X = \{(x, y, z, w) \in \mathbb{C}^4 \mid f(x, y, z, w) = 0\}$  of a polynomial  $f \in \mathbb{C}[x, y, z, w]$  of degree 5. Let  $X$  be a quintic threefold defined by a randomly chosen polynomial  $f$ . Then,  $X$  contains finitely many lines. Compute them. How many are there?
- (7) Consider three Gaussian random variables  $X_1, X_2, X_3$  with means  $\mu_1, \mu_2, \mu_3$  and variances  $\sigma_1^2, \sigma_2^2, \sigma_3^2$ . The density of  $X_i$  is

$$\phi_i(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}.$$

A mixture of the three random variables is a random variable with density

$$\psi(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x), \quad \text{for } a_1 + a_2 + a_3 = 1.$$

The method of moments recovers  $\psi$  from the moments

$$m_k = \int x^k \psi(x) dx.$$

Since we have 9 unknowns plus one equation  $a_1 + a_2 + a_3 = 1$ , we expect to need at least 8 moments to recover  $\psi$ . Set up a system  $F$  with variables  $\mu_1, \mu_2, \mu_3, \sigma_1^2, \sigma_2^2, \sigma_3^2$  and  $a_1, a_2, a_3$  that computes the first 8 moments of  $\psi$ . Then, randomly sample values for the 9 variables, evaluate  $F$  at those values. This gives a vector  $m \in \mathbb{C}^8$ . Solve the system  $F - m = 0$  (see here, if you need help).

- (8) Describe the solution set to the system

$$\begin{aligned} & (4s_4^2 - 2s_4s_6 + 4s_4s_7 + 6s_4s_8 - 2s_4 - 2s_6s_8 + s_6 + 4s_7s_8 + 2s_8^2 - s_8)y + \\ & (-4s_2s_4 - 4s_2s_8 + s_2 - 4s_4^2 - 4s_4s_6 - 8s_4s_8 + 3s_4 - 4s_6s_8 + 2s_6 + 2s_7 - \\ & \quad 4s_8^2 + 4s_8 - 1)z = 0. \\ & (2s_6 + 4s_7 + 2s_8 - 1)z^2 + (8s_2s_4^2 + 16s_2s_4s_8 - 4s_2s_4 - s_2s_6 - 2s_2s_7 + \\ & \quad 8s_2s_8^2 - 5s_2s_8 + s_2 + 8s_4^3 + 16s_4^2s_7 + 16s_4^2s_8 - 8s_4^2 + 3s_4s_6 + \\ & \quad 32s_4s_7s_8 - 2s_4s_7 + 8s_4s_8^2 - 9s_4s_8 + s_4 - s_6^2 - 2s_6s_7 + 2s_6s_8 + \\ & \quad 16s_7s_8^2 - 2s_7s_8 - s_8^2) = 0. \end{aligned}$$

in the variables  $s_2, s_4, s_6, s_7, s_8, y, z$  restricted to the locally closed set

$$\mathcal{C} = V_1 - (V_2 \cap V_3) \cup (V_4 \cap V_5),$$

where  $V_1$  is the solution set to

$$s_2^2 + 2s_2s_4 + s_2s_6 + 2s_2s_7 + s_2s_8 - s_2 + s_4^2 + s_4s_6 + 2s_4s_7 + s_4s_8 - s_4 + 2s_6s_7 + 2s_7s_8 = 0,$$

and

$$V_2 = \{2s_6 + 4s_7 + 2s_8 - 1 = 0\},$$

$$V_3 = \{s_2 + s_4 - 2s_7 = 0 = 0\},$$

$$V_4 = \{4s_4^2 - 2s_4s_6 + 4s_4s_7 + 6s_4s_8 - 2s_4 - 2s_6s_8 + s_6 + 4s_7s_8 + 2s_8^2 - s_8 = 0\},$$

$$V_5 = \{s_2 - s_4 + s_6 - s_8 = 0\},$$

(submitted by Ernesto Álvarez González.)

