Interpolation

Numerical Method to approximate numbers or functions over a given interval $[a \ b]$

Newton's Divided Difference Interpolation Method

Given Data

$$(x_0, f_0)\ (x_1, f_1)\ (x_2, f_2) \text{ ---- } (x_n, f_n)$$

$n+1$ data points

- Approximate a function from a given set of data points
- function will be a polynomial of degree 'n'. That is,

$$f(x) = P_n(x)$$

- Formula

$$f(x) = P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) \text{ ----}$$
$$\text{----}\ a_n(x-x_0)(x-x_1)(x-x_2) \text{ ---- } (x-x_{n-1}) \rightarrow (1)$$

In (1) a_i's for $i = 0, 1, 2, \text{ ---- } n$ are ~~#~~ unknown. ~~for~~ These are called Newton's divided difference Coefficients.

Task is to find a_i's and substitute them in (1) to get the polynomial.

We know Interpolation Conditions

$$P_n(x_i) = f_i \qquad i=0,1,2,\ldots n.$$

Eg. $P_n(x_0)=f_0$; $P_n(x_1)=f_1$; $P_n(x_2)=f_2$; --- $P_n(x_n)=f_n$

using $P_n(x_0)=f_0$ in (1)

$$P_n(x_0)=a_0+a_1\cancel{(x_0-x_0)}^{0}+a_2\cancel{(x_0-x_0)}^{0}(x_0-x_1)+a_3\cancel{(x_0-x_0)}^{0}(x_0-x_1)(x_0-x_2)+\cdots$$
$$a_n\cancel{(x_0-x_0)}^{0}(x_0-x_1)(x_0-x_2)\cdots(x_0-x_{n-1}) = f_0$$

$$\boxed{a_0=f_0.}$$

Similarly $P_n(x_1)=f_1$ in (1)

$$P_n(x_1)=a_0+a_1(x_1-x_0)+a_2(x_1-x_0)\cancel{(x_1-x_1)}^{0}+a_3(x_1-x_0)\cancel{(x_1-x_1)}^{0}(x_1-x_2)+\cdots$$
$$a_n(x_1-x_0)\cancel{(x_1-x_1)}^{0}(x_1-x_2)\cdots(x_1-x_{n-1}) = f_1$$

$$a_0+a_1(x_1-x_0)=f_1$$

$$\boxed{a_1=\frac{f_1-f_0}{x_1-x_0}} \qquad \text{where } a_0=f_0$$

Now $P_n(x_2)=f_2$

$$P_n(x_2)=a_0+a_1(x_2-x_0)+a_2(x_2-x_0)(x_2-x_1)+a_3(x_2-x_0)(x_2-x_1)\cancel{(x_2-x_2)}^{0}+$$
$$\cdots a_n(x_2-x_0)(x_2-x_1)\cancel{(x_2-x_2)}^{0}\cdots(x_2-x_{n-1}) = f_2$$

$$a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f_2$$

$$\boxed{a_2 = \frac{\dfrac{f_2 - f_1}{x_2 - x_1} - \dfrac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0}}$$

Similarly one can find $a_3, a_4 \cdots a_n$. However process is difficult and computationally expensive. Therefore we use the difference brackets to find the a_i's.

Given

$$(x_0, f_0)\ (x_1, f_1)\ (x_2, f_2)\ ---\ (x_n, f_n)$$

$$f(x) = P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) +$$
$$\cdots\cdots a_n(x-x_0)(x-x_1)(x-x_2) \cdots (x-x_{n-1}) \quad \sim (1)$$

where

$$a_0 = f_0$$
$$a_1 = f[x_1, x_0]$$
$$a_2 = f[x_2, x_1, x_0]$$
$$a_3 = f[x_3, x_2, x_1, x_0]$$
$$\vdots$$
$$a_{n-1} = f[x_{n-1}, x_{n-2}, \cdots x_0]$$
$$a_n = f[x_n, x_{n-1}, x_{n-2}, \cdots x_0]$$

General formula

$$a_m = f[x_m, x_{m-1} \cdots x_0]$$
$$= \frac{f[x_m, x_{m-1} --- x_0] - f[x_{m-1}, x_{m-2}, \cdots x_0]}{x_m - x_0}$$

Example

(1, 1.54) (2, 0.58) (3, 0.01) (4, 0.35)

Construct divided difference polynomial to approximate $x = 2.6$.

x	f	$f[x_1, x_0]$	$f[x_2, x_1, x_0]$
$x_0 \leftarrow 1$	$f_0 \leftarrow 1.54 \rightarrow a_0$	$\frac{f_1 - f_0}{x_1 - x_0} = \frac{0.58 - 1.54}{2 - 1} = -0.96 \rightarrow a_1$	$\frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0} = \cancel{\frac{-0.96-(-0.57)}{3-1}}$ $\frac{-0.57-(-0.96)}{3-1} = 0.2 \rightarrow a_2$
$x_1 \leftarrow 2$	$f_1 \leftarrow 0.58$	$\frac{f_2 - f_1}{x_2 - x_1} = \frac{0.01 - 0.58}{3 - 2} = -0.57$	$\frac{\frac{f_3 - f_2}{x_3 - x_2} - \frac{f_2 - f_1}{x_2 - x_1}}{x_3 - x_1} = \frac{0.29 - (-0.57)}{4 - 2} = 0.43$
$x_2 \leftarrow 3$	$f_2 \leftarrow 0.01$	$\frac{f_3 - f_2}{x_3 - x_2} = \frac{0.3 - 0.01}{4 - 3} = 0.29$	
$x_3 \leftarrow 4$	$f_3 \leftarrow 0.3$		

$f[x_3, x_2, x_1, x_0]$

$$\frac{0.43 - 0.2}{4 - 1} = 0.0767 \rightarrow a_3$$

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \rightarrow (1)$$

using values from table and given data.

$$P_3(x) = 1.54 - 0.96(x-1) + 0.2(x-1)(x-2) + 0.0767(x-1)(x-2)(x-3)$$

Ans.

$$P_3(2.6) = f(2.6) = 1.54 - 0.96(2.6-1) + 0.2(2.6-1)(2.6-2) + 0.0767(2.6-1)(2.6-2)(2.6-3)$$

$$= \boxed{\quad}$$ Ans.

Example 2

The upward velocity of a rocket is given as a function of time

t (sec)	0	10	15	20	22.5	30
v (m/sec)	0	227.04	362.78	517.35	602.97	901.67

a) Determine the velocity at $t = 16$ sec using third order polynomial Interpolation.

b) Find the distance covered by rocket from $t = 11$ to $t = 16$ sec.

c) Find acceleration of the rocket at $t = 16$ sec.

Solution

t	v
0	0
10	227.04
15	362.78
→ t = 16	→ $\boxed{}$
20	517.35
22.5	602.97
30	901.67

t	v
$t_0 \leftarrow 10$	$227.04 \rightarrow v_0$
$t_1 \leftarrow 15$	$362.78 \rightarrow v_1$
$t_2 \leftarrow 20$	$517.35 \rightarrow v_2$
$t_3 \leftarrow 22.5$	$602.97 \rightarrow v_3$

t	v	$v[t_1, t_0]$	$v[t_2, t_1, t_0]$	$v[t_3, t_2, t_1, t_0]$
10	227.04 (a_0)	$\frac{362.78-227.04}{15-10} = 27.148$ (a_1)	$\frac{30.914-27.148}{20-15} = 0.37660$ (a_2)	$\frac{0.44453-0.37660}{22.5-10} = 5.4347\times10^{-3}$ (a_3)
15	362.78	$\frac{517.35-362.78}{20-15} = 30.914$	$\frac{34.248-30.914}{22.5-20} = 0.44453$	
20	517.35	$\frac{602.97-517.35}{22.5-20} = 34.248$		
22.5	602.97			

$$v = f(x) = P_3(x) = a_0 + a_1(t-t_0) + a_2(t-t_0)(t-t_1) + a_3(t-t_0)(t-t_1)(t-t_2)$$

$$v(t) = 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15) + 5.4347\times10^{-3}(t-10)(t-15)(t-20)$$

Ans.

