Kinematics of the slider-crank mechanism.

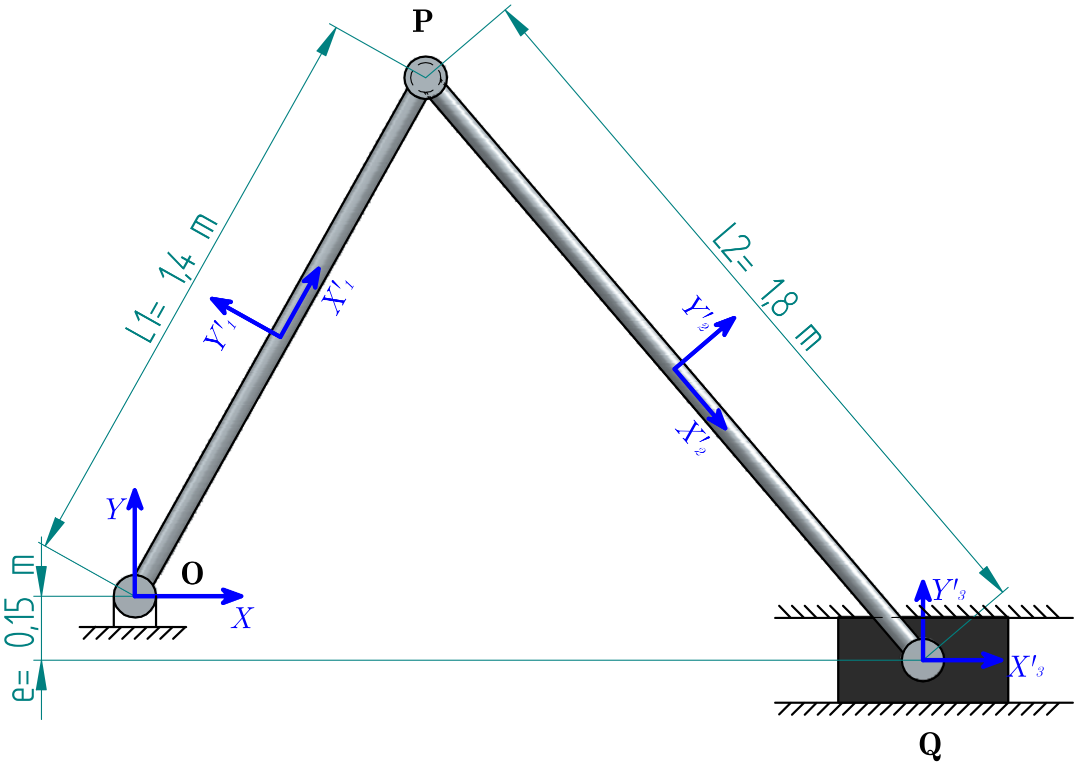

FIGURE 1: PLANAR SLIDER-CRANK MECHANISM.

The system shown in the figure is a planar slider-crank mechanism, composed of three bodies: the crank (body 1), the connecting rod (body 2) and the slider (body 3). Model the system using reference point coordinates in 2-D, $\mathbf{q}=\left[\mathbf{r}_1^{\mathrm{T}},\varphi_1,\mathbf{r}_2^{\mathrm{T}},\varphi_2,\mathbf{r}_3^{\mathrm{T}},\varphi_3\right]^{\mathrm{T}}$ and calculate the following:

1. Constraints vector $\mathbf{\Phi}(\mathbf{q},t)$, adding the driving constraint: $\varphi_1=\pi/3+0.5\,t+t^2$.
2. Velocity equation: $\dot{\mathbf{\Phi}}(\mathbf{q},\dot{\mathbf{q}},t)=\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}+\mathbf{\Phi}_t=\mathbf{0}$.
3. Acceleration equation: $$\ddot{\mathbf{\Phi}}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},t)=\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}}+\left(\mathbf{\Phi}_{\mathbf{q}}\dot{\mathbf{q}}\right)_{\mathbf{q}}\dot{\mathbf{q}}+2\mathbf{\Phi}_{t\mathbf{q}}+\mathbf{\Phi}_{tt}$$ $$\ddot{\mathbf{\Phi}}(\mathbf{q},\dot{\mathbf{q}},\ddot{\mathbf{q}},t)=\mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}}+\dot{\mathbf{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}}+\dot{\mathbf{\Phi}}_t=\mathbf{0}$$
4. Using Matlab and the theoretical results of 1, solve the position problem using the following initial guess: $\mathbf{q}=\left[0.34,0.61,\pi/3,1.26,0.53,-\pi/3,2.0,-0.16,0.01\right]^{\mathrm{T}}$.
5. Using the results of 2 and 4, calculate the vector of generalized velocities of the mechanism in the initial position $t=0$.
6. Using the results of 3, 4 and 5, calculate the vector of generalized accelerations in the initial position $t=0$.

Multibody Dynamics
Homework 1
Peter Racioppo

1.) Contraint Vectors

$$\boldsymbol{q} = (x_1, y_1, \varphi_1, x_2, y_2, \varphi_2, x_3, y_3, \varphi_3)^T$$

$$\boldsymbol{\Phi}^{r(i,j)} = \boldsymbol{r}_i + \boldsymbol{s}_i^P - \boldsymbol{r}_j - \boldsymbol{s}_j^P = \boldsymbol{r}_j + \mathbb{A}_i \boldsymbol{s}'^P_i - \boldsymbol{r}_j - \mathbb{A}_j \boldsymbol{s}'^P_j = \boldsymbol{0}$$

$\boldsymbol{s}'^Q_1 = [\frac{-1}{2} l_1 \ \ 0]^T$, $\boldsymbol{s}'^Q_4 = [0 \ \ 0]^T$,
$\boldsymbol{s}'^P_1 = [\frac{1}{2} l_1 \ \ 0]^T$, $\boldsymbol{s}'^P_2 = [\frac{-1}{2} l_2 \ \ 0]^T$,
$\boldsymbol{s}'^Q_2 = [\frac{1}{2} l_2 \ \ 0]^T$, $\boldsymbol{s}'^Q_3 = [0 \ \ 0]^T$

$$\mathbb{A}_1 = \begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix}, \ \mathbb{A}_2 = \begin{pmatrix} \sin\varphi_2 & \cos\varphi_2 \\ -\cos\varphi_2 & \sin\varphi_2 \end{pmatrix}, \ \mathbb{A}_3 = \mathbb{A}_4 = \mathbb{I}_2$$

