Global Warming I: The Science and Modeling of Climate Change Peer-graded Assignment: What is heat and how can you warm up something in space?

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1 Question 1

1.a

Suppose $|\Psi\rangle$ is a product state, then we have

$$\begin{aligned}|\Psi\rangle &= \frac{1}{\sqrt{3}}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \\ &= |\alpha\rangle \otimes |\beta\rangle \\ &= (\alpha_0 |0\rangle + \alpha_1 |1\rangle) \otimes (\beta_0 |0\rangle + \beta_1 |1\rangle)\end{aligned}$$

This means that the following equations have to be satisfied:

$$\begin{cases} \alpha_0\beta_0 = \frac{1}{\sqrt{3}} \\ \alpha_0\beta_1 = \frac{1}{\sqrt{3}} \\ \alpha_1\beta_0 = \frac{1}{\sqrt{3}} \\ \alpha_1\beta_1 = 0 \end{cases}$$

We can see that it is impossible for satisfying all the equations. Equation 4 suggests that either α_1 or β_1 is 0, while equation 2 and 3 requires both of them not equal to 0. We see a contradiction. Thus, $|\Psi\rangle$ is an entangled state.

1.b

Suppose $|\Psi\rangle$ is a product state, then we have

$$\begin{aligned}|\Psi\rangle &= \frac{1}{3}(\sqrt{2}\,|0\rangle \otimes |0\rangle + i\,|0\rangle \otimes |1\rangle + 2\,|1\rangle \otimes |0\rangle + i\sqrt{2}\,|1\rangle \otimes |1\rangle) \\ &= |\alpha\rangle \otimes |\beta\rangle \\ &= (\alpha_0\,|0\rangle + \alpha_1\,|1\rangle) \otimes (\beta_0\,|0\rangle + \beta_1\,|1\rangle)\end{aligned}$$

This means that the following equations have to be satisfied:

$$\begin{cases} \alpha_0\beta_0 = \frac{\sqrt{2}}{3} \\ \alpha_0\beta_1 = \frac{i}{3} \\ \alpha_1\beta_0 = \frac{2}{3} \\ \alpha_1\beta_1 = \frac{i\sqrt{2}}{3} \end{cases}$$

Solving the equations, we have

$$\begin{cases} \alpha_0 = \frac{1}{\sqrt{3}} \\ \alpha_1 = \frac{\sqrt{2}}{\sqrt{3}} \\ \beta_0 = \frac{\sqrt{2}}{\sqrt{3}} \\ \beta_1 = \frac{i}{\sqrt{3}} \end{cases}$$

Hence, $|\Psi\rangle$ is a product state.

1.c

Suppose $|\Psi\rangle$ is a product state, then we have

$$\begin{aligned}|\Psi\rangle &= \frac{1}{2\sqrt{2}}((1+i)\,|0\rangle \otimes |0\rangle + (1-i)\,|0\rangle \otimes |1\rangle + (1-i)\,|1\rangle \otimes |0\rangle + (1+i)\,|1\rangle \otimes |1\rangle) \\ &= |\alpha\rangle \otimes |\beta\rangle \\ &= (\alpha_0\,|0\rangle + \alpha_1\,|1\rangle) \otimes (\beta_0\,|0\rangle + \beta_1\,|1\rangle)\end{aligned}$$

This means that the following equations have to be satisfied:

$$\begin{cases} \alpha_0\beta_0 = \frac{1+i}{2\sqrt{2}} \\ \alpha_0\beta_1 = \frac{1-i}{2\sqrt{2}} \\ \alpha_1\beta_0 = \frac{1-i}{2\sqrt{2}} \\ \alpha_1\beta_1 = \frac{1+i}{2\sqrt{2}} \end{cases}$$

Dividing equation 1 by equation 2, and equation 3 by equation 4, we have

$$\begin{cases} \frac{\beta_0}{\beta_1} = \frac{1+i}{1-i} \\ \frac{\beta_0}{\beta_1} = \frac{1-i}{1+i} \end{cases}$$

These euqations could not be satisfied at the same time. Thus, $|\Psi\rangle$ is an entangled state.

