HW 5: ODEs

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ENSC-481

5.1.1

Given: A couple of methods to approximate solutions to ODEs.

Determine: The region of stability for each method if the ODE is the following:

$$\dot{x} = \lambda x$$

a) Trapezoidal Rule

$$x_{n+1} = x_n + \frac{h}{2}\left(f(t_n, x_n) + f(t_{n+1}, x_{n+1})\right)$$

b) Runge-Kutta 2 $\quad \alpha = \left\{ 1/4,\ 1/2,\ 2/3 \right\}$

$$\begin{array}{c|cc} 0 & & \\ \alpha & \alpha & \\ \hline & 1-\frac{1}{2\alpha} & \frac{1}{2\alpha} \end{array}$$

Solution:

We start by solving for stability analytically by using a discrete method with some initial condition x_0.

$$x_i = \sigma^i x_0 \qquad (1)$$

a) $x_{n+1} = x_n + \frac{h}{2}\left(f(t_n, x_n) + f(t_{n+1}, x_{n+1})\right)$

Substituting a) with (1)

$$\sigma^{n+1} \cancel{x_0} = \sigma^n \cancel{x_0} + \frac{h}{2}\left(\lambda \cdot \overset{(x_n)}{\sigma^n \cancel{x_0}} + \lambda \sigma^{n+1} \cancel{x_0}\right)$$

Divide x_0 out

$$\sigma^{n+1} - \frac{h}{2}\lambda \sigma^{n+1} = \sigma^n + \frac{h}{2}\lambda \sigma^n$$

Divide by σ^n

$$* \quad \frac{\sigma^{n+1}}{\sigma^n} = \sigma^{n+1-n} = \sigma \quad *$$

$$\sigma\left(1 - \frac{h}{2}\lambda\right) = 1 + \frac{h}{2}\lambda$$

$$\sigma = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} \qquad (2)$$

Now to determine stability we can say the system will be stable if:

