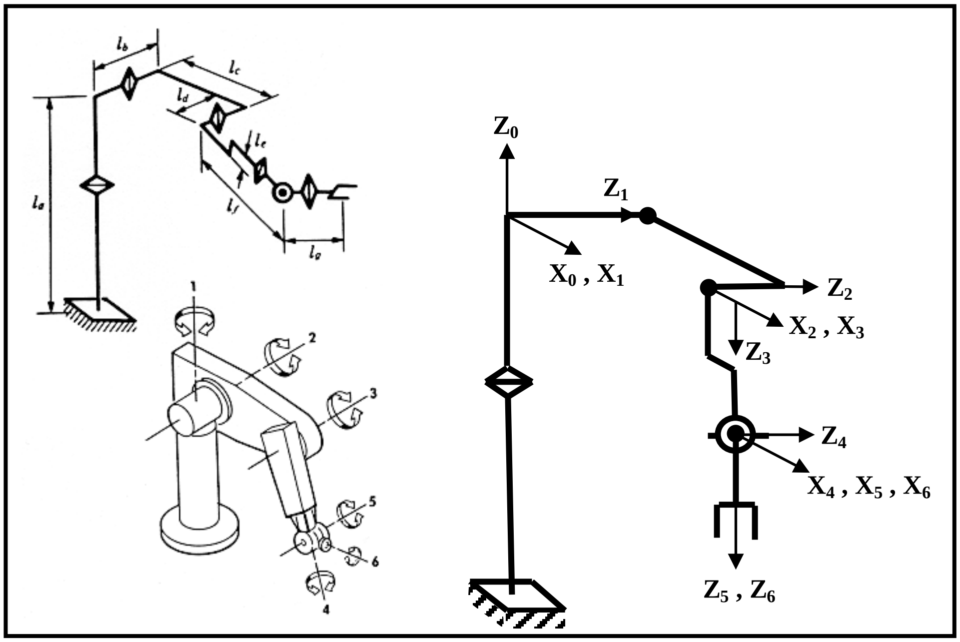
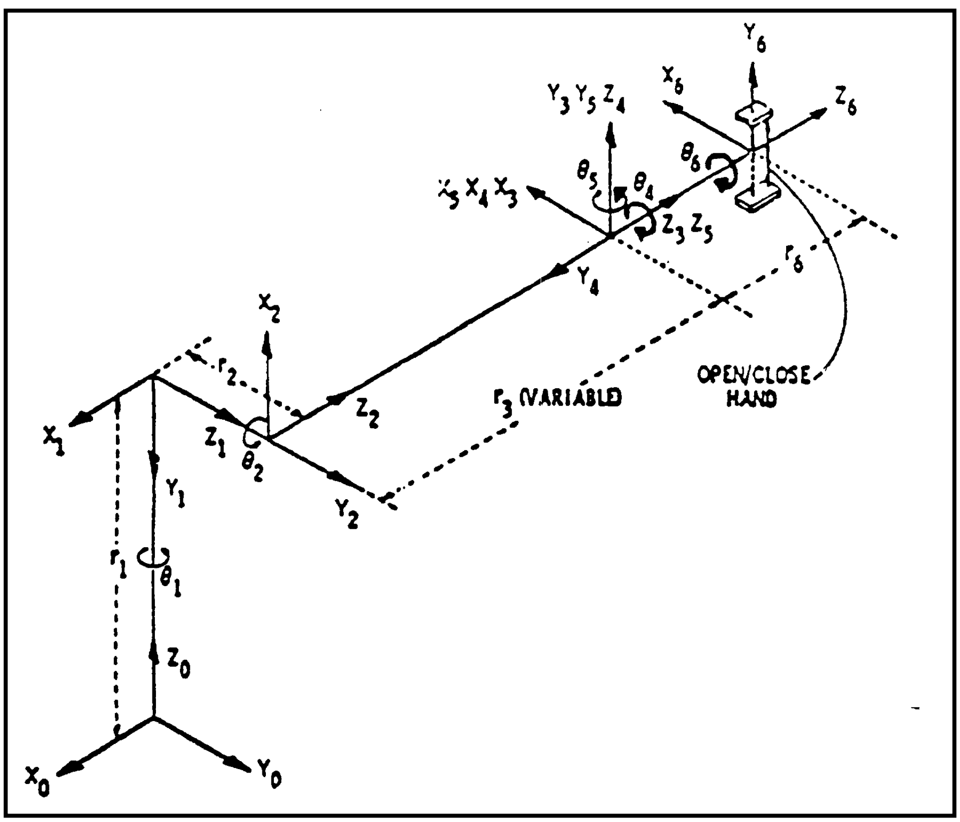
ME4524 – Robotics and Automation
Exercise # 3 - Solutions

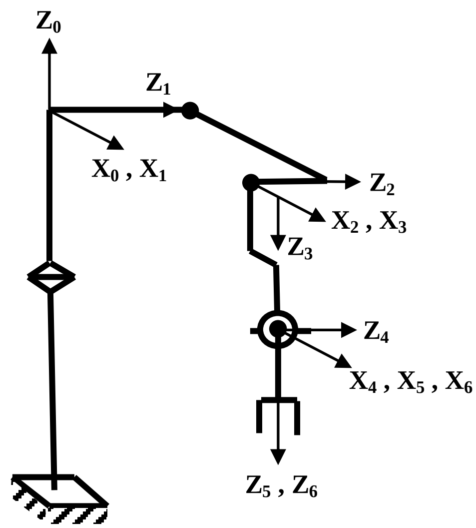
1. Obtain the Jacobian $J_0\,(q)$ for: **(i)** the PUMA 560 manipulator; and **(ii)** the Stanford manipulator shown in the Figure below.





