MATH-351 **Numerical Methods** [redacted]

Lecture/Week 5

Interpolation

Numerical approximation addresses both

a) Approximating numbers

b) Approximating functions

The mathematical concept of interpolation is concerned with the problem of approximating a function $f(x)$ defined over a closed interval $[a \ b]$.

- The function $f(x)$ to be approximated is usually defined through a set of its values a $n+1$ distinct points lying in the interval $[a \ b]$. That is

$$(x_i, f(x_i)) \quad \text{for } i = 0, 1, 2, \dots n$$

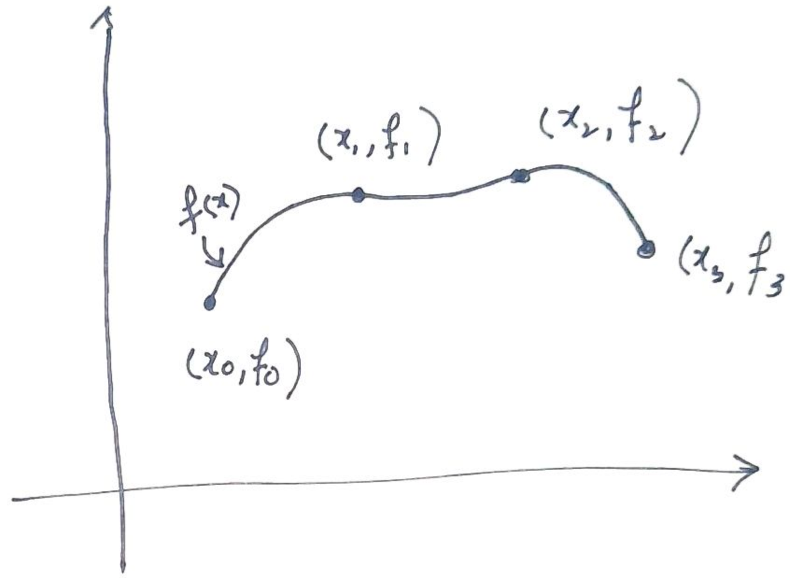
where $x_0 = a$ and $x_n = b$.

Given Data points

$$(x_0, f(x_0)) \ (x_1, f(x_1)) \ (x_2, f(x_2)) \ \dots \ (x_n, f(x_n))$$

or $$(x_0, f_0) \ (x_1, f_1) \ (x_2, f_2) \ \dots \ (x_n, f_n)$$

The problem is to determine a function $f(x)$ that passes through the distinct points (x_0, f_0) (x_1, f_1) (x_2, f_2) ----- (x_n, f_n).



Two types of interpolation

1) Polynomial Interpolation

2) Spline Interpolation
(Piece-wise continuous function)

Polynomial Interpolation

a) Lagrange Interpolation Method

b) Newton's Divided Difference Interpolation Method

- → Newton's Forward Difference Interpolation Method
- → Newton's Backword Difference Interpolation Method

Lagrange Interpolation Method.

Given $(x_0, f_0)\ (x_1, f_1)\ (x_2, f_2)\ \cdots\cdots\ (x_n, f_n)$

Problem is to find

$f(x) = P_n(x)$ → that is polynomial of degree 'n'.

This means if we have $n+1$ points the polynomial is of degree 'n'.

Ex:- If we have 10 points, polynomial of degree is 9.

For 100 points, polynomial degree is 99.

" 5 " " " " 4.

Formula.

$$P_n(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2 \cdots\cdots L_n(x) f_n$$

where $L_i(x)$ for $i = 0, 1, 2, \cdots n$

are called the Lagrange Interpolation coefficients.

$$\cancel{L_i(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{}}$$

$$\boxed{L_i(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\cdots\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}}$$

For example

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)\cdots\cdots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)\cdots(x_2-x_n)}$$

$$L_5(x) = \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_6)(x-x_7)\cdots(x-x_n)}{(x_5-x_0)(x_5-x_1)(x_5-x_3)(x_5-x_4)(x_5-x_6)(x_5-x_7)\cdots(x_5-x_n)}$$

Example 1

Given $(1, 1.54)$ $(2, 0.58)$ $(3, 0.01)$ $(4, 0.35)$

x	f
$x_0 = 1$	$f_0 = 1.54$
$x_1 = 2$	$f_1 = 0.58$
$x_2 = 3$	$f_2 = 0.01$
$x_3 = 4$	$f_3 = 0.35$

Question Construct the Lagrange Interpolation polynomial to approximate the value of the function at $x = 2.6$

Solution

Step I: Approximate polynomial function

$f(x) = P_3(x)$ → polynomial of degree 3.

