Trasformata di Fourier

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Linear Systems

- Definitions & Properties
- Shift Invariant Linear Systems
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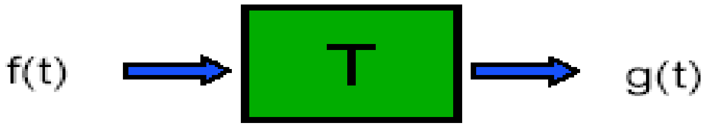
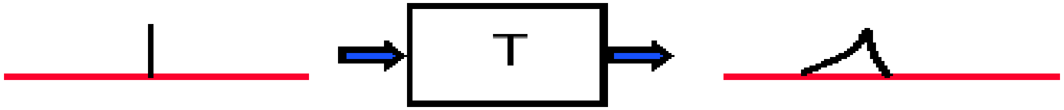
- Linear Systems - Frequency Response

Linear System

- A linear system T gets an input f(t) and produces an output g(t):



$$g(t)=T\{f(t)\}$$

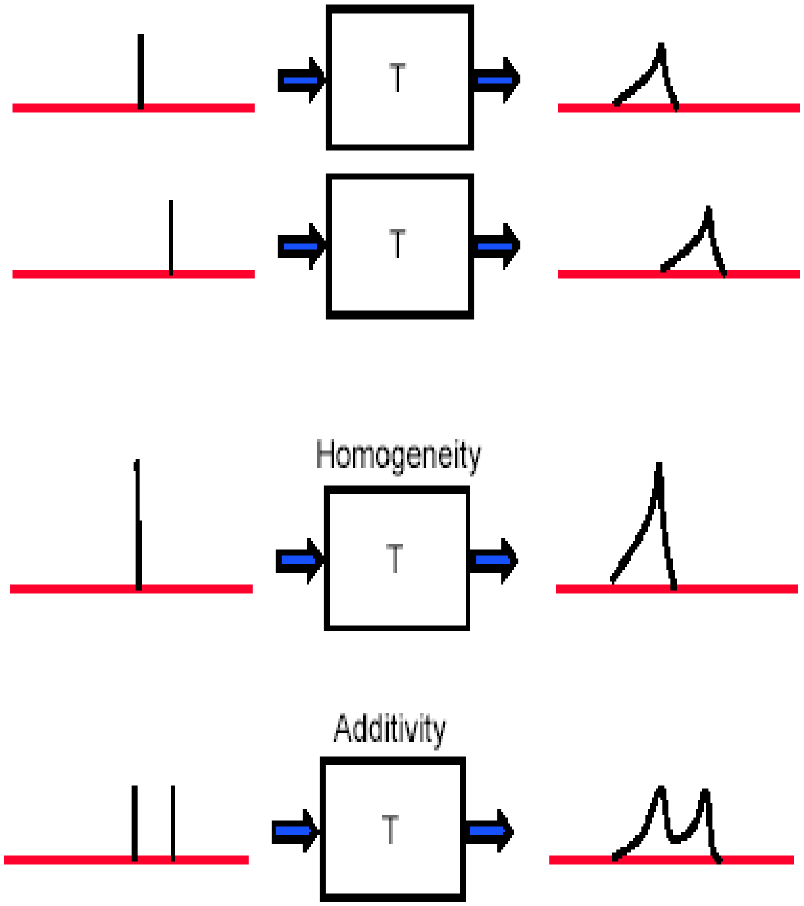


- In the discrete caes:
  - input : f[n] , n = 0,1,2,...
  - output: g[n] , n = 0,1,2,...

$$g[n]=T\{f[n]\}$$

Linear System Properties

- A linear system must satisfy two conditions:
  - Homogeneity: $T\{a\,f[n]\} = aT\{f[n]\}$
  - Additivity: $T\{f_1[n] + f_2[n]\} = T\{f_1[n]\} + T\{f_2[n]\}$



Linear Shift Invariant Systems (LSIS)

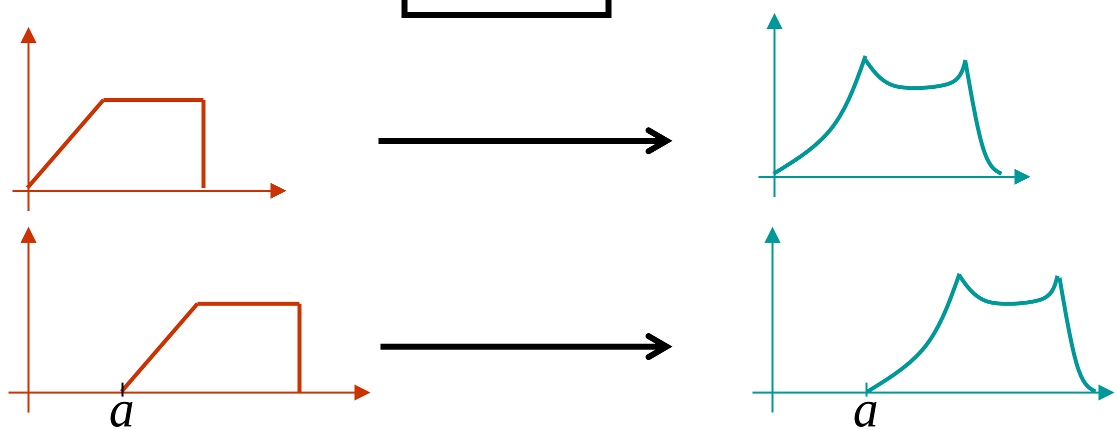
Linearity:

$$f_1 \rightarrow \boxed{} \rightarrow g_1 \qquad f_2 \rightarrow \boxed{} \rightarrow g_2$$

$$\alpha f_1 + \beta f_2 \rightarrow \boxed{} \rightarrow \alpha g_1 + \beta g_2$$

