PROBLEMS AND EXERCISES

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Contents

1. Coding exercises

(1) A conic is the zero set of a quadratic polynomial

$$c(x,y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$$

with $a_i \in \mathbb{C}$.

Emiris and Tzoumas showed that there are 184 complex circles that are tangent to 3 general conics C_1, C_2 and C_3. This means, that there are 184 complex solutions (a_1, a_2, r) such that there exists some $(x, y) \in \mathbb{C}^2$ with

$(x - a_1)^2 + (y - a_2)^2 = r$

$(x, y) \in C_i$ for $\leq i \leq 3$

$(x - a_1, y - a_2)$ spans the normal space of C_i at (x, y) for $1 \leq i \leq 3$.

(a) Define the polynomial system for 3 general conics and verify that this system has indeed 184 solutions. Use certification.

(b) Consider the three conics

$$C_1 = \{y = -x^2 + 2x + 5\},$$
$$C_2 = \{y = 2x^2 + 5x - 8\},$$
$$C_3 = \{y = 8x^2 - 3x - 2\}.$$

How many circles are tangent to these 3 conics? How many of them are real?

(c) Find a configuration of 3 conics with as many real solutions as possible. It is possible to find 184 real solutions?

(2) A real algebraic variety is the common zero set of polynomials $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$ denoted by $X = V(f_1, \ldots, f_m)$. A bottleneck of X is defined to be a pair of distinct points $x, y \in X$ such that $x - y$ is orthogonal to the tangent space T_xX and to T_yX.

It was recently shown that a generic plane curve of degree d has $d^4 - 5d^2 + 4d$ bottleneck pairs. This is called the bottleneck degree of the curve.

Consider the curve $X = V(f)$ defined by

$$f = (x^4 + y^4 - 1)(x^2 + y^2 - 2) + x^5y.$$

(a) Write down definining equations for computing all bottlenecks.
(b) What is the Bottleneck degree of X? How many real bottlenecks does it have?
(c) What are the coordinates smallest bottleneck pair?
(d) What effect do different start systems have on the number of paths necessary to track?

(3) Consider a general quartic surface $X \in \mathbb{C}^3$. This is defined by a random polynomial $f \in \mathbb{C}[x, y, z]$ of degree 4. HC.jl provides functions to sample random polynomials. We want to count the number of planes in three-space which are tangent to $f = 0$ in at least 3 points.
(a) Set up polynomial systems to compute all tritangent planes of a general quartic surface. (*Hint: you should obtain a polynomial system in 11 variables*).
(b) Use monodromy to solve the system from (a).

(4) Extend the triangulation example from two to three (or more) cameras.

(5) Verify that the configurations of 5 conics at this link has 3264 real conics, which are simultaneously tangent to all 5 of them. Use certification methods to obtain a proof!

(6) A quintic threefold is the zero set $X = \{(x, y, z, w) \in \mathbb{C}^4 \mid f(x, y, z, w) = 0\}$ of a polynomial $f \in \mathbb{C}[x, y, z, w]$ of degree 5. Let X be a quintic threefold defined by a randomly chosen polynomial f. Then, X contains finitely many lines. Compute them. How many are there?

(7) Consider three Gaussian random variables X_1, X_2, X_3 with means μ_1, μ_2, μ_3 and variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$. The density of X_i is

$$\phi_i(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\mu_i)^2}{2\sigma_i^2}}.$$

A mixture of the three random variables is a random variable with density

$$\psi(x) = a_1\phi_1(x) + a_2\phi_2(x) + a_3\phi_3(x), \quad \text{for} \quad a_1 + a_2 + a_3 = 1.$$

The method of moments recovers ψ from the moments

$$m_k = \int x^k \psi(x) \mathrm{d}x.$$

Since we have 9 unknowns plus one equation $a_1 + a_2 + a_3 = 1$, we expect to need at least 8 moments to recover ψ. Set up a system F with variables μ_1, μ_2, μ_3, $\sigma_1^2, \sigma_2^2, \sigma_3^2$ and a_1, a_2, a_3 that computes the first 8 moments of ψ. Then, randomly sample values for the 9 variables, evaluate F at those values. This gives a vector $m \in \mathbb{C}^8$. Solve the system $F - m = 0$ (see here, if you need help).