2 Question 2

2.a

$$
\begin{aligned}
U_t U_t^\dagger &= \left[\cos\left(\frac{t}{2}\right) I \otimes I - i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right] \left[\cos\left(\frac{t}{2}\right) I \otimes I - i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right]^\dagger \\
&= \left[\cos\left(\frac{t}{2}\right) I \otimes I - i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right] \left[\cos\left(\frac{t}{2}\right) I \otimes I + i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}^\dagger\right] \\
&= \cos^2\left(\frac{t}{2}\right) I \otimes I + \sin^2\left(\frac{t}{2}\right) \mathrm{SWAP}\ \mathrm{SWAP}^\dagger \\
&= \left(\cos^2\left(\frac{t}{2}\right) + \sin^2\left(\frac{t}{2}\right)\right) I \otimes I \\
&= I \otimes I
\end{aligned}
$$

Therefore, U_t is a unitary gate for every $t \in \mathbb{R}$.

2.b

$$
\begin{aligned}
U_{\pi/2}(|0\rangle \otimes |1\rangle) &= \left[\cos\left(\frac{\pi}{4}\right) I \otimes I - i \sin\left(\frac{\pi}{4}\right) \mathrm{SWAP}\right] (|0\rangle \otimes |1\rangle) \\
&= \left(\cos\left(\frac{\pi}{4}\right) I \otimes I\right) (|0\rangle \otimes |1\rangle) - i \sin\left(\frac{\pi}{4}\right) (|1\rangle \otimes |0\rangle) \\
&= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}(|1\rangle \otimes |0\rangle)
\end{aligned}
$$

2.c

For both Alice and Bob measure in $\{|0\rangle, |1\rangle\}$, we have four different outcomes: 00, 01, 10, 11. Denote probability for outcome m by p_m, then we have

$$
\begin{aligned}
p_{00} &= \left|(\langle 0| \otimes \langle 0|) U_{\pi/2}(|0\rangle \otimes |1\rangle)\right|^2 \\
&= \left|(\langle 0| \otimes \langle 0|) \left(\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}(|1\rangle \otimes |0\rangle)\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}(\langle 0| \otimes \langle 0|)(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}(\langle 0| \otimes \langle 0|)(|1\rangle \otimes |0\rangle)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}} \langle 0 \mid 0\rangle \langle 0 \mid 1\rangle - \frac{i}{\sqrt{2}} \langle 0 \mid 1\rangle \langle 0 \mid 0\rangle\right|^2 \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
p_{01} &= \left|(\langle 0| \otimes \langle 1|)U_{\pi/2}(|0\rangle \otimes |1\rangle)\right|^2 \\
&= \left|(\langle 0| \otimes \langle 1|)\left(\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}(|1\rangle \otimes |0\rangle)\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}((\langle 0| \otimes \langle 1|)(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}((\langle 0| \otimes \langle 1|)(|1\rangle \otimes |0\rangle)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\langle 0 \,|\, 0\rangle \langle 1 \,|\, 1\rangle - \frac{i}{\sqrt{2}}\langle 0 \,|\, 1\rangle \langle 1 \,|\, 0\rangle\right|^2 \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
p_{10} &= \left|(\langle 1| \otimes \langle 0|)U_{\pi/2}(|0\rangle \otimes |1\rangle)\right|^2 \\
&= \left|(\langle 1| \otimes \langle 0|)\left(\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}(|1\rangle \otimes |0\rangle)\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}((\langle 1| \otimes \langle 0|)(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}((\langle 1| \otimes \langle 0|)(|1\rangle \otimes |0\rangle)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\langle 1 \,|\, 0\rangle \langle 0 \,|\, 1\rangle - \frac{i}{\sqrt{2}}\langle 1 \,|\, 1\rangle \langle 0 \,|\, 0\rangle\right|^2 \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
p_{11} &= \left|(\langle 1| \otimes \langle 1|)U_{\pi/2}(|0\rangle \otimes |1\rangle)\right|^2 \\
&= \left|(\langle 1| \otimes \langle 1|)\left(\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}(|1\rangle \otimes |0\rangle)\right)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}((\langle 1| \otimes \langle 1|)(|0\rangle \otimes |1\rangle) - \frac{i}{\sqrt{2}}((\langle 1| \otimes \langle 1|)(|1\rangle \otimes |0\rangle)\right|^2 \\
&= \left|\frac{1}{\sqrt{2}}\langle 1 \,|\, 0\rangle \langle 1 \,|\, 1\rangle - \frac{i}{\sqrt{2}}\langle 1 \,|\, 1\rangle \langle 1 \,|\, 0\rangle\right|^2 \\
&= 0
\end{aligned}
$$

