

Définitions et Preuves

Les mesures

$VaR_k(X)$, $TVaR_k(X)$, $LTVaR_k(X)$, $RVaR_k(X)$

Def 1:

$$VaR_k(X) = F^{-1}(X)$$

Preuve:

Def 2:

$$TVaR_k(X) = \frac{1}{1-k} \int_k^1 VaR_u(X) du$$

Def 3:

$$TVaR_k(X) = \frac{1}{1-k} \pi_x(VaR_k(X)) + VaR_k(X)$$

Preuve:

$$\begin{aligned}
TVaR_k(X) &= \frac{1}{1-k} \int_k^1 VaR_u(X) \, du \\
&= \frac{1}{1-k} \int_k^1 (VaR_u(X) - VaR_k(X) - VaR_k(X)) \, du \\
&= \frac{1}{1-k} \left(\int_k^1 (VaR_u(X) - VaR_k(X)) \, du + \int_k^1 VaR_k(X) \, du \right) \\
&= \frac{1}{1-k} \left(\int_k^1 F_X^{-1}(u) - VaR_k(X) \, du + (1-k)VaR_k(X) \right) \\
&= \frac{1}{1-k} \int_k^1 F_X^{-1}(u) - VaR_k(X) \, du + VaR_k(X) \\
&= \frac{1}{1-k} E[F_X^{-1}(u) - VaR_k(X); 0] + VaR_k(X) \\
&= \frac{1}{1-k} E[X - VaR_k(X); 0] + VaR_k(X) \\
&= \frac{1}{1-k} \pi_x(VaR_k(X)) + VaR_k(X)
\end{aligned}$$

Def 4:

$$TVaR_k(X) = \frac{1}{1-k} [E[X \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) (F_X(VaR_k(X)) - k)]$$

Preuve:

$$\begin{aligned}
TVaR_k(X) &= \frac{1}{1-k} E[X - VaR_k(X); 0] + VaR_k(X) \\
&= \frac{1}{1-k} E[(X - VaR_k(X)) \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) \\
&= \frac{1}{1-k} \{E[(X - VaR_k(X)) \times 1_{\{x > VaR_k(X)\}}] + (1-k)VaR_k(X)\} \\
&= \frac{1}{1-k} \{E[X \times 1_{\{x > VaR_k(X)\}}] - VaR_k(X)E[1 \times 1_{\{x > VaR_k(X)\}}] + (1-k)VaR_k(X)\} \\
&= \frac{1}{1-k} \left\{ E[X \times 1_{\{x > VaR_k(X)\}}] - VaR_k(X) \int_{VaR_k(X)}^1 1 \, du + (1-k)VaR_k(X) \right\} \\
&= \frac{1}{1-k} \{E[X \times 1_{\{x > VaR_k(X)\}}] - VaR_k(X)(1 - F_x(VaR_k(X))) + (1-k)VaR_k(X)\} \\
&= \frac{1}{1-k} \{E[X \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) (F_x(VaR_k(X)) - 1 + 1 - k)\} \\
&= \frac{1}{1-k} \{E[X \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) (F_x(VaR_k(X)) - k)\}
\end{aligned}$$

Propriété: (En cas continue)

$$TVaR_k(X) = \frac{1}{1-k} E[X \times 1_{\{x > VaR_k(X)\}}]$$

Preuve:

$$\begin{aligned} &= \frac{1}{1-k} \{ E[X \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) (F_x(VaR_k(X)) - k) \} \\ &= \frac{1}{1-k} \{ E[X \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) (F_x(F_x^{-1}(k)) - k) \} \\ &= \frac{1}{1-k} \{ E[X \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) (k - k) \} \\ &= \frac{1}{1-k} \{ E[X \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) (0) \} \\ &= \frac{1}{1-k} E[X \times 1_{\{x > VaR_k(X)\}}] \end{aligned}$$

Def 5:

$$LTVaR_k(X) = \frac{1}{k} \int_0^k VaR_u(X) du$$

Def 6:

$$LTVaR_k(X) = \frac{1}{k} [E[x] - (1-k)TVaR_k(X)]$$

Preuve:

$$\begin{aligned} LTVaR_k(X) &= \frac{1}{k} \int_0^k VaR_u(x) du \\ &= \frac{1}{k} \left(\int_0^1 F^{-1}(u) du - \int_1^k VaR_u(x) du \right) \\ &= \frac{1}{k} (E[x] - (1-k)TVaR_k(x)) \end{aligned}$$

