MPEPDEs
Autumn 2018
Assignment
Due date: 23rd November 2018. Email e-copies of your solutions to T.Pryer@reading.ac.uk and copy in craig.smith@imperial.ac.uk

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1. (10 points) Draw the characteristics for the following equations and write the general solution.

$$\begin{aligned} u_t + 7u_x &= 0 \\ u_t - 2u_x + 3u &= 0. \end{aligned} \tag{1}$$

2. (10 points) Let g be the gravitational constant and $h : \mathbb{R} \times \mathbb{R}^+$ be the height of a fluid. Further, let $v : \mathbb{R} \times \mathbb{R}^+$ be the horizontal velocity. The shallow water equations are given by

$$\begin{aligned} \phi_t + (v\phi)_x &= 0 \\ v_t + \left(\frac{v^2}{2} + \phi\right)_x &= 0. \end{aligned} \tag{2}$$

Let $q = q(\phi, v)$ be an entropy for the shallow water equations. Prove that it must satisfy

$$\frac{\partial^2 q}{\partial v^2} = \frac{\partial^2 q}{\partial \phi^2}. \tag{3}$$

3. (20 points) The Barotropic compressible Euler equations are given when the internal energy of the system is constant. The equations are given by conservation of mass and momentum

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho v)_t + \left(\rho v^2 + p\right)_x &= 0, \end{aligned} \tag{4}$$

with $p = p(\rho)$ for a smooth function p. Assume $p'(\rho) > 0$ then show that

$$q(\rho, v) = \frac{1}{2}\rho v^2 + P(\rho) \tag{5}$$

is an entropy when $P''(\rho) = \frac{p'(\rho)}{\rho}$ for $\rho > 0$. What is the associated entropy flux?

4. Consider the following partial differential equation (PDE)

$$u_t = u_{xxx} + 6uu_x, \quad u = u(x,t) \tag{6}$$

known as the Korteweg-de Vries equation (or KdV) and which describes water waves in shallow waters.

(a) (10 points) Consider a scaling transformation, on both the independent and dependent variables, of the form

$$S : (t, x, u) \mapsto (T, X, U) = (at, bx, cu) \tag{7}$$

where a, b and c are nonzero constants. Find conditions for the parameters a, b and c such that the transformation S is a symmetry transformation of the KdV equation. Use the obtained symmetry to deduce that if $u = f(x,t)$ is a solution of the KdV equation then $u = \epsilon^2 f(\epsilon x, \epsilon^3 t)$ is also a solution for all nonzero ϵ.

(b) (15 points) Consider solutions of the KdV equation of the special form

$$u(x,t) = f(\xi), \quad \xi(x,t) = x - vt \tag{8}$$

where v is a real constant and substitute it in (6). Such type of solutions are known as *travelling wave* solutions. Use the chain rule for partial derivatives to obtain an ordinary differential equation (ODE) for $f(\xi)$. Can you solve the obtained differential equation? (Hint: You might want to revise about *elliptic functions* and in particular Weierstrass $\wp$-function). If not, assume that f and its derivatives go to zero as $\xi \to \pm\infty$. In this way derive the solution

$$u(x,t) = 2v^2 \operatorname{sech}^2(vx + 4v^3 t) \tag{9}$$

known as the *1-soliton* solution of KdV.

(c) (5 points) Apply the symmetry argument of part (a) to the travelling wave form (8) to deduce that higher amplitude waves travel faster. This is a purely nonlinear effect, can you explain this? This shows that even when an exact solution cannot be found, we can still find properties of the solutions by studying the symmetries of the equation.

(d) (10 points) Verify that the quantities

$$x^2 u, \text{ and } x^3 t^{-1} \tag{10}$$

are invariant under the symmetry obtained in part (a). Deduce that the general form of those solutions of the KdV equation which are invariant under the scaling symmetry is

$$u(x,t) = \frac{1}{x^2} f(\xi), \quad \xi(x,t) = x^3 t^{-1}. \tag{11}$$

Using again the chain rule obtain an ODE for $f(\xi)$. Try to find simple special solutions for the obtained ODE and then find the corresponding solutions for the KdV equation. For example, start with the assumption that $f(\xi)$ is a constant function. Can you find a rational solution $f(\xi)$? (Hint: There exists one rational solution where both the numerator and denominator are quadratic polynomials and $f(0) = 0$. Try to use a computer algebra package.)

(e) (5 points) The KdV equation admits infinitely many solutions of solitonic, rational and mixed type. All of them can be obtained recursively in an algebraic manner (no integrations are required!!). Can you find more?

(f) (10 points) Using energy arguments, show that any solution of the problem (6) satisfies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} u &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} u^2 &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}} u^3 - \frac{1}{2} u_x^2 &= 0. \end{aligned} \tag{12}$$

(g) (5 points) Use these *invariants* to derive stability bounds in appropriate norms for the solution in terms of the initial condition.