2.d

After applying U, we have

$$\begin{aligned}
U(|0\rangle \otimes |1\rangle) &= \left[\cos\left(\frac{t}{2}\right) I \otimes I - i\sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right](|0\rangle \otimes |1\rangle) \\
&= \cos\left(\frac{t}{2}\right)(|0\rangle \otimes |1\rangle) - i\sin\left(\frac{t}{2}\right)(|1\rangle \otimes |0\rangle)
\end{aligned}$$

Measuring in the basis $\{|nm\rangle\}$ for $n, m \in \{0, 1\}$, we have the probabilities

$$\begin{aligned}
p_{00} &= |(\langle 0| \otimes \langle 0|)U(|0\rangle \otimes |1\rangle)|^2 \\
&= \left|(\langle 0| \otimes \langle 0|)\left[\cos\left(\frac{t}{2}\right)(|0\rangle \otimes |1\rangle) - i\sin\left(\frac{t}{2}\right)(|1\rangle \otimes |0\rangle)\right]\right|^2 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
p_{01} &= \left|(\langle 0| \otimes \langle 1|)U_{\pi/2}(|0\rangle \otimes |1\rangle)\right|^2 \\
&= \left|(\langle 0| \otimes \langle 1|)\left[\cos\left(\frac{t}{2}\right)(|0\rangle \otimes |1\rangle) - i\sin\left(\frac{t}{2}\right)(|1\rangle \otimes |0\rangle)\right]\right|^2 \\
&= \left|\cos\left(\frac{t}{2}\right)\right|^2 \\
&= \cos^2\left(\frac{t}{2}\right)
\end{aligned}$$

$$\begin{aligned}
p_{10} &= |(\langle 1| \otimes \langle 0|)U(|0\rangle \otimes |1\rangle)|^2 \\
&= \left|(\langle 1| \otimes \langle 0|)\left[\cos\left(\frac{t}{2}\right)(|0\rangle \otimes |1\rangle) - i\sin\left(\frac{t}{2}\right)(|1\rangle \otimes |0\rangle)\right]\right|^2 \\
&= \left|-i\sin\left(\frac{t}{2}\right)\right|^2 \\
&= \sin^2\left(\frac{t}{2}\right)
\end{aligned}$$

$$\begin{aligned}
p_{11} &= |(\langle 1| \otimes \langle 1|)U(|0\rangle \otimes |1\rangle)|^2 \\
&= \left|(\langle 1| \otimes \langle 1|)\left[\cos\left(\frac{t}{2}\right)(|0\rangle \otimes |1\rangle) - i\sin\left(\frac{t}{2}\right)(|1\rangle \otimes |0\rangle)\right]\right|^2 \\
&= 0
\end{aligned}$$

Therefore, the state of the two qubits could only be $|0\rangle \otimes |1\rangle$ or $|1\rangle \otimes |0\rangle$. Applying U to both states, we have

$$\begin{aligned} U(|0\rangle \otimes |1\rangle) &= \left[\cos\left(\frac{t}{2}\right) I \otimes I - i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right] (|0\rangle \otimes |1\rangle) \\ &= \cos\left(\frac{t}{2}\right) (|0\rangle \otimes |1\rangle) - i \sin\left(\frac{t}{2}\right) (|1\rangle \otimes |0\rangle) \end{aligned}$$

and

$$\begin{aligned} U(|1\rangle \otimes |0\rangle) &= \left[\cos\left(\frac{t}{2}\right) I \otimes I - i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right] (|1\rangle \otimes |0\rangle) \\ &= \cos\left(\frac{t}{2}\right) (|1\rangle \otimes |0\rangle) - i \sin\left(\frac{t}{2}\right) (|0\rangle \otimes |1\rangle) \end{aligned}$$

