Chapter- 7

Systems of Particle and Rotational Motion

Particle :- an object whose size is negligible and whose internal structure can be neglected is called particle.

Example :- Proton, electron

System :- an arrangement of particles which mutually interacts.

~~Example :-~~

Internal force :- The forces exerted by the particles of a system when they mutually interact.

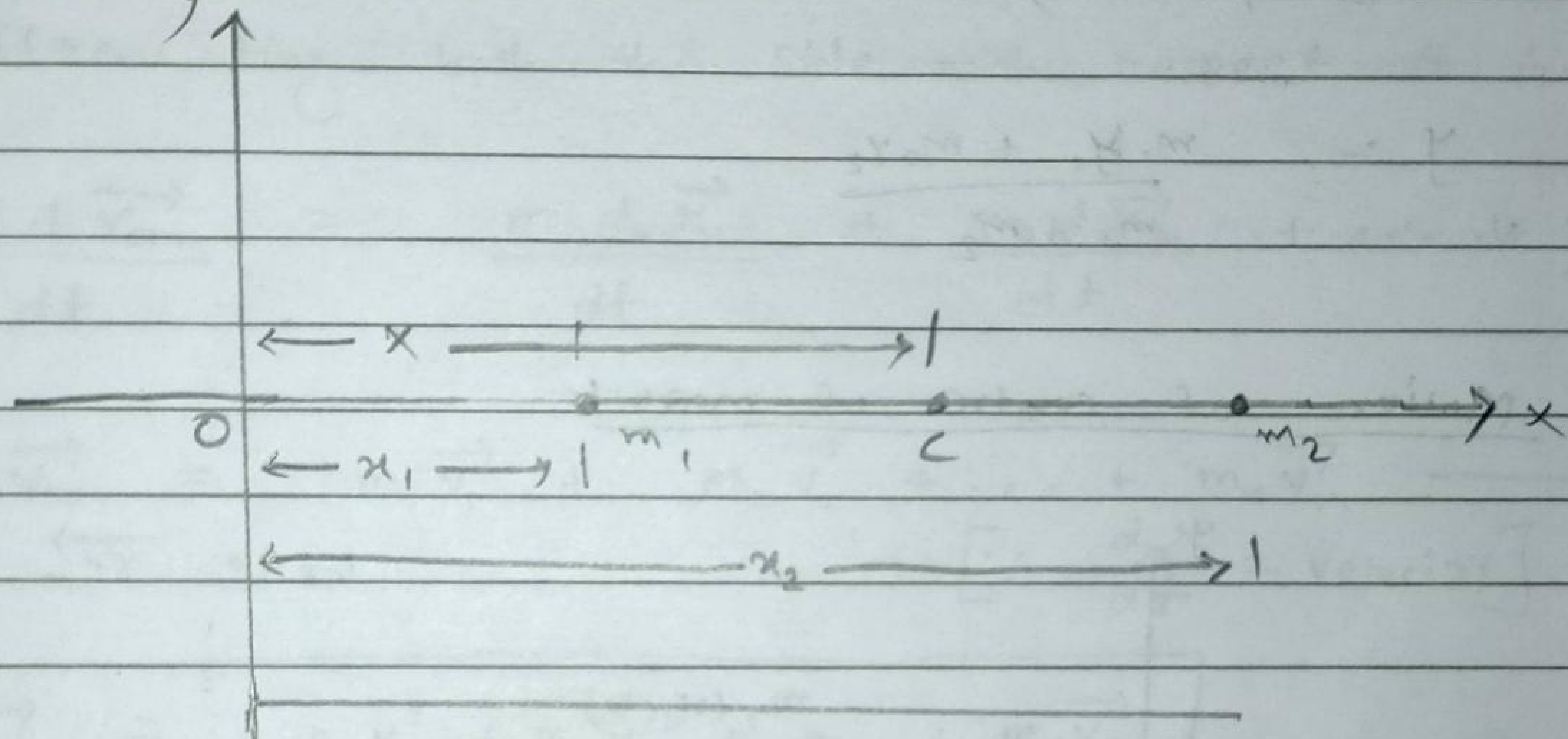
External force :- The force exerted by particles, objects or system ~~when they~~ on a system is called external force.

Isolated system :- A system on which there is no external force is called isolated system.

Centre of Mass :- Centre of mass of a system is that point, on which the entire mass of the system can be considered to be concentrated.

$M = M_1 + M_2 + M_3 + \dots + M_n$

for 2 Particle,



$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$\Rightarrow \quad m_1 = m_2 = m$

$$\Rightarrow \quad X = \frac{m x_1 + m x_2}{m + m}$$

$$\Rightarrow \quad X = \frac{m(x_1 + x_2)}{2m}$$

$$\boxed{X = \frac{x_1 + x_2}{2}}$$

for n Particle,

$$X = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \dots + m_n x_n}{m_1 + m_2 + m_3 + \dots + m_n}$$

