Continuity And Differentiability

undefined Value :- $\frac{0}{0}$, $\frac{1}{0}$, ∞, $\sqrt{-ve}$, 0^0.

formula of limits :-

① $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

② $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

③ $\lim_{\theta \to 0} \cos\theta = 1$

④ $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$

⑤ $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

⑥ $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

⑦ $\lim_{x \to 0} \sin\frac{1}{x}$ = A finite value.

Continuity :-

(i) a function f(x) is called continous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c)$$

(ii) a function f(x) is continous at $x = c$ when

$$L.H.L = f(c) = R.H.L$$

where,

$$L.H.L = \lim_{x \to c^-} f(x)$$

$$= \lim_{x \to 0} f(c-x)$$

$$R.H.L = \lim_{x \to c^+} f(x)$$

$$= \lim_{x \to 0} f(c+x)$$

Now,

Find all points of discontinuity of f, where f is defined by {from Q.6 to Q.12}.

(6) $f(x) = \begin{cases} 2x+3 & , x \leq 2 \\ 2x-3 & , x > 2 \end{cases}$

Sol.

$$f(x) = \begin{cases} 2x+3 & , x \leq 2 \\ 2x-3 & , x > 2 \end{cases}$$

at $x = 2$,

$$f(2) = 2(2)+3$$
$$= 7$$

L.H.L. $= \lim\limits_{x \to 2^-} f(x)$

$= \lim\limits_{x \to 0} f(2-x)$

$= \lim\limits_{x \to 0} 2(2-x)+3$

$= 4+3$

$= 7$

R.H.L $= \lim\limits_{x \to 2^+} f(x)$

$$= \lim_{x\to 0} f(2+x)$$

$$= \lim_{x\to 0} \{2(2+x)-3\}$$

$$= 2(2+0)-3$$

$$= 4-3$$

$$R.H.L. = 1$$

Here,

$$\lim_{x\to 2^-} f(x) = f(2) \neq \lim_{x\to 2^+} f(x)$$

$\therefore$ $f(x)$ is discontinous at $x = 2$.

Theorem 1:- suppose f and g be two real functions continous at a real no. c, Then.

(i) $f+g$ is continous at $x=c$

(ii) $f-g$ is continous at $x=c$

(iii) $f \cdot g$ is continous at $x=c$

(iv) f/g is continous at $x=c$.

(1) First Principle :-

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \frac{dy}{dx}.$$

Differentiation :-

$\frac{dy}{dx}$ represent slope of the tangent.

$$y = mx + c$$

$$\frac{dy}{dx} = m \rightarrow \text{slope.}$$

Notes

① $x^4 + x^2 + 1 = (x^2+x+1)(x^2-x+1)$

Sol.

$$x^4 + x^2 + 1$$
$$= x^4 + 2x^2 + 1 - x^2$$
$$= (x^2+1)^2 - x^2$$
$$= (x^2+x+1)(x^2-x+1)$$

Standard Formulae

1. $\frac{d}{dx} x^n = n x^{n-1}$
2. $\frac{d}{dx} e^x = e^x$ $2 < e < 3$
3. $\frac{d}{dx} e^{-x} = -e^{-x}$
4. $\frac{d}{dx} a^x = a^x \log x$, a = constant.
5. $\frac{d}{dx} a^{-x} = -a^{-x} \log a$
6. $\frac{d}{dx} a = 0$, a = constant.
7. $\frac{d}{dx} \log x = \frac{1}{x}$.
8. $\frac{d}{dx} \log_a x = \frac{1}{x} \log_a e$
9. $\frac{d}{dx} \sin x = \cos x$
10. $\frac{d}{dx} \cos x = -\sin x$
11. $\frac{d}{dx} \tan x = \sec^2 x$.
12. $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x . \cot x$
13. $\frac{d}{dx} \sec x = \sec x . \tan x$
14. $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$
15. $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$
17. $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$
18. $\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$
19. $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$
20. $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$

Rules of Differentiation :-

① $\frac{d}{dx}\left\{f(x) + g(x)\right\} = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$

② $\frac{d}{dx}\left\{f(x) - g(x)\right\} = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$

③ $\frac{d}{dx}\left\{f(x) \cdot g(x)\right\} = f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x)$

④ $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{\{g(x)\}^2}$

⑤ $\frac{d}{dx} C = 0$, where $c \rightarrow$ constant.