Answer:

(i) Here are the frame assignments and joint varaibles for Standard D-H convention.



Joint	a_i	α_i	d_i	θ_i
1	0	-90°	0	θ_1 (0°)
2	l_c	0°	(l_b-l_d)	θ_2 (0°)
3	l_e	-90°	0	θ_3 (0°)
4	0	90°	l_f	θ_4 (0°)
5	0	-90°	0	θ_5 (0°)
6	0	0°	0	θ_6 (0°)

$$H_{01} = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{12} = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_C c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & l_c s\theta_2 \\ 0 & 0 & 1 & l_b - l_d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{23} = \begin{bmatrix} c\theta_3 & 0 & -s\theta_3 & l_e c\theta_3 \\ s\theta_3 & 0 & c\theta_3 & l_e s\theta_3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{34} = \begin{bmatrix} c\theta_4 & 0 & s\theta_4 & 0 \\ s\theta_4 & 0 & -c\theta_4 & 0 \\ 0 & 1 & 0 & l_f \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{45} = \begin{bmatrix} c\theta_5 & 0 & -s\theta_5 & 0 \\ s\theta_5 & 0 & c\theta_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{56} = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{02} = \begin{bmatrix} c\theta_1 c\theta_2 & -c\theta_1 s\theta_2 & -s\theta_1 & l_c c\theta_1 c\theta_2 - (l_b - l_d) s\theta_1 \\ s\theta_1 c\theta_2 & -s\theta_1 s\theta_2 & c\theta_1 & l_c s\theta_1 c\theta_2 + (l_b - l_d) c\theta_1 \\ -s\theta_2 & -c\theta_2 & 0 & -l_c s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{03} = \begin{bmatrix} c\theta_1 c\theta_{23} & s\theta_1 & -c\theta_1 s\theta_{23} & l_e c\theta_1 c\theta_{23} + l_c c\theta_1 c\theta_2 - (l_b - l_d) s\theta_1 \\ s\theta_1 c\theta_{23} & -c\theta_1 & -s\theta_1 s\theta_{23} & l_e s\theta_1 c\theta_{23} + l_c s\theta_1 c\theta_2 + (l_b - l_d) c\theta_1 \\ -s\theta_{23} & 0 & -c\theta_{23} & -l_e s\theta_{23} - l_c s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{04} = \begin{bmatrix} c\theta_1 c\theta_{23} c\theta_4 + s\theta_1 s\theta_4 & -c\theta_1 s\theta_{23} & c\theta_1 c\theta_{23} s\theta_4 - s\theta_1 c\theta_4 & l_e c\theta_1 c\theta_{23} + l_c c\theta_1 c\theta_2 - (l_b - l_d) s\theta_1 - l_f c\theta_1 s\theta_{23} \\ s\theta_1 c\theta_{23} c\theta_4 - c\theta_1 s\theta_4 & -s\theta_1 s\theta_{23} & s\theta_1 c\theta_{23} s\theta_4 + c\theta_1 c\theta_4 & l_e s\theta_1 c\theta_{23} + l_c s\theta_1 c\theta_2 + (l_b - l_d) c\theta_1 - l_f s\theta_1 s\theta_{23} \\ -s\theta_{23} c\theta_4 & -c\theta_{23} & -s\theta_{23} s\theta_4 & -l_e s\theta_{23} - l_c s\theta_2 - l_f c\theta_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{05} = \begin{bmatrix} (c\theta_1 c\theta_{23} c\theta_4 + s\theta_1 s\theta_4) c\theta_5 - c\theta_1 s\theta_{23} s\theta_5 & s\theta_1 c\theta_4 - c\theta_1 c\theta_{23} s\theta_4 & (c\theta_1 c\theta_{23} c\theta_4 + s\theta_1 s\theta_4)(-s\theta_5) - c\theta_1 s\theta_{23} c\theta_5 & X \\ (s\theta_1 c\theta_{23} c\theta_4 - c\theta_1 s\theta_4) c\theta_5 - s\theta_1 s\theta_{23} s\theta_5 & -s\theta_1 c\theta_{23} s\theta_4 - c\theta_1 c\theta_4 & (s\theta_1 c\theta_{23} c\theta_4 - c\theta_1 s\theta_4)(-s\theta_5) - s\theta_1 s\theta_{23} c\theta_5 & Y \\ -s\theta_{23} c\theta_4 c\theta_5 - c\theta_{23} s\theta_5 & s\theta_{23} s\theta_4 & s\theta_{23} c\theta_4 s\theta_5 - c\theta_{23} c\theta_5 & Z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where X,Y and Z do not change from T_{04}. We now have enough information to construct J.