2.e

After applying U, we have

$$\begin{aligned} U(|0\rangle \otimes |1\rangle) &= \left[\cos\left(\frac{t}{2}\right) I \otimes I - i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right] (|0\rangle \otimes |1\rangle) \\ &= \cos\left(\frac{t}{2}\right) (|0\rangle \otimes |1\rangle) - i \sin\left(\frac{t}{2}\right) (|1\rangle \otimes |0\rangle) \end{aligned}$$

Applying U again, we have

$$\begin{aligned} UU(|0\rangle \otimes |1\rangle) &= \left[\cos\left(\frac{t}{2}\right) I \otimes I - i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right]^2 (|0\rangle \otimes |1\rangle) \\ &= \left[\cos\left(\frac{t}{2}\right) I \otimes I - i \sin\left(\frac{t}{2}\right) \mathrm{SWAP}\right] \left[\cos\left(\frac{t}{2}\right) (|0\rangle \otimes |1\rangle) - i \sin\left(\frac{t}{2}\right) (|1\rangle \otimes |0\rangle)\right] \\ &= \cos^2\left(\frac{t}{2}\right) (|0\rangle \otimes |1\rangle) - 2i \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right) (|1\rangle \otimes |0\rangle) - \sin^2\left(\frac{t}{2}\right) (|0\rangle \otimes |1\rangle) \\ &= \left[\cos^2\left(\frac{t}{2}\right) - \sin^2\left(\frac{t}{2}\right)\right] (|0\rangle \otimes |1\rangle) - 2i \sin\left(\frac{t}{2}\right) \cos\left(\frac{t}{2}\right) (|1\rangle \otimes |0\rangle) \\ &= \left[\frac{1 + \cos t - 1 + \cos t}{2}\right] (|0\rangle \otimes |1\rangle) - i \sin t (|1\rangle \otimes |0\rangle) \\ &= \cos t (|0\rangle \otimes |1\rangle) - i \sin t (|1\rangle \otimes |0\rangle) \end{aligned}$$

3 Question 3

3.a

Alice measures qubit A in the $\{|0\rangle \otimes |1\rangle\}$ basis.
For the outcome of qubit A is $|0\rangle$, the state of Bob's system is

$$\begin{aligned}\frac{(\langle 0| \otimes I_B)\,|\Psi\rangle}{\|(\langle 0| \otimes I_B)\,|\Psi\rangle\|} &= \frac{\frac{1}{\sqrt{2}}(\langle 0\,|\,0\rangle\,|1\rangle)}{\left\|\frac{1}{\sqrt{2}}(\langle 0\,|\,0\rangle\,|1\rangle)\right\|} \\ &= |1\rangle\end{aligned}$$

For the outcome of qubit A is $|1\rangle$, the state of Bob's system is

$$\begin{aligned}\frac{(\langle 1| \otimes I_B)\,|\Psi\rangle}{\|(\langle 1| \otimes I_B)\,|\Psi\rangle\|} &= \frac{\frac{1}{\sqrt{2}}(-i\,\langle 1\,|\,1\rangle\,|0\rangle)}{\left\|\frac{1}{\sqrt{2}}(-i\,\langle 1\,|\,1\rangle\,|0\rangle)\right\|} \\ &= |0\rangle\end{aligned}$$

3.b

Bob measures qubit B in the $\{|+\rangle \otimes |-\rangle\}$ basis.
For the outcome of qubit B is $|+\rangle$, the state of Alice's system is

$$\begin{aligned}\frac{(I_A \otimes \langle +|)\,|\Psi\rangle}{\|(I_A \otimes \langle +|)\,|\Psi\rangle\|} &= \frac{\frac{1}{2}[I_A \otimes (\langle 0| + \langle 1|)](|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle)}{\left\|\frac{1}{2}[I_A \otimes (\langle 0| + \langle 1|)](|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle)\right\|} \\ &= \frac{-i\,|1\rangle + |0\rangle}{\|-i\,|1\rangle + |0\rangle\|} \\ &= \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}}\end{aligned}$$