Similarly,

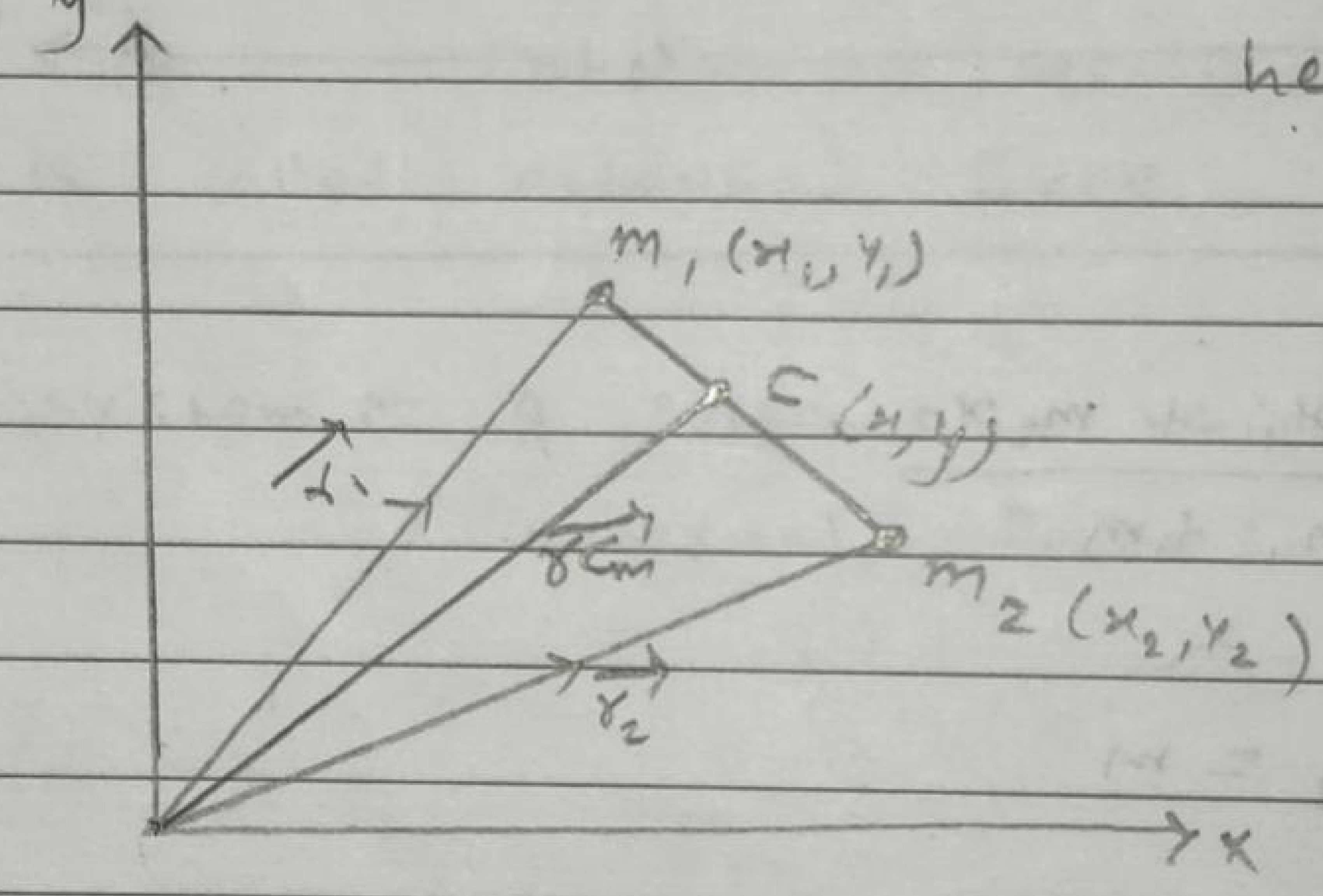
for n Particle,

$$y = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 + \dots + m_n y_n}{m_1 + m_2 + m_3 + \dots + m_n}$$

for 2 particle,

$$y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

Motion of centre of mass :-



here, $\vec{r_{cm}} \Rightarrow$ centre of mass

$$\Rightarrow X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

$$\Rightarrow \vec{r_{cm}} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2}}{m_1 + m_2}$$

for n Particle,

$$\vec{r_{cm}} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r_3} + \dots + m_n \vec{r_n}}{m_1 + m_2 + m_3 + \dots + m_n}$$

since, $M = m_1 + m_2 + m_3 + \dots + m_n$

So,

$$\vec{r}_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \dots + m_n\vec{r_n}}{\cancel{m_1 + m_2 + m_3}\ M}$$

$$\Rightarrow M\vec{r}_{cm} = m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3} + \dots + m_n\vec{r_n}$$

differenciating both the side with respect to time,

$$\Rightarrow \frac{M\,d\vec{r}_{cm}}{dt} = \frac{m_1\,d\vec{r_1}}{dt} + \frac{m_2\,d\vec{r_2}}{dt} + \dots + \frac{m_n\,d\vec{r_n}}{dt}$$

$$\Rightarrow M\vec{v}_{cm} = m_1\vec{v_1} + m_2\vec{v_2} + \dots + m_n\vec{v_n} \quad \text{--- (i)}$$

$$\left[\because \frac{dr}{dt} = \text{velocity}\right]$$

$$\Rightarrow \boxed{\vec{v}_{cm} = \frac{m_1\vec{v_1} + m_2\vec{v_2} + \dots + m_n\vec{v_n}}{M}}$$

where,

$\vec{v}_{cm}$ = velocity of centre of mass

→ from eq^n (i),

$$\Rightarrow M\vec{v}_{cm} = m_1\vec{v_1} + m_2\vec{v_2} + \dots + m_n\vec{v_n}$$

differenciating both side with respect to time,

$$\Rightarrow \frac{M\,d\vec{v}_{cm}}{dt} = m_1\frac{d\vec{v_1}}{dt} + m_2\frac{d\vec{v_2}}{dt} + \dots\ m_n\frac{\vec{v_n}}{dt}$$

$$\Rightarrow M\vec{a}_{cm} = m_1\vec{a_1} + m_2\vec{a_2} + \dots + m_n\vec{a_n}$$

$$\left[\because \frac{dv}{dt} = \text{acceleration}\right]$$

$$\Rightarrow \boxed{\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \dots + m_n\vec{a}_n}{M}}$$

<u>where,</u>

$\vec{a}_{cm}$ = acceleration of centre of mass.

~~Linear Momentum of A System of Particles :-~~

<u>Vector Product of 2 vectors :-</u>

Vector

↓

Product of vector

Scalar Product (dot Product)

$\vec{a} \cdot \vec{b} = ab\cos\theta$.

Vector Product (cross Product)

$\vec{a} \times \vec{b} = ab\sin\theta\,\hat{a}$

<u>Example 7.4.></u> find scalar product and vector product of 2 vectors $a = (3\hat{i} - 4\hat{j} + 5\hat{k})$ and $b = (-2\hat{i} + \hat{j} - 3\hat{k})$

<u>Sol.</u>

$$\vec{a} = 3\hat{i} - 4\hat{j} + 5\hat{k}$$
$$\vec{b} = -2\hat{i} + \hat{j} - 3\hat{k}$$

$$\Rightarrow \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix}$$

$$\Rightarrow \vec{a} \times \vec{b} = \hat{i}\begin{vmatrix} -4 & 5 \\ 1 & -3 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 5 \\ -2 & -3 \end{vmatrix}$$