(8) Describe the solution set to the system

$$\begin{aligned}
&(4s_4^2 - 2s_4s_6 + 4s_4s_7 + 6s_4s_8 - 2s_4 - 2s_6s_8 + s_6 + 4s_7s_8 + 2s_8^2 - s_8)y+\\
&(-4s_2s_4 - 4s_2s_8 + s_2 - 4s_4^2 - 4s_4s_6 - 8s_4s_8 + 3s_4 - 4s_6s_8 + 2s_6 + 2s_7-\\
&\qquad 4s_8^2 + 4s_8 - 1)z = 0.\\
&(2s_6 + 4s_7 + 2s_8 - 1)z^2 + (8s_2s_4^2 + 16s_2s_4s_8 - 4s_2s_4 - s_2s_6 - 2s_2s_7+\\
&\qquad 8s_2s_8^2 - 5s_2s_8 + s_2 + 8s_4^3 + 16s_4^2s_7 + 16s_4^2s_8 - 8s_4^2 + 3s_4s_6+\\
&\qquad 32s_4s_7s_8 - 2s_4s_7 + 8s_4s_8^2 - 9s_4s_8 + s_4 - s_6^2 - 2s_6s_7 + 2s_6s_8+\\
&\qquad 16s_7s_8^2 - 2s_7s_8 - s_8^2) = 0.
\end{aligned}$$

in the variables $s_2, s_4, s_6, s_7, s_8, y, z$ restricted to the locally closed set

$$\mathcal{C} = V_1 - (V_2 \cap V_3) \cup (V_4 \cap V_5),$$

where V_1 is the solution set to

$$s_2^2 + 2s_2s_4 + s_2s_6 + 2s_2s_7 + s_2s_8 - s_2 + s_4^2 + s_4s_6 + 2s_4s_7 + s_4s_8 - \\ s_4 + 2s_6s_7 + 2s_7s_8 = 0,$$

and

$V_2 = \{2s_6 + 4s_7 + 2s_8 - 1 = 0\}$,
$V_3 = \{s_2 + s_4 - 2s_7 = 0 = 0\}$,
$V_4 = \{4s_4^2 - 2s_4s_6 + 4s_4s_7 + 6s_4s_8 - 2s_4 - 2s_6s_8 + s_6 + 4s_7s_8 + 2s_8^2 - s_8 = 0\}$,
$V_5 = \{s_2 - s_4 + s_6 - s_8 = 0\}$,

(submitted by Ernesto Álvarez González.)

2. Witness Sets

(1) Prove the trace test for plane curves. i.e. prove
- if $C = \mathcal{V}(f(x,y))$ is an irreducible curve, and L_t a generic parallel family of lines, then the sum of the points $C \cap L_t$ moves linearly in t.
(Hint: assume L_t are the lines $\mathcal{V}(x - t)$)
- if S_t is a proper subset of $C \cap L_t$, then the sum of the points in S_t moves nonlinearly as t moves.
(Hint: the monodromy of the points $C \cap L_t$, as t moves, is the full symmetric group)

(2) How many maximal dimensional irreducible components does

$$\mathrm{HSO}(4) = \{M \in \mathrm{Mat}_{\mathbb{C}}(4,4) \mid MM^T = \mathrm{id}, \det(M) = 1, M_{i,i} = 0 \text{ for all } i\}$$

have? What are their degrees? How do they intersect?