$$\frac{\partial P_{06}}{\partial \theta_1} = \begin{bmatrix} -l_e s\theta_1 c\theta_{23} - l_c s\theta_1 c\theta_2 - (l_b - l_d) c\theta_1 + l_f s\theta_1 s\theta_{23} \\ l_e c\theta_1 c\theta_{23} + l_c c\theta_1 c\theta_2 - (l_b - l_d) s\theta_1 - l_f c\theta_1 s\theta_{23} \\ 0 \end{bmatrix} \qquad \frac{\partial P_{06}}{\partial \theta_2} = \begin{bmatrix} -l_e c\theta_1 s\theta_{23} - l_c c\theta_1 s\theta_2 - l_f c\theta_1 c\theta_{23} \\ -l_e s\theta_1 s\theta_{23} - l_c s\theta_1 s\theta_2 - l_f s\theta_1 c\theta_{23} \\ -l_e c\theta_{23} - l_c c\theta_2 + l_f s\theta_{23} \end{bmatrix}$$

$$\frac{\partial P_{06}}{\partial \theta_3} = \begin{bmatrix} -l_e c\theta_1 s\theta_{23} - l_f c\theta_1 c\theta_{23} \\ -l_e s\theta_1 s\theta_{23} - l_f s\theta_1 c\theta_{23} \\ -l_e c\theta_{23} + l_f s\theta_{23} \end{bmatrix} \qquad \frac{\partial P_{06}}{\partial \theta_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial P_{06}}{\partial \theta_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\partial P_{06}}{\partial \theta_6} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} -s\theta_1 \\ c\theta_1 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} -s\theta_1 \\ c\theta_1 \\ 0 \end{bmatrix} \quad z_3 = \begin{bmatrix} -c\theta_1 s\theta_{23} \\ -s\theta_1 s\theta_{23} \\ -c\theta_{23} \end{bmatrix} \quad z_4 = \begin{bmatrix} c\theta_1 c\theta_{23} s\theta_4 - s\theta_1 c\theta_4 \\ s\theta_1 c\theta_{23} s\theta_4 + c\theta_1 c\theta_4 \\ -s\theta_{23} s\theta_4 \end{bmatrix}$$

$$z_5 = \begin{bmatrix} -(c\theta_1 c\theta_{23} c\theta_4 + s\theta_1 s\theta_4) s\theta_5 - c\theta_1 s\theta_{23} c\theta_5 \\ -(s\theta_1 c\theta_{23} c\theta_4 - c\theta_1 s\theta_4) s\theta_5 - s\theta_1 s\theta_{23} c\theta_5 \\ s\theta_{23} c\theta_4 s\theta_5 - c\theta_{23} c\theta_5 \end{bmatrix}$$

$$J_0 = \begin{bmatrix} \frac{\partial P_{06}}{\partial \theta_1} & \frac{\partial P_{06}}{\partial \theta_2} & \frac{\partial P_{06}}{\partial \theta_3} & \frac{\partial P_{06}}{\partial \theta_4} & \frac{\partial P_{06}}{\partial \theta_5} & \frac{\partial P_{06}}{\partial \theta_6} \\ z_0 & z_1 & z_2 & z_3 & z_4 & z_5 \end{bmatrix}$$

Part (ii): The diagram given above for the Stanford manipulator is consistent with standard DH convention. Note that the end effector coordinate frame is assigned in a slightly different manner than what is in your notes.

Joint	a_i	α_i	d_i	θ_i
1	0	-90°	r_1	θ_1
2	0	90°	r_2	θ_2-90°
3	0	0	r_3	-90°
4	0	-90°	0	θ_4
5	0	90°	0	θ_5
6	0	0	r_6	θ_6

$$T_{01} = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & 0 \\ s\theta_1 & 0 & c\theta_1 & 0 \\ 0 & -1 & 0 & r_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} c(\theta_2 - 90) & 0 & s(\theta_2 - 90) & 0 \\ s(\theta_2 - 90) & 0 & -c(\theta_2 - 90) & 0 \\ 0 & 1 & 0 & r_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s\theta_2 & 0 & -c\theta_2 & 0 \\ -c\theta_2 & 0 & -s\theta_2 & 0 \\ 0 & 1 & 0 & r_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{23}=\begin{bmatrix}0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 1 & r_3\\ 0 & 0 & 0 & 1\end{bmatrix} T_{34}=\begin{bmatrix}c\theta_4 & 0 & -s\theta_4 & 0\\ s\theta_4 & 0 & c\theta_4 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} T_{45}=\begin{bmatrix}c\theta_5 & 0 & s\theta_5 & 0\\ s\theta_5 & 0 & -c\theta_5 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$T_{56}=\begin{bmatrix}c\theta_6 & -s\theta_6 & 0 & 0\\ s\theta_6 & c\theta_6 & 0 & 0\\ 0 & 0 & 1 & r_6\\ 0 & 0 & 0 & 1\end{bmatrix} T_{02}=\begin{bmatrix}c\theta_1 s\theta_2 & -s\theta_1 & -c\theta_1 c\theta_2 & -r_2 s\theta_1\\ s\theta_1 s\theta_2 & c\theta_1 & -s\theta_1 c\theta_2 & r_2 c\theta_1\\ c\theta_2 & 0 & s\theta_2 & r_1\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$T_{03}=\begin{bmatrix}s\theta_1 & c\theta_1 s\theta_2 & -c\theta_1 c\theta_2 & -r_2 s\theta_1 - r_3 c\theta_1 c\theta_2\\ -c\theta_1 & s\theta_1 s\theta_2 & -s\theta_1 c\theta_2 & r_2 c\theta_1 - r_3 s\theta_1 c\theta_2\\ 0 & c\theta_2 & s\theta_2 & r_1 + r_3 s\theta_2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$T_{04}=\begin{bmatrix}s\theta_1 c\theta_4 + c\theta_1 s\theta_2 s\theta_4 & c\theta_1 c\theta_2 & -s\theta_1 s\theta_4 + c\theta_1 s\theta_2 c\theta_4 & -r_2 s\theta_1 - r_3 c\theta_1 c\theta_2\\ -c\theta_1 c\theta_4 + s\theta_1 s\theta_2 s\theta_4 & s\theta_1 c\theta_2 & c\theta_1 s\theta_4 + s\theta_1 s\theta_2 c\theta_4 & r_2 c\theta_1 - r_3 s\theta_1 c\theta_2\\ c\theta_2 s\theta_4 & -s\theta_2 & c\theta_2 c\theta_4 & r_1 + r_3 s\theta_2\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$T_{05} = \begin{bmatrix} (s\theta_1 c\theta_4 + c\theta_1 s\theta_2 s\theta_4)c\theta_5 + c\theta_1 c\theta_2 s\theta_5 & -s\theta_1 s\theta_4 + c\theta_1 s\theta_2 c\theta_4 & (s\theta_1 c\theta_4 + c\theta_1 s\theta_2 s\theta_4)s\theta_5 - c\theta_1 c\theta_2 c\theta_5 & -r_2 s\theta_1 - r_3 c\theta_1 c\theta_2 \\ (-c\theta_1 c\theta_4 + s\theta_1 s\theta_2 s\theta_4)c\theta_5 + s\theta_1 c\theta_2 s\theta_5 & c\theta_1 s\theta_4 + s\theta_1 s\theta_2 c\theta_4 & (-c\theta_1 c\theta_4 + s\theta_1 s\theta_2 s\theta_4)s\theta_5 - s\theta_1 c\theta_2 c\theta_5 & r_2 c\theta_1 - r_3 s\theta_1 c\theta_2 \\ c\theta_2 s\theta_4 c\theta_5 - s\theta_2 s\theta_5 & c\theta_2 c\theta_4 & c\theta_2 s\theta_4 s\theta_5 + s\theta_2 c\theta_5 & r_1 + r_3 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The math gets messy from this point if we assume that r_6 has a non-zero length. Since the manipulator has a spherical wrist we can simplify things by assuming that $r_6=0$.

