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ECE 236B - HW 3

A 2.21

(a.) Let $S = \sum_i \theta_i P_i$, $0 \le \theta_i \le 1$, $\sum \theta_i = 1$, and P_i permutation matrices.

$\Rightarrow f(Sx) = f(\theta_1 P_1 x + \dots + \theta_n P_n x)$

$\le \theta_1 f(P_1 x) + \dots + \theta_n f(P_n x)$
(by convexity of f)

$= \theta_1 f(x) + \dots + \theta_n f(x)$
(by symmetry of f)

$= ((\sum_i \theta_i) f(x)) = f(x)$.

(b.) Let $Y = Q\,diag(\lambda)\,Q^T$, Q ortho

$QQ^T = Q^TQ = I$

Let $S_{ij} = Q_{ij}^2$

Then, $\sum_i S_{ij} = \sum_i Q_{ij}^2 = (Q^TQ)_{jj} = 1$.

and $\sum_j S_{ij} = \sum_j Q_{ij}^2 = (QQ^T)_{ii} = 1$.

Thus, S is doubly stochastic

$Y_{ii} = (Q\,diag(\lambda)\,Q^T)_{ii} = \sum_j \lambda_j Q_{ij}^2 = \sum_j \lambda_j S_{ij}$

Thus, $diag(Y) = S\lambda$.

(2)

(c.). Let $\lambda(X) = (\lambda_1(X), \lambda_2(X), \ldots, \lambda_n(X))$.

$diag(V^T X V) = (V^T U^T \lambda(X) U V)$

for some $U^T U = U U^T = I$.

Let $Q = V^T U^T$. Then, $Q^T Q = Q Q^T = I$.

Let $Y = Q \lambda(X) Q^T$.

$diag(Y) = S\lambda$, where $S_{ij} = Q_{ij}^2$ is doubly symmetric, by part (b).

Thus, for all $V \in \mathcal{V}$, we can write $f(diag(V^T X V)) = f(S\lambda(X))$ for some doubly symmetric S.

Suppose $U = V$. Then,

$f(diag(V^T X V)) = f(\lambda(X))$.

By part (a.), $f(S\lambda(X)) \leq f(\lambda(X))$.

It follows that $f(diag(V^T X V)) \leq f(\lambda(X))$.

$\Leftrightarrow \quad f(\lambda(X)) = \sup_{V \in \mathcal{V}} f(diag(V^T X V))$.

Thus, $f(\lambda(\theta_1 P_1 X) + \cdots + \lambda(\theta_n P_n X))$

$= f(\theta_1 \lambda(P_1 X) + \cdots + \theta_n \lambda(P_n X))$

$= f((\theta_1 + \cdots + \theta_n) S X)$ for some S.

$\leq f(\lambda(X))$ by part (c.).

$\theta_1 f(\lambda(P_1 X)) + \cdots + \theta_n f(\lambda(P_n X)) = f(\lambda(X))$

QED.

③

A2.17

Let $g(x) = \inf_{x_1,\dots,x_m} \{ f_1(x_1) + \dots + f_m(x_m) \mid x_1 + \dots + x_m = x \}$.

<u>THRM.</u> If $f(x,y)$ convex in (x,y), C convex, $g(x) = \inf_{y \in C} f(x,y)$ is convex.

(a.)
In this case, we can rewrite g as:

$$g(x) = \inf_{x_1, \dots x_{m-1}} \{ f_1(x_1) + f(x_2) + \dots + f(x_{m-1}) + f(x - \sum_{i=1}^{m-1} x_i) \}$$

Each of the first m-1 functions are convex, and the last function is jointly convex in x, Σx_i. Thus, the sum of these functions is jointly convex in x and $x_1, \dots, x_{m-1}$. Thus, by the theorem, $g(x)$ is convex.

(b.) Show that $g^* = f_1^* + \dots + f_m^*$.