(3) Given a surface $X \subset \mathbb{R}^3$, the *flecnodal surface* $\mathcal{F}(X)$ is the union of all lines L with contact order 4 at a point of X. It is one of five *event surfaces* described in "Changing views on curves and surfaces" by Kohn, Sturmfels, and Trager.
- A quintic polynomial

$$f = c_5 + c_4t + \cdots + c_1t^4 + c_0t^5$$

has a root of multiplicity four whenever the coefficients satisfy

$$20c_0c_4 - 8c_1c_3 + 3c_2^2 = 0$$
$$50c_0c_5 - 6c_1c_4 + c_2c_3 = 0$$
$$20c_1c_5 - 8c_2c_4 + 3c_3^2 = 0.$$

Compute a witness set for the above equations. For each witness point, solve the corresponding polynomial and observe that one root comes with multiplicity four.
- Consider the quintic surface

$$f = x_1^5 + x_2^5 + x_3^5 + 1 + (x_1 + x_2 + x_3 + 1)^5 + x_1x_2x_3(x_1 + x_2 + x_3 + 1)$$

in $\mathbb{R}^3$. Find a suitable parametrization for lines in $\mathbb{R}^3$., Compute a witness set for the Flecnodal surface of $X = \mathcal{V}(f)$. What is its degree?

- We can pass from $\mathbb{R}^3$ to projective space $\mathbb{P}^3$. Here, lines can be parametrized using Plücker coordinates $\{q_{1,2}, q_{1,3}, q_{1,4}, q_{2,3}, q_{2,4}, q_{3,4}\}$:

$$z(t) = (q_{1,2} : tq_{1,2} : tq_{1,3} - q_{2,3} : tq_{1,4} - q_{2,4})$$

Homogenize $f(x_1, x_2, x_3)$ to $f_{\text{hom}}(x_0, x_1, x_2, x_3)$ and compute a witness set for the Flecnodal surface of $X = \mathcal{V}(f_{\text{hom}})$ in the coordinates q (remember to include the Plücker relation $q_{1,2}q_{3,4} - q_{1,3}q_{2,4} + q_{1,4}q_{2,3} = 0$.) What is its degree?

3. Total degree and polyhedral homotopies

(1) (Conics in the plane). Consider the total degree family $\mathcal{F}(2,2)$, i.e. $n = 2$ and $(d_1, d_2) = (2,2)$:

$$\mathcal{F}(2,2) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix}.$$

What is $\mathcal{N}_{\text{Béz}}$ in this example? Verify this by solving a random member of this family

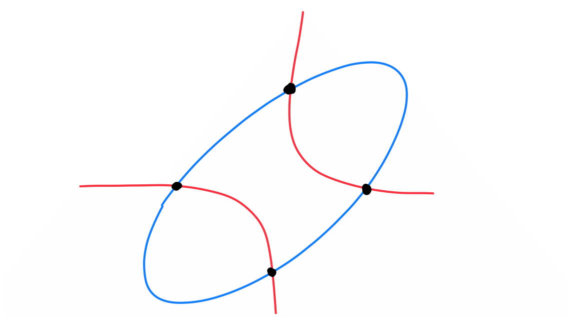

FIGURE 1. Two *generic* conics in the plane.

using `HomotopyContinuation.jl`.

There are strictly less than $\mathcal{N}_{\text{Béz}}$ solutions in the following two scenarios.

- Two or more solutions *coincide*. This happens if there are solutions to the overdetermined system

$$f_1 = f_2 = \det \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} = 0 \quad \text{has a solution.}$$

Here $f_{ix} = \partial f_i / \partial x$ and likewise for f_{iy}. Prove (possibly using a computer algebra system) that this is equivalent to the vanishing of a nonzero polynomial in the coefficients of f_1, f_2. This polynomial is called the *discriminant*.

- There are solutions *at infinity*. To make this precise, we homogenize the equations:

$$\begin{pmatrix} a_{00}z^2 + a_{10}xz + a_{01}yz + a_{20}x^2 + a_{11}xy + a_{02}y^2 \\ b_{00}z^2 + b_{10}xz + b_{01}yz + b_{20}x^2 + b_{11}xy + b_{02}y^2 \end{pmatrix},$$

and consider solutions with $z = 0$. Geometrically, we replace our conics by their closures in $\mathbb{P}^2$. Show that there are solutions 'at infinity' if and only if

$$\det A_\infty = \det \begin{pmatrix} a_{20} & a_{11} & a_{02} & \\ & a_{20} & a_{11} & a_{02} \\ b_{20} & b_{11} & b_{02} & \\ & b_{20} & b_{11} & b_{02} \end{pmatrix} = 0.$$

What about the case where $f_1 = f_2 = 0$ has infinitely many solutions?

