MATH-351 Numerical Methods 14/12/2020 ①

Lecture/Week-7

Numerical Differentiation

Our focus will be on
→ 1st Order Derivative.
2nd " " .

First Derivative

Note :- Derivative of a function $f(x)$ is another function $f'(x)$.

Example $f(x) = x^3 + 4x^2 - 1$

$$\boxed{f'(x) = 3x^2 + 8x}$$

or $f(x) = \sin x + x^2$

$f'(x) = \cos x + 2x$

Can we approximate/find a function numerically?

Numerically we can only approximate a number ~~number~~.

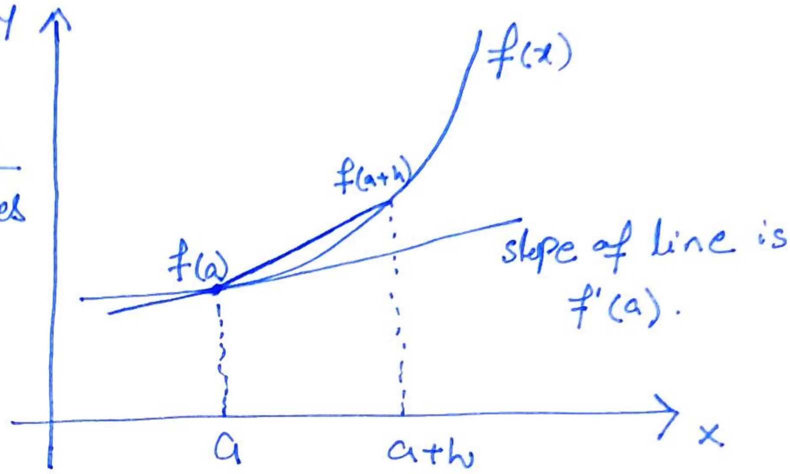
This means we can find $f'(a)$
Derivative at a point.

First derivative

The aim is to approximate the slope of a curve '$f(x)$' at a particular point '$x=a$'.

$$m = f'(a) \approx \frac{\text{difference in y-values}}{\text{difference in x-values}}$$

slope



$$f'(a) \approx \frac{f(a+h) - f(a)}{\cancel{a}+h - \cancel{a}}$$

$$\boxed{f'(a) \approx \frac{f(a+h) - f(a)}{h}} \rightsquigarrow 1$$

Exact formula.

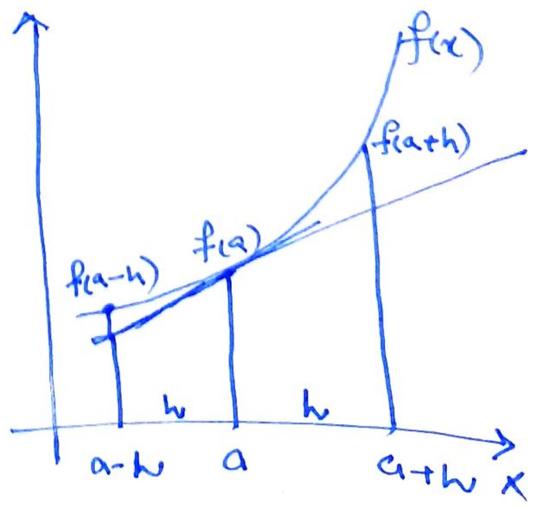
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

↳ Forward difference approximation to the derivative of f at $x=a$.

$$m = f'(a) \approx \frac{f(a) - f(a-h)}{\cancel{a} - \cancel{a} + h}$$

$$\boxed{f'(a) \approx \frac{f(a) - f(a-h)}{h}} \rightsquigarrow 2$$

↳ Backward difference approximation of $f'(a)$



Taking average of 1 and 2

$$f'(a) \approx \frac{\dfrac{f(a+h)-f(a)}{h} + \dfrac{f(a)-f(a-h)}{h}}{2}$$

$$= \frac{f(a+h) - \cancel{f(a)} + \cancel{f(a)} - f(a-h)}{2h}$$

$$\boxed{f'(a) = \frac{f(a+h) - f(a-h)}{2h}} \rightarrow 3$$

↳ Central difference approximation of $f'(a)$.

Example (1)

Approximate derivative of $\cos x$ at $x = \pi/3$

a) $h = 0.1$ b) $h = 0.01$ c) $h = 0.001$ d) $h = 0.0001$

Use forward difference method.

Solution

$$f(x) = \cos x \qquad a = \pi/3$$

a) $h = 0.1$

$$f'(a) = \frac{f(a+h) - f(a)}{h} = \frac{\cos(a+h) - \cos(a)}{h}$$

$$= \frac{\cos(\pi/3 + 0.1) - \cos(\pi/3)}{0.1} = \frac{0.41104381 - 0.5}{0.1} = -0.88956192$$

b) $h = 0.01$

$$f'(a) = \frac{f(a+h) - f(a)}{h} = \frac{\cos(a+h) - \cos a}{h}$$

$$= \frac{\cos(\pi/3 + 0.01) - \cos(\pi/3)}{0.01} = \frac{0.49131489 - 0.5}{0.01} = -0.86851095$$

c) $h = 0.001$

$$\frac{\cos(\pi/3 + 0.001) - \cos(\pi/3)}{0.001} = \frac{0.49913372 - 0.5}{0.001} = -0.86627526$$

d) $h = 0.0001$

$$= -0.86605040$$

Exact Solution

$$f'(x) = -\sin x$$

$$f'(\pi/3) = -\sin(\pi/3) = -0.86602540$$

Smaller the value of 'h' better will be our approximations.

