Peter Racioppo
103953689

ECE 236B - HW6

$\boxed{A\,4.33}$ $\quad \phi(u) = \begin{cases} 0 & u \le 0 \\ u^2/2 & 0 < u \le 1 \\ u - 1/2 & u > 1 \end{cases}$

(a.) minimize $\sum_i \phi(y_i)$
s.t. $Ax + b = y$

$$\mathcal{L}(x, y, \nu) = \sum_i \phi(y_i) + \sum_i \nu_i (Ax + b - y)_i$$

$$\nu^T (Ax + b - y) = \nu^T b + (A^T \nu)^T x - \nu^T y$$

If $(A^T\nu)^T \neq 0$, we can let $x = -\alpha A^T \nu$, and let $\alpha > 0$ grow arbitrarily large. Otherwise,

$$g(\nu) = \inf_{x,y} \mathcal{L} = \inf_y \mathbf{1}^T \phi(y) - \nu^T y + \nu^T b$$

$$\phi(y_i) - \nu_i y_i = \begin{cases} -\nu_i y_i & y_i \le 0 \\ y_i^2/2 - \nu_i y_i & 0 \le y_i < 1 \\ (1-\nu_i) y_i - 1/2 & y_i > 1 \end{cases}$$

If $1 - \nu_i < 0$ we can make $\mathcal{L}$ arbitrarily small by making y_i arbitrarily large.
If $\nu_i < 0$, $-\nu_i y_i$ can be arbitrarily large.
Thus, we need $0 \le \nu_i \le 1 \quad \forall\, i$
In this case, if $y_i \le 0$, $-\nu_i y_i \ge 0$ so choose $y_i = 0$
If $0 \le y_i \le 1$, $\frac{d}{dy_i}(y_i^2/2 - \nu_i y_i) = y_i - \nu_i = 0$
when $y_i = \nu_i \Rightarrow \nu_i^2/2 - \nu_i^2 = -\nu_i^2/2 < 0$
$y_i = 0 \Rightarrow 0^2/2 - 0 = 0$, $\quad y_i = 1 \Rightarrow 1/2 - \nu_i$.
If $y_i > 1$, $1 - \nu_i \ge 0 \Rightarrow \quad \mathcal{L} > 1 - \nu_i - \frac{1}{2} = \frac{1}{2} - \nu_i$
Also, $-\nu_i^2/2 \le 1/2 - \nu_i \quad \forall \quad 0 \le \nu_i \le 1$.

Thus,
$$\boxed{g(\nu) = \begin{cases} \sum_i \nu_i b_i - \frac{1}{2}\nu_i^2 & \text{if } 0 \le \nu_i \le 1, \; A^T\nu = 0 \\ -\infty & \text{otherwise} \end{cases}}$$

②

(b.) minimize $\phi(\|y\|_2)$
s.t. $Ax + b = y$.

$$g(\nu) = \inf_{x,y} \phi(\|y\|_2) + \nu^T(Ax+b-y)$$

Again, $A^T\nu = 0$.

$$\phi(\|y\|) - \nu^T y = \begin{cases} 0 & \|y\| = 0 \\ \|y\|^2/2 - \nu^T y & \|y\| \le 1 \\ \|y\| - \frac{1}{2} - \nu^T y & \|y\| > 1 . \end{cases}$$

$$\frac{d}{dy} y^T y/2 - \nu^T y = y - \nu = 0 \Rightarrow y = \nu$$

Let $y = \nu$, we have $\|\nu\|^2/2 - \|\nu\|^2 = -\|\nu\|^2/2$.
If $\|\nu\| > 1$, we can let $y = \alpha\nu$, $\alpha > 0$
and $\|y\| - \frac{1}{2} - \nu^T y = \alpha\|\nu\| - \alpha\|\nu\|^2 - \frac{1}{2} \to -\infty$
as $\alpha \to \infty$.
Since $\|\nu\| \le 1$, $\Rightarrow \|y\| - \frac{1}{2} - \nu^T y \ge \|y\| - \|\nu\|\|y\| - \frac{1}{2}$
$> 1/2 - \|\nu\| \ge -1/2$.

$$\Rightarrow \inf \mathcal{L} = \min(-\|\nu\|^2/2,\ 1/2 - \|\nu\|)$$
$$= -\|\nu\|^2/2 \quad \text{for } \|\nu\| \le 1.$$

$$\boxed{g(\nu) = \begin{cases} -\|\nu\|_2^2/2 + \nu^T b, & \|\nu\|_2 \le 1,\ A^T\nu = 0 \\ -\infty & \text{otherwise} \end{cases}}$$

③

T5.18 $P_1 = \{x \mid Ax \preceq b\}$, $P_2 = \{x \mid Cx \preceq d\}$.