a) $v(16) = 227.04 + 27.148(16-10) + 0.37660(16-10)(16-15) + 5.4347\times10^{-3}(16-10)(16-15)(16-20) = 392.06 \text{ m/sec}$

b) $$s(16) - s(11) = \int_{11}^{16} v(t)\,dt$$

$$= \int_{11}^{16} 227.04 + 27.148(t-10) + 0.37660(t-10)(t-15) + 5.4347\times10^{-3}(t-10)(t-15)(t-20)\,dt$$

$$= 1605 \text{ m}$$

c) $$a(16) = \cancel{\frac{d}{dt}} \left.\frac{dv}{dt}\right|_{t=16}$$

$$= 29.664 \text{ m/sec}^2$$

Newton's Forward Difference Interpolation

- Data points are equally spaced

$$x_n - x_{n-1} = h$$

Eg.

x	f	t	v
2	6	0	10
4	5	3	15
6	-8	6	20
8	1	9	60

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= f_0 + \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) + \frac{\frac{f_2-f_1}{x_2-x_1} - \frac{f_1-f_0}{x_1-x_0}}{x_2-x_0}(x-x_0)(x-x_1)$$

$$= f_0 + \frac{f_1 - f_0}{h}(x-x_0) + \frac{(f_2-f_1)-(f_1-f_0)}{2h^2}(x-x_0)(x-x_1) + \cdots$$

General formula.

$$\cancel{P_n(x) = f_0 + \frac{\Delta f_0}{h} + \frac{\Delta^2 f_0}{2!\,h^2}}$$

General formula

$$P_n(x) = f_0 + \frac{\Delta f_0}{1!\,h}(x-x_0) + \frac{\Delta^2 f_0}{2!\,h^2}(x-x_0)(x-x_1) + \frac{\Delta^3 f_0}{3!\,h^3}(x-x_0)(x-x_1)(x-x_2)$$

$$+ \frac{\Delta^4 f_0}{4!\,h^4}(x-x_0)(x-x_1)(x-x_2)(x-x_3) + \text{———} \quad \text{———}$$

Example

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
	f_0	Δf_0	$\Delta^2 f_0$	$\Delta^3 f_0$	$\Delta^4 f_0$	$\Delta^5 f_0$
2	-7					
		$-3-(-7)=4$				
4	-3		$9-4=5$			
		$6-(-3)=9$		$10-5=5$		
6	6		$19-9=10$		$8-5=3$	
		$25-6=19$		$18-10=8$		$4-3=1$
8	25		$37-19=18$		$12-8=4$	
		$62-25=37$		$30-18=12$		
10	62		$67-37=30$			
		$12-62=67$				
12	129					

$h=2$

$$P_5(x) = f_0 + \frac{\Delta f_0}{2}(x-x_0) + \frac{\Delta^2 f_0}{2!\,(2)^2}(x-x_0)(x-x_1) + \frac{\Delta^3 f_0}{3!\,(2)^3}(x-x_0)(x-x_1)(x-x_2) +$$

$$\frac{\Delta^4 f_0}{4!\,(2)^4}(x-x_0)(x-x_1)(x-x_2)(x-x_3) + \frac{\Delta^5 f_0}{5!\,(2)^5}(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

$$= -7 + \frac{4}{2}(x-2) + \frac{5}{16}(x-2)(x-4) + \frac{5}{3!\,(2)^3}(x-2)(x-4)(x-6) +$$

$$\frac{3}{4!\,(2)^4}(x-2)(x-4)(x-6)(x-8) + \frac{1}{5!\,(2)^5}(x-2)(x-4)(x-6)(x-8)(x-10)$$

Ans.

Example (4)

Construct the Newton's Divided Difference interpolating Polynomial that passes through the points

$$\underset{x_0}{(-1}, \underset{f_0}{-3)}\quad \underset{x_1}{(0}\ \ \underset{f_1}{5)}\quad \underset{x_2}{(2}, \underset{f_2}{17)} \text{ and } \underset{x_3}{(3}, \underset{f_3}{21)}$$

Solution

x	f	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_3]$
-1	$-3 \nearrow a_0$			
		$\frac{5-(-3)}{0-(-1)} = 8 \nearrow a_1$		
0	5		$\frac{6-8}{2-(-1)} = -2/3 \nearrow a_2$	
		$\frac{17-5}{2-0} = 6$		$\frac{\frac{-2}{3} - \left(\frac{-2}{3}\right)}{3-(-1)} = 0 \nearrow a_3$
2	17		$\frac{4-6}{3-(0)} = -2/3$	
		$\frac{21-17}{3-2} = 4$		
3	21			

$$f(x) = P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= -3 + 8(x-(-1)) - \frac{2}{3}(x-(-1))(x-0) + 0$$

$$\boxed{f(x) = -\frac{2}{3}x^2 + \frac{22}{3}x + 5}$$

Example (5)

For what values of x, the function $f(x)$ passing through the points $(-1, -9)$ $(1, -1)$ $(3, 23)$ and $(4, 71)$, attains it extreme value. x_0 f_0 x_1 f_1 x_2 f_2 x_3 f_3

x	$f(x)$	$f[x_0,x_1]$	$f[x_0,x_1,x_2]$	$f[x_0,x_1,x_2,x_3]$
-1	$-9 \to a_0$			
		$4 \to a_1$		
1	-1		$2 \to a_2$	
		12		$2 \to a_3$
3	23		12	
		48		
4	71			

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= -9 + 4(x+1) + 2(x+1)(x-1) + 2(x+1)(x-1)(x-3)$$

$$f(x) = 2x^3 - 4x^2 + 2x - 1$$

For extreme point

$f'(x) = 0 \Rightarrow 6x^2 - 8x + 2 = 0$

$x = 1/3$ and $x = 1$

At $x = 1/3 \longrightarrow f(x) = -4 < 0$ maximum at $x = 1/3$

$x = 1 \longrightarrow f(x) = 4 > 0$ minimum at $x = 1$

Interpolation for Equally Spaced Data Points.

* Newton's Forward Difference Interpolation
** Newton's Backward Difference Interpolation

Given $n+1$ data points (~~[illegible]~~)

$$(x_0, f_0)\ (x_1, f_1)\ (x_2, f_2)\ \text{------}\ (x_n\ f_n)$$

Now $\quad x_i - x_{i-1} = h \longrightarrow$ equally spaced

$\quad i = 1, 2, 3 \text{---} n$

Now we define the forward differences as

$$\Delta f_i = f_{i+1} - f_i \qquad \Delta^2 f_i = \Delta(\Delta f_i) = \Delta f_{i+1} - \Delta f_i$$

$$\Delta^3 f_i = \Delta(\Delta^2 f_i) = \Delta^2 f_{i+1} - \Delta^2 f_i$$

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i \longrightarrow \text{general formula.}$$

Thus divided differences can be expressed as:

$$f[x_0] = f_0$$

$$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0} = \frac{\Delta f_0}{h}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{\Delta f_1}{h} - \frac{\Delta f_0}{h}}{2h} = \frac{1}{2!\,h^2}\Delta^2 f_0$$

Similarly

$$f[x_0, x_1, x_2, x_3] = \frac{1}{3!\,h^3}\Delta^3 f_0 \quad \text{------} \quad f[x_0, x_1 \text{---} x_n] = \frac{1}{n!\,h^n}\Delta^n f_0$$

This reduces the Newton's divided difference Interpolation formula to

$$f(x) = P_n(x) = f_0 + \frac{1}{h}(x-x_0)\Delta f_0 + \frac{1}{2!h^2}(x-x_0)(x-x_1)\Delta^2 f_0 + \frac{1}{3!h^3}(x-x_0)(x-x_1)(x-x_2)\Delta^3 f_0$$

$$+ \cdots \frac{1}{n!h^n}(x-x_0)(x-x_1)\cdots(x-x_{n-1})\Delta^n f_0 \rightarrow (A)$$

→ forward difference formula.