$$|\sigma| < 1$$

From (2), $h\gamma = z$

$$\left| \frac{1 + \frac{z}{2}}{1 - \frac{z}{2}} \right| < 1$$

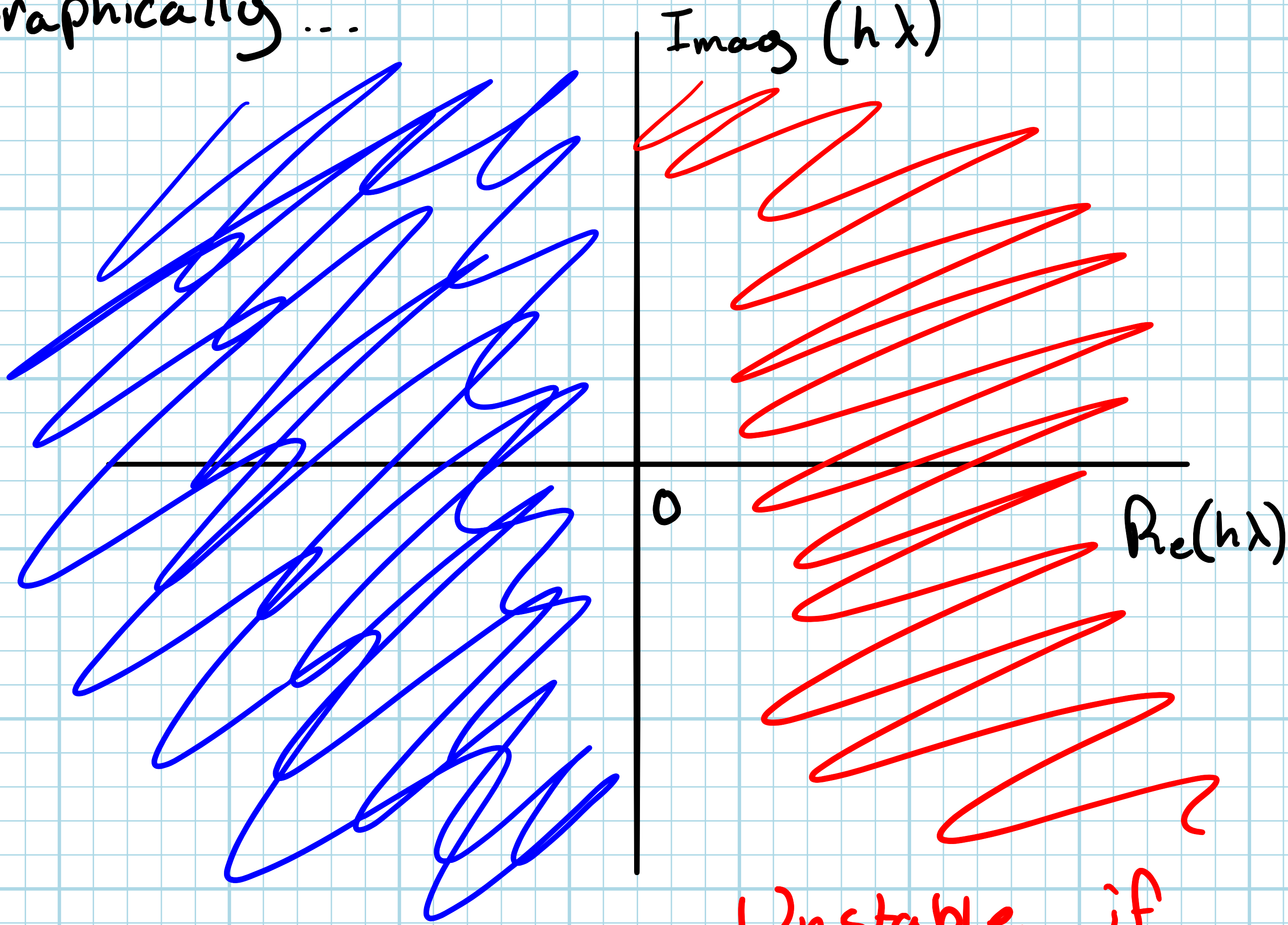
$$\left|1+\frac{z}{2}\right| < \left|1-\frac{z}{2}\right|$$

$$\left|\frac{2z}{2}\right| < |0|$$

$$|z| < 0$$

* Our system is stable if $h\lambda$ is negative. So the real part of λ must be negative.

Graphically ...



Stable as long as $Re(h\lambda) < 0$

Unstable if $Re(h\lambda) > 0$

b)

$$\begin{array}{c|cc} 0 & & \\ \alpha & \alpha & \\ \hline & 1-\frac{1}{2\alpha} & \frac{1}{2\alpha} \end{array}$$

"Butcher Table for Runge Kutta 2"

We know the following from class (4/2):

$$\begin{array}{c|cc} 0 & & \\ c_2 & a_{21} & \\ \hline & b_1 & b_2 \end{array}$$

$$x_{n+1} = x_n + h(b_1 k_1 + b_2 k_2) \quad (1)$$

We know everything except k_1, k_2:

$$k_1 = f(t_n, x_n)$$

$$k_2 = f(t_n + c_2 \cdot h,\ x_n + h \cdot a_{21} \cdot k_1)$$

Let's write these out with our givens and assuming $x_i = \sigma^i x_0$

$$k_1 = \lambda \sigma^n x_0$$

$$k_2 = \lambda(\sigma^n x_0 + h \cdot \alpha \cdot \lambda \sigma^n x_0)$$

$$(1) \quad \sigma^{n+1} \cancel{x_0} = \sigma^n \cancel{x_0} + h\left[\left(1-\tfrac{1}{2\alpha}\right)(\lambda \sigma^n \cancel{x_0}) + \tfrac{\lambda}{2\alpha}\left(\sigma^n \cancel{x_0} + h\alpha\lambda\sigma^n \cancel{x_0}\right)\right]$$