Def 7:

$$LTVaR_k(X) = \frac{1}{k} \left[E[X \times 1_{\{x \leq VaR_k(X)\}}] + VaR_k(X) (k - F_X(VaR_k(X))) \right]$$

Preuve:

$$\begin{aligned}
LTVaR_k(X) &= \frac{1}{k} (E[X] - (1 - k)TVaR_k(x)) \\
&= \frac{1}{k} \left(E[X] - (1 - k) \left[\frac{1}{1 - k} [E[X \times 1_{\{x > VaR_k(X)\}}] + VaR_k(X) (F_X(VaR_k(X)) - 1) \right] \right) \\
&= \frac{1}{k} \left(E[X] - E[X \times 1_{\{x > VaR_k(X)\}}] - VaR_k(X) (F_X(VaR_k(X)) - k) \right) \\
&= \frac{1}{k} \left(E[X \times 1_{\{x \leq VaR_k(X)\}}] + VaR_k(X) (k - F_X(VaR_k(X))) \right)
\end{aligned}$$

Def 8:

$$RVaR_{k_1, k_2}(X) = \frac{1}{k_2 - k_1} \int_{k_1}^{k_2} VaR_u(X) du$$

Def 9:

$$RVaR_{k_1, k_2}(X) = \frac{1}{k_2 - k_1} \left((1 - k_1)TVaR_{k_1}(X) - (1 - k_2)TVaR_{k_2}(X) \right)$$

Preuve:

$$\begin{aligned}
RVaR_{k_1, k_2}(X) &= \frac{1}{k_2 - k_1} \int_{k_1}^{k_2} VaR_u(X) du \\
&= \frac{1}{k_2 - k_1} \left(\left(\frac{1 - k_1}{1 - k_1} \right) \int_{k_1}^1 VaR_u(X) du - \left(\frac{1 - k_2}{1 - k_2} \right) \int_{k_2}^1 VaR_u(X) du \right) \\
&= \frac{1}{k_2 - k_1} \left((1 - k_1)TVaR_{k_1}(X) - (1 - k_2)TVaR_{k_2}(X) \right)
\end{aligned}$$

Espérance tronquée

Def 1:

$$E[X] = E[X \times 1_{\{X \leq d\}}] + E[X \times 1_{\{X > d\}}]$$

Def 2: (La loi Weibull)

$$E[X \times 1_{\{X \leq d\}}] = \frac{1}{\beta} \Gamma \left(1 + \frac{1}{\tau} \right) \overline{H} \left(d^\tau, 1 + \frac{1}{\tau}, \beta^\tau \right)$$

Preuve:

$$E[X \times 1_{\{X > d\}}] = \int_d^\infty x \beta \tau (\beta x)^{\tau-1} e^{-(\beta x)^\tau} dx$$

$$u = (x\beta)^\tau$$
$$du = \tau \beta (\beta x)^{\tau-1} dx$$

$$\begin{aligned} &= \frac{1}{\beta} \int_{(\beta d)^\tau}^\infty u^{1/\tau} e^{-u} du \\ &= \frac{\Gamma\left(\frac{1}{\tau} + 1\right)}{\beta} \int_{(\beta d)^\tau}^\infty \frac{u^{(1/\tau+1)-1} e^{-u}}{\Gamma\left(\frac{1}{\tau} + 1\right)} du \\ &= \frac{\Gamma\left(\frac{1}{\tau} + 1\right)}{\beta} \bar{H}\left((\beta d)^\tau, \frac{1}{\tau} + 1, 1\right) \\ &= \frac{\Gamma\left(\frac{1}{\tau} + 1\right)}{\beta} \bar{H}\left(d^\tau, \frac{1}{\tau} + 1, \beta^\tau\right) \end{aligned}$$

Def 3: (La loi Burr)

$$E[X \times 1_{\{X \leq d\}}] = \frac{1}{\Gamma(\alpha)} \lambda^{1/\tau} \Gamma\left(1 + \frac{1}{\tau}\right) \Gamma\left(\alpha - \frac{1}{\tau}\right) B\left(\frac{d^\tau}{\lambda + d^\tau}; 1 + \frac{1}{\tau}, \alpha - \frac{1}{\tau}\right)$$

