MATH-351 — Numerical Methods — [redacted]

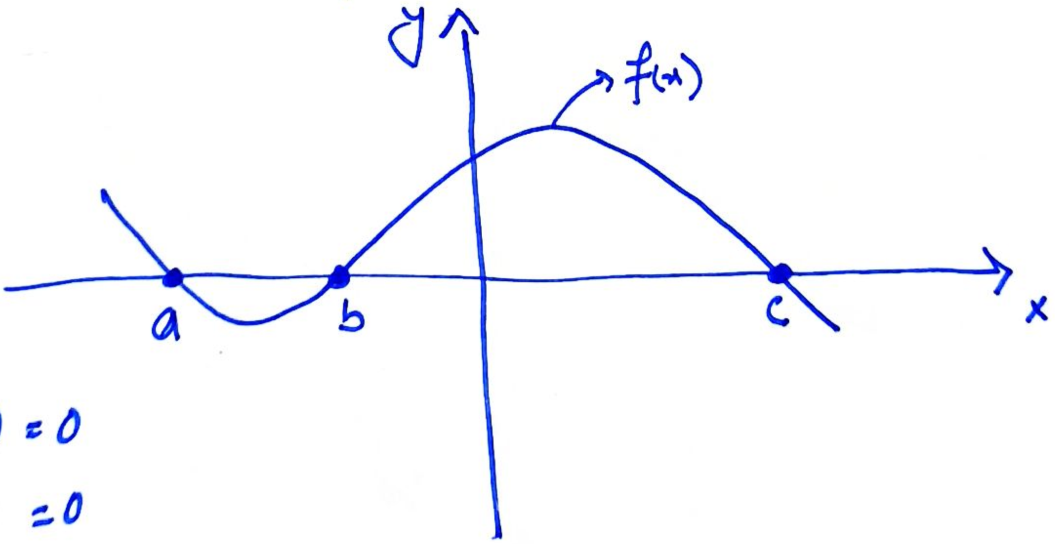
Lecture/Week [redacted] 2

Roots of non-linear functions

Zeros " " "

Solution of non-linear equations.

For a single variable function $f(x)$, a point $x=a$ is called a root or a zero of $f(x)$ if $f(a)=0$. That is, the value of function $f(x)$ is zero at $x=a$.



$f(a)=0$
$f(b)=0$
$f(c)=0$

Therefore a, b, c are roots or zeros of given function $f(x)$.

Example 1

$$f(x) = x^2 + 4x - 10$$

$$x^2 + 4x - 10 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 4(-10)}}{2(1)} = \frac{-4 \pm \sqrt{16+40}}{2}$$

$$x = \frac{-4 \pm \sqrt{56}}{2}$$

$x = 1.74166$

$x = -5.74166$ Ans.

Example 2

$$f(x) = \cos 2x$$

Example 3

$$T(x) = x - \frac{1+3x^2}{4+x^3}$$

Example 4

$$G(x) = x^{1/3} - x^2 - 8$$

Numerical Methods

1) Bisection Method
2) Regula Falsi Method (Method of False Position)
3) Secant Method
4) Newton's Method

Bisection Method

The bisection Method is based on the intermediate value theorem (Mean value theorem)

i) If $f(x)$ is continuous on an Interval $[a\ b]$

ii) $f(a)$ and $f(b)$ differ in sign, that is

$$f(a) * f(b) < 0 .$$

Then there exist a point 'c' in the interval $[a\ b]$ for which $f(x) = 0$

$$\text{or} \quad f(c) = 0$$

'c' is a root.

In bisection method, we assume

$$c = \frac{a+b}{2}$$

midpoint of the Interval $[a\ b]$.

Example

Approximate the positive root of $f(x) = x^2+4x-10$. Correct to 2-decimal places by using the Bisection Method.

Solution

$$f(x) = x^2+4x-10$$

Step I :- Finding Interval $[a \quad b]$

$f(0) = -10$ $\qquad$ $f(1) = -5$

$f(-1) = -13$ $\qquad$ $f(2) = 2$

$[0 \quad 2]$ $\qquad$ $\underbrace{[1 \quad 2]}_{\text{Interval.}}$ $\qquad$ $[-1 \quad 2]$

$a = 1$ $\qquad$ $b = 2$

Step II :-

n	a	$f(a)$	b	$f(b)$	$c = \frac{a+b}{2}$	$f(c)$
1	1	$\boxed{-5}$	2	$\boxed{2}$	$c = \frac{1+2}{2} = 1.5$	$\boxed{f(1.5) = -1.75}$
2	1.5	$\boxed{-1.75}$	2	$\boxed{2}$	$c = \frac{1.5+2}{2} = 1.75$	$\boxed{f(1.75) = 0.0625}$
3	1.5	-1.75	1.75	0.0625	$c = \frac{1.5+1.75}{2} = 1.625$	-0.859375
4	1.625	-0.859375	1.75	0.0625	$c = 1.6875$	-0.402344
5	1.6875	-0.402344	1.75	0.0625	$c = 1.71875$	-0.170898
6	1.71875	-0.170898	1.75	0.0625	$c = 1.734375$	-0.054443
7	1.734375	-0.054443	1.75	0.0625	$\boxed{c = 1.742188}$	0.003971

$c = 1.742188 \rightarrow$ Root.