<u>Def.</u> $f^*(y) = \sup_{x \in \mathrm{dom} f} (y^T x - f(x))$

$$g^*(y) = \sup_{x \mid g(x) < \infty} (y^T x - g(x))$$

$$= \sup_x (y^T x - \inf_x \{ f_1(x_1) + \dots + f_m(x_m) \mid \Sigma x_i = x \}$$

$$= \sup_x \left(y^T x + \sup_x \{ -f_1(x_1) - \dots - f_m(x_m) \mid \Sigma x_i = x \} \right)$$

$$= \sup_x \left[\sum_{i=1}^m (y^T x_i - f_i(x_i)) \right]$$

$$= \sum_{i=1}^m \sup_{x_i} (y^T x_i - f_i(x_i)) = \sum_{i=1}^m f_i^*(y).$$

④

$\boxed{A2.5}$ Let f be convex, $f(0) \leq 0$.

Let $g(x,t) = t f(x/t)$, with

$\operatorname{dom}(g) = \{(x,t) \mid t > 0,\ x/t \in \operatorname{dom} f\}$.

(a.) Show g is nonincreasing wrt t.

By convexity of f,

$t f\left(\frac{1}{t}x_1 + (1-\frac{1}{t})x_2\right) \leq t\left(\frac{1}{t} f(x_1) + (1-\frac{1}{t}) f(x_2)\right)$

$= f(x_1) + (t-1) f(x_2)$, where $t \geq 1$.

Let $x_2 = 0$. Then,

$t f(x/t) \leq f(x) + (t-1) f(0)$.

and $f(0) \leq 0$. Thus,

$g(x,t)$ is a nonincreasing function of t for all $t \geq 1$.

For $0 < t < 1$, we have that

$f(t x_1 + (1-t) x_2) \leq t f(x_1) + (1-t) f(x_2)$

?

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(b.) Let g be concave & $g > 0$.
Show that $h(x) = g(x)\, f(x/g(x))$,
$\text{dom } h = \{x \in \text{dom } g \mid x/g(x) \in \text{dom } f\}$ is convex.

Let $q(x,t) = t\, f(x/t)$.
$h(x) = q(x, g(x)) = q(g'(x))$,
where $g'(x) = [x \;\; g(x)]$.

Now $g(x)$ is concave, so $g'(x)$ is as well (since a linear function is concave).

Since q is convex and nonincreasing, by part (a.), $h(x)$ is convex.

$\boxed{\text{A 2.48}}$ Explain why the following are convex.

(a.) $f(x) = \cosh(\|x\|)$, where $\cosh(u) = (\exp(u) + \exp(-$

We can write $f(x) = h(g(x)) = K(\ell(x)) + K(-\ell(x))$, and use composition rules.
$\|x\|$ is convex.
$\exp(u)$ and $\exp(-u)$ are both convex, so their sum is as well, so $\cosh(u)$ is convex. However, this doesn't help.
Also, $e^{-\|x\|}$ is concave, so we can't consider the two terms one by one.

Instead let's consider $x \geq 0$ & $x \leq 0$ separately.

For $x \geq 0$, $\cosh(x)$ is convex and non-decreasing and $\|x\|$ is convex. Thus, $f(x)$ is convex for $x \geq$

Likewise, $\cosh(x)$ is convex and non-increasing for $x \leq 0$, and $-\|x\|$ is concave, so $f(x)$ is convex for $x \leq 0$.

Finally $f(x)$ is a continuous function. I believe this suffices to show that $f(x)$ is convex for all x.

$\boxed{\text{T 3.55}}$ Let $g(t) = \exp(-h(t))$ (a differentiable log-concave pdf).

Let $f(x) = \int_{-\infty}^{x} g(t)\,dt = \int_{-\infty}^{x} e^{-h(t)}\,dt$ be its cdf.

Show that $f''(x) f(x) \leq (f'(x))^2$, $\forall x$ (and thus f is log concave).

