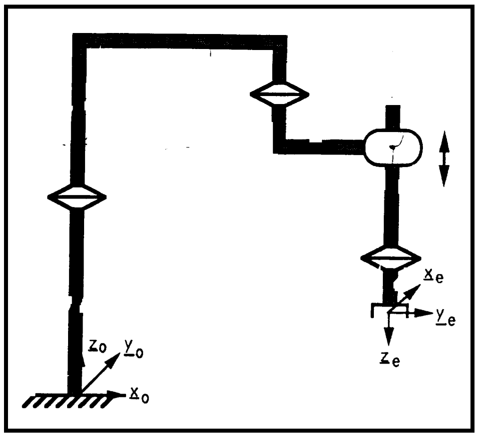
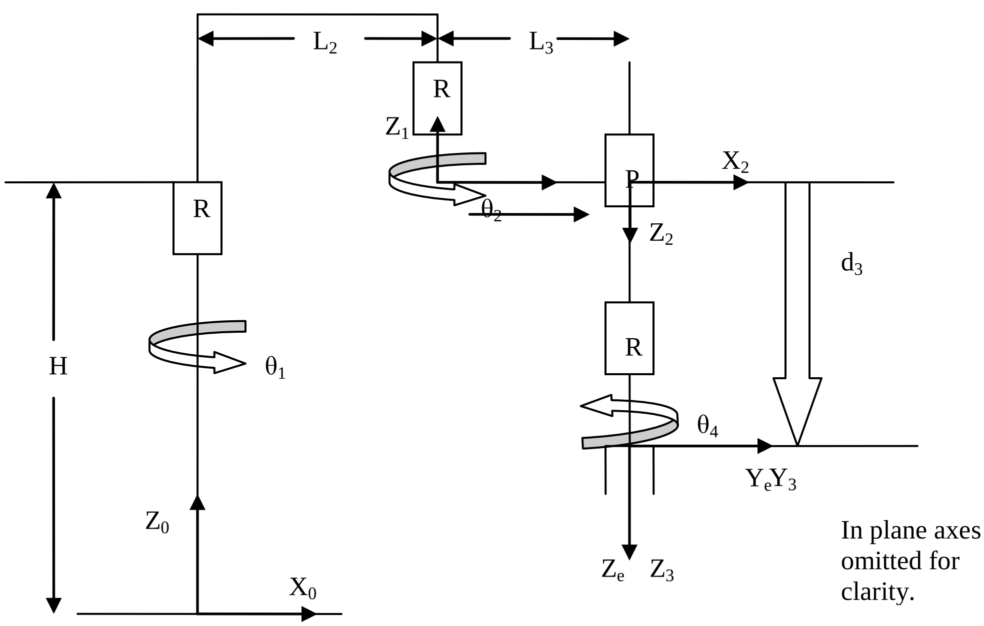
ME4524 – Robotics and Automation

Exercise # 2 - Solutions

1. Develop the kinematic model for obtaining the location of the end-effector as a function of the generalized coordinates for the SCARA-type robot shown below.



Answer:



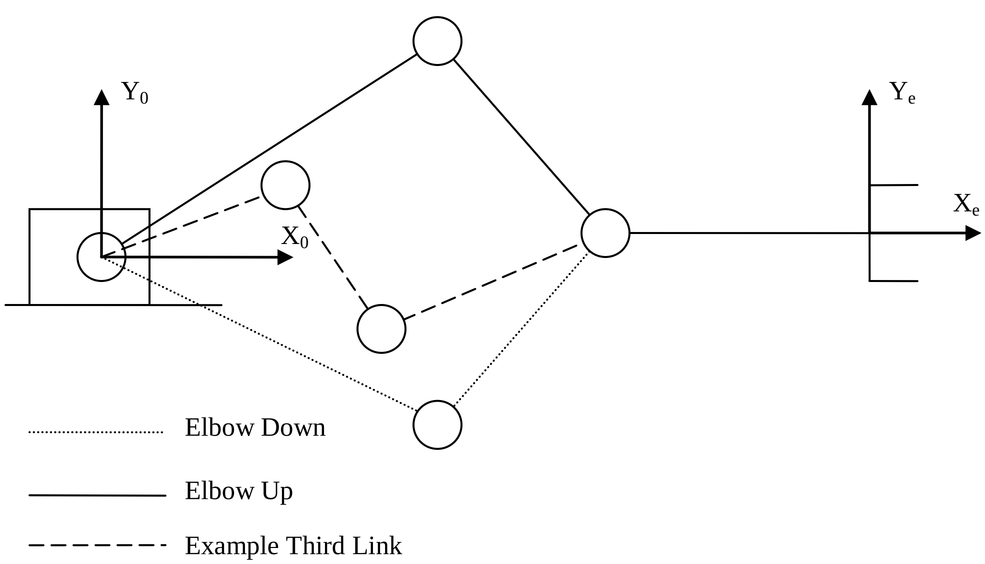
The solution to this manipulator can be computed without using the D-H convention. By inspection, motion in the X-Y plane is equivalent to a two-link planar robot, and the position in the z direction is easy to compute if H is chosen so that it is equal to the height of O_3 in the fixed frame.

