Homework Set 4

Dynamics: equations of motion

Revision: 13–Feb–2020

4.1 An old friend

Consider the system of Figure 4.1.

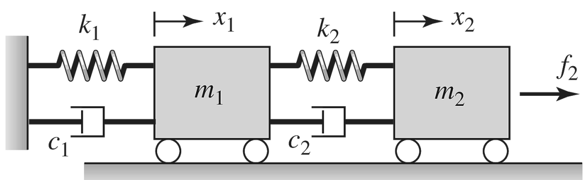


Figure 4.1: Two-mass lumped-model

4.1.1 Building equations of motion

1. Determine kinetic energy, potential energy, and dissipation function of the system.
2. Use Lagrange's equation to construct the equations of motion.
3. Use Hamilton's equation to construct the equations of motion.

4.1.2 Control

Let's control the system from Figure 4.1, where $m_1 = m_2 = 1\,\text{kg}$, $k_1 = 20\,\text{N/m}$, $k_2 = 10\,\text{N/m}$, $c_1 = 0.4\,\text{Ns/m}$, and $c_2 = 0.2\,\text{Ns/m}$. The input to the system is the force f_2, and the output is x_2.

1. Determine a state-space representation of the system.

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u \tag{4.1}$$
$$y = \mathbf{C}\mathbf{z} + Du \tag{4.2}$$

   Be clear on your choice of state-variables.

2. Determine if the system is controllable and observable. Hint: it should be or else we need to change the input and/or the output. Build the controllability matrix $\mathbf{C_M}$ and the observability matrix $\mathbf{O_M}$. Check the rank of each using the following methods

   (a) the determinant

(b) the singular value decomposition

(c) the eigenvalues

(d) a built-in `rank` function.[1]

3. If we want the closed-loop response to have a settle time less than 2 s, pick some desired poles as well as poles for the observer.

4. Design a state feedback controller. Test and verify.

5. Design a state observer.

6. Test the observer-based controller on system performance for a step input with (reasonable) random initial conditions.

7. *Experimental Bonus*: Let's imagine that this vibrating control system is isolated and where energy is at a premium. For example, a robotic space vehicle. The goal now is reduce the total energy needed to go from rest to a unit-step reference. The energy used to actuate the input is $\int u \, \mathrm{d}x_2$. If we assume that both pushing and pulling consume energy (there is no recovery), then we can track the energy usage z_u as

$$z_u = \int |u(t)| \, \mathrm{d}x_2(t) = \int \left| u(t) \frac{\mathrm{d}x_2}{\mathrm{d}t} \right| \mathrm{d}t = \int |u(t)\dot{x}_2(t)| \, \mathrm{d}t \tag{4.3}$$

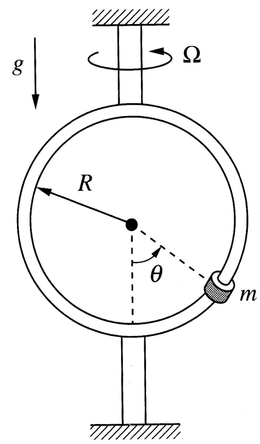
$$\dot{z}_u = |u\dot{x}_2| \tag{4.4}$$

Augment the set of equations we are integrating to include (4.4), and calculate the total energy used for a particular choice of the closed-loop poles. Modify your choice of where the closed-loop poles are placed to lower the value of $z_u(t_{\text{end}})$, while maintaining the other performance goals.

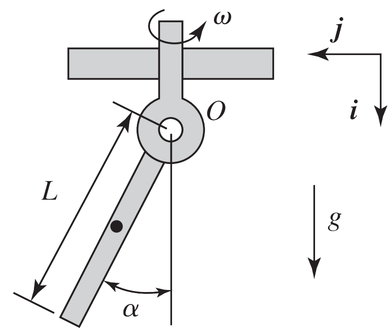
[1]The rank function in Matlab is part of its base `https://www.mathworks.com/help/matlab/ref/rank.html`, and Julia has one in its standard library LinearAlgebra `https://docs.julialang.org/en/v1/stdlib/LinearAlgebra/#LinearAlgebra.rank`

