

PROBLEMS AND EXERCISES

P. BREIDING, T. BRYSEWICZ, S. TELEN, S. TIMME

CONTENTS

1. Coding exercises	1
2. Witness Sets	3
3. Total degree and polyhedral homotopies	4
4. Monodromy	6

1. CODING EXERCISES

- (1) A conic is the zero set of a quadratic polynomial

$$c(x, y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$$

with $a_i \in \mathbb{C}$.

Emiris and Tzoumas showed that there are 184 complex circles that are tangent to 3 general conics C_1 , C_2 and C_3 . This means, that there are 184 complex solutions (a_1, a_2, r) such that there exists some $(x, y) \in \mathbb{C}^2$ with

$$(x - a_1)^2 + (y - a_2)^2 = r$$

$$(x, y) \in C_i \text{ for } 1 \leq i \leq 3$$

$$(x - a_1, y - a_2) \text{ spans the normal space of } C_i \text{ at } (x, y) \text{ for } 1 \leq i \leq 3.$$

- (a) Define the polynomial system for 3 general conics and verify that this system has indeed 184 solutions. Use certification.
 (b) Consider the three conics

$$C_1 = \{y = -x^2 + 2x + 5\},$$

$$C_2 = \{y = 2x^2 + 5x - 8\},$$

$$C_3 = \{y = 8x^2 - 3x - 2\}.$$

How many circles are tangent to these 3 conics? How many of them are real?

- (c) Find a configuration of 3 conics with as many real solutions as possible. It is possible to find 184 real solutions?
 (2) A real algebraic variety is the common zero set of polynomials $f_1, \dots, f_m \in \mathbb{R}[x_1, \dots, x_n]$ denoted by $X = V(f_1, \dots, f_m)$. A bottleneck of X is defined to be a pair of distinct points $x, y \in X$ such that $x - y$ is orthogonal to the tangent space $T_x X$ and to $T_y X$.

It was recently shown that a generic plane curve of degree d has $d^4 - 5d^2 + 4d$ bottleneck pairs. This is called the bottleneck degree of the curve.

Consider the curve $X = V(f)$ defined by

$$f = (x^4 + y^4 - 1)(x^2 + y^2 - 2) + x^5y.$$

- (a) Write down defining equations for computing all bottlenecks.
 - (b) What is the Bottleneck degree of X ? How many real bottlenecks does it have?
 - (c) What are the coordinates smallest bottleneck pair?
 - (d) What effect do different start systems have on the number of paths necessary to track?
- (3) Consider a general quartic surface $X \in \mathbb{C}^3$. This is defined by a random polynomial $f \in \mathbb{C}[x, y, z]$ of degree 4. `HC.jl` provides functions to sample random polynomials. We want to count the number of planes in three-space which are tangent to $f = 0$ in at least 3 points.
- (a) Set up polynomial systems to compute all tritangent planes of a general quartic surface. (*Hint: you should obtain a polynomial system in 11 variables*).
 - (b) Use monodromy to solve the system from (a).
- (4) Extend the triangulation example from two to three (or more) cameras.
- (5) Verify that the configurations of 5 conics at this link has 3264 real conics, which are simultaneously tangent to all 5 of them. Use certification methods to obtain a proof!
- (6) A quintic threefold is the zero set $X = \{(x, y, z, w) \in \mathbb{C}^4 \mid f(x, y, z, w) = 0\}$ of a polynomial $f \in \mathbb{C}[x, y, z, w]$ of degree 5. Let X be a quintic threefold defined by a randomly chosen polynomial f . Then, X contains finitely many lines. Compute them. How many are there?
- (7) Consider three Gaussian random variables X_1, X_2, X_3 with means μ_1, μ_2, μ_3 and variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$. The density of X_i is

$$\phi_i(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}.$$

A mixture of the three random variables is a random variable with density

$$\psi(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x), \quad \text{for } a_1 + a_2 + a_3 = 1.$$

The method of moments recovers ψ from the moments

$$m_k = \int x^k \psi(x) dx.$$

Since we have 9 unknowns plus one equation $a_1 + a_2 + a_3 = 1$, we expect to need at least 8 moments to recover ψ . Set up a system F with variables $\mu_1, \mu_2, \mu_3, \sigma_1^2, \sigma_2^2, \sigma_3^2$ and a_1, a_2, a_3 that computes the first 8 moments of ψ . Then, randomly sample values for the 9 variables, evaluate F at those values. This gives a vector $m \in \mathbb{C}^8$. Solve the system $F - m = 0$ (see here, if you need help).

