Définitions et Preuves

Les mesures $VaR_k(X),\ TVaR_k(X),\ LTVaR_k(X),\ RVaR_k(X)$

Def 1:

$$VaR_k(X) = F^{-1}(X)$$

Preuve:

Def 2:

$$TVaR_k(X) = \frac{1}{1-k}\int_k^1 VaR_u(X)\,du$$

Def 3:

$$TVaR_k(X) = \frac{1}{1-k}\pi_x(VaR_k(X)) + VaR_k(X)$$

Preuve:

$$
\begin{aligned}
TVaR_k(X) &= \frac{1}{1-k}\int_k^1 VaR_u(X)\,du \\
&= \frac{1}{1-k}\int_k^1 (VaR_u(X) - VaR_k(X) - VaR_k(X))\,du \\
&= \frac{1}{1-k}\left(\int_k^1 (VaR_u(X) - VaR_k(X))\,du + \int_k^1 VaR_k(X)\,du\right) \\
&= \frac{1}{1-k}\left(\int_k^1 F_X^{-1}(u) - VaR_k(X)\,du + (1-k)VaR_k(X)\right) \\
&= \frac{1}{1-k}\int_k^1 F_X^{-1}(u) - VaR_k(X)\,du + VaR_k(X) \\
&= \frac{1}{1-k}E[F_X^{-1}(u) - VaR_k(X);\ 0] + VaR_k(X) \\
&= \frac{1}{1-k}E[X - VaR_k(X);\ 0] + VaR_k(X) \\
&= \frac{1}{1-k}\pi_x(VaR_k(X)) + VaR_k(X)
\end{aligned}
$$

Def 4:

$$
TVaR_k(X) = \frac{1}{1-k}\left[E[X \times_{1\{x>VaR_k(X)\}}] + VaR_k(X)\,(F_X(VaR_k(X)-k))\right]
$$

Preuve:

$$
\begin{aligned}
TVaR_k(X) &= \frac{1}{1-k}E[X - VaR_k(X);\ 0] + VaR_k(X) \\
&= \frac{1}{1-k}E[(X - VaR_k(X)) \times_{1\{x>VaR_k(X)\}}] + VaR_k(X) \\
&= \frac{1}{1-k}\left\{E[(X - VaR_k(X)) \times_{1\{x>VaR_k(X)\}}] + (1-k)VaR_k(X)\right\} \\
&= \frac{1}{1-k}\left\{E[X \times_{1\{x>VaR_k(X)\}}] - VaR_k(X)E[1 \times_{1\{x>VaR_k(X)\}}] + (1-k)VaR_k(X)\right\} \\
&= \frac{1}{1-k}\left\{E[X \times_{1\{x>VaR_k(X)\}}] - VaR_k(X)\int_{VaR_{k(X)}}^1 1\,du + (1-k)VaR_k(X)\right\} \\
&= \frac{1}{1-k}\left\{E[X \times_{1\{x>VaR_k(X)\}}] - VaR_k(X)(1 - F_x(VaR_k(X))) + (1-k)VaR_k(X)\right\} \\
&= \frac{1}{1-k}\left\{E[X \times_{1\{x>VaR_k(X)\}}] + VaR_k(X)\,(F_x(VaR_k(X)) - 1 + 1 - k)\right\} \\
&= \frac{1}{1-k}\left\{E[X \times_{1\{x>VaR_k(X)\}}] + VaR_k(X)\,(F_x(VaR_k(X)) - k)\right\}
\end{aligned}
$$

Propriété: (En cas continue)

$$TVaR_k(X) = \frac{1}{1-k} E[X \times_{1\{x > VaR_k(X)\}}]$$

Preuve:

$$\begin{aligned}
&= \frac{1}{1-k}\left\{E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X)\left(F_x(VaR_k(X)) - k\right)\right\} \\
&= \frac{1}{1-k}\left\{E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X)\left(F_x(F_x^{-1}(k)) - k\right)\right\} \\
&= \frac{1}{1-k}\left\{E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X)\left(k - k\right)\right\} \\
&= \frac{1}{1-k}\left\{E[X \times_{1\{x > VaR_k(X)\}}] + VaR_k(X)\left(0\right)\right\} \\
&= \frac{1}{1-k} E[X \times_{1\{x > VaR_k(X)\}}]
\end{aligned}$$

