Homework Set 2

Linear Algebra

Revision: 13–Jan–2019

2.1 Analysis

2.1.1 Uniqueness

Let $\mathbf{A}$ be a given square matrix. Show that if $\mathbf{Ax} = \mathbf{b}$ has *at least* one solution for any $\mathbf{b}$, then it has *exactly one* solution for any $\mathbf{b}$.

2.1.2 Companion matrix

A *companion matrix* has the form

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_k & -a_{k-1} & -a_{k-2} & \cdots & -a_1 \end{bmatrix} \tag{2.1}$$

1. Show that the characteristic equation is

   $$\lambda^k + a_1\lambda^{k-1} + \cdots + a_k = 0 \tag{2.2}$$

   where λ is an eigenvalue.

2. If $k = 3$ show that eigenvectors $\mathbf{v}$ have the form

   $$\mathbf{v} = \begin{bmatrix} 1 & \lambda & \lambda^2 \end{bmatrix}^T \tag{2.3}$$

2.1.3 Similarity

Consider the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, and the orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$.

- Show that $\mathbf{B} = \mathbf{Q}^T\mathbf{AQ}$ has the same eigenvalues as $\mathbf{A}$.
- Also, comment on the eigenvectors of $\mathbf{B}$.

2.1.4

Show that the linear independence of eigenvectors given by

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0} \tag{2.4}$$

is true by making use of the orthogonality property of eigenvectors.

2.2 Some hand calculations

2.2.1 Determinant

Compute (by hand) the determinant of

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \tag{2.5}$$

check your result using Matlab.

2.2.2 Rank

Determine the rank of the following matrix by elimination

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \tag{2.6}$$

2.2.3 Inverse

Using Gaussian Elimination, compute the inverse of the following matrices

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \tag{2.7}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \tag{2.8}$$

2.3 Some programming

2.3.1 BLAS

Here are some general questions about BLAS implementations

1. What is the BLAS function(s) that multiplies a general matrix and a vector (in double precision)? Also, explain each input/output of the routine.
2. What does `dsymv` do?
3. Write a `daxpy`-equivalent function in Matlab.

2.3.2 Matrix Multiplication

Implement a naïve (simple loop-based) function that multiplies two rectangular matrices. Check the performance against the built-in matrix multiply on a range of random matrices of increasing size.

2.3.3 Determinant

Find the largest determinant of a 6 by 6 matrix whose entries are 1's and -1's.

2.4 Eigenvalues and differential equations

Initial value problems in LTI differential equations, without an input, can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{2.9}$$
$$\mathbf{x}(0) = \mathbf{x}_0. \tag{2.10}$$

The general solution to this involves the so-called matrix exponential $e^{\mathbf{A}t}$:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 \tag{2.11}$$

which can be computed in many ways. One such method is the use of the Laplace transform which results in

$$e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]. \tag{2.12}$$

Another approach (if the $\mathbf{A}$ is diagonalizable) is to use the eigenvalues λ_i and the matrix of eigenvectors $\mathbf{V}$ of $\mathbf{A}$ as

$$e^{\mathbf{A}t} = \mathbf{V}\begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n t} \end{bmatrix}\mathbf{V}^{-1} \tag{2.13}$$

1. Using the Laplace transform and (2.11), show (2.12).
2. Given the matrix $\mathbf{A}$ in (2.14), compute $e^{\mathbf{A}t}$ using (2.12).
3. Given the matrix $\mathbf{A}$ in (2.14), compute $e^{\mathbf{A}t}$ using (2.13).
4. Are the solutions from the previous two steps the same? Should they be?

$$\mathbf{A} = \begin{bmatrix} -4 & 1 & -4 \\ 2 & -5 & 2 \\ -3 & 3 & -3 \end{bmatrix} \tag{2.14}$$

2.5 Some mechanics

At a certain point the stress in a body made of 6061-T6 aluminum is known, and is given as the following matrix

$$\boldsymbol{\sigma} = \begin{bmatrix} 2 & 5 & 3 \\ 5 & 1 & 4 \\ 3 & 4 & 3 \end{bmatrix} 10\text{MPa} \tag{2.15}$$

1. Determine the principal stresses: σ_1, σ_2, σ_3.
2. Determine the directions of the principal stresses.
3. Determine the maximum shear stress $\tau_{\max}$.
4. Determine the factor of safety (compared to yield strength) using the von Mises yield criterion. Compute the von Mises stress σ_v in three ways:

(a) Using the values in $\boldsymbol{\sigma}$

$$\sigma_v = \sqrt{\frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)\right]} \tag{2.16}$$

(b) Using the principal stresses

$$\sigma_v = \sqrt{\frac{1}{2}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]} \tag{2.17}$$

(c) Using the second invariant of the deviatoric part of the stress tensor.

$$\boldsymbol{\sigma}_{\text{dev}} = \boldsymbol{\sigma} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})\,\mathbf{I} \tag{2.18a}$$

$$J_2 = \frac{1}{2}\operatorname{tr}\left(\boldsymbol{\sigma}_{\text{dev}}^2\right) \tag{2.18b}$$

$$\sigma_v = \sqrt{3J_2} \tag{2.18c}$$

2.6 Some dynamics

Given the mass-spring-damper system in Figure 2.1, assume that the contact forces are viscous friction.