- (8) Describe the solution set to the system

$$\begin{aligned} & (4s_4^2 - 2s_4s_6 + 4s_4s_7 + 6s_4s_8 - 2s_4 - 2s_6s_8 + s_6 + 4s_7s_8 + 2s_8^2 - s_8)y + \\ & (-4s_2s_4 - 4s_2s_8 + s_2 - 4s_4^2 - 4s_4s_6 - 8s_4s_8 + 3s_4 - 4s_6s_8 + 2s_6 + 2s_7 - \\ & \quad 4s_8^2 + 4s_8 - 1)z = 0. \\ & (2s_6 + 4s_7 + 2s_8 - 1)z^2 + (8s_2s_4^2 + 16s_2s_4s_8 - 4s_2s_4 - s_2s_6 - 2s_2s_7 + \\ & \quad 8s_2s_8^2 - 5s_2s_8 + s_2 + 8s_4^3 + 16s_4^2s_7 + 16s_4^2s_8 - 8s_4^2 + 3s_4s_6 + \\ & \quad 32s_4s_7s_8 - 2s_4s_7 + 8s_4s_8^2 - 9s_4s_8 + s_4 - s_6^2 - 2s_6s_7 + 2s_6s_8 + \\ & \quad 16s_7s_8^2 - 2s_7s_8 - s_8^2) = 0. \end{aligned}$$

in the variables $s_2, s_4, s_6, s_7, s_8, y, z$ restricted to the locally closed set

$$\mathcal{C} = V_1 - (V_2 \cap V_3) \cup (V_4 \cap V_5),$$

where V_1 is the solution set to

$$\begin{aligned} s_2^2 + 2s_2s_4 + s_2s_6 + 2s_2s_7 + s_2s_8 - s_2 + s_4^2 + s_4s_6 + 2s_4s_7 + s_4s_8 - \\ s_4 + 2s_6s_7 + 2s_7s_8 = 0, \end{aligned}$$

and

$$V_2 = \{2s_6 + 4s_7 + 2s_8 - 1 = 0\},$$

$$V_3 = \{s_2 + s_4 - 2s_7 = 0 = 0\},$$

$$V_4 = \{4s_4^2 - 2s_4s_6 + 4s_4s_7 + 6s_4s_8 - 2s_4 - 2s_6s_8 + s_6 + 4s_7s_8 + 2s_8^2 - s_8 = 0\},$$

$$V_5 = \{s_2 - s_4 + s_6 - s_8 = 0\},$$

(submitted by Ernesto Álvarez González.)

- (9) Compute the Maximum Likelihood Estimate of discrete models and Gaussian Models.
(submitted by Aida Maraj.)

2. WITNESS SETS

- (1) Prove the trace test for plane curves. i.e. prove
- if $C = \mathcal{V}(f(x, y))$ is an irreducible curve, and L_t a generic parallel family of lines, then the sum of the points $C \cap L_t$ moves linearly in t .
(Hint: assume L_t are the lines $\mathcal{V}(x - t)$)
 - if S_t is a proper subset of $C \cap L_t$, then the sum of the points in S_t moves nonlinearly as t moves.
(Hint: the monodromy of the points $C \cap L_t$, as t moves, is the full symmetric group)
- (2) How many maximal dimensional irreducible components does

$$\text{HSO}(4) = \{M \in \text{Mat}_{\mathbb{C}}(4, 4) \mid MM^T = \text{id}, \det(M) = 1, M_{i,i} = 0 \text{ for all } i\}$$

have? What are their degrees? How do they intersect?