Step II:

Find $f(2.6) = P_3(2.6) = \boxed{\ldots\ldots}$

Step I

$$f(x) = P_3(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2 + L_3(x) f_3 \rightsquigarrow (1)$$

Now

$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = \frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)}$$

$$= -\frac{1}{6}(x-2)(x-3)(x-4) \rightsquigarrow (2)$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)}$$

$$= \frac{1}{2}(x-1)(x-3)(x-4) \rightsquigarrow (3)$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4}$$

$$= \frac{-1}{2}(x-1)(x-2)(x-4) \rightsquigarrow (4)$$

$$L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_3)} = \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)}$$

$$= \frac{1}{6}(x-1)(x-2)(x-3) \rightsquigarrow (5)$$

Using (2) — (5) into 1

$$P_3(x) = \frac{-1}{6}(x-2)(x-3)(x-4)[1.54] + \frac{1}{2}(x-1)(x-3)(x-4)[0.58]$$
$$-\frac{1}{2}(x-1)(x-2)(x-4)[0.01] + \frac{1}{6}(x-1)(x-2)(x-3)[0.35]$$

Interpolating Polynomial.

Interpolation Conditions

$$P_n(x_i) = f_i \qquad i = 0, 1, 2, \ldots n.$$

Check for our example

$$P_3(x_i) = f_i \qquad i = 0, 1, 2, 3.$$

Let $i = 1$

$$P_3(x_1) = f_1$$

$$P_3(x_1) = P_3(2) = \frac{-1}{6}(2-2)^{\to 0}(2-3)(2-4)[1.54] + \frac{1}{2}(2-1)(2-3)(2-4)[0.58]$$
$$-\frac{1}{2}(2-1)(2-2)^{\to 0}(2-4)[0.01] + \frac{1}{6}(2-1)(2-2)^{\to 0}(2-3)[0.35]$$
$$= \frac{1}{2}(1)(-1)(-2)[0.58] = 0.58 = f_1$$

Similarly one can check

$$P_3(x_3) = f_3 \quad \text{or} \quad P_3(x_0) = f_0$$

or

$$P_2(x_2) = f_2 .$$

Step II :

$$f(x) = P_3(x)$$

$$f(2.6) = P_3(2.6) = -\frac{1}{6}(2.6-2)(2.6-3)(2.6-4)[1.54] +$$

$$\frac{1}{2}(2.6-1)(2.6-3)(2.6-4)[0.58] - \frac{1}{2}(2.6-1)(2.6-2)(2.6-4)[0.01]$$

$$+\frac{1}{6}(2.6-1)(2.6-2)(2.6-3)[0.35] = \underline{\qquad}$$

Ans.

Example 2

	x	f
$x_0 \rightarrow$	0.1	$0.91 \rightarrow f_0$
$x_1 \rightarrow$	0.2	$0.70 \rightarrow f_1$
$x_2 \rightarrow$	0.3	$0.43 \rightarrow f_2$
$x_3 \rightarrow$	0.5	$0.54 \rightarrow f_3$

find $f(0.4) = ?$

Solution

$$f(x) = P_3(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2 + L_3(x) f_3$$

$$= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f_2$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3 .$$

$$= \frac{(x-0.2)(x-0.3)(x-0.5)}{(0.1-0.2)(0.1-0.3)(0.1-0.5)} (0.91) + \frac{(x-0.1)(x-0.3)(x-0.5)}{(0.2-0.1)(0.2-0.3)(0.2-0.5)} (0.70) +$$

$$\frac{(x-0.1)(x-0.2)(x-0.5)}{(0.3-0.1)(0.3-0.2)(0.3-0.5)} (0.43) + \frac{(x-0.1)(x-0.2)(x-0.3)}{(0.5-0.1)(0.5-0.2)(0.5-0.3)} (0.54)$$

$f(0.4) = P_3(0.4) =$ ______ ______ . . .

