Notes on Control Theory

Introduction

Control theory deals with the control and analysis of systems of differential equations or difference equations. I here give a brief outline of some of the most important topics, starting with some basic results on existence and uniqueness, basic definitions of stability, and important results from linear systems theory. I then discuss some introductory topics from nonlinear systems theory, optimal control, and stochastic estimation.

Local Existence and Uniqueness

Let $f(t, x)$ be piecewise continuous in t and satisfy the Lipschitz condition $\|f(t, x) - f(t, y)\| \le L\|x - y\|$, $\forall\ x, y \in B = \{x \in R^n |\ \|x - x_0\| \le r\ \}$, $\forall\ t \in [t_0, t_1]$. Then, $\exists\ \delta > 0$ such that $x'(t) = f(t, x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

Global Existence and Uniqueness

If $f(t, x)$ satisfies the Lipschitz condition for $\forall\ x, y \in R,\ \forall\ t \in [t_0, t_1]$, then $x'(t) = f(t, x)$ with $x(t_0) = x_0$ has a unique solution over $[t_0, t_1]$.

Definition: Stability

Consider the nonlinear system $x'(t) = f(t, x)$, where $f(t, x)$ is piecewise continuous in t and locally Lipschitz in x . Let $x = 0$ be an equilibrium point.
The equilibrium point is

- Stable if, for each $\varepsilon > 0$, $\exists\ \delta(\varepsilon,\ t_0) > 0$ such that $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon,\ \forall t \ge t_0 \ge 0$
- Uniformly stable if, for each $\varepsilon > 0$ in the previous equation, the choice of δ can be made independent of t_0 .
- Asymptotically stable if it is stable and $\delta(t_0)$ can be chosen such that $\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$.
- Exponentially stable if $\exists\ c, k, \lambda$ such that $\|x(t)\| \le k\|x(t_0)\|e^{-\lambda(t - t_0)}$,

$\forall\ \|x(t_0)\| < c$.

- Unstable if not stable.

Linear Systems

We begin with the simplest class of dynamical systems. A continuous-time linear system is defined, in state-space form, by:

$x'(t) = A(t)x(t) + B(t)u(t)$

$y(t) = C(t)x(t) + D(t)u(t)$

The signals $x(t) : [0, \infty) \to R^k,\ u(t) : [0, \infty) \to R^n,\ y(t) : [0, \infty) \to R^m$ are called the input, state, and output of the system.

Letting $x(t_0) = x_0,$ the unique solution to the foregoing equations is given by:

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

$$y(t) = C(t)\Phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$, where $\Phi(t, t_0)$, the state-transition matrix, is the unique solution to $\frac{d}{dt}\Phi(t, t_0) = A(t)\Phi(t, t_0)$.

Linear Time Invariant Systems

A linear time invariant (LTI) system is the special case of a linear system in which the matrices $A(t) = A, B(t) = B, C(t) = C, D(t) = D$ are not functions of time. LTI systems have several useful properties.

The linear state space ODE $x'(t) = Ax(t)$ is asymptotically stable if A is Hurwitz, that is if every eigenvalue of A has strictly negative real part. Note that linear systems can have no more than one equilibrium point.

The output of an LTI system is the convolution of the input with the impulse response: $y(t) = (x * h)(t),$ where the impulse response $h(t)$ can be found using Laplace transforms as $h(t) = L^{-1}[C(sI - A)^{-1}B + D]$. (Here, L is the Laplace transform operator). $H(s)\ =\ C(sI - A)^{-1}B + D$ is called the transfer function.

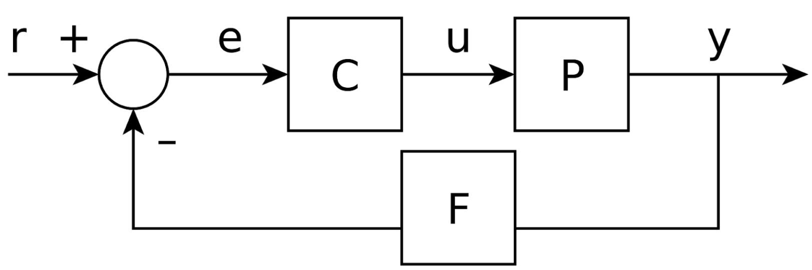
In an LTI system, $\Phi(t, t_0) = e^{A(t-t_0)}$, where e^M is the matrix exponential $e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$. It follows that the unique solution to an LTI system is:

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-t_0)} Bu(\tau) d\tau$$

$$y(t) = Ce^{A(t-t_0)} x_0 + \int_{t_0}^{t} Ce^{A(t-t_0)} Bu(\tau) d\tau + Du(t)$$

Feedback of a Single-Input-Single-Output (SISO) System

Control systems are often represented visually with block diagrams. An example SISO system is represented in the block diagram below, where *C*, *P*, *F* are referred to as the controller, plant, and observer, respectively.



Assuming that the system above is LTI, we may take Laplace transforms of the dynamic equations to obtain:
$Y(s) = P(s)U(s)$, $U(s) = C(s)E(s)$, $E(s) = R(s) - F(s)Y(s)$, where $E(s)$ is the Laplace transform of the error signal. Solving for $Y(s)$ in terms of $R(s)$ gives:
$Y(s) = \left(\frac{P(s)C(s)}{1+P(s)C(s)F(s)}\right)R(s) = H(s)R(s)$, where $H(s)$ is referred to as the closed-loop transfer function of the system. The system above is asymptotically stable provided all of the poles of the closed-loop transfer function have strictly negative real part.

PID Control

One simple and very common type of feedback control is PID, or Proportional-Derivative-Integral control, which is a control input of the form:
$u(t) = k_P e(t) + k_I \int_{t_0}^{\tau} e(\tau) d\tau + k_D \frac{d}{dt} e(t)$. The proportional and derivative control terms act like a virtual spring-damper. Derivative action improves the

responsiveness of the controller, but is sensitive to measurement noise. Integral control helps to ensure zero steady-state error, but can cause instability in certain situations. These problems can be avoided by applying a moving time window or sigmoidal activation to the integral term. PID control does not guarantee stability or optimal performance, but is widely used for SISO systems, both for its simplicity and because it requires no knowledge of the system's underlying dynamical model.

Full State Feedback (Pole Placement)

Consider again the multi-input-multi-output (MIMO) LTI system:

$x'(t) = Ax(t) + Bu(t)$

$y(t) = Cx(t) + Du(t)$

Setting the input vector to be proportional to the state vector: $u(t) = -Kx(t)$, the system becomes:

$x'(t) = (A - BK)x(t)$

$y(t) = (C - DK)x(t)$

The closed-loop system is stable provided that the eigenvalues of $A - BK$ have strictly negative real part. Moving the closed-loop eigenvalues in the left-hand complex plane can be used to achieve desired closed-loop system properties, such as settling time, overshoot, and steady-state error.

Observability & Controllability

Definition (Observable LTI): An LTI system is *observable*, i.e. "(A, C) observable", if for any unknown initial condition $x(0) \in \mathbf{C}_n$, there exists a finite time $t_1 > 0$ such that measurement of the output y over the interval $[0, t_1]$ suffices to *uniquely* determine $x(0)$. If the system is not observable then it is *unobservable*.

Definition (Controllable LTI): An LTI system is *controllable*, i.e. "(A, B) controllable", if for any terminal state $x_1 \in \mathbf{C}_n$, there exists an input $\widehat{u}$ defined on the interval $[0, t_1]$,
$t_1 > 0$, that transfers the state from the origin at $t = 0$ to x_1 at t_1.

LTI Observability Conditions: For an LTI system, the following statements are equivalent (similar conditions hold for the linear time-invariant case).

1.) The system is (A,C) observable.

2.) $W_O(t) = \int_{t_0}^{t} e^{A^T\tau} C^T C e^{A\tau} d\tau$ is nonsingular for any $t > t_0$.

3.) The "observability matrix" $[C,\ CA,\ CA^2, ..., CA^{n-1}]^T$ is full-rank.

On the other hand, if A is Hurwitz (the system is asymptotically stable), and the unique solution of the Lyapunov equation $A^T W_O + W_O A = -C^T C$ is positive definite, the system is observable. The solution to this equation is called the "observability Grammian" and can be expressed as $W_O = \lim_{t\to\infty} W_O(t)$.