(a) $\frac{d}{dx} \int_{g(x)}^{f(x)} h(t)\,dt = h(f(x))f'(x) - h(g(x))g'(x)$. (Fundamental thm. of calculus)

$$f'(x) = \frac{d}{dx} \int_{-\infty}^{x} e^{-h(t)}\,dt = e^{-h(x)}.$$

$$f''(x) = \frac{d}{dx} e^{-h(x)} = -h'(x)\, e^{-h(x)}.$$

$$f''(x) f(x) = -h'(x)\, e^{-h(x)} \int_{-\infty}^{x} e^{-h(t)}\,dt \leq 0 \text{ if } h'(x) \geq 0.$$

$$(f'(x))^2 = e^{-2h(x)} \geq 0.$$

Thus, $f''(x) f(x) \leq (f'(x))^2$ if $h'(x) \geq 0$.

⑦

(b.) Assume that $h'(x) < 0$. Use the inequality $h(t) \geq h(x) + h'(x)(t-x)$ (which follows from convexity of h), to show that $\int_{-\infty}^{x} e^{-h(t)}\,dt \leq \frac{e^{-h(x)}}{-h'(x)}$. Verify that $f''(x)f(x) \leq (f'(x))^2$.

Using the inequality,

$e^{-h(t)} \leq e^{-h(x)} e^{-h'(x)(t-x)}$ (Multiplying by -1 and exponentiating)

$$\int_{-\infty}^{x} e^{-h(t)}\,dt \leq e^{-h(x)} \int_{-\infty}^{x} e^{-h'(x)(t-x)}\,dt$$

$$= e^{-h(x)} e^{h'(x)x} \int_{-\infty}^{x} e^{-h'(x)t}\,dt = e^{-h(x)} e^{h'(x)x} \frac{1}{-h'(x)} e^{-h'(x)t}\Big|_{-\infty}^{x}$$

$$= e^{-h(x)} \cancel{e^{h'(x)x}} \frac{1}{-h'(x)} \cancel{e^{-h'(x)x}} = -\frac{e^{-h(x)}}{h'(x)}$$

That is, $\int_{-\infty}^{x} e^{-h(t)}\,dt \leq \frac{e^{-h(x)}}{-h'(x)}$.

$$\Rightarrow -h'(x)e^{-h(x)} \int_{-\infty}^{x} e^{-h(t)}\,dt \leq e^{-2h(x)}$$

(Since $h'(x) < 0$).

$\Rightarrow f''(x)f(x) \leq (f'(x))^2$ if $h'(x) < 0$.

⑧

A 3.17 Minimum-fuel optimal control.

Consider a linear dynamical system

$X(t+1) = AX(t) + bu(t)$, $t = 0, \dots, N-1$,
$x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $x(0) = 0$.

Find $u(0), \dots, u(N-1)$ to minimize
$F = \sum_{t=0}^{N-1} f(u(t))$ s.t. $X(N) = X_{des}$,

$$f(a) = \begin{cases} |a| , & |a| \leq 1 \\ 2|a| - 1 , & |a| > 1 . \end{cases}$$

Formulate this as an LP.

$$\min_u \sum_{t=0}^{N-1} \max\{|u(t)|, 2|u(t)| - 1\}$$

$$\text{s.t. } X(t+1) = AX(t) + bu(t), \forall t \in [0, N]$$
$$X(N) = X_{des}$$

$$\Updownarrow$$

$$\min_u \sum_{t=0}^{N-1} a(t)$$

$$\text{s.t. } \left.\begin{array}{l} |u(t)| \leq a(t) \\ 2|u(t)| - 1 \leq a(t) \end{array}\right\} \forall t \in [0, N].$$
$$X(t+1) = Ax(t) + bu(t)$$
$$X(N) = X_{des}$$

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0.3 \end{bmatrix}, \quad X_{des} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \quad N = 30.$$

In [1]:
```
import numpy as np
import cvxpy as cp
import matplotlib.pyplot as plt
```

In [35]:
```
# Data for control problem.
A = np.array([[-1, 0.4, 0.8], [1,0,0],[0,1,0]])
B = np.array([1, 0, 0.3])
x_des = np.array([7, 2, -6])
x_0 = np.zeros(3)
N = 30
n = 3
```