$$\frac{\partial P_{06}}{\partial \theta_1} = \begin{bmatrix} -r_2 c\theta_1 + r_3 s\theta_1 c\theta_2 \\ -r_2 s\theta_1 - r_3 c\theta_1 c\theta_2 \\ 0 \end{bmatrix} \quad \frac{\partial P_{06}}{\partial \theta_2} = \begin{bmatrix} r_3 c\theta_1 s\theta_2 \\ r_3 s\theta_1 s\theta_2 \\ r_3 c\theta_2 \end{bmatrix} \quad \frac{\partial P_{06}}{\partial r_3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\text{, since the link is prismatic, } \frac{\partial P_{06}}{\partial \theta_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \frac{\partial P_{06}}{\partial \theta_5} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \frac{\partial P_{06}}{\partial \theta_6} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} -s\theta_1 \\ c\theta_1 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} -c\theta_1 c\theta_2 \\ -s\theta_1 c\theta_2 \\ s\theta_2 \end{bmatrix} \quad z_3 = \begin{bmatrix} -c\theta_1 c\theta_2 \\ -s\theta_1 c\theta_2 \\ s\theta_2 \end{bmatrix} \quad z_4 = \begin{bmatrix} -s\theta_1 s\theta_4 + c\theta_1 s\theta_2 c\theta_4 \\ c\theta_1 s\theta_4 + s\theta_1 s\theta_2 c\theta_4 \\ c\theta_2 c\theta_4 \end{bmatrix}$$

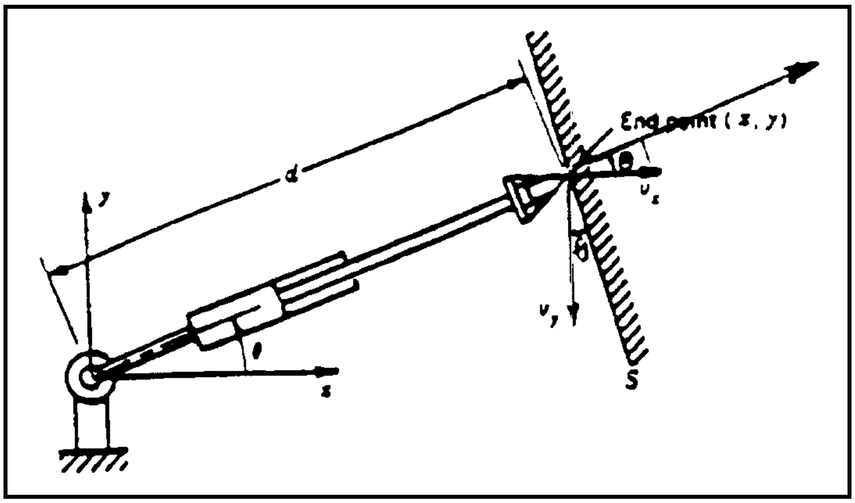
$$z_5 = \begin{bmatrix} (s\theta_1 c\theta_4 + c\theta_1 s\theta_2 s\theta_4)s\theta_5 - c\theta_1 c\theta_2 c\theta_5 \\ (-c\theta_1 c\theta_4 + s\theta_1 s\theta_2 s\theta_4)s\theta_5 - s\theta_1 c\theta_2 c\theta_5 \\ c\theta_2 s\theta_4 s\theta_5 + s\theta_2 c\theta_5 \end{bmatrix}$$

$$J_0 = \begin{bmatrix} \frac{\partial P_{06}}{\partial \theta_1} & \frac{\partial P_{06}}{\partial \theta_2} & z_2 & \frac{\partial P_{06}}{\partial \theta_4} & \frac{\partial P_{06}}{\partial \theta_5} & \frac{\partial P_{06}}{\partial \theta_6} \\ z_0 & z_1 & 0 & z_3 & z_4 & z_5 \end{bmatrix}$$

2. A planar manipulator consisting of a revolute and a prismatic joint is shown in the figure below.

(a) Derive the Jocobian matirx $J_0\,(q)$ associated with the coordinate transformation from joint displacements θ and d to the end-point position x and y.