Construct two members of $\mathcal{F}(2,2)$ with 3 solutions, one with a solution at infinity, and one with a solution of multiplicity 2. Verify using `HomotopyContinuation.jl`.

(2) (Systems supported on the square). Consider the subfamily

$$\mathcal{F}_Q = \begin{pmatrix} f_1 = a_{00} + a_{10}x + a_{10}y + a_{11}xy \\ f_2 = b_{00} + b_{10}x + b_{10}y + b_{11}xy \end{pmatrix} \subset \mathcal{F}(2,2).$$

Use the previous exercise to show that $\mathcal{N}(Q) < \mathcal{N}_{\text{Béz}}$. Verify the formula $\text{MV}(P_1, P_2) = \text{Vol}_2(P_1 + P_2) - \text{Vol}_2(P_1) - \text{Vol}_2(P_2)$ for $P_1 = P_2 = [0,1]^2 \subset \mathbb{R}^2$. More generally, compute $\text{MV}(P_1, \ldots, P_n)$ with $P_i = [0,1]^n \subset \mathbb{R}^n$ for all i. This corresponds to a sparse family $\mathcal{F}_Q \subset \mathcal{F}(n, \ldots, n)$. Compare $\mathcal{N}(Q)$ for these systems with their Bézout number. For some n, solve a generic member of $\mathcal{F}_Q$ using a total degree and a polyhedral start system in `HomotopyContinuation.jl`.

(3) (Asymptotic BKK and Bézout numbers). This is an example taken from "A polyhedral method for solving sparse polynomial systems" by Birkett Huber and Bernd Sturmfels. Consider the family

$$\begin{pmatrix} a_1 + a_2 x + a_3 x^k y^k \\ b_1 + b_2 y + b_3 x^k y^k \end{pmatrix} \subset \mathcal{F}(2k, 2k).$$

Show that $\lim_{k\to\infty}(\mathcal{N}_{\text{BKK}}/\mathcal{N}_{\text{Béz}}) = 0$. Compare the computation time for the function `solve` in `HomotopyContinuation.jl` using the default option (`start_system = :polyhedral`) and the option `start_system = :total_degree` for random coefficients a, b and increasing values of k.

(4) (Toric varieties and the BKK theorem). Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_r\} \subset \mathbb{N}^n$ be a set of exponents such that $P = \text{Conv}(\mathcal{A})$ has dimension n. The *projective toric variety* $X_\mathcal{A}$ associated to $\mathcal{A}$ is the Zariski closure of the image of the monomial map

$$(x_1, \ldots, x_n) \mapsto (x^{\alpha_1} : \cdots : x^{\alpha_r}) \in \mathbb{P}^{r-1}.$$

Use the BKK theorem to relate the degree of $X_\mathcal{A}$ to the volume of P. The statement you obtain is known as *Kushnirenko's theorem*, which can be seen as a specialized version of the BKK theorem for *unmixed systems of equations*, for which $\mathcal{A}_i = \mathcal{A}, i = 1, \ldots, n$.

(5) (Puiseux series solutions). Consider the polynomial $f = tx^3 + 2x^2 + t \in K[x]$, where K is the field of Puiseux series with complex coefficients in the variable t. Compute the leading term of all solutions $x \in K$ to $f = 0$. That is, compute all possible $X \in \mathbb{C} \setminus \{0\}$ and $e \in \mathbb{Q}$ such that there is a solution $x(t) = Xt^e +$ higher order terms satisfying $f(x(t), t) = 0$.

Hint: substitute $x(t) = Xt^e +$ higher order terms in $f(x(t), t)$ and look for all exponents e for which at least two terms of $f(x(t), t)$ are of lowest order in t. Obtain X from the condition that these lowest order terms cancel.

Can you give a graphical interpretation of the numbers e in terms of the Newton polygon of f?