Example 2

$$f(x) = e^{-x}\left(-1.35\cos x - 1.85\sin x\right)$$

Approximate the critical point of $f(x)$ using Bisection method with initial interval $[3 \quad 4]$. Perform three iterations.

Solution For critical point $f'(x) = 0$

$$f'(x) = e^{-x}\left[3.25\sin x - 0.5\cos x\right]$$

To approximate zero of $f'(x)$, let

$$\boxed{F(x) = e^{-x}\left[3.25\sin x - 0.5\cos x\right]}$$

Given $a = 3$ $\qquad$ $b = 4$

n	a	F(a)	b	F(b)	$c = \frac{a+b}{2}$	F(c)
1	3	$\boxed{0.047127}$	4	$\boxed{-0.038372}$	3.5	$\boxed{-0.0202871}$
2	3	0.047127	3.5	-0.0202871	3.25	0.00563896
3	3.25	0.00563896	3.5	-0.0202871	$\boxed{3.375}$	~~-0.0060~~ -0.0090768

Ans.

The Regula Falsi Method
Method of False Position

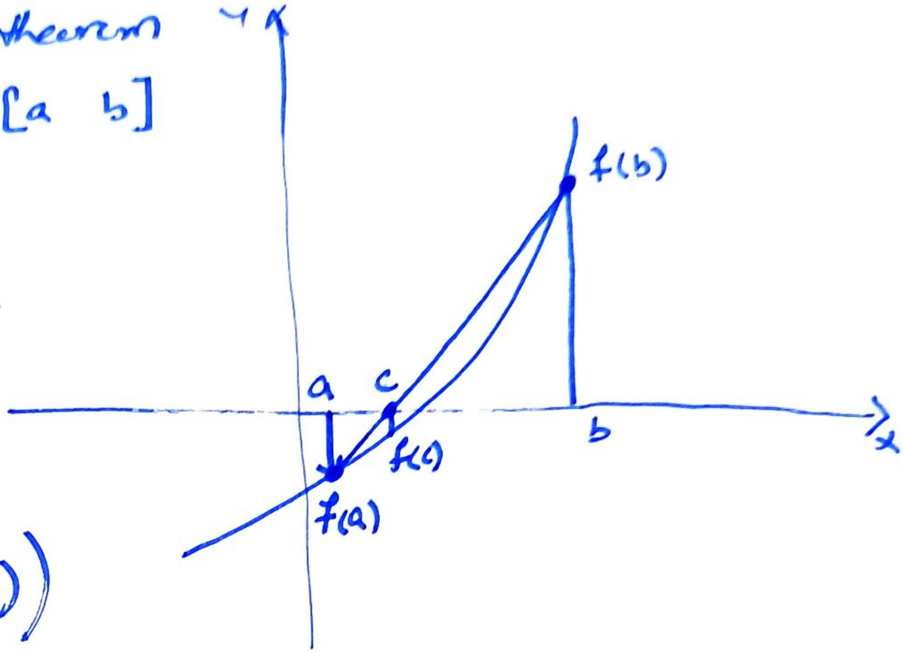
1) Based on mean value theorem

continuous on interval [a b]

$f(a) * f(b) < 0$

There exist 'c' such that

$f(c) = 0$



$(a, f(a))$ $(b, f(b))$

Equation of line

$y - y_1 = m(x - x_1)$ $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$\boxed{y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)}$$

The point where line cuts the x-axis

$y = 0$ and $x = c$

$$0 - f(a) = \frac{f(b) - f(a)}{b - a}(c - a)$$

$$\boxed{c = a - \frac{b - a}{f(b) - f(a)} f(a)}$$ Formula.

Example $f(x) = x^2 + 4x - 10$

Solution

$f(0) = -10 \quad f(1) = -5$

$f(-1) = -13 \quad f(2) = 2$

$[0 \quad 2] \quad \underbrace{[1 \quad 2]} \quad [-1 \quad 2]$

$a = 1 \quad b = 2 \quad c = a - \dfrac{b-a}{f(b)-f(a)} f(a)$

n	a	f(a)	b	f(b)	c	f(c)
1	1	-5	2	2	$c = 1 - \dfrac{(2-1)(-5)}{(2-(-5))}$ $= 1.714286$	$f(1.714286)$ $= -0.204080$
2	1.714286	-0.204080	2	2	1.740741	-0.006857
3	1.740741	-0.006857	2	2	1.741627	-0.000227
4	1.741627	-0.000227	2	2	$\boxed{1.741656}$ Root.	-0.000010

Examples Bisection Method.