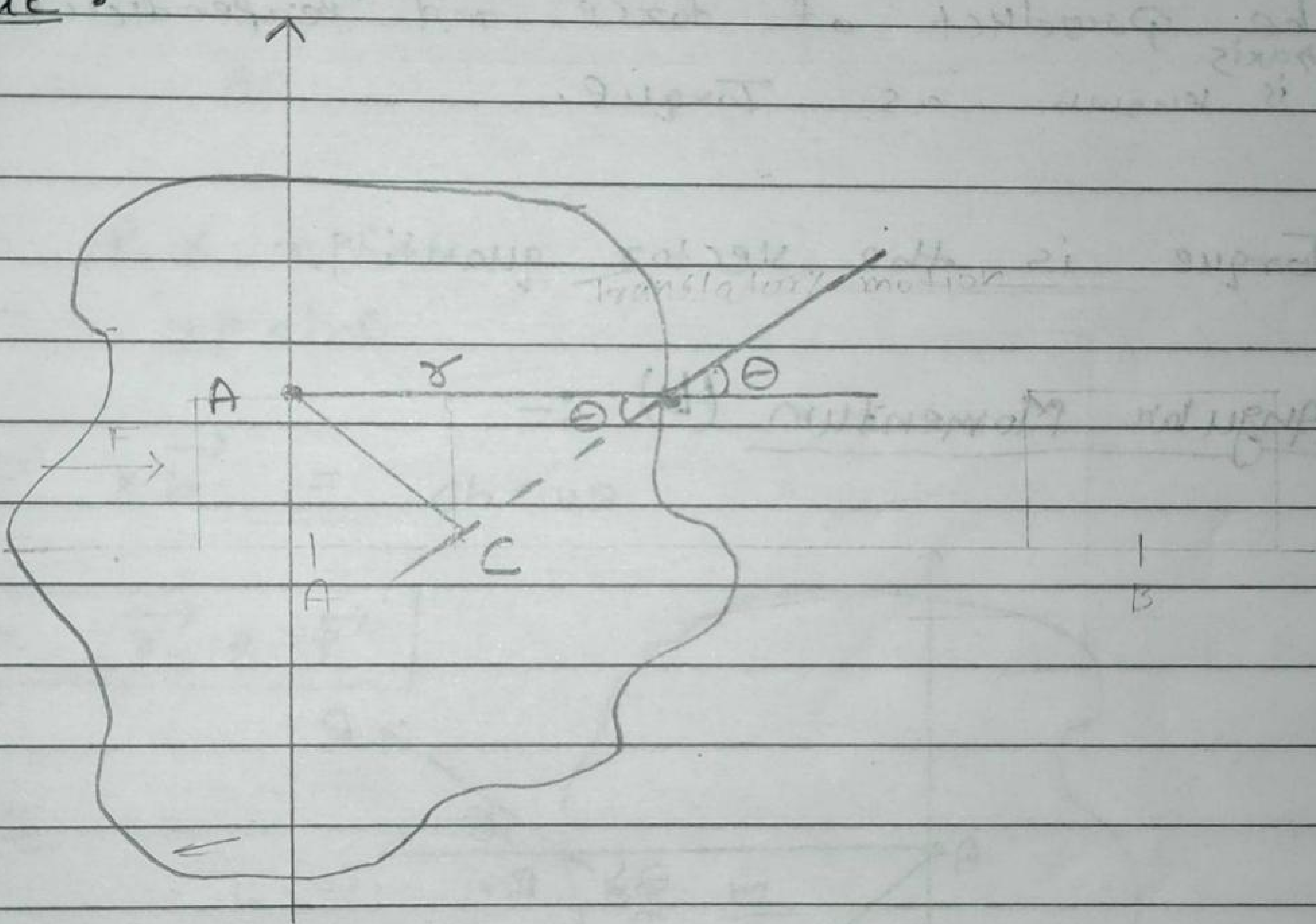
$\Rightarrow \vec{a} \times \vec{b} = \hat{i}\begin{vmatrix} -4 & 5 \\ 1 & -3 \end{vmatrix} - \hat{j}\begin{vmatrix} 3 & 5 \\ -2 & -3 \end{vmatrix} + \hat{k}\begin{vmatrix} 3 & -4 \\ -2 & 1 \end{vmatrix}$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}(12-5) - \hat{j}(-9-(-10)) + \hat{k}(3-8)$