(b) The end-effector is required to move along the flat surface **S** at a constant velocity $(v_x,\ v_y)$. Compute the corresponding joint velocities and accelerations in terms of the joint displacements.



<u>Answer:</u> Part a)

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} d\cos\theta \\ d\sin\theta \end{bmatrix}$$

Take the differential with respect to time (note that both d and θ change with respect to time.)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \dot{d}\cos\theta - d\,\dot{\theta}\sin\theta \\ \dot{d}\sin\theta + d\,\dot{\theta}\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -d\sin\theta & \cos\theta \\ d\cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{d} \end{bmatrix}$$

$$\text{Therefore } J_0 = \begin{bmatrix} -d\sin\theta & \cos\theta \\ d\cos\theta & \sin\theta \end{bmatrix}$$

Part b): Velocity:

$$\begin{bmatrix} \dot{\theta} \\ \dot{d} \end{bmatrix} = J_0^{-1} \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} \dfrac{-\sin\theta}{d} & \dfrac{\cos\theta}{d} \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

The signs on $\begin{bmatrix} v_x \\ -v_y \end{bmatrix}$ come from the sign convention in the diagram.

Acceleration:

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{d} \end{bmatrix} = \dot{J}_0^{-1} \begin{bmatrix} v_x \\ -v_y \end{bmatrix} + J_0^{-1} \begin{bmatrix} \dot{v}_x \\ -\dot{v}_y \end{bmatrix}$$

Because we have constant velocity, $\begin{bmatrix} \dot{v}_x \\ -\dot{v}_y \end{bmatrix} = 0$.

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{d} \end{bmatrix} = \begin{bmatrix} \dfrac{-\dot{\theta}\cos\theta}{d} + \dfrac{\dot{d}\sin\theta}{d^2} & \dfrac{-\dot{\theta}\sin\theta}{d} - \dfrac{\dot{d}\cos\theta}{d^2} \\ -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta} \\ \ddot{d} \end{bmatrix} = \begin{bmatrix} \dfrac{-\dot{\theta}\cos\theta}{d} + \dfrac{\dot{d}\sin\theta}{d^2} & \dfrac{-\dot{\theta}\sin\theta}{d} - \dfrac{\dot{d}\cos\theta}{d^2} \\ -\dot{\theta}\sin\theta & \dot{\theta}\cos\theta \end{bmatrix} \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

We can separate this into two equations.

$$\begin{bmatrix} \ddot{\theta} \end{bmatrix} = \left[\begin{bmatrix} \frac{-\cos\theta}{d} & \frac{\sin\theta}{d^2} \\ \frac{-\sin\theta}{d} & -\frac{\cos\theta}{d^2} \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{d} \end{bmatrix} \right]^T \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

$$\begin{bmatrix} \ddot{\theta} \end{bmatrix} = \left[\begin{bmatrix} \frac{-\cos\theta}{d} & \frac{\sin\theta}{d^2} \\ \frac{-\sin\theta}{d} & -\frac{\cos\theta}{d^2} \end{bmatrix} \begin{bmatrix} \frac{-\sin\theta}{d} & \frac{\cos\theta}{d} \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} v_x \\ -v_y \end{bmatrix} \right]^T \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

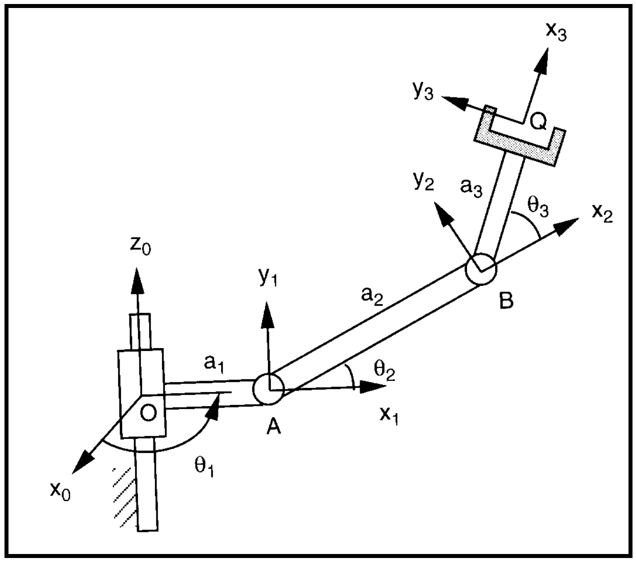
$$\begin{bmatrix} \ddot{\theta} \end{bmatrix} = \begin{bmatrix} v_x & -v_y \end{bmatrix} \begin{bmatrix} \frac{-\sin\theta}{d} & \cos\theta \\ \frac{\cos\theta}{d} & \sin\theta \end{bmatrix} \begin{bmatrix} \frac{-\cos\theta}{d} & \frac{-\sin\theta}{d} \\ \frac{\sin\theta}{d^2} & -\frac{\cos\theta}{d^2} \end{bmatrix} \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

$$\begin{bmatrix} \ddot{d} \end{bmatrix} = \left[\begin{bmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{d} \end{bmatrix} \right]^T \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

$$\begin{bmatrix} \ddot{d} \end{bmatrix} = \left[\begin{bmatrix} -\sin\theta & 0 \\ \cos\theta & 0 \end{bmatrix} \begin{bmatrix} \frac{-\sin\theta}{d} & \frac{\cos\theta}{d} \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} v_x \\ -v_y \end{bmatrix} \right]^T \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