- (9) A *Gaussian conditional independence model* is a semialgebraic set of *real, positive definite*  $n \times n$  matrices given by a set of “CI statements”. Such a statement is of the form  $(ij|K)$  where  $i, j \in [n]$  are different and  $K \subseteq [n] \setminus \{i, j\}$ . It prescribes the following relation on the entries of a positive definite  $\Sigma$ :

$$\det \Sigma_{ij|K} := \det \Sigma_{\{i\} \cup K, \{j\} \cup K} = 0,$$

i.e., the “almost-principal submatrix” with rows  $i$  and  $K$  and columns  $j$  and  $K$  is singular. The roles of  $i$  and  $j$  in this statement are symmetric, so we identify  $(ij|K)$  with  $(ji|K)$ . There are  $\binom{n}{2} 2^{n-2}$  possible CI statements. For any set of them, the Gaussian model is the set of real, positive definite matrices which satisfy the indicated equations and *no other* CI statements. Decide whether the following Gaussian CI models are empty or not:

- (a)  $n = 3$ :  $(12|), (12|3)$ .
  - (b)  $n = 4$ :  $(12|), (12|34), (34|1), (34|2)$ .
  - (c)  $n = 5$ :  $(15|23), (15|234), (23|15), (23|145), (34|12), (34|15), (34|125), (45|12), (45|23), (45|123)$ .
  - (d)  $n = 5$ :  $(12|345), (13|25), (14|2), (15|24), (23|1), (24|15), (25|), (34|), (35|14), (45|123)$ .
- Note:**  $(23|145)$  should be read as  $i = 2, j = 3, K = \{1, 4, 5\}$ .

## 2. WITNESS SETS

- (1) Prove the trace test for plane curves. i.e. prove
  - if  $C = \mathcal{V}(f(x, y))$  is an irreducible curve, and  $L_t$  a generic parallel family of lines, then the sum of the points  $C \cap L_t$  moves linearly in  $t$ .  
(Hint: assume  $L_t$  are the lines  $\mathcal{V}(x - t)$ )
  - if  $S_t$  is a proper subset of  $C \cap L_t$ , then the sum of the points in  $S_t$  moves nonlinearly as  $t$  moves.  
(Hint: the monodromy of the points  $C \cap L_t$ , as  $t$  moves, is the full symmetric group)
- (2) How many maximal dimensional irreducible components does
 
$$\text{HSO}(4) = \{M \in \text{Mat}_{\mathbb{C}}(4, 4) \mid MM^T = \text{id}, \det(M) = 1, M_{i,i} = 0 \text{ for all } i\}$$
 have? What are their degrees? How do they intersect?
- (3) Given a surface  $X \subset \mathbb{R}^3$ , the *flecnodal surface*  $\mathcal{F}(X)$  is the union of all lines  $L$  with contact order 4 at a point of  $X$ . It is one of five *event surfaces* described in “Changing views on curves and surfaces” by Kohn, Sturmfels, and Trager.

- A quintic polynomial

$$f = c_5 + c_4t + \cdots + c_1t^4 + c_0t^5$$

has a root of multiplicity four whenever the coefficients satisfy

$$20c_0c_4 - 8c_1c_3 + 3c_2^2 = 0$$

$$50c_0c_5 - 6c_1c_4 + c_2c_3 = 0$$

$$20c_1c_5 - 8c_2c_4 + 3c_3^2 = 0.$$

Compute a witness set for the above equations. For each witness point, solve the corresponding polynomial and observe that one root comes with multiplicity four.

- Consider the quintic surface

$$f = x_1^5 + x_2^5 + x_3^5 + 1 + (x_1 + x_2 + x_3 + 1)^5 + x_1x_2x_3(x_1 + x_2 + x_3 + 1)$$

in  $\mathbb{R}^3$ . Find a suitable parametrization for lines in  $\mathbb{R}^3$ . Compute a witness set for the Flecnodal surface of  $X = \mathcal{V}(f)$ . What is its degree?

