MATH-351 Numerical Methods [redacted]

Lecture 4 / Week [redacted] 3

Roots of non-linear function

i) Bisection Method

Step I $\rightarrow$ Choose initial Interval

[a   b]

$f(a)$ $f(b)$

$-ve$ $+ve$

Step II $\rightarrow$ (Assume root) $\longrightarrow$ $c = \frac{a+b}{2}$

$\downarrow$

$f(c)$

$-ve$

[c = a   b]

$f(c) = f(a)$ $f(b)$

$-ve$ $+ve$

c'

$\downarrow$

$f(c')$

$+ve$

[a $c' = b$]

$f(a)$ $f(c) = f(b)$

$-ve$ $+ve$

c''

$f(c'')$

$-ve$

[[   ]

. . . . . .

ii) Regula Falsi Method

Step I Assume Initial Interval

[a b]

a → f(a) -ve

b → f(b) +ve

Step II → (Assume Root) → $C = a - \frac{b-a}{f(b)-f(a)} f(a)$

↓

f(c) +ve

[a c = b]

a → f(a)

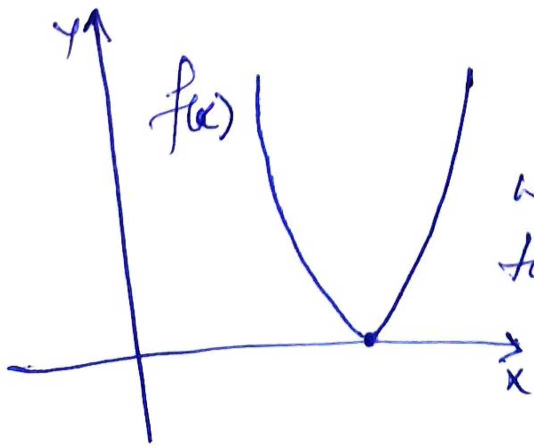
c = b → f(c) = f(b)

↓

c - - - - -

Short Coming

Two initial inputs



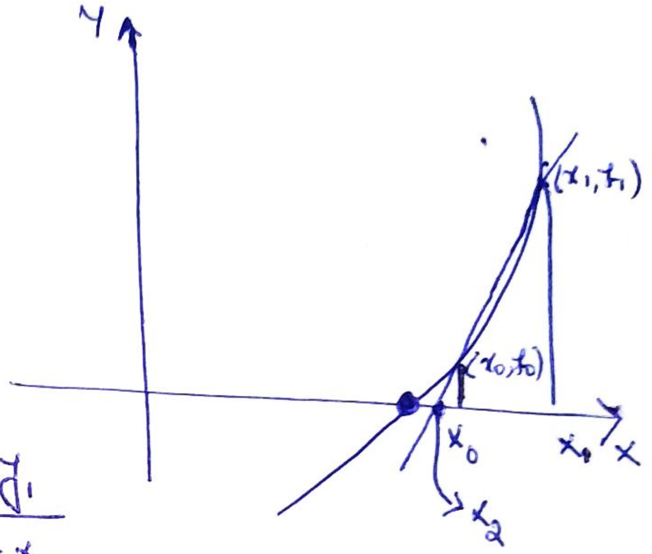
Cannot find the root where graph of f(x) only touches the x-axis.

iii) Secant Method

Step I → choose any two points

$(x_0, f(x_0))$ $(x_1, f(x_1))$

(x_0, f_0) (x_1, f_1)



Eq. of secant line

$y - y_1 = m(x - x_1)$ $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\boxed{y - f_0 = \frac{f_1 - f_0}{x_1 - x_0}(x - x_0)}$$

The point where secant line cuts the x-axis

$y = 0$ $x = x_2$

$$0 - f_0 = \frac{f_1 - f_0}{x_1 - x_0}(x_2 - x_0)$$

$$\boxed{x_2 = x_0 - \frac{x_1 - x_0}{f_1 - f_0} f_0}$$ and $f(x_2) = f_2 = \ldots$

Now ignore first point and take

(x_1, f_1) (x_2, f_2)