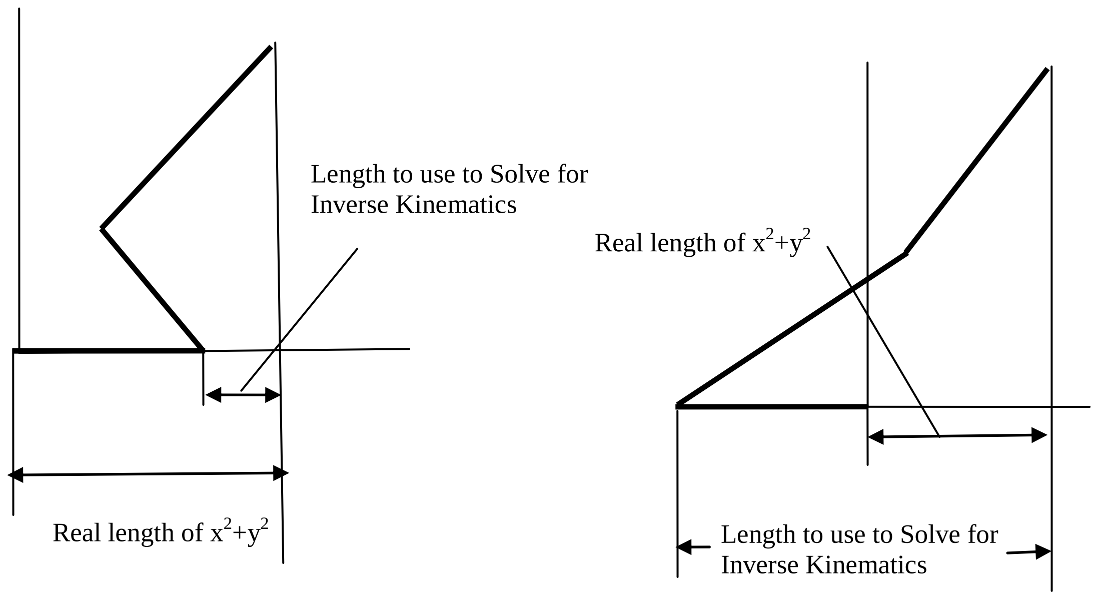
$$\begin{bmatrix} \ddot{d} \end{bmatrix} = \begin{bmatrix} v_x & -v_y \end{bmatrix} \begin{bmatrix} \frac{-\sin\theta}{d} & \cos\theta \\ \frac{\cos\theta}{d} & \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ -v_y \end{bmatrix}$$

3. Find the joint angles needed to bring the end-effector of the manipulator shown in the figure below to a given position. Corresponding to a given position, how many number of possible arm configuartions exist?



Answer: We have some point (x,y,z) that we want to get to.

By inspection of the diagram, we have $\cos\theta_1 = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta_1 = \frac{y}{\sqrt{x^2+y^2}}$, which gives $\theta_1 = a\tan(\frac{y}{x})$. There will be two valid solutions to this equation that are 180° apart from each other. If x = y = 0, there are infinite solutions for θ_1. The remaining two links are planar and so we can use a variation of the planar manipulator solution to solve for these equations. The difference is that we must take into account the effect of a_1 so that we measure the proper length of the horizontal distance. In general, there will be two solutions, one where a_1 is subtracted and one where a_1 is added. We take the absolute value of this term to ensure we get a positive value for cases when $x^2 + y^2 < a_1$. This effectively represents a "shift" of the origin.



$$c\theta_3 = \frac{\left[abs(\sqrt{x^2+y^2} \pm a_1)\right]^2 + z^2 - a_2^2 - a_3^2}{2a_2a_3} = D$$

$$\theta_3 = \tan^{-1}\left(\frac{\pm\sqrt{1-D^2}}{D}\right)$$

We can then solve for θ_2

$$\theta_2 = \tan^{-1}\left(\frac{z}{abs(\sqrt{x^2+y^2} \pm a_1)}\right) - \tan^{-1}\left(\frac{a_3 s\theta_3}{a_2 + a_3 c\theta_3}\right)$$

We have 4 solutions for this system (assuming x and y are not 0). Two valid angles for θ_1, and an elbow up and elbow down configuration for each of those angles.

4. A planar manipulator with three revolute joints is shown in the figure below. $a_1 = 50\sqrt{2} - 50$ cm, $a_2 = 25\sqrt{2}$ cm, $a_3 = 10$ cm.

   (a) Compute the end-point velocities $\dot{x}$, $\dot{y}$, and $\dot{\alpha}$, given that the joint angeles and angular velocities are $\theta_1 = 120$ deg., $\dot{\theta}_1 = 5$ deg/s, $\theta_2 = -120$ deg, $\dot{\theta}_2 = 120$ deg/sec, $\theta_3 = -90$ deg, and $\dot{\theta}_3 = 20$ deg/sec.