- We can pass from  $\mathbb{R}^3$  to projective space  $\mathbb{P}^3$ . Here, lines can be parametrized using Plücker coordinates  $\{q_{1,2}, q_{1,3}, q_{1,4}, q_{2,3}, q_{2,4}, q_{3,4}\}$ :

$$z(t) = (q_{1,2} : tq_{1,2} : tq_{1,3} - q_{2,3} : tq_{1,4} - q_{2,4})$$

Homogenize  $f(x_1, x_2, x_3)$  to  $f_{\text{hom}}(x_0, x_1, x_2, x_3)$  and compute a witness set for the Flecnodal surface of  $X = \mathcal{V}(f_{\text{hom}})$  in the coordinates  $q$  (remember to include the Plücker relation  $q_{1,2}q_{3,4} - q_{1,3}q_{2,4} + q_{1,4}q_{2,3} = 0$ .) What is its degree?

### 3. TOTAL DEGREE AND POLYHEDRAL HOMOTOPIES

- (1) (Conics in the plane). Consider the total degree family  $\mathcal{F}(2, 2)$ , i.e.  $n = 2$  and  $(d_1, d_2) = (2, 2)$ :

$$\mathcal{F}(2, 2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix}.$$

What is  $\mathcal{N}_{\text{Béz}}$  in this example? Verify this by solving a random member of this family

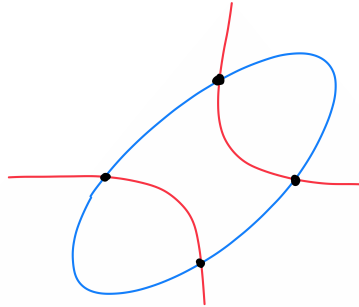


FIGURE 1. Two *generic* conics in the plane.

using `HomotopyContinuation.jl`.

There are strictly less than  $\mathcal{N}_{\text{Béz}}$  solutions in the following two scenarios.

- Two or more solutions *coincide*. This happens if there are solutions to the overdetermined system

$$f_1 = f_2 = \det \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} = 0 \quad \text{has a solution.}$$

Here  $f_{ix} = \partial f_i / \partial x$  and likewise for  $f_{iy}$ . Prove (possibly using a computer algebra system) that this is equivalent to the vanishing of a nonzero polynomial in the coefficients of  $f_1, f_2$ . This polynomial is called the *discriminant*.

- There are solutions *at infinity*. To make this precise, we homogenize the equations:

$$\begin{pmatrix} a_{00}z^2 + a_{10}xz + a_{01}yz + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{00}z^2 + b_{10}xz + b_{01}yz + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix},$$

and consider solutions with  $z = 0$ . Geometrically, we replace our conics by their closures in  $\mathbb{P}^2$ . Show that there are solutions ‘at infinity’ if and only if

$$\det A_\infty = \det \begin{pmatrix} a_{20} & a_{11} & a_{02} \\ & a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} \\ & b_{20} & b_{11} & b_{02} \end{pmatrix} = 0.$$

What about the case where  $f_1 = f_2 = 0$  has infinitely many solutions?

Construct two members of  $\mathcal{F}(2, 2)$  with 3 solutions, one with a solution at infinity, and one with a solution of multiplicity 2. Verify using `HomotopyContinuation.jl`.