Now ~~If~~ for any point $x \in [x_0 \; x_n]$ or $x \in [a=x_0 \; b=x_n]$ can be written as $\cancel{x = x_0 + h}$ $\boxed{x = x_0 + \alpha h}$ for $0 \le \alpha \le n$

For example $x_1 = x_0 + (1)h \qquad x_2 = x_0 + (2)h$ ------

$$\Rightarrow \boxed{\alpha = \frac{x - x_0}{h}}$$

Thus $x - x_0 = \alpha h$; $(x-x_0)(x-x_1) = \alpha h\,(x-(x_0+h))$

$$= \alpha h\,(x - x_0 - h)$$

$$= \alpha h\,(\alpha h - h) = \alpha(\alpha-1)h^2$$

$$(x-x_0)(x-x_1)(x-x_2) = \alpha(\alpha-1)(\alpha-2)h^3$$

In general

$$(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1}) = \alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdots(\alpha-(n-1))h^n$$

The above reduces the forward difference formula to

$$f(x) = f_0 + \alpha\Delta f_0 + \frac{\alpha(\alpha-1)}{2!}\Delta^2 f_0 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Delta^3 f_0 + \cdots \frac{(\alpha)(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!}$$

Backward Difference Interpolation Formula.

$$f(x) = P_n(x) = f_n + \beta \nabla f_n + \frac{\beta(\beta+1)}{2!} \nabla^2 f_n + \frac{\beta(\beta+1)(\beta+2)}{3!} \nabla^3 f_n + \cdots$$

$$\cdots \frac{\beta(\beta+1)(\beta+2)\cdots(\beta+(n-1))}{n!} \nabla^n f_n$$

where $\boxed{\beta = \frac{x - x_n}{h}}$

Example (1) Construct Newton's forward difference Interpolation polynomial through the points $(-1, -3)$ $(1, 5)$ $(3, 17)$ and $(5, 2)$ and approximate data at $x = 2$

Solution

x	f	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
$-1 \to x_0$	$-3 \to f_0$			
		$5-(-3) = 8 \to \Delta f_0$		
$1 \to x_1$	$5 \to f_1$		$12-8 = 4 \to \Delta^2 f_0$	
		$17-5 = 12$		$-8-4 = -12 \to \Delta^3 f_0$
$3 \to x_2$	$17 \to f_2$		$4-12 = -8$	
		$21-17 = 4$		
$5 \to x_3$	$2 \to f_3$			

$h = 2$

$\alpha = \frac{x - x_0}{h}$

$x_0 = 1$

$\boxed{\alpha = \frac{x+1}{2}}$

$$f(x) = P_3(x) = f_0 + \alpha \Delta f_0 + \frac{\alpha(\alpha-1)}{2!} \Delta^2 f_0 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} \Delta^3 f_0$$

$$= -3 + \frac{8}{2}(x+1) + \frac{4}{8}(x+1)(x-1) - \frac{12}{48}(x+1)(x-1)(x-3)$$

$f(2) = $ ——— Ans.

Example (2)

Given $\cos(0.1) = 0.9950$, $\cos(0.4) = 0.9211$, $\cos(0.7) = 0.7648$, $\cos(1.0) = 0.5403$, ~~to~~ estimate $\cos(0.5) = ?$

Solution

x	f	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0.1	$0.9950 \nearrow f_0$			
		$-0.0739 \nearrow \Delta f_0$		
0.4	0.9211		$-0.0824 \nearrow \Delta^2 f_0$	
		-0.1563		$0.0142 \nearrow \Delta^3 f_0$
0.7	0.7648		-0.0682	
		-0.2245		
1.0	0.5403			

$h = 0.3$

$\alpha = \frac{x - x_0}{h}$

$x_0 = 0.1$

$$\boxed{\alpha = \frac{(x-0.1)}{0.3}}$$

$$f(x) = f_0 + \alpha \Delta f_0 + \frac{\alpha(\alpha-1)}{2!}\Delta^2 f_0 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Delta^3 f_0$$

$$f(x) = 0.9950 + \frac{(-0.0739)}{0.3}(x-0.1) + \frac{(-0.0824)}{(0.3)^2\, 2!}(x-0.1)(x-0.4) + \frac{0.0142}{(0.3)^3\, 3!}(x-0.1)$$

$(x-0.4)(x-0.7)$

$f(0.5) = \text{------}$ Ans.

Example (3) Construct the Newton's Backward difference Interpolating Polynomial using the nodes $x_0 = 1.25$, $x_1 = 1.50$, $x_2 = 1.75$ $x_3 = 2$ for the function $f(x) = \frac{1}{x^2}$. Also find $f(1.3)$, $f(1.85)$ and $f(1.95)$

Solution

$\boxed{h = 0.25}$ $\boxed{x_n = x_3 = 2}$

x	f	∇f_i	$\nabla^2 f_i$	$\nabla^3 f_i$
$1.25 \nearrow x_0$	$0.6400 \nearrow f_0$			
		$0.4444 - 0.6400 = -0.1956$		
$1.50 \nearrow x_1$	$0.4444 \nearrow f_1$		$-0.1179 + 0.1956 = 0.0777$	
		$0.3265 - 0.4444 = -0.1179$		$0.0414 - 0.0777 = -0.0363$
$1.75 \nearrow x_2$	$0.3265 \nearrow f_2$		$-0.0765 + 0.1179 = 0.0414$	
		$0.2500 - 0.3265 = -0.0765$		
$2 \nearrow x_3$	$0.2500 \nearrow f_3$			
	f_3	∇f_3	$\nabla^2 f_3$	$\nabla^3 f_3$

$$f(x) = P_3(x) = f_3 + \beta \nabla f_3 + \frac{\beta(\beta+1)}{2!} \nabla^2 f_3 + \frac{\beta(\beta+1)(\beta+2)}{3!} \nabla^3 f_3 \quad (1)$$

$$\beta = \frac{x - x_n}{h} = \frac{x - x_3}{h} = \boxed{\frac{x-2}{0.25}}$$

$$f(x) = 0.2500 - \frac{0.0765}{0.25}(x-2) + \frac{0.0414}{(0.25)^2\, 2!}(x-2)(x-1.75) - \frac{0.0363}{(0.25)^3\, 3!}(x-2)(x-1.75)(x-1.5)$$

$f(1.3) = 0.5929$ $\quad f(1.85) = 0.2929$ $\quad f(1.95) = 0.2636$

Ans.

Interpolation

Lagrange Interpolation:

Let us suppose that the given data points $(x_i, y_i),\ i = 0, 1, 2...n$ is coming from a function $f(x)$. Let us assume that this function $y = f(x)$ takes the values $y_0, y_1.......y_n$ at $x_0, x_1, ...x_n$. Since there are $(n+1)$ data points (x_i, y_i), we can represent the function $f(x)$ by a polynomial of degree n

$$\therefore f(x) = C_n x^n + C_{n-1} x^{n-1} + ... + C_1 x + C_0 \tag{1}$$

As we have assumed that $f(x_i) = y_i, i = 0, 1, 2.......n$ i.e. the function $f(x)$ passes through (x_i, y_i), (1) can be rewritten as:

$$y = f(x) = a_0(x - x_1)(x - x_2)...(x - x_n) + a_1(x - x_0)(x - x_2)...(x - x_n) +$$

$$a_2(x - x_0)(x - x_1)(x - x_3)...(x - x_n) + ... + a_n(x - x_0)...(x - x_{n-1}) \tag{2}$$

$$But, y_i = f(x_i) \quad i = 0, 1, ...n \tag{3}$$

Using (3) for i=0, in (2) we get

$$y_0 = f(x_0) = a_0(x_0 - x_1)...(x_0 - x_n)$$

$$\therefore a_0 = \frac{y_0}{(x_0 - x_1)...(x_0 - x_n)} \tag{4.1}$$

For $i = 1$, we get

$$y_1 = f(x_1) = a_1(x_1 - x_o)(x_1 - x_2)...(x_1 - x_n)$$

$$\therefore a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)...(x_1 - x_n)} \tag{4.2}$$

Similarly for $i = 2.......n-1$, we get

$$a_i = \frac{y_i}{(x_i - x_0)(x_i - x_1)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)}$$

and for $i = n$, we get

$$a_n = \frac{y_n}{(x_n - x_0)...(x_n - x_{n-1})} \qquad (4.3)$$

Using (4.1)-(4.3) in (2) we get

$$y = f(x) = \frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)}y_1 + ... \qquad (5)$$

$$+\frac{(x-x_0)(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)(x_i-x_1)..(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)}y_i + ... + \frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)...(x_n-x_{n-1})}y_n \qquad (6)$$

(5) can be rewritten in a compact form as:

$$\begin{aligned} y = f(x) &= L_0(x)y_0 + L_1(x)y_1 + ... + L_n(x)y_n \\ &= \sum_{i=0}^{n} L_i(x)y_i \\ &= \sum_{i=0}^{n} L_i(x)f(x_i) \qquad (7.1) \end{aligned}$$

where

$$L_i(x) = \frac{(x-x_0)(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_n)}{(x_i-x_0)(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_n)} \qquad (7.2)$$

It can be easily noted that

$$L_i(x_j) = \begin{cases} 1 \; if \; i = j \\ 0 \; if \; i \neq j \end{cases} \qquad 7.3$$

Let us introduce the product notation as **:**

$$\prod(x) = \prod_{i=0}^{n}(x - x_i) = (x-x_0)(x-x_1)....(x-x_n) \qquad (8.1)$$

$$\therefore \quad L_k(x) = \frac{\prod_{i=0, i\neq k}^{n} (x - x_i)}{\prod_{i=0, i\neq k}^{n} (x_k - x_i)} \tag{8.2}$$

Therefore, Lagrange interpolation polynomial of degree n can be written as

$$y = f(x) = \sum_{k=0}^{n} L_k(x) y_k \tag{9}$$

Example 1:
Given the following data table, construct the Lagrange interpolation

polynomial $f(x)$, to fit the data and find $f(1.25)$:

i	0	1	2	3
x_i	0	1	2	3
$y_i = f(x_i)$	1	2.25	3.75	4.25

Solution:

Here $n=3$.