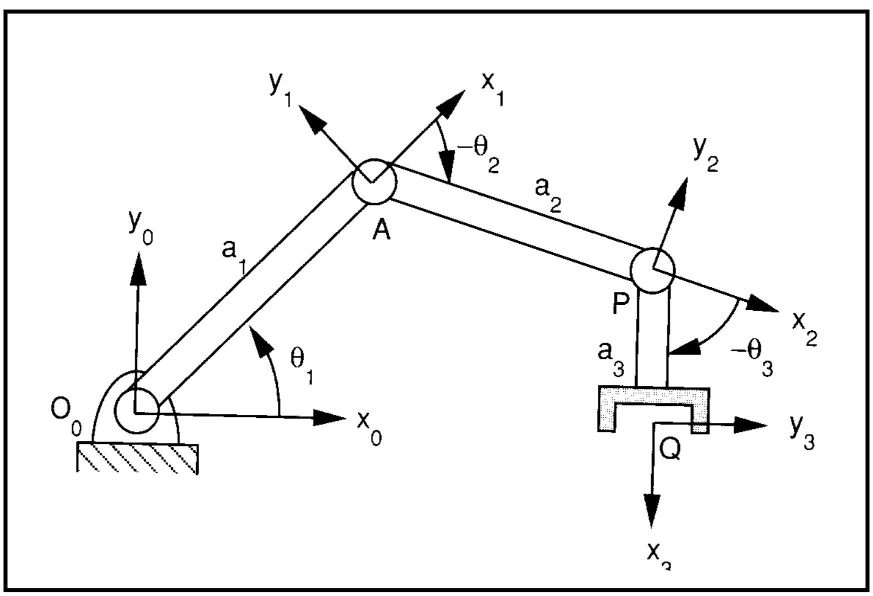
   (b) The end-point is required to move from point A(35, $50\sqrt{2} - 25$) to point B(35, $50\sqrt{2} - 75$) along the y-axis at a constant speed of 10 cm/s. Assuming that link 3 is kept parallel to the x-axis, compute the required angular velocities when the end-point is at A.

   (c) If no condition is imposed on the orientation of link 3 during the motion from A to B, determine the joint velocities at point A that minimizes the following squared norm:

   $$v^2 = \dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + \dot{\theta}_3^{\,2}$$

   Assume that link 3 is parallel to x-axis at A.

   (d) In the motion of part (b), does the manipulator Jacobian become singular? If so, determine the singular configuration and the direction along which the arm cannot move.



Answer: This is a planar 3 link manipulator, and because of how the coordinate frames are assigned it is relatively easy to derive the position equations.

$$x = a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123}$$
$$y = a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123}$$
$$\alpha = \theta_1 + \theta_2 + \theta_3$$

Where α represents the global angular position of the end effector. Take the derivatives with respect to time:

$$\dot{x} = -a_1 \dot{\theta}_1 s\theta_1 - a_2 \dot{\theta}_1 s\theta_{12} - a_3 \dot{\theta}_1 s\theta_{123} - a_2 \dot{\theta}_2 s\theta_{12} - a_3 \dot{\theta}_2 s\theta_{123} - a_3 \dot{\theta}_3 s\theta_{123}$$
$$\dot{y} = a_1 \dot{\theta}_1 c\theta_1 + a_2 \dot{\theta}_1 c\theta_{12} + a_3 \dot{\theta}_1 c\theta_{123} + a_2 \dot{\theta}_2 c\theta_{12} + a_3 \dot{\theta}_2 c\theta_{123} + a_3 \dot{\theta}_3 c\theta_{123}$$
$$\dot{\alpha} = \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3$$

The Jacobian matrix can be extracted from the above equations.

$$J_0 = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_{12} - a_3 s\theta_{123} & -a_2 s\theta_{12} - a_3 s\theta_{123} & -a_3 s\theta_{123} \\ a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} & a_2 c\theta_{12} + a_3 c\theta_{123} & a_3 c\theta_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

a) $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = J_0(\theta)\dot{\theta}.$$

$$\theta_1 = 120^\circ \quad \dot{\theta}_1 = \frac{\pi}{36}$$
$$\theta_2 = -120^\circ \quad \dot{\theta}_2 = \frac{2\pi}{3}$$
$$\theta_3 = -90^\circ \quad \dot{\theta}_3 = \frac{\pi}{9}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} (50-50\sqrt{2})s(120) - 25\sqrt{2}s(0) - 10s(-90) & -25\sqrt{2}s(0) - 10s(-90) & -10s(-90) \\ (50\sqrt{2}-50)c(120) + 25\sqrt{2}c(0) + 10c(-90) & 25\sqrt{2}c(0) + 10c(-90) & 10c(-90) \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\pi}{36} \\ \frac{2\pi}{3} \\ \frac{\pi}{9} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} (50-50\sqrt{2})\frac{\sqrt{3}}{2} - 25\sqrt{2} + 10 & 10 & 10 \\ (50\sqrt{2}-50)(-0.5) + 25\sqrt{2} & 25\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\pi}{36} \\ \frac{2\pi}{3} \\ \frac{\pi}{9} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} (50-50\sqrt{2})\frac{\sqrt{3}}{2} - 25\sqrt{2} + 10 & 10 & 10 \\ (50\sqrt{2}-50)(-0.5) + 25\sqrt{2} & 25\sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\pi}{36} \\ \frac{2\pi}{3} \\ \frac{\pi}{9} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} 20.66 \\ 76.23 \\ 2.53 \end{bmatrix}$$

b) For link 3 to be parallel to the x axis, we require $\theta_{123} = 0$.