4.2 An old enemy

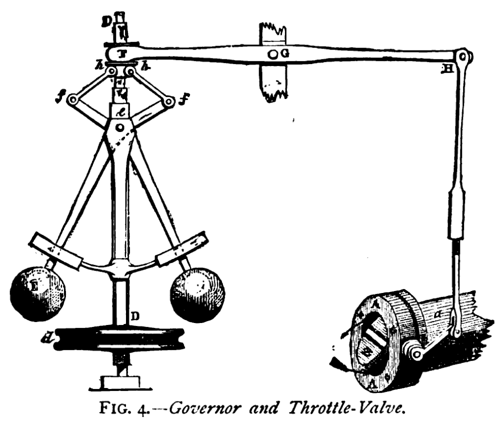
Consider the system of Figure 4.2. These are all models that share similar dynamics, and have been used in a variety of mechanisms over the past century. Using Lagrange's equation, determine the equation of motion of the system in Figure 4.2b. Assume the rod of length L is uniform, and has mass m.



(a) A bead on a spinning hoop.



(b) A rotating pendulum.



(c) A centrifugal governor. Image from *Discoveries & Inventions of the Nineteenth Century* by R. Routledge, 13th edition, published 1900.

Figure 4.2: The greatest dynamics problem in the universe 🚀.

4.3 Simplified car model

Consider the rigid bar shown in Figure 4.3, which can rotate (pitch) about the $\hat{\boldsymbol{k}}$ direction and translate (bounce) along the $\hat{\boldsymbol{j}}$ direction. We can assume the motion to be small enough that the spring/damper forces remain vertical. The motion is described by the generalized coordinates y and θ at the center of gravity of the beam. The coordinates are not measured from the equilibrium position, but when the forces in the springs are zero. This model provides ~~a good~~ an interesting representation for describing certain types of motions of motorcycles, automobiles, and other vehicles.

1. Determine the equations of motion in terms of y and θ.
2. Determine the equilibrium equations for the system, and the conditions if the car is to be horizontal at rest.
3. Determine the linear equations of motion about equilibrium.
4. Bonus: determine the natural frequencies.

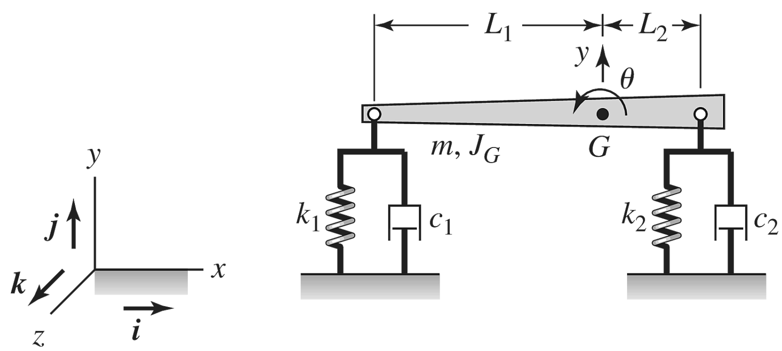


Figure 4.3: Rigid body in the plane constrained by springs and dampers

4.4 Pendulum Absorber

Consider the pendulum attached to a cart, as depicted in Figure 4.4.

1. Use Newton's Laws to derive the equations of motion.
2. Use Lagrange's equation to derive the equations of motion.
3. Compare the equations from the previous two steps, and show they are equivalent.
4. Use Hamilton's equation to derive a set of first-order equations of motion.

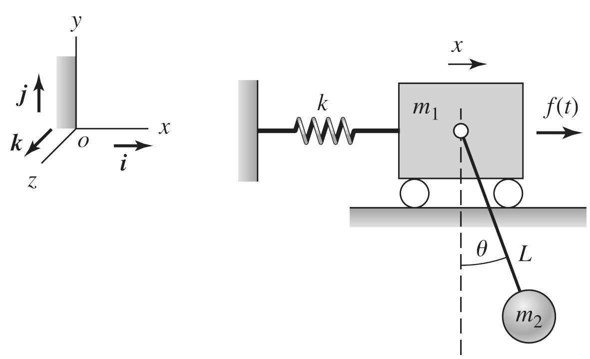


Figure 4.4: Pendulum absorber

4.5 Airfoil

An airfoil section to be tested in a wind tunnel is supported by a linear spring k and a torsional spring k_t, as shown in Figure 4.5. The center of gravity of the section is a distance l ahead of the point of support. Assume that the spring k remains vertical.