LTI Controllability Conditions: Similar results hold for controllability. For an LTI system, the following statements are equivalent:

1.) The system is (A,B) controllable.

2.) $W_C(t) = \int_{t_0}^{t} e^{A\tau} B B^T e^{A^T\tau} d\tau$ is nonsingular for any $t > t_0$.

3.) The "controllability matrix" $[B,\ AB,\ A^2B, ..., A^{n-1}B]$ is full-rank.

If A is Hurwitz and the unique solution of the Lyapunov equation $AW_C + W_C A^T = -BB^T$ is positive definite, the system is controllable. The solution to this equation is called the "controllability Grammian" and can be expressed as $W_C = \lim_{t\to\infty} W_C(t)$.

Duality: A duality relation holds between controllability and observability. In particular, an LTI system with matrices (A, B, C, D) is controllable if and only if the system (A', C, B', D) is observable.

Definition (Exact Controllability): Given a matrix A, its exact controllability $n_D(A)$ is the minimum rank of all possible B matrices in which the corresponding LTI is (A, B) controllable.

Definition (Structural Controllability): Two LTI systems $x'(t) = Ax(t) + Bu(t)$ and $x'(t) = \overline{A}x(t) + \overline{B}u(t)$ are said to have the same structure (structural sparsity) if A and $\overline{A}$ and B and $\overline{B}$ are of the same dimension and have zero/nonzero entries in the same locations. An LTI system $x'(t) = Ax(t) + Bu(t)$ is said to be structurally controllable if and only if, for $\forall \varepsilon > 0$, there exists some controllable LTI of the same structure $x'(t) = \overline{A}x(t) + \overline{B}u(t)$ such that $\|A - \overline{A}\|, \|B - \overline{B}\| < \varepsilon$.

It has been shown that the exact controllability of a complex network described by the LTI $x'(t) = Ax(t) + Bu(t)$, in which A represents the graph adjacency matrix, is determined by the maximum matching in the network.

Stability of Nonlinear Systems

Establishing stability of nonlinear systems is more difficult. In fact, nonlinear systems may have multiple equilibria, or even non-isolated equilibria. The following definitions are useful for analyzing this larger class of systems.

Theorem (Lyapunov Stability)

Let $x = 0$ be an equilibrium point for the autonomous system $x'(t) = f(x)$ and $x \in D \subset R^n$. Suppose there exists a continuously differentiable function $V : D \to R$ such that $V(0) = 0$ and $V(x) > 0$ in $D - \{0\}$ and $V'(x) \leq 0$ in D. Then, $x = 0$ is stable. If $V'(x) \leq 0$ in $D - \{0\}$, then $x = 0$ is asymptotically stable. Note: this result can be extended to nonautonomous systems.

Definition: Invariant and Positively Invariant Sets

A set M is said to be invariant with respect to $x'(t) = f(x)$ if $x(0) \in M \Rightarrow x(t) \in M \;\forall t \in R$. M is said to be *positively* invariant if $x(0) \in M \Rightarrow x(t) \in M \;\forall t \geq 0$. In other words, all trajectories that start in M will stay in M for all future time.

Theorem (LaSalle's Invariance Principle)

Let $\Omega \subset D$ be a compact set that is positively invariant with respect to $x'(t) = f(x)$. Suppose there exists a continuously differentiable function $V : D \to R$ such that $V'(x) \leq 0$ in Ω. Let $E = \{x \in \Omega \mid V'(x) = 0\}$ and let M be the largest invariant set in E. Then, every solution starting in Ω approaches M as $t \to \infty$.

Theorem (Lyapunov's Indirect Method)

Let $x = 0$ be an equilibrium point for the autonomous nonlinear system $x' = f(x)$ and let $A = \frac{\partial}{\partial x} f(x)|_{x=0}$, so that an approximation of the nonlinear system about the origin is $x' = Ax$. Then the origin is asymptotically stable if A is Hurwitz (that is, $Re(\lambda_i) > 0$ for all eigenvalues of A) and unstable otherwise.