$\therefore$ Lagrange interpolation polynomial is given by

$$\text{y=} f(x) = \sum_{i=0}^{3} L_i(x) y_i$$

$$L_0(x) = \frac{\prod_{i=0, i\neq 0}^{3} (x - x_i)}{\prod_{i=0, i\neq 0}^{3} (x_0 - x_i)}$$

$$= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$= \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)}$$

$$= \frac{x^3 - 6x^2 + 11x - 6}{-6}$$

$$L_1(x) = \frac{\prod_{i=0, i\neq 1}^{3}(x - x_i)}{\prod_{i=0, i\neq 1}^{3}(x_1 - x_i)}$$

$$= \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$= \frac{(x - 0)(x - 2)(x - 3)}{(1 - 0)(1 - 2)(1 - 3)}$$

$$= \frac{x^3 - 5x^2 + 6x}{2}$$

$$L_2(x) = \frac{\prod_{i=0, i\neq 2}^{3}(x - x_i)}{\prod_{i=0, i\neq 2}^{3}(x_2 - x_i)}$$

$$= \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$= \frac{(x - 0)(x - 1)(x - 3)}{(2 - 0)(2 - 1)(2 - 3)}$$

$$= \frac{x^3 - 4x^2 + 3x}{-2}$$

$$L_3(x) = \frac{\prod_{i=0, i\neq 3}^{3}(x - x_i)}{\prod_{i=0, i\neq 3}^{3}(x_3 - x_i)}$$

$$= \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

$$= \frac{(x - 0)(x - 1)(x - 2)}{(3 - 0)(3 - 1)(3 - 2)}$$

$$= \frac{x^3 - 3x^2 + 2x}{6}$$

$$\therefore \quad f(1.25) = \sum_{i=0}^{3} L_i(1.25)y_i$$

$$= L_0(1.25)y_0 + L_1(1.25)y_1 + L_2(1.25)y_2 + L_3(1.25)y_3$$

$$= (-0.546875).1 + (0.8203125)2.25 + (0.2734375)3.75 + (-0.0390625)4.25$$

$$= 2.650390625$$

Example 2:

Given the following data table, construct the Lagrange interpolation polynomial f(x), to fit the data and find $f(1998)$:

i	0	1	2	3	4	5
x_i	1980	1985	1990	1995	2000	2005
$y_i = f(x_i)$	440	510	525	571	500	600

Solution:

Here $n = 6$, $\quad x_k = 1998$

$\therefore$ Lagrange interpolation polynomial is given by

$$y = f(x) = \sum_{i=0}^{5} L_i(x)y_i; \quad y_k = f(x_k) = \sum_{i=0}^{5} L_i(x_k)y_i$$

$$L_0(x) = \frac{\prod_{i=0, i\neq 0}^{5}(x - x_i)}{\prod_{i=0, i\neq 0}^{5}(x_0 - x_i)} = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}$$

$$= \frac{(x-1985)(x-1990)(x-1995)(x-2000)(x-2005)}{(1980-1985)(1980-1990)(1980-1995)(1980-2000)(1980-2005)}$$

$$L_0(x_k) = L_0(1998) = \frac{(1998-1985)(1998-1990)(1998-1995)(1998-2000)(1998-2005)}{(-5)(-10)(-15)(-20)(-25)}$$

$$= \frac{13.8.3.(-2).(-7)}{-(375000)}$$

$$= -\frac{4368}{375000} = -0.011648$$

$$L_1(x_k) = \frac{\prod_{i=0, i\neq 1}^{5}(x_k - x_i)}{\prod_{i=0, i\neq 1}^{5}(x_1 - x_i)} = \frac{(x_k-x_0)(x_k-x_2)(x_k-x_3)(x_k-x_4)(x_k-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}$$

$$= \frac{(1998-1980)(1998-1990)(1998-1995)(1998-2000)(1998-2005)}{(1985-1980)(1985-1990)(1985-1995)(1985-2000)(1985-2005)}$$

$$= \frac{18.8.3.(-2).(-7)}{5(-5)(-10)(-15)(-20)} = 0.08064$$

$$\prod^{5} (x_k - x_i)$$

$$L_3(x_k) = \frac{\prod_{i=0, i\neq 3}^{5} (x_k - x_i)}{\prod_{i=0, i\neq 3}^{5} (x_3 - x_i)} = \frac{(1998-1980)(1998-1985)(1998-1990)(1998-2000)(1998-2005)}{(1995-1980)(1995-1985)(1995-1990)(1995-2000)(1995-2005)}$$

$$= \frac{18.13.8.(-2)(-7)}{15.10.5(-5)(-10)}$$

$$= 0.69888$$

$$L_4(x_k) = \frac{\prod_{i=0, i\neq 4}^{5} (x_k - x_i)}{\prod_{i=0, i\neq 4}^{5} (x_4 - x_i)} = \frac{(1998-1980)(1998-1985)(1998-1990)(1998-1995)(1998-2005)}{(2000-1980)(2000-1985)(2000-1990)(2000-1995)(2000-2005)}$$

$$= \frac{18.13.8.3.(-7)}{20.15.10.5(-5)}$$

$$= 0.52416$$

$$L_5(x_k) = \frac{\prod_{i=0, i\neq 5}^{5} (x_k - x_i)}{\prod_{i=0, i\neq 5}^{5} (x_5 - x_i)} = \frac{(1998-1980(1998-1985)(1998-1990)(1998-1995)(1998-2000)}{(2005-1980)(2005-1985)(2005-1990)(2005-1995)(2005-2000)}$$

$$= \frac{18.13.8.3.(-2)}{25.20.15.10.5} = -0.029952$$

$$\therefore \qquad f(1998) = \sum_{i=0}^{5} L_i(1998) y_i$$

$$= -0.011648 \times 440 + 0.08064 \times 510 + (-0.26208) \times 525 + 0.69888 \times 571 + 0.52416 \times 500 + (-0.029952) \times 600$$
$$= 541.578560$$

Note: Given a set of data points (x_i, y_i) , $i = 1, ...n$. Suppose we are interested in evaluating $f(x)$ at some intermediate point x to a desired level of accuracy. Directly using the entire data set of size n may not only be computationally economical but may also turn out to be redundant. Naturally one would like to use an interpolating polynomial of optimal degree. Since this is not known apriori, one may start with $P_o(x)$ and if it was enough then move onto $P_1(x)$ and so on i.e. slowly increase the no. of the interpolating points (or) data points $x_0, x_1..x_k$ so that $P_{k-1}(x)$ will be close to $f(x)$. In this context the biggest disadvantage with Lagrange Interpolation is that we cannot use the work that has already been done i.e. we cannot make use of $P_{k-1}(x)$ while evaluating $P_k(x)$. With the addition of each new data point, calculations have to be repeated. Newton Interpolation polynomial overcomes this drawback.

Exercise: Using the following data, construct the Lagrange Interpolation Polynomial

(1) $f(x) = \cos x + \sin x$, $x_0 = 0$, $x_1 = 0.25$, $x_2=0.5$, $x_3 = 1.0$, $n=3$

(2) $f(x) = \sin(\ln x)$, $x_0 = 2.0$, $x_1 = 2.4$, $x_2 = 2.6$, $n = 2$

Newton Interpolation polynomial with equidistant points:

Gregory-Newton Forward Difference Approach:

Very often it so happens in practice that the given data set $(x_i, y_i), i = 0, 1, ., n$ correspond to a sequence $\{x_i\}$ of equally spaced points. Here we can assume that

$$x_i = x_0 + ih, \qquad i = 0, 1, 2 \ldots n \tag{1}$$

where x_0 is the starting point (sometimes, for convenience, the middle data point is taken as x_0 and in such a case the integer i is allowed to take both negative and positive values.) and h is the step size. Further it is enough to calculate simple differences rather than the divided differences as in the non-uniformly placed data set case. These simple differences can be forward differences (Δf_i) or backward differences (∇f_i). We will first look at forward differences and the interpolation polynomial based on forward differences.