$$O_e = \begin{bmatrix} L_2 c\theta_1 + L_3 c\theta_{12} \\ L_2 s\theta_1 + L_3 s\theta_{12} \\ H - d_3 \end{bmatrix}$$

Also, if we just want to measure the absolute angle through which the manipulator has turned, we can say that $\theta_e = \theta_1 + \theta_2 - \theta_4$, since all rotations occur about an axis normal to the x-y plane.

2. Given the desired position and orientation of the end effector of a 3-link planar rotary jointed manipulator, there are two possible solutions. Sketch the two solutions. If we add one more rotational joint (in such a way that it is still planar), how many solutions are there?

<u>Answer:</u> The two solutions are the “elbow-up” and “elbow down” positions, shown below. A fourth rotational joint will allow for infinite solutions that fall in between the elbow up and elbow down configurations.

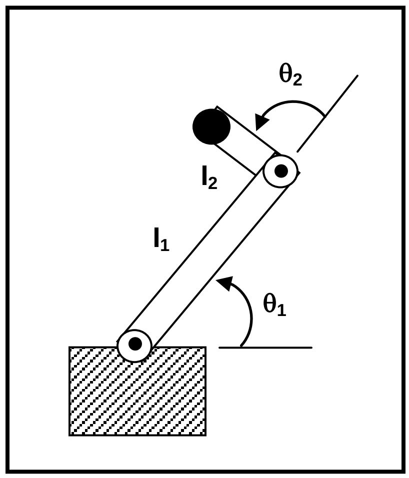


3. The Figure below shows a 2-link planar arm with rotary joints. For this arm, the second link is half as long as the first, that is $l_1 = 2l_2$. Also the joint limits are:

$$0 < \theta_1 < 180^0$$

$$-90^0 < \theta_2 < 180^0$$

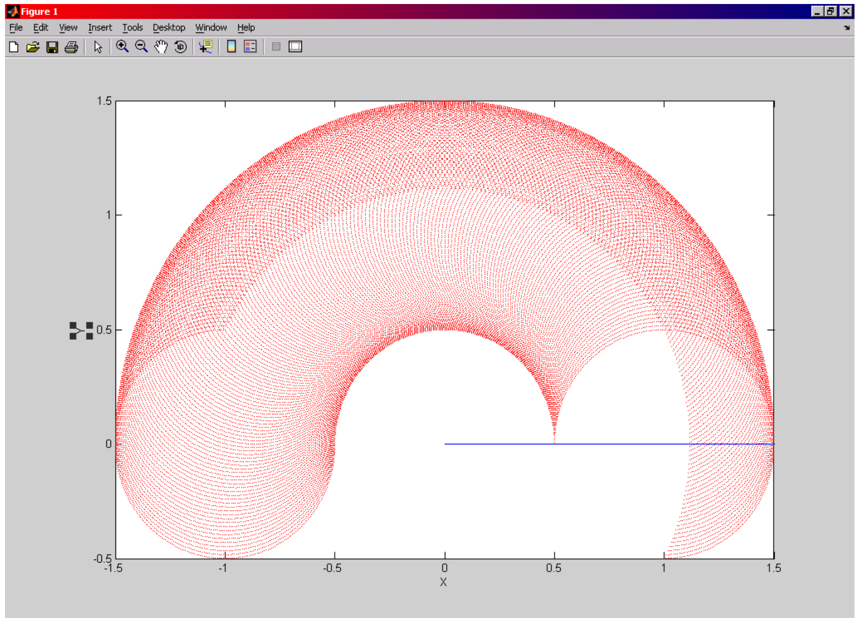
Sketch the approximate reachable workspace of the tip of link 2.