Example ③

A slider in a machine move along a fixed straight rod. Its velocity 'v' is given below for variables values of time. Compute the time for which the velocity of the slider is maximum.

t	3 $\to t_0$	4 $\to t_1$	5 $\to t_2$	6 $\to t_3$
v	0.205 $\to v_0$	0.240 $\to v_1$	0.259 $\to v_2$	0.262 $\to v_3$

$$V(t) = L_0(t)V_0 + L_1(t)V_1 + L_2(t)V_2 + L_3(t)V_3 \quad \text{①}$$

$$L_0(t) = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)} = \frac{(t-4)(t-5)(t-6)}{(3-4)(3-5)(3-6)} = \frac{(t-4)(t-5)(t-6)}{-6} \quad \text{②}$$

$$L_1(t) = \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)} = \frac{(t-3)(t-5)(t-6)}{(4-3)(4-5)(4-6)} = \frac{(t-3)(t-5)(t-6)}{2} \quad \text{③}$$

$$L_2(t) = \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)} = \frac{(t-3)(t-4)(t-6)}{(5-3)(5-4)(5-6)} = \frac{(t-3)(t-4)(t-6)}{-2} \quad \text{④}$$

$$L_3(t) = \frac{(t-t_0)(t-t_1)(t-t_2)}{(t_3-t_0)(t_3-t_1)(t_3-t_2)} = \frac{(t-3)(t-4)(t-5)}{(6-3)(6-4)(6-5)} = \frac{(t-3)(t-4)(t-5)}{6} \quad \text{⑤}$$

using (2) — (5) into ①

$$\boxed{V(t) = \frac{0.205}{(-6)}(t-4)(t-5)(t-6) + \frac{0.240}{2}(t-3)(t-5)(t-6) - \frac{0.259}{2}(t-3)(t-4)(t-6) + \frac{0.262}{6}(t-3)(t-4)(t-5).}$$

For maximum velocity find zero of $v'(t)$

Complete yourself.

Example (4)

Find the point where $f(x)=0$.
That is, the root of $f(x)$

x	$f(x)$
$x_0 \leftarrow 1.1$	$-0.09 \to f_0$
$x_1 \leftarrow 1.2$	$0.15 \to f_1$
$x_2 \leftarrow 1.5$	$1.15 \to f_2$

$$f(x) = P_2(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$= \frac{(x-1.2)(x-1.5)}{(1.1-1.2)(1.1-1.5)}(-0.09) + \frac{(x-1.1)(x-1.5)}{(1.2-1.1)(1.2-1.5]}(0.15) + \frac{(x-1.1)(x-1.2)}{(1.5-1.1)(1.5-1.2)}(1.15)$$

$$= 25(x-1.2)(x-1.5)(-0.09) - 33.33(x-1.1)(x-1.5)(0.15) + 8.33(x-1.1)(x-1.2)(1.15)$$

Simplify.

$$f(x) = 2.321x^2 - 2.955x + 0.332$$

Now to find root

$$2.321x^2 - 2.955x + 0.332 = 0$$

use quadratic formula

$x_1 = 1.1486$ $\qquad$ $\underline{x_2 = 0.1245}$

out of range.

Therefore $\boxed{x_1 = 1.1486}$

Ans.

Assignment # 4

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Q. 1 The following data give the percentage age of criminals for different age groups.

Age	25	30	40	50
% of criminals	52	67.3	84.1	94.4

Use Lagrange formula to find the percentage age of criminals under the age of 35.

Q 2 **A)** Find the third degree Taylor polynomial for the function $f(x) = \frac{2}{x+2}$ expanded about $x_0 = 2$. Then use it to approximate $f(2.3)$.

B) Use the Lagrange's interpolation formula based on the points $x_0 = 0$, $x_1 = 1$ $x_2 = 1.5$ and $x_3 = 3$ to find the equation of the cubic polynomial to approximate $f(x) = \frac{2}{x+2}$ at $x = 2.3$.

Q. 3 The velocity of a particle at time t from a point on its path is given by the table

Time (t)	0	10	20	30	40	50	60
Velocity (v)	47	58	64	65	61	52	38

Estimate velocity at t = 40 sec using five data points in interpolation method.

Q. 4 Torque-speed data for an electric motor is given in the first two rows of the table below.
Find the velocity when $t = 0.35\ sec$? Use appropriate interpolation method.

t (sec)	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**
S (cm)	**3.162**	**3.287**	**3.364**	**3.395**	**3.381**