- (2) (Systems supported on the square). Consider the subfamily

$$\mathcal{F}_Q = \begin{pmatrix} f_1 = a_{00} + a_{10}x + a_{10}y + a_{11}xy \\ f_2 = b_{00} + b_{10}x + b_{10}y + b_{11}xy \end{pmatrix} \subset \mathcal{F}(2, 2).$$

Use the previous exercise to show that  $\mathcal{N}(Q) < \mathcal{N}_{\text{Béz}}$ . Verify the formula  $\text{MV}(P_1, P_2) = \text{Vol}_2(P_1 + P_2) - \text{Vol}_2(P_1) - \text{Vol}_2(P_2)$  for  $P_1 = P_2 = [0, 1]^2 \subset \mathbb{R}^2$ . More generally, compute  $\text{MV}(P_1, \dots, P_n)$  with  $P_i = [0, 1]^n \subset \mathbb{R}^n$  for all  $i$ . This corresponds to a sparse family  $\mathcal{F}_Q \subset \mathcal{F}(n, \dots, n)$ . Compare  $\mathcal{N}(Q)$  for these systems with their Bézout number. For some  $n$ , solve a generic member of  $\mathcal{F}_Q$  using a total degree and a polyhedral start system in `HomotopyContinuation.jl`.

- (3) (Asymptotic BKK and Bézout numbers). This is an example taken from “A polyhedral method for solving sparse polynomial systems” by Birkett Huber and Bernd Sturmfels. Consider the family

$$\begin{pmatrix} a_1 + a_2x + a_3x^ky^k \\ b_1 + b_2y + b_3x^ky^k \end{pmatrix} \subset \mathcal{F}(2k, 2k).$$

Show that  $\lim_{k \rightarrow \infty} (\mathcal{N}_{\text{BKK}} / \mathcal{N}_{\text{Béz}}) = 0$ . Compare the computation time for the function `solve` in `HomotopyContinuation.jl` using the default option (`start_system = :polyhedral`) and the option `start_system = :total_degree` for random coefficients  $a, b$  and increasing values of  $k$ .

- (4) (Toric varieties and the BKK theorem). Let  $\mathcal{A} = \{\alpha_1, \dots, \alpha_r\} \subset \mathbb{N}^n$  be a set of exponents such that  $P = \text{Conv}(\mathcal{A})$  has dimension  $n$ . The *projective toric variety*  $X_{\mathcal{A}}$  associated to  $\mathcal{A}$  is the Zariski closure of the image of the monomial map

$$(x_1, \dots, x_n) \mapsto (x^{\alpha_1} : \dots : x^{\alpha_r}) \in \mathbb{P}^{r-1}.$$

Use the BKK theorem to relate the degree of  $X_{\mathcal{A}}$  to the volume of  $P$ . The statement you obtain is known as *Kushnirenko's theorem*, which can be seen as a specialized version of the BKK theorem for *unmixed systems of equations*, for which  $\mathcal{A}_i = \mathcal{A}, i = 1, \dots, n$ .

- (5) (Puisseux series solutions). Consider the polynomial  $f = tx^3 + 2x^2 + t \in K[x]$ , where  $K$  is the field of Puiseux series with complex coefficients in the variable  $t$ . Compute the leading term of all solutions  $x \in K$  to  $f = 0$ . That is, compute all possible  $X \in \mathbb{C} \setminus \{0\}$  and  $e \in \mathbb{Q}$  such that there is a solution  $x(t) = Xt^e + \text{higher order terms}$  satisfying  $f(x(t), t) = 0$ .

*Hint: substitute  $x(t) = Xt^e + \text{higher order terms}$  in  $f(x(t), t)$  and look for all exponents  $e$  for which at least two terms of  $f(x(t), t)$  are of lowest order in  $t$ . Obtain  $X$  from the condition that these lowest order terms cancel.*

Can you give a graphical interpretation of the numbers  $e$  in terms of the Newton polygon of  $f$ ?