Def 5:

$$LTVaR_k(X) = \frac{1}{k} \int_0^k VaR_u(X)\, du$$

Def 6:

$$LTVaR_k(X) = \frac{1}{k}\left[E[x] - (1-k) TVaR_k(X)\right]$$

Preuve:

$$\begin{aligned}
LTVaR_k\,(X) &= \frac{1}{k} \int_0^k VaR_u\,(x)\; du \\
&= \frac{1}{k}\left(\int_0^1 F^{-1}\,(u)\; du - \int_k^1 VaR_u\,(x)\; du\right) \\
&= \frac{1}{k}\left(E[x] - (1-k) TVaR_k(x)\right)
\end{aligned}$$

Def 7:

$$LTVaR_k(X) = \frac{1}{k}\left[E[X \times_{1\{x \leq VaR_k(X)\}}] + VaR_k(X)\big(k - F_X(VaR_k(X))\big)\right]$$

Preuve:

$$\begin{aligned}
LTVaR_k(X) &= \frac{1}{k}\big(E[X] - (1-k)TVaR_k(x)\big) \\
&= \frac{1}{k}\Bigg(E[X] - (1-k)\Bigg[\frac{1}{1-k}\left[E[X \times_{1\{x > VaR_k(X)\}}\right] + VaR_k(X)\left(F_X(VaR_k(X) - 1\right. \\
&= \frac{1}{k}\Bigg(E[X] - E[X \times_{1\{x > VaR_k(X)\}}] - VaR_k(X)\left(F_X(VaR_k(X) - k)\right)\Bigg) \\
&= \frac{1}{k}\Bigg(E[X \times_{1\{x \leq VaR_k(X)\}}] + VaR_k(X)\left(k - F_X(VaR_k(X))\right)\Bigg)
\end{aligned}$$

Def 8:

$$RVaR_{k_1,k_2}(X) = \frac{1}{k_2 - k_1}\int_{k_1}^{k_2} VaR_u(X)\,du$$

Def 9:

$$RVaR_{k_1,k_2}(X) = \frac{1}{k_2 - k_1}\Bigg((1-k_1)TVaR_{k_1}(X) - (1-k_2)TVaR_{k_2}(X)\Bigg)$$

Preuve:

$$\begin{aligned}
RVaR_{k_1,k_2}(X) &= \frac{1}{k_2 - k_1}\int_{k_1}^{k_2} VaR_u(X)\,du \\
&= \frac{1}{k_2 - k_1}\Bigg(\left(\frac{1-k_1}{1-k_1}\right)\int_{k_1}^{1} VaR_u(X)\,du - \left(\frac{1-k_2}{1-k_2}\right)\int_{k_2}^{1} VaR_u(X)\,du\Bigg) \\
&= \frac{1}{k_2 - k_1}\Bigg((1-k_1)TVaR_{k_1}(X) - (1-k_2)TVaR_{k_2}(X)\Bigg)
\end{aligned}$$

Espérance tronquée

Def 1:

$$E[X] = E[X \times_{1\{X \leq d\}}] + E[X \times_{1\{X > d\}}]$$

Def 2: (La loi Weibull)

$$E[X \times_{1\{X \leq d\}}] = \frac{1}{\beta}\Gamma\left(1 + \frac{1}{\tau}\right)\overline{H}\left(d^\tau, 1 + \frac{1}{\tau}, \beta^\tau\right)$$

Preuve:

$$E[X \times_{1\{X>d\}}] = \int_d^\infty x\beta\tau(\beta x)^{\tau-1} e^{-(\beta x)^\tau}\, dx$$

$$\begin{aligned} u &= (x\beta)^\tau \\ du &= \tau\beta(\beta x)^{\tau-1} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{\beta}\int_{(\beta d)^\tau}^\infty u^{1/\tau} e^{-u}\, du \\ &= \frac{\Gamma\left(\frac{1}{\tau}+1\right)}{\beta}\int_{(\beta d)^\tau}^\infty \frac{u^{(1/\tau+1)-1}e^{-u}}{\Gamma\left(\frac{1}{\tau}+1\right)}\, du \\ &= \frac{\Gamma\left(\frac{1}{\tau}+1\right)}{\beta}\bar{H}\left((\beta d)^\tau, \frac{1}{\tau}+1, 1\right) \\ &= \frac{\Gamma\left(\frac{1}{\tau}+1\right)}{\beta}\bar{H}\left(d^\tau, \frac{1}{\tau}+1, \beta^\tau\right) \end{aligned}$$

Def 3: (La loi Burr)

$$E\left[X \times 1_{\{X\leq d\}}\right] = \frac{1}{\Gamma(\alpha)}\lambda^{1/\tau}\Gamma\left(1+\frac{1}{\tau}\right)\Gamma\left(\alpha-\frac{1}{\tau}\right)B\left(\frac{d^\tau}{\lambda+d^\tau}; 1+\frac{1}{\tau}, \alpha-\frac{1}{\tau}\right)$$