The first order forward difference Δf_i is defined as

$$\Delta f_i = f_{i+1} - f_i \tag{7.1}$$

The second order forward difference $\Delta^2 f_i$ is defined as

$$\Delta^2 f_i = \Delta f_{i+1} - \Delta f_i \tag{7.2}$$

The k^{th} order forward difference $\Delta^k f_i$ is defined as

$$\Delta^k f_i = \Delta^{k-1} f_{i+1} - \Delta^{k-1} f_i \tag{7.3}$$

Since we already know Newton interpolation polynomial in terms of divided differences, to derive or generate Newton interpolation polynomial in terms of forward differences it is enough to express forward differences in terms of divided differences.

Recall the definition of first divided difference $f[x_0, x_1]$,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f_1 - f_0}{h} = \frac{\Delta f_0}{h}$$

$$\therefore \Delta f_0 = h f[x_0, x_1] \tag{8.1}$$

Similarly we can get

$$\Delta f_1 = h f[x_1, x_2] \tag{8.2}$$

By the definition of second order forward difference $\Delta^2 f_0$, we get

$$\begin{aligned}
\Delta^2 f_0 &= \Delta f_1 - \Delta f_0 \\
&= \quad h f[x_1, x_2] - h f[x_0, x_1] \qquad\qquad (using (8.1) \& (8.2)) \\
&= h\{f[x_1, x_2] - f[x_0, x_1]\} \\
&= h.2h\{(f[x_1, x_2] - f[x_0, x_1])/2h\} \\
&= \quad 2h^2\{(f[x_1, x_2] - f[x_0, x_1])/(x_2 - x_o)\} \\
&= 2h^2 f[x_0, x_1, x_2]
\end{aligned}$$

In a similar way, in general, we can show that

$$\Delta^k f_i = k! h^k f[x_i, x_{i+1}, x_{i+2} \ldots x_{i+k}] \tag{8.4}$$

$$\therefore f[x_i, x_{i+1}, \ldots x_{i+k}] = \frac{\Delta^k f_i}{k! h^k} \tag{8.5}$$

For $i = 0$,

$$f[x_0, x_1 \ldots x_k] = \frac{\Delta^k f_0}{k! h^k} \tag{8.6}$$

Now using (6.1) & (8.6) the Newton forward difference interpolation polynomial may be written as follows:

$$P_n(x) = \sum_{k=0}^{n} \frac{\Delta^k f_0}{k! h^k} \prod_{i=0}^{k-1} (x - x_i) \tag{9}$$

To rewrite (9) in a simpler way let us set

$$x = x_0 + sh, \quad P_n(s) = P_n(x)$$

$$\because x_k = x_0 + kh$$

$$x - x_k = (s - k)h$$

$$\therefore \quad P_n(s) = \sum_{k=0}^{n} \frac{\Delta^k f_0}{k! h^k} \prod_{i=0}^{k-1} (s - i)h$$

$$= \sum_{k=0}^{n} \frac{\Delta^k f_0}{k! h^k} [s(s-1) \ldots\ldots (s-k+1)] h^k$$

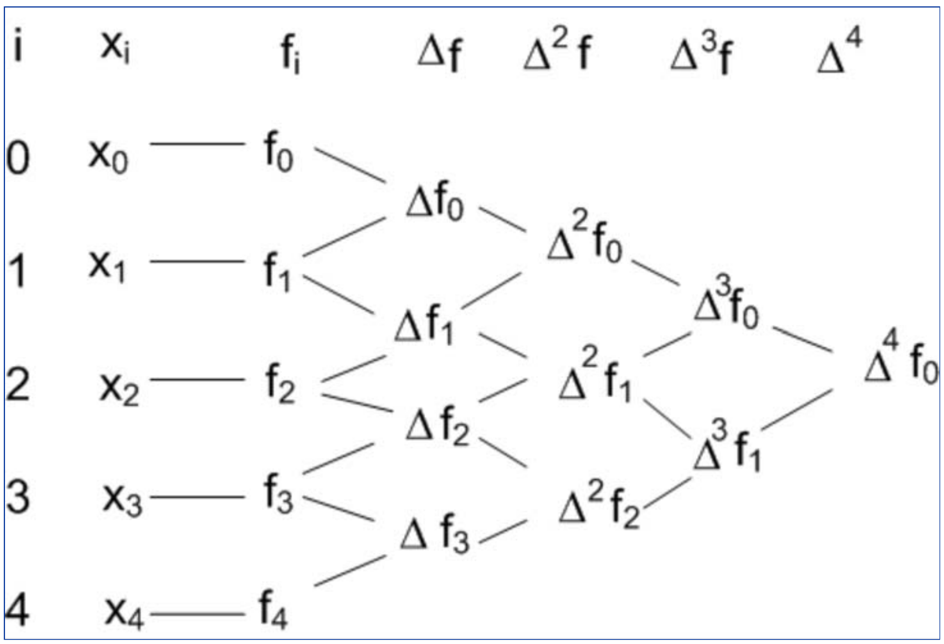
i.e $$P_n(s) = \sum_{k=0}^{n} \binom{s}{k} \Delta^k f_0 \tag{10}$$

where

$$\binom{s}{k} = \frac{s(s-1)......(s-k+1)}{k!}$$

This is known as Newton-Gregory forward difference interpolation polynomial. For convenience while constructing (10) one can first generate a forward difference table and use the $\Delta^k f_i$ from the table. Suppose we have data set $(x_i f_i)$, $i = 0, 1, 2, 3, 4$ then

forward difference table looks as follows:



Example 1:

Given the following data, estimate $f(1.83)$using Newton-Gregory forward difference interpolation polynomial:

i	0	1	2	3	4
x_i	1.0	3.0	5.0	7.0	9.0
f_i	0	1.0986	1.6094	1.9459	2.1972

Solution:

Here we have five data points i.e $i = 0, 1, 2, 3, 4$. Let us first generate the forward difference table.

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	1.0	.0				
			1.0986			
1	3.0	1.0986		-0.5878		
			0.5108		0.4135	
2	5.0	1.6094		-0.1743		−0.3244
			0.3365		0.0891	
3	7.0	1.9459		-0.0852		
			0.2513			
4	9.0	2.1972				

$$h = 2, \quad x = 1.83, \quad x_0 = 1.0, \quad s = \frac{x - x_0}{h} = 0.415$$

∴ Newton Gregory forward difference interpolation polynomial is given by:

