# **Définitions et Preuves**

# Les mesures

 $VaR_k(X)$ ,  $TVaR_k(X)$ ,  $LTVaR_k(X)$ ,  $RVaR_k(X)$ 

## **Def 1:**

$$VaR_k(X) = F^{-1}(X)$$

Preuve:

#### Def 2:

$$TVaR_k(X) = rac{1}{1-k} \int_k^1 VaR_u(X) \, du$$

### Def 3:

$$TVaR_k(X) = rac{1}{1-k}\pi_x(VaR_k(X)) + VaR_k(X)$$

Preuve:

$$\begin{split} TVaR_k(X) &= \frac{1}{1-k} \int_k^1 VaR_u(X) \ du \\ &= \frac{1}{1-k} \int_k^1 (VaR_u(X) - VaR_k(X) - VaR_k(X)) \ du \\ &= \frac{1}{1-k} \left( \int_k^1 (VaR_u(X) - VaR_k(X)) \ du + \int_k^1 VaR_k(X) \ du \right) \\ &= \frac{1}{1-k} \left( \int_k^1 F_X^{-1}(u) - VaR_k(X) \ du + (1-k)VaR_k(X) \right) \\ &= \frac{1}{1-k} \int_k^1 F_X^{-1}(u) - VaR_k(X) \ du + VaR_k(X) \\ &= \frac{1}{1-k} E[F_X^{-1}(u) - VaR_k(X); \ 0] + VaR_k(X) \\ &= \frac{1}{1-k} E[X - VaR_k(X); \ 0] + VaR_k(X) \\ &= \frac{1}{1-k} \pi_x(VaR_k(X)) + VaR_k(X) \end{split}$$

## Def 4:

$$TVaR_k(X) = rac{1}{1-k}igl[E[X imes_{1\{x>VaR_k(X)\}}] + VaR_k(X)\left(F_X(VaR_k(X)-k)
ight)igr]$$

Preuve:

$$\begin{split} TVaR_k(X) &= \frac{1}{1-k} E[X - VaR_k(X); \ 0] + VaR_k(X) \\ &= \frac{1}{1-k} E[(X - VaR_k(X)) \times_{1\{x > VaR_k(X)\}}] + VaR_k(X) \\ &= \frac{1}{1-k} \left\{ E[(X - VaR_k(X)) \times_{1\{x > VaR_k(X)\}}] + (1-k)VaR_k(X) \right\} \\ &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] - VaR_k(X)E[1 \times_{1\{x > VaR_k(X)\}}] + (1-k)VaR_k(X) \right\} \\ &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] - VaR_k(X) \int_{VaR_k(X)}^{1} 1 \ du + (1-k)VaR_k(X) \right\} \\ &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] - VaR_k(X)(1 - F_x(VaR_k(X))) + (1-k)VaR_k(X) \right\} \\ &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X) \left( F_x(VaR_k(X)) - 1 + 1 - k \right) \right\} \\ &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X) \left( F_x(VaR_k(X)) - k \right) \right\} \end{split}$$

Propriété: (En cas continue)

$$TVaR_k(X) = rac{1}{1-k} E[X imes_{1\{x>VaR_k(X)\}}]$$

$$\begin{split} &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X) \left( F_x(VaR_k(X)) - k \right) \right\} \\ &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X) \left( F_x(F_x^{-1}(k)) - k \right) \right\} \\ &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X) \left( k - k \right) \right\} \\ &= \frac{1}{1-k} \left\{ E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X) \left( 0 \right) \right\} \\ &= \frac{1}{1-k} E[X \times_{1\{x > VaR_k(X)\}}] \end{split}$$

#### Def 5:

$$LTVaR_k(X) = rac{1}{k} \int_0^k VaR_u(X) \; du$$

#### Def 6:

$$LTVaR_k(X) = rac{1}{k}igl[E[x] - (1-k)TVaR_k(X)igr]$$

Preuve:

$$egin{aligned} LTVaR_{k}\left(X
ight) &= rac{1}{k} \int_{0}^{k} VaR_{u}\left(x
ight) \, du \ &= rac{1}{k} igg( \int_{0}^{1} F^{-1}\left(u
ight) \, du - \int_{k}^{1} VaR_{u}\left(x
ight) \, du igg) \ &= rac{1}{k} ig( E[x] - (1-k)TVaR_{k}(x) ig) \end{aligned}$$