For the outcome of qubit B is $|-\rangle$, the state of Alice's system is

$$\begin{aligned}\frac{(I_A \otimes \langle -|)\,|\Psi\rangle}{\|(I_A \otimes \langle -|)\,|\Psi\rangle\|} &= \frac{\frac{1}{2}[I_A \otimes (\langle 0| - \langle 1|)](|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle)}{\left\|\frac{1}{2}[I_A \otimes (\langle 0| - \langle 1|)](|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle)\right\|} \\ &= \frac{-i\,|1\rangle - |0\rangle}{\|-i\,|1\rangle - |0\rangle\|} \\ &= \frac{-\,|0\rangle - i\,|1\rangle}{\sqrt{2}}\end{aligned}$$

3.c

Denote probability of outcome nm for $n \in \{0,1\}, m \in \{+,-\}$ by p_{nm}, we have

$$\begin{aligned} p_{0+} &= |(\langle 0| \otimes \langle +|)\,|\Psi\rangle|^2 \\ &= \frac{1}{4}\,|(\langle 0| \otimes \langle 0| + \langle 0| \otimes \langle 1|)(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle)|^2 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} p_{0-} &= |(\langle 0| \otimes \langle -|)\,|\Psi\rangle|^2 \\ &= \frac{1}{4}\,|(\langle 0| \otimes \langle 0| - \langle 0| \otimes \langle 1|)(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle)|^2 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} p_{1+} &= |(\langle 1| \otimes \langle +|)\,|\Psi\rangle|^2 \\ &= \frac{1}{4}\,|(\langle 1| \otimes \langle 0| + \langle 1| \otimes \langle 1|)(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle)|^2 \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} p_{1-} &= |(\langle 1| \otimes \langle -|)\,|\Psi\rangle|^2 \\ &= \frac{1}{4}\,|(\langle 1| \otimes \langle 0| - \langle 1| \otimes \langle 1|)(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle)|^2 \\ &= \frac{1}{4} \end{aligned}$$

3.d

Denote the system state after Alice applys unitary gate U by $|\Psi_{UI}\rangle$. Then, we have system states after Alice applying different gates as follows

$$\begin{aligned} |\Psi_{II}\rangle &= [(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes I_B]\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle) \\ &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle) = |\Psi\rangle \end{aligned}$$

$$\begin{aligned} |\Psi_{XI}\rangle &= [(|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes I_B]\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle) \\ &= \frac{1}{\sqrt{2}}(|1\rangle \otimes |1\rangle - i\,|0\rangle \otimes |0\rangle) \end{aligned}$$

$$
\begin{aligned}
|\Psi_{YI}\rangle &= [(i\,|0\rangle\langle 1| - i\,|1\rangle\langle 0|) \otimes I_B]\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle) \\
&= \frac{1}{\sqrt{2}}(-i\,|1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle)
\end{aligned}
$$

$$
\begin{aligned}
|\Psi_{ZI}\rangle &= [(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes I_B]\frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - i\,|1\rangle \otimes |0\rangle) \\
&= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + i\,|1\rangle \otimes |0\rangle)
\end{aligned}
$$

Obviously, all the 4 states are normalised:

$$
\begin{aligned}
\|\,|\Psi_{II}\rangle\,\| &= \sqrt{\langle \Psi_{II} \,|\, \Psi_{II} \rangle} \\
&= \frac{1}{\sqrt{2}}\sqrt{2} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\|\,|\Psi_{XI}\rangle\,\| &= \sqrt{\langle \Psi_{XI} \,|\, \Psi_{XI} \rangle} \\
&= \frac{1}{\sqrt{2}}\sqrt{2} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\|\,|\Psi_{YI}\rangle\,\| &= \sqrt{\langle \Psi_{YI} \,|\, \Psi_{YI} \rangle} \\
&= \frac{1}{\sqrt{2}}\sqrt{2} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\|\,|\Psi_{ZI}\rangle\,\| &= \sqrt{\langle \Psi_{ZI} \,|\, \Psi_{ZI} \rangle} \\
&= \frac{1}{\sqrt{2}}\sqrt{2} \\
&= 1
\end{aligned}
$$