Preuve:

$$\begin{aligned}
E[X \times 1_{\{x \leq d\}}] &= \int_0^d x \times \frac{\alpha \tau \lambda^\alpha x^{\tau-1}}{(\lambda + x^\tau)^{\alpha+1}} dx \\
&= \int_0^d x \times \frac{\alpha \tau \lambda^{\alpha+1-1} x^{\tau-1}}{(\lambda + x^\tau)^{\alpha+2-1}} dx \\
&= \int_0^d \frac{x \alpha \tau \lambda x^{\tau-1}}{(\lambda + x^\tau)^2} \left(\frac{\lambda}{\lambda + x^\tau} \right)^{\alpha-1} dx
\end{aligned}$$

$$\begin{aligned}
u &= \left(\frac{\lambda}{\lambda + x^\tau} \right) \\
du &= -\frac{\lambda}{(\lambda + x^\tau)^2} \times \tau x^{\tau-1} dx
\end{aligned}$$

$$\begin{aligned}
&= -\alpha \int_{\frac{\lambda}{\lambda+d^\tau}}^{\frac{\lambda}{\lambda+d^\tau}} \left(\frac{\lambda}{u} - \lambda \right)^{1/\tau} u^{\alpha-1} du \\
&= \lambda^{1/\tau} \alpha \int_{\frac{\lambda}{\lambda+d^\tau}}^1 \left(\frac{1-u}{u} \right)^{1/\tau} u^{\alpha-1} du \\
&= \lambda^{1/\tau} \alpha \int_{\frac{\lambda}{\lambda+d^\tau}}^1 (1-u)^{1/\tau} u^{\alpha-1/\tau-1} du
\end{aligned}$$

$$\begin{aligned}
v &= 1 - u \\
dv &= -du
\end{aligned}$$

$$\begin{aligned}
&= -\lambda^{1/\tau} \alpha \int_{\frac{d^\tau}{\lambda+d^\tau}}^0 v^{1/\tau} (1-v)^{\alpha-1/\tau-1} dv \\
&= \lambda^{1/\tau} \alpha \int_0^{\frac{d^\tau}{\lambda+d^\tau}} v^{1/\tau+1-1} (1-v)^{\alpha-1/\tau-1} dv
\end{aligned}$$

Completion avec la loi Beta

$$\begin{aligned}
&= \lambda^{1/\tau} \alpha \times \frac{I\left(\frac{d^\tau}{\lambda+d^\tau}, \frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right)}{I\left(\frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right)} \\
&= \lambda^{1/\tau} \alpha I\left(\frac{d^\tau}{\lambda+d^\tau}, \frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right) \times \frac{\Gamma\left(1 + \frac{1}{\tau}\right) \Gamma\left(\alpha - \frac{1}{\tau}\right)}{\Gamma(\alpha + 1)} \\
&= \frac{1}{\Gamma(\alpha)} \lambda^{1/\tau} \Gamma\left(1 + \frac{1}{\tau}\right) \Gamma\left(\alpha - \frac{1}{\tau}\right) I\left(\frac{d^\tau}{\lambda+d^\tau}; 1 + \frac{1}{\tau}, \alpha - \frac{1}{\tau}\right)
\end{aligned}$$

La stop-loss

Def Bonus 2:

$$E[X] = \sum_{k=0}^{\infty} (1 - F_X(k))$$

Preuve:

$$\begin{aligned} E[X] &= \sum_{k=0}^{\infty} x f_X(x) \\ &= 0 \times f_X(0) + 1 \times f_X(1) + 2 \times f_X(2) + 3 \times f_X(3) + \dots \\ &= f_X(1) + (f_X(2) + f_X(2)) + (f_X(3) + f_X(3) + f_X(3)) + \dots \\ &= (f_X(1) + f_X(2) + f_X(3) + \dots) + (f_X(2) + f_X(3) + \dots) + (f_X(3) + \dots) \\ &= (1 - F_X(0)) + (1 - F_X(1)) + (1 - F_X(2)) + (1 - F_X(3)) + \dots \\ &= \sum_{k=0}^{\infty} (1 - F_X(k)) \end{aligned}$$