$(P_1 \cap P_2 = \emptyset .)$

Find a, γ s.t. $a^T x > \gamma$, $x \in P_1$, $a^T x < \gamma$, $x \in P_2$.

We need: $\inf_{x \in P_1} a^T x > \gamma > \sup_{x \in P_2} a^T x$.

$$\begin{array}{ll} \text{minimize} & 0^T y + 0^T z \\ \text{s.t.} & \inf a^T y > \gamma \\ & Ay \preceq b \\ & \sup a^T z < \gamma \\ & Cz \preceq d . \end{array}$$

$$\Updownarrow$$

$$\boxed{\begin{array}{ll} \text{minimize} & 0^T y + 0^T z \\ \text{s.t.} & -a^T y + \gamma \le 0 \\ & Ay - b \preceq 0 \\ & a^T z - \gamma \le 0 \\ & Cz - d \preceq 0 \end{array}}$$

Dual

$$g(\lambda_1, \lambda_2, \rho_1, \rho_2) = \inf_{y, z} \lambda_1(-a^T y + \gamma) + \rho_1^T (Ay - b) + \lambda_2 (a^T z - \gamma) + \rho_2^T (Cz - d).$$

$$= \inf_{y, z} (-\lambda_1 a + A^T \rho_1)^T y + (\lambda_2 a + C^T \rho_2)^T z + \lambda_1 \gamma - \rho_1^T b - \lambda_2 \gamma - \rho_2^T d$$

If $-\lambda_1 a + A^T \rho_1 \neq 0$ or $\lambda_2 a + C^T \rho_2 \neq 0$, g is unbounded below

$$\boxed{g(\lambda_1,\lambda_2,\rho_1,\rho_2) = \begin{cases} \gamma\lambda_1 - \gamma\lambda_2 - b^T\rho_1 - d^T\rho_2 \\ \quad \text{if } -\lambda_1 a + A^T\rho_1 = 0 \\ \quad \text{and } \lambda_2 a + C^T\rho_2 = 0 \\ -\infty \quad \text{otherwise.} \end{cases}}$$

We have:

$$\boxed{\begin{aligned} \underset{\lambda_1,\lambda_2,\rho_1,\rho_2}{\text{maximize}} \quad & \gamma\lambda_1 - \gamma\lambda_2 - b^T\rho_1 - d^T\rho_2 \\ \text{s.t.} \quad & -\lambda_1 a + A^T\rho_1 = 0 \\ & \lambda_2 a + C^T\rho_2 = 0. \end{aligned}}$$

5

$\boxed{\text{A 4.32}}$ minimize x_1

s.t. $\sqrt{x_1^2 + x_2^2} \le x_2$

$-x_1 \le 1$

(a.)

Suppose $x_2 \ge 0$. Then,

$\sqrt{x_1^2 + x_2^2} \le x_2 \Leftrightarrow x_1^2 + x_2^2 \le x_2^2 \Leftrightarrow x_1^2 \le 0$.

$\Leftrightarrow x_1 = 0$, which satisfies $0 \le 1$.

If $x_2 < 0$, $\sqrt{x_1^2 + x_2^2} \le x_2$ is always false.

Thus, the soln. for $x_2 \ge 0$ is unique.

$\boxed{\text{The optimal value is } 0.}$

(b.) $g(\lambda_1, \lambda_2) = \inf_{x_1, x_2} \left[x_1 + \lambda_1 (\sqrt{x_1^2 + x_2^2} - x_2) + \lambda_2(-x_1 - 1) \right]$

$= \inf_{x_1, x_2} (1 - \lambda_2) x_1 - \lambda_1 x_2 + \lambda_1 \sqrt{x_1^2 + x_2^2} - \lambda_2)$

Suppose $|1 - \lambda_2| > \lambda_1$. (and $\lambda_1 > 0$).