Here is some MATLAB code that can solve the problem fairly accuratly. The code simply plots the position of the end effector through all possible ranges of motion assuming that the length of link 1 is 1. :

```
for i=0:1:180
for j=-90:1:180
plot(cos(pi*i/180)+0.5*cos((i+j)*pi/180),sin(pi*i/180)+0.5*sin((i+j)*pi/180), ‘b.’);
hold on;
end
end
```

Here is a picture of what the code should generate:



4. Solve the inverse kinematics problem of the SCARA-type robot in problem 1. Assume that the end-effector position is given by a four-dimentional vector consisting of three dimentional position and the orientation of the end-effector about the $\underline{z}_0$ axis (or $\underline{z}_e$ axis).

<u>Answer:</u> From 1, we have $O_e = \begin{bmatrix} L_2 c\theta_1 + L_3 c\theta_{12} \\ L_2 s\theta_1 + L_3 s\theta_{12} \\ H - d_3 \end{bmatrix}, \theta_e = \theta_1 + \theta_2 - \theta_4$ and we want to solve

for some $O_e = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \theta_e = \Theta$.

Let's start with the easy part. H-d_3=z, so d_3=H-z.

We will need to use some trigonometry to solve the two equations. $\begin{aligned} x &= L_2 c\theta_1 + L_3 c\theta_{12} \\ y &= L_2 s\theta_1 + L_3 s\theta_{12} \end{aligned}$

From the law of cosines for a planar robot we have $x^2 + y^2 = L_2^2 + L_3^2 + 2L_2L_3\cos\theta_2$.

Rather than trying to take the inverse cosine of this solution, we will use an identity in order to take the inverse tangent since the atan2 function will give us answer(s) that take into consideration the signs on y and x.

$$\cos\theta = \frac{x^2 + y^2 - L_2^2 - L_3^2}{2L_2 L_3} = D$$

Noting that $\sin\theta = \pm\sqrt{1-\cos^2\theta} = \pm\sqrt{1-D^2}$ we have

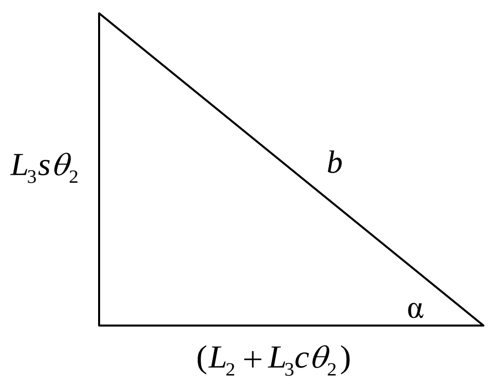
$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1-D^2}}{D}$$

Note there will be two solutions, corresponding to the elbow up and elbow down configuration.

To find θ_1, we rewrite the equations in the following way:

$$x = (L_2 + L_3 c\theta_2)c\theta_1 - L_3 s\theta_1 s\theta_2$$
$$y = (L_2 + L_3 c\theta_2)s\theta_1 + L_3 s\theta_2 c\theta_1$$

We now define a right angle triangle in the following way:



let $b = \sqrt{(L_2 + L_3 c\theta_2)^2 + (L_3 s\theta_2)^2}$

Then
$$c\alpha = \frac{L_2 + L_3 c\theta_2}{b}$$
$$s\alpha = \frac{L_3 s\theta_2}{b}$$

We now have

$$\frac{x}{b} = c\alpha c\theta_1 - s\alpha s\theta_1 = \cos(\alpha + \theta_1)$$
$$\frac{y}{b} = c\alpha s\theta_1 + s\alpha c\theta_1 = \sin(\alpha + \theta_1)$$

