Lecture/Week 8

Numerical Integration

Introduction

$$\int f(x)\,dx$$

Example $f(x) = x^2$

$$\int f(x)\,dx = \int x^2\,dx = \frac{x^3}{3} + C$$

$$f(x) = \sin x$$

$$\int f(x)\,dx = \int \sin x\,dx = -\cos x + C$$

* Integration of a given function is another function

** Numerically it is not possible.

When we Integrate a function and answer is a number?

→ In the case of definite Integral
Integration with a limit

$$\int_a^b f(x)\,dx$$

Example $f(x) = x^2$

$a = 0 \quad b = 2$

$$\int_a^b f(x)\,dx = \int_0^2 x^2\,dx = \left|\frac{x^3}{3}\right|_0^2 = \frac{(2)^3}{3} - 0$$

$$= \frac{8}{3} \text{ Ans.}$$

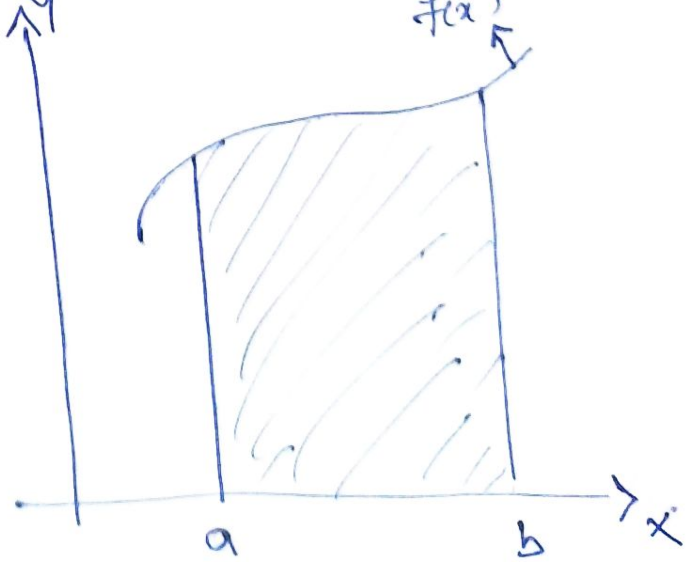
Numerical Integration is possible for definite integrals.

Methods

1) Rectangular Rule
2) Trapezoidal Rule
3) Simpson's Rule.

$\int_a^b f(x)\,dx$ = Area under the curve $f(x)$ from point 'a' to 'b'

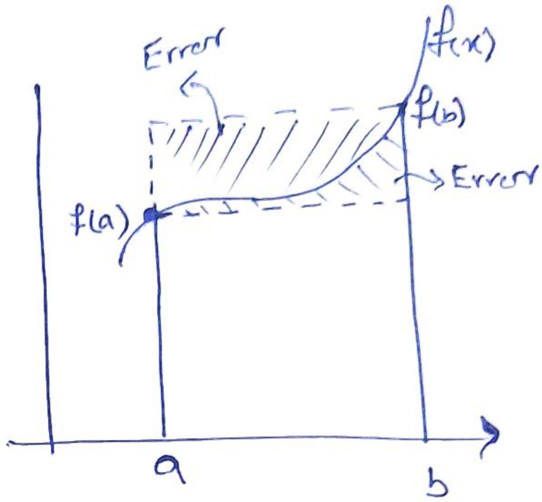
[~~shaded~~ Shaded Area]



Task is to approximate the area.