$\Rightarrow \vec{a} \times \vec{b} = \hat{i}(7) - \hat{j}(1) + \hat{k}(-5)$

$\Rightarrow \vec{a} \times \vec{b} = 7\hat{i} - \hat{j} - 5\hat{k}$

Torque :-



$\Rightarrow \tau$ = Force × perpendicular distance

$\Rightarrow \tau = F \times AC$

ΔABC,

$\Rightarrow \sin\theta = \frac{AC}{AB}$

$\Rightarrow \sin\theta = \frac{AC}{r}$

$\Rightarrow r\sin\theta = AC$

$$\tau = F \times r\sin\theta$$

$$\boxed{\tau = Fr\sin\theta}$$

$\because \vec{a} \times \vec{b} = ab\sin\theta$

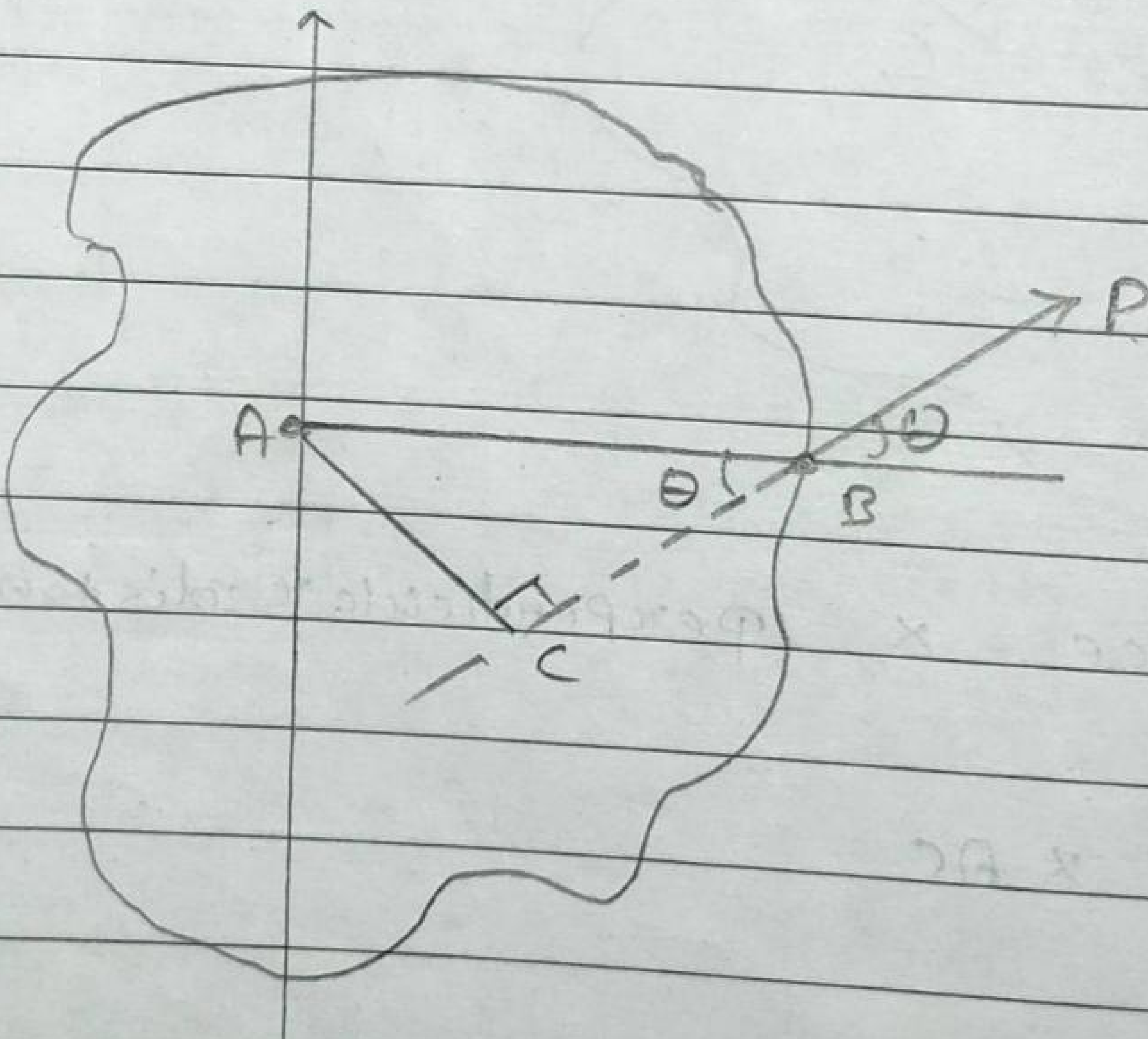
$$\Rightarrow \boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

S.I unit of Torque (τ) = Newton metre,

The product of force and Perpendicular distance from the rotational axis is known as Torque.

Torque is the vector quantity.

Angular Momentum (L) :-



The Product of linear momentum and Perpendicular distance from the rotational axis is called angular momentum.

$\Rightarrow$ L = linear momentum x perpendicular distance

= P x AC

$\triangle ABC$,

$\Rightarrow \sin\theta = \frac{AC}{AB}$

$\Rightarrow \sin\theta = \frac{AC}{r}$

$\Rightarrow r\sin\theta = AC$

$\Rightarrow L = P \times r\sin\theta$

$\Rightarrow L = rP\sin\theta$.

$\because \vec{a} \times \vec{b} = ab\sin\theta$

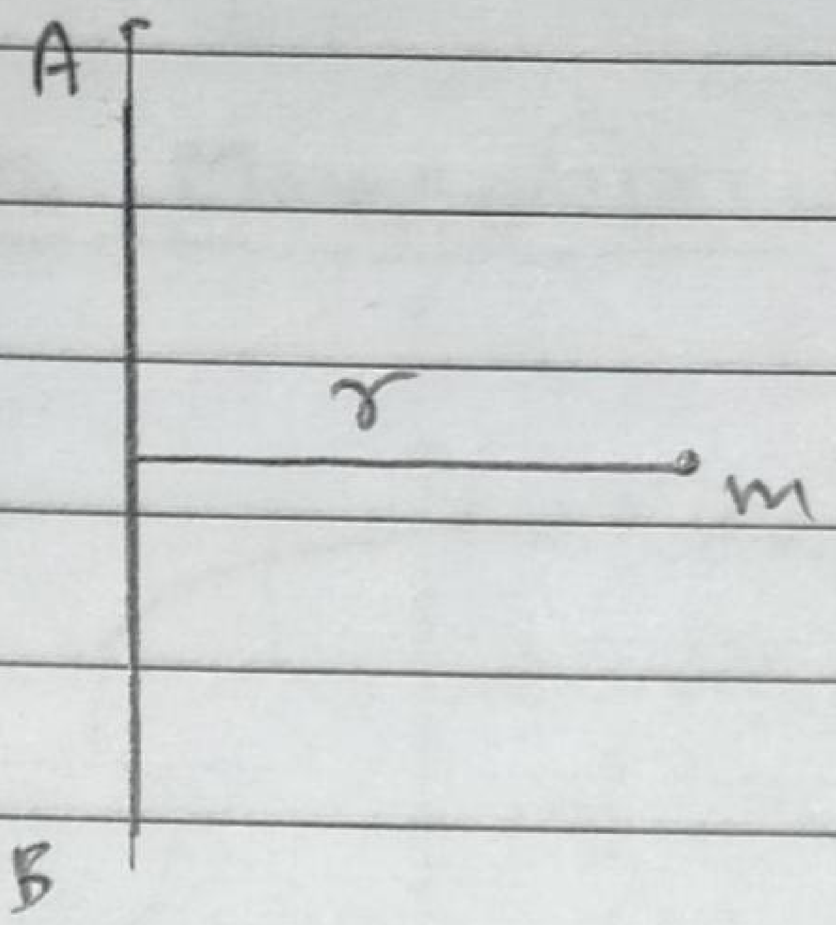
$$\boxed{\vec{L} = \vec{r} \times \vec{P}}$$