$$\boldsymbol{\Phi}^r = \begin{pmatrix} \boldsymbol{r}_1 + \begin{pmatrix} c\varphi_1 & -s\varphi_1 \\ s\varphi_1 & c\varphi_1 \end{pmatrix}\begin{pmatrix} \frac{-1}{2} l_1 \\ 0 \end{pmatrix} - \boldsymbol{r}_4 \\ \boldsymbol{r}_1 + \begin{pmatrix} c\varphi_1 & -s\varphi_1 \\ s\varphi_1 & c\varphi_1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} l_1 \\ 0 \end{pmatrix} - \boldsymbol{r}_2 - \begin{pmatrix} s\varphi_2 & c\varphi_2 \\ -c\varphi_2 & s\varphi_2 \end{pmatrix}\begin{pmatrix} \frac{-1}{2} l_2 \\ 0 \end{pmatrix} \\ \boldsymbol{r}_2 + \begin{pmatrix} s\varphi_2 & c\varphi_2 \\ -c\varphi_2 & s\varphi_2 \end{pmatrix}\begin{pmatrix} \frac{1}{2} l_2 \\ 0 \end{pmatrix} - \boldsymbol{r}_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 - \frac{1}{2} l_1 c\varphi_1 - x_4 \\ y_1 - \frac{1}{2} l_1 s\varphi_1 - y_4 \\ x_1 + \frac{1}{2} l_1 c\varphi_1 - x_2 + \frac{1}{2} l_2 s\varphi_2 \\ y_1 + \frac{1}{2} l_1 s\varphi_1 - y_2 - \frac{1}{2} l_2 c\varphi_2 \\ x_2 + \frac{1}{2} l_2 s\varphi_2 - x_3 \\ y_2 - \frac{1}{2} l_2 c\varphi_2 - y_3 \end{pmatrix}$$

$$\boldsymbol{\Phi}^{t(i,j)} = \begin{pmatrix} \boldsymbol{v}'^T_i \mathbb{B}_i (\boldsymbol{r}_j - \boldsymbol{r}_i) - \boldsymbol{v}'^T_i \mathbb{B}_{ij} \boldsymbol{s}'^P_j - v'^T_i \mathbb{R}^T \boldsymbol{s}'^P_i \\ \boldsymbol{v}'^T_i \mathbb{B}_{ij} \boldsymbol{v}'_i \end{pmatrix} = \boldsymbol{0}$$

$$\mathbb{B}_{ij} = \mathbb{B}_i \mathbb{B}_j = (\tfrac{d}{d\varphi_i} \mathbb{A}_i)(\tfrac{d}{d\varphi_j} \mathbb{A}_j)$$

$\boldsymbol{v}'_3 = \boldsymbol{v}'_4 = [1 \ \ 0]^T$
$\boldsymbol{s}'^P_3 = [0 \ \ -e]^T$, $\boldsymbol{s}'^P_4 = [0 \ \ 0]^T$
$$\mathbb{B}_{34} = -\begin{pmatrix} s\varphi_3 & -c\varphi_3 \\ c\varphi_3 & s\varphi_3 \end{pmatrix}$$

$$\boldsymbol{\Phi}^{t(3,4)} = \begin{pmatrix} (1 \ 0)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x_4 - x_3 \\ y_4 - y_3 \end{pmatrix} - (1 \ 0)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} - (1 \ 0)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -e \end{pmatrix} \\ (1 \ 0)\begin{pmatrix} s\varphi_3 & -c\varphi_3 \\ c\varphi_3 & s\varphi_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} y_4 - e - y_3 \\ \varphi_3 \end{pmatrix}$$

$(x_4, y_4) = (0, 0)$
$$\Phi^D = (\varphi_1 - \pi/3 - 0.5 - t^2)^T$$

Thus, $$\Phi = \begin{pmatrix} x_1 - \frac{1}{2}l_1 c\varphi_1 \\ y_1 - \frac{1}{2}l_1 s\varphi_1 \\ x_1 + \frac{1}{2}l_1 c\varphi_1 - x_2 + \frac{1}{2}l_2 s\varphi_2 \\ y_1 + \frac{1}{2}l_1 s\varphi_1 - y_2 - \frac{1}{2}l_2 c\varphi_2 \\ x_2 + \frac{1}{2}l_2 s\varphi_2 - x_3 \\ y_2 - \frac{1}{2}l_2 c\varphi_2 - y_3 \\ y_3 + e \\ \varphi_3 \\ \varphi_1 - \pi/3 - 0.5 - t^2 \end{pmatrix}$$

2.) Velocity Equations

(calculations performed with Mathematica)

$$\Phi_q = \begin{pmatrix} 1 & 0 & 0.5\ell 1\sin(\phi_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -0.5\ell 1\cos(\phi_1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -0.5\ell 1\sin(\phi_1) & -1 & 0 & 0.5\ell 2\cos(\phi_2) & 0 & 0 & 0 \\ 0 & 1 & 0.5\ell 1\cos(\phi_1) & 0 & -1 & 0.5\ell 2\sin(\phi_2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0.5\ell 2\cos(\phi_2) & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5\ell 2\sin(\phi_2) & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Phi_q \dot{\boldsymbol{q}} = \begin{pmatrix} x_1' + 0.5\ell 1\phi_1'\sin(\phi_1) \\ y_1' - 0.5\ell 1\phi_1'\cos(\phi_1) \\ x_1' - x_2' - 0.5\ell 1\phi_1'\sin(\phi_1) + 0.5\ell 2\phi_2'\cos(\phi_2) \\ y_1' - y_2' + 0.5\ell 1\phi_1'\cos(\phi_1) + 0.5\ell 2\phi_2'\sin(\phi_2) \\ x_2' - x_3' + 0.5\ell 2\phi_2'\cos(\phi_2) \\ y_2' - y_3' + 0.5\ell 2\phi_2'\sin(\phi_2) \\ y_3' \\ \phi_3' \\ \phi_1' \end{pmatrix}$$

$$\Phi_t = (0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 2t + 0.5)^T$$

$$\Phi(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t) = \Phi_q \dot{\boldsymbol{q}} + \Phi_t$$
$$= \begin{pmatrix} x_1' + 0.5\ell 1\phi_1'\sin(\phi_1) \\ y_1' - 0.5\ell 1\phi_1'\cos(\phi_1) \\ x_1' - x_2' - 0.5\ell 1\phi_1'\sin(\phi_1) + 0.5\ell 2\phi_2'\cos(\phi_2) \\ y_1' - y_2' + 0.5\ell 1\phi_1'\cos(\phi_1) + 0.5\ell 2\phi_2'\sin(\phi_2) \\ x_2' - x_3' + 0.5\ell 2\phi_2'\cos(\phi_2) \\ y_2' - y_3' + 0.5\ell 2\phi_2'\sin(\phi_2) \\ y_3' \\ \phi_3' \\ \phi_1' + 2t + 0.5 \end{pmatrix} = 0$$