Letting $x_1 = -\alpha(1 - \lambda_2)$, $\alpha > 0$, we have:

$\mathcal{L} = -\alpha(1-\lambda_2)^2 - \lambda_1 x_2 + \lambda_1 \sqrt{\alpha^2 (1-\lambda_2)^2 + x_2^2})$. Letting $\alpha \to \infty$,

we have $\mathcal{L} = -\alpha(1 - \lambda_2)^2 + \alpha \lambda_1 |1 - \lambda_2|$

$= \alpha |1 - \lambda_2| (-|1 - \lambda_2| + \lambda_1) \to -\infty$.

$\frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda_2 + \frac{1}{2} \lambda_1 (x_1^2 + x_2^2)^{-1/2} \cdot 2x_1 = 0$

$\frac{\partial \mathcal{L}}{\partial x_2} = -\lambda_1 + \frac{1}{2} \lambda_1 (x_1^2 + x_2^2)^{-1/2} \cdot 2x_2 = 0$

$\Rightarrow \left.\begin{array}{l} (1-\lambda_2)\sqrt{x_1^2 + x_2^2} + \lambda_1 x_1 = 0 \\ -(1-\lambda_2)\sqrt{x_1^2 + x_2^2} + (1-\lambda_2) x_2 = 0 \end{array}\right\} \Rightarrow \lambda_1 x_1 + (1-\lambda_2) x_2 = 0 \Rightarrow x_2 = \left(\frac{\lambda_1}{\lambda_2 - 1}\right) x_1$

$\Rightarrow (1-\lambda_2) x_1 \sqrt{1 + \frac{\lambda_1^2}{(1-\lambda_2)^2}} + \lambda_1 x_1 = 0 \Rightarrow (1 + \lambda_1) x_1 = 0$

$\Rightarrow x_1 = 0, x_2 = 0$.

⑥

$$\mathcal{L} = (1-\lambda_2)x_1 - \lambda_1 x_2 + \lambda_1\sqrt{x_1^2+x_2^2} - \lambda_2$$

Suppose $\lambda_1 \geq 0$.

$\lambda_1\sqrt{x_1^2+x_2^2} - \lambda_1 x_2 \geq \lambda_1|x_2| - \lambda_1 x_2 \geq 0$.

If $|1-\lambda_2| \leq \lambda_1$,

$\lambda_1\sqrt{x_1^2+x_2^2} + (1-\lambda_2)x_1 \geq \lambda_1|x_1| + (1-\lambda_2)x_1$
$\geq \lambda_1|x_1| - |1-\lambda_2||x_1| \geq 0$.

Suppose $\lambda_1 < 0$.

$\lambda_1\sqrt{x_1^2+x_2^2} < 0$ so we can choose
$x_2 = \alpha > 0$ and as $\alpha \to \infty$, $-\lambda_1 x_2 \to -\infty$.

If $\lambda_1 \geq 0$ & $|1-\lambda_2| \leq \lambda_1$,
$(1-\lambda_1)x_1 - \lambda_1 x_2 + \lambda_1\sqrt{x_1^2+x_2^2}$
Obviously, choose $x_2 \geq 0$.
Since $|1-\lambda_2| \leq \lambda_1$, choose $|x_1| \leq |x_2|$
we can do no better that choosing $x_1 = 0$
$\Rightarrow (1-\lambda_2)x_1 - \lambda_1 x_2 + \lambda_1\sqrt{x_1^2+x_2^2} \geq 0$
$\Rightarrow \inf[(1-\lambda_2)x_2 - \lambda_1 x_2 + \lambda_1\sqrt{x_1^2+x_2}] = 0$
(when $x_1 = x_2 = 0$).
$\Rightarrow \inf \mathcal{L} = -\lambda_2$

$$\boxed{g(\lambda_1,\lambda_2) = \begin{cases} -\lambda_2 & \text{if } \lambda_1 \geq 0,\ |1-\lambda_2| \leq \lambda_1 \\ -\infty & \text{otherwise} \end{cases}}$$

we have:

(c.) $$\underset{\lambda_1, \lambda_2}{\text{maximize}} \; -\lambda_2 \iff \min \lambda_2$$

$$\text{s.t.} \quad \lambda_1 \geq 0$$
$$|1-\lambda_2| \leq \lambda_1$$

This is an LP $\Rightarrow$ optimal pt is on boundary.

