System Identification in the frequency domain

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System Identification

- System Identification involves the estimation of relationships between signals.
- Noise is the common enemy: Introduces uncertainty in terms of variance (random errors).
- Learning goals;
  - How to estimate relationships
  - How to minimize uncertainty

Systems and Models

A system N transforms input $\mathbf{u}(t)$ into output $\mathbf{y}(t)$:

$$\mathbf{y}(t) = N(\mathbf{u}(t)) \tag{1}$$

A model M estimates output $\hat{\mathbf{y}}(t)$ from input $\mathbf{u}(t)$ based on its model parameters $\boldsymbol{\theta}$:

$$\hat{\mathbf{y}}(t) = M(\boldsymbol{\theta}, \mathbf{u}(t)) \tag{2}$$

- Parametricmodels: few parameters, in many cases with a physical meaning.
- Nonparametricmodels: many parameters with no physical meaning.

Assumptions

The following assumptions are made for the rest of the analysis.

- System is time invariant:

$$\mathbf{y}(t) = N(\mathbf{u}(t)) \;\rightarrow\; \mathbf{y}(t-\tau) = N(\mathbf{u}(t-\tau)) \quad , \; \forall \tau \in R \qquad (3)$$

- No process noise $\mathbf{w}(t)$, e.i. the system is deterministic.
- Observer noise $\mathbf{n}(t)$ is assumed to be Gaussian white noise.

Correlation functions

Correlation functions reveal structures of signals that are not apparently detectable in the time series.

- Correlation function:

$$\Phi_{uy}(\tau) = E\left[u(t-\tau)y(t)\right] \tag{4}$$

- Covariance function:

$$C_{uy}(\tau) = E\left[\left(u(t-\tau)-\mu_u\right)\left(y(t)-\mu_y\right)\right] = \Phi_{uy}(\tau) - \mu_y\mu_y \tag{5}$$

- Correlation coefficient:

$$r_{uy}(\tau) = E\left[\left(\frac{u(t-\tau)-\mu_u}{\sigma_u}\right)\left(\frac{y(t)-\mu_y}{\sigma_y}\right)\right] = \frac{C_{uy}(\tau)}{\sqrt{C_{uu}(0)C_{yy}(0)}} \tag{6}$$

Fourier transformation

Maps time domain signals into the frequency domain.

- Discrete Fourier transform:

$$\mathbf{u}(\omega) = \mathfrak{F}(\mathbf{u}(t)) = 2\pi \sum_{t=1}^{N} \mathbf{u}(t) e^{-j\omega\pi\frac{t}{N}} \tag{7}$$

- Inverse discrete Fourier transform

$$\mathbf{u}(t) = \mathfrak{F}^{-1}(\mathbf{u}(\omega)) = \frac{1}{2\pi N} \sum_{f=1}^{N} \mathbf{u}(\omega) e^{j\omega\pi\frac{t}{N}} \tag{8}$$

Spectral functions

Next we take the Fourier transformation of the correlation functions. By doing so we obtain:

- Spectral density:

$$\hat{S}_{uy}(\omega) = \mathfrak{F}\{\Phi_{uy}\} = \frac{1}{N} u^*(\omega) y(\omega) \tag{9}$$

- Coherence:

$$\hat{\gamma}_{uy}^2(\omega) = \frac{\left|\hat{S}_{uy}(\omega)\right|^2}{\hat{S}_{uu}(\omega)\hat{S}_{yy}(\omega)} \tag{10}$$

Overview signal functions

The following table gives an overview of the correlatation and spectral functions and how they can be obtained:

Time domain		FT		Frequency domain
Input, output	$u(t),\ y(t)$	$\leftrightarrow$	$u(\omega),\ y(\omega)$	Input, output
	$\downarrow$		$\downarrow$	
Cross-correlation	$\Phi_{uy}(\tau)$	$\leftrightarrow$	$S_{uy}(\omega)$	Cross-spectral density
	$\downarrow$		$\downarrow$	
Cross-covariance	$C_{uy}(\tau)$		$\downarrow$	
	$\downarrow$		$\downarrow$	
Correlation coefficient	$r_{uy}(\tau)$		$\gamma_{uy}(\omega)$	Coherency