Which gives us

$$\frac{y}{x} = \tan(\alpha + \theta_1)$$

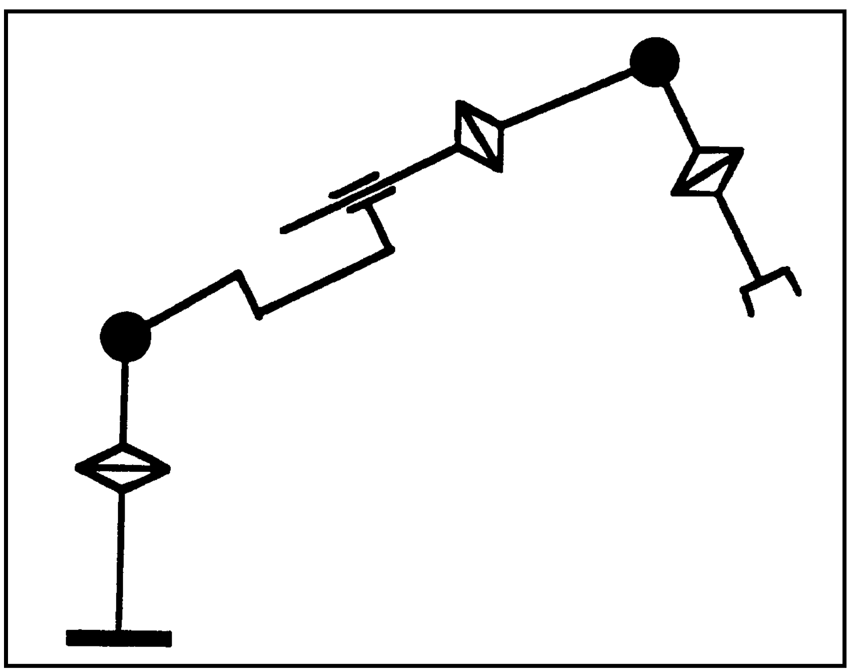
$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \alpha$$

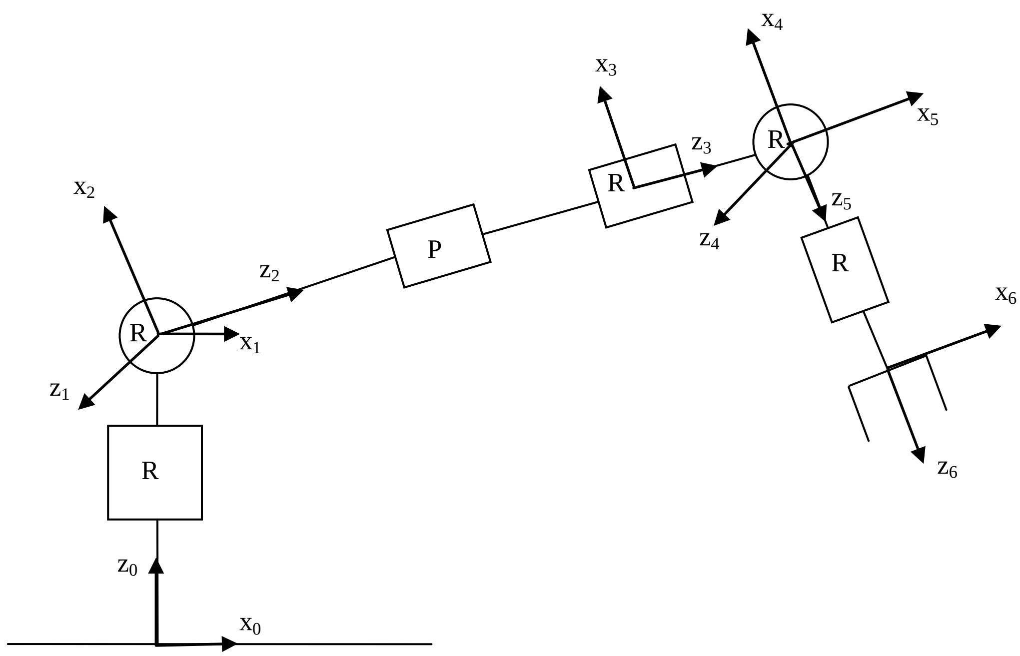
To solve for α, use $\alpha = \tan^{-1}\left(\frac{L_3 s\theta_2}{L_2 + L_3 c\theta_2}\right)$. Solve this equation for the two values of θ_2 found above and then plug in to find the corresponding value of θ_1.

Finally, to find θ_4, solve $\theta_4 = -\Theta + \theta_1 + \theta_2$ using the two previously calculated solutions.

5. (a) Develop the kinematic model of the manipulator in the Figure below and obtain H_{06}.