Letting $\lambda_2 = \alpha$, we have that
$|1-\lambda_2| = |1-\alpha| \leq \lambda_1$ provided that
$\lambda_1 \geq \beta$ where $|1-\alpha| \leq \beta$.
Letting $\lambda_1 \to \infty$, we can let λ_2 equal any number

$\Rightarrow \min \lambda_2 = -\infty$. Thus,

Strong duality is not satisfied.

The result does not violate duality because the solution of the dual problem is less than or equal to that of the primal.

$$\text{minimize } x_1$$
$$\text{s.t.} \quad |x_1| + |x_2| \leq x_2$$
$$-x_1 \leq 1.$$

$x_2 \geq 0 \Rightarrow |x_1| + |x_2| \leq x_2 \iff |x_1| \leq 0$
$\Rightarrow x_1 = 0$ and $0 \leq 1$ is satisfied.
$\Rightarrow x_1 = 0$ is the solution.

8

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = x_1 + \lambda_1(|x_1| + |x_2| - x_2) + \lambda_2(-x_1 - 1)$$

$$= (1-\lambda_2)x_1 + \lambda_1|x_1| - \lambda_1 x_2 + \lambda_1|x_2| - \lambda_2$$

Now, $\lambda_1(|x_2| - x_2) \geq 0$

Also, $\lambda_1|x_1| + (1-\lambda_2)x_1 \geq 0 \quad \forall\, x_1, x_2$

$\Rightarrow \quad |1-\lambda_2| \leq \lambda_1$ and $\lambda_1 \geq 0$.

Otherwise, choose $x_1 = -\alpha(1-\lambda_2)$ and let $\alpha \to \infty$, to get $\mathcal{L} \to -\infty$.

If $|1-\lambda_2| \leq \lambda_1$ and $\lambda_1 \geq 0$ we have:

$$\mathcal{L} \geq -|1-\lambda_2||x_1| + \lambda_1|x_1| - \lambda_1|x_2| + \lambda_1|x|_2 - \lambda_2$$

$$= (\lambda_1 - |1-\lambda_2|)|x_1| - \lambda_2$$

$$\geq -\lambda_2$$

and $\mathcal{L} = -\lambda_2$ when $x_1 = x_2 = 0$

$\Rightarrow \inf_{x_1, x_2} \mathcal{L} = -\lambda_2$

$$\boxed{g(\lambda_1, \lambda_2) = \begin{cases} -\lambda_2, & \lambda_1 \geq 0,\ |1-\lambda_2| \leq \lambda_1 \\ -\infty, & \text{otherwise} \end{cases}}$$

We have:

$$\max_{\lambda_1, \lambda_2} \; -\lambda_2$$

$$\text{s.t.} \quad \lambda_1 \geq 0$$

$$|1-\lambda_2| \leq \lambda_1$$

which has solution

$\boxed{A\ 4.14}$ Kantorovich Inequality

(a.) Let $a \in \mathbb{R}^n$, $a_1 \geq \dots \geq a_n > 0$, $b \in \mathbb{R}^n$,
$b_k = 1/a_k$

KKT Conditions

1. $f_i(x) \leq 0$, $h_i(x) = 0$
2. $\lambda \succeq 0$ 3. $\lambda_i f_i(x) = 0$
4. x minimizer of $\mathcal{L}$. 4.' $\nabla f_0(x) + \sum \lambda_i \nabla f_i(x) + \sum \nu_i \nabla h_i(x) = 0$

$$\text{minimize} \quad -\log(a^T x) - \log(b^T x)$$
$$\text{s.t.} \quad x \succeq 0, \ 1^T x = 1.$$

Dual:

$$\mathcal{L}(x, \lambda, \nu) = -\log(a^T x) - \log(b^T x) + \lambda^T(-x) + \nu(1^T x - 1)$$