unit :-

$L \Rightarrow m \cdot kg \cdot \frac{m}{s}$

$$\boxed{L \Rightarrow kg\ m^2/s}$$

Moment of inertia :-



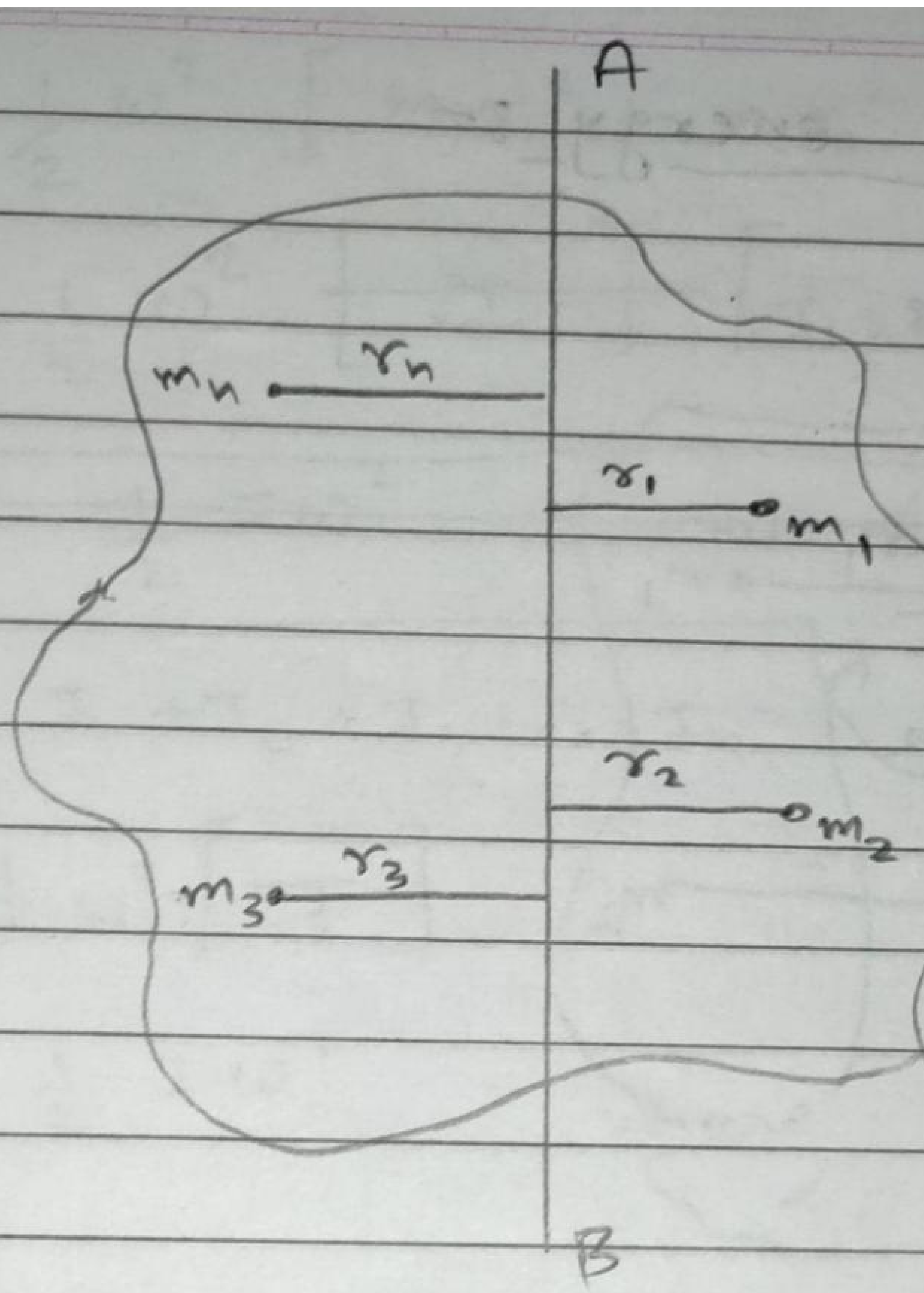
$\Rightarrow$ $\boxed{I = mr^2}$

where,

I = moment of inertia

m = mass

r = distance



for 'n' mass,

$\Rightarrow I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2$

$\Rightarrow$ unit :-

$I = mr^2$

$$\boxed{I = kg\ m^2}$$

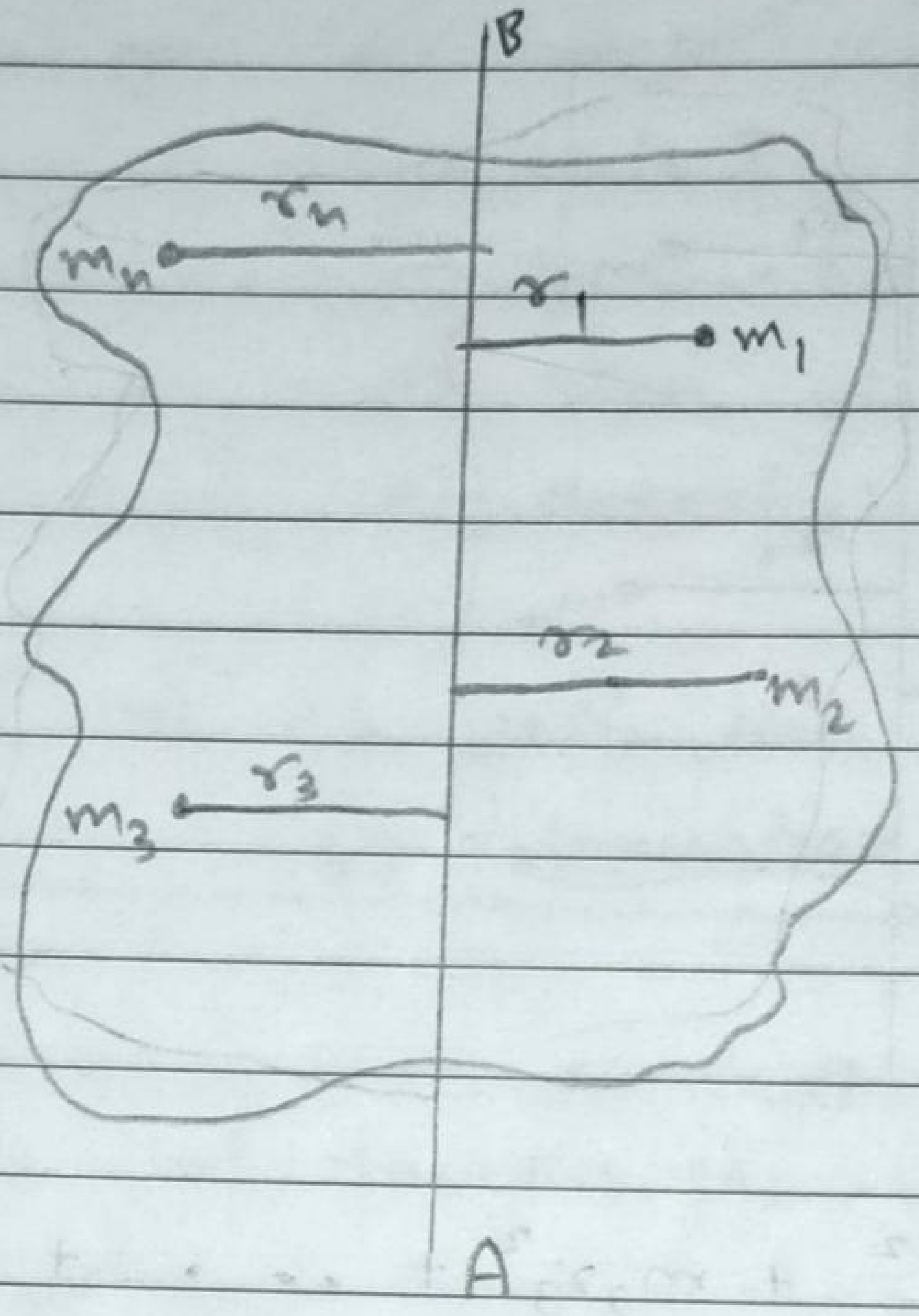
Dimension :-

$I = mr^2$

$I = [ML^2]$

Defination :- The Product of mass and square of distance is known as moment of inertia.

Rotational kinetic energy :-



$\Rightarrow \quad KE_1 = \frac{1}{2} m_1 v_1^2$

$\Rightarrow \quad KE_2 = \frac{1}{2} m_2 v_2^2$

$\Rightarrow \quad KE_n = \frac{1}{2} m_n v_n^2$

$\Rightarrow \quad KE = KE_1 + KE_2 + \dots + KE_n$