Shift invariance:

$$f(x-a) \rightarrow \boxed{} \rightarrow g(x-a)$$



Linear Shift-Invariant systems

$\delta(n_1, n_2)$

0	0	0	0	0
0	0	0	0	0
0	0	1	0	0
0	0	0	0	0
0	0	0	0	0

Impulse signal

$\xrightarrow{\text{LSI}}$

$h(n_1, n_2)$

0	0	0	0	0
0	0	1	0	0
0	1	1	1	0
0	0	1	0	0
0	0	0	0	0

Impulse response function

$x(n_1, n_2)$

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

$\xrightarrow{\text{LSI}}$

$y(n_1, n_2)$

0	0	0	0	0	0	0	0
0	0	1	0	0	0	0	0
0	1	1	1	0	0	0	0
0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0
0	0	0	0	1	1	1	0
0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0

Shift Invariant Linear System (cont.)

Shift-Invariant Linear System - Example

- **Contrast change** by grayscale stretching around 0:

$$T\{f(x)\} = af(x) = g(x)$$

  - Shift Invariant:

$$T\{f(x-x_0)\} = af(x-x_0) = g(x-x_0)$$

- **Convolution**:

$$T\{f(x)\} = f(x)*a = g(x)$$

  - Shift Invariant:

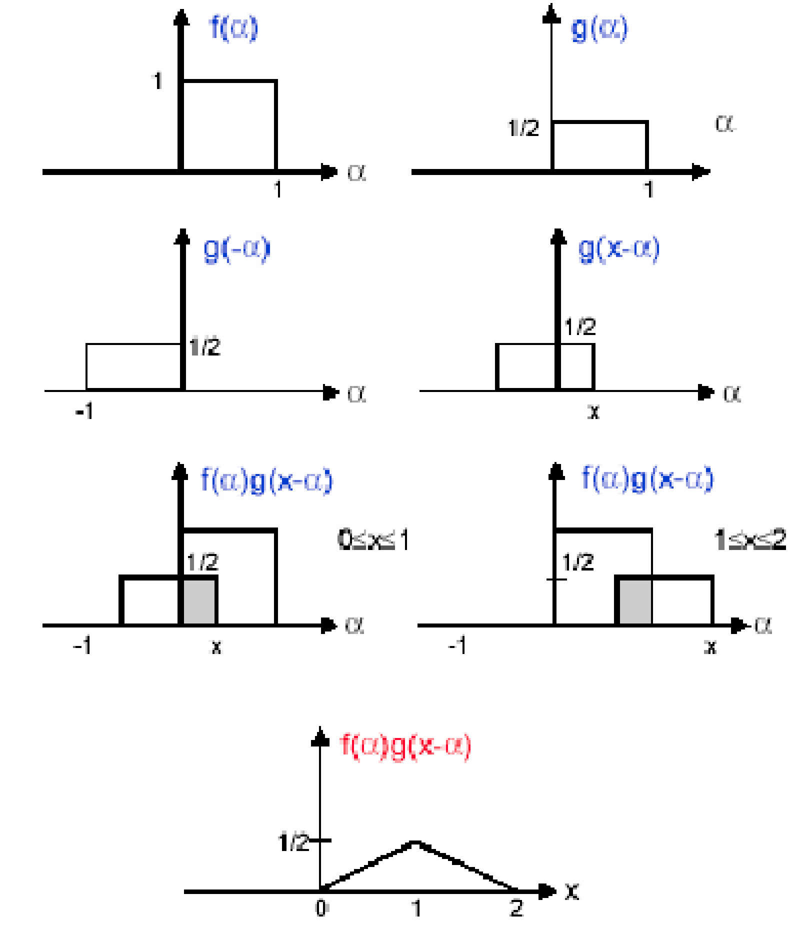
$$T\{f(x-x_0)\} = f(x-x_0)*a$$

$$= \sum_i f(i - x_0)a(x - i) = \sum_j f(j)a(x - j - x_0)$$

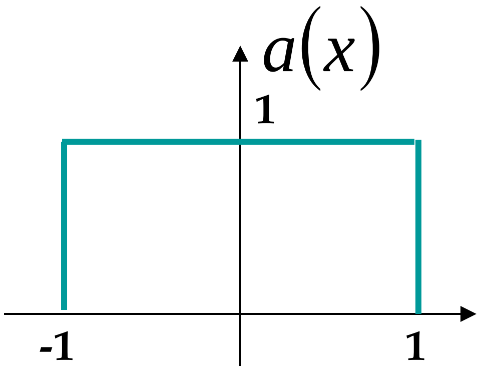
$$= g(x-x_0)$$

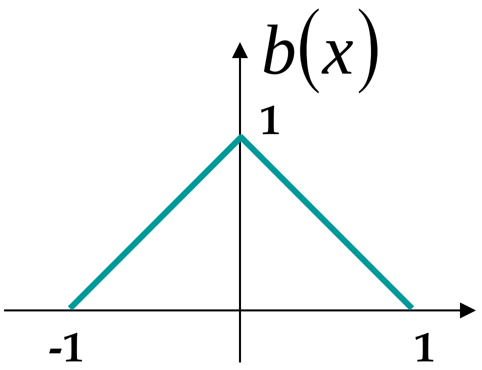
Convolution in Continous Case

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\alpha)\, g(x-\alpha)\, d\alpha$$

