Lab Assignment 6
Simplex Method for Linear Programming Problems

ACS II

Spring 2020

Assigned: February 20, 2020

Due: February 20, 2020

Introduction

In the lectures, you've seen an overview for how one solves a typical linear programming problem using the simplex method. A typical linear programming minimization problem can be written as

$$\begin{aligned} \text{Minimize: } & z = \mathbf{c}^T \mathbf{x} \\ \text{Subject to: } & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

m = # constraint
n = # variable

Here $\mathbf{x}, \mathbf{c} \in \mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ and $A \in \mathbb{R}^m \times \mathbb{R}^n$ with $n > m$. One method for solving this kind of problem is the *simplex method*. Many introductory textbooks (e.g. Chvatal 1983) begin by teaching the simplex method using either dictionaries or tableaus. The *revised simplex method*, as derived in the course notes makes use of linear algebra. Though arguably less intuitive, the revised simplex algorithm can be considerably faster on large problems and almost all modern implementation of the simplex method are actually the revised simplex method. Without repeating the derivation, the steps are as follows:

1. Given a basis $B \in \mathbb{R}^m \times \mathbb{R}^m$ (chosen from the columns of A) for the non-zero entries in $\mathbf{x}$ and a corresponding cost $\mathbf{c}_B$ compute $\mathbf{x}_B$ and $\mathbf{z}$ according to:

$$B\mathbf{x}_B = \mathbf{b}, \qquad B^T\mathbf{z} = \mathbf{c}_B.$$

2. The remaining columns in the matrix A form $N \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, a basis for the zero entries in $\mathbf{x}$. This basis has a corresponding cost $\mathbf{c}_N$. Compute $\mathbf{r} = \mathbf{c}_N - N^T\mathbf{z}$.

3. If $\mathbf{r}$ has only positive values we are done, otherwise the entering variable i is the index in $\mathbf{r}$ with the smallest negative value.

4. $\mathbf{e}_i$ is the ith column of the $(n-m)\times(n-m)$ identity matrix. Compute $\mathbf{w}$ from $B\mathbf{w} = N\mathbf{e}_i$.

5. Compute a vector $\mathbf{f}$ where the jth entry of $\mathbf{f}$ is given by

$$f_j = \frac{\mathbf{e}_j^T \mathbf{x}_B}{\mathbf{e}_j^T \mathbf{w}},$$

   where now $\mathbf{e}_j$ is the jth row of the $m \times m$ identity matrix. The leaving variable k is the index of the smallest positive value in $\mathbf{f}$.

6. Permute the basis by exchanging the kth column of B with the ith column of N. Return to step 1.

You'll want to keep track of the columns of A in both B and N. In the end (once $\mathbf{r}$ has no negative entries), $\mathbf{x}$ can be constructed by inserting $\mathbf{x}_B$ into the rows of an $n \times 1$ vector in the appropriate places. For example if B contains columns 2, 3 and 8 of A, then $\mathbf{x}$ could be constructed by inserting $\mathbf{x}_B$ into rows 2,3 and 8 of an $n \times 1$ array of zeros.

Simplex Code

Implement the revised simplex method. For this lab you may use any language of your choice, including Python or Matlab. Your code should be structured as follows:

- a function `simplex_method` that takes as arguments:
  - A, $\mathbf{b}$, $\mathbf{c}$
  - the columns to form an initial basis B

  and returns:
  - the optimal solution $\mathbf{x}$
  - the optimal cost $\mathbf{c}^T\mathbf{x}$
- a helper function `simplex_step` that takes as arguments:
  - A, $\mathbf{b}$, $\mathbf{c}$

- a list of columns currently in B
- a list of columns currently in N

and returns:

- a new list of columns in B
- a new list of columns in N
- $\mathbf{r}$
- $\mathbf{x}_B$

The code is worth 20 points.