Second, check if all states are orthogonal to each other,

$$
\begin{aligned}
\langle \Psi_{II} \,|\, \Psi_{XI} \rangle &= \frac{1}{2}(\langle 0| \otimes \langle 1| + i\,\langle 1| \otimes \langle 0|)(|1\rangle \otimes |1\rangle - i\,|0\rangle \otimes |0\rangle) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \Psi_{II} \,|\, \Psi_{YI} \rangle &= \frac{1}{2}(\langle 0| \otimes \langle 1| + i\,\langle 1| \otimes \langle 0|)(-i\,|1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \Psi_{II} \mid \Psi_{ZI} \rangle &= \frac{1}{2}(\langle 0| \otimes \langle 1| + i \langle 1| \otimes \langle 0|)(|0\rangle \otimes |1\rangle + i |1\rangle \otimes |0\rangle) \\
&= \frac{1}{2} \times (1 - 1) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \Psi_{XI} \mid \Psi_{YI} \rangle &= \frac{1}{2}(\langle 1| \otimes \langle 1| + i \langle 0| \otimes \langle 0|)(-i |1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle) \\
&= \frac{1}{2} \times (-i + i) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \Psi_{XI} \mid \Psi_{ZI} \rangle &= \frac{1}{2}(\langle 1| \otimes \langle 1| + i \langle 0| \otimes \langle 0|)(|0\rangle \otimes |1\rangle + i |1\rangle \otimes |0\rangle) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\langle \Psi_{YI} \mid \Psi_{ZI} \rangle &= \frac{1}{2}(i |1\rangle \otimes |1\rangle + |0\rangle \otimes |0\rangle)(|0\rangle \otimes |1\rangle + i |1\rangle \otimes |0\rangle) \\
&= 0
\end{aligned}
$$

Therefore, $\{|\Psi_{II}\rangle, |\Psi_{XI}\rangle, |\Psi_{YI}\rangle, |\Psi_{ZI}\rangle\}$ is an orthonormal basis for $\mathbb{C}^4$.
Then, we could define the following unitary gate

$$
U = (|0\rangle \otimes |0\rangle) \langle \Psi_{II}| + (|0\rangle \otimes |1\rangle) \langle \Psi_{XI}| + (|1\rangle \otimes |0\rangle) \langle \Psi_{YI}| + (|1\rangle \otimes |1\rangle) \langle \Psi_{ZI}|
$$

Bob can put the two qubits through U and mearsure on the ONB $\{|n\rangle \otimes |m\rangle\}$ for $n, m \in \{0, 1\}$, the following table shows the outcomes and corresponding gate that Alice applied.

Outcome	State	Gate
00	Ψ_{II}	I
01	Ψ_{XI}	X
10	Ψ_{YI}	Y
11	Ψ_{ZI}	Z

4 Question 4

4.a

Denote the states as follows

$$|\Psi_1\rangle = |1\rangle \otimes |1\rangle$$
$$|\Psi_2\rangle = |0\rangle \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|\Psi_3\rangle = |0\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$
$$|\Psi_4\rangle = |2\rangle \otimes \frac{|1\rangle + |2\rangle}{\sqrt{2}}$$
$$|\Psi_5\rangle = |2\rangle \otimes \frac{|1\rangle - |2\rangle}{\sqrt{2}}$$
$$|\Psi_6\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |2\rangle$$
$$|\Psi_7\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \otimes |2\rangle$$
$$|\Psi_8\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}} \otimes |0\rangle$$
$$|\Psi_9\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}} \otimes |0\rangle$$

First, we can easily find that all states are normalised

$$\| \, |\Psi_1\rangle \, \| = 1$$
$$\| \, |\Psi_2\rangle \, \| = \sqrt{\frac{1+1}{2}} = 1$$
$$\| \, |\Psi_3\rangle \, \| = \sqrt{\frac{1+1}{2}} = 1$$
$$\| \, |\Psi_4\rangle \, \| = \sqrt{\frac{1+1}{2}} = 1$$
$$\| \, |\Psi_5\rangle \, \| = \sqrt{\frac{1+1}{2}} = 1$$
$$\| \, |\Psi_6\rangle \, \| = \sqrt{\frac{1+1}{2}} = 1$$
$$\| \, |\Psi_7\rangle \, \| = \sqrt{\frac{1+1}{2}} = 1$$
$$\| \, |\Psi_8\rangle \, \| = \sqrt{\frac{1+1}{2}} = 1$$
$$\| \, |\Psi_9\rangle \, \| = \sqrt{\frac{1+1}{2}} = 1$$

For orthogonality, we can build the following table indicating which basis do that state has/have.