- (3) Given a surface $X \subset \mathbb{R}^3$, the *flecnodal surface* $\mathcal{F}(X)$ is the union of all lines L with contact order 4 at a point of X . It is one of five *event surfaces* described in “Changing views on curves and surfaces” by Kohn, Sturmfels, and Trager.
- A quintic polynomial

$$f = c_5 + c_4t + \cdots + c_1t^4 + c_0t^5$$

has a root of multiplicity four whenever the coefficients satisfy

$$20c_0c_4 - 8c_1c_3 + 3c_2^2 = 0$$

$$50c_0c_5 - 6c_1c_4 + c_2c_3 = 0$$

$$20c_1c_5 - 8c_2c_4 + 3c_3^2 = 0.$$

Compute a witness set for the above equations. For each witness point, solve the corresponding polynomial and observe that one root comes with multiplicity four.

- Consider the quintic surface

$$f = x_1^5 + x_2^5 + x_3^5 + 1 + (x_1 + x_2 + x_3 + 1)^5 + x_1 x_2 x_3 (x_1 + x_2 + x_3 + 1)$$

in \mathbb{R}^3 . Find a suitable parametrization for lines in \mathbb{R}^3 . Compute a witness set for the Flecnodal surface of $X = \mathcal{V}(f)$. What is its degree?

- We can pass from \mathbb{R}^3 to projective space \mathbb{P}^3 . Here, lines can be parametrized using Plücker coordinates $\{q_{1,2}, q_{1,3}, q_{1,4}, q_{2,3}, q_{2,4}, q_{3,4}\}$:

$$z(t) = (q_{1,2} : tq_{1,2} : tq_{1,3} - q_{2,3} : tq_{1,4} - q_{2,4})$$

Homogenize $f(x_1, x_2, x_3)$ to $f_{\text{hom}}(x_0, x_1, x_2, x_3)$ and compute a witness set for the Flecnodal surface of $X = \mathcal{V}(f_{\text{hom}})$ in the coordinates q (remember to include the Plücker relation $q_{1,2}q_{3,4} - q_{1,3}q_{2,4} + q_{1,4}q_{2,3} = 0$.) What is its degree?

3. TOTAL DEGREE AND POLYHEDRAL HOMOTOPIES

- (1) (Conics in the plane). Consider the total degree family $\mathcal{F}(2, 2)$, i.e. $n = 2$ and $(d_1, d_2) = (2, 2)$:

$$\mathcal{F}(2, 2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix}.$$

What is $\mathcal{N}_{\text{Béz}}$ in this example? Verify this by solving a random member of this family

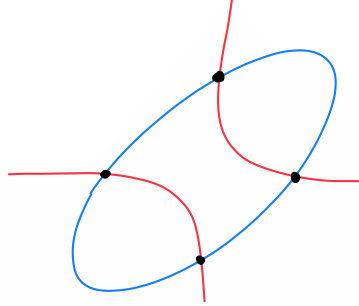


FIGURE 1. Two *generic* conics in the plane.

using `HomotopyContinuation.jl`.

There are strictly less than $\mathcal{N}_{\text{Béz}}$ solutions in the following two scenarios.

- Two or more solutions *coincide*. This happens if there are solutions to the overdetermined system

$$f_1 = f_2 = \det \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} = 0 \quad \text{has a solution.}$$

Here $f_{ix} = \partial f_i / \partial x$ and likewise for f_{iy} . Prove (possibly using a computer algebra system) that this is equivalent to the vanishing of a nonzero polynomial in the coefficients of f_1, f_2 . This polynomial is called the *discriminant*.

- There are solutions *at infinity*. To make this precise, we homogenize the equations:

$$\begin{pmatrix} a_{00}z^2 + a_{10}xz + a_{01}yz + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{00}z^2 + b_{10}xz + b_{01}yz + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix},$$

and consider solutions with $z = 0$. Geometrically, we replace our conics by their closures in \mathbb{P}^2 . Show that there are solutions ‘at infinity’ if and only if

$$\det A_\infty = \det \begin{pmatrix} a_{20} & a_{11} & a_{02} & \\ & a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} & \\ & b_{20} & b_{11} & b_{02} \end{pmatrix} = 0.$$

What about the case where $f_1 = f_2 = 0$ has infinitely many solutions?