3.) Acceleration Equations

$$\Phi_q \ddot{\boldsymbol{q}} = \begin{pmatrix} x_1' + 0.5\ell 1 \phi_1' \sin(\phi_1) \\ y_1' - 0.5\ell 1 \phi_1' \cos(\phi_1) \\ x_1' - x_2' - 0.5\ell 1 \phi_1' \sin(\phi_1) + 0.5\ell 2 \phi_2' \cos(\phi_2) \\ y_1' - y_2' + 0.5\ell 1 \phi_1' \cos(\phi_1) + 0.5\ell 2 \phi_2' \sin(\phi_2) \\ x_2' - x_3' + 0.5\ell 2 \phi_2' \cos(\phi_2) \\ y_2' - y_3' + 0.5\ell 2 \phi_2' \sin(\phi_2) \\ y_3' \\ \phi_3' \\ \phi_1' \end{pmatrix}$$

$$(\Phi_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} = \begin{pmatrix} 0.5\ell 1 \phi_1'^2 \cos(\phi_1) \\ 0.5\ell 1 \phi_1'^2 \sin(\phi_1) \\ -0.5\ell 1 \phi_1'^2 \cos(\phi_1) - 0.5\ell 2 \phi_2'^2 \sin(\phi_2) \\ 0.5\ell 2 \phi_2'^2 \cos(\phi_2) - 0.5\ell 1 \phi_1'^2 \sin(\phi_1) \\ -0.5\ell 2 \phi_2'^2 \sin(\phi_2) \\ 0.5\ell 2 \phi_2'^2 \cos(\phi_2) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Phi_{tq} = \mathbf{0}_9$$

$$\Phi_{tt} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$\ddot{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t) = \Phi_q \ddot{\boldsymbol{q}} + (\Phi_q \dot{\boldsymbol{q}})_q \dot{\boldsymbol{q}} + 2\Phi_{tq} + \Phi_{tt}$$

$$= \begin{pmatrix} x_1' + 0.5\ell 1 \phi_1'^2 c\phi_1 + 0.5\ell 1 \phi_1' s\phi_1 \\ y_1' + 0.5\ell 1 \phi_1'^2 s\phi_1 - 0.5\ell 1 \phi_1' c\phi_1 \\ x_1' - x_2' - 0.5\ell 1 \phi_1'^2 c\phi_1 - 0.5\ell 1 \phi_1' s\phi_1 - 0.5\ell 2 \phi_2'^2 s\phi_2 + 0.5\ell 2 \phi_2' c\phi_2 \\ y_1' - y_2' - 0.5\ell 1 \phi_1'^2 s\phi_1 + 0.5\ell 1 \phi_1' c\phi_2 + 0.5\ell 2 \phi_2'^2 c\phi_2 + 0.5\ell 2 \phi_2' s\phi_2 \\ x_2' - x_3' - 0.5\ell 2 \phi_2'^2 s\phi_2 + 0.5\ell 2 \phi_2' c\phi_2 \\ y_2' - y_3' + 0.5\ell 2 \phi_2'^2 \cos(\phi_2) + 0.5\ell 2 \phi_2' s\phi_2 \\ y_3' \\ \phi_3' \\ \phi_1' + 2 \end{pmatrix} = \mathbf{0}$$

$$\dot{\Phi}_q \dot{\boldsymbol{q}} = \begin{pmatrix} 0.5\ell 1 \phi_1'^2 \cos(\phi_1) \\ 0.5\ell 1 \phi_1'^2 \sin(\phi_1) \\ -0.5\ell 1 \phi_1'^2 \cos(\phi_1) - 0.5\ell 2 \phi_2'^2 \sin(\phi_2) \\ 0.5\ell 2 \phi_2'^2 \cos(\phi_2) - 0.5\ell 1 \phi_1'^2 \sin(\phi_1) \\ -0.5\ell 2 \phi_2'^2 \sin(\phi_2) \\ 0.5\ell 2 \phi_2'^2 \cos(\phi_2) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dot{\Phi}_t = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$\ddot{\Phi}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t) = \Phi_q \ddot{\boldsymbol{q}} + \dot{\Phi}_q \dot{\boldsymbol{q}} + \dot{\Phi}_t = \mathbf{0}$$

$$= \begin{pmatrix} x_1' + 0.5\ell 1 \phi_1'^2 c\phi_1 + 0.5\ell 1 \phi_1' s\phi_1 \\ y_1' + 0.5\ell 1 \phi_1'^2 s\phi_1 - 0.5\ell 1 \phi_1' c\phi_1 \\ x_1' - x_2' - 0.5\ell 1 \phi_1'^2 c\phi_1 - 0.5\ell 1 \phi_1' s\phi_1 - 0.5\ell 2 \phi_2'^2 s\phi_2 + 0.5\ell 2 \phi_2' c\phi_2 \\ y_1' - y_2' - 0.5\ell 1 \phi_1'^2 s\phi_1 + 0.5\ell 1 \phi_1' c\phi_2 + 0.5\ell 2 \phi_2'^2 c\phi_2 + 0.5\ell 2 \phi_2' s\phi_2 \\ x_2' - x_3' - 0.5\ell 2 \phi_2'^2 s\phi_2 + 0.5\ell 2 \phi_2' c\phi_2 \\ y_2' - y_3' + 0.5\ell 2 \phi_2'^2 \cos(\phi_2) + 0.5\ell 2 \phi_2' s\phi_2 \\ y_3' \\ \phi_3' \\ \phi_1' + 2 \end{pmatrix} = \mathbf{0}$$

which is the same result.

```
(* Peter Racioppo, Multibody Dynamics, HW1 *)

ClearAll["Global`*"]

Φ = {x1[t] - 0.5 * ℓ1 * Cos[φ1[t]],
      y1[t] - 0.5 * ℓ1 * Sin[φ1[t]],
      x1[t] + 0.5 * ℓ1 * Cos[φ1[t]] - x2[t] + 0.5 * ℓ2 * Sin[ φ2[t]],
      y1[t] + 0.5 * ℓ1 * Sin[φ1[t]] - y2[t] - 0.5 * ℓ2 * Cos[ φ2[t]],
      x2[t] + 0.5 * ℓ2 * Sin[ φ2[t]] - x3[t],
      y2[t] - 0.5 * ℓ2 * Cos[ φ2[t]] - y3[t],
      y3[t] + e,
      φ3[t],
      φ1[t] - (Pi / 3) - 0.5 * t - t^2} ;
Φ // MatrixForm;

q = {x1[t], y1[t], φ1[t] , x2[t], y2[t], φ2[t], x3[t], y3[t], φ3[t]};

qp = D[q, t];

Φq = D[Φ, {q}];
Φq // MatrixForm;

Φ1 = {x1 - 0.5 * ℓ1 * Cos[φ1],
      y1 - 0.5 * ℓ1 * Sin[φ1],
      x1 + 0.5 * ℓ1 * Cos[φ1] - x2 + 0.5 * ℓ2 * Sin[ φ2],
      y1 + 0.5 * ℓ1 * Sin[φ1] - y2 - 0.5 * ℓ2 * Cos[ φ2],
      x2 + 0.5 * ℓ2 * Sin[ φ2] - x3,
      y2 - 0.5 * ℓ2 * Cos[ φ2] - y3,
      y3 - e,
      φ3,
      φ1 - (Pi / 3) + 0.5 * t + t^2} ;

Φt = D[Φ1, t];
Φt // MatrixForm;

qpp = D[D[q, t], t];

T1 = Φq.qp;
T1 // MatrixForm;
Tq = D[T1, {q}];
Tq // MatrixForm;

T2 = Tq.qp;
T2 // MatrixForm;

Φtq = D[Φt, {q}];
2 * Φtq // MatrixForm;

Φtt = D[Φt, t];
Φtt // MatrixForm;


Φp = Φq.qp + Φt;
Φp // MatrixForm;

Φq.qpp // MatrixForm;
```

```
Φpp = Φq.qpp + Tq.qp + 2 * Φtq.qp + Φtt;
Φpp // MatrixForm;

Φa = D[Φp, {q}].qp;
Φa // MatrixForm;

Φpt = {0, 0, 0, 0, 0, 0, 0, 0, 2};

Φpp2 = Φq.qpp + Φa + Φpt;
Φpp2 // MatrixForm;

(* FullSimplify[Φqi = Inverse[Φq ]]//MatrixForm;
(* Check: *) Simplify[Inverse[Φq ].Φq ]//MatrixForm; *)
```