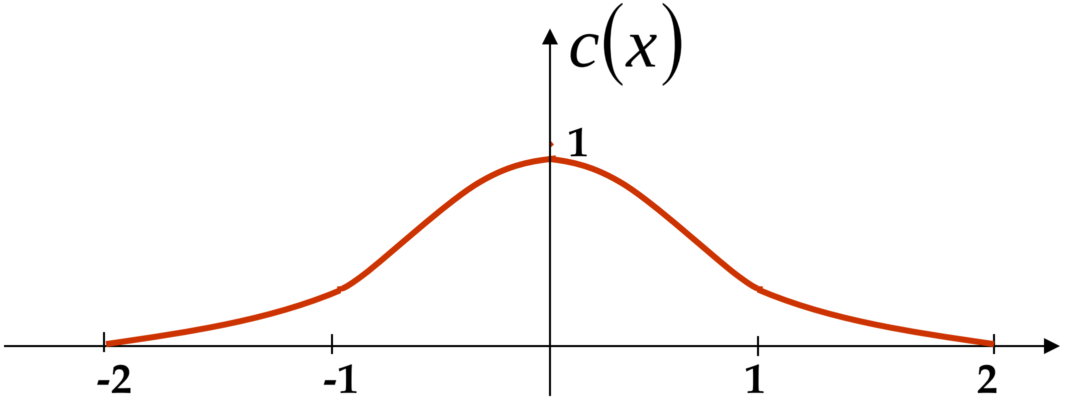


Convolution - Example





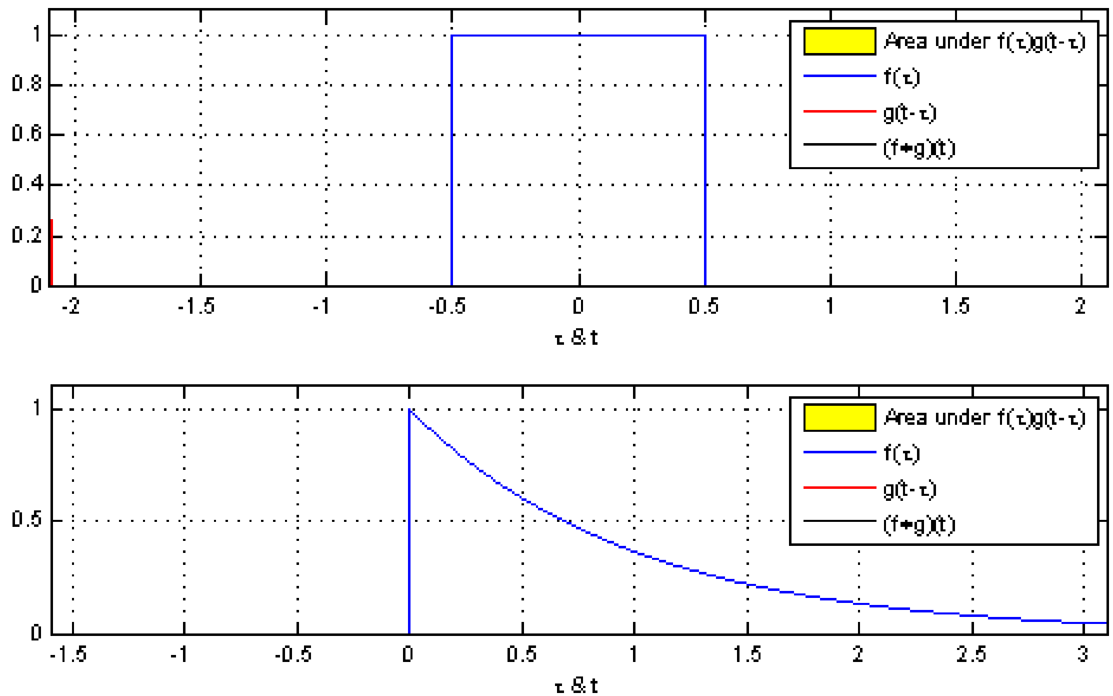
$$c = a * b$$



Convoluzione tra 2 Funzioni

La **convoluzione** è un'operazione tra due funzioni di una variabile che consiste nell'integrare il prodotto tra la prima e la seconda traslata di un certo valore

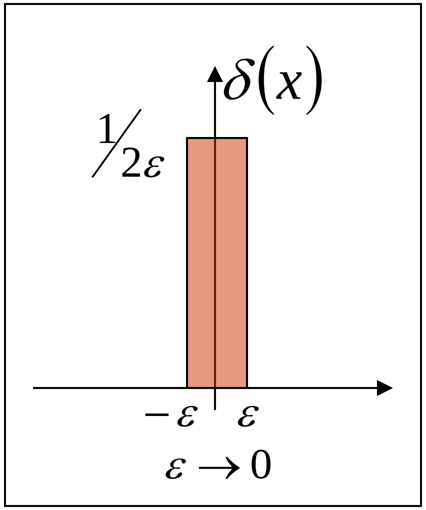
$$(f * g)[n] \stackrel{\mathrm{def}}{=} \sum_{m=-\infty}^{\infty} f[m]\, g[n-m] = \sum_{m=-\infty}^{\infty} f[n-m]\, g[m]$$



Convolution Kernel – Impulse Response

$$f \longrightarrow \boxed{h} \longrightarrow g \qquad\qquad g = f * h$$

- What h will give us $g = f$?



