Lab Assignment 13
Integration methods for Differential Equations

ACS II
Spring2020

Assigned April 9, 2020
Due April 16, 2020

1 Simple Integration Schemes for ODEs

In this lab, we will briefly consider the implementation and stability qualities of simple integration methods for systems of ordinary differential equations, introduce the concept of stiffness, and use it to motivate the IMEX method for solving partial differential equations.

1.1 Euler Methods

As you have seen in ACSI, by taking the Taylor series expansion of function of time centered around $t = t_n$, we can derive a method to approximate the time evolution of this function, thereby integrating the ODE this function is the solution for. This Taylor expansion is

$$y(t_n + h) \equiv y_{n+1} = y(t_n) + h\frac{\partial y}{\partial t}|_{t_n} + O(h^2)$$

From this, the forward Euler method is immediately suggested

$$y_{n+1} = y_n + hf(y_n t_n)$$

Where $y' = f(y,t)$.

Reversing the direction of the Taylor series expansion allows us to form an implicit method for ODE integration.

$$y_n \equiv y(t_{n+1} - h) = y(t_{n+1}) - h\frac{\partial y}{\partial t}|_{t_{n+1}} + O(h^2)$$

Suggesting the backward Euler method

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1})$$

As the resultant equation for y_{n+1}, such a method must be paired with a nonlinear root-finding algorithm. Through analysis of the typical linear test problem, $y' = ky$ It can be shown that while the Backward Euler is A-stable (and thus stable for all real time-steps), the Forward Euler method is stable on the region $hk = z, z \in C, |z + 1| \leq 1$

1.2 Task 1

In a compiled langauge, implement both the forward and Backward Euler methods for ODE systems (along with the necessary nonlinear equation solver). Test the stability of your methods using the IVP

$$y' = -10y$$

With the initial value $y(0) = 1$ and the exact solution $y = e^{-10t}$. Use $h = [0.21, 0.2, 0.05, 0.001]$, $t_f = 1.0$ and discuss your results for both methods. Perform a convergence study and show that the theoretical order of the method is recovered.

2 Implicit-Explicit Methods

As you saw above, each of the above methods had its benefits and drawbacks; explicit methods are simpler to implement, but suffer from stability issues, while implicit methods require more effort to implement but are free of stability concerns. Now consider the first order ODE with stiff and nonstiff components.

$$y' = f_N(t, y) + f_S(t, y), y(t_0) = y_0$$

Such systems are very common when modeling physical phenomena. While discretization both stiff and nonstiff terms view an implicit method would require many nonlinear systems solves, work not needed for accurate solutions to the nonstiff portion, using an explicit method for both terms would lead to severe time-step restrictions due to the stiff term.

The logical solution for such a problem is the solving of the stiff term via and implicit method, the nonstiff term via an explicit method, and then combining the two, yielding an Implicit-Explicit (IMEX Method). A very popular method IMEX method is known as the Crank-Nicolson Leapfrog method and is described as

$$y^{n+1} = y^{n-1} + 2\Delta t F_N(t_n, y^n) + \frac{\Delta t}{2}[F_S(t_{n+1}, y^{n+1}) + F_S(t_{n-1}, y^{n-1})]$$

2.1 Task 2

Consider the PDE

$$u_t + \sin(2\pi x)u_x = \nu u_{xx}$$

On the interval $[0, 1]$, with initial conditions $u(x, 0) = \sin(2\pi x)$ and periodic boundary conditions. Using a second order centered difference discretization in space, generate stable solutions for this problem with $\nu = 0.002$ using Forward Euler, Backward Euler, and the CNLF method for the time-discretization. Report on the required timesteps for stability, as well as the total wall-clock time for these methods. Use the same tolerance for all nonlinear system solves between your methods.

Submission and Grading

To get credit for this assignment, you must submit the following information to the lab instructor and course instructor by 11:59pm, April 16, 2020

- your source code files
- Your report as a single file in pdf format, including results from your work and relevant discussion of your observations, results, and conclusions.

This information must be received by 11:59pm, April 16, 2020. Send the required documents as email attachments to both the course instructor and the lab TA. As stated in the course syllabus, late assignment submissions will be subject to a 10% point penalty per 24 hours past the due date at time of submission, to a maximum reduction of 50%, according to the formula:

*[final score] = [raw score] - min(0.5, 0.1 * [# of days past due]) * [maximum score]*