$$\int_a^b f(x)\,dx \approx \text{Area under the curve}$$

Rectangular Rule



$$\int_a^b f(x)\,dx \approx \text{Area of a rectangle}$$

$$= (b-a) * f(a)$$

or

$$= (b-a) * f(b)$$

Example $\int_1^2 e^{-x^2/2}\,dx$

Solution $f(x) = e^{-x^2/2}$ $\quad a=1 \quad b=2$

$$\int_1^2 e^{-x^2/2}\,dx = (b-a) * f(a) = (2-1)\,e^{-1/2} = 0.606531$$

$$\int_1^2 e^{-x^2/2}\,dx = (b-a) * f(b) = (2-1)\,e^{-2} = 0.135335$$

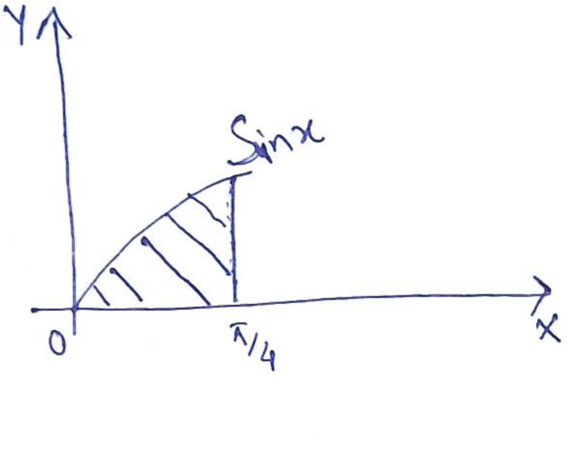
Example (2)

$$\int_0^{\pi/4} \sin x \, dx$$

$f(x) = \sin x \qquad a = 0 \qquad b = \pi/4$

$$\int_0^{\pi/4} \sin x \, dx = (b-a) \times f(a) = (\pi/4 - 0) * \sin(0) = 0$$

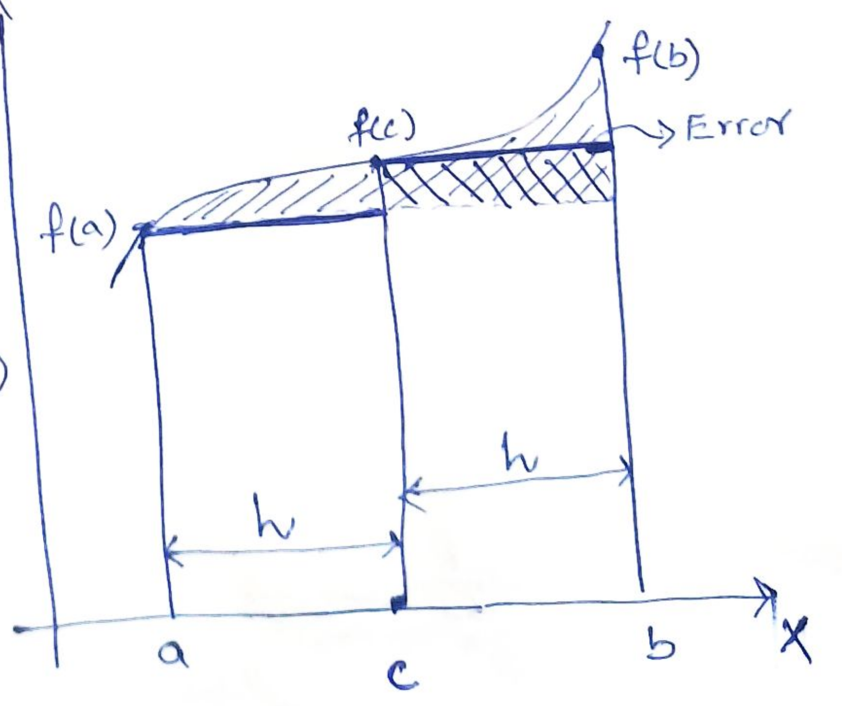
$$\int_0^{\pi/4} \sin x \, dx = (b-a) \times f(b) = (\pi/4 - 0) \sin \pi/4 = \frac{\pi}{4} \cdot \frac{1}{\sqrt{2}} = \ldots$$