Dirac Delta Function (Unit Impulse)

A frequently used concept in Fourier theory is that of the *Dirac Delta Function*, which is somewhat abstractly defined as:

$$\begin{aligned} \delta(x) &= 0 \qquad \text{for } x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x &= 1 \end{aligned} \tag{1}$$

This can be thought of as a very "*tall*-and-*thin*" spike with unit area located at the origin, as shown in figure 1.

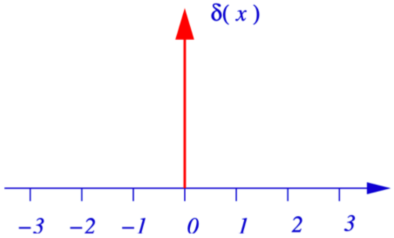


Figure 1: The δ-function.

Impulse Sequence

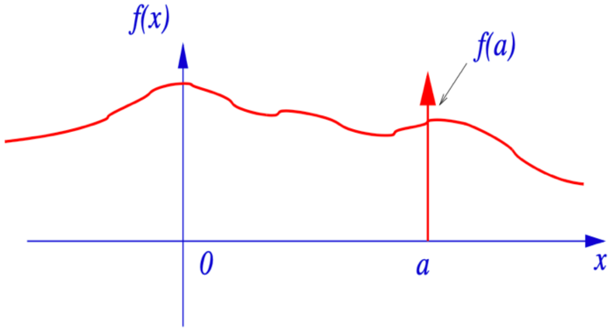
Since the *Dirac Delta Function* is used extensively, and has some useful, and slightly perculiar properties, it is worth considering these are this point. For a function $f(x)$, being integrable, then we have that

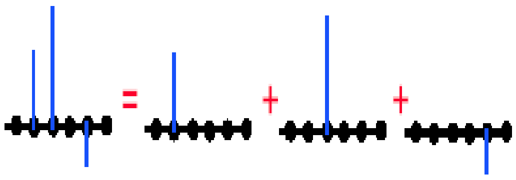
$$\int_{-\infty}^{\infty} \delta(x) f(x) \, \mathrm{d}x = f(0) \tag{6}$$

which is often taken as an alternative definition of the Delta function. This says that integral of any function multiplied by a δ-function located about zero is just the value of the function at zero. This concept can be extended to give the *Shifting Property*, again for a function $f(x)$, giving,

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) \, \mathrm{d}x = f(a) \tag{7}$$

where $\delta(x-a)$ is just a δ-function located at $x = a$ as shown in figure 2.





- An impulse signal is defined as follows:

$$\delta[n-k] = \begin{cases} 0 & \text{where} \quad n \neq k \\ 1 & \text{where} \quad n = k \end{cases}$$

- Any signal can be represented as a linear sum of scales and shifted impulses:

$$f[n] = \sum_{j=-\infty}^{\infty} f[j]\,\delta[n-j]$$

Shift-Invariant Linear System and Convolution

Shift-Invariant Linear System is a Convolution

Proof:

- f[n] input sequence
- g[n] output sequence
- h[n] the system impulse response:

$$h[n]=T\{\delta[n]\}$$

$$g[n]=T\{f[n]\}=T\left\{\sum_{j=-\infty}^{\infty} f[j]\delta[n-j]\right\}$$

$$=\sum_{j=-\infty}^{\infty} f[j]T\{\delta[n-j]\} \quad (\textit{from linearity})$$

$$=\sum_{j=-\infty}^{\infty} f[j]h[n-j] \quad (\textit{from shift-inariancce})$$

$$= f*h$$

The output is a sum of scaled and shifted copies of impulse responses.

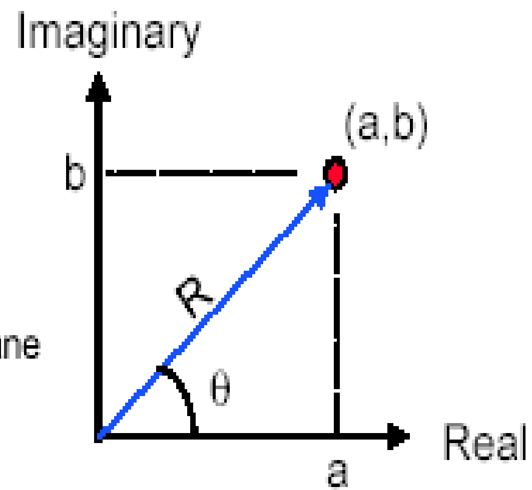
Complex Number

- Two kind of representations for a point (a,b) in the complex plane
  - The Cartesian representation:

$$Z = a + ib \qquad \text{where} \quad i^2 = -1$$

  - The Polar representation:

$$Z = Re^{i\theta} \qquad \text{(Complex exponential)}$$

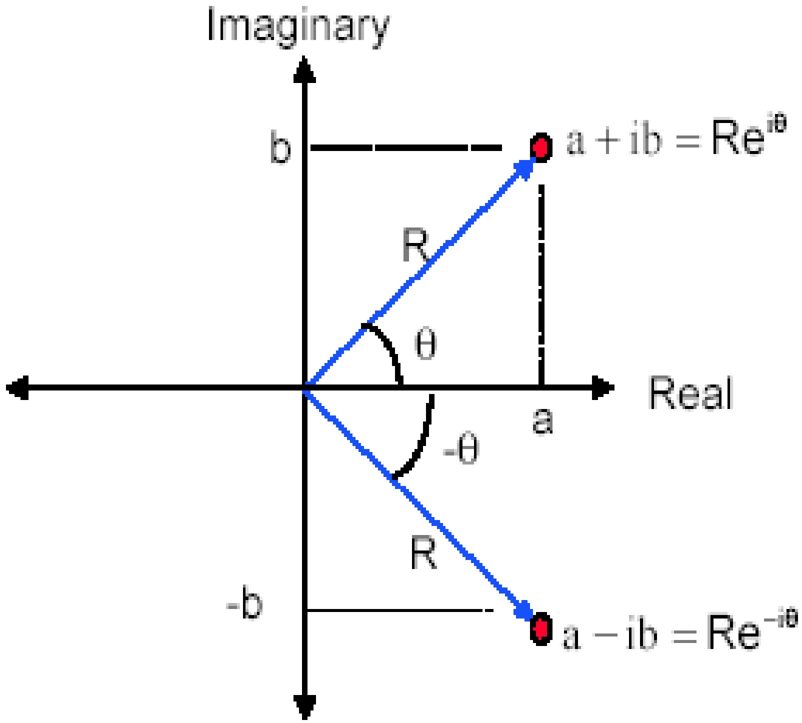


The Complex Plane

- Conversions:
  - Polar to Cartesian: $Re^{i\theta} = R\cos(\theta) + iR\sin(\theta)$
  - Cartesian to Polar $a + ib = \sqrt{a^2 + b^2}\; e^{i\tan^{-1}(b/a)}$

Complex Number (cont.)

- Conjugate of Z is Z^*:
  - Cartesian rep. $(a+ib)^* = a - ib$
  - Polar rep. $(Re^{i\theta})^* = Re^{-i\theta}$



Complex Number (cont.)

Algebraic operations:

- addition/subtraction:

$$(a+ib)+(c+id)=(a+c)+i(b+d)$$

- multiplication:

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)$$

$$Ae^{i\alpha}\ Be^{i\beta}=ABe^{i(\alpha+\beta)}$$

- Norm:

$$\|a+ib\|^2=(a+ib)^*(a+ib)=a^2+b^2$$

$$\|Re^{i\theta}\|^2=(Re^{i\theta})^*Re^{i\theta}=Re^{-i\theta}Re^{i\theta}=R^2$$

The (Co-)Sinusoid

- The (Co-)Sinusoid as complex exponential:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

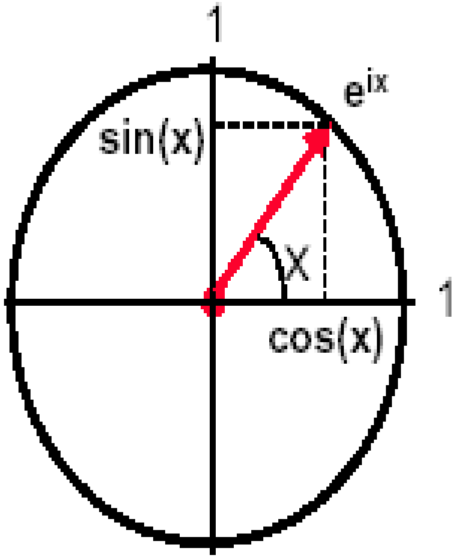
$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Or

$$\cos(x) = \mathrm{Real}(e^{ix})$$

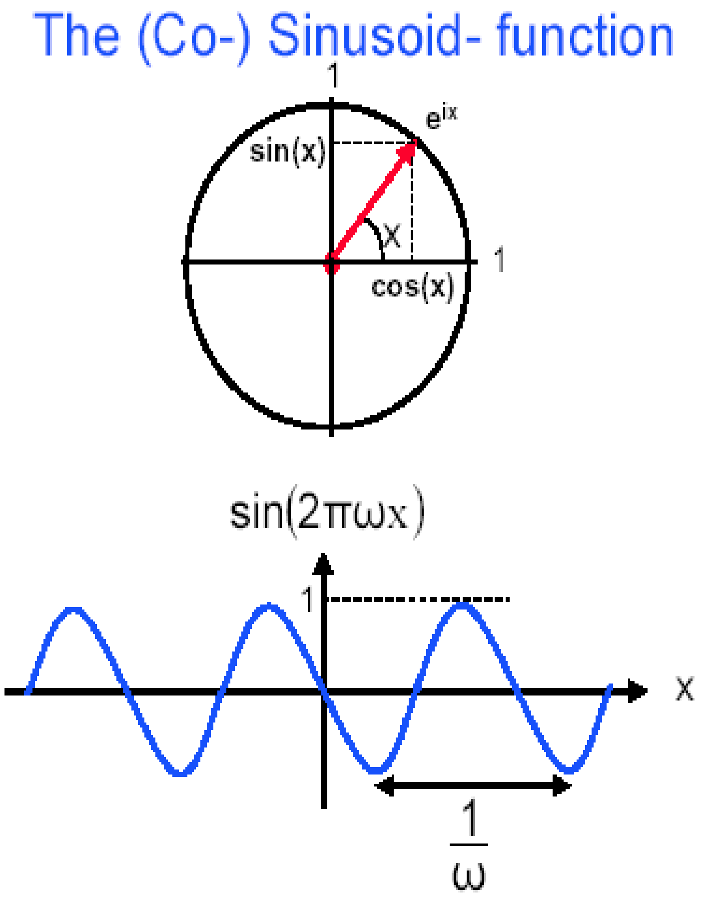
$$\sin(x) = \mathrm{Imag}(e^{ix})$$

The (Co-) Sinusoid



The (Co-)Sinusoid (cont.)