Eq. of secant line

$$y - y_1 = m(x - x_1)$$

$$\boxed{y - f_1 = \frac{f_2 - f_1}{x_2 - x_1}(x - x_1)}$$

The point where line cuts the x-axis

$$y = 0 \qquad x = x_3$$

$$0 - f_1 = \frac{f_2 - f_1}{x_2 - x_1}(x_3 - x_1)$$

$$\boxed{x_3 = x_1 - \frac{x_2 - x_1}{f_2 - f_1} f_1} \qquad f(x_3) = f_3 = \cdots$$

Now new points are

$$(x_2, f_2) \quad (x_3, f_3)$$

$$\boxed{x_4 = x_2 - \frac{x_3 - x_2}{f_3 - f_2} f_2}$$

$$\vdots$$

$$\boxed{x_{n+1} = x_{n-1} - \frac{x_n - x_{n-1}}{f_n - f_{n-1}} f_{n-1}}$$

Formula.

Example 1

Starting with $x_0=1$ and $x_1=2$ apply Regula-Falsi method on the function $f(x)=x^2+4x-10$ to obtain an approximate root correct to 4-D places.

Solution

$$f(x)=x^2+4x-10$$

$x_0 = 1$ $x_1 = 2$ $x_{n+1} = x_{n-1} - \dfrac{x_n - x_{n-1}}{f_n - f_{n-1}} f_{n-1}$

$f_0 = -5$ $f_1 = 2$

n	x_{n-1}	$f(x_{n-1})$	x_n	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	$x_0=1$	$f(x_0)=f_0=-5$	$x_1=2$	$f(x_1)=f_1=2$	$x_2=x_0-\dfrac{x_1-x_0}{f_1-f_0}f_0$ $x_2=1-\dfrac{(2-1)(-5)}{(2-(-5))}$ $=1.714286$	$f(x_2)=f_2$ $=-0.204080$
2	$x_1=2$	$f_1=2$	$x_2=1.714286$	$f_2=-0.204080$	$x_3=x_1-\dfrac{x_2-x_1}{f_2-f_1}f_1$ $=1.740741$	-0.006857
3	$x_2=1.714286$	$f_2=-0.204080$	$x_3=1.740741$	$f_3=-0.006857$	$x_4=1.741627$	-0.000227
4	$x_3=1.740741$	$f_3=-0.006857$	$x_4=1.741627$	$f_4=-0.000227$	$\boxed{x_5=1.741656}$	-0.000010

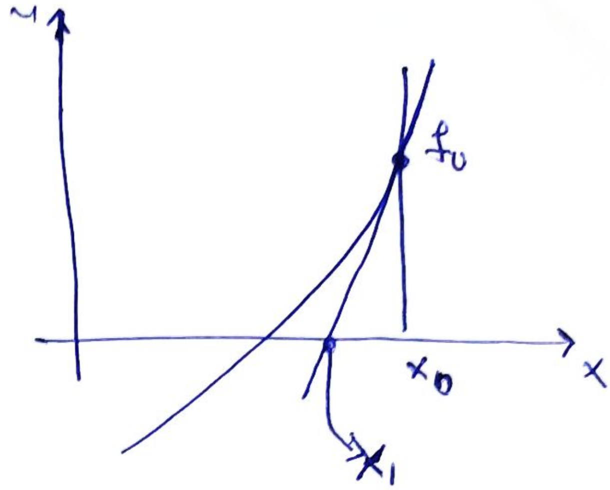
Ans.

iv) Newton's Method

Choose one point

$(x_0, f(x_0))$

(x_0, f_0).



Eq. of tangent line

$y - y_1 = m(x - x_1)$ $\quad m = f'(a)$.

$$\boxed{y - f_0 = f'(x_0)(x - x_0)}$$

The point where tangent line cuts the x-axis

$y = 0 \quad x = x_1$

$$\boxed{0 - f_0 = f'(x_0)(x_1 - x_0)}$$

$$\boxed{x_1 = x_0 - \frac{f_0}{f'(x_0)}} \qquad f(x_1) = f_1 = \text{-----}$$

Now new point (x_1, f_1)

$y - y_1 = m(x - x_1)$

$$\boxed{y - f_1 = f'(x_1)(x - x_1)}$$

The point where line cuts the x-axis

$y = 0 \qquad x = x_2$

$$0 - f_1 = f'(x_1)(x_2 - x_1)$$

$$\boxed{x_2 = x_1 - \frac{f_1}{f'(x_1)}} \longrightarrow f(x_2) = f_2 = \text{——}$$

New point (x_2, f_2)

$$\boxed{x_3 = x_2 - \frac{f_2}{f'(x_2)}}$$

$$\vdots$$

$$\boxed{x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}}$$ formula.