$$\int_a^b f(x)dx \approx \text{Area of a rectangles}$$

$= h \times f(a) + h \times f(c)$

$= h[f(a) + f(c)]$



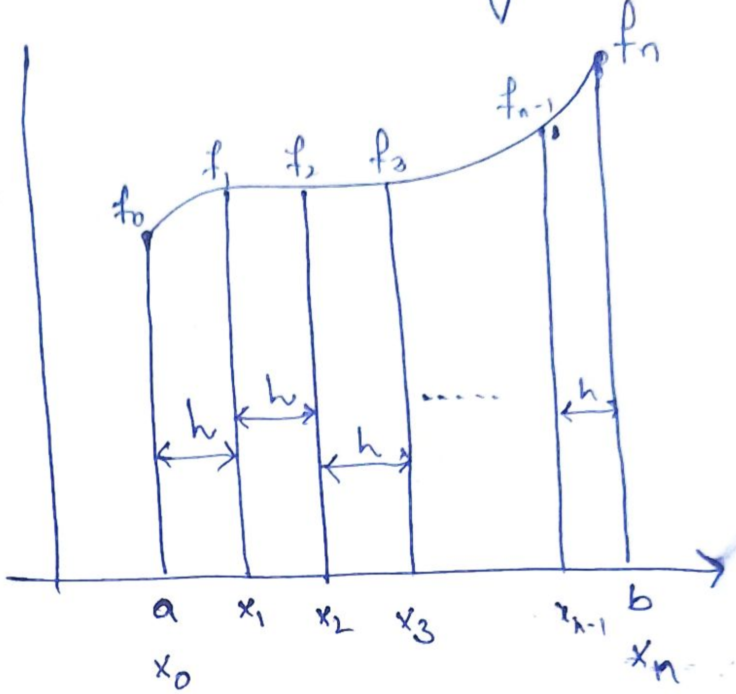
Dividing area into 'n' number of rectangles.

$$\int_a^b f(x)dx = h\times f_0 + h\times f_1 + h\times f_2 + \cdots h\times f_{n-1}$$

$$= h\left[f_0 + f_1 + f_2 \cdots f_{n-1}\right]$$

General formular

or



where $f(x_0) = f_0$; $f(x_1) = f_1$
$f(x_2) = f_2$ ..... $f(x_n) = f_n$

$$h = \frac{b-a}{n}$$
$$= \frac{x_n - x_0}{n}$$

$$\boxed{\int_a^b f(x)dx = h\left[f_1 + f_2 \cdots f_n\right]}$$

$$\boxed{\int_a^b f(x)dx = h\left[f_0 + f_1 + f_2 \cdots f_{n-1}\right]}$$

Example Approximate $\int_0^2 \cosh x \, dx$ by using 4 subintervals in the rectangular rule.

Solution $f(x) = \cosh x \quad x_0 = 0 \quad x_n = 2 \quad n = 4$

$$h = \frac{x_n - x_0}{n} = \frac{2-0}{4} = 0.5$$

x	$f(x)$
$x_0 = 0$	$f_0 = f(x_0) = \cosh(0) = 1$
$x_1 = 0.5$	$f_1 = f(x_1) = \cosh(0.5) = 1.127626$
$x_2 = 1.0$	$f_2 = f(x_2) = \cosh(1.0) = 1.543081$
$x_3 = 1.5$	$f_3 = f(x_3) = \cosh(1.5) = 2.352410$
$x_4 = 2.0$	$f_4 = f(x_4) = \cosh(2) = 3.762196$

$$\int_0^2 \cosh x \, dx = h\left[f_0 + f_1 + f_2 + f_3\right]$$

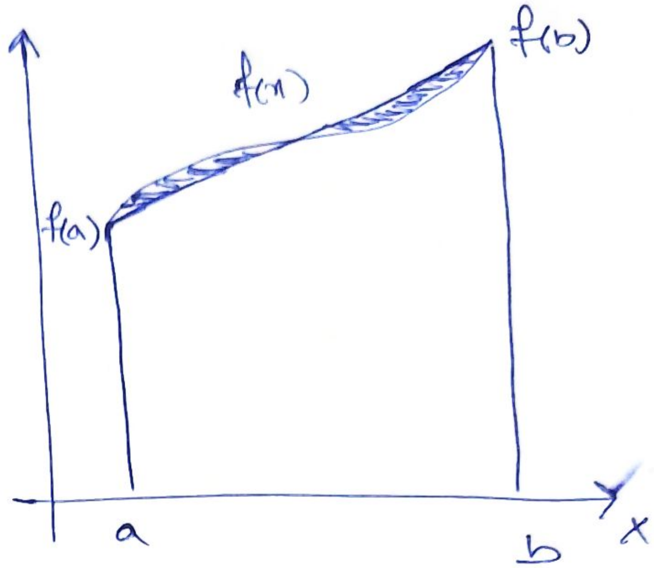
$$= 0.5\left[1 + 1.127626 + 1.543081 + 2.352410\right]$$