- The wavelength of $\sin(2\pi\omega x)$ is $1/\omega$.
- The frequency is ω .

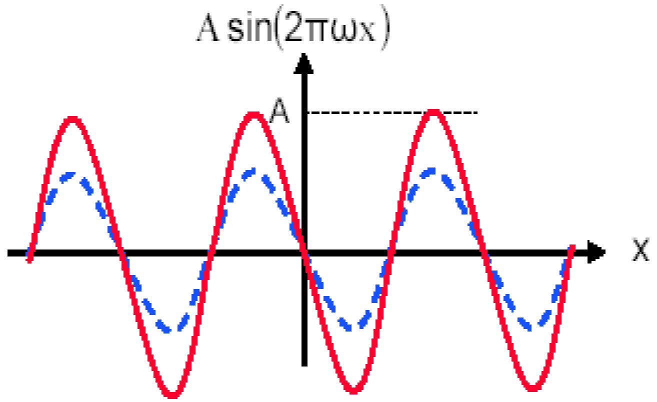


The (Co-)Sinusoid (cont.)

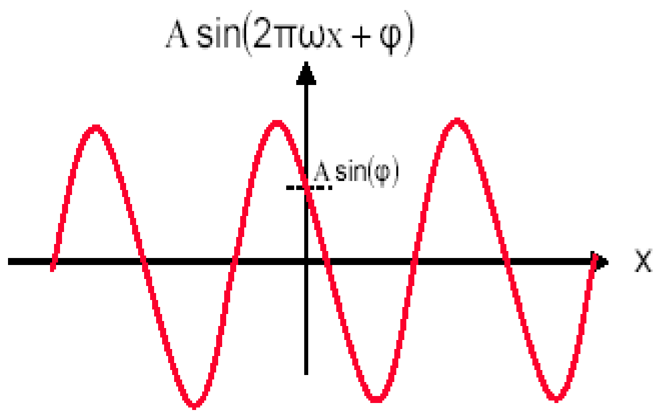
Scaling and shifting can be represented as a multiplication with $Ae^{i\varphi}$

$$A\sin(2\pi\omega x+\varphi)=\mathrm{Imag}(Ae^{i\varphi}\,e^{i2\pi\omega x})$$

– Changing Amplitude:



– Changing Phase:



The (Co-)Sinusoid (cont.)

- If we add a Sine wave to a Cosine wave with the same frequency we get a scaled and shifted (Co-)Sine wave with the same frequency

$$a\sin(\omega x)+b\cos(\omega x)=R\sin(\omega x+\phi)$$

$$\text{where } R=\sqrt{a^2+b^2} \text{ and } \phi=\tan^{-1}\left(\frac{b}{a}\right)$$

$$a\sin(\omega x)+b\cos(\omega x)$$

$$\Updownarrow$$

$$\mathrm{Re}^{i(\omega x+\phi)}$$

Two alternatives to represent a scaled and shifted Sine wave.

The (Co-)Sinusoid (cont.)

Combining Sine and Cosine - Proof

Linear Combination of sin(ωx) and cos(ωx) produces sin(ωx) with a change in Phase and Amplitude.

$$a \sin(\omega x) + b \cos(\omega x) = R \sin(\omega x + \theta)$$

Proof :

$$\mathrm{asin}(\omega x) + \mathrm{bcos}(\omega x) = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin(\omega x) + \frac{b}{\sqrt{a^2+b^2}}\cos(\omega x)\right] \quad (1)$$

Since

$$\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2 = 1$$

There exists θ such that :

$$\frac{a}{\sqrt{a^2+b^2}} = \cos(\theta) \text{ and } \frac{b}{\sqrt{a^2+b^2}} = \sin(\theta)$$

Thus from (1) we obtain :

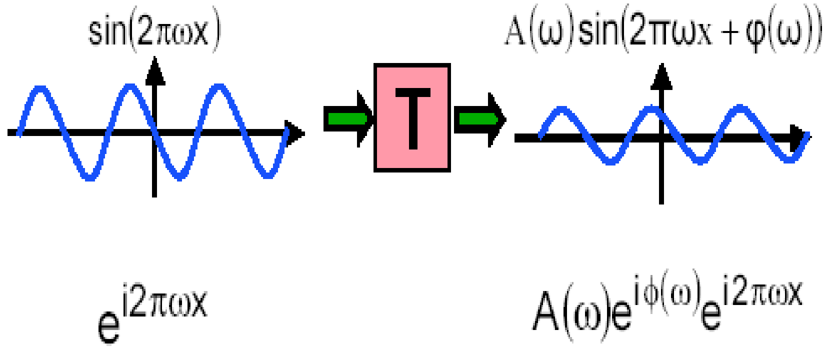
$$\sqrt{a^2+b^2}\left[\cos(\theta)\sin(\omega x) + \sin(\theta)\cos(\omega x)\right] = \sqrt{a^2+b^2}\,\sin(\omega x + \theta)$$

