Problem 5: Bayes Math

A. Prove that:

$$P(A|B,C) = \frac{P(B|A,C) \cdot P(A|C)}{P(B|C)} \tag{1}$$

As the definition of conditional probability:

$$P(A|B,C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} \tag{2}$$

By applying the chain rule, we can express the numerator as:

$$P(A \cap B \cap C) = P(B|A,C) \cdot P(A \cap C) \tag{3}$$

It is also useful to express $P(A \cap C)$ using the definition of conditional probability:

$$P(A \cap C) = P(A|C) \cdot P(C) \tag{4}$$

Therefore:

$$P(A \cap B \cap C) = P(B|A,C) \cdot P(A|C) \cdot P(C) \tag{5}$$

Similarly,:

$$P(B \cap C) = P(B|C) \cdot P(C) \tag{6}$$

Taking back into the original expression:

$$P(A|B,C) = \frac{P(B|A,C) \cdot P(A|C) \cdot P(C)}{P(B|C) \cdot P(C)} \tag{7}$$

Thus, we can cancel $P(C)$ from the numerator and denominator:

$$P(A|B,C) = \frac{P(B|A,C) \cdot P(A|C)}{P(B|C)} \tag{8}$$

B.

Setup:

- A denote the event that the coin is fair.
- B denote the event that the coin is double-headed.
- Prior probabilities: $P(A) = \frac{F}{F+1}$ and $P(B) = \frac{1}{F+1}$.
- For fair coin, $P(H|A) = \frac{1}{2}$.
- For double-headed coin, $P(H|B) = 1$.

Applying Bayes' Theorem:

We want to find $P(B|n \text{ heads})$, after observing n heads.

As per Bayes' theorem:

$$P(B|n \text{ heads}) = \frac{P(n \text{ heads}|B) \cdot P(B)}{P(n \text{ heads})} \tag{9}$$

Compute the likelihoods:

$$P(n \text{ heads}|B) = 1^n = 1 \tag{10}$$

$$P(n \text{ heads}|A) = \left(\frac{1}{2}\right)^n \tag{11}$$

Using the law of total probability:

$$P(n \text{ heads}) = P(n \text{ heads}|A) \cdot P(A) + P(n \text{ heads}|B) \cdot P(B) \tag{12}$$

$$= \left(\frac{1}{2}\right)^n \cdot \frac{F}{F+1} + 1 \cdot \frac{1}{F+1} \tag{13}$$

$$= \frac{1}{F+1}\left[\frac{F}{2^n} + 1\right] \tag{14}$$

Thus,

$$P(B|n \text{ heads}) = \frac{1 \cdot \frac{1}{F+1}}{\frac{1}{F+1}\left[\frac{F}{2^n} + 1\right]} \tag{15}$$

$$= \frac{1}{\frac{F}{2^n} + 1} \tag{16}$$

Find the smallest n such that $P(B|n \text{ heads}) > 0.5$:

We need to solve:

$$\frac{1}{\frac{F}{2^n} + 1} > 0.5 \tag{17}$$

Solving this inequality:

$$n > \log_2(F) \tag{18}$$

Since n must be an integer, we take the ceiling:

$$n \geq \lceil \log_2(F) \rceil \tag{19}$$