$$= 3.01156$$

Ans.

Trapezoidal Rule

$$\int_a^b f(x)\,dx \simeq \text{Area of a trapezium}$$



$$\int_a^b f(x)\,dx \simeq \frac{1}{2}(b-a)\left[f(a)+f(b)\right]$$

Example $\displaystyle\int_1^2 e^{-x^2/2}\,dx$ $\quad f(x) = e^{-x^2/2} \quad a=1 \quad b=2$

$$\int_1^2 e^{-x^2/2}\,dx = \frac{1}{2}(b-a)\left[f(a)+f(b)\right] = \frac{1}{2}(2-1)\left[e^{-1/2} - e^{-2}\right] = 0.37093$$

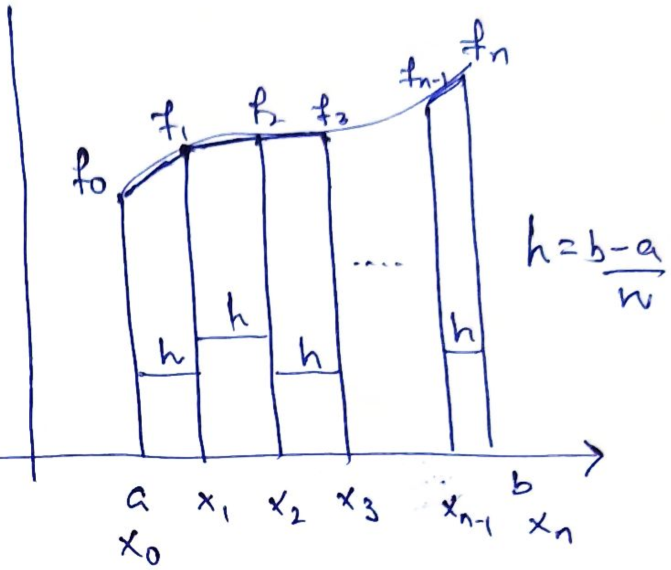
Ans.

$$\int_0^2 \cosh x\,dx = \frac{1}{2}(b-a)\left[f(a)+f(b)\right] = \frac{1}{2}(2-0)\left[\cosh(0) - \cosh(2)\right]$$

$$= 4.76220$$

Divide the area into 'n' numbers of trapeziums.

$$\int_a^b f(x)\,dx = \text{Area of } 1^{st} \text{ trapezium} + \text{Area of } 2^{nd} \text{ trapezium} + \text{Area of } 3^{rd} \text{ trapezium} + \cdots + \text{Area of } n^{th} \text{ trapezium}$$



$h = \frac{b-a}{n}$

$$\int_a^b f(x) = \frac{1}{2}[h][f_0 + f_1] + \frac{1}{2}h[f_1 + f_2] + \frac{1}{2}h[f_2 + f_3] + \cdots + \frac{1}{2}h[f_{n-1} + f_n]$$

$$= \frac{h}{2}\left[f_0 + f_n + 2(f_1 + f_2 + f_3 \cdots f_{n-1})\right]$$

Formula.

Example

Approximate $\int_0^2 \cosh x \, dx$ using 4 subintervals in the trapezoidal rule.