Multibody Dynamics

Homework 2

Peter Racioppo

Last time, we found that:

$$\Phi = \begin{pmatrix} x_1 - \frac{1}{2} l_1 c\varphi_1 \\ y_1 - \frac{1}{2} l_1 s\varphi_1 \\ x_1 + \frac{1}{2} l_1 c\varphi_1 - x_2 + \frac{1}{2} l_2 c\varphi_2 \\ y_1 + \frac{1}{2} l_1 s\varphi_1 - y_2 + \frac{1}{2} l_2 s\varphi_2 \\ x_2 + \frac{1}{2} l_2 s\varphi_2 - x_3 \\ y_2 + \frac{1}{2} l_2 c\varphi_2 - y_3 \\ y_3 + e \\ \varphi_3 \\ \varphi_1 - \pi/3 - \frac{1}{2} t - t^2 \end{pmatrix}$$

$$q = [x1 \quad y1 \quad \phi1 \quad x2 \quad y2 \quad \phi2 \quad x3 \quad y3 \quad \phi3]^T$$

$$\Phi_q = \begin{pmatrix} 1 & 0 & \frac{7\sin(\phi1)}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{7\cos(\phi1)}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -\frac{7\sin(\phi1)}{10} & -1 & 0 & -\frac{9\sin(\phi2)}{10} & 0 & 0 & 0 \\ 0 & 1 & \frac{7\cos(\phi1)}{10} & 0 & -1 & \frac{9\cos(\phi2)}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\frac{9\sin(\phi2)}{10} & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{9\cos(\phi2)}{10} & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos(\phi3) \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Phi_t = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -2t - \frac{1}{2}]^T$$

$\dot{\boldsymbol{q}} = -\Phi_q^{-1}\Phi_t$

$\Rightarrow$

$$\dot{\boldsymbol{q}} = \begin{pmatrix} -\dfrac{7\sin(\phi 1)\left(2t+\frac{1}{2}\right)}{10} \\ \dfrac{\left(7\cos(\phi 1)\left(2t+\frac{1}{2}\right)\right)}{10} \\ 2t+\dfrac{1}{2} \\ \dfrac{\left(7\left(2t+\frac{1}{2}\right)(\cos(\phi 1)\sin(\phi 2)-2\cos(\phi 2)\sin(\phi 1))\right)}{10\cos(\phi 2)} \\ \dfrac{\left(7\cos(\phi 1)\left(2t+\frac{1}{2}\right)\right)}{10} \\ -\dfrac{7\cos(\phi 1)\left(2t+\frac{1}{2}\right)}{9\cos(\phi 2)} \\ \dfrac{\left(7\left(2t+\frac{1}{2}\right)(\cos(\phi 1)\sin(\phi 2)-\cos(\phi 2)\sin(\phi 1))\right)}{5\cos(\phi 2)} \\ 0 \\ 0 \end{pmatrix}$$

and similarly,

$\ddot{\boldsymbol{q}} = -\Phi_q^{-1}[(\Phi_q\dot{\boldsymbol{q}})_q\dot{\boldsymbol{q}} + 2\Phi_{tq} + \Phi_{tt}]$

$\Rightarrow$

$\ddot{\boldsymbol{q}} = [a \;\; b \;\; c \;\; d \;\; e \;\; f \;\; g \;\; h \;\; i]^T$

where,

$a = -0.18\cos(\phi 1) - \dfrac{7\sin(\phi 1)}{5}$

$b = \dfrac{7\cos(\phi 1)}{5} - 0.18\sin(\phi 1)$

$c = 2$

$$d = \frac{7(c(\phi 1)s(\phi 2)-2c(\phi 2)s(\phi 1))}{5c(\phi 2)} - 0.08c(\phi 2) - 0.35c(\phi 1) - \frac{0.04s(\phi 2)^2}{c(\phi 2)} - \frac{s(\phi 2)(0.18s(\phi 1)+0.08s(\phi 2))}{2c(\phi 2)} - \frac{0.09s(\phi 1)s(\phi 2)}{c(\phi 2)}$$

$e = \dfrac{7\cos(\phi 1)}{5} - 0.18\sin(\phi 1)$

$$f = \frac{5(0.18s(\phi 1)+0.08s(\phi 2))}{9c(\phi 2)} - \frac{14c(\phi 1)}{9c(\phi 2)} + \frac{0.10s(\phi 1)}{c}(\phi 2) + \frac{0.04s(\phi 2)}{c}(\phi 2)$$

$$g = \frac{14(c(\phi 1)s(\phi 2)-c(\phi 2)s(\phi 1))}{5c(\phi 2)} - 0.16c(\phi 2) - 0.35c(\phi 1) - \frac{0.08s(\phi 2)^2}{c(\phi 2)} - \frac{s(\phi 2)(0.18s(\phi 1)+0.08s(\phi 2))}{c(\phi 2)} - \frac{0.18s(\phi 1)s(\phi 2)}{c(\phi 2)}$$

$h = 0$

$i = 0$

we take as an initial guess for $t = 0$:

$q_0 = \left[0.34 \;\; 0.61 \;\; \frac{\pi}{3} \;\; 1.26 \;\; 0.53 \;\; -\frac{\pi}{3} \;\; 2.0 \;\; -0.16 \;\; 0.01\right]^T$

using *fsolve* to solve for position and plugging in values, we obtain $\boldsymbol{q}, \dot{\boldsymbol{q}}$, and $\ddot{\boldsymbol{q}}$ as

$\boldsymbol{q} = [\;0.35 \;\; 0.61 \;\; 1.05 \;\; 1.29 \;\; 0.53 \;\; -0.86 \;\; 1.88 \;\; -0.15 \;\; 0]^T$

$\dot{\boldsymbol{q}} = [-0.30 \;\; 0.18 \;\; 0.50 \;\; -0.81 \;\; 0.18 \;\; -0.30 \;\; -1.01 \;\; 0 \;\; 0]^T$

$\ddot{\boldsymbol{q}} = [-1.30 \;\; 0.55 \;\; 2.00 \;\; -3.36 \;\; 0.55 \;\; -1.03 \;\; -4.11 \;\; 0 \;\; 0]^T$