$$\mathrm{R} = \sqrt{a^2+b^2} \quad \text{Amplitude}$$

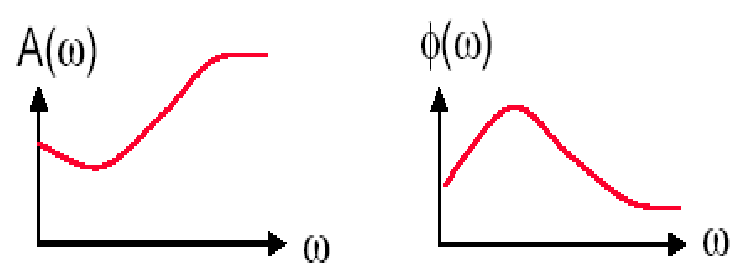
$$\theta = \mathrm{tg}^{-1}\left(\frac{b}{a}\right) \quad \text{Phase}$$

The response of Shift-Invariant Linear System to a Sine wave

- The response of a shift-invariant linear system to a sine wave is a shifted and scaled sine wave with the same frequency.



- The frequency response (or Transfer Function) of a system:



The response of Shift-Invariant Linear System to a Sine wave (cont.)

$$\delta(x) \Rightarrow \boxed{T} \Rightarrow h(x)$$

Impulse response

$$A(\omega)e^{i\varphi(\omega)} = H(\omega)$$

$$e^{i2\pi\omega x} \Rightarrow \boxed{T} \Rightarrow H(\omega)e^{i2\pi\omega x}$$

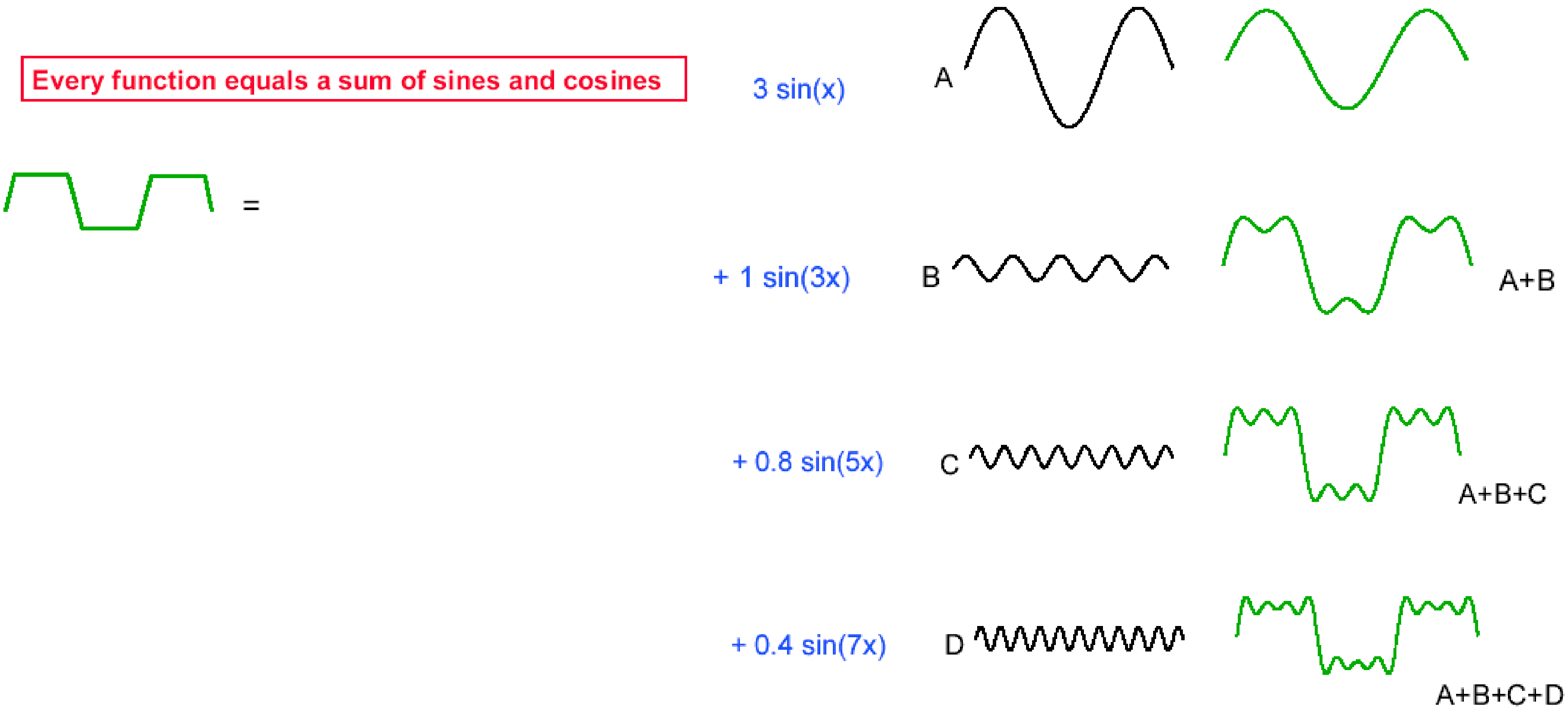
Frequency response

Jean Baptiste Joseph Fourier (1768-1830)