$$P_n(s) = \sum_{k=0}^{n} \binom{s}{k} \Delta^k f_0$$

$$P_1(0.415) = \binom{0.415}{0} f_0 + \binom{0.415}{1} \Delta f_0$$

$$= 0 + (0.415)(1.0986) = 0.455919$$

$$P_2(0.415) = P_1(0.415) + \binom{0.415}{2} \Delta^2 f_0$$

$$= 0.455919 + \frac{0.415(0.415 - 1)}{2}(-0.5878)$$

$$= 0.455919 + 0.071352$$

$$= 0.527271$$

$$P_3(0.415) = P_2(0.415) + \binom{0.415}{3} \Delta^3 f_0$$

$$= 0.527271 + \frac{0.415(0.415-1)(0.415.2)}{6}(0.4135)$$

$$= 0.527271 + 0.026519$$

$$= 0.554157$$

$$P_4(0.415) = P_3(0.415) + \binom{0.415}{4} \Delta^4 f_0$$

$$= 0.554157 + \frac{0.415(0.415-1)(0.415-2)(0.415-3)}{24}(-0.3244)$$

$$= 0.554157 + 0.013445$$

$$= 0.567602$$

$$\therefore f(1.83) = 0.567602$$

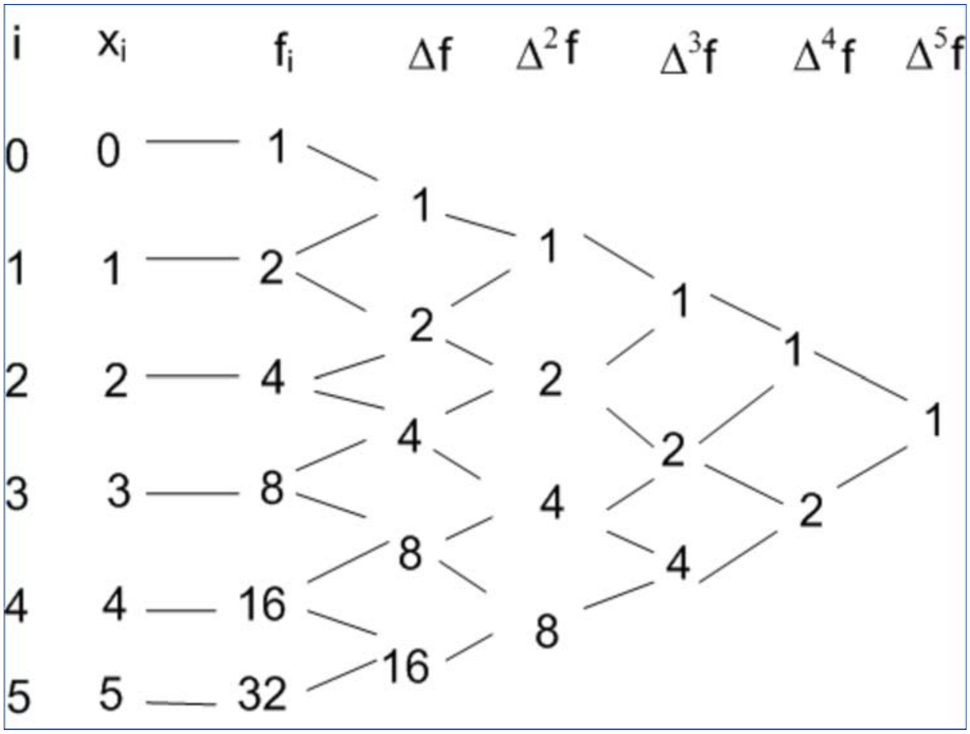
Example 2:

Given the following data estimate f(4.12) using Newton-Gregory forward difference interpolation polynomial:

i	0	1	2	3	4	5
x_i	0	1	2	3	4	5
f_i	1	2	4	8	16	32

Solution:

Let us first generate the Newton-Gregory forward difference table:



Here $i = 0, 1, 2, 3, 4, 5;$

$$x = 4.12, \quad x_0 = 0, \quad h = 1, \quad s = \frac{x - x_0}{h} = \frac{4.12}{1} = 4.12$$

We know that the forward difference interpolation polynomial is given by:

$$P_5(s) = \sum_{k=0}^{5} \binom{s}{k} \Delta^k f_0$$

$$\begin{aligned} \therefore P_1(4.12) &= \binom{4.12}{0} \Delta^0 f_0 + \binom{4.12}{1} \Delta f_0 \\ &= f_0 + 4.12 \quad \Delta f_0 \\ &= 1 + (4.12)(1) = 5.12 \qquad (10a.1) \end{aligned}$$

$$P_2(4.12) = P_1(4.12) + \binom{4.12}{2} \Delta^2 f_0$$

$$= 5.12 + \frac{4.12(4.12-1)}{2} 1$$

$$= 5.12 + 6.4272 = 11.5472 \tag{10a.2}$$

$$P_3(4.12) = P_2(4.12) + \binom{4.12}{3} \Delta^3 f_0$$

$$= 11.5472 + \frac{4.12(4.12-1)(4.12-2)}{6} 1$$

$$= 11.5472 + 4.5419$$

$$= 16.0891 \tag{10a.3}$$

$$P_4(4.12) = P_3(4.12) + \binom{4.12}{4} \Delta^4 f_0$$

$$= 16.0891 + \frac{4.12(4.12-1)(4.12-2)(4.12-3)}{24} 1$$

$$= 16.0891 + 1.2717$$

$$= 17.3608 \tag{10a.4}$$

$$P_5(4.12) = P_4(4.12) + \frac{4.12(4.12-1)(4.12-2)(4.12-3)(4.12-4)}{120}$$

$$= 17.3608 + 0.0305$$

$$= 17.3913 \tag{10a.5}$$

$$\therefore f(4.12) = 17.3913$$

Exercise: Calculate $f(\bar{x})$ using Newton-Gregory forward difference formula for the following data

1)

x	10	20	30	40	50
f(x)	0.1736	0.3420	0.5000	0.6428	0.7660

and $\bar{x} = 25$

2)

x	1.0	2.0	3.0	4.0
f(x)	0.0	0.6931	1.0986	1.3863

and $\bar{x} = 3.5$

Newton-Gregory Backward Difference Interpolation polynomial:

If the data size is big then the divided difference table will be too long. Suppose the desired intermediate value $(\widetilde{x})$ at which one needs to estimate the function $(i.e. f(\widetilde{x}))$ falls towards the end or say in the second half of the data set then it may be better to start the estimation process from the last data set point. For this we need to use backward-differences and backward difference table.

Let us first define backward differences and generate backward difference table, say for the data set $(x_i, f_i), i = 0, 1, 2, 3, 4.$

First order backward difference ∇f_i is defined as:

$$\nabla f_i = f_i - f_{i-1} \tag{11.1}$$

Second order backward difference $\nabla^2 f_i$ is defined as:

$$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1} \tag{11.2}$$

In general, the k^{th} order backward difference is defined as

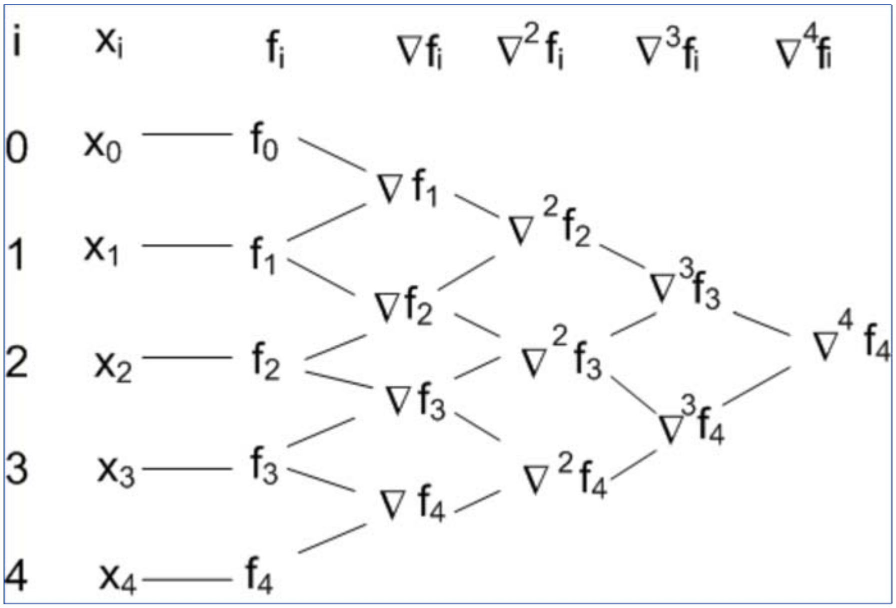
$$\nabla^k f_i = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1} \tag{11.3}$$

In this case the reference point is x_n and therefore we can derive the Newton-Gregory backward difference interpolation polynomial as:

$$P_n(\mathrm{S}) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \ldots\ldots + \frac{s(s+1)\ldots(s+n-1)}{n!}\nabla^n f_n \tag{12}$$

Where $s = \frac{x - x_n}{h}$

For constructing $P_n(s)$ as given in $Eqn.(12)$ it will be easier if we first generate backward-difference table. The backward difference table for the data $(x_i, f_i), \quad i = 0, 1, 2, 3, 4$ is given below:



The rounded differences appear in Eqn (12)

Newton Backward Difference Table:
Now let us apply Newton Backward difference approach to the second example solved earlier following the Newton forward difference approach i.e.