Example

Starting with $x_0 = 1$ approximate the root of the function $f(x) = x^2+4x-10$ correct to 6-D places.

Solution

$f(x) = x^2+4x-1$ $\quad x_0 = 1$

$f'(x) = 2x+4$ $\quad f(x_0) = f_0 = -5$

$$\boxed{x_n = x_{n-1} - \frac{f_{n-1}}{f'_{n-1}}}$$

n	x_{n-1}	$f(x_{n-1})$	$f'(x_{n-1})$	x_n	$f(x_n)$
1	$x_0 = 1$	$f(x_0) = f_0 = -5$	$f'(x_0) = f'(1) = 6$	$x_1 = x_0 - \frac{f_0}{f'_0} = 1 - \frac{(-5)}{6} = 1.8333333$	$f(x_1) = 0.6944442$
2	$x_1 = 1.8333333$	$f_1 = 0.6944442$	$f'(x_1) = 7.66666667$	$x_2 = 1.7427536$	0.0082045
3	$x_2 = 1.7427536$	$f_2 = 0.0082045$	$f'(x_2) = 7.4855072$	$x_3 = 1.7416575$	0.000000062...

Ans.

Examples

Secant and Newton's Methods

1) An astronaut walked on the moon's surface and hit a golf ball upward following a trajectory $f(x) = e^{-x}(-1.35\cos x - 1.85\sin x)$. Use Secant method with initial points $x_0 = 3$ and $x_1 = 4$ to estimate the critical point of ball's trajectory. Show sequence of iterations and poform 3 iterations.

Solution Given $f(x) = e^{-x}(-1.35\cos x - 1.85\sin x)$

For critical point

$$f'(x) = 0$$

$$\Rightarrow \quad f'(x) = g(x) = e^{-x}(-0.5\cos x + 3.2\sin x)$$

We have to find the root of $g(x)$.

Formula

$$x_{n+1} = x_{n-1} - \frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})}\, g(x_{n-1})$$

n	x_{n-1}	x_n	$g(x_{n-1})$	$g(x_n)$	x_{n+1}
1	$x_0 = 3$	$x_1 = 4$	$g(x_0) = g(3)$ 0.04713	$g(x_1) = g(4)$ -0.038370	$x_2 = 3 - \frac{(4-3)}{(-0.038370 - 0.04713)}(0.04713)$ $= 3.55121$
2	$x_1 = 4$	$x_2 = 3.55121$	$g(x_1) = -0.038370$	$g(x_2) =$ -0.02340	$x_3 = 2.84933$
3	$x_2 = 3.55121$	$x_3 = 2.84933$	—	—	$x_4 = 3.393920$

Ans.

(2) $f_1(x) = e^{-x}(1.35\cos x)$ $\qquad f_2(x) = x^2 - 2x + 1$

Point of intersection $f_1(x) = f_2(x)$

$$1.35\, e^{-x} \cos x = x^2 - 2x + 1$$

$$\Rightarrow \underbrace{1.35\, e^{-x} \cos x - x^2 + 2x - 1}_{\hookrightarrow f(x)} = 0$$

$$x_{n+1} = x_n - \left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right] f(x_{n-1})$$

Let $x_0 = 1$; $x_1 = 2$

n	x_{n-1}	x_n	$f(x_{n-1})$	$f(x_n)$	x_{n+1}	$f(x_{n+1})$
1	$x_0 = 1$	$x_1 = 2$	$f(x_0) = f(1)$ 0.268334	$f(x_1) = f(2)$ -1.07603	$x_2 = 1.1996$	0.10771
2	$x_1 = 2$	$x_2 = 1.1996$	-1.07603	0.10771	$x_3 = 1.27243$	0.036958
3	1.1996	1.27243	0.10771	0.036958	$x_4 = 1.310474$	-0.002681

Point $(1.310474, -0.002681)$.