Divide by σ^n & x_0

$$\sigma = 1 + h\left[\left(1-\frac{1}{2\alpha}\right)(\lambda) + \frac{\lambda}{2\alpha}(1 + h\alpha\lambda)\right]$$

$$\sigma = 1 + h\left[\lambda - \cancel{\frac{\lambda}{2\alpha}} + \cancel{\frac{\lambda}{2\alpha}} + \frac{h\alpha\lambda^2}{2\alpha}\right]$$

$$\sigma = 1 + h\lambda + \frac{h^2\lambda^2}{2}$$ * $h\lambda = z$

$$\sigma = \frac{1}{2}z^2 + z + 1$$

Stable when $|\sigma| < 1$

$$\boxed{\left|\frac{1}{2}z^2 + z + 1\right| < 1}$$

Let's Plot this:

we need to parameterize our σ equation.

$$\sigma - \frac{1}{2}z^2 - z - 1 = 0$$

Sub: $\sigma \rightarrow \cos\theta + i\sin\theta$

$z \rightarrow z_x + i z_y$

$$\cos\theta + i\sin\theta - \frac{1}{2}(z_x + iz_y)^2 - z_x - iz_y - 1 = 0$$

$$\cos\theta - \frac{1}{2}z_x^2 - z_x - 1 + \frac{1}{2}z_y^2 \ldots$$

$$+ i\sin\theta - i\frac{1}{2}z_y^2 - i\frac{1}{\cancel{2}}\cdot\cancel{2}\, z_y z_x = 0$$

Setting real & imag. parts to zero

Real: $\cos\theta - \frac{1}{2} z_x^2 - z_x - 1 + \frac{1}{2} z_y^2 = 0$

$$-z_x^2 - 2z_x - 2 + z_y^2 + 2\cos\theta = 0$$

$$z_y^2 = z_x^2 + 2z_x + 2 - 2\cos\theta$$

Imag: $\sin\theta - \frac{1}{2} z_y^2 - z_y z_x = 0$

$$z_y^2 = 2\sin\theta - 2z_y z_x$$

$$z_x^2 + 2z_x + 2 - 2\cos\theta = 2\sin\theta - 2z_y z_x$$

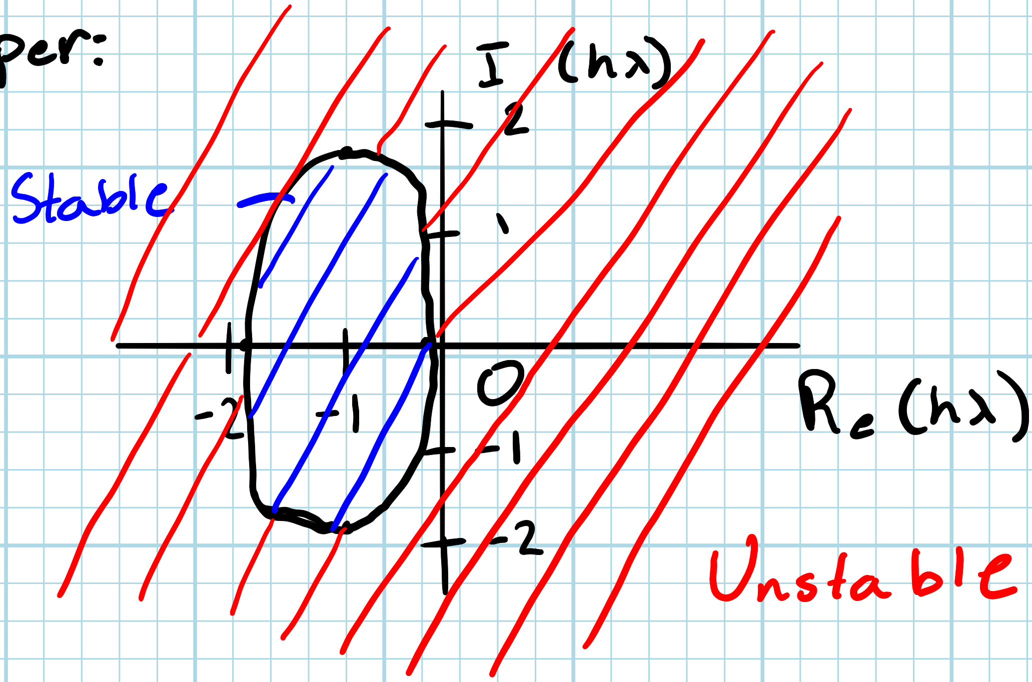
$$z_x^2 + (2 + 2z_y) z_x + (2 - 2\cos\theta - 2\sin\theta) = 0$$