Hint: draw the Newton polygon of f as a polynomial in x, t.

(6) (Solving binomial systems is easy). Consider the system of equations over the field K of complex Puiseux series in t:

$$F = \begin{pmatrix} 1 + 2x^2y + 3xy^2 \\ 5 + 2tx + 4ty + 6txy \end{pmatrix} = 0.$$

How many solutions $(x(t), y(t))$ do you expect? Check that there exists a solution of the form $x(t) = Xt^{-1} +$ higher order terms and $y(t) = Yt^2 +$ higher order terms, where (X, Y) is the solution of

$$1 + 2X^2Y = 5 + 2X = 0.$$

Find $e_1, e_2 \in \mathbb{Q}$ such that $x(t) = Xt^{e_1} +$ higher order terms and $y(t) = Yt^{e_2} +$ higher order terms gives a solution for each $(X, Y) \in (\mathbb{C} \setminus \{0\})^2$ satisfying

$$2X^2Y + 3XY^2 = 5 + 6XY = 0.$$

To solve this *system of binomial equations*, we write it in the form

$$XY^{-1} = -3/2, \quad XY = -5/6. \tag{3.1}$$

We collect the exponent vectors in the columns of a matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. There exist matrices $P, Q \in \mathbb{Z}^{2\times 2}$ with an inverse defined over Z (i.e. P and Q are *unimodular*) which diagonalize A:

$$PAQ = \begin{pmatrix} s_1 & \\ & s_2 \end{pmatrix}. \tag{3.2}$$

This diagonal matrix is called the *Smith normal form* of A. Denote p_{ij}, q_{ij} for the entries of P and Q respectively. Show that the map $(\mathbb{C} \setminus \{0\})^2 \to (\mathbb{C} \setminus \{0\})^2$ given by

$$(U, V) \mapsto (U^{p_{11}}V^{p_{21}}, U^{p_{12}}V^{p_{22}})$$

is invertible. Use this change of coordinates on $(\mathbb{C} \setminus \{0\})^2$ and the identity (3.2) to reduce (3.1) to an equivalent system of equations

$$U^{s_1} = c_1, \qquad V^{s_2} = c_2.$$

Deduce that the number of solutions of (3.1) is $\det A$. Can you write down an algorithm for solving a system of binomial equations in the form (3.1) with exponent matrix $A \in \mathbb{Z}^{n\times n}$?

We have now found the leading term of 3 solutions to $F = 0$. Can you find the missing solution(s) as well?

4. Monodromy

Let $F_c(x)$ be a zero-dimensional parametrized polynomial system with variables $x_1, \ldots, x_n$ and parameters $c_1, \ldots, c_k$. Let $Z \xrightarrow{\pi} \mathbb{C}^k$ be the branched cover where $Z = \{(x, p) | F_p(x) = 0\}$ and $\pi : Z \to \mathbb{C}^k$ is the projection onto the parameters. Let d be the degree of this branched cover. Let U be the set of regular values of π and G_π the monodromy group based at some point $p \in U$.

(1) Show G_π is a group and it doesn't depend on the choice of $p \in U$ where you base monodromy loops.

(2) Show G_π is transitive if and only if Z has a unique irreducible component of maximal dimension. Explain why G_π being transitive is exactly the condition which allows `monodromy solve` to find all solutions to $\pi^{-1}(p)$.
(3) Suppose $F_c(x)$ is defined over the real numbers.
- Is it possible for a real path in U to produce a nontrivial monodromy permutation?
- Give an example.
- What does this tell you about the branch locus?

(4) Consider the sparse polynomial system

$$\begin{pmatrix} a_0 + a_1x^2 + a_2y^2 + a_3x^2y^2 \\ b_0x + b_1x^3 + b_2xy^2 \end{pmatrix},$$

- For 1000 parameter values, solve this system and count how many real solutions you find.
- Compute the Galois group over the parameters $\{a_0, a_1, a_3, a_4, b_0, b_1, b_2\}$.
- What is the structure of this system that explains what you've observed?