#### **Def 7:**

$$LTVaR_k(X) = rac{1}{k}igg[E[X imes_{1\{x\leq VaR_k(X)\}}] + VaR_k(X)ig(k-F_X(VaR_k(X))ig)igg]$$

Preuve:

$$egin{aligned} LTVaR_k(X) &= rac{1}{k}ig(E[X] - (1-k)TVaR_k(x)ig) \ &= rac{1}{k}igg(E[X] - (1-k)igg[rac{1}{1-k}ig[E[X imes_{1\{x>VaR_k(X)\}}] + VaR_k(X)\left(F_X(VaR_k(X) - 1)
ight) \ &= rac{1}{k}igg(E[X] - E[X imes_{1\{x>VaR_k(X)\}}] - VaR_k(X)\left(F_X(VaR_k(X) - k)
ight)igg) \ &= rac{1}{k}igg(E[X imes_{1\{x\leq VaR_k(X)\}}] + VaR_k(X)\left(k - F_X(VaR_k(X))
ight)igg) \end{aligned}$$

Def 8:

$$RVaR_{k_1,k_2}(X) = rac{1}{k_2-k_1} \int_{k_1}^{k_2} VaR_u(X) \; du$$

Def 9:

$$RVaR_{k_1,k_2}(X) = rac{1}{k_2-k_1}igg((1-k_1)TVaR_{k_1}(X) - (1-k_2)TVaR_{k_2}(X)igg)$$

Preuve:

$$egin{aligned} RVaR_{k_1,k_2}(X) &= rac{1}{k_2-k_1} \int_{k_1}^{k_2} VaR_u(X) \ du \ &= rac{1}{k_2-k_1} igg( igg( rac{1-k_1}{1-k_1} igg) \int_{k_1}^1 VaR_u(X) \ du - igg( rac{1-k_2}{1-k_2} igg) \int_{k_2}^1 VaR_u(X) \ du igg) \ &= rac{1}{k_2-k_1} igg( (1-k_1)TVaR_{k_1}(X) - (1-k_2)TVaR_{k_2}(X) igg) \end{aligned}$$

# Espérance tronquée

Def 1:

$$E[X] = E[X \times_{1\{X \leq d\}}] + E[X \times_{1\{X > d\}}]$$

Def 2: (La loi Weibull)

$$E[X imes_{1\{X\leq d\}}] = rac{1}{eta}\Gamma\left(1+rac{1}{ au}
ight)\overline{H}\left(d^ au,1+rac{1}{ au},eta^ au
ight)$$

$$egin{aligned} E[X imes_{1\{X>d\}}] &= \int_d^\infty xeta au(eta x)^{ au-1}e^{-(eta x)^ au}\,dx \ u &= (xeta)^ au \ du &= aueta(eta x)^{ au-1} \ &= rac{1}{eta}\int_{(eta d)^ au}^\infty u^{1/ au}e^{-u}\,du \ &= rac{\Gamma\left(rac{1}{ au}+1
ight)}{eta}\int_{(eta d)^ au}^\infty rac{u^{(1/ au+1)-1}e^{-u}}{\Gamma\left(rac{1}{ au}+1
ight)}\,du \ &= rac{\Gamma\left(rac{1}{ au}+1
ight)}{eta}ar{H}\left((eta d)^ au,rac{1}{ au}+1,1
ight) \ &= rac{\Gamma\left(rac{1}{ au}+1
ight)}{eta}ar{H}\left(d^ au,rac{1}{ au}+1,eta^ au
ight) \end{aligned}$$

## Def 3: (La loi Burr)

$$E\left[X imes 1_{\{X\leq d\}}
ight] = rac{1}{\Gamma(lpha)} \lambda^{1/ au} \Gamma\left(1+rac{1}{ au}
ight) \Gamma\left(lpha-rac{1}{ au}
ight) B\left(rac{d^ au}{\lambda+d^ au};1+rac{1}{ au},lpha-rac{1}{ au}
ight)$$