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% Modeling and Simulation
% Homework 2
% Peter Racioppo


function hw2
clc, close all

guess = [0.34;0.61;pi/3;1.26;0.53;-pi/3;2.0;-0.16;0.01];
x = fsolve(@position,guess)

syms x1 y1 phi1 x2 y2 phi2 x3 y3 phi3 t

q=[x1; y1; phi1; x2; y2; phi2; x3; y3; phi3];
r1=[x1;y1]; r2=[x2;y2]; r3=[x3;y3];

L1=1.4; L2=1.8; e=0.15;

A1= [cos(phi1) -sin(phi1); sin(phi1) cos(phi1)];
A2= [cos(phi2) -sin(phi2); sin(phi2) cos(phi2)];
A3= [cos(phi3) -sin(phi3); sin(phi3) cos(phi3)];

s1O=[-L1/2;0]; s1P=[L1/2;0]; s2P=[-L2/2;0]; s2Q=[L2/2;0];
v3X=[1;0]; r0R=[0;-e]; v0Y= [0;1];

PHI=[r1+A1*s1O; r1+A1*s1P-r2-A2*s2P; r2+A2*s2Q-r3; v0Y'*(r3-r0R);
    v0Y'*(A3*v3X); phi1-pi/3-0.5*t-t^2];

PHIq=jacobian(PHI,q);

PHIt=diff(PHI,t);

iPHIq=inv(PHIq);

v=iPHIq*(-PHIt);

v=vpa(subs(v,x1,x(1)));
v=vpa(subs(v,t,0));
v=vpa(subs(v,y1,x(2)));
v=vpa(subs(v,phi1,x(3)));
v=vpa(subs(v,x2,x(4)));
v=vpa(subs(v,y2,x(5)));
v=vpa(subs(v,phi2,x(6)));
v=vpa(subs(v,x3,x(7)));
v=vpa(subs(v,y3,x(8)));
v=vpa(subs(v,phi3,x(9)));
v


term2=jacobian(PHIq*v,q)*v;
term4=diff(PHIt,t);
```

```
a=iPHIq*-(term2+term4);
a=vpa(subs(a,x1,x(1)));
a=vpa(subs(a,t,0));
a=vpa(subs(a,y1,x(2)));
a=vpa(subs(a,phi1,x(3)));
a=vpa(subs(a,x2,x(4)));
a=vpa(subs(a,y2,x(5)));
a=vpa(subs(a,phi2,x(6)));
a=vpa(subs(a,x3,x(7)));
a=vpa(subs(a,y3,x(8)));
a=vpa(subs(a,phi3,x(9)));
a

end


function PHI=position(q)

L1=1.4; L2=1.8; e=0.15; t=0;

r1=[q(1);q(2)]; r2=[q(4);q(5)]; r3=[q(7);q(8)];
phi1=q(3); phi2=q(6); phi3=q(9);

s1O=[-L1/2;0]; s1P=[L1/2;0]; s2P=[-L2/2;0]; s2Q=[L2/2;0];
v3X=[1;0]; r0R=[0;-e]; v0Y= [0;1];
% s3Q=[0;0]; r0O=[0;0];

A1= [cos(phi1) -sin(phi1); sin(phi1) cos(phi1)];
A2= [cos(phi2) -sin(phi2); sin(phi2) cos(phi2)];
A3= [cos(phi3) -sin(phi3); sin(phi3) cos(phi3)];

PHI=[r1+A1*s1O; r1+A1*s1P-r2-A2*s2P; r2+A2*s2Q-r3; v0Y'*(r3-r0R);
    v0Y'*(A3*v3X); phi1-pi/3-0.5*t-t^2];

end
```

```
Equation solved.

fsolve completed because the vector of function values is near zero
as measured by the default value of the function tolerance, and
the problem appears regular as measured by the gradient.




x =

    0.3500
    0.6062
    1.0472
    1.2882
    0.5312
   -0.8586
```

```
    1.8763
   -0.1500
         0


v =

 -0.303108891324553526367303109763353
                                0.175
                                  0.5
 -0.808902989295672918276329312222216
                                0.175
 -0.297533674810305840686804577576 75
  -1.01158819594223878381805240491 73
                                    0
                                    0


a =

 -1.29993556529821410546921243905 41
 0.548445554337723236816348445118 24
                                 2.0
 -3.35699591560597688740988040665 26
 0.548445554337723236816348445118 24
 -1.03499422926106404767141255949 26
 -4.11412070061552556388133593519 71
                                   0
                                   0
```

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Multibody Dynamics

Homework 3

Peter Racioppo

We define four frames:

Frame 1: the wheel on the left

Frame 2: the right on the right

Frame 3: the wheel in the middle

Frame 4: the ground frame

Our generalized coordinate vector thus has 12 components:

$q = [x_1, y_1, \varphi_1, x_2, y_2, \varphi_2, x_3, y_3, \varphi_3, x_4, y_4, \varphi_4]^T$

There is one driving constraint on φ_3.

The initial velocity is 0.25 rev/s and the acceleration is 0.5 rev/s^2.

Thus, $\varphi_3 = 0.25(2\pi)t + 0.25(2\pi)t^2 = \frac{\pi}{2}t + \frac{\pi}{2}t^2$.

Our 11 body constraints can be written down by inspection, by noting that the x and y distances between the wheels must remain constant, that wheels 1 and 3 must roll through equal arc lengths as they rotate, and that wheels 1 and 2 must rotate by the same amount.

$$x_2 - x_1 - 900 = 0$$
$$x_3 - x_1 - 450 = 0$$
$$y_1 - 300 = 0$$
$$y_2 - 300 = 0$$
$$y_3 - \left(\sqrt{(r_1 + r_3)^2 - 450^2} + 300\right) = 0 \quad \Rightarrow \quad y_3 - 650.10 \approx 0$$
$$x_1 - r_1\varphi_1 = 0 \quad \Rightarrow \quad x_1 - 300\varphi_1 = 0$$
$$\alpha_1 r_1 - \alpha_3 r_3 = 0$$
$$\varphi_1 - \varphi_2 = 0$$
$$x_4 - 450 = 0$$
$$y_4 - 300 = 0$$
$$\varphi_4 = 0$$

where $\alpha_1 = \varphi_1 + \theta_1 - \theta$ and $\alpha_3 = -(\varphi_3 + \theta_3 - \theta - \pi)$

Our constraint vector is thus:

$$\Phi = \begin{bmatrix} x_2 - x_1 - 900 \\ x_3 - x_1 - 450 \\ y_1 - 300 \\ y_2 - 300 \\ y_3 - 650.10 \\ x_1 - 300\varphi_1 \\ \alpha_1 r_1 - \alpha_3 r_3 \\ \varphi_1 - \varphi_2 \\ x_4 - 450 \\ y_4 - 300 \\ \varphi_4 \\ \varphi_3 - \frac{\pi}{2}t - \frac{\pi}{2}t^2 \end{bmatrix}$$

As it's written, it makes no difference if we remove the fourth frame. However, a more systematic way to formulate the distance constraints would have been to write that:

$$\begin{bmatrix} r_4^P - r_1^P \\ r_4^Q - r_2^Q \\ r_4^R - r_3^R \end{bmatrix} = 0$$

This approach is only possible with a fourth frame. Otherwise, we are forced to write the constraints in the way in which they're written above.

The Jacobian is:

$$\Phi_q = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 300 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 300 & 0 & 0 & 0 & 0 & 0 & 270.15 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

We can now solve for the velocities and accelerations using the following formulae:

$$\Phi(q,\dot{q},\ddot{q},t) = \Phi_q \dot{q} + \Phi_t = 0$$
$$\Phi_q \dot{q} = -\Phi_t = v$$
$$\dot{q} = -\Phi_q^{-1}\Phi_t$$

$$\ddot{\Phi}(q,\dot{q},\ddot{q},t) = \Phi_q \ddot{q} + (\Phi_q \dot{q})_q \dot{q} + 2\Phi_{tq} + \Phi_{tt} = 0$$
$$\Phi_q \ddot{q} = -(\Phi_q \dot{q})_q \dot{q} - 2\Phi_{tq} - \Phi_{tt} = \gamma$$
$$\ddot{q} = -\Phi_q^{-1}((\Phi_q \dot{q})_q \dot{q} + 2\Phi_{tq} + \Phi_{tt})$$

Using Mathematica, we find that:

$$\dot{q} = [424.35 + 848.70t, 0, -1.41 - 2.83t, 424.35 + 848.70t, 0, -1.41 + 3.14t, 424.35 + 848.70t, 0, 1.57 + 3.14t, 424.35 + 848.70t, 0, 0]^T$$

$$\ddot{q} = [848.70, 0, -2.83, 848.70, 0, -2.83, 848.70, 0, 3.14, 848.70, 0, 0]^T$$

Thus, the linear acceleration of the wheels is 848.70 units/s^2 and the angular acceleration of wheels 1 and 2 is -2.83 rad/s^2.

Plots of position, velocity, and acceleration for 5 seconds are given below, after the Mathematica code.