Example ② Approximate derivative of $f(x) = \cos x$ at $x = \pi/3$ using central difference formula for

a) $h = 0.1$ b) $h = 0.01$ c) $h = 0.001$ d) $h = 0.0001$

Solution

$$f(x) = \cos x \qquad a = \pi/3$$

a) $h = 0.1$

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} = \frac{\cos(a+h) - \cos(a-h)}{2h}$$

$$= \frac{\cos(\pi/3 + 0.1) - \cos(\pi/3 - 0.1)}{2(0.1)} = \frac{0.41104381 - 0.58396036}{2(0.1)}$$

$$= \underline{\underline{-0.86458275}}$$

b) $h = 0.01$

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h} = \frac{\cos(\pi/3 + 0.01) - \cos(\pi/3 - 0.01)}{2(0.01)}$$

$$= \frac{0.49131489 - 0.50863511}{0.02} = -0.86601097$$

c) $h = 0.001$ $\qquad = -0.86602526$

d) $h = 0.0001$ $\qquad = -0.86602540$

Central Difference Method give better approximation when compared with forward or backward difference method.

Second Derivative

$$f''(a) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2}$$

Central difference formula for 2^{nd} derivative

Example

The distance 's' of runner from a fixed point is measured at intervals of half a second.

t	0	0.5	1.0	1.5	2.0
s	0	3.65	6.80	9.90	12.15

Obtain runner's velocity at $t = 0.5$ sec, $t = 1.25$ sec, $t = 0.7$ sec $t = 2.0$ sec.

Solution $v(0.5) = s'(0.5) = ?$ $v(1.25) = s'(1.25) = ?$

$v(0.7) = s'(0.7) = ?$ $v(2.0) = s'(2.0) = ?$

$$v(0.5) = s'(0.5) = \frac{s(a+h) - s(a-h)}{2h}$$

$h = 0.5$
$a = 0.5$

$$= \frac{s(0.5+0.5) - s(0.5-0.5)}{2(0.5)} = \frac{s(1.0) - s(0.0)}{2(0.5)}$$

$$= \frac{6.80 - 0}{1} = 6.80 \text{ m/sec.}$$

$$v(1.25) = s'(1.25) = \frac{s(a+h) - s(a-h)}{2h} \qquad a = 1.25,\ h = 0.25$$

$$= \frac{s(1.25+0.25) - s(1.25-0.25)}{2(0.25)}$$

$$= \frac{s(1.5) - s(1.0)}{2(0.25)} = \frac{9.90 - 6.80}{2(0.25)}$$

$$= 6.20 \text{ m/sec.}$$

$v(0.7) = s'(0.7)$ Not possible with this method
Have to use Interpolation

→ $s(0.7)$ not given

→ h is not same.

$v(2.0) = s'(2.0)$ cannot use central and forward difference method.

Backward difference.

$$s'(2.0) = \frac{s(2.0) - s(2.0 - 0.5)}{0.5} \qquad h = 0.5$$

$$= \frac{12.15 - 9.90}{0.5} = \text{———}$$

Ans:

Approximate runner's acceleration at $t = 1.5$

$$a(1.5) = f''(1.5) = \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} \qquad a = 1.5,\; h = 0.5$$

$$= \frac{f(1.5+0.5) - 2f(1.5) + f(1.5-0.5)}{(0.5)^2}$$

$$= \frac{f(2.0) - 2f(1.5) - f(1.0)}{(0.5)^2} = \frac{12.15 - 2(9.90) + (6.80)}{(0.5)^2}$$

$$= -3.40 \text{ m/sec}^2$$

Ans.

Ex:- 2

Approximate the first derivative of $C(x)$ at $x=-1$, with $\boxed{h=1}$ using central difference method

$$C(x) = \begin{cases} \frac{2}{3} - \frac{1}{2}x^2(2-|x|)^3, & 0 \le |x| \le 1 \\ \frac{1}{6}(2-|x|)^3, & 1 \le |x| \le 2 \\ 0, & |x| > 2 \end{cases}$$

Solution

$$C(x) = \begin{cases} 0 & x < -2 \\ 1/6\,(2+x)^3 & -2 \le x \le -1 \\ 2/3 - \frac{x^2}{2}(2+x) & -1 \le x \le 0 \\ 2/3 - \frac{x^2}{2}(2-x) & 0 \le x \le 1 \\ 1/6\,(2-x)^3 & 1 \le x \le 2 \\ 0 & x > 2 \end{cases}$$

$h = -1$

$$C'(-1) = \frac{C(a+h) - C(a-h)}{2h} = \frac{C(-1+1) - C(-1-1)}{2(1)}$$

$$= \frac{C(0) - C(-2)}{2} = \frac{2/3 - 0}{2} = \frac{2}{6} = \frac{1}{3}$$

Ans.