Test Problems

1. Test your code on the example in the notes. Report the optimal solution $\mathbf{x}$ as well as B, N and $\mathbf{r}$ at each step. Take the first two columns of A as your initial basis for $\mathbf{x}_B$. (15 points)

2. Use your simplex function to solve the following problem:

   Minimize: $z = 4x_1 + 2x_2 + x_3 + 4x_4 - 2x_5$.

   Subject to:

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 + x_4 + 2x_5 &= 4, \\ -2x_1 + x_2 + 2x_4 + 3x_5 &= 2, \\ x_1, x_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

Again, use the first two columns as A as your initial basis for x_B. (15 points)

Inequality Constraints

It is common to see linear programs with inequality constraints, i.e.

$$\begin{aligned} \text{Minimize: } z &= \mathbf{c}^T\mathbf{x}. \\ \text{Subject to: } A\mathbf{x} &\leq \mathbf{b}. \\ \mathbf{x} &\geq 0 \end{aligned}$$

This problem can to turned into an equivalent problem with equality constraints by introducing *slack variables* $\mathbf{s}$ which are equal to the difference between $A\mathbf{x}$ and $\mathbf{b}$. For example, consider the problem:

Minimize: $-4x_1 - 3x_2 - x_3 - x_4$

$$\begin{aligned} x_1 + 2x_2 - x_4 &\leq 3, \\ 2x_1 + x_2 - x_3 + x_4 &\leq 10, \\ x_2 + x_3 &\leq 2, \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

We can introduce three slack variables s_1, s_2 and s_3 such that

$$s_1 = 3 - x_1 - 2x_2 + x_4, \qquad s_2 = 10 - 2x_1 - x_2 + x_3 - x_4, \qquad s_3 = 2 - x_2 - x_3.$$

Then the following equalities hold:

$$\begin{aligned} x_1 + 2x_2 - x_4 + s_1 &= 3, \\ 2x_1 + x_2 - x_3 + x_4 + s_2 &= 10, \\ x_2 + x_3 + s_3 &= 2, \\ x_1, x_2, x_3, x_4, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

Note that this procedure turns the problem $A\mathbf{x} \leq \mathbf{b}$ into

$$\begin{aligned} [A \qquad I] \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} &= \mathbf{b}, \\ \mathbf{x}, \mathbf{s} &\geq 0. \end{aligned}$$

Finally, recall that to turn a greater than or equal to into a less than or equal to, we can simply multiply both sides by -1, i.e.

$$x \geq y \Leftrightarrow -x \leq -y.$$

Gasoline Blending

This problem is taken from Richard Anstee's course page. We wish to blend aviation gasoline from three available components: Toluene, Alkylate and Pentane. There are certain constraints that must be satisfied as summarized in the following table:

constraint	Toluene (T)	Alkylate (A)	Pentane (P)	Product specification
% aromatics	100	0	0	5 (minimum)
Reid vapor pressure PSI	2.0	4.8	19.7	between 5.5 and 7.0
Performance no.	100	125	125	115 (minimum)
Cost per barrel	$45	$30	$30	

The goal is obviously to minimize the cost. We wish to produce a single barrel of fuel (i.e. $T + A + P = 1$). Please complete the following tasks:

1. Write out the appropriate linear program that must be solved. Define any slack variables used. (25 points)

2. Using your simplex code from earlier solve this problem and report the amounts of Toluene, Alkylate and Pentane needed as well as the optimal cost. You may need to choose your initial basis carefully. (25 points)

Submission and Grading

To get credit for this assignment, you must submit the following information to the lab instructor by 11:59pm, February 20, 2020:

- Your code.
- Your report as a single file in pdf format, including your plots, results from your work and relevant discussion of your observations, results, and conclusions.

This information must be received by 11:59pm, February 20, 2020. As stated in the course syllabus, late assignment submissions will be subject to a 10% point penalty per 24 hours past the due date at time of submission, to a maximum reduction of 50%, according to the formula:

*[final score] = [raw score] - min(0.5, 0.1 * [# of days past due]) * [maximum score]*