   (b) Solve the inverse kinematics for this manipulator.





x4
x3
R
x5
z3
R
z5
x2
z4
R
P
z2
x6
R
x1
z1
z6
R
z0
x0

Answer:

Joint	a	d	α	θ
1	0	d_1	90°	$\boldsymbol{\theta_1}$
2	0	0	90°	$\boldsymbol{\theta_2}$
3	0	$\mathbf{d_3}$	0	0
4	0	d_4	-90°	$\boldsymbol{\theta_4}$
5	0	0	90°	$\boldsymbol{\theta_5}$
6	0	d_6	0	$\boldsymbol{\theta_6}$

$$H_{01} = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & 0 \\ s\theta_1 & 0 & -c\theta_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{12} = \begin{bmatrix} c\theta_2 & 0 & s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{34} = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{45} = \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{56} = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} H_{02} = \begin{bmatrix} c\theta_1 c\theta_2 & s\theta_1 & c\theta_1 s\theta_2 & 0 \\ s\theta_1 c\theta_2 & -c\theta_1 & s\theta_1 s\theta_2 & 0 \\ s\theta_2 & 0 & -c\theta_2 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{03} = \begin{bmatrix} c\theta_1 c\theta_2 & s\theta_1 & c\theta_1 s\theta_2 & d_3 c\theta_1 s\theta_2 \\ s\theta_1 c\theta_2 & -c\theta_1 & s\theta_1 s\theta_2 & d_3 s\theta_1 s\theta_2 \\ s\theta_2 & 0 & -c\theta_2 & d_1 - d_3 c\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{04} = \begin{bmatrix} c\theta_1 c\theta_2 c\theta_4 + s\theta_1 s\theta_4 & -c\theta_1 s\theta_2 & -c\theta_1 c\theta_2 s\theta_4 + s\theta_1 c\theta_4 & d_3 c\theta_1 s\theta_2 + d_4 c\theta_1 s\theta_2 \\ s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_4 & -s\theta_1 s\theta_2 & -s\theta_1 c\theta_2 c\theta_4 - c\theta_1 c\theta_4 & d_3 s\theta_1 s\theta_2 + d_4 s\theta_1 s\theta_2 \\ s\theta_2 c\theta_4 & c\theta_2 & -s\theta_2 s\theta_4 & d_1 - d_3 c\theta_2 - d_4 c\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{05} = \begin{bmatrix} (c\theta_1 c\theta_2 c\theta_4 + s\theta_1 s\theta_4) c\theta_5 - c\theta_1 s\theta_2 s\theta_5 & -c\theta_1 c\theta_2 s\theta_4 + s\theta_1 c\theta_4 & (c\theta_1 c\theta_2 c\theta_4 + s\theta_1 s\theta_4) s\theta_5 + c\theta_1 s\theta_2 c\theta_5 & d_3 c\theta_1 s\theta_2 + d_4 c\theta_1 s\theta_2 \\ (s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_4) c\theta_5 - s\theta_1 s\theta_2 s\theta_5 & -s\theta_1 c\theta_2 c\theta_4 - c\theta_1 c\theta_4 & (s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_4) s\theta_5 + s\theta_1 s\theta_2 c\theta_5 & d_3 s\theta_1 s\theta_2 + d_4 s\theta_1 s\theta_2 \\ s\theta_2 c\theta_4 c\theta_5 + c\theta_2 s\theta_5 & -s\theta_2 s\theta_4 & s\theta_2 c\theta_4 s\theta_5 - c\theta_2 c\theta_5 & d_1 - d_3 c\theta_2 - d_4 c\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{06} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & x \\ a_{21} & a_{22} & a_{23} & y \\ a_{31} & a_{32} & a_{33} & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$a_{11} = ((c_1 c_2 c_4 + s_1 s_4) c_5 - c_1 s_2 s_5)) c_6 + (s_1 c_4 - c_1 c_2 s_4) s_6$

$a_{12} = ((c_1 c_2 c_4 + s_1 s_4) c_5 - c_1 s_2 s_5))(-s_6) + (s_1 c_4 - c_1 c_2 s_4) c_6$

$a_{13} = (c\theta_1 c\theta_2 c\theta_4 + s\theta_1 s\theta_4) s\theta_5 + c\theta_1 s\theta_2 c\theta_5$

$a_{21} = ((s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_4) c\theta_5 - s\theta_1 s\theta_2 s\theta_5) c\theta_6 - (s\theta_1 c\theta_2 c\theta_4 + c\theta_1 c\theta_4) s\theta_6$

$a_{22} = ((s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_4) c\theta_5 - s\theta_1 s\theta_2 s\theta_5)(-s\theta_6) - (s\theta_1 c\theta_2 c\theta_4 + c\theta_1 c\theta_4) c\theta_6$

$a_{23} = (s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_4) s\theta_5 + s\theta_1 s\theta_2 c\theta_5$

$a_{31} = (s\theta_2 c\theta_4 c\theta_5 + c\theta_2 s\theta_5) c\theta_6 - s\theta_2 s\theta_4 s\theta_6$