Construct two members of $\mathcal{F}(2, 2)$ with 3 solutions, one with a solution at infinity, and one with a solution of multiplicity 2. Verify using `HomotopyContinuation.jl`.

- (2) (Systems supported on the square). Consider the subfamily

$$\mathcal{F}_Q = \left(\begin{array}{l} f_1 = a_{00} + a_{10}x + a_{10}y + a_{11}xy \\ f_2 = b_{00} + b_{10}x + b_{10}y + b_{11}xy \end{array} \right) \subset \mathcal{F}(2, 2).$$

Use the previous exercise to show that $\mathcal{N}(Q) < \mathcal{N}_{\text{Béz}}$. Verify the formula $\text{MV}(P_1, P_2) = \text{Vol}_2(P_1 + P_2) - \text{Vol}_2(P_1) - \text{Vol}_2(P_2)$ for $P_1 = P_2 = [0, 1]^2 \subset \mathbb{R}^2$. More generally, compute $\text{MV}(P_1, \dots, P_n)$ with $P_i = [0, 1]^n \subset \mathbb{R}^n$ for all i . This corresponds to a sparse family $\mathcal{F}_Q \subset \mathcal{F}(n, \dots, n)$. Compare $\mathcal{N}(Q)$ for these systems with their Bézout number. For some n , solve a generic member of \mathcal{F}_Q using a total degree and a polyhedral start system in `HomotopyContinuation.jl`.

- (3) (Asymptotic BKK and Bézout numbers). This is an example taken from “A polyhedral method for solving sparse polynomial systems” by Birkett Huber and Bernd Sturmfels. Consider the family

$$\left(\begin{array}{l} a_1 + a_2x + a_3x^ky^k \\ b_1 + b_2y + b_3x^ky^k \end{array} \right) \subset \mathcal{F}(2k, 2k).$$

Show that $\lim_{k \rightarrow \infty} (\mathcal{N}_{\text{BKK}} / \mathcal{N}_{\text{Béz}}) = 0$. Compare the computation time for the function `solve` in `HomotopyContinuation.jl` using the default option (`start_system = :polyhedral`) and the option `start_system = :total_degree` for random coefficients a, b and increasing values of k .

- (4) (Toric varieties and the BKK theorem). Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_r\} \subset \mathbb{N}^n$ be a set of exponents such that $P = \text{Conv}(\mathcal{A})$ has dimension n . The *projective toric variety* $X_{\mathcal{A}}$ associated to \mathcal{A} is the Zariski closure of the image of the monomial map

$$(x_1, \dots, x_n) \mapsto (x^{\alpha_1} : \dots : x^{\alpha_r}) \in \mathbb{P}^{r-1}.$$

Use the BKK theorem to relate the degree of $X_{\mathcal{A}}$ to the volume of P . The statement you obtain is known as *Kushnirenko’s theorem*, which can be seen as a specialized version of the BKK theorem for *unmixed systems of equations*, for which $\mathcal{A}_i = \mathcal{A}$, $i = 1, \dots, n$.

- (5) (Puiseux series solutions). Consider the polynomial $f = tx^3 + 2x^2 + t \in K[x]$, where K is the field of Puiseux series with complex coefficients in the variable t . Compute the leading term of all solutions $x \in K$ to $f = 0$. That is, compute all possible $X \in \mathbb{C} \setminus \{0\}$ and $e \in \mathbb{Q}$ such that there is a solution $x(t) = Xt^e + \text{higher order terms}$ satisfying $f(x(t), t) = 0$.

Hint: substitute $x(t) = Xt^e + \text{higher order terms}$ in $f(x(t), t)$ and look for all exponents e for which at least two terms of $f(x(t), t)$ are of lowest order in t . Obtain X from the condition that these lowest order terms cancel.

Can you give a graphical interpretation of the numbers e in terms of the Newton polygon of f ?