```
(* Peter Racioppo, Multibody Dynamics, HW3 *)

ClearAll["Global`*"]


r1 = 300;
r2 = 300;
r3 = 270.15;
L = 900;

α1 = φ1[t] + θ1 - θ;
α3 = - (φ3[t] + θ3 - θ - Pi);

q = {x1[t], y1[t], φ1[t] , x2[t],
    y2[t], φ2[t], x3[t], y3[t], φ3[t], x4[t], y4[t], φ4[t]};

Φ = {x2[t] - x1[t] - L,
    x3[t] - x1[t] - L / 2,
    y1[t] - r1,
    y2[t] - r2,
    y3[t] - 650.10,
    x4[t] - x1[t] - 450,
    y4[t] - 300,
    φ4[t],
    x1[t] + r1 φ1[t],
    α1 r1 - α3 r3,
    φ1[t] - φ2[t],
    φ3[t] - 0.25 * (2 * Pi) * t - 0.25 * (2 * Pi) * t^2 };

(*
θA=(r1(φ1[t]+θ1)+r3(φ3[t]+θ3+Pi))/(r1+r3);
θB=(r3(φ3[t]+θ3)+r2(φ2[t]+θ2+Pi))/(r3+r2);
(x3[t]-x1[t])*Sin[θA]-(y3[t]-y1[t])*Cos[θA],
(x2[t]-x3[t])*Sin[θB]-(y2[t]-y3[t])*Cos[θB]
*)

Φ // MatrixForm;

qp = D[q, t];
```

```
Φq = D[Φ, {q}];
Φq // MatrixForm;

α1 a = φ1 + θ1 - θ;
α3 a = -(φ3 + θ3 - θ - Pi);

Φ1 = {x2 - x1 - L,
    x3 - x1 - L / 2,
    y1 - r1,
    y2 - r3,
    y3 - 650.10,
    x4 - x1 - 450,
    y4 - 300,
    φ4,
    x1 - r1 φ1,
    α1 a r1 - α3 a r3,
    φ1 - φ2,
    φ3 - 0.25 * (2 * Pi) * t - 0.25 * (2 * Pi) * t ^ 2 };

Φt = D[Φ1, t];
Φt // MatrixForm;

qpp = D[D[q, t], t];

T1 = Φq.qp;
T1 // MatrixForm;
Tq = D[T1, {q}];
Tq // MatrixForm;

T2 = Tq.qp;
T2 // MatrixForm;

Φtq = D[Φt, {q}];
2 * Φtq // MatrixForm;

Φtt = D[Φt, t];
Φtt // MatrixForm;

Φp = Φq.qp + Φt;
Φp // MatrixForm;

Φq.qpp // MatrixForm;

(* Φpp=Φq.qpp+Tq.qp+2*Φtq.qp+Φtt;
Φpp//MatrixForm; *)

(* FullSimplify[Φqi = Inverse[Φq ]]//MatrixForm;
(* Check: *) Simplify[Inverse[Φq ].Φq ]//MatrixForm; *)

γ = Tq.qp + 2 * Φtq.qp + Φtt;
γ // MatrixForm;

A = Inverse[Φq];
A // MatrixForm;
```

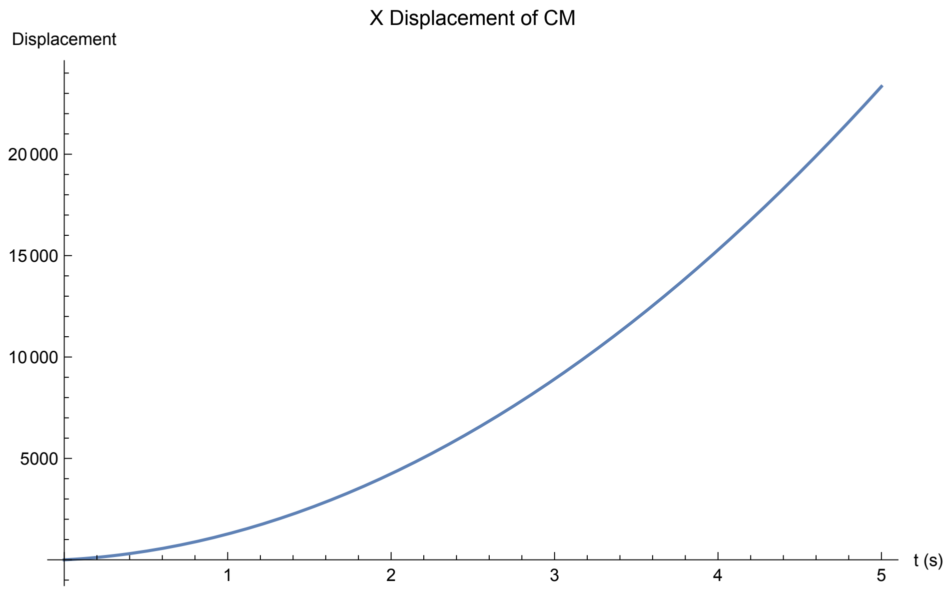
```
vel = Simplify[-A.Φt];
vel // MatrixForm

