CS337 Assignment-1

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1 Question-1

1.1 Part a

Let $P_X(Y)$ be the probability of Liam earning \$X and finishing the game given initially he has a \$Y with him. So we can write a recursive function in $P_X(Y)$ such as,

$$P_X(Y) = pP_X(Y+1) + (1-p)P_X(Y-1) \tag{1}$$

Explained as, if we initially have \$Y, then we can earn \$1 with probability p and loss \$1 with probability 1-p. We can rewrite the equation 1 as,

$$\beta = \frac{P_X(Y+1) - P_X(Y)}{P_X(Y) - P_X(Y-1)} \tag{2}$$

where $\beta = \frac{1-p}{p}$. So we can say that $P_X(Y+1) - P_X(Y)$ are in G.P. Let $a = P_X(1) - P_X(0)$. Therefore,

$$P_X(Y+1) - P_X(Y) = a\beta^Y \tag{3}$$

Now we know that $P_X(0) = 0$ and $P_X(X+Y) = 1$. Using equation 3, we can write $P_X(Y) - P_X(0) = a\frac{1-\beta^Y}{1-\beta}$ or alternatively, $P_X(Y) = a\frac{1-\beta^Y}{1-\beta}$. Now $P_X(1) = 1$, Therefore,

$$P_X(Y) = \frac{1-\beta^Y}{1-\beta^{X+Y}} \tag{4}$$

For Liam to lose everything, we can model a function $Q_X(Y)$ similar to $P_X(Y)$ which models the losing everything. The recursion with $Q_X(Y)$ will be same as $P_X(Y)$. Only difference will be $Q_X(0) = 1$ and $Q_X(Y+X) = 0$. Solving for $Q_X(Y)$, we will get,

$$Q_X(Y) = 1 - \frac{1-\beta^Y}{1-\beta^{X+Y}} \tag{5}$$

Therefore the probability that the game continues forever will be $1 - (P_X(Y) + Q_X(Y)) = 0$.

For the expected gain of Liam when X=1, we have

$$0 * Q_1(Y) + (Y+1)P_1(Y) - Y = (Y+1)\frac{1-\beta^Y}{1-\beta^{1+Y}} - Y \tag{6}$$

Now we need to prove,

$$\frac{Y+1}{\beta} - Y > (Y+1)\frac{1-\beta^Y}{1-\beta^{1+Y}} - Y$$

$$\frac{1}{\beta} > \frac{1-\beta^Y}{1-\beta^{1+Y}}$$

$$1 > \frac{\beta-\beta^{Y+1}}{1-\beta^{1+Y}}$$

$$0 > \frac{\beta-1}{1-\beta^{1+Y}}$$

This is true because $\beta > 1$ as game is favoured in the side of casino.

1.2 Part b

The only way Liam can lose is that he does not win in any of the first Y trials. Therefore, Probability of Liam winning is $H(Y) = 1 - (1-p)^Y$. Liam expected gain of is $Gain = 2^Y(1 - (1-p)^Y) - (2^Y - 1)$.

$$Gain = 1 - (2(1-p))^Y$$

Now as $p < 0.5$, therefore expected gain is negative.

2 Q2

In long-run we can expect that half of the coins would have been heads and half would be tails. Therefore, fraction of the original money after 2k tosses is, where k is very large

$$(1-p)^k(1+\epsilon p)^k$$

Differentiating the same would yield

$$p = \frac{1-\epsilon}{2\epsilon}$$

Here $\epsilon = 1.4$, therefore $p = \frac{1}{7}$

3 Q3

3.1 Part a

We know that trace(AB) = trace(AB). Also trace(AB-BA) = 0, but $trace(I) \neq 0$. Therefore, the statement is wrong.