1. A blood pathology analysis model of drug concentration "C(t)" in the blood stream is given by the formula

$$C(t) = t \tanh\left[\frac{t}{2}\right] - 1$$

a) Find the two points which bracket the zero of C(t)?

b) Find the interval in which the drug concentration is maximum?

Solution a) $C(t) = t \tanh(t/2) - 1$

$C(0) = -1$; $C(1) = -0.453698$; $C(-1) = -0.453698$

$C(2) = 2.11482$

$[-1 \quad 2]$ $\boxed{[1 \quad 2]}$ $[0 \quad 2]$

→ Ans.

b) For maximum $c'(t) = \tanh[t/2] + \frac{1}{2} t \operatorname{sech}^2[t/2]$

$c'(t) = f(t)$

$f(0) = 0$; $f(1) = 0.855341$; $f(-1) = -0.85534$

$\boxed{[-1 \quad 0]}$ $[-1 \quad 1]$

→ Ans.

(2) Find the point of intersection (perform 3 iterations).

$f_1(t) = 16t^{-3} + 45$ $\qquad$ $f_2(t) = 300t$

For point of intersection $f_1(t) = f_2(t)$

$16t^{-3} + 45 = 300t$

$\Rightarrow \boxed{f(t) = 16t^{-3} - 300t + 45}$

$f(0)$ = undefined ; $f(0.5) = 23$; $f(1) = -239$

Interval $[0.5 \quad 1]$

Bisection Method

n	a	b	f(a)	f(b)	$c = \frac{a+b}{2}$	f(c)
1	0.5	1	23	-239	$c = \frac{0.5+1}{2} = 0.75$	$f(0.75) = -142.074$
2	0.5	0.75	23	-142.074	c = 0.625	$f(0.625) = -76.964$
3	0.5	0.625	23	-76.964	c = 0.5625	$f(0.5625) = -33.8515$
4	0.5	0.5625	23	-33.8515	c = 0.53125	$f(0.53125) = -7.66057$
5	0.5	0.53125	23	-7.66057	c = 0.515625	$f(0.515625) = 7.02519$
6	0.515625	0.53125	7.02519	-7.66057	c = 0.523438	$f(0.523438) = -0.467358$
7	0.515625	0.523438	7.02519	-0.467358	c = 0.519532	$f(0.519532) = 3.23972$

False Position Method

n	a	b	f(a)	f(b)	$c = a - \frac{(b-a)}{f(b)-f(a)} f(a)$	f(c)
1	0.5	1	23	-239	c = 0.543893	-18.7238
2	0.5	0.543893	23	-18.7238	c = 0.524196	-1.17787
3	0.5	0.524196	23	-1.17787	c = 0.523017	-0.07152
4	~~0.523017~~ 0.5	~~0.524196~~ 0.523017	23	-0.07152	$\boxed{c = 0.522946}$	-0.00432699

Ans.

Point of Intersection
(0.522946 -0.00432699).

Assignment # 2

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NOTE: Submission is only required for question number 1, 2, and 5.

Q. 1 The single positive zero of the function $f(x) = x\tanh(\frac{x}{2}) - 1$ models the wavenumber of the water wave at a certain frequency in the water of depth 0.5 (don't worry about the units here).

a) Use the Bisection Method to approximate the zero of $f(x)$ to four decimal places

c) Wirte a Matlab code for the Bisection Method and approximate the zero of $f(x)$ to six decimal places.

Note: Attach hard copy of Matlab code and print result and error after each iteration

Q. 2 Approximate solution of $x = (13)^{\frac{1}{3}}$ to three decimal places by applying Bisection Method.

Q. 3 In 1971 astronaut Alan Shepard walked on the moon's surface. He hit a golf ball, which was launched upward and followed the trajectory $f(x) = \sqrt[3]{x^2}\,(2x - 1)$. The trajectory of the ball differed from what it would have been on earth because the acceleration due to gravity on the moon is about six times smaller than that on earth. Use False Position Method with initial point $a = 1$ to estimate the critical point of the ball's trajectory. Show the sequence of iterations and perform 3 iterations

Q.4 Can you use false position method to find a zero of $x^3 - 3x + 1 = 0$, in the interval $[0 \quad 2]$. Give mathematical justification of your answer.

Q.5 Use false position method with $x_0 = 0.5$ to find the point where temperature is maximum in the pressurized vessel

$$T(x) = x - P(x), \qquad where \qquad P(x) = \frac{1 + 3x^2}{4 + x^3}.$$

Show the sequence of iterations and perform five iterations.