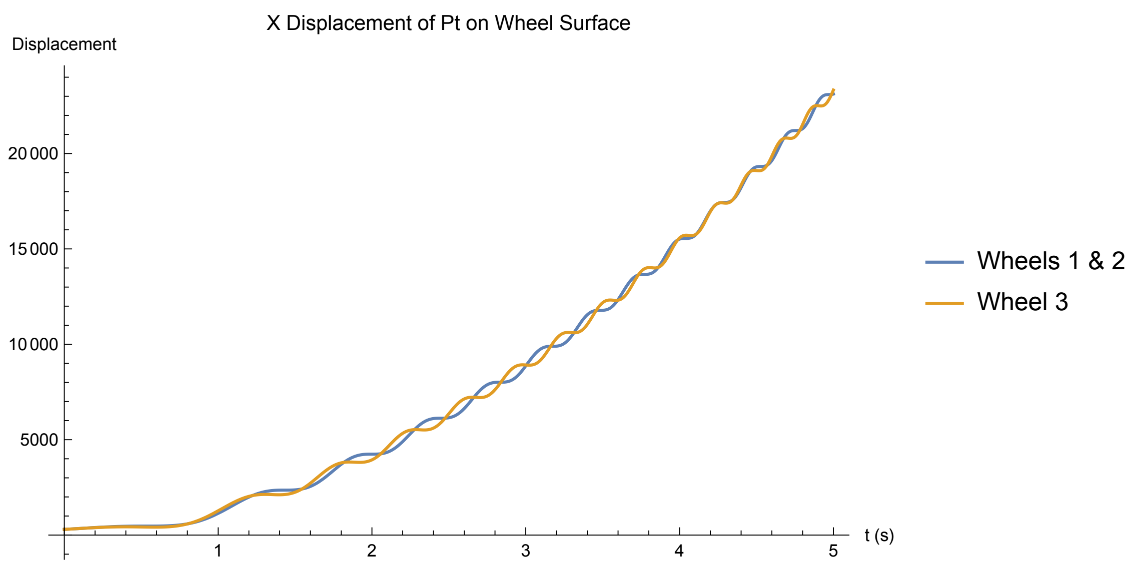
accel = -A.γ;
accel // MatrixForm
```

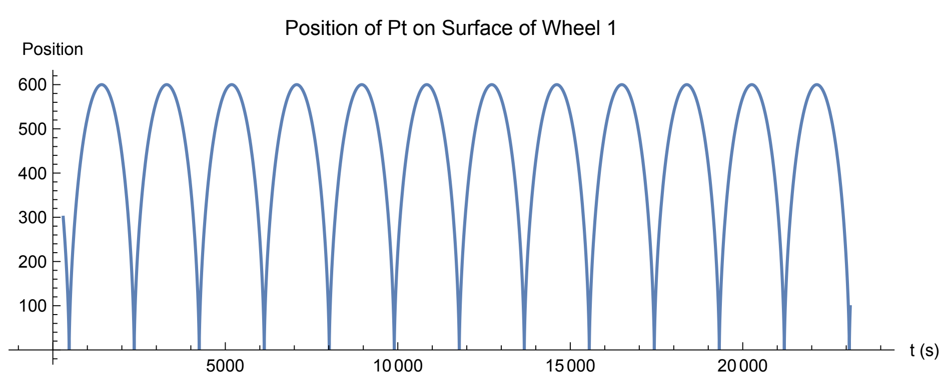
$$\begin{pmatrix} 424.351+848.701\,t \\ 0. \\ -1.4145-2.829\,t \\ 424.351+848.701\,t \\ 0. \\ -1.4145-2.829\,t \\ 424.351+848.701\,t \\ 0. \\ 1.5708+3.14159\,t \\ 424.351+848.701\,t \\ 0. \\ 0. \end{pmatrix}$$

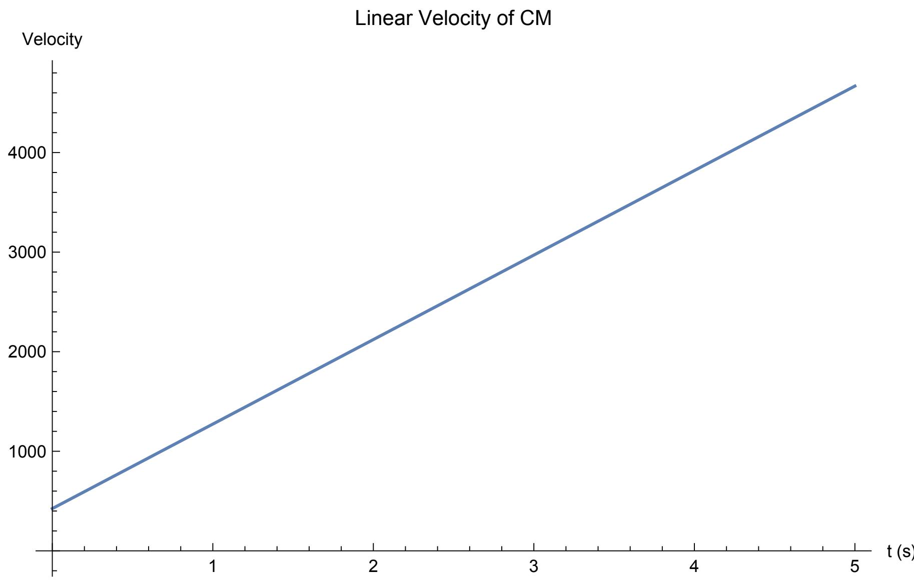
$$\begin{pmatrix} 848.701 \\ 0. \\ -2.829 \\ 848.701 \\ 0. \\ -2.829 \\ 848.701 \\ 0. \\ 3.14159 \\ 848.701 \\ 0. \\ 0. \end{pmatrix}$$

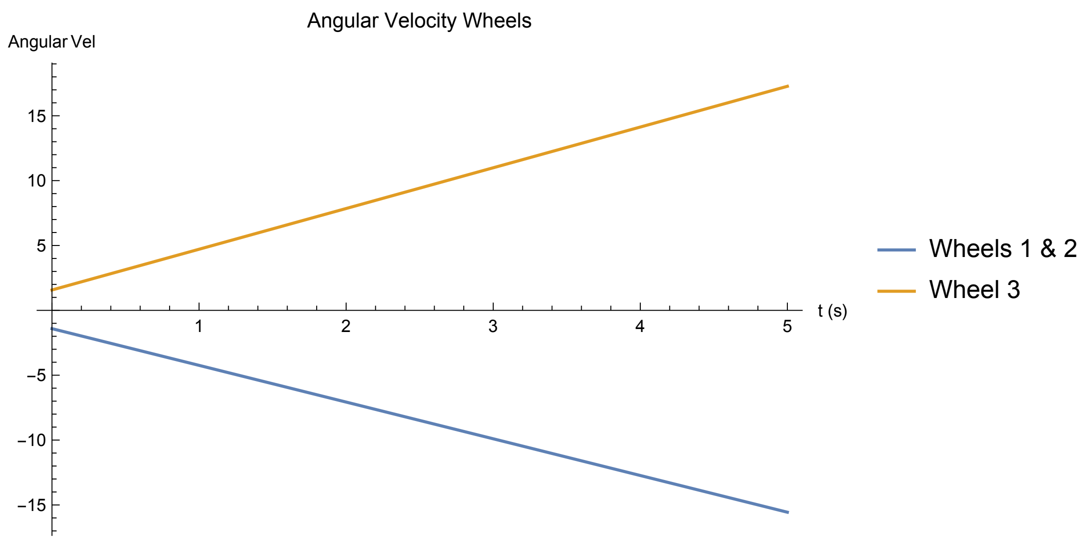
```
Plot[424.35 + 848.70 x, {x, 0, 5},
 AxesLabel → {"t  (s)", Velocity}, PlotLabel → "Linear Velocity of CM"]
Plot[{-1.41 - 2.83 x, 1.57 + 3.14 x}, {x, 0, 5}, AxesLabel → {"t  (s)", Angular Vel},
 PlotLegends → {"Wheels 1 & 2", "Wheel 3"}, PlotLabel → "Angular Velocity Wheels"]
Plot[424.35x + 848.70 x^2, {x, 0, 5}, AxesLabel → {"t  (s)", Displacement},
 PlotLabel → "X Displacement of CM"]
Plot[{(424.35x + 848.70 x^2) + 300 * Cos[(-1.41 - 2.83 x) x],
   (424.35x + 848.70 x^2) + 300 * Cos[(1.57 + 3.14 x) x]}, {x, 0, 5},
 AxesLabel → {"t  (s)", Displacement}, PlotLegends → {"Wheels 1 & 2", "Wheel 3"},
 PlotLabel → "X Displacement of Pt on Wheel Surface"]
ParametricPlot[{(424.35x + 848.70 x^2) + 300 * Cos[(-1.41 - 2.83 x) x],
   300 * (1 + Sin[(-1.41 - 2.83 x) x])}, {x, 0, 5},
 AspectRatio → 1 / 3, AxesLabel → {"t  (s)", Position},
 PlotLabel → "Position of Pt on Surface of Wheel 1"]
Plot[848.70, {x, 0, 5}, AxesLabel → {"t  (s)", Accel},
 PlotLabel → "Linear Acceleration of CM"]
Plot[{-2.83, 3.14}, {x, 0, 5}, AxesLabel → {"t  (s)", Angular Accel},
 PlotLegends → {"Wheels 1 & 2", "Wheel 3"},
 PlotLabel → "Angular Acceleration of Wheels"]
```

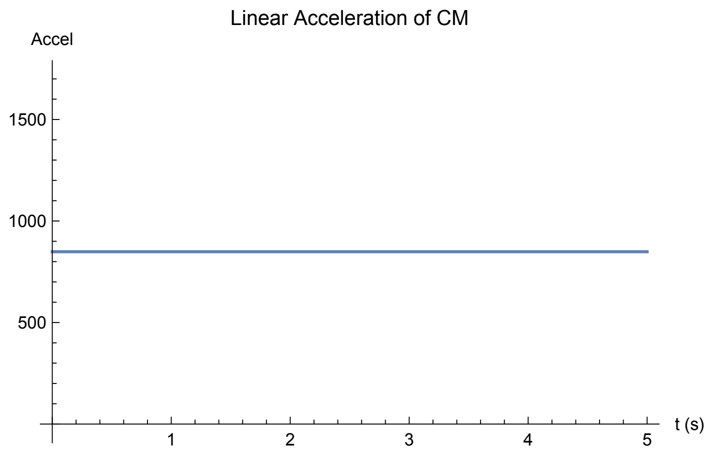

X Displacement of CM
Displacement
20000
15000
10000
5000
1
2
3
4
5
t (s)


X Displacement of Pt on Wheel Surface
Displacement
20000
15000
10000
5000
1
2
3
4
5
t (s)
Wheels 1 & 2
Wheel 3

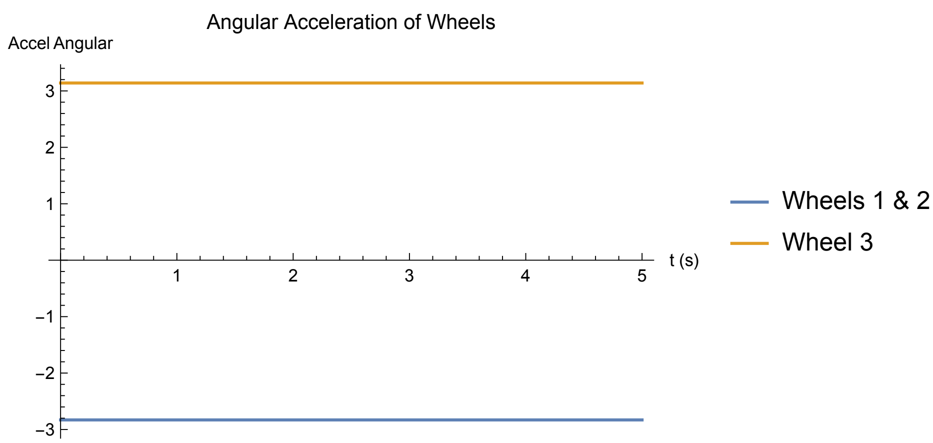

Position of Pt on Surface of Wheel 1
Position
600
500
400
300
200
100
5000
10000
15000
20000
t (s)


Linear Velocity of CM
Velocity
4000
3000
2000
1000
1
2
3
4
5
t (s)


Angular Velocity Wheels
Angular Vel
15
10
5
−5
−10
−15
1
2
3
4
5
t (s)
Wheels 1 & 2
Wheel 3



Linear Acceleration of CM
Accel
1500
1000
500
1
2
3
4
5
t (s)



Angular Acceleration of Wheels
Accel Angular
3
2
1
−1
−2
−3
1
2
3
4
5
t (s)
Wheels 1 & 2
Wheel 3