$\Rightarrow \quad KE = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \dots + \frac{1}{2} m_n v_n^2$

$\Rightarrow \quad KE = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 + \dots + m_n v_n^2 \right)$

$\because \quad v = r\omega$

$\Rightarrow \quad KE = \frac{1}{2}\left[m_1 (r_1\omega)^2 + m_2 (r_2\omega)^2 + \dots + m_n (r_n\omega)^2 \right]$

$\Rightarrow KE = \frac{1}{2}\omega^2 \left[m_1 r_1^2 + m_2 r_2^2 + \dots + m_n r_n^2 \right]$

$\Rightarrow KE = \frac{1}{2}\omega^2 \cancel{\left[I \right]} \left[I_1 + I_2 + \dots + I_n \right]$

$\cancel{\Rightarrow KE = \frac{1}{2} I \omega^2}$, where,

$I \rightarrow$ moment of inertia

$\omega^2 \rightarrow$ angular velocity

$\because I = I_1 + I_2 + I_3 \dots + I_n$

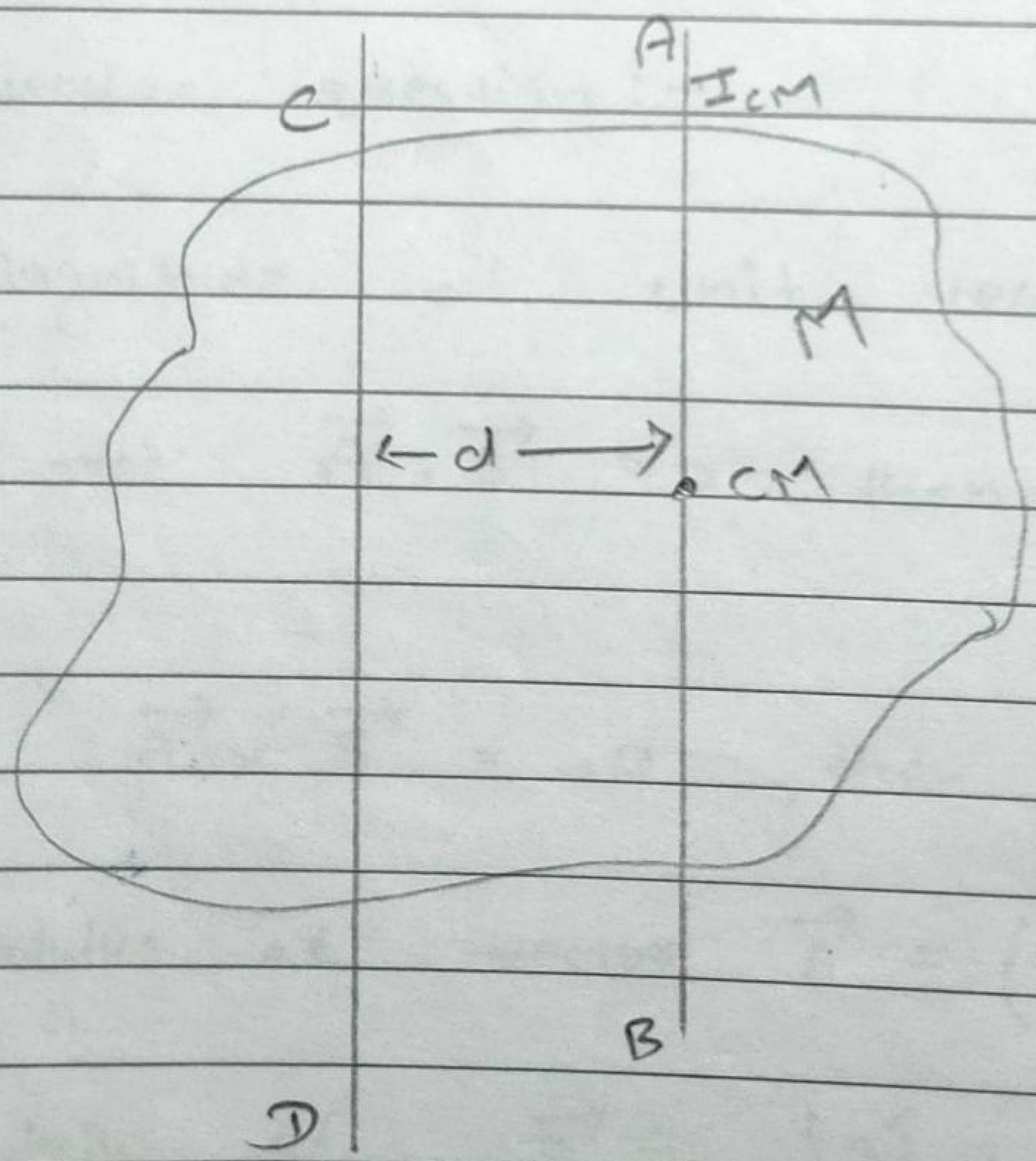
$\Rightarrow KE = \frac{1}{2}\omega^2 \boxed{I}$