Preuve:

$$
\begin{aligned}
E[X \times 1_{\{x \leq d\}}] &= \int_0^d x \times \frac{\alpha\tau\lambda^\alpha x^{\tau-1}}{(\lambda + x^\tau)^{\alpha+1}}\, dx \\
&= \int_0^d x \times \frac{\alpha\tau\lambda^{\alpha+1-1} x^{\tau-1}}{(\lambda + x^\tau)^{\alpha+2-1}}\, dx \\
&= \int_0^d \frac{x\alpha\tau\lambda x^{\tau-1}}{(\lambda + x^\tau)^2} \left(\frac{\lambda}{\lambda + x^\tau}\right)^{\alpha-1} dx
\end{aligned}
$$

$$
\begin{aligned}
u &= \left(\frac{\lambda}{\lambda + x^\tau}\right) \\
du &= -\frac{\lambda}{(\lambda + x^\tau)^2} \times \tau x^{\tau-1}\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= -\alpha \int_1^{\frac{\lambda}{\lambda + d^\tau}} \left(\frac{\lambda}{u} - \lambda\right)^{1/\tau} u^{\alpha-1}\, du \\
&= \lambda^{1/\tau}\alpha \int_{\frac{\lambda}{\lambda + d^\tau}}^1 \left(\frac{1-u}{u}\right)^{1/\tau} u^{\alpha-1}\, du \\
&= \lambda^{1/\tau}\alpha \int_{\frac{\lambda}{\lambda + d^\tau}}^1 (1-u)^{1/\tau} u^{\alpha-1/\tau-1}\, du
\end{aligned}
$$

$$
\begin{aligned}
v &= 1 - u \\
dv &= -du
\end{aligned}
$$

$$
\begin{aligned}
&= -\lambda^{1/\tau}\alpha \int_{\frac{d^\tau}{\lambda + d^\tau}}^0 v^{1/\tau}(1-v)^{\alpha-1/\tau-1}\, dv \\
&= \lambda^{1/\tau}\alpha \int_0^{\frac{d^\tau}{\lambda + d^\tau}} v^{1/\tau+1-1}(1-v)^{\alpha-1/\tau-1}\, dv
\end{aligned}
$$

Completion avec la loi Beta

$$
\begin{aligned}
&= \lambda^{1/\tau}\alpha \times \frac{I\left(\frac{d^\tau}{\lambda + d^\tau}, \frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right)}{I\left(\frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right)} \\
&= \lambda^{1/\tau}\alpha\, I\left(\frac{d^\tau}{\lambda + d^\tau}, \frac{1}{\tau} + 1, \alpha - \frac{1}{\tau}\right) \times \frac{\Gamma\left(1 + \frac{1}{\tau}\right)\Gamma\left(\alpha - \frac{1}{\tau}\right)}{\Gamma(\alpha + 1)} \\
&= \frac{1}{\Gamma(\alpha)}\lambda^{1/\tau}\Gamma\left(1 + \frac{1}{\tau}\right)\Gamma\left(\alpha - \frac{1}{\tau}\right) I\left(\frac{d^\tau}{\lambda + d^\tau}; 1 + \frac{1}{\tau}, \alpha - \frac{1}{\tau}\right)
\end{aligned}
$$

Def 4: (La loi Log-logistique)

La stop-loss

Def Bonus 2:

$$E[X] = \sum_{k=0}^{\infty}(1 - F_X(k))$$

Preuve:

$$\begin{aligned}
E[X] &= \sum_{k=0}^{\infty} x f_X(x) \\
&= 0 \times f_X(0) + 1 \times f_X(1) + 2 \times f_X(2) + 3 \times f_X(3) + \cdots \\
&= f_X(1) + (f_X(2) + f_X(2)) + (f_X(3) + f_X(3) + f_X(3)) + \cdots \\
&= (f_X(1) + f_X(2) + f_X(3) + \cdots) + (f_X(2) + f_X(3) + \cdots) + (f_X(3) + \cdots) \\
&= (1 - F_X(0)) + (1 - F_X(1)) + (1 - F_X(2)) + (1 - F_X(3)) + \cdots) \\
&= \sum_{k=0}^{\infty}(1 - F_X(k))
\end{aligned}$$