Solution $f(x) = \cosh x \quad x_0 = 0 \quad x_n = 2 \quad n = 4$

$$h = \frac{b-a}{n} = \frac{x_n - x_0}{n} = \frac{2-0}{4} = 0.5$$

X	$f(x)$
$x_0 = 0$	$f_0 = 1$
$x_1 = 0.5$	$f_1 = 1.127626$
$x_2 = 1.0$	$f_2 = 1.543081$
$x_3 = 1.5$	$f_3 = 2.352410$
$x_4 = 2.0$	$f_4 = 3.762196$

$$\int_a^b f(x) = \int_0^2 \cosh(x)\, dx$$

$$= \frac{h}{2}\left[f_0 + f_4 + 2(f_1 + f_2 + f_3)\right]$$

$$= \frac{0.5}{2}\left[1 + 3.76216 + 2(1.127626 + 1.543081 + 2.352410)\right]$$

$$= 3.452107$$

Ans.

Example

$$\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx \qquad h = \frac{1}{3}$$

Solution

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad a = x_0 = 0 \qquad h = \frac{1}{3}$$

$$b = x_n = 1$$

$$h = \frac{b-a}{n} = \frac{x_n - x_0}{n}$$

$$n = \frac{x_n - x_0}{h} = \frac{1-0}{1/3} = 3$$

X	$f(x)$
$x_0 = 0$	$f_0 = 0.39894$
$x_1 = 1/3$	$f_1 = 0.37738$
$x_2 = 2/3$	$f_2 = 0.31945$
$x_3 = 1$	$f_3 = 0.24197$

$$\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = \frac{h}{2}\left[f_0 + f_3 + 2(f_1 + f_2)\right]$$

$$= \frac{1/3}{2}\left[0.39894 + 0.24197 + 2(0.37738 + 0.31945)\right]$$

$$= 0.33910$$

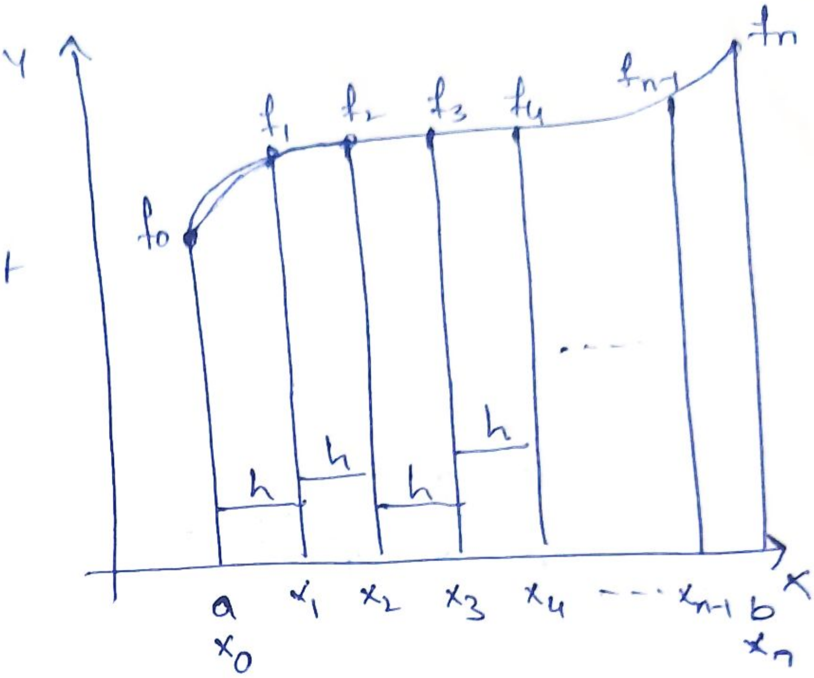
Ans.