Preuve:

$$egin{align*} E[X imes 1_{\{x\leq d\}}] &= \int_0^d x imes rac{lpha au\lambda^lpha x^{ au-1}}{(\lambda+x^ au)^{lpha+1}} \, dx \ &= \int_0^d x imes rac{lpha au\lambda^{lpha+1-1}x^{ au-1}}{(\lambda+x^ au)^{lpha+2-1}} \, dx \ &= \int_0^d rac{xlpha au\lambda x^{ au-1}}{(\lambda+x^ au)^2} igg(rac{\lambda}{\lambda+x^ au}igg)^{lpha-1} \, dx \ &u = igg(rac{\lambda}{\lambda+x^ au}igg) \ du &= -rac{\lambda}{(\lambda+x^ au)^2} imes au x^{ au-1} \, dx \ &= -lpha \int_1^{rac{\lambda}{\lambda+d^ au}} igg(rac{\lambda}{u} - \lambdaigg)^{1/ au} u^{lpha-1} \, du \ &= \lambda^{1/ au} lpha \int_{rac{\lambda}{\lambda+d^ au}}^1 igg(1-uigg)^{1/ au} u^{lpha-1} \, du \ &= \lambda^{1/ au} lpha \int_{rac{\lambda}{\lambda+d^ au}}^1 igg(1-uigg)^{1/ au} u^{lpha-1/ au-1} \, du \ &= -\lambda^{1/ au} lpha \int_{rac{\lambda}{\lambda+d^ au}}^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} lpha \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} lpha \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} lpha \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} lpha \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} lpha \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} lpha \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} lpha \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} lpha \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} (1-v)^{lpha-1/ au-1} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda^{1/ au} a \int_0^0 v^{1/ au} \, dv \ &= -\lambda$$

$$egin{aligned} &= -\lambda^{1/ au} lpha \int_{rac{d^ au}{\lambda + d^ au}}^0 v^{1/ au} (1-v)^{lpha - 1/ au - 1} \; dv \ &= \lambda^{1/ au} lpha \int_0^{rac{d^ au}{\lambda + d^ au}} v^{1/ au + 1 - 1} (1-v)^{lpha - 1/ au - 1} \; dv \end{aligned}$$

Completion avec la loi Beta

$$\begin{split} &= \lambda^{1/\tau} \alpha \times \frac{I\left(\frac{d^\tau}{\lambda + d^\tau}, \frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right)}{I\left(\frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right)} \\ &= \lambda^{1/\tau} \alpha \ I\left(\frac{d^\tau}{\lambda + d^\tau}, \frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right) \times \frac{\Gamma\left(1 + \frac{1}{\tau}\right)\Gamma\left(\alpha - \frac{1}{\tau}\right)}{\Gamma(\alpha + 1)} \\ &= \frac{1}{\Gamma(\alpha)} \lambda^{1/\tau} \Gamma\left(1 + \frac{1}{\tau}\right)\Gamma\left(\alpha - \frac{1}{\tau}\right) I\left(\frac{d^\tau}{\lambda + d^\tau}; 1 + \frac{1}{\tau}, \alpha - \frac{1}{\tau}\right) \end{split}$$

# La stop-loss

#### **Def Bonus 2:**

$$E[X] = \sum_{k=0}^\infty (1-F_X(k))$$

Preuve:

$$egin{aligned} E[X] &= \sum_{k=0}^\infty x f_X(x) \ &= 0 imes f_X(0) + 1 imes f_X(1) + 2 imes f_X(2) + 3 imes f_X(3) + \cdots \ &= f_X(1) + (f_X(2) + f_X(2)) + (f_X(3) + f_X(3) + f_X(3)) + \cdots \ &= (f_X(1) + f_X(2) + f_X(3) + \cdots) + (f_X(2) + f_X(3) + \cdots) + (f_X(3) + \cdots) \ &= (1 - F_X(0)) + (1 - F_X(1)) + (1 - F_X(2)) + (1 - F_X(3)) + \cdots ) \ &= \sum_{k=0}^\infty (1 - F_X(k)) \end{aligned}$$