X position: $$\begin{aligned} 35 &= (50\sqrt{2}-50)c\theta_1 + 25\sqrt{2}c\theta_{12} + 10 \\ 25 &= (50\sqrt{2}-50)c\theta_1 + 25\sqrt{2}c\theta_{12} \end{aligned}$$

Y Position: $50\sqrt{2} - 25 = (50\sqrt{2}-50)s\theta_1 + 25\sqrt{2}s\theta_{12}$

$$c\theta_2 = \frac{25^2 + (50\sqrt{2}-25)^2 - (50\sqrt{2}-50)^2 - (25\sqrt{2})^2}{2(50\sqrt{2}-50)(25\sqrt{2})} = \frac{1}{\sqrt{2}}$$

$$\theta_2 = \pm 45^\circ$$

$$\theta_1 = \tan^{-1}\left(\frac{50\sqrt{2}-25}{25}\right) - \tan^{-1}\left(\frac{25\sqrt{2}s\theta_2}{(50\sqrt{2}-50)+25\sqrt{2}c\theta_2}\right)$$
$$\theta_1 = 90^\circ$$
$$\theta_1 = 32.65^\circ$$

There are two solutions:

$$\theta_1 = 90^\circ \quad \theta_2 = -45^\circ \ \theta_3 = -45^\circ$$
$$\theta_1 = 32.65^\circ \ \theta_2 = 45^\circ \quad \theta_3 = -77.65^\circ$$

$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = J_0(\theta)\dot{\theta}$. Here $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix}$. You can use either angle solution to calculate the answer, but since the first solution is computationally easier we will use that one.

$$\begin{bmatrix} 0 \\ -10 \\ 0 \end{bmatrix} = \begin{bmatrix} -50\sqrt{2}+25 & -25 & 0 \\ 35 & 35 & 10 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix}$$
$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0.48 \\ -0.88 \\ 0.4 \end{bmatrix}$$

c) The equations for $\dot{x}$ and $\dot{y}$ remain the same, but now the angular velocity is controlled by minimizing $v^2 = \dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2$. We will use the method of Lagrange multipliers to solve these equations.

$$L = \dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + \dot{\theta}_3^{\,2} + \lambda_1((-50\sqrt{2}+25)\dot{\theta}_1 - 25\dot{\theta}_2) + \lambda_2(35\dot{\theta}_1 + 35\dot{\theta}_2 + 10\dot{\theta}_3 + 10)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = 2\dot{\theta}_1 + \lambda_1(-50\sqrt{2}+25) + \lambda_2(35)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = 2\dot{\theta}_2 + \lambda_1(25\sqrt{2}) + \lambda_2(35)$$

$$\frac{\partial L}{\partial \dot{\theta}_3} = 2\dot{\theta}_3 + \lambda_2(10)$$

$$\frac{\partial L}{\partial \lambda_1} = (-50\sqrt{2}+25)\dot{\theta}_1 - 25\sqrt{2}\,\dot{\theta}_2$$

$$\frac{\partial L}{\partial \lambda_2} = 35\dot{\theta}_1 + 35\dot{\theta}_2 + 10\dot{\theta}_3 + 10$$

Set each of the derivatives equal to zero and solve for the unknowns.

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} .227 \\ -0.416 \\ -.341 \end{bmatrix}$$

d)

$$J_0 = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_{12} - a_3 s\theta_{123} & -a_2 s\theta_{12} - a_3 s\theta_{123} & -a_3 s\theta_{123} \\ a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} & a_2 c\theta_{12} + a_3 c\theta_{123} & a_3 c\theta_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

We need to see under what conditions $\det(J_0)=0$. .

$$\begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_{12} - a_3 s\theta_{123} & -a_2 s\theta_{12} - a_3 s\theta_{123} & -a_3 s\theta_{123} \\ a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123} & a_2 c\theta_{12} + a_3 c\theta_{123} & a_3 c\theta_{123} \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det(J_0) = (-a_2 s\theta_{12} - a_3 s\theta_{123})a_3 c\theta_{123} + (a_2 c\theta_{12} + a_3 c\theta_{123})a_3 s\theta_{123} + (a_1 s\theta_1 + a_2 s\theta_{12} + a_3 s\theta_{123})a_3 c\theta_{123} -$$
$$(a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123})a_3 s\theta_{123} + (a_2 c\theta_{12} + a_3 c\theta_{123})(-a_1 s\theta_1 - a_2 s\theta_{12} - a_3 s\theta_{123}) +$$
$$(a_1 c\theta_1 + a_2 c\theta_{12} + a_3 c\theta_{123})(a_2 s\theta_{12} + a_3 s\theta_{123})$$

Multiplying out this equation and cancelling terms leaves

$$\det(J_0) = a_1 a_2 c\theta_{12} s\theta_1 - a_1 a_2 s\theta_{12} c\theta_1$$
$$\det(J_0) = a_1 a_2 s(\theta_1 - \theta_2 - \theta_1) = a_1 a_2 \sin(-\theta_2)$$

Therefore we have no solution when $\sin(-\theta_2) = 0$. Therefore $\theta_2 = 0$ or $\theta_2 = 180^{\circ}$.

The following table shows the angles of the manipulator as it makes the desired motion.

		Solution 1			Solution 2		
X	Y	Theta 1	Theta 2	Theta 3	Theta 1	Theta 2	Theta 3
35	45.71068	32.6499	45	-77.6499	90	-45	-45
35	35.71068	1.708894	81.30776	-83.0167	108.3016	-81.3078	-26.9939
35	25.71068	-25.9557	105.5622	-19.8058	117.5615	-105.562	-11.9993
35	15.71068	-55.4751	123.4444	-67.9693	119.7678	-123.444	3.676529
35	5.710678	-85.9849	134.2189	-48.234	111.7192	-134.219	22.49968
35	-4.28932	-109.471	135	-25.5288	90	-135	45

The table shows that the manipulator will not reach the singularity condition during the desired motion.