```
(* Deriving the constraints in a more general way. *)

r1 = {x1, y1};
r2 = {x2, y2};
r3 = {x3, y3};
r4 = {x4, y4};

A1 = {{Cos[ϕ1], -Sin[ϕ1]}, {Sin[ϕ1], Cos[ϕ1]}};
A2 = {{Cos[ϕ2], -Sin[ϕ2]}, {Sin[ϕ2], Cos[ϕ2]}};
A3 = {{Cos[ϕ3], -Sin[ϕ3]}, {Sin[ϕ3], Cos[ϕ3]}};
A4 = {{1, 0}, {0, 1}};

sp1 = {0, 0};
sp4 = {-450, 0};
sq4 = {450, 0};
sq2 = {0, 0};
sr4 = {0, ((300 + 270.15) ^2 - 450^2) ^0.5};
sr3 = {0, 0};

rp4 = r4 + A4.sp4;
rp1 = r1 + A1.sp1;
rq4 = r4 + A4.sq4;
rq2 = r2 + A2.sq2;
rr4 = r4 + A4.sr4;
rr3 = r3 + A3.sr3;

rp4 - rp1
rq4 - rq2
rr4 - rr3
{-450 - x1 + x4, -y1 + y4}

{450 - x2 + x4, -y2 + y4}

{0. - x3 + x4, 350.101 - y3 + y4}
```