Example: Stability of the Hopfield Neural Network

A Hopfield artificial neural network with N neurons can be represented with the first order ODE:

$$x_i'(t) = \tfrac{1}{C_i} h_i(x_i(t)) \left(\sum_j T_{ij} x_j(t) - \tfrac{1}{R_j} g_i^{-1}(x_i(t)) + I_i \right), \; i \in 1, \ldots, N$$

where T is a symmetric matrix of neural connection weights, $R_i > 0$, $C_i > 0$, and I_i are constants, $g_i(\cdot) : R \to (-V_M, V_M)$ is a sigmoid function, and $h_i(x_i(t)) = \frac{dg_i}{du_i}(u_i(t))|_{u_i = g_i^{-1}(x_i(t))}$. Note that $x_i \in H = \{x \in R^n | -V_M < x_i < V_M\}$ and $h_i(x_i) > 0$, $\forall x_i \in (-V_M, V_M)$, $i \in 1, \ldots, N$. The equilibrium points of the system are the solutions of $\sum_j T_{ij} x_j - \frac{1}{R_j} g_i^{-1}(x_i) + I_i = 0$. To apply LaSalle's Theorem, we need to find a function $V(x)$ with a negative semi-definite derivative. Observe that $x_i'(t) = -\frac{1}{C_i} h_i(x_i(t)) \nabla_i V(x(t))$, where

$V(x(t)) = -\frac{1}{2} \sum_i \sum_j T_{ij} x_i x_j + \sum_i \frac{1}{R_i} \int_0^{x_i} g_i^{-1}(y) dy - \sum_i I_i x_i$. Now

$V'(x(t)) = \sum_i \nabla_i V(x(t)) x_i'(t) = -\sum_i \frac{1}{C_i} h_i(x_i(t)) [\nabla_i V(x(t))]^2 \leq 0$ and

$V'(x(t)) = 0 \Rightarrow \nabla_i V(x(t)) = 0 \Rightarrow x(t) = 0$.

To apply LaSalle's Invariance Principle, we must construct a positive invariant compact set $\Omega(\varepsilon)$. Consider $\Omega(\varepsilon) = \{x \in R^m | -(V_M - \varepsilon) \le x_i \le (V_M - \varepsilon)$. For simplicity, let us consider an arctangent sigmoidal function:

$g_i(u_i) = A \arctan(\lambda u_i)$, $\lambda > 0$. Then, $x_i' = \frac{1}{C_i} h_i(x_i(t)) \left[\sum_j T_{ij} x_j - \frac{A}{\lambda R_i} tan(x_i/A) + I_i \right]$.

Since x_i and I_i are bounded, the bracketed term can be made strictly negative by choosing a sufficiently small value of ε. For $V_M - \varepsilon < |x_i(t)| < V_M$, it is then easy to show that $\frac{d}{dt}(x_i)^2 < 0 \ \forall \ i$, from which it follows that $\Omega(\varepsilon)$ is positively invariant. In fact, all $x \in H$ will converge to $\Omega(\varepsilon)$. Under the assumption that all equilibrium points are isolated, it can be shown that there are a finite number of equilibrium points in $\Omega(\varepsilon)$, and thus by LaSalle's Invariance Principle, any trajectory for this system must approach one of these equilibria as $t \to \infty$.

Input-Output Stability

An alternative way to characterize the stability of a dynamical system is to relate the output of the system directly to the input, without reference to the dynamic model of the system.

Definition: Class K Functions

Class *K* (and class *KL*) functions are comparison functions that prove useful in generalizing the foregoing theorems to nonautonomous systems, and proving results on boundedness, input-to-state stability, and input-output stability.
A continuous function $f_K : [0, a) \to [0, \infty)$ is said to belong to class *K* if it is strictly increasing and $f_K(0) = 0$. It is said to belong to class K_∞ if $a = \infty$ and $\lim_{r \to \infty} f_K(r) = \infty$.

Definition: *L*-Norm

The L_p norm of a piecewise-continuous vector function $u(t)$ is defined as

$\|u\|_{L_p} = \left(\int_0^\infty \|u(t)\|^p dt \right)^{1/p}$, where $\|\cdot\|$ is any norm. The space $L_p{}^m$ is defined as the set of all piecewise continuous functions $u : [0, \infty) \to R^m$ such that $\|u\|_{L_p} < \infty$.