$\Rightarrow KE = \frac{1}{2} I \omega^2$, where;

Theorems on moment of Inertia :-

(i) Parrallel axis theorem :- According to this theorem the moment of inertia of a body about the given axis is equal to the sum of the moment of inertia of the body about an axis through the COM and Parrallel to the given axis and the Product of the mass of the body and the square of distance between the 2 axis that is -

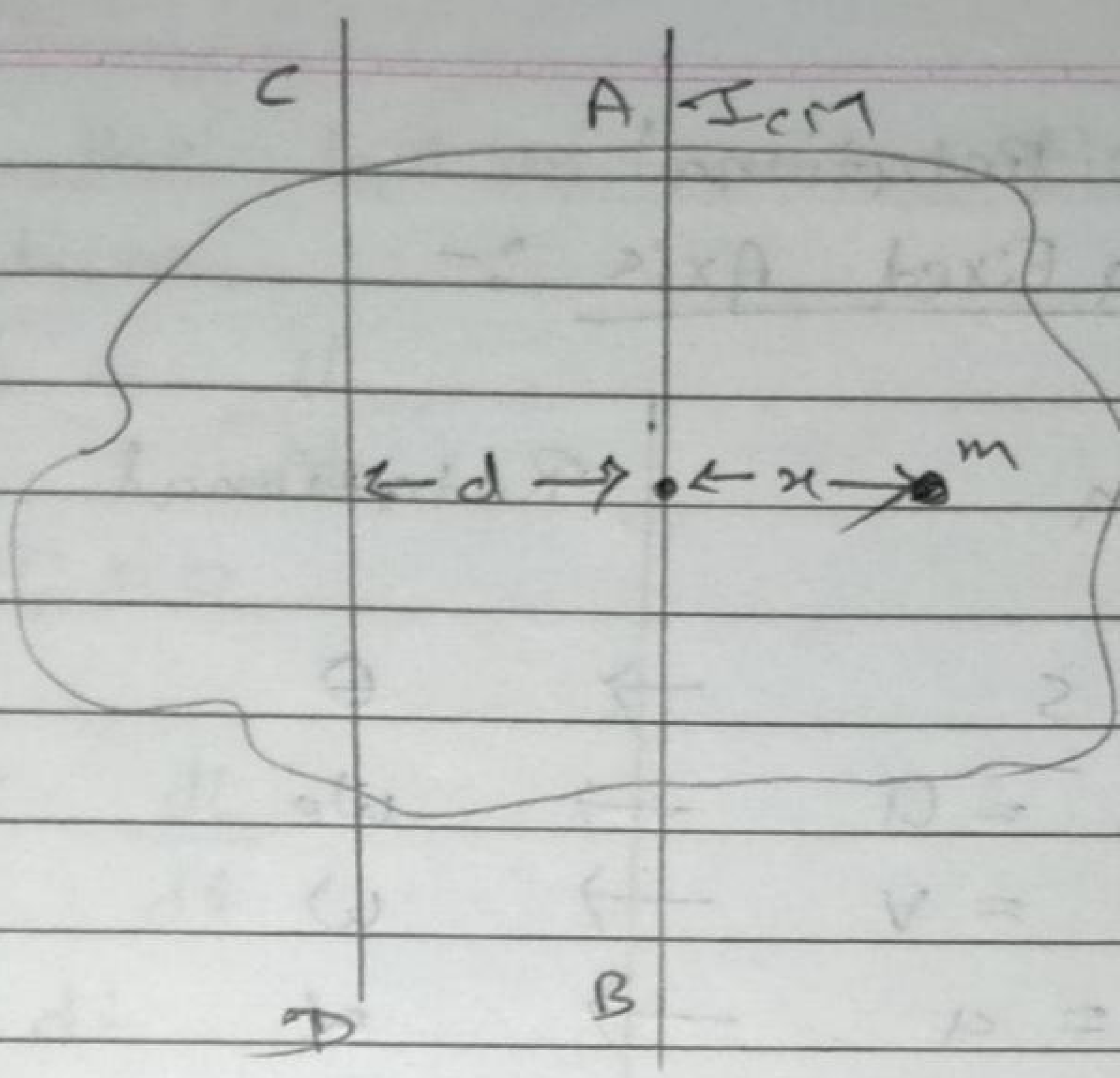
$$\boxed{I = I_{CM} + Ma^2}$$



Moment of Inertia w.r.t. CD,

$$\boxed{I = I_{CM} + Md^2}$$

Proof :-



We know that,

$$I = mr^2$$

moment of inertia of the Particle w.r.t. CD.

$$= m(x+d)^2$$

The moment of inertia of the system w.r.t CD,

$\Rightarrow I = \Sigma m(d+x)^2$

$\Rightarrow I = \Sigma m(d^2 + x^2 + 2dx)$

$\Rightarrow I = \Sigma md^2 + \Sigma mx^2 + \Sigma 2dxm$

$\Rightarrow I = d^2 \Sigma m + \Sigma mx^2 + \Sigma mdx \times 2$

$\Rightarrow I = d^2 M + I_{CM} + \Sigma 2mdx$

$\Rightarrow I = I_{CM} + Md^2 + 2d \Sigma mx$

$\Rightarrow I = I_{CM} + Md^2 + 0 \quad [\because \Sigma mx = 0]$

$\boxed{I = I_{CM} + Md^2}$, Hence Proved.

* The moment of the masses of the particles of the body centre axis through COM will be zero. that is $\Sigma mx = 0$.