Simpson's Rule

Condition

No. of subintervals must be even

n = even number



$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + \int_{x_4}^{x_6} f(x)dx + \cdots$$

$$+ \cdots \int_{x_{n-2}}^{x_n} f(x)dx \quad (1)$$

$$\approx \int_{x_0}^{x_2} P_2^1(x)dx + \int_{x_2}^{x_4} P_2^2(x)dx + \int_{x_4}^{x_6} P_2^3(x)dx + \cdots \int_{x_{n-2}}^{x_n} P_2^{n-2}(x)dx \quad (2)$$

where $P_2(x)$ are Lagrange polynomials

Take

$$\int_{x_0}^{x_2} P_2(x)\,dx$$

$$\int_{x_0}^{x_2} \left[L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2 \right] dx$$

$$\int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2 \right] dx \quad (3)$$

Integration by substitution

Let $\boxed{x - x_1 = sh}$

$\boxed{dx = h\, ds}$

$x_0 \overset{h}{\longleftrightarrow} x_1 \overset{h}{\longleftrightarrow} x_2$

when $x \to x_0$ $\quad \boxed{s \to -1}$

$x \to x_2$ $\quad \boxed{s \to 1}$

$$\begin{aligned} x - x_2 &= x - (x_1 + h) \\ &= x - x_1 - h \\ &= sh - h \end{aligned}$$

$$\boxed{x - x_2 = h(s-1)}$$

$$\begin{aligned} x - x_0 &= x - (x_1 - h) \\ &= x - x_1 + h \\ &= sh + h \end{aligned}$$

$$\boxed{x - x_0 = h(s+1)}$$

use all in (3)

$$\int_{-1}^{1} \frac{(sh)\, h(s-1)}{(-h)(-2h)} f_0\, h\, ds + \int_{-1}^{1} \frac{h(s+1)\, h(s-1)}{(h)(-h)} f_1\, h\, ds + \int_{-1}^{1} \frac{h(s+1)\, sh}{(2h)(h)} f_2\, h\, ds$$

$$= \frac{h f_0}{2} \int_{-1}^{1} s(s-1)\, ds - h f_1 \int_{-1}^{1} (s+1)(s-1)\, ds + \frac{h f_2}{2} \int_{-1}^{1} (s+1) s\, ds$$

$$\int_{x_0}^{x_2} P_2^1(x)\,dx = \frac{h}{3}\left[f_0 + 4f_1 + f_2\right]$$

Now

$$\int_{x_2}^{x_4} P_2^2(x)\,dx = \frac{h}{3}\left[f_2 + 4f_3 + f_4\right]$$

$$\int_{x_4}^{x_6} P_2^3(x)\,dx = \frac{h}{3}\left[f_4 + 4f_5 + f_6\right]$$

$$\vdots \qquad\qquad \vdots$$

$$\int_{x_{n-1}}^{x_n} P_2(x)\,dx = \frac{h}{3}\left[f_{n-2} + 4f_{n-1} + f_n\right]$$

Therefore from (1)

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f_0 + f_n + 4\left(f_1 + f_3 + f_5 \cdots f_{n-1}\right) + 2\left(f_2 + f_4 + f_6 + \cdots f_{n-2}\right)\right].$$

formula.

Example Approximate $\int_0^2 \cosh x \, dx$ using 4 Subintervals in the Simpson's rule.

Solution $f(x) = \cosh x \quad a=0 \quad b=2 \quad n=4$

$h = \frac{b-a}{n} = 0.5$

x	f
$x_0 = 0$	$f_0 = 1$
$x_1 = 0.5$	$f_1 = 1.127626$
$x_2 = 1.0$	$f_2 = 1.543081$
$x_3 = 1.5$	$f_3 = 2.352410$
$x_4 = 2.0$	$f_4 = 3.762196$

$$\int_0^2 \cosh x \, dx = \frac{h}{3}\left[f_0 + f_4 + 4(f_1 + f_3) + 2(f_2)\right]$$

$$= \frac{0.5}{3}\left[1 + 3.762196 + 4(1.127626 + 2.352410) + 2(1.543081)\right]$$

$$= \underline{\qquad\qquad}$$

Ans.

