Problem 5

Assume the two sets are linearly separable and their convex hulls intersect.

Since A and B are linearly separable, there exists a hyperplane defined by a vector $\mathbf{w} \in \mathbb{R}^d$ and a scalar $b \in \mathbb{R}$ such that:

$$\mathbf{w}^\top \mathbf{x}_i + b > 0 \quad \text{for all } i = 1, \dots, n,$$
$$\mathbf{w}^\top \mathbf{y}_j + b < 0 \quad \text{for all } j = 1, \dots, m.$$

Then, suppose the convex hulls of A and B intersect. This means there exist points $\mathbf{x}^* \in \text{conv}(A)$ and $\mathbf{y}^* \in \text{conv}(B)$ such that:

$$\mathbf{z} = \sum_{i=1}^{n} \alpha_i \mathbf{x}_i = \sum_{j=1}^{m} \beta_j \mathbf{y}_j,$$

where $\alpha_i \geq 0$ and $\beta_j \geq 0$ are the coefficients of the convex combinations, and $\sum_{i=1}^{n} \alpha_i = 1$ and $\sum_{j=1}^{m} \beta_j = 1$.

Applying the hyperplane condition to $\mathbf{z}$:

$$\mathbf{w}^\top \mathbf{z} + b = \mathbf{w}^\top \left(\sum_{i=1}^{n} \alpha_i \mathbf{x}_i \right) + b = \sum_{i=1}^{n} \alpha_i (\mathbf{w}^\top \mathbf{x}_i) + b > 0,$$

since each $\mathbf{x}_i$ satisfies $\mathbf{w}^\top \mathbf{x}_i + b > 0$, and the coefficients α_i are non-negative and sum to 1.

Similarly,

$$\mathbf{w}^\top \mathbf{z} + b = \mathbf{w}^\top \left(\sum_{j=1}^{m} \beta_j \mathbf{y}_j \right) + b = \sum_{j=1}^{m} \beta_j (\mathbf{w}^\top \mathbf{y}_j) + b < 0,$$

since each $\mathbf{y}_j$ satisfies $\mathbf{w}^\top \mathbf{y}_j + b < 0$, and the coefficients β_j are non-negative and sum to 1.

Thus, we have:

$$\mathbf{w}^\top \mathbf{z} + b > 0 \quad \text{and} \quad \mathbf{w}^\top \mathbf{z} + b < 0,$$

which is a contradiction.

Therefore, the assumption that the convex hulls of A and B intersect must be false. This means that the convex hulls of A and B do not intersect, confirming that the two sets are linearly separable.