ASP4200/ASP4020 Stars

Assignment 1

Perform all calculations and provide results using cgs units or solar units where appropriate. Final results for long time scales may be converted to "yr". Provide an ***appropriate*** *number of significant digits.*

Submit a single PDF file with at least 11pt font (12 pt preferred for Moolde online marking) for all text and figures (legends, axis labels, etc.). Use of scientific notation of results and proper (cgs or astronomical) units is required. Summarise your results, even if you use a IPython notebook (which, of course, could be part of a formatted text printed by the notebook). I recommend to use Python modules with classes, including auto-reload, not just `Jupyter` cells.

1. **Simple Stellar Models**

   Write a "program" to solve the Lane-Emden Equation.

   Some of you may have seen or done this in ASP2062, ASP3012, or ASP3162; in the question below, however, we will build an extended version that adds nuclear reactions. Don't just use the code from back then.

   What I have done for my fun version - and this may be very educational to see - is to implement this in Google docs. It turns out for a resolution of 0.01 or even 0.001 in ξ this is still fast enough. one need to code a hand full of cells, then just replicate these a thousand times. Plus another hand full or two zones for finding and managing results. Plotting is natural. (Being able to solve complex problems using Google Docs may impress any business managers. Maybe even more so if you can do it using Excel.)

   Suggested steps are:

   i) Convert Lane-Emden Equation into a system of 1st order linear differential equations.

   [2 marks]

   ii) find inner boundary condition as needed for integration; formally the term is 0/0, so you have to find the right limit by Taylor expansion or other method.

   $$\lim_{\xi \to 0} \frac{\mathrm{d}^2\theta(\xi)}{\mathrm{d}\xi^2} = \ldots$$

   [2 marks]

iii) Write an integration program using finite steps. Don't just use pre-compiled solver package. As staed above, it is quite reasonably possible to do this just with a spreadsheet like Google Documents (which is what I did for a proof of concept).

[2 marks]

iv) Find value of ξ where θ crosses 0. In the lecture we called this value ξ_1. If you use finite steps you could use interpolation to refine the estimate. Also obtain

$$\left.\frac{\mathrm{d}\theta(\xi)}{\mathrm{d}\xi}\right|_{\xi_1} .$$

[2 marks]

Show and document you solution steps and program (submit program or formulas used in spreadsheet).

Tasks:

(a) Compute a model for $n = 1.5$, $M = 1\,\mathrm{M}_\odot$, central density $\rho_\mathrm{c} = 160\,\mathrm{g\,cm^{-3}}$. Use ideal gas with $\mu = 0.61$.

What is the radius of the star, what are central pressure, P_c and central temperature, T_c, of the star?

[8 marks]

(b) **Compare your solar model from above with key data of the actual sun. What might be reasons for the discrepancy?**

[2 marks]

(c) **Compute a model for $n = 1.5$, $M = 15\,\mathrm{M}_\odot$, $\rho_\mathrm{c} = 6\,\mathrm{g\,cm^{-3}}$, and a mixture of $71.5\,\%$ hydrogen ($^{1}\mathrm{H}$), $27.1\,\%$ helium ($^{4}\mathrm{He}$) and the rest of nitrogen ($^{14}\mathrm{N}$), all percentages by mass fraction. Use ideal gas *with radiation.***

This is not trivial, you have to solve a 4^{th} order equation, you may need to use maths tool for that. Assume the gas is fully ionized.

What is the radius of the star, what are central pressure, P_c and central temperature, T_c, of the star? What is the ratio of gas pressure to total pressure,

$$\beta = \frac{P_\mathrm{gas}}{P} ,$$

at the center of the star?

[8 marks]

2. **Stellar Model with Thermonuclear Burning**

Based on the Lane Emden Equation solver developed in Questions 1, construct stellar models that include nuclear burning.

Assume a composition that is a mixture 71.5 % hydrogen ($^{1}\mathsf{H}$), 27.1 % helium ($^{4}\mathsf{He}$) and the rest of nitrogen ($^{14}\mathsf{N}$), all percentages by mass fraction.

(a) Compute a model with $n = 1.5$ and $M = 1\,\mathsf{M}_{\odot}$. Assume the star generates its energy according to the pp chains. Use formula 18.63 from Kippenhahn, Weigert, & Weiss (2012). For simplicity, assume $\psi = 1$ and $f_{11} = 2$. Assume the star is in thermal and hydrostatic equilibrium, i.e., nuclear energy generation balances luminosity from the surface.

Find central density such that the luminosity of star equal the solar luminosity. What is the central temperature of the star? Plot specific nuclear energy generation rate and the luminosity as a function of mass coordinate. What is the radius of the star? Why does it deviate from that of the sun?

[12 marks]

(b) Compute a model with $n = 3$ and $M = 100,000\,\mathsf{M}_{\odot}$. Assume the star generates its energy according to the CNO cycle. Use formula 18.65 from Kippenhahn, Weigert, & Weiss (2012). Assume the star is in thermal and hydrostatic equilibrium, i.e., nuclear energy generation balances luminosity from the surface.

What is the luminosity, radius, and central density of the star for a central temperature of $2 \times 10^{7}\,\mathsf{K}$, $2.5 \times 10^{7}\,\mathsf{K}$, $3 \times 10^{7}\,\mathsf{K}$, $3.5 \times 10^{7}\,\mathsf{K}$, and $4 \times 10^{7}\,\mathsf{K}$? Please also plot temperature and density as a function of enclosed mass. Please provide results for luminosity and radius in solar units. Consider using an appropriate equation of state, confirm by computing the ratio of gas pressure to total pressure including radiation, β.

HINT: Solve for ρ_{c} from T and P_{c} using EOS.

[12 marks]

(c) You now have from the two parts above the energy generations for PP chain and for CNO cycle. **For the models from above , at which central temperature would the PP energy generation from the PP chains dominate? Identify the key physics why this is so.**

[8 marks]

(d) SPECIAL CHALLENGE: **Show that for an $n = 3$ polytrope β is constant throughout the star, i.e., $\frac{\mathsf{d}}{\mathsf{d}r}\beta = 0$.**

[Note: This is a bonus question you do not need to do.]

Hint: For this problem, find a combination of P, P_{gas}, and P_{rad} that only depends on ρ and P, but not on T, such that the result is a non-trivial function of β alone, $f(\beta)$. When β is constant, then $f(\beta)$ is constant, and you can compute the derivative of that instead. Also use the polytropic relation for your solution, with $\gamma = 1 + 1/n$. You may want to look at KWW12, §19.5.

[10 bonus marks]

(e) SPECIAL CHALLENGE: **For a polytrope, show that for given μ and M, β_{c} does not depend on ρ_{c}**

[Note: This is a bonus question you do not need to do.]

[10 bonus marks]