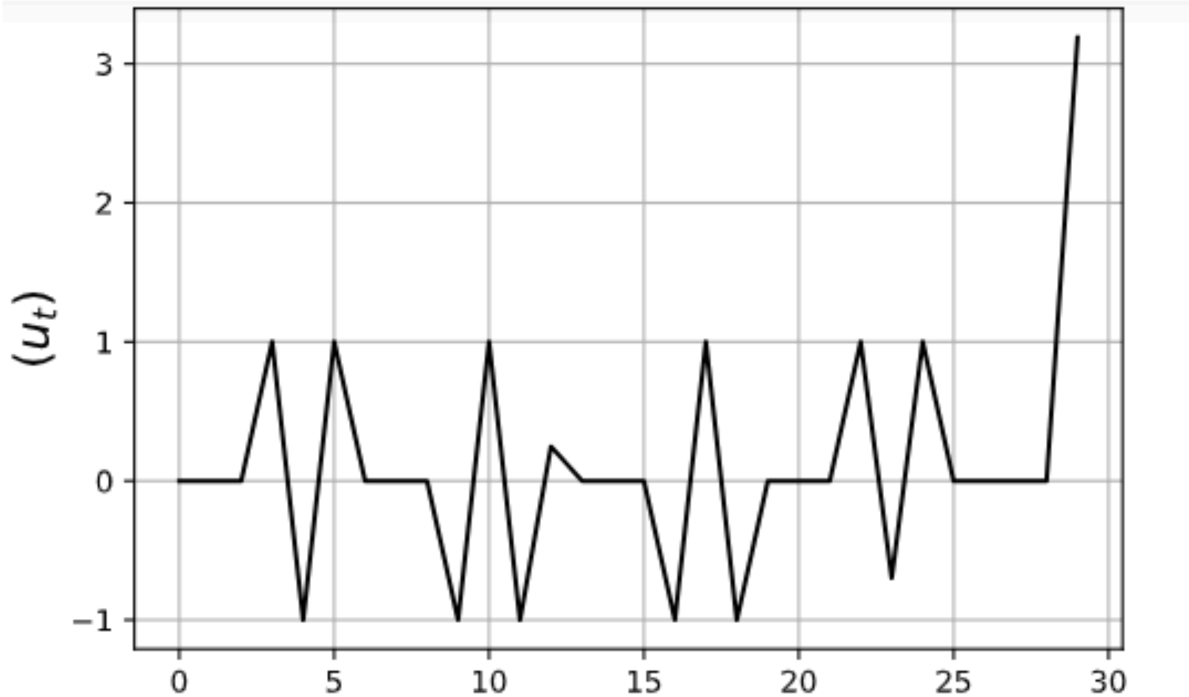
In [45]:
```
# Form and solve control problem (formulation 1)

x = cp.Variable((n, N+1))
u = cp.Variable((N))

cost = 0
constr = []
for t in range(N):
    cost += cp.maximum(cp.norm(u[t],1),2*cp.norm(u[t],1)-1)
    constr += [x[:,t+1] == A@x[:,t] + B*u[t]]
# sums problem objectives and concatenates constraints.
constr += [x[:,N] == x_des, x[:,0] == x_0]
problem = cp.Problem(cp.Minimize(cost), constr)
problem.solve(solver=cp.ECOS)
```

