PROBLEMS AND EXERCISES

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1. Monodromy

Let $F_c(x)$ be a zero-dimensional parametrized polynomial system with variables $x_1, \dots, x_n$ and parameters $c_1, \dots, c_k$. Let $Z \xrightarrow{\pi} \mathbb{C}^k$ be the branched cover where $Z = \{(x,p) | F_p(x) = 0\}$ and $\pi : Z \to \mathbb{C}^k$ is the projection onto the parameters. Let d be the degree of this branched cover. Let U be the set of regular values of π and G_π the monodromy group based at some point $p \in U$.

(1) Show G_π is a group.
(2) Show G_π doesn't depend on the choice of $p \in U$ where you base monodromy loops.
(3) Show G_π is transitive if and only if Z has a unique irreducible component of maximal dimension.
(4) Explain why G_π being transitive is exactly the condition which allows `monodromy solve` to find all solutions to $\pi^{-1}(p)$.
(5) Suppose G_π is transitive. Show that G_π is 2-transitive if and only if the variety

$$\{(x_1, x_2, p) \mid x_1, x_2 \in \pi^{-1}(p), p \in \mathbb{C}^k\}$$

has two maximal dimensional irreducible components
(6) Suppose $F_c(x)$ is defined over the real numbers. Is it possible for a real path in U to produce a nontrivial monodromy permutation? Under which conditions can this happen?
(7) As explained in previous lectures, solving a system $G(x) = 0$ using homotopy methods requires one to embed $G(x)$ into a family of polynomial systems $F_c(x)$. Does the ability to solve $G(x)$ using monodromy depend on which family is chosen?

2. Witness Sets

(1) Prove the trace test for plane curves.
Hint: for one direction, it is useful to know that the monodromy group

$$\{(x, L) \mid x \in X \cap L\} \xrightarrow{\pi} \mathrm{Gr(2,3)}$$

$$(x, L) \mapsto L$$

is the full symmetric group
(2) Compute a witness set for

$$\mathrm{SO}(5) = \{M \in \mathrm{Mat}_{\mathbb{C}}(5,5) \mid MM^T = \mathrm{id}, \det(M) = 1\}$$

(3) How many maximal dimensional irreducible components does

$$\mathrm{HSO}(4) = \{M \in \mathrm{Mat}_{\mathbb{C}}(4,4) \mid MM^T = \mathrm{id}, \det(M) = 1, M_{i,i} = 0 \text{ for all } i\}$$

have? What are their degrees? How do they intersect?

(4) The Lüroth hypersurface $\mathfrak{L}$ is the hypersurface in the space of plane quartics parameterized by

$$(\mathbb{C}^3)^5 \to \mathbb{P}^{15}$$
$$(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5) \mapsto \sum_{i=1}^{5} \prod_{j \neq i} \ell_j \text{ where } \ell_i = a_i x + b_i y + c_i$$

Compute a witness set for $\mathfrak{L}$. What is its degree?

3. Other

(1) A conic is the zero set of a quadratic polynomial

$$c(x, y) = a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x + a_5 y + a_6$$

with $a_i \in \mathbb{C}$.

Emiris and Tzoumas showed that there are 184 complex circles that are tangent to 3 general conics C_1, C_2 and C_3. This means, that there are 184 complex solutions (a_1, a_2, r) such that there exists some $(x, y) \in \mathbb{C}^2$ with

$$(x - a_1)^2 + (y - a_2)^2 = r$$
$$(x, y) \in C_i \text{ for } \leq i \leq 3$$
$$(x - a_1, y - a_2) \text{ spans the normal space of } C_i \text{ at } (x, y) \text{ for } 1 \leq i \leq 3.$$

(a) Define the polynomial system for 3 general conics and verify that this system has indeed 184 solutions. Use certification.

(b) Consider the three conics

$$C_1 = \{y = -x^2 + 2x + 5\},$$
$$C_2 = \{y = 2x^2 + 5x - 8\},$$
$$C_3 = \{y = 8x^2 - 3x - 2\}.$$

How many circles are tangent to these 3 conics? How many of them are real?

(c) Find a configuration of 3 conics with as many real solutions as possible. It is possible to find 184 real solutions?

(2) A real algebraic variety is the common zero set of polynomials $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$ denoted by $X = V(f_1, \ldots, f_m)$. A bottleneck of X is defined to be a pair of distinct points $x, y \in X$ such that $x - y$ is orthogonal to the tangent space $\mathrm{T}_x X$ and to $\mathrm{T}_y X$.

It was recently shown that a generic plane curve of degree d has $d^4 - 5d^2 + 4d$ bottleneck pairs. This is called the bottleneck degree of the curve.

Consider the curve $X = V(f)$ defined by

$$f = (x^4 + y^4 - 1)(x^2 + y^2 - 2) + x^5 y.$$

(a) Write down definining equations for computing all bottlenecks.

(b) What is the Bottleneck degree of X? How many real bottlenecks does it have?

(c) What are the coordinates smallest bottleneck pair?

(d) What effect do different start systems have on the number of paths necessary to track?

(3) Consider a general quartic surface $X \in \mathbb{C}^3$. This is defined by a random polynomial $f \in \mathbb{C}[x, y, z]$ of degree 4. HC.jl provides functions to sample random polynomials. We want to count the number of planes in three-space which are tangent to $f = 0$ in at least 3 points.
   (a) Set up polynomial systems to compute all tritangent planes of a general quartic surface. (Hint you should obtain a polynomial system in 11 variables).
   (b) Use monodromy to solve the system from (a).
(4) Extend the triangulation example from two to three (or more) cameras.
(5) Verify that the configurations of 5 conics at this link has 3264 real conics, which are simultaneously tangent to all 5 of them. Use certification methods to obtain a proof!