#### **Def Bonus 2:**

$$\pi_X(0) = E[X] = \sum_{k=0}^{\infty} (1 - F_X(k))$$

Preuve:

$$E[X] = \sum_{x=k+1}^{\infty} x f_X(x)$$
 $= (k+1) imes f_X(k+1) + (k+2) imes f_X(k+2) + (k+3) imes f_X(k+3) + \cdots$ 
 $= f_X(k+1) + (f_X(k+2) + f_X(k+2)) + (f_X(k+3) + f_X(k+3) + f_X(k+3)) + \cdots$ 
 $= (f_X(k+1) + f_X(k+2) + f_X(k+3) + \cdots) + (f_X(k+2) + f_X(k+3) + \cdots) + (f_X(k+2) + f_X(k+3)) + \cdots) + (f_X(k+2) + f_X(k+3)) + \cdots$ 
 $= \sum_{l=0}^{\infty} (1 - F_X(k+l)) \text{ ou } k = 0$ 

#### **Def Bonus 3:**

$$F_X(k) = 1 - \pi_X(k) + \pi_X(k+1)$$

on sait que:

$$egin{aligned} \pi_X(k) &= \sum_{l=k}^\infty (1-F_X(l)) \quad ext{et} \quad \pi_X(k+1) = \sum_{l=k+1}^\infty (1-F_X(l)) \ \pi_X(k) - \pi_X(k+1) &= \sum_{l=k}^\infty (1-F_X(l)) - \sum_{l=k+1}^\infty (1-F_X(l)) \ &= (1-F_X(l)) + \sum_{l=k+1}^\infty (1-F_X(l)) - \sum_{l=k+1}^\infty (1-F_X(l)) \ &= 1-F_X(l) \ dots \cdot \colon F_X(k) = 1 - \pi_X(k) + \pi_X(k+1) \end{aligned}$$

# L'inégalité de Markov

#### **Def 1:**

$$P(Z \ge a) \le \frac{E[Z]}{a}$$

Preuve:

$$egin{aligned} E[Z] &= \int_0^\infty z f_Z(z) \, dz \ &= \int_0^a z f_Z(z) \, dz + \int_a^\infty z f_Z(z) \, dz \ &\geq \int_a^\infty z f_Z(z) \, dz \ &\geq \int_a^\infty a f_Z(z) \, dz \ &= a \int_a^\infty f_Z(z) \, dz \ &= a P(Z \geq a) \ rac{E[Z]}{a} \geq P(Z \geq a) \end{aligned}$$

# ### L'inégalité de Chebychev

### **Def 1:**

$$Pr\left(|Z-E[Z]|>k\sqrt{Var(Z)}
ight)\leq rac{1}{k^2}$$

On introduit la  $v.\,a.\,Y=rac{(Z-E[Z])^2}{Var(Z)}$  avec E[Z]=1

$$egin{aligned} P(Z \geq k^2) & \leq rac{E[Z]}{k^2} \ & o P\left(rac{(Z-E[Z])^2}{Var(Z)} \geq k^2
ight) & \leq rac{1}{k^2} \ & o P\left(\left|rac{(Z-E[Z])}{\sqrt{Var(Z)}}
ight| \geq k
ight) & \leq rac{1}{k^2} \ & o P\left(|(Z-E[Z])| \geq k\sqrt{Var(Z)}
ight) \leq rac{1}{k^2} \end{aligned}$$

# Convergence en Distribution

#### **Def 1:**

$$\lim_{n o\infty}F_{W_n}(x)=F_Z(x)$$

## Def 2:

Juste si  $Pr(Z \leq E[Z]) = 1$ 

$$Y_n 
ightarrow^{\mathcal{D}} Z \Rightarrow Y_n 
ightarrow^{\mathcal{P}} Z$$

# Convergence en Probabilité

## **Def 1:**

$$\lim_{n o \infty} Pr(|Y_N - Z| > \epsilon) = 0$$

#### Def 2:

$$Y_n 
ightarrow^{\mathcal{P}} Z \Rightarrow Y_n 
ightarrow^{\mathcal{D}} Z$$