Out[45]:
```
17.323567851898538
```

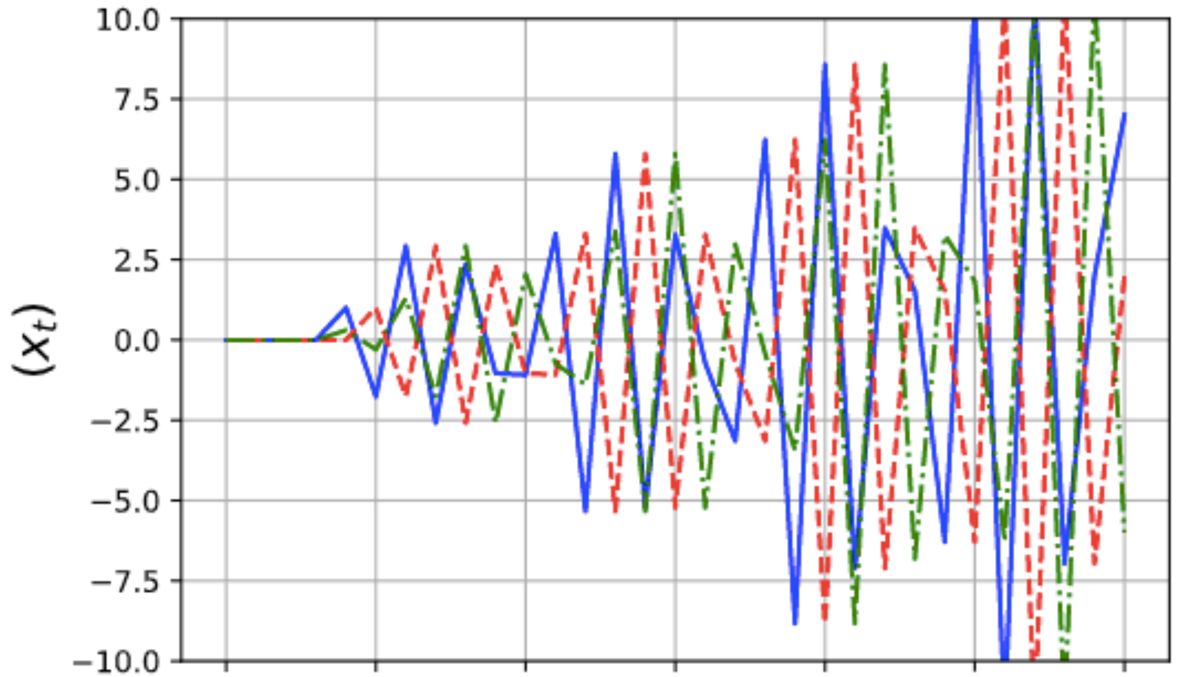
In [42]:
```
# Form and solve control problem (formulation 2)

x = cp.Variable((n, N+1))
u = cp.Variable((N))
a = cp.Variable((N+1))

cost = 0
constr = []
for t in range(N):
    cost += a[t]
    constr += [cp.norm(u[t],1) <= a[t],
               2*cp.norm(u[t],1)-1 <= a[t],
               x[:,t+1] == A@x[:,t] + B*u[t]]
# sums problem objectives and concatenates constraints.
constr += [x[:,N] == x_des, x[:,0] == x_0]
problem = cp.Problem(cp.Minimize(cost), constr)
problem.solve(solver=cp.ECOS)
```

Out[42]:
```
17.323567851898535
```

In [68]:
```
# Plot results.
import matplotlib.pyplot as plt

# Plot (u_t)_1.
plt.plot(u.value,'k')
plt.ylabel(r"$(u_t)$", fontsize=16)
plt.grid()
plt.show()

# Plot (x_t)_1.
x1 = x[0,:].value
plt.plot(x1,'b')
plt.ylabel(r"$(x_t)_1$", fontsize=16)
plt.ylim([-10, 10])
plt.grid()

# Plot (x_t)_2.
x2 = x[1,:].value
plt.plot(x2,'r',linestyle='--')
plt.ylabel(r"$(x_t)_2$", fontsize=16)
plt.ylim([-10, 10])
plt.grid()

# Plot (x_t)_3.
x3 = x[2,:].value
plt.plot(x3,'g',linestyle='-.')
plt.ylabel(r"$(x_t)_3$", fontsize=16)
plt.ylim([-10, 10])
plt.grid()
plt.show()
```



(u_t)
3
2
1
0
−1
0
5
10
15
20
25
30



(x_t)
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0