Basis	00	01	02	10	11	12	20	21	22
$\|\Psi_1\rangle$					✓				
$\|\Psi_2\rangle$	✓	✓							
$\|\Psi_3\rangle$	✓	✓							
$\|\Psi_4\rangle$								✓	✓
$\|\Psi_5\rangle$								✓	✓
$\|\Psi_6\rangle$			✓			✓			
$\|\Psi_7\rangle$			✓			✓			
$\|\Psi_8\rangle$				✓			✓		
$\|\Psi_9\rangle$				✓			✓		

Then, states with no overlapping basis must be orthogonal. For those overlap ones, we could check by doing some maths

$$\begin{aligned}
\langle \Psi_2 \mid \Psi_3 \rangle &= \frac{1}{\sqrt{2}}(1-1) = 0 \\
\langle \Psi_4 \mid \Psi_5 \rangle &= \frac{1}{\sqrt{2}}(1-1) = 0 \\
\langle \Psi_6 \mid \Psi_7 \rangle &= \frac{1}{\sqrt{2}}(1-1) = 0 \\
\langle \Psi_8 \mid \Psi_9 \rangle &= \frac{1}{\sqrt{2}}(1-1) = 0
\end{aligned}$$

So, we have the set $\{\Psi_n\}$ for $n \in [1, ..., 9]$ is an ONB.
Then, they can directly measure two qutrits in that ONB.
Alternatively, we could define the following unity gate

$$\begin{aligned}
U =& (|0\rangle \otimes |0\rangle) \langle \Psi_1| + (|0\rangle \otimes |1\rangle) \langle \Psi_2| + (|0\rangle \otimes |2\rangle) \langle \Psi_3| + \\
& (|1\rangle \otimes |0\rangle) \langle \Psi_4| + (|1\rangle \otimes |1\rangle) \langle \Psi_5| + (|1\rangle \otimes |2\rangle) \langle \Psi_6| + \\
& (|2\rangle \otimes |0\rangle) \langle \Psi_7| + (|2\rangle \otimes |1\rangle) \langle \Psi_8| + (|2\rangle \otimes |2\rangle) \langle \Psi_9|
\end{aligned}$$

Putting the two qutrits through this gate, and measure both qutrits in the $\{|0\rangle, |1\rangle\}$. The combined outcome indicate the state of the system before applying the gate as below.

Combined Outcome	State
00	Ψ_1
01	Ψ_2
02	Ψ_3
10	Ψ_4
11	Ψ_5
12	Ψ_6
20	Ψ_7
21	Ψ_8
22	Ψ_9

Therefore, they can discover the state of the two qutrits without any error if the two qutrits are brought together.

4.b

For identifying the state without bringing the qutrits togehter, Alice and Bob can only do measurements on their own qutrits. Let say Alice measures in the ONB $\{|\alpha_n\rangle\}_{n\in\{0,1,2\}} \in \mathbb{C}^3$, and Bob measures in the ONB $\{|\beta_m\rangle\}_{m\in\{0,1,2\}} \in \mathbb{C}^3$. Then, we have the probabilities given by

$$p(n,m,k) = |(\langle\alpha_n| \otimes \langle\beta_m|)\,|\Psi_k\rangle|^2 = |\langle M_{nm} \,|\, \Psi_k\rangle|^2$$

Alice and Bob could make a guess using the two outcomes, namely n, m. Then, the guess could be any function $f(n,m) = k_{\text{guess}}$. For the guessing to be perfect, the probability of the guessing must be 1, namely

$$p(n,m,k) = 1 \qquad \forall f(n,m) = k \tag{1}$$

As there are 9 possible states and 9 combinations of the two ONB, the function f must be bijective. Then, we could rewrite Eq. 1 as

$$p(f^{-1}(k),k) = \left|\left\langle M_{f^{-1}(k)} \,\middle|\, \Psi_k\right\rangle\right| = 1 \tag{2}$$