1. Using Lagrange's equation, determine the equations of motion.
2. Using Hamilton's equation, determine the equations of motion.

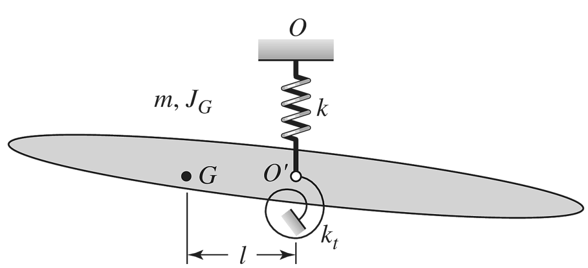


Figure 4.5: Elastically mounted airfoil

4.6 Rolling pendulum

Determine the equations of motion of the system in Figure 4.6. Assume that the length of the pendulum is L.

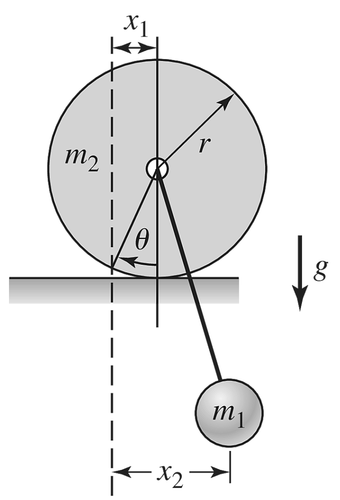


Figure 4.6: Rolling pendulum

4.7 Rotating pendulum of doom

Consider the rotary pendulum in Figure 4.7. Here, $\Omega = \Omega(t)$ is the variable angular velocity of frame. This frame has a rotary inertia J_G about the Z axis. Assume there is an input torque $T(t)$ acting on the frame (same direction as Ω). The deformed length of the spring is $L = L(t)$, in the x–z plane. The unstretched length of the spring is L_0. The mass of point P is m_p. Derive the equations of motion for this system.

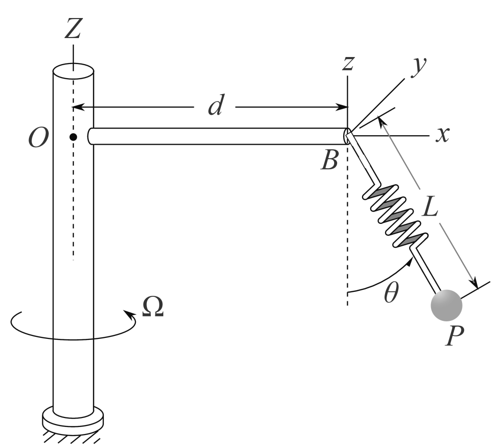


Figure 4.7: A rotary elastic pendulum

4.8 The double pendulum

The double-pendulum shown in Figure 4.8 is a classic problem in the study of chaos. The parameters of this system are $\{g, m_1, m_2, l_1, l_2\}$.

1. Derive equations of motion for this system. Use θ_1 and θ_2 as the generalized coordinates.
2. Derive the equation of motion in terms of the x–y locations of the centers of mass using the Udwadia-Kalaba equation.

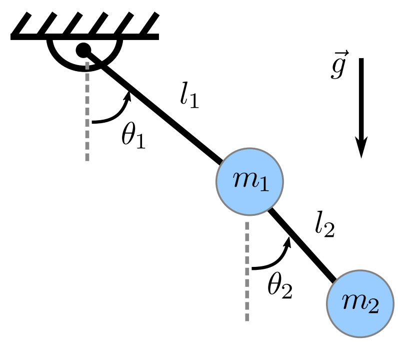


Figure 4.8: A planar lumped-mass ideal double-pendulum.