Hint: draw the Newton polygon of f as a polynomial in x, t .

- (6) (Solving binomial systems is easy). Consider the system of equations over the field K of complex Puiseux series in t :

$$F = \begin{pmatrix} 1 + 2x^2y + 3xy^2 \\ 5 + 2tx + 4ty + 6txy \end{pmatrix} = 0.$$

How many solutions $(x(t), y(t))$ do you expect? Check that there exists a solution of the form $x(t) = Xt^{-1} + \text{higher order terms}$ and $y(t) = Yt^2 + \text{higher order terms}$, where (X, Y) is the solution of

$$1 + 2X^2Y = 5 + 2X = 0.$$

Find $e_1, e_2 \in \mathbb{Q}$ such that $x(t) = Xt^{e_1} + \text{higher order terms}$ and $y(t) = Yt^{e_2} + \text{higher order terms}$ gives a solution for each $(X, Y) \in (\mathbb{C} \setminus \{0\})^2$ satisfying

$$2X^2Y + 3XY^2 = 5 + 6XY = 0.$$

To solve this *system of binomial equations*, we write it in the form

$$XY^{-1} = -3/2, \quad XY = -5/6. \quad (3.1)$$

We collect the exponent vectors in the columns of a matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. There exist matrices $P, Q \in \mathbb{Z}^{2 \times 2}$ with an inverse defined over \mathbb{Z} (i.e. P and Q are *unimodular*) which diagonalize A :

$$PAQ = \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix}. \quad (3.2)$$

This diagonal matrix is called the *Smith normal form* of A . Denote p_{ij}, q_{ij} for the entries of P and Q respectively. Show that the map $(\mathbb{C} \setminus \{0\})^2 \rightarrow (\mathbb{C} \setminus \{0\})^2$ given by

$$(U, V) \mapsto (U^{p_{11}}V^{p_{21}}, U^{p_{12}}V^{p_{22}})$$

is invertible. Use this change of coordinates on $(\mathbb{C} \setminus \{0\})^2$ and the identity (3.2) to reduce (3.1) to an equivalent system of equations

$$U^{s_1} = c_1, \quad V^{s_2} = c_2.$$

Deduce that the number of solutions of (3.1) is $\det A$. Can you write down an algorithm for solving a system of binomial equations in the form (3.1) with exponent matrix $A \in \mathbb{Z}^{n \times n}$?

We have now found the leading term of 3 solutions to $F = 0$. Can you find the missing solution(s) as well?

4. MONODROMY

Let $F_c(x)$ be a zero-dimensional parametrized polynomial system with variables x_1, \dots, x_n and parameters c_1, \dots, c_k . Let $Z \xrightarrow{\pi} \mathbb{C}^k$ be the branched cover where $Z = \{(x, p) | F_p(x) = 0\}$ and $\pi : Z \rightarrow \mathbb{C}^k$ is the projection onto the parameters. Let d be the degree of this branched cover. Let U be the set of regular values of π and G_π the monodromy group based at some point $p \in U$.

- (1) Show G_π is a group and it doesn't depend on the choice of $p \in U$ where you base monodromy loops.

- (2) Show G_π is transitive if and only if Z has a unique irreducible component of maximal dimension. Explain why G_π being transitive is exactly the condition which allows `monodromy_solve` to find all solutions to $\pi^{-1}(p)$.
- (3) Suppose $F_c(x)$ is defined over the real numbers.
- Is it possible for a real path in U to produce a nontrivial monodromy permutation?
 - Give an example.
 - What does this tell you about the branch locus?
- (4) Consider the sparse polynomial system

$$\begin{pmatrix} a_0 + a_1x^2 + a_2y^2 + a_3x^2y^2 \\ b_0x + b_1x^3 + b_2xy^2 \end{pmatrix},$$

- For 1000 parameter values, solve this system and count how many real solutions you find.
- Compute the Galois group over the parameters $\{a_0, a_1, a_3, a_4, b_0, b_1, b_2\}$.
- What is the structure of this system that explains what you've observed?