3.2 Part b

Now every entry in the diagonal is 4 with probability 0.2 and 5 with probability 0.8.

$$\lim_{n\to\infty} \frac{tr(A_n)}{n} = 0.2(4) + 0.8(5) = 4.8$$

Also,

$$\lim_{n\to\infty} det(A_n) = 4^{0.5n}5^{0.8n}$$

$$\therefore \lim_{n\to\infty} \frac{log(det(A_n))}{n} = n(0.2log4 + 0.8log5)/n = 0.2log4 + 0.8log5 \approx 1.565$$

4 Q4

For T to satisfy the model, The condition must be,

$$\forall i, \sum_{j\in V} T(i,j) = 1$$

The T that would satisfy the graph,

T(v,w)	n_1	n_2	n_3
n_1	$\frac{1}{2}$	$\frac{1}{2}$	0
n_2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
n_3	0	$\frac{1}{2}$	$\frac{1}{2}$

We can diagonalize the matrix and then calculate the calculate the power to n where $\lim_{n\to\infty}$.

$$T = PAP^{-1}$$
$$T^n = PA^nP^{-1}$$

Therefore we will have

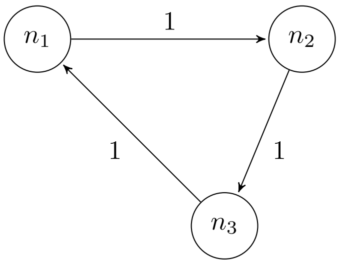
$$\lim_{n\to\infty} T^n = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -\frac{4}{3} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & \frac{1}{2}^n & 0 \\ 0 & 0 & \frac{1}{6}^n \end{bmatrix} \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{14} & -\frac{3}{7} & \frac{3}{14} \end{bmatrix}$$

$$\lim_{n\to\infty} T^n = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -\frac{4}{3} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{14} & -\frac{3}{7} & \frac{3}{14} \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \end{bmatrix}$$

Let Set $A = \{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\}$. Let $a \in A$, then we have

$$aT^n = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{2}{7} \end{bmatrix}, \forall a \in A$$

The following graph will never converge starting from any initial point.



5 Q5

We have,

$$P(\epsilon) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\epsilon^2}{\sigma^2}}$$

For maximizing the maximum likelihood estimate we take,

$$\prod_{(x_i, y_i)\in M} P(\epsilon) = (\frac{1}{\sqrt{2\pi}\sigma})^n e^{-\frac{\sum(y_i - ax_i)^2}{\sigma^2}}$$

Therefore the correct loss function would be,

$$Loss = \frac{1}{n}\sum_{i=0}^{n}(y_i - ax_i)^2$$

The maximum likelihood estimate of a would be

$$0 = \frac{1}{n}\sum_{i=0}^{n} -x_i(y_i - ax_i)$$

$$a = \frac{\sum x_i y_i}{\sum x_i^2}$$

6 Q6

We make the assumption that $\epsilon \sim N(0, \sigma^2)$. So minimum the loss greater the likelihood. For estimating the values of (a,b), we can differentiate the loss function. Here n=52

$$Loss = \frac{1}{n}\sum_{i=1}^{n}(t_i - a - br_i)^2$$

After differentiating we get two equations,

$$\sum t_i = na + b\sum r_i$$

$$\sum r_i t_i = a\sum r_i + b\sum r_i^2$$

Solving the above two equations we get,

$$a = \frac{\sum t_i \sum r_i^2 - \sum r_i \sum r_i t_i}{(\sum r_i)^2 - n\sum r_i^2}$$
$$b = \frac{\sum r_i \sum t_i - n\sum r_i t_i}{(\sum r_i)^2 - n\sum r_i^2}$$

Using the given means, standard deviations and correlation, we get a = 6.49 , b = 0.039 .

Now for part (c), for all the values of (r_{53}, t_{53}) we have reasonable values except the value of $(r_{53}, t_{53}) = (2000, 30)$. therefore, this value will change the parameters the most.

For part (d), we will add the priors of the a and b to the maximum likelihood function of ϵ.

$$Max - Likelihood = \lambda^n e^{-\lambda(\sum(t_i - a - br_i))}\frac{1}{\sqrt{2\pi}\sigma_a}e^{-\frac{(a-\mu_a)^2}{\sigma_a^2}}\frac{1}{\sqrt{2\pi}\sigma_b}e^{-\frac{(b-\mu_b)^2}{\sigma_b^2}}$$

Therefore, we will model the loss function as

$$Loss = \frac{1}{n}(\lambda\sum(t_i - a - br_i) + \frac{(a-\mu_a)^2}{\sigma_a^2} + \frac{(b-\mu_b)^2}{\sigma_b^2})$$

Which evaluates to

$$Loss = \frac{1}{52}(2\sum(t_i - a - br_i) + \frac{(a-30)^2}{6} + \frac{b^2}{2})$$