Definition: *L*-Stability

Let $u_\tau = u(t)$ for $0 \le t \le \tau$ and $u_\tau = 0$ for $t > \tau$. A mapping is L stable if there exist a class *K* function α and a constant β such that for all *p*-integrable functions *u* and all $\tau \in [0, \infty)$, $\|(Hu)_\tau\|_L \le \alpha(\|u_\tau\|_L) + \beta$.

For causal, *L*-stable systems, $u \in L^m \Rightarrow Hu \in L^q$. The special case of $L\infty$ stability corresponds to the familiar notion of bounded-input-bounded-output (BIBO) stability; that is, if a system is $L\infty$ stable, then for every bounded input $u(t)$, the output $Hu(t)$ is also bounded.

Example: Sigmoidal Activation Function

Consider $h(u) = a + b\tanh(cu) = a + b(e^{cu} - e^{-cu})/(e^{cu} + e^{-cu})$, $a, b, c \ge 0$. Using the fact that $h'(u) = 4bc/(e^{cu} + e^{-cu})^2 \le bc$, we have that $|h(u)| \le a + bc|u|$. Thus, $h(\cdot)$ is finite-gain L_∞ stable.

Optimal Control

Consider the nonlinear dynamic system $x'(t) = f(x(t), u(t), t),\ x(t_0) = x_0$, where $x(\cdot)$ is an *n*-dimensional state vector, $u(\cdot)$ is an *m*-dimensional control input, and $t \in [t_0, t_f]$ is time. Assume that $u(\cdot) \in U_B$, the set of bounded, piecewise-continuous *m*-dimensional vector functions on $[t_0, t_f]$, and that a unique solution for x exists on $[t_0, t_f]$. The objective of the optimal control problem is to find the control function $u^o(\cdot)$ that minimizes the objective function

$J(u(t), x_0) = \phi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t), t)dt$, where $L(\cdot)$ is the Lagrangian and $\phi(\cdot)$ is a boundary constraint. Assume that $L(\cdot)$, $f(\cdot)$, and $\phi(\cdot)$ are once continuously differentiable in x and that $L(\cdot)$ and $f(\cdot)$ are continuous in $u(\cdot)$ and t for $t \in [t_0, t_f]$.

Theorem (Pontryagin's Principle)

Suppose the control function $u^o(\cdot)$ minimizes the objective function $J(u(t), x_0)$ over all $u(\cdot) \in U_B$ and let $x^o(\cdot)$ be the state trajectory generated by $u^o(\cdot)$. Define the variational Hamiltonian as:

$H(x(t), u(t), \lambda(t), t) = L(x(t), u(t), t) + \lambda^T(t) f(x(t), u(t), t)$, where λ is a vector of Lagrange multipliers, generated by

$\frac{d}{dt}\lambda^T(t) = -H_x(x^o(t), u^o(t), \lambda(t), t), \quad \lambda^T(t_f) = \phi_x(x(t_f))$.

Then, the Hamiltonian satisfies the Weierstrass condition:

$H(x^o(t), u^o(t), \lambda(t), t) \le H(x^o(t), u(t), \lambda(t), t)$ for all $t \in [t_0, t_f]$.

In other words, the optimal control minimizes the Hamiltonian.

Furthermore, if $H(x^o(t), u(t), \lambda(t), t)$ is twice differentiable with respect to u, the generalized Legendre–Clebsch condition, $H_u(x^o(t), u^o(t), \lambda(t), t) = 0$ and $H_{uu}(x^o(t), u^o(t), \lambda(t), t) \ge 0$, is a *necessary* condition for optimality, and strict inequality, $H_{uu}(x^o(t), u^o(t), \lambda(t), t) > 0$ (the Hessian of the Hamiltonian is greater than zero along the trajectory of the solution), is a *sufficient* condition for *local* optimality.