$a_{32} = (s\theta_2 c\theta_4 c\theta_5 + c\theta_2 s\theta_5)(-s\theta_6) - s\theta_2 s\theta_4 c\theta_6$

$a_{33} = s\theta_2 c\theta_4 s\theta_5 - c\theta_2 c\theta_5$

$x = d_3 c\theta_1 s\theta_2 + d_4 c\theta_1 s\theta_2 + d_6 ((c\theta_1 c\theta_2 c\theta_4 + s\theta_1 s\theta_4) s\theta_5 + c\theta_1 s\theta_2 c\theta_5)$

$y = d_3 s\theta_1 s\theta_2 + d_4 s\theta_1 s\theta_2 + d_6 ((s\theta_1 c\theta_2 c\theta_4 - c\theta_1 s\theta_4) s\theta_5 + s\theta_1 s\theta_2 c\theta_5)$

$z = d_1 - d_3 c\theta_2 - d_4 c\theta_2 + d_6 (s\theta_2 c\theta_4 s\theta_5 - c\theta_2 c\theta_5)$

b) The geometry of this problem is such that we cannot use any simple trigonometry to solve for the position of the manipulator. We will need to use inverse transform matrices to derive equations for the solutions of the angles.

It helps to work with numbers in this example. We use the following angle vector:

θ=[0 -45 0 0 -90 -90]

The tranform matrix works out to

$$H_{06} = \begin{bmatrix} 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}(d_3 + d_4 + d_6) \\ 1 & 0 & 0 & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}(d_6 - d_4 - d_3) + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H_{06}H_{56}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}}s_6 & \frac{-1}{\sqrt{2}}c_6 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}(d_3 + d_4) \\ c_6 & s_6 & 0 & 0 \\ \frac{1}{\sqrt{2}}s_6 & \frac{-1}{\sqrt{2}}c_6 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}(-d_4 - d_3) + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = H_{05}$$

$$H_{01}^{-1}H_{06}H_{56}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}}s_6c_1 + s_1c_6 & \frac{-1}{\sqrt{2}}c_6c_1 + s_1s_6 & \frac{-1}{\sqrt{2}}c_1 & \frac{-1}{\sqrt{2}}(d_3 + d_4)c_1 \\ \frac{1}{\sqrt{2}}s_6 & \frac{1}{\sqrt{2}}c_6 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}(-d_4 - d_3) \\ \frac{1}{\sqrt{2}}s_6s_1 - c_1c_6 & \frac{-1}{\sqrt{2}}c_6s_1 - c_1s_6 & \frac{-1}{\sqrt{2}}s_1 & \frac{1}{\sqrt{2}}(-d_4 - d_3)s_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = H_{15}$$

If you calculate H_{15} using H_{12}, $H_{23,}$ $H_{34,}$ $H_{45,}$ you will find that the fourth column is:

$$\begin{bmatrix} s_2(d_4+d_3) \\ c_2(-d_4-d_3) \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}}(d_3+d_4)c_1 \\ \frac{1}{\sqrt{2}}(-d_4-d_3) \\ \frac{1}{\sqrt{2}}(-d_4-d_3)s_1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} s_2 \\ c_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{\sqrt{2}}c_1 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}(-d_4-d_3)s_1 \end{bmatrix}$$

There is enough information here to determine θ_2, θ_1, and d_3.

$$\theta_2 = \tan^{-1}\frac{\pm\sqrt{1-(\frac{1}{\sqrt{2}})^2}}{\frac{1}{\sqrt{2}}} \qquad c\theta_1 = -\tan\theta_2 \qquad \theta_1 = \tan^{-1}\frac{\pm\sqrt{1-(-\tan\theta_2)^2}}{-\tan\theta_2} \qquad 0 = (-d_3-d_4)s\theta_1$$

We now know all of the variables in H_{03}.

$$H_{34}^{-1}H_{03}^{-1}H_{06} = H_{45}H_{56}$$

$$\begin{bmatrix} -s_4 & 0 & -c_4 & -c_4d_6 \\ 0 & -1 & 0 & 0 \\ -c_4 & 0 & s_4 & d_6s_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_5c_6 & -c_5s_6 & s_5 & d_6s_5 \\ s_5c_6 & -s_5s_6 & -c_5 & -d_6c_5 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use the equations $\sin\theta_4 = 0$, $\cos\theta_6 = 0$, $-\cos\theta_5 = 0$ to solve for the remaining joint angles.