⑥ $\frac{d}{dx}\{c \cdot f(x)\} = c \frac{d}{dx} f(x)$, where $c \rightarrow$ constant.

Differentiation :-

(1) $\sin^3 x = f(x)$

Sol. D. w. r. to x

$\Rightarrow \frac{d}{dx}(\sin^3 x) = f'(x)$

$\Rightarrow 3\sin^2 x \cdot \cos x = f'(x)$

$\Rightarrow \boxed{f' = 3\sin^2 x \cdot \cos x}$

(2) $\sin x^3 = f(x)$

Sol. D. w. r. to x.

$\Rightarrow f'(x) = \frac{d}{dx} \sin x^3$

$\Rightarrow f'(x) = \cos x^3 \cdot 3x^2$

$\Rightarrow \boxed{f'(x) = 3x^2 \cos x^3}$

Differentiability :-

a function $f(x)$ is differentiable at $x = c$, when

$\Rightarrow$ $\boxed{Lf'(c) = Rf'(c)}$, where, $Lf'(c) \rightarrow$ left hand derivative (L.H.D)

$Rf'(c) \rightarrow$ Right hand derivative (R.H.D)

Here,

$$Lf'(c) = \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h}$$

$$Rf'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Note :-

(i) If a function $f(x)$ is differentiable at $x = c$ then it is continous also at the same point.

~~(ii)~~ but if it is continous at $x = c$ then it is not necessary that it is differentiable also.

Q. Show that the function $f(x) = \begin{cases} x-1 & , x < 2 \\ 2x-3 & , x \geq 2 \end{cases}$

if not differentiable at $x = 2$.

~~Sol.~~
Sol. $f(x) = \begin{cases} x-1 & , x < 2 \\ 2x-3 & , x \geq 2 \end{cases}$

at $x = 2$,

$\Rightarrow$ $$Lf'(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{(2-h-1) - \{2(2)-3\}}{-h}$$

$$= \lim_{h \to 0} \frac{(1-h) - 1}{-h}$$

$$= \lim_{h \to 0} \frac{\cancel{-h}}{\cancel{-h}}$$

$$= \lim_{h \to 0} 1$$

$$= 1.$$

$\Rightarrow$ $$Rf'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{\{2(2+h)-3\} - \{2(2)-3\}}{h}$$

$$= \lim_{h \to 0} \frac{\{4+2h-3\} - 1}{h}$$

$$= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2$$

$$= 2$$

$\Rightarrow$ Here,

$Lf'(2) \neq Rf'(2)$

Hence $f(x)$ is not differentiable at $x = 2$.

Chain Rule:-

(i) $\frac{d^n}{dx^n} x^n = \lfloor n$

(ii) $\frac{d^n}{dx^n} \sin(ax+b) = a^n \sin\left\{\frac{n\pi}{2} + ax + b\right\}$.

(iii) $\frac{d^n}{dx^n} \cos(ax+b) = a^n \cos\left\{\frac{n\pi}{2} + ax + b\right\}$

$$y = f \circ g$$

$$y = f\{g(x)\}.$$

diff. w.r. to "x"

$$\frac{dy}{dx} = \frac{d}{dx} f\{g(x)\}$$

put $g(x) = t$

$$\frac{dy}{dx} = \frac{d}{dx} f(t)$$

$$= f'(t) \frac{d}{dx} g(x)$$

$$\boxed{\frac{dy}{dx} = f'\{g(x)\}\, g'(x)}$$

Formulas of logarithm :-

① $\log_a a = 1$

② $\log_b a = \frac{1}{\log_a b}$

③ $\log_e m.n = \log_e m + \log_e n$

④ $\log_e \frac{m}{n} = \log_e m - \log_e n$

⑤ $\log_e m^n = n \log m$

Rolle's Theorem :- if $f(x)$ is a real value function such that

(i) $f(x)$ is continous on the close interval $[a,b]$.
(ii) $f(x)$ is differentiable on open interval (a,b).
(iii) $f(a) = f(b)$

then their exict 1 value 'C' of x in the open interval (a,b) is such that

$$f'(c) = 0$$

Q. verify Rolle's theorem for the following function-
$f(x) = x^2 - 2x - 15$, $[-3, 5]$

Sol.