*Hint: draw the Newton polygon of  $f$  as a polynomial in  $x, t$ .*

- (6) (Solving binomial systems is easy). Consider the system of equations over the field  $K$  of complex Puiseux series in  $t$ :

$$F = \begin{pmatrix} 1 + 2x^2y + 3xy^2 \\ 5 + 2tx + 4ty + 6txy \end{pmatrix} = 0.$$

How many solutions  $(x(t), y(t))$  do you expect? Check that there exists a solution of the form  $x(t) = Xt^{-1} + \text{higher order terms}$  and  $y(t) = Yt^2 + \text{higher order terms}$ , where  $(X, Y)$  is the solution of

$$1 + 2X^2Y = 5 + 2X = 0.$$

Find  $e_1, e_2 \in \mathbb{Q}$  such that  $x(t) = Xt^{e_1} + \text{higher order terms}$  and  $y(t) = Yt^{e_2} + \text{higher order terms}$  gives a solution for each  $(X, Y) \in (\mathbb{C} \setminus \{0\})^2$  satisfying

$$2X^2Y + 3XY^2 = 5 + 6XY = 0.$$

To solve this *system of binomial equations*, we write it in the form

$$XY^{-1} = -3/2, \quad XY = -5/6. \quad (3.1)$$

We collect the exponent vectors in the columns of a matrix  $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ . There exist matrices  $P, Q \in \mathbb{Z}^{2 \times 2}$  with an inverse defined over  $\mathbb{Z}$  (i.e.  $P$  and  $Q$  are *unimodular*) which diagonalize  $A$ :

$$PAQ = \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix}. \quad (3.2)$$

This diagonal matrix is called the *Smith normal form* of  $A$ . Denote  $p_{ij}, q_{ij}$  for the entries of  $P$  and  $Q$  respectively. Show that the map  $(\mathbb{C} \setminus \{0\})^2 \rightarrow (\mathbb{C} \setminus \{0\})^2$  given by

$$(U, V) \mapsto (U^{p_{11}} V^{p_{21}}, U^{p_{12}} V^{p_{22}})$$

is invertible. Use this change of coordinates on  $(\mathbb{C} \setminus \{0\})^2$  and the identity (3.2) to reduce (3.1) to an equivalent system of equations

$$U^{s_1} = c_1, \quad V^{s_2} = c_2.$$

Deduce that the number of solutions of (3.1) is  $\det A$ . Can you write down an algorithm for solving a system of binomial equations in the form (3.1) with exponent matrix  $A \in \mathbb{Z}^{n \times n}$ ?

We have now found the leading term of 3 solutions to  $F = 0$ . Can you find the missing solution(s) as well?

#### 4. MONODROMY

Let  $F_c(x)$  be a zero-dimensional parametrized polynomial system with variables  $x_1, \dots, x_n$  and parameters  $c_1, \dots, c_k$ . Let  $Z \xrightarrow{\pi} \mathbb{C}^k$  be the branched cover where  $Z = \{(x, p) | F_p(x) = 0\}$  and  $\pi : Z \rightarrow \mathbb{C}^k$  is the projection onto the parameters. Let  $d$  be the degree of this branched cover. Let  $U$  be the set of regular values of  $\pi$  and  $G_\pi$  the monodromy group based at some point  $p \in U$ .

- (1) Show  $G_\pi$  is a group and it doesn't depend on the choice of  $p \in U$  where you base monodromy loops.
- (2) Show  $G_\pi$  is transitive if and only if  $Z$  has a unique irreducible component of maximal dimension. Explain why  $G_\pi$  being transitive is exactly the condition which allows `monodromy_solve` to find all solutions to  $\pi^{-1}(p)$ .
- (3) Suppose  $F_c(x)$  is defined over the real numbers.
  - Is it possible for a real path in  $U$  to produce a nontrivial monodromy permutation?
  - Give an example.
  - What does this tell you about the branch locus?
- (4) Consider the sparse polynomial system

$$\begin{pmatrix} a_0 + a_1x^2 + a_2y^2 + a_3x^2y^2 \\ b_0x + b_1x^3 + b_2xy^2 \end{pmatrix},$$

- For 1000 parameter values, solve this system and count how many real solutions you find.
- Compute the Galois group over the parameters  $\{a_0, a_1, a_3, a_4, b_0, b_1, b_2\}$ .
- What is the structure of this system that explains what you've observed?