Experiment considerations

- Measurement time T: Determines frequency resolution:

$$\Delta f = \frac{1}{T} \tag{11}$$

- Sampling frequency f_s: Determines frequency bandwidth:

$$f_n = \frac{f_s}{2} \tag{12}$$

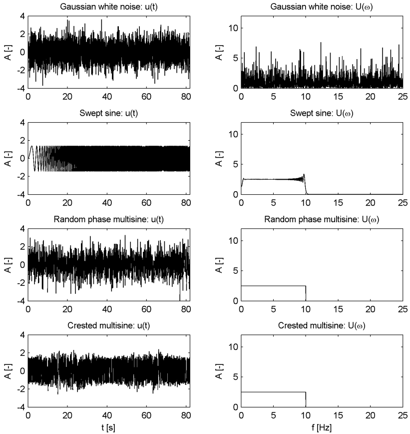
- Input signal $u(t)$

Input signal design

Commonly used input signals for frequency domain estimation:

	White noise	Swept sine	Multisine	Crested
Mean (μ_u):	0	0	0	0
Correlation (R_{uu}):	1	1	1	1
Spectral density (S_{uu}) :	1.0145	2.4912	2.5000	2.5000
Crest factor (C):	3.6129	1.4026	3.2857	2.3943
Predictable:	no	yes	no	no

Input signal design



Frequency averaging

Welch method

- Divide data in multiple segments
- Calculate spectral density for each segment
- Average over the segments
- D-time segments
- Drawback: reduced spectral resolution with factor D:

$$\hat{S}_{uy}(\omega) = \frac{1}{D}\sum_{d=1}^{D} S_{uy}(\omega) \tag{13}$$

Frequency averaging

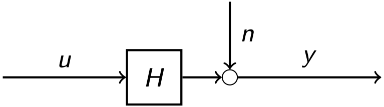
- Calculate the raw spectral density
- Average over adjacent frequencies over bandwidth D

$$\hat{S}_{uy}(\omega_c) = \frac{1}{D}\sum_{d=1}^{D} \hat{S}_{uy}(\omega_d) \tag{14}$$

- Drawback: Introduces bias at sharp transitions in FRF.

Open loop SISO Identification

Considering the following SISO system $H(\omega)$ with white observation noise n:



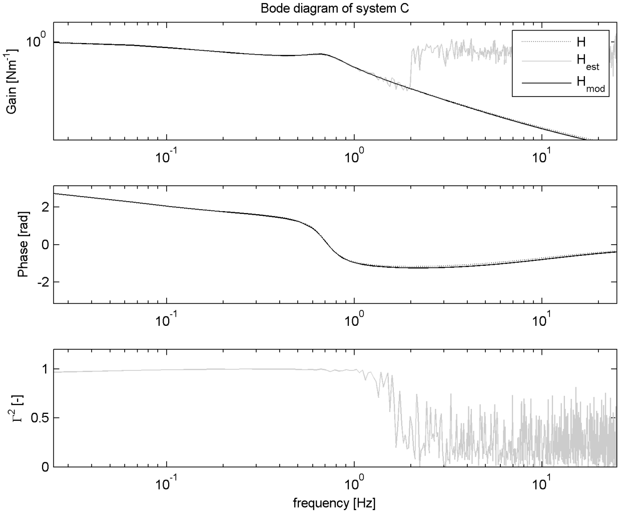
Solving the block diagram for the unknown system $H(\omega)$ gives:

$$Hu = y - n \tag{15}$$

Unfortunately we cant solve for H, since the noise term is unknown. However, assuming the noise n is uncorrelated with u we can estimate the system:

$$\begin{aligned} u^* Hu &= u^*(y - n) \\ H &= \frac{u^*(y-n)}{u^* u} = \frac{\hat{S}_{uy} - \hat{S}_{un}}{\hat{S}_{uu}} \\ \hat{H} &= \frac{\hat{S}_{uy}}{\hat{S}_{uu}} \quad \text{, if } \hat{S}_{un} \approx 0 \end{aligned} \tag{16}$$