1. Derive the equations of motion and state them in matrix notation.

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) \tag{2.19}$$

2. Determine the (undamped) natural frequencies of the system. Recall that eigenvalue problem starts from assuming a harmonic free-response $\mathbf{x}(t) = e^{j\omega t}\mathbf{v}$, which simplifies the equation of motion to

$$-\omega^2\mathbf{M}\mathbf{v} + \mathbf{K}\mathbf{v} = \mathbf{0} \tag{2.20}$$

where the eigenvalues are ω^2, and their associated mode shape is $\mathbf{v}$.

3. What is the physical significance of the values of ω?

4. *Bonus:* What are the modeshapes of the system? Be careful here, this system is degenerate, and some eigenvalues are repeated, which means we need to find generalized eigenvectors.

5. If the system parameters are given in Table 2.1, numerically check your expressions for the eigenvalues (and eigenvectors if applicable) from the previous steps of the problem.

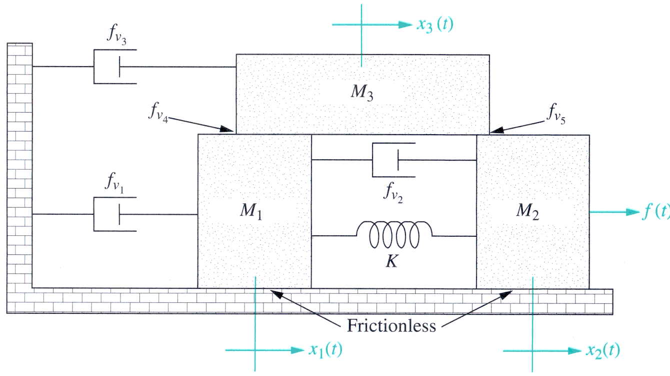

Figure 2.1: A three degree of freedom mass spring damper system.

Table 2.1: Parameter values for §2.6

Quantity	Value	
m_1	0.5	kg
m_2	0.5	kg
m_3	2.0	kg
k	20.0	N/m
$f_{v1} = f_{v2} = f_{v3}$	0.1	Ns/m
$f_{v4} = f_{v5}$	2.0	Ns/m

2.7 Some controls

Consider the system of Figure 2.2 where $m_1 = m_2 = 1\,\mathrm{kg}$, $k_1 = 20\,\mathrm{N/m}$, $k_2 = 10\,\mathrm{N/m}$, $c_1 = 0.4\,\mathrm{Ns/m}$, $c_2 = 0.2\,\mathrm{Ns/m}$.

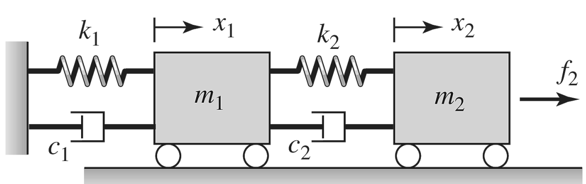

Figure 2.2: Two-mass lumped-model

1. Determine the equation of motion in matrix notation $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t)$, where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$.
2. Determine the linear state-space representation of the system

$$\dot{\mathbf{z}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \mathbf{z} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{b} \end{bmatrix} u \tag{2.21}$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{z} \tag{2.22}$$

where $\mathbf{z} = \begin{bmatrix} x_1 & x_2 & \dot{x}_1 & \dot{x}_2 \end{bmatrix}^T$, the output y is x_2, the input u is f_2 so $\mathbf{b} = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$.

3. Determine the coefficients of the desired characteristic equation, if we want the system to have poles at $\{-2 \pm 2j, -4 \pm 4j\}$

$$q(s) = s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0 \tag{2.23}$$

4. Show that the system is controllable. Compute the controllability matrix $\mathbf{C_M}$, and determine its rank using the following ways:

   (a) the determinant

   (b) the QR decomposition

   (c) the singular value decomposition

   (d) the eigenvalue decomposition

   (e) the `rank` function in Matlab.

   Recall that $\mathbf{C_M}$ is

$$\mathbf{C_M} = \left[\begin{array}{ccccc} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{array} \right] \tag{2.24}$$

5. Design a state-feedback controller $u = -\mathbf{kz}$ using Ackermann's formula. This was originally from (Ackermann, 1972) but it's likely easier to read in English from a text like (Åstörm and Murray, 2008) or Wikipedia. It is a handy tool to directly compute the gains in $\mathbf{k}$ without needing to transform the system. Ackermann's formula is stated, in general, as

$$\mathbf{k} = \left[\begin{array}{ccccc} 0 & 0 & \cdots & 0 & 1 \end{array} \right] \mathbf{C_M}^{-1}\mathbf{P} \tag{2.25}$$

   where $\mathbf{P}$ is the desired characteristic polynomial evalulated at $\mathbf{A}$

$$\mathbf{P} = \mathbf{A}^n + d_{n-1}\mathbf{A}^{n-1} + \cdots + d_1\mathbf{A} + d_0\mathbf{I} \tag{2.26}$$

6. Demonstrate that your controlled system meets the desired specifications.

Bibliography

Ackermann, J. (1972). "Der Entwurf linearer Regelungssysteme im Zustandsraum". *Regelungstechnik und Prozess-Datenverarbeitung*, 7, pp. 297–300. DOI: `10.1524/auto.1972.20.112.297`.

Åstörm, K. J. and R. M. Murray (2008). *Feedback Systems: An Introduction for Scientists and Engineers*. Princeton University Press. URL: `http://www.cds.caltech.edu/~murray/amwiki/index.php`.