To verify Rolle's Theorem

(i) $f(x) = x^2 - 2x - 15$, is a polynomial therefore it is continous on $[-3, 5]$

(ii) $f'(x) = \frac{d}{dx}(x^2 - 2x - 15)$

$f'(x) = 2x - 2$, $(-3, 5)$
$f'(x) = 2x - 2$ exist for every value of x on $(-3, 5)$.

Hence it is differentiable.

(iii) $f(-3) = (-3)^2 - 2(-3) - 15 = 0$
$f(5) = (5)^2 - 2(5) - 15 = 0$

$\therefore f(-3) = f(5)$

Now we assume a real no. c such that

$\Rightarrow f'(c) = 0$

$\Rightarrow 2c - 2 = 0$

$\Rightarrow c = 1 \in (-3, 5)$

Hence Rolle's theorem is verified.

Mean Value Theorem :-

or

Leingraj Mean Value Theorem :-

if $f(x)$ be a function such that

(i) $f(x)$ is continous on $[a, b]$
(ii) $f(x)$ is differentiable on ~~(a,b)~~ (a, b).
(iii) $f(a) \neq f(b)$

then their exist atleast one $c \in (a, b)$ such that ~~f(b)~~

$$\Rightarrow \boxed{f'(c) = \frac{f(b) - f(a)}{(b-a)}}$$

Q. verify leingraj mean value theorem for the function $f(x) = 2x^2 - 3x + 1$ on $[1, 3]$.

Sol. To verify leingraj mean value theorem.

(i) $f(x)$ is a polynomial hence $f(x)$ is continous on $[1, 3]$

(ii) $f(x) = 2x^2 - 3x + 1$

D.w.r. to "x"

$$\Rightarrow f'(x) = \frac{d}{dx}(2x^2 - 3x + 1)$$

$$\Rightarrow f'(x) = 4x - 3$$

$f'(x)$ exist for every value of x in $(1, 3)$

(iii) $f(1) = 2(1)^2 - 3(1) + 1 = 0$

$f(3) = 2(3)^2 - 3(3) + 1 = 10$

$\therefore \quad f(1) \neq f(3)$

~~Now,~~

Therefore there exist atleast one value of C such that ~~$f'(c)$~~

$$\Rightarrow \quad f'(c) = \frac{f(b) - f(a)}{3 - 1}$$

$$\Rightarrow \quad 4c - 3 = \frac{10 - 0}{2}$$

$$\Rightarrow \quad 4c - 3 = 5$$

$$\Rightarrow \quad c = 2 \in (1,3)$$

Hence leingraj mean value theorem verified.

Assignment :-

D.W.r. to x^y

1. $\cos^2 x^2$
2. $\log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$
3. $\sin x^\circ$
4. prove that every constant function is continuous
5. $(\sin x)^{\cos x} + (\cos x)^{\sin x}$ find $\frac{dy}{dx}$. ~~if y=~~
6. if $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots}}}$ then prove that $\frac{dy}{dx} = \frac{\cos x}{2y-1}$

~~7. If $y = e^x +$~~

7. If $y = e^{x+e^{x+e^{x+\ldots}}}$ Then prove that $\frac{dy}{dx} = \frac{y}{1-y}$
8. If $y = e^{-kt}\cos(pt+c)$ then prove that $\frac{d^2y}{dt^2} + k\frac{dy}{dt} + n^2y = 0$ where $n^2 = p^2+k^2$
9. If $x^y = e^{x-y}$ then prove that $\frac{dy}{dx} = \frac{\log x}{(1+\log_e x)^2}$
10. If $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$ then prove that $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$