(3) Why Newton's method is not applicable to approximate the root of $f(x) = 2x - 2\sin x + 3$ with $x_0 = 0$

Solution Newton's Method $\quad x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$f'(x) = 2 - 2\cos x$$

$$f'(x_0) = f(0) = 2 - 2\cos(0) = 2 - 2 = 0$$

The requirement for Newton's method $f'(x_n) \neq 0$.

(4) $f_1(t) = 16t^{-3} + 45$ $\qquad f_2(t) = 300t$

Point of intersection $\quad f_1(t) = f_2(t)$

$$16t^{-3} + 45 = 300t$$

$$16t^{-3} + 45 - 300t = 0$$

$$\boxed{f(t) = 16t^{-3} + 45 - 300t} \qquad \boxed{f'(t) = -48t^{-4} - 300}$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})} \quad \text{or}; \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

n	t_{n-1}	$f(t_{n-1})$	$f'(t_{n-1})$	t_n	$f(t_n)$
1	$t_0 = 1$	$f(t_0) = f(1)$ -239	$f'(t_0) = f'(1)$ -348	$1 - f(1)/f'(1)$ $= 1 - (-239)/348$ $= 0.313218$	471.725
2	$t_1 = 0.313218$	471.725	-5287.172485	0.402439	169.750
3	$t_2 = 0.402439$	169.750	-2129.96	0.482136	43.120

Point $(0.482136, 43.120)$

(5)

$$f(x) = x \log_{10} x - 1.2 \qquad x_0 = 3$$

find root correct to 4-D places.

Formula $$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$f'(x) = \log_{10}(x) + \frac{\ln(x)}{\log_{10}(x)} = \frac{1+\ln x}{\log_{10}(x)}$$

n	x_{n-1}	f_{n-1}	f'_{n-1}	x_n	$f_n = f(x_n)$
1	$x_0 = 3$	0.231364	0.911416	2.74615	0.00480261
2	$x_1 = 2.7465$	0.00480	0.873019	2.74065	3.40689×10^{-6}

<u>Assignment # 3</u>

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Q. 1 Approximate solution of $x = (13)^{\frac{1}{3}}$ to three decimal places by applying Secant Method.

Q. 2 Consider a simple electronic circuit with an input voltage, a resistor and a diode. It can be shown that the voltage across the diode can be found as the single positive zero of

$$\boldsymbol{x = 3^{-x}}.$$

Use the Newton-Raphson method to approximate the zero by taking the initial value $\boldsymbol{x_0 = 0.5}$. Perform three iterations using 8 decimal arithmetic with rounding.

Q. 3 In 1971 astronaut Alan Shepard walked on the moon's surface. He hit a golf ball, which was launched upward and followed the trajectory $\boldsymbol{f(x) = \sqrt[3]{x^2}\,(2x - 1)}$. The trajectory of the ball differed from what it would have been on earth because the acceleration due to gravity on the moon is about six times smaller than that on earth. Use Secant Method with initial point $\boldsymbol{x_0 = 1}$ to estimate the critical point of the ball's trajectory. Show the sequence of iterations and perform 5 iterations

Q. 4 Consider a simple electronic circuit with an input voltage of 2.0 V, a resister of resistance 1000 Ω and a diode. It can be shown that the voltage across the diode can be found as the single positive zero of

$$\boldsymbol{f(x) = 1 \times 10^{-14} e^{\left(\frac{x}{0.026}\right)} - \frac{2 - x}{1000}}.$$

Apply Newton's method to compute the root of $\boldsymbol{f(x)}$, taking the initial value of $\boldsymbol{x_0 = 0.75}$.

Do three iterations using 8 decimal arithmetic.

Q. 5 Consider a simple electronic circuit with an input voltage, a resistor and a diode. It can be shown that the voltage across the diode can be found as the single positive zero of

$$\boldsymbol{f(x) = 2^x - 3}.$$

Use the Newton-Raphson method to approximate the zero of $\boldsymbol{f(x)}$, taking the initial value of $\boldsymbol{x_0 = 1.0}$. Do three iterations using 8 decimal arithmetic.