1. $x \succeq 0$, $1^T x = 1$
2. $\lambda \succeq 0$
3. $\lambda^T x = 0$

$$\nabla(-\log(a^T x) - \log(b^T x)) = -\frac{a^T}{a^T x} - \frac{b^T}{b^T x}.$$

$\nabla f_i(x) = \nabla(-x) = -I$
$\nabla h_i(x) = \nabla 1^T x - 1 = 1^T$

$$\boxed{4.' \quad \frac{a}{a^T x} + \frac{b}{b^T x} + \lambda - \nu 1 = 0}$$

Let $x = (\frac{1}{2}, 0, \dots, 0, \frac{1}{2})$.
$x \succeq 0$ and $1^T x = \frac{1}{2} + \frac{1}{2} = 1$ so x satisfies ①.
By ②, $\lambda \succeq 0$. By ③, $\lambda^T x = 0 \Rightarrow \lambda_1 = \lambda_n = 0$

By 4',

$$\frac{a}{a^T x} + \frac{b}{b^T x} + \lambda - \nu \mathbb{1} = 0$$

$$\Rightarrow \frac{a}{\frac{1}{2}(a_1+a_n)} + \frac{b}{\frac{1}{2}(b_1+b_n)} + \lambda - \nu \mathbb{1} = 0$$

$$\frac{b}{b_1+b_n} = \frac{1/a}{1/a_1 + 1/a_n} = \frac{a_1/a}{1+a_1/a_n}$$

$$\frac{a}{a_1+a_n} = \frac{a/a_n}{1+a_1/a_n}$$

$$\Rightarrow \frac{2(a/a_n + a_1/a)}{1+a_1/a_n} + \lambda - \nu \mathbb{1} = 0$$

$$\Rightarrow 2(a/a_n + a_1/a) + (1+a_1/a_n)(\lambda - \nu \mathbb{1}) = 0.$$

$$a_1/a = a_1\left(\frac{1}{a_1}, \dots, \frac{1}{a_n}\right)$$

$$\Rightarrow 2(a + a_1 a_n/a) + (a_1+a_n)(\lambda - \nu \mathbb{1}) = 0.$$

Let's consider each element in turn:

$$\frac{a_i}{\frac{1}{2}(a_1+a_n)} + \frac{b_i}{\frac{1}{2}(b_1+b_n)} + \lambda_i - \nu = 0$$

$$\frac{a_i}{\frac{1}{2}(a_1+a_n)} + \frac{1/a_i}{\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_n}\right)} + \lambda_i - \nu = 0 \quad . \text{ Let } i=1.$$

$$\frac{2\, a_1/a_n}{1+a_1/a_n} + \frac{2}{1+a_1/a_n} = \frac{2(1+a_1/a_n)}{1+a_1/a_n}$$

$\Rightarrow 2 + \lambda_1 - \nu = 0$ and likewise, $2 + \lambda_n - \nu = 0$.

But $\lambda_1 = \lambda_n = 0 \Rightarrow \nu = 2$.

$$\frac{1/a_i}{1/a_1 + 1/a_n} = \frac{a_n/a_i}{1+a_n/a_1}, \quad \frac{a_i}{a_1+a_n} = \frac{a_i/a_1}{1+a_n/a_1}$$

$$\frac{2a}{a_1+a_n} + \frac{2b}{b_1+b_n} = \frac{2(a_i/a_1 + a_n/a_i)}{1+a_n/a_1}$$

$a_i/a_1 + a_n/a_i \leq 1 + a_n/a_i \leq 1 + a_n/a_1$ for $i \neq 1, n$

$$\Rightarrow \frac{a_i}{\frac{1}{2}(a_1+a_n)} + \frac{b_i}{\frac{1}{2}(b_1+b_n)} \leq 2$$

11

It follows that $\frac{a_i}{\frac{1}{2}(a_1+a_n)} + \frac{b_i}{\frac{1}{2}(b_1+b_n)} + \lambda_i - \nu$
$= 2 - \varepsilon_i + \lambda_i - \nu$ (where $\varepsilon_i \geq 0$, for $i \neq 1, n$)
$= \lambda_i - \varepsilon_i$ (since $\nu = 2$).
Letting $\lambda_i = \varepsilon_i \geq 0$, we have that
$a_i / a^T x + b_i / b^T x + \lambda_i - \nu I = 0$ for $\forall\, i \in [1, n]$.
Thus, $\lambda \succeq 0$.