Example

$$\int_1^2 \frac{1}{x}\,dx \qquad n = 10$$

Solution

$f(x) = \frac{1}{x}$ $\quad a = 1 \quad b = 2 \quad n = 10$

$h = 0.1$

x	$f(x)$
$x_0 = 1$	$f_0 = 1$
$x_1 = 1.1$	$f_1 = 0.9091$
$x_2 = 1.2$	$f_2 = 0.8333$
$x_3 = 1.3$	$f_3 = 0.7692$
$x_4 = 1.4$	$f_4 = 0.7143$
$x_5 = 1.5$	$f_5 = 0.6667$
$x_6 = 1.6$	$f_6 = 0.6250$
$x_7 = 1.7$	$f_7 = 0.5882$
$x_8 = 1.8$	$f_8 = 0.5556$
$x_9 = 1.9$	$f_9 = 0.5263$
$x_{10} = 2.0$	$f_{10} = 0.5$

$$\int_1^2 f(x)\,dx = \frac{h}{3}\left[f_0 + f_{10} + 4(f_1 + f_3 + f_5 + f_7 + f_9) + 2(f_2 + f_4 + f_6 + f_8)\right]$$

$$= \frac{0.1}{3}\left[1 + 0.5 + 4(0.9091 + 0.7692 + 0.6667 + 0.5882 + 0.5263) + 2(0.8333 + 0.7143 + 0.6250 + 0.5556)\right]$$

$$= 0.693147$$

Ans.

Example

The solid of revolution obtained by rotating the region under the curve $y = f(x)$ $a \le x \le b$ about the x-axis. The surface area of the region is given by the formula

$$\text{Area} = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)}\, dx$$

Find the area using $f(x) = x \sin x$, $1 \le x \le 2$ in the Simpson's rule with six sub intervals.

Solution $a = 1$; $b = 2$ $n = 6$ $h = \frac{b-a}{n} = \frac{2-1}{6} = 1/6$

$$\text{Area} = \pi \int_1^2 (x \sin x) \sqrt{1 + \sin x + x \cos x}\, dx \qquad f'(x) = \sin x + x \cos x$$

$$\text{Function} = F(x) = x \sin x \sqrt{1 + \sin x + x \cos x}$$

x	$F(x)$
$x_0 \leftarrow 1$	$1.298642 \to f_0$
$x_1 \leftarrow 7/6$	$1.654233 \to f_1$
$x_2 \leftarrow 4/3$	$1.959189 \to f_2$
$x_3 \leftarrow 3/2$	$2.170119 \to f_3$
$x_4 \leftarrow 5/3$	$2.247867 \to f_4$
$x_5 \leftarrow 11/6$	$2.161134 \to f_5$
$x_6 \leftarrow 2$	$1.867316 \to f_6$

$$\text{Area} = \pi \left[\frac{h}{3} \left[f_0 + f_6 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) \right] \right]$$

$$= \pi \left[1.974556 \right]$$

$$= 6.203252$$

If same question with trapezoidal rule

$$\text{Area} = \pi \left[\frac{h}{2} \left[f_0 + f_6 + 2(f_1 + f_2 + f_3 + f_4 + f_5) \right] \right]$$

$=$ ———

Date: # Assignment # 7

Degree / Syndicate: ______________________ NAME: __________________ REGISTRATION No:

NOTE: Submission is required for questions 1 and 2

Q. 1 Estimate the integral $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$ using six sub-intervals in the Trapezoidal rule

Q. 2 The solid of revolution obtained by rotating the region under the curve $y = f(x),\ a \le x \le b,$ about the x-axis has the surface area given by $Area = a\pi \int_a^b f(x)\sqrt{1 + f'(x)}dx$. Find the area using

$f(x) = x\,Cosx \quad 1 \le x \le 2.$

a) Use the Rectangular rule with six subintervals.

b) Use Trapezoidal rule with six subintervals.

c) Use Simpson's rule with six subintervals.

Q. 3 Perform the Integration for the following in order to find area under the curve over an interval [0, 10] with step size of 1.25.

$$\int_0^{10} \frac{300\,x}{1 + e^x} dx$$

a) Use Rectangular rule.

b) Use Trapezoidal rule.

c) Use Simpson's rule.