Example:

Given the following data estimate $f(4.12)$ using Newton-Gregory backward difference interpolation polynomial:

i	0	1	2	3	4	5
x_i	0	1	2	3	4	5
f_i	1	2	4	8	16	32

Solution:

Here

$$x_n = 5, \quad x = 4.12, \qquad h = 1$$

$$\therefore s = \frac{x - x_n}{h} = \frac{4.12 - 5}{1} = -0.88$$

$\therefore$ Newton Backward Difference polynomial $P_5(x)$ is given by

$$P_5(s) = f_5 + s\nabla f_5 + \frac{s(s+1)}{2!}\nabla^2 f_5 + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_5 + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f_5 +$$

$$\frac{s(s+1)(s+2)(s+3)(s+4)}{5!}\nabla^5 f_5$$

Let us first generate backward difference table:

i	x_i	f_i	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$	$\nabla^5 f$
0	0	1					
			1				
1	1	2		1			
			2		1		
2	2	4		2		1	
			4		2		1
3	3	8		4		2	
			8		4		
4	4	16		8			
			16				
5	5	32					

$$\therefore P_5(-0.88) = 32 + (-0.88)16 + \frac{(-0.88)(-0.88+1)}{2}8 + \frac{(-0.88)(-0.88+1)(-0.88+2)}{6}(4)$$

$$+\frac{(-0.88)(-0.88+1)(-0.88+2)(-0.88+3)}{24}(2)\frac{+(-0.88)(-0.88+1)(-0.88+2)(-0.88+3}{120}$$

$$= 32 - 14.08 - 0.4224 - 0.07885 - 0.0209 - 0.0065$$

$$= 17.92 - 0.4229 - 0.7885 - 0.0209 - 0.0065$$

$$= 17.4976 - 0.07885 - 0.0209 - 0.0065$$

$$= 17.41875 - 0.0209 - 0.0065$$

=17.39135 (13.5)

Now for comparison with the earlier solution i.e. the one obtained by forward Newton Divided Difference approach we may look at the above solution in stages similar to that provided earlier i.e.

$$P_1(-0.88) = f_5 + s\nabla f_5$$

$$= 32 + (-0.88)16 = 17.92 \qquad (13.1)$$

$$P_2(-0.88) = P_1(-0.88) + \frac{(-0.88)(-0.88+1)}{2!}8$$

$$= 17.92 - 0.4224$$

$$= 17.4976 \qquad (13.2)$$

$$P_3(-0.88) = P_2(-0.88) + \frac{(-0.88)(-0.88+1)(-0.88+2)}{3!}$$

$$= 17.4976 - 0.07885$$

$$= 17.41875 \qquad (13.3)$$

$$P_4(-0.88) = P_3(-0.88) + \frac{(-0.88)(-0.88+1)(-0.88+2)(-0.88+3)}{4!}$$

$$= 17.41875 - 0.0209$$

$$= 17.39785 \qquad (13.4)$$

$$P_5(-0.88) = 17.39785 - \frac{(-0.88)(-0.88+1)(-0.88+2)(-0.88+3)(-0.88+4)}{5!}$$

$$= 17.39785 - 0.0065$$

$$= 17.39135 \qquad (13.5)$$

Now one may note from (13.2) and (10a.2) that it is definitely advantageous of use backward difference approach here, as in exactly the same number of steps we are relatively more close to the approximate solution.

Exercise:

1) Given

x	1	2	3	4	5	6	7	8
f(x)	1	8	27	64	125	216	343	512

Estimate f(7.5) **using Newton-Gregory Backward difference**

interpolation formula.

2) Given

x	1.0	2.0	3.0	4.0
ln x	0.0	0.6931	1.0986	1.3863

Estimate ln(3.5) **using Newton-Gregory Backward difference interpolation formula.**

Newton Interpolation polynomial:

Suppose that we are given a data set $(x_i, f_i), i = 0, 1 ... n-1$. Let us assume that these are interpolating points of Newton form of interpolating polynomial $P_n(x)$ of degree n i.e

$$P_n(x_i) = f_i, \qquad i = 0, 1, ..n-1$$

The Newton form of the interpolating polynomial $P_n(x)$ is given by

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + ... + a_n(x - x_0)(x -$$

For i=0, from (1) & (2) we get

$$f_0 = P_n(x_0) = a_0$$

For $i = 1$, from (1) & (2) we get

$$f_1 = p_n(x_1) = a_0 + a_1(x_1 - x_0)$$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} \qquad (3$$

For i=2, from(1) & (2) we get

$$f_2 = P_n(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

Using (3.1) & (3.2), we get

$$a_2 = \frac{[(f_2 - f_1)/(x_2 - x_1)] - [(f_1 - f_0)/(x_1 - x_0)]}{(x_2 - x_0)} \qquad (3.3)$$

Similarly we can find $a_3.....a_{n-1}$. To express $a_i, i = 0.....n-1$ in a compact manner let us first define the following notation called divided differences:

$$f[x_k] = f_k$$

$$f[x_k, x_{k+1}] = \frac{f[x_{k+1}] - f[x_k]}{x_{k+1} - x_k} \qquad (4.2)$$

$$f[x_k, x_{k+1}, x_{k+2}] = \frac{f[x_{k+1}, x_{k+2}] - f[x_k, x_{k+1}]}{x_{k+2} - x_k} \qquad (4.3)$$

$$f[x_k, x_{k+1} ....x_i, x_{i+1}] = \frac{f[x_{k+1} ...x_{i+1}] - f[x_k ...x_i]}{x_{i+1} - x_k} \qquad (4.4)$$

Now the co-efficients a_i, $i=0,1,....n-1$ can be expressed in terms of divided differences as follows:

$$a_0 = f_0 = f[x_0] \qquad (5.1)$$

$$a_1 = \frac{f_1 - f_0}{x_1 - x_0} = f[x_0, x_1] \qquad (5.2)$$

$$a_2 = \frac{\frac{f_2 - f_1}{x_2 - x_1} - \frac{f_1 - f_0}{x_1 - x_0}}{x_2 - x_0} = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= f[x_0, x_1, x_2] \qquad (5.3)$$

$$a_n = f[x_0, x_1 ... x_n] \quad (5.4)$$

Note that a_1 is called as the first divided difference, a_2 as the second divided difference and so on. Now the polynomial (2) can be rewritten as:

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + ... + f[x_0 x_1 ...... x_n](x - x_0)(x -$$

i.e. $$P_n(x) = \sum_{k=0}^{n} f[x_0 ...... x_k] \prod_{i=0}^{k-1} (x - x_i) \quad (6.1)$$

This is called as **Newton's Divided Difference** interpolation polynomial.

Example:

Given the following data table, evaluate $f(2.4)$ using 3^{rd} order Newton's Divided Difference interpolation polynomial.

i	0	1	2	3	4
x_i	0	1	2	3	4
$y_i = f(x_i)$	1	2.25	3.75	4.25	5.81

Solution:

Here $n=5$. For constructing 3^{rd} order Newton Divided Difference polynomial we need only four points. Let us use the first four points. The 3^{rd} Newton Divided Difference polynomial is given by:

$$p_3(x) = \sum_{k=0}^{3} a_k \prod_{j=0}^{k-1} (x - x_j) = \sum_{k=0}^{3} f[x_0 ... x_k] \prod_{j=0}^{k-1} (x - x_j)$$

$$= f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x -$$

$$\therefore a_0 = f[x_0] = 1$$

$$a_1 = f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \frac{2.25 - 1}{1 - 0} = 1.25$$

$$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = \frac{3.75 - 2.25}{2 - 1} = 1.5$$

$$\therefore a_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1.5 - 1.25}{2 - 0} = 0.125$$

$$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2} = \frac{4.25 - 3.75}{3 - 2} = \frac{0.5}{1} = 0.5$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{0.5 - 1.5}{3 - 1} = -0.5$$

$$\therefore a_3 = f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}$$

$$= \frac{-0.5 - 0.125}{3 - 0} = \frac{-0.625}{3} = -0.20833$$

$$\therefore p_3(x) = 1 + 1.25(x - 0) + 0.125(x - 0)(x - 1) + (-0.20833)(x - 0)(x - 1)(x - 2)$$

$$\therefore f(2.4) = p_3(2.4)$$

$$= 1 + 1.25(2.4 - 0) + 0.125(2.4 - 0)(2.4 - 1) + (-0.20833)(2.4 - 0)(2.4 - 1)(2.4 - 2)$$

$$= 1 + (1.25)(2.4) + 0.125(2.4)(1.4) - 0.20833(2.4)(1.4)(0.4)$$

$$= 4.2200032$$

In this example it may be noted that for calculating the $3rd$ order polynomial, we first start with $P_0 = f[x_0] = 1$. To it we add $a_1(x - x_0)$ to get P_1 and to P_1 we add $a_2(x - x_0)(x - x_1)$ to get P_2. Finally on adding $a_3(x - x_0)(x - x_1)(x - x_2)$ to P_2 we get P_3.

Some remarks on the Error in the interpolation approach:

Suppose that the given data points (x_i, y_i), $i = 0, 1, ..n$ correspond to a real valued function $f(x)$ defined on the internal $I = [a, b]$. Let $P_n(x)$ be the interpolating polynomial of degree $\leq n.\ i.e.\ P_n(x_i) = f(x_i) = y_i \quad i = 0, 1...n$. Then the interpolation error $e_n(x)$ due to interpolation by $P_n(x)$ is given by

$$e_n(x) = f(x) - P_n(x). \tag{6.2}$$

An estimate of the error is provided in the following theorem.