In summary, x satisfies all four KKT conditions.

(b.) Let $A \in S^n_{++}$, with eigenvals $\lambda_1 \geq \ldots \geq \lambda_n$.

(13)

A4.22 minimize $x^T A y$

s.t. $\prod_{i=1} x_i^{c_i} = 1$

$\prod_{j=1} y_j^{d_j} = 1$

$x, y \in \mathbb{R}^n$, $x, y \succ 0$, $A \in \mathbb{R}_+^{n \times n}$, $\mathbf{1}^T c = \mathbf{1}^T d = 1$.

let $B = \frac{1}{x^T A y} \operatorname{diag}(x) A \operatorname{diag}(y)$.

$z_i = \log x_i$, $q_j = \log y_j$

$$\prod x_i^{c_i} = \prod e^{c_i z_i} = e^{c_1 z_1} e^{c_2 z_2} \cdots e^{c_n z_n}$$

$= e^{\sum c_i z_i} = e^{c^T z} = 1$. Likewise, $e^{d^T q} = 1$

$$x^T A y = \sum_{i,j} A_{ij} x_i y_j = \sum_{i,j} e^{z_i + q_j + \log(A_{ij})}$$

Convex Form:

$$\boxed{\begin{array}{ll} \text{minimize} & \log\left(\sum_{i,j} e^{z_i + q_j + a_{ij}}\right) \\ \text{s.t.} & c^T z = 0 \\ & d^T q = 0 \end{array}}$$

(where $a_{ij} = \log(A_{ij})$, $z = \log x$, $q = \log y$

The dual is:

$$\mathcal{L}(z, q, \nu_1, \nu_2) = \log\left(\sum_{i,j} e^{z_i + q_j + a_{ij}}\right) + \nu_1 c^T z + \nu_2 d^T q.$$

KKT

① $c^T z = 0$, $d^T q = 0$

② & ③ are automatically satisfied since $\lambda = 0$.

(14)

$\frac{\partial}{\partial z_i} f_0(z,q) = \frac{\partial}{\partial z_i} \log \sum_{ij} e^{z_i + q_j + a_{ij}}$

$\frac{1}{\sum_{ij} e^{z_i+q_j+a_{ij}}} \frac{\partial}{\partial z_i} \sum_{ij} e^{z_i+q_j+a_{ij}}$

$\partial/\partial z_i \sum(\;) = \sum_j \frac{\partial}{\partial z_i} e^{z_i+q_j+a_{ij}} = \sum_j e^{z_i+q_j+a_{ij}}$

Likewise, $\partial/\partial q_i f_0(z,q) = \frac{1}{\sum_{ij}(\;)} \sum_j e^{z_j+q_i+a_{ji}}$

$\frac{\partial}{\partial z_i} c^T z = c_i$, $\frac{\partial}{\partial q_i} d^T q = d_i$. Thus,

$$\frac{\sum_j e^{z_i+q_j+a_{ij}}}{\sum_{ij} e^{z_i+q_j+a_{ij}}} + \nu_1 c_i = 0$$

$$\frac{\sum_j e^{z_j+q_i+a_{ji}}}{\sum_{ij} e^{z_i+q_j+a_{ij}}} + \nu_2 d_i = 0$$

for $\forall\; i \in [1,n]$.

Let $B = \frac{1}{x^TAy} \operatorname{diag}(x) A \operatorname{diag}(y)$.

$B\mathbf{1} = \frac{1}{x^TAy} \operatorname{diag}(x) Ay$, $B^T\mathbf{1} = \frac{1}{x^TAy} \operatorname{diag}(y) A^T x$

$$\sum_j e^{z_i+q_j+a_{ij}} + \nu_1 c_i \sum_{i,j} e^{z_i+q_j+a_{ij}} = 0$$

$\Rightarrow x_i \sum_j A_{ij} y_j + \nu_1 c_i \sum_{i,j} A_{ij} x_i y_j = 0, \forall i$