Def Bonus 2:

$$\pi_X(0) = E[X] = \sum_{k=0}^{\infty}(1 - F_X(k))$$

Preuve:

$$\begin{aligned}
E[X] = &\sum_{x=k+1}^{\infty} x f_X(x) \\
&= (k+1) \times f_X(k+1) + (k+2) \times f_X(k+2) + (k+3) \times f_X(k+3) + \cdots \\
&= f_X(k+1) + (f_X(k+2) + f_X(k+2)) + (f_X(k+3) + f_X(k+3) + f_X(k+3)) + \cdots \\
&= (f_X(k+1) + f_X(k+2) + f_X(k+3) + \cdots) + (f_X(k+2) + f_X(k+3) + \cdots) + (f_X(k + \\
&= (1 - F_X(k)) + (1 - F_X(k+1)) + (1 - F_X(k+2)) + (1 - F_X(k+3)) + \cdots) \\
&= \sum_{l=0}^{\infty}(1 - F_X(k+l)) \text{ ou } k = 0
\end{aligned}$$

Def Bonus 3:

$$F_X(k) = 1 - \pi_X(k) + \pi_X(k+1)$$

Preuve:
on sait que:

$$\pi_X(k) = \sum_{l=k}^{\infty}(1 - F_X(l)) \quad \text{et} \quad \pi_X(k+1) = \sum_{l=k+1}^{\infty}(1 - F_X(l))$$

$$\begin{aligned} \pi_X(k) - \pi_X(k+1) &= \sum_{l=k}^{\infty}(1 - F_X(l)) - \sum_{l=k+1}^{\infty}(1 - F_X(l)) \\ &= (1 - F_X(l)) + \sum_{l=k+1}^{\infty}(1 - F_X(l)) - \sum_{l=k+1}^{\infty}(1 - F_X(l)) \\ &= 1 - F_X(l) \end{aligned}$$

$$\therefore F_X(k) = 1 - \pi_X(k) + \pi_X(k+1)$$

L'inégalité de Markov

Def 1:

$$P(Z \geq a) \leq \frac{E[Z]}{a}$$

Preuve:

$$\begin{aligned} E[Z] &= \int_0^{\infty} z f_Z(z)\, dz \\ &= \int_0^{a} z f_Z(z)\, dz + \int_a^{\infty} z f_Z(z)\, dz \\ &\geq \int_a^{\infty} z f_Z(z)\, dz \\ &\geq \int_a^{\infty} a f_Z(z)\, dz \\ &= a\int_a^{\infty} f_Z(z)\, dz \\ &= aP(Z \geq a) \\ \frac{E[Z]}{a} &\geq P(Z \geq a) \end{aligned}$$

L'inégalité de Chebychev

Def 1:

$$Pr\left(|Z-E[Z]|>k\sqrt{Var(Z)}\right)\leq\frac{1}{k^2}$$

Preuve:

On introduit la $v.a. Y=\frac{(Z-E[Z])^2}{Var(Z)}$ avec $E[Z]=1$

$$\begin{aligned}P(Z\geq k^2)\leq\frac{E[Z]}{k^2}\\ \rightarrow P\left(\frac{(Z-E[Z])^2}{Var(Z)}\geq k^2\right) &\leq\frac{1}{k^2}\\ \rightarrow P\left(\left|\frac{(Z-E[Z])}{\sqrt{Var(Z)}}\right|\geq k\right) &\leq\frac{1}{k^2}\\ \rightarrow P\left(|(Z-E[Z])|\geq k\sqrt{Var(Z)}\right) &\leq\frac{1}{k^2}\end{aligned}$$

Convergence en Distribution

Def 1:

$$\lim_{n\to\infty}F_{W_n}(x)=F_Z(x)$$

Def 2:

Juste si $Pr(Z\leq E[Z])=1$

$$Y_n\rightarrow^{\mathcal{D}}Z\Rightarrow Y_n\rightarrow^{\mathcal{P}}Z$$

Convergence en Probabilité

Def 1:

$$\lim_{n\to\infty}Pr(|Y_N-Z|>\epsilon)=0$$

Def 2:

$$Y_n\rightarrow^{\mathcal{P}}Z\Rightarrow Y_n\rightarrow^{\mathcal{D}}Z$$

Principe de l'écart-type

Def 1:

$$\Psi(X)=E[X]+\theta\sqrt{Var(X)}$$

Preuve: on utilise l'inégalité de Cantelli

$$Pr(X > c) \leq \frac{Var(X)}{Var(X) + (c - E[X])^2}$$

$$(1 - \kappa) = \frac{Var(X)}{Var(X) + (c - E[X])^2}$$

$$c = E[X] + \sqrt{\frac{k}{1-k}}\sqrt{Var(X)}$$

$$\Psi_\theta(X) = E[X] + \theta\sqrt{Var(X)}$$

ou $\theta = \sqrt{\frac{k}{1-k}}$