SISO example



Open loop MIMO Identification

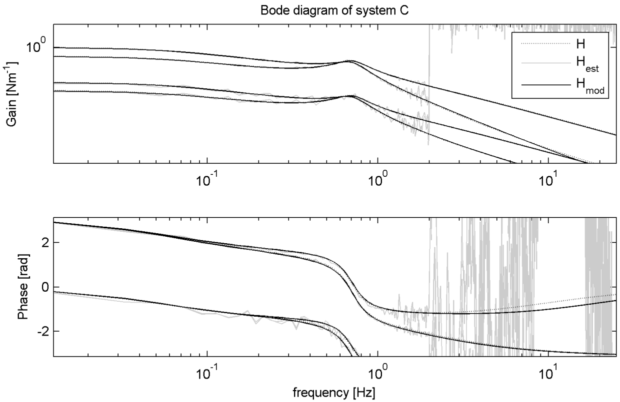
Similar to the SISO case, we can use the same strategy to solve for the MIMO case:

$$
\begin{aligned}
\mathbf{HU} &= \mathbf{Y} - \mathbf{N} \\
\mathbf{U}^*\mathbf{H}^* &= \mathbf{Y}^* - \mathbf{N}^* \\
\mathbf{UU}^*\mathbf{H}^* &= \mathbf{U}\left(\mathbf{Y}^* - \mathbf{N}^*\right) \\
\mathbf{H}^* &= \left(\mathbf{UU}^*\right)^{-1}\mathbf{U}\left(\mathbf{Y}^* - \mathbf{N}^*\right) \\
\mathbf{H} &= \left(\mathbf{Y} - \mathbf{N}\right)\mathbf{U}^*\left(\mathbf{UU}^*\right)^{-1}
\end{aligned}
\tag{17}
$$

Next we define the estimated transferfunction $\hat{\mathbf{H}}$ to be the transferfunction that linearly correlates the input/output data, assuming that the input is uncorrelated with the noise.

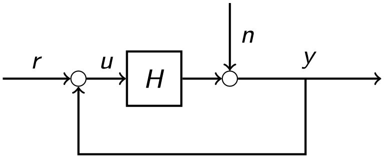
$$
\hat{\mathbf{H}} = \mathbf{YU}^*\left(\mathbf{UU}^*\right)^{-1} = \hat{\mathbf{S}}_{yu}\hat{\mathbf{S}}_{uu}^{-1} \tag{18}
$$

MIMO example



Close loop identification

Suppose we have a closed loop system shown in the block diagram below:



Than the open loop estimator $\hat{H} = S_{uy}/S_{uu}$ no longer holds, because the noise is correlated with the input signal through the feedback loop:

$$\hat{H} \neq \frac{S_{uy}}{S_{uu}} \tag{19}$$

Not discussed

- Windowing techniques: prevents or reduces spectral leaking.
- MIMO coherence: seperating coherent inputs/outputs.
- Closed loop system estimation
- Parameter Estimation in the frequency domain

Conclusions

- Frequency domain estimation techniques are well suited for linear dynamic deterministic systems.
- Proper input signal design can boost the signal to noise ratio.
- Frequency averaging should be applied to average out the noise.
- Don't use openloop SISO estimator when feedback is present in the system!

Further information

Some references about system identification and parameter estimation.

- System Identification: A frequency domain approach - Pintelon and Schoukens
- Filtering and Identification - Verhaegen and Verdult