$\Rightarrow x_i (Ay)_i + \nu_1 c_i\, x^TAy = 0, \forall i$

$\Rightarrow \frac{1}{x^TAy} x_i (Ay)_i = -\nu_1 c_i$

$\Rightarrow \frac{1}{x^TAy} \operatorname{diag}(x) Ay = -\nu_1 c$

Likewise, $\sum_j e^{z_j + q_i + a_{ji}} + \nu_2 d_i \sum_{ij} e^{z_i + q_j + a_{ij}} = 0.$

$\Rightarrow \; y_i \sum_j A_{ji} x_j + \nu_2 d_i \sum_{ij} A_{ij} x_i y_j = 0 \quad , \forall i$

$\Rightarrow \; y_i (A^T x)_i + \nu_2 d_i \, x^T A y = 0 \quad , \forall i$

$\Rightarrow \; \frac{1}{x^T A y} \operatorname{diag}(y) A^T x + \nu_2 d = 0$

In summary, the KKT conditions are:

$c^T z = 0, \; d^T q = 0,$

$\frac{1}{x^T A y} \operatorname{diag}(x) A y = -\nu_1 c$

$\frac{1}{x^T A y} \operatorname{diag}(y) A^T x = -\nu_2 d$

$B\mathbb{1} = \frac{1}{x^T A y} \operatorname{diag}(x) A y = -\nu_1 c$

$B^T \mathbb{1} = \frac{1}{x^T A y} \operatorname{diag}(y) A^T x = -\nu_2 c$

Let $\nu_1 = -1$ and $\nu_2 = -1$, since there are no restrictions on ν_1 & ν_2, and we obtain the result.

(16)

$\boxed{7\ 5.29.}$

$$\text{minimize} \quad -3x_1^2 + x_2^2 + 2x_3^2 + 2(x_1 + x_2 + x_3)$$
$$\text{s.t.} \quad x_1^2 + x_2^2 + x_3^2 = 1$$

KKT Conditions:

① $x_1^2 + x_2^2 + x_3^2 - 1 = 0$

② & ③ are automatically satisfied since there are no inequality constraints.

Dual: $\mathcal{L}(x_1, x_2, x_3, \nu) = -3x_1^2 + x_2^2 + 2x_3^2 + 2(x_1 + x_2 + x_3) + \nu(x_1^2 + x_2^2 + x_3^2 - 1)$

$$\nabla f_0(x) = \begin{bmatrix} -6x_1 + 2 \\ 2x_2 + 2 \\ 4x_3 + 2 \end{bmatrix}, \quad \nabla h(x) = 2\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\nabla f_0(x) + \nu h(x) = 0$$

$$\Rightarrow \begin{bmatrix} (2\nu - 6)x_1 \\ (2\nu + 2)x_2 \\ (2\nu + 4)x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} \Rightarrow \begin{bmatrix} (\nu - 3)x_1 \\ (\nu + 1)x_2 \\ (\nu + 2)x_3 \end{bmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So we have 4 eqs. in 4 unknowns:

$x_1 = -1/(\nu - 3)$, $x_2 = -1/(\nu + 1)$, $x_3 = -1/(\nu + 2)$, $x_1^2 + x_2^2 + x_3^2 = 1$

$$\Rightarrow \frac{1}{(\nu-3)^2} + \frac{1}{(\nu+1)^2} + \frac{1}{(\nu+2)^2} - 1 = 0$$

$$\Rightarrow (\nu+1)^2(\nu+2)^2 + (\nu-3)^2(\nu+2)^2 + (\nu-3)^2(\nu+1)^2 - (\nu+1)^2(\nu+2)^2(\nu-3)^2 = 0$$

$$\Rightarrow \nu^6 - 17\nu^4 - 12\nu^3 + 49\nu^2 + 48\nu - 13 = 0.$$

(17)

The four real roots are $\nu = -3.14, 0.22, 1.89, 4.04$.

$$f_0(\nu) = \frac{-3}{(\nu-3)^2} + \frac{1}{(\nu+1)^2} + \frac{2}{(\nu+2)^2} - 2\left(\frac{1}{\nu-3} + \frac{1}{\nu+1} + \frac{1}{\nu+2}\right)$$

$f_0(-3.14) \approx 4.69$, $f_0(0.22) \approx -1.13$, $f_0(1.89) \approx -1.59$, $f_0(4.04) \approx -5.33$.