Problem 3

Let X be a binary random variable, which takes values $X \in \{0, 1\}$, and Y be a random variable representing the label. The mutual information between X and Y is given by:

$$I(X, Y) = H(Y) - H(Y|X)$$

Entropy of a Binary Variable X

For a binary variable X, the entropy $H(X)$ is given by the binary entropy function:

$$H(X) = -p \log_2 p - (1-p) \log_2 (1-p)$$

where $p = P(X = 0)$ and $1 - p = P(X = 1)$, with $0 \leq p \leq 1$.

The maximum value of this entropy occurs when $p = 0.5$, i.e., when the two outcomes are equally likely. In this case:

$$H(X) = -0.5 \log_2 0.5 - 0.5 \log_2 0.5 = 1$$

Thus, for a binary variable X, we have:

$$H(X) \leq 1$$

Information Gain from a Binary Split

Now consider the mutual information between X and Y.

Since $H(X) \leq 1$ for any binary variable X, it follows that:

$$I(X; Y) = H(X) - H(X|Y) \leq H(X) \leq 1$$

Thus, the maximum possible information gain (entropy reduction) from a split on X is at most 1 bit.