Theorem: Let $f(x)$ be a real-valued function define on $[a,b]$ and $n+1$ times differentiable on (a,b). If $P_n(x)$ is the polynomial of degree $\leq n$ which interpolates $f(x)$ at the (n+1) distinct points $x_0..x_n \epsilon [a,b]$, then for all $\overline{x} \epsilon [a,b]$, there exists $\xi = \xi(\overline{x}) \epsilon (a,b) s.t.$

$$e_n(\overline{x}) = f(\overline{x}) - P_n(\overline{x})$$

$$= \frac{f^{n+1}(\xi)}{(n+1)!} \prod_{j=0}^{n} (\overline{x} - x_j) \tag{6.3}$$

Note:

i) $\xi = \xi(\overline{x})$ i.e. ξ depends on the point $\overline{x}$ at which the error estimate is required.

ii) Since $f^{n+1}(x)$ i.e. $(n+1)^{th}$ derivative is seldom known the error formula $(\mathbf{6.3})$ in the above theorem is of limited practical value. But when a bound on $|f^{n+1}(x)|$ is known over the entire interval $[a,b]$, then the formula $(\mathbf{6.3})$ may be used to get a bound on the interpolation error.

Newton Divided Difference Table:

It may also be noted for calculating the higher order divided differences we have used lower order divided differences. In fact starting from the given zeroth order differences $f[x_i]$; $i = 0,1,...,n,$ one can systematically arrive at any of higher order divided differences. For clarity the entire calculation may be depicted in the form of a table called

Newton Divided Difference Table.

i	x_i	$f[x_i]$	First order differences	Second order differences	Third order differences	Fourth order differences	Fifth order differences
0	x_0	$f[x_0]$					
			$f[x_0, x_1]$				
1	x_1	$f[x_1]$		$f[x_0,x_1,x_2,]$			
			$f[x_1,x_2]$		$f[x_0,x_1,x_2,x_3]$		
2	x_2	$f[x_2]$		$f[x_1,x_2,x_3]$		$f[x_0,x_1,x_2,x_3,x_4]$	
			$f[x_2,x_3]$		$f[x_1,x_2,x_3,x_4]$		$f[x_0,x_1,x_2,x_3,x_4,x_5]$
3	x_3	$f[x_3]$		$f[x_2,x_3,x_4]$		$f[x_1,x_2,x_3,x_4,x_5]$	
			$f[x_3,x_4]$		$f[x_2,x_3,x_4,x_5]$		
4	x_4	$f[x_4]$		$f[x_3,x_4,x_5]$			
			$f[x_4,x_5]$				
5	x_5	$f[x_5]$					

Again suppose that we are given the data set $(x_i, f_i),\ i = 0.......5$ and that we are interested in finding the 5^{th} order Newton Divided Difference interpolynomial. Let us first construct the Newton Divided Difference Table. Wherein one can clearly see how the lower order differences are used in calculating the higher order Divided Differences:

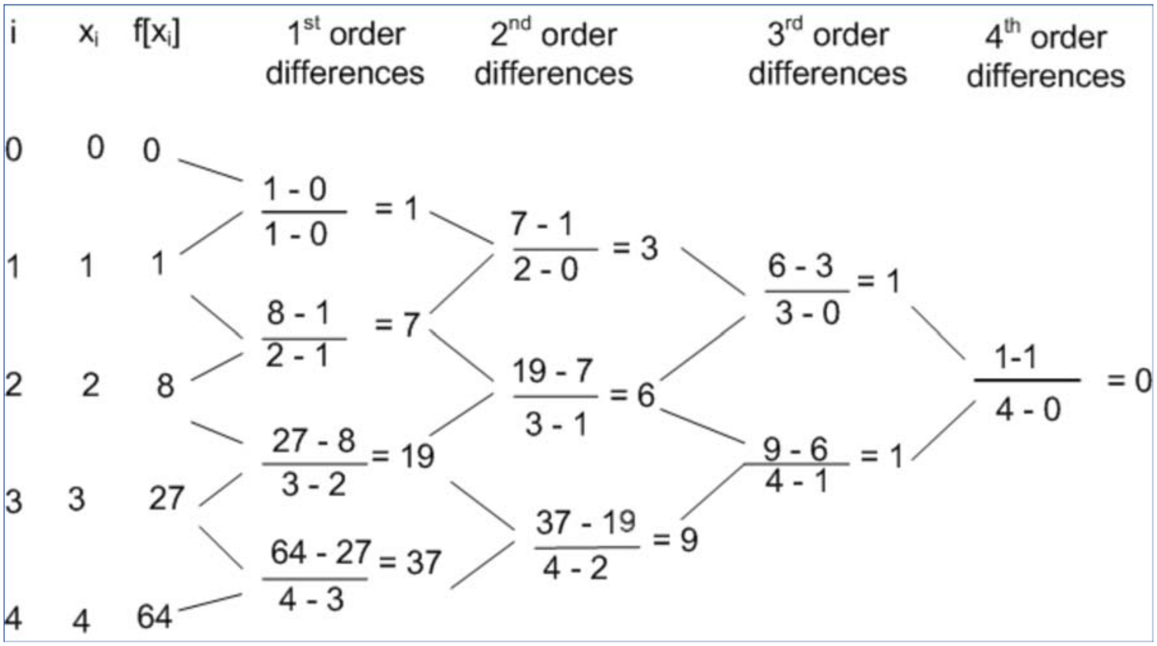
Example:

Construct the Newton Divided Difference Table for generating Newton interpolation polynomial with the following data set:

i	0	1	2	3	4
x_i	0	1	2	3	4
$y_i = f(x_i)$	0	1	8	27	64

Solution:

Here $n=5$. One can fit a fourth order Newton Divided Difference interpolation polynomial to the given data. Let us generate Newton Divided Difference Table; as requested.



Note: One may note that the given data corresponds to the cubic polynomial x^3. To fit such a data 3^{rd} order polynomial is adequate. From the Newton Divided Difference table we notice that the fourth order difference is zero. Further the divided differences in the table can be directly used for constructing the Newton Divided Difference interpolation polynomial that would fit the data.

Exercise: Using Newton divided difference interpolation polynomial , construct polynomials of degree two and three for the following data:

(1) *f(8.1) = 16.94410, f(8.3)=17.56492 , f(8.6) = 18.50515, f(8.7) = 18.82091.*

Also approximate *f(8.4).*

(2) *f(0.6) = -0.17694460 , f(0.7) = 0.01375227 , f(0.8) = 0.22363362 , f(1.0) = 0.65809197.*

Also approximate *f(0.9).*

Assignment # 5

Degree / Syndicate: ____________________ NAME: ________________ REGISTRATION No:

Q. 1 The velocity of a particle at time ***t*** from a point on its path is given by the table

Time (t)	0	10	20	30	40	50	60
Velocity (v)	47	58	64	65	61	52	38

Estimate velocity at t = 45 sec using five data points in the Newton's forward difference interpolation method.

Q. 2 Torque-speed data for an electric motor is given in the first two rows of the table below. Find the velocity when $\boldsymbol{t = 0.35}\ sec$? Use Newton's backward difference interpolation method.

t (sec)	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
S (cm)	**3.162**	**3.287**	**3.364**	**3.395**	**3.381**

Q. 3 The following values of $\boldsymbol{x}$ and *f(x)* were obtained in an experiment. The values of $\boldsymbol{x}$ are exact; the values of $\boldsymbol{y}$ are correct to 2 decimal places. It is required to estimate $\boldsymbol{\alpha}$, the value of $\boldsymbol{x}$ for which $\boldsymbol{f(x) = 0}$.

x	**0.9**	**1.1**	**1.2**	**1.4**	**1.5**
f(x)	**-0.43**	**-0.09**	**0.15**	**0.78**	**1.15**

Obtain an estimate of $\boldsymbol{\alpha}$ by fitting a cubic polynomial.

Q.4 The velocity ***v*** of an airplane which starts from rest is given at fixed intervals of time as shown

t (min)	**0**	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
v (km/min)	**0**	**8**	**17**	**24**	**28**	**30**	**20**	**12**	**6**	**2**	**0**

Estimate the acceleration at ***t* = *13*** minutes. Use both Newton's forward difference interpolation and the Newton's backward difference interpolation numerical methods.