Theorem (Hamilton-Jacobi-Bellman Equation)

Suppose there exists a function $V(x(t), t)$, which is once continuously differentiable in terms of $x(t)$ and t, and which satisfies the Hamilton-Jacobi-Bellman equation:

$-V_t(x(t), t) = min_{u(t) \in U_b}[L(x(t), u(t), t) + V_x(x(t), t) f(x(t), u(t), t)]$, $V(x(t_f), t_f) = \phi(x(t_f))$

Suppose that the control $u^o(x^o(t), t)$ is a solution of the minimization problem $min_{u(t) \in U_b}[L(x(t), u(t), t) + V_x(x(t), t) f(x(t), u(t), t)]$. Then $u^o(x^o(t), t)$ minimizes the objective function $J(u(t), x_0)$ over all $u(x(t), t) \in U_B$ and furthermore $V(x_0, t_0) = inf_{u(t) \in U_b} J(u(t), x_0)$. This condition is a necessary and sufficient condition for a global optimum.

Linear Quadratic Regulation

A problem of particular importance in optimal control theory is that of controlling an LTI system defined on $[t_0, t_f]$ while minimizing a quadratic cost functional, typically of the form:

$J = x^T(t_f)Fx(t_f) + \int_{t_0}^{t_f} \{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt$, where *F*, *Q*, and *R* are positive, semi-definite matrices. It can be shown that the feedback control law that minimizes J is $u(t) = -Kx(t)$, where $K = R^{-1}BP(t)$, with $P(t)$ given by the solution of the continuous-time Ricatti differential equation (backwards in time from the final condition):

$A^T P(t) + P(t)A - P(t)BR^{-1}B^T P(t) + Q = -P'(t)$, with the boundary condition $P(t_f) = F(t_f)$. In the so-called "infinite horizon" case,

$J = \int_{t_0}^{\infty} \{x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\}dt$, and *P* is a constant function of time, which can be found by solving the algebraic Ricatti equation:

$A^T P + PA - PBR^{-1}B^T P(t) + Q = 0$. Thus, for this problem, a *linear* control law is optimal among the set of *all* control laws.

Stochastic Estimation

The problem of stochastic estimation or filtering is to find a best estimate for a signal from a noisy and/or incomplete set of observations.

The Discrete-Time Kalman Filter

Consider the linear discrete-time dynamic system:

$x_{k+1} = \Phi_k x_k + \Gamma_k w_k$ (state equation)

$z_k = H_k x_k + v_k$ (measurement equation)

where w_k and v_k are the process noise and measurement noise at the *k*th time step, both assumed to be zero-mean, Gaussian, white-noise processes, $w_k \sim N(0, W_k)$, $v_k \sim N(0, V_k)$. Additionally define the generated covariance M_k and take $x_0 \sim N(\bar{x}_0, M_0)$. We make the following definitions:

- The measurement vector $Z_k = [z_0\ z_1\ \ldots\ z_k]^T$.
- The conditional mean $\widehat{x}_k = E\left[x_k | Z_k\right]$.

- The estimated state $\bar{x}_k = E\left[x_k | Z_{k-1}\right]$.
- The conditional covariance $P_k = E\left[(x_k - E[x_k])\,(x_k - E[x_k])^T | Z_k\right]$.

The discrete-time Kalman filter algorithm is as follows:

$\bar{x}_k = \Phi_k \widehat{x}_k$

$K_k = M_k H_k^{\,T} (H_k M_k H_k^{\,T} + V_k)^{-1}$ (traditionally called the Kalman gain)

$P_k = M_k - K_k H_k M_k$

$M_{k+1} = \Phi_k P_k \Phi_k^{\,T} + \Gamma_k W_k \Gamma_k^{\,T}$

$\widehat{x}_k = \bar{x}_k + P_k H_k^{\,T} V_k^{\,-1} (z_k - H_k \bar{x}_k)$

The Kalman filter can be altered to handle nonlinear systems. The standard in the field of nonlinear state estimation is the so called extended Kalman filter, which linearizes about estimates of the current mean and covariance.

Stochastic Optimal Control

The problem of stochastic optimal control generalizes the optimal control problem to the case of stochastic state variables. Of special importance is the linear-quadratic Gaussian control problem, which concerns linear systems driven by additive white Gaussian noise, with the objective of minimizing a quadratic cost functional. In the case of an LTI system, the separation principle holds: if a stable observer and a stable feedback controller are separately designed, the combined observer-feedback controller is guaranteed to be stable. The unique solution of this control problem is simply the combination of a Kalman filter (a linear-quadratic state estimator (LQE)), together with the linear-quadratic regulator (LQR).