- Had crazy idea (1807):
- **Any** periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies.
- Don't believe it?
  - Neither did Lagrange, Laplace, Poisson and other big wigs
  - Not translated into English until 1878!
- But it's true!
  - called Fourier Series
  - Possibly the greatest tool used in Engineering

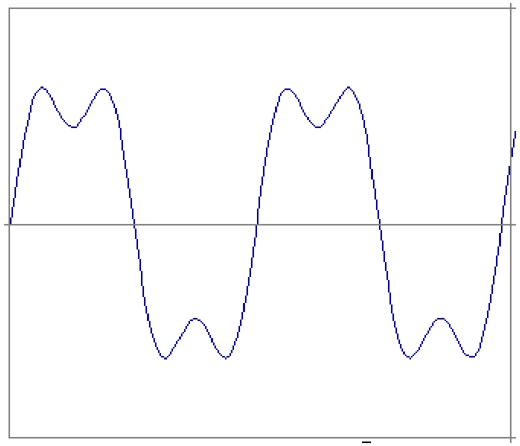
The Fourier Transform (cont.)

Every function equals a sum of sines and cosines



Time and Frequency

- example : $g(t) = \sin(2pf\, t) + (1/3)\sin(2p(3f)\, t)$



Time and Frequency

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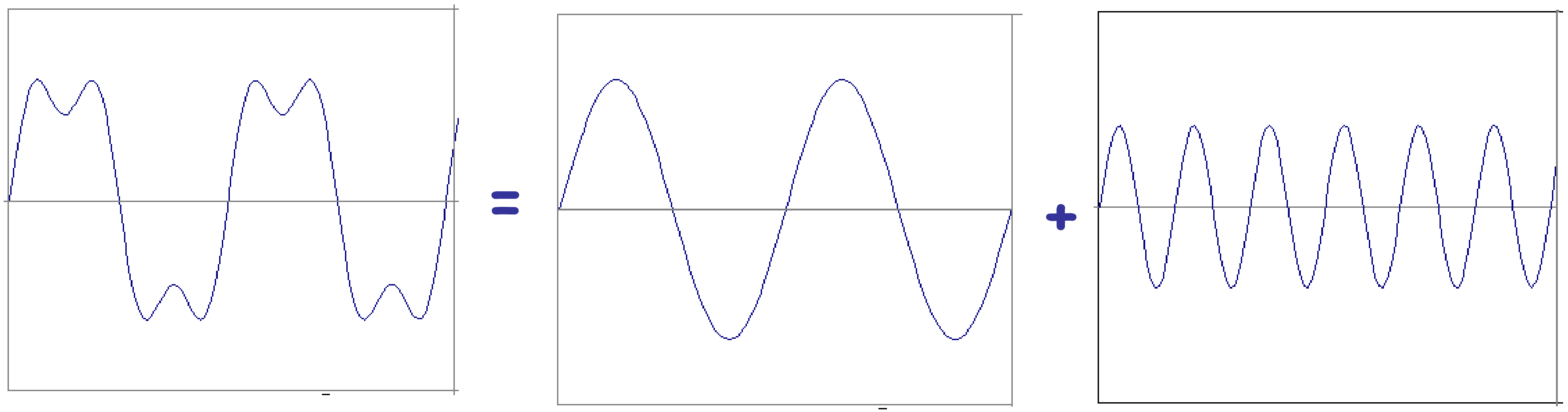
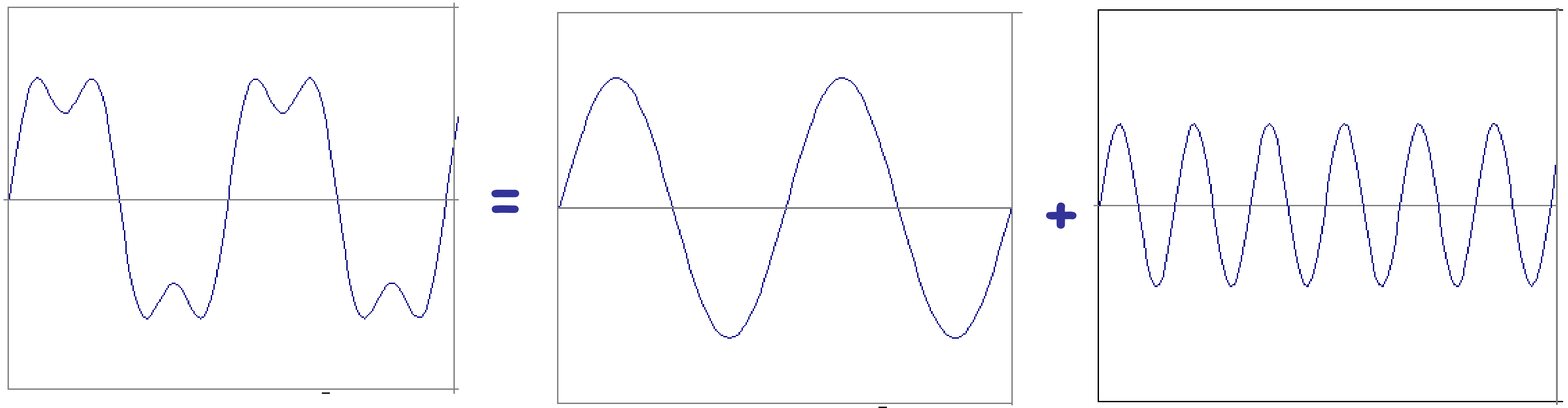