Kinematics of Rotational Motion about a Fixed Axis :-

Linear Motion	Rotational motion
1) displacement $= S$	$\rightarrow$ θ
2) initial velocity $= u$	$\rightarrow$ ω_0
3) final velocity $= v$	$\rightarrow$ ω
4) acceleration $= a$	$\rightarrow$ α
5) $v = u + at$	$\rightarrow$ $\omega = \omega_0 + \alpha t$
6) $v^2 = u^2 + 2aS$	$\rightarrow$ $\omega^2 = \omega_0^2 + 2\alpha\theta$
7) ~~$S = u^2 + 2aS$~~ $S = ut + \frac{1}{2}at^2$	$\rightarrow$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

Q. derive $\omega = \omega_0 + \alpha t$ by integration.

Sol.

$$\alpha = \frac{d\omega}{dt}$$

$\Rightarrow \alpha\, dt = d\omega$

integrating,

$$\Rightarrow \int_0^t \alpha\, dt = \int_{\omega_0}^{\omega} d\omega$$

$$\Rightarrow \alpha \left[t\right]_0^t = \left[\omega\right]_{\omega_0}^{\omega}$$

$$\Rightarrow \alpha\left[t - 0\right] = \left[\omega - \omega_0\right]$$

$$\Rightarrow \alpha t = \omega - \omega_0$$

$$\Rightarrow \boxed{\omega = \omega_0 + \alpha t}$$ Hence Proved.

Q. Write the law of conservation of Angular momentum.

Ans. $\tau = \frac{dL}{dt}$

If $\tau = 0$

$\Rightarrow \quad 0 = \frac{dL}{dt}$

$\Rightarrow (0)dt = dL$

$\Rightarrow \quad 0 = dL$

$\Rightarrow \quad dL = 0$

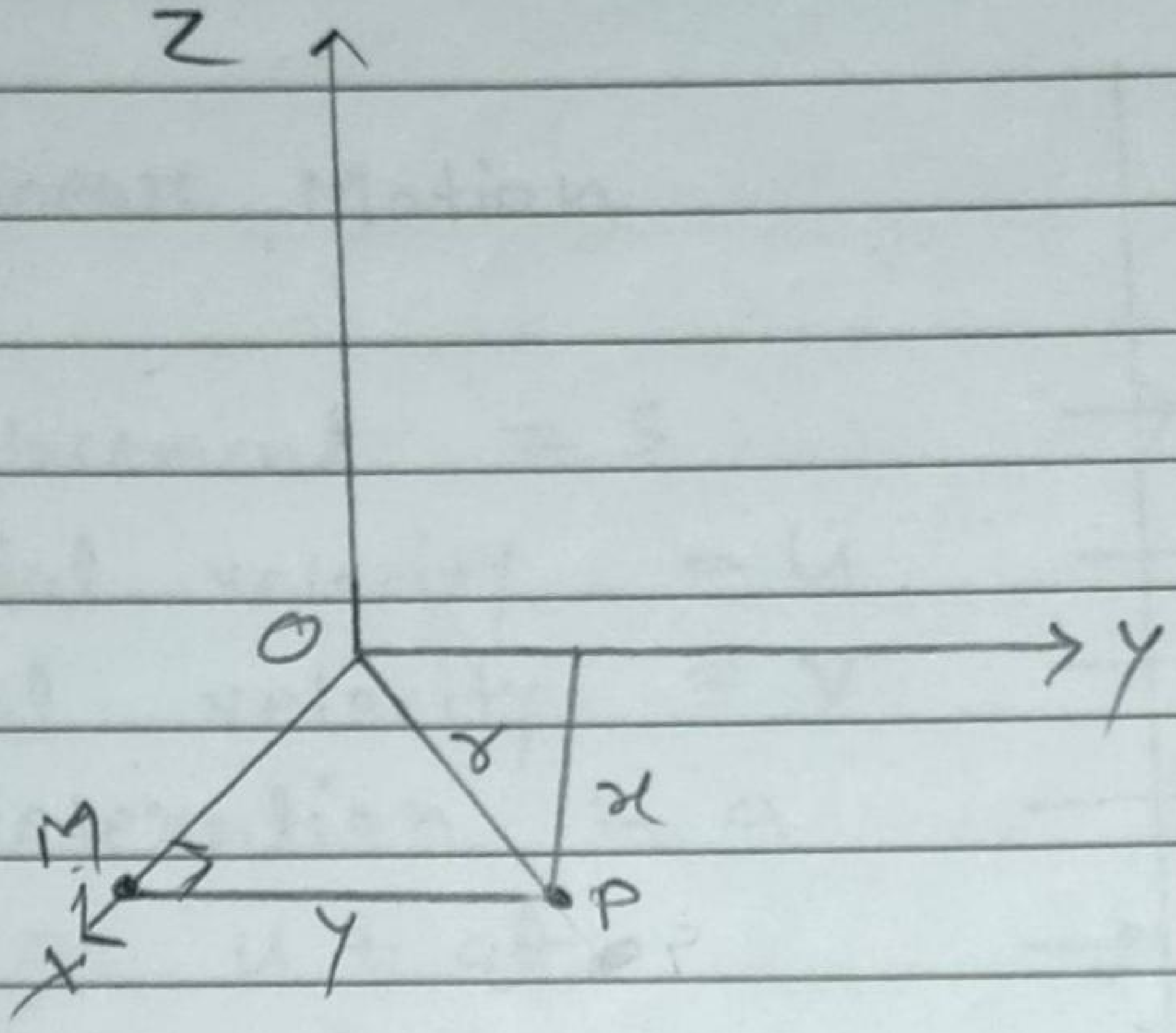
$\Rightarrow \quad \Delta L = 0$

$\Rightarrow \quad \Delta L =$ constant.

Perpendicular axis Theorem :-

To Prove :- $\boxed{I_z = I_x + I_y}$

Proof :-



$$\Rightarrow \quad I_z = \sum m r^2$$
$$I_x = \sum m x^2$$
$$I_y = \sum m y^2$$

$\Rightarrow$ In ΔMOP,

ΔMOP,

$$\Rightarrow r^2 = x^2 + y^2$$
$$\Rightarrow m r^2 = m(x^2 + y^2)$$
$$\Rightarrow m r^2 = m x^2 + m y^2$$
$$\Rightarrow \sum m r^2 = \sum m x^2 + \sum m y^2$$
$$\Rightarrow \boxed{I_z = I_x + I_y}$$