Def Bonus 2:

$$\pi_X(0) = E[X] = \sum_{k=0}^{\infty} (1 - F_X(k))$$

Preuve:

$$\begin{aligned} E[X] &= \sum_{x=k+1}^{\infty} x f_X(x) \\ &= (k+1) \times f_X(k+1) + (k+2) \times f_X(k+2) + (k+3) \times f_X(k+3) + \dots \\ &= f_X(k+1) + (f_X(k+2) + f_X(k+2)) + (f_X(k+3) + f_X(k+3) + f_X(k+3)) + \dots \\ &= (f_X(k+1) + f_X(k+2) + f_X(k+3) + \dots) + (f_X(k+2) + f_X(k+3) + \dots) + (f_X(k+3) + \dots) \\ &= (1 - F_X(k)) + (1 - F_X(k+1)) + (1 - F_X(k+2)) + (1 - F_X(k+3)) + \dots \\ &= \sum_{l=0}^{\infty} (1 - F_X(k+l)) \text{ ou } k=0 \end{aligned}$$

Def Bonus 3:

$$F_X(k) = 1 - \pi_X(k) + \pi_X(k+1)$$

Preuve:

on sait que:

$$\begin{aligned}\pi_X(k) &= \sum_{l=k}^{\infty} (1 - F_X(l)) \quad \text{et} \quad \pi_X(k+1) = \sum_{l=k+1}^{\infty} (1 - F_X(l)) \\ \pi_X(k) - \pi_X(k+1) &= \sum_{l=k}^{\infty} (1 - F_X(l)) - \sum_{l=k+1}^{\infty} (1 - F_X(l)) \\ &= (1 - F_X(k)) + \sum_{l=k+1}^{\infty} (1 - F_X(l)) - \sum_{l=k+1}^{\infty} (1 - F_X(l)) \\ &= 1 - F_X(k) \\ \therefore F_X(k) &= 1 - \pi_X(k) + \pi_X(k+1)\end{aligned}$$

L'inégalité de Markov

Def 1:

$$P(Z \geq a) \leq \frac{E[Z]}{a}$$

Preuve:

$$\begin{aligned}E[Z] &= \int_0^{\infty} z f_Z(z) dz \\ &= \int_0^a z f_Z(z) dz + \int_a^{\infty} z f_Z(z) dz \\ &\geq \int_a^{\infty} z f_Z(z) dz \\ &\geq \int_a^{\infty} a f_Z(z) dz \\ &= a \int_a^{\infty} f_Z(z) dz \\ &= a P(Z \geq a) \\ \frac{E[Z]}{a} &\geq P(Z \geq a)\end{aligned}$$

L'inégalité de Chebychev

Def 1:

$$Pr \left(|Z - E[Z]| > k\sqrt{Var(Z)} \right) \leq \frac{1}{k^2}$$

Preuve:

On introduit la v. a. $Y = \frac{(Z-E[Z])^2}{Var(Z)}$ avec $E[Z] = 1$

$$\begin{aligned} P(Z \geq k^2) &\leq \frac{E[Z]}{k^2} \\ &\rightarrow P \left(\frac{(Z - E[Z])^2}{Var(Z)} \geq k^2 \right) \leq \frac{1}{k^2} \\ &\rightarrow P \left(\left| \frac{(Z - E[Z])}{\sqrt{Var(Z)}} \right| \geq k \right) \leq \frac{1}{k^2} \\ &\rightarrow P \left(|(Z - E[Z])| \geq k\sqrt{Var(Z)} \right) \leq \frac{1}{k^2} \end{aligned}$$

Convergence en Distribution

Def 1:

$$\lim_{n \rightarrow \infty} F_{W_n}(x) = F_Z(x)$$

Def 2:

Juste si $Pr(Z \leq E[Z]) = 1$

$$Y_n \rightarrow^D Z \Rightarrow Y_n \rightarrow^P Z$$

Convergence en Probabilité

Def 1:

$$\lim_{n \rightarrow \infty} Pr(|Y_N - Z| > \epsilon) = 0$$

Def 2:

$$Y_n \rightarrow^P Z \Rightarrow Y_n \rightarrow^D Z$$