Both $|M\rangle$ and $|\Psi\rangle$ are in unit length. By the Cauchy-Schwarz inequality, they must be proportional, i.e.

$$e^{i\phi}\left|M_{f^{-1}(k)}\right\rangle = |\Psi_k\rangle = |a_k\rangle \otimes |b_k\rangle$$

for some $\phi \in \mathbb{R}$ and $|\Psi_k\rangle = |a_k\rangle \otimes |b_k\rangle$.

Also, from Eq. 2, we have $\forall k$,

$$\left|\left\langle \alpha_{n(f^{-1}(k))} \,\middle|\, a_k\right\rangle\right| \left|\left\langle \beta_{m(f^{-1}(k))} \,\middle|\, b_k\right\rangle\right| = 1 \tag{3}$$

By Cauchy-Schwarz inequality, we have

$$\begin{cases} \left|\left\langle \alpha_{n(f^{-1}(k))} \,\middle|\, a_k\right\rangle\right| \leq \left\|\left|\alpha_{n(f^{-1}(k))}\right\rangle\right\| \, \|\, |a_k\rangle \,\| \\ \left|\left\langle \beta_{m(f^{-1}(k))} \,\middle|\, b_k\right\rangle\right| \leq \left\|\left|\beta_{m(f^{-1}(k))}\right\rangle\right\| \, \|\, |b_k\rangle \,\| \end{cases}$$

All $|\alpha_n\rangle, |\beta_m\rangle, |a_k\rangle, |b_k\rangle \, \forall n,m,k$ are normalised, we have

$$\begin{cases} \left|\left\langle \alpha_{n(f^{-1}(k))} \,\middle|\, a_k\right\rangle\right| \leq \left\|\left|\alpha_{n(f^{-1}(k))}\right\rangle\right\| \, \|\, |a_k\rangle \,\| = 1 \\ \left|\left\langle \beta_{m(f^{-1}(k))} \,\middle|\, b_k\right\rangle\right| \leq \left\|\left|\beta_{m(f^{-1}(k))}\right\rangle\right\| \, \|\, |b_k\rangle \,\| = 1 \end{cases}$$

Therefore, together with Eq. 3, we have

$$\begin{cases} \left|\left\langle \alpha_{n(f^{-1}(k))} \,\middle|\, a_k\right\rangle\right| = 1 \\ \left|\left\langle \beta_{m(f^{-1}(k))} \,\middle|\, b_k\right\rangle\right| = 1 \end{cases}$$

Again, by Cauchy-Schwarz inequality, we have $\forall k$,

$$\begin{cases} \left|\alpha_{n(f^{-1}(k))}\right\rangle = e^{i\phi_{a_k}} \left|a_k\right\rangle \\ \left|\beta_{m(f^{-1}(k))}\right\rangle = e^{i\phi_{b_k}} \left|b_k\right\rangle \end{cases}$$

for some $\phi_{a_k}, \phi_{b_k} \in \mathbb{R}$.

Then, as $\{|\alpha_n\rangle\}$ is an ONB, for every k and k', we have

$$\begin{aligned} |\left\langle a_k \,|\, a_{k'}\right\rangle| &= \left|\left\langle \alpha_{n(f^{-1}(k))} \,\middle|\, \alpha_{n(f^{-1}(k'))}\right\rangle\right| \\ &= \delta_{n(f^{-1}(k)),n(f^{-1}(k'))} \end{aligned}$$

Therefore, $|\left\langle a_k \,|\, a_{k'}\right\rangle|$ be either be 1 or 0. This creates a contradiction. For $k = 1, k' = 6$, we have

$$\begin{aligned} |\left\langle a_1 \,|\, a_6\right\rangle| &= \frac{1}{\sqrt{2}} |\left\langle 1 \,|\, 0\right\rangle + \left\langle 1 \,|\, 1\right\rangle| \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Hence, the assumption that the guessing is perfect is incorrect, Alice and Bob cannot discover the state of the two qutrits without error.