Hmmm... This doesn't get clean or easy, so I'm going to jump to Joey's MIT Paper and the big idea.

<u>α doesn't matter for stability</u> with this equation $\dot{x} = \lambda x$.

$\boxed{\sigma = \frac{1}{2} z^2 + z + 1}$ $\times$ No α

MIT Paper:



5.2

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Given: An ODE and two cases for A.

$$\dot{x} = Ax$$

Determine: The maximum time step for two different methods.

a) Two-Stage Adams-Basthforth
b) Implicit Mid point Method

Solution:

Case 1: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Case 2: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -23 & -304 & -732 \end{bmatrix}$

a) 2-Stage AB

$$x_{n+2} = x_{n+1} + \frac{3}{2}h\,f(t_{n+1}, x_{n+1}) - \frac{1}{2}h\,f(t_n, x_n) \quad (1)$$

Using the I.C. Stability condition $x_i = \sigma^i x_0$

$$(1) \quad \sigma^{n+2}\cancel{x_0} = \sigma^{n+1}\cancel{x_0} + \frac{3}{2}h \cdot A \cdot \sigma^{n+1}\cancel{x_0} - \frac{1}{2}hA\sigma^n\cancel{x_0}$$

Divide by σ^n & x_0

$$\sigma^2 = \sigma + \frac{3}{2}hA\sigma - \frac{1}{2}hA$$

A is some $n \times n$ matrix. We can evaluate this by using the eigenvalues of A like our λ in the previous problems.

Case 1: $\text{eig}(A) = \begin{bmatrix} 0+i \\ 0-i \end{bmatrix}$

Case 2: $\text{eig}(A) = \begin{bmatrix} -731.58 \\ -0.0995 \\ -0.3160 \end{bmatrix}$

Let's Say $hA = z$

$$\sigma^2 = \sigma + \frac{3}{2} z\sigma - \frac{1}{2} z$$

$$\sigma^2 - \left(1 + \frac{3}{2} z\right)\sigma + \frac{1}{2} z = 0$$

Parameterize:

$$\sigma \to \cos\theta + i\sin\theta$$

$$z \to z_x + i z_y$$

$$(\cos\theta + i\sin\theta)^2 - \left(1 + \frac{3}{2} z_x + i\frac{3}{2} z_y\right)(\cos\theta + i\sin\theta) + \frac{1}{2}(z_x + i z_y)$$

Wolfram...

$$-\frac{3}{2} z_x \cos(\theta) + \frac{3}{2} z_y \sin(\theta) \overbrace{-\sin^2(\theta) + \cos^2(\theta)}^{\cos(2\theta)} - \cos(\theta) + \frac{z_x}{2} \ldots$$

$$+ i\left(-\frac{3}{2} z_x \sin(\theta) - \frac{3}{2} z_y \cos(\theta) - \sin(\theta) + \underbrace{2\sin(\theta)\cos(\theta)}_{\sin(2\theta)} + \frac{z_y}{2}\right) = 0$$

Real & Imag. parts set equal to zero

Real: $\cos(2\theta) - \frac{3}{2} z_x \cos(\theta) + \frac{3}{2} z_y \sin(\theta) - \cos(\theta) + \frac{z_x}{2} = 0 \quad (2)$

Imag: $\sin(2\theta) - \frac{3}{2} z_x \sin(\theta) - \frac{3}{2} z_y \cos(\theta) - \sin(\theta) + \frac{z_y}{2} = 0 \quad (3)$

(2) $$z_x\left(\frac{3}{2}\cos(\theta)-\frac{1}{2}\right)=\cos(2\theta)+\frac{3}{2}z_y\sin(\theta)-\cos(\theta)$$

$$z_x=\frac{\cos(2\theta)+\frac{3}{2}z_y\sin(\theta)-\cos(\theta)}{\frac{3}{2}\cos(\theta)-\frac{1}{2}}$$

(3) $$z_y\left(\frac{3}{2}\cos(\theta)-\frac{1}{2}\right)=\sin(2\theta)-\frac{3}{2}z_x\sin(\theta)-\sin(\theta)$$

$$z_y=\frac{\sin(2\theta)-\frac{3}{2}z_x\sin(\theta)-\sin(\theta)}{\frac{3}{2}\cos(\theta)-\frac{1}{2}}$$

$$z_x=\frac{\cos(2\theta)+\frac{3}{2}\left(\frac{\sin(2\theta)-\frac{3}{2}z_x\sin(\theta)-\sin(\theta)}{\frac{3}{2}\cos(\theta)-\frac{1}{2}}\right)\sin(\theta)-\cos(\theta)}{\frac{3}{2}\cos(\theta)-\frac{1}{2}}$$

$$z_x\left(\frac{3}{2}\cos(\theta)-\frac{1}{2}\right)^2=\left(\frac{3}{2}\cos(\theta)-\frac{1}{2}\right)\left(\cos(2\theta)-\cos(\theta)\right)\ldots$$
$$+\frac{3}{2}\sin(2\theta)\sin(\theta)-\frac{9}{4}z_x\sin^2(\theta)-\frac{3}{2}\sin^2(\theta)$$

$$z_x\left[\left(\frac{3}{2}\cos(\theta)-\frac{1}{2}\right)^2+\frac{9}{4}\sin^2(\theta)\right]=\left(\frac{3}{2}\cos(\theta)-\frac{1}{2}\right)\left(\cos(2\theta)-\cos(\theta)\right)\ldots$$
$$+\frac{3}{2}\sin(2\theta)\sin(\theta)-\frac{3}{2}\sin^2(\theta)$$

$$z_x=\frac{\left(\frac{3}{2}\cos(\theta)-\frac{1}{2}\right)\left(\cos(2\theta)-\cos(\theta)\right)+\frac{3}{2}\sin(2\theta)\sin(\theta)-\frac{3}{2}\sin^2(\theta)}{\left(\frac{3}{2}\cos(\theta)-\frac{1}{2}\right)^2+\frac{9}{4}\sin^2(\theta)}$$

Now to MATLAB...

b) Implicit Midpoint Method

Wikipedia:

$$x_{n+1} = x_n + h\, f\left(t_n + \frac{h}{2}, \frac{1}{2}(x_n + x_{n+1})\right)$$

I.C, Stability $\quad x_i = \sigma^i x_0$

$$\sigma^{n+1} \cancel{x_0} = \sigma^n \cancel{x_0} + hA\left(\tfrac{1}{2}\sigma^n \cancel{x_0} + \tfrac{1}{2}\sigma^{n+1}\cancel{x_0}\right)$$

Divide by x_0 & σ^n , $hA = z$

$$\sigma = 1 + z\left(\tfrac{1}{2} + \tfrac{1}{2}\sigma\right)$$

$$\sigma - \tfrac{1}{2} z \sigma = 1 + \tfrac{1}{2} z$$

$$\sigma\left(1 - \tfrac{1}{2} z\right) = 1 + \tfrac{1}{2} z$$

$$\sigma = \frac{1 + \frac{1}{2} z}{1 - \frac{1}{2} z}$$

$|\sigma| < 1 \quad$ * Stability condition *

$$\left|\frac{1 + \frac{1}{2} z}{1 - \frac{1}{2} z}\right| < 1$$

$$1 + \tfrac{1}{2} z < 1 - \tfrac{1}{2} z$$

$$\boxed{z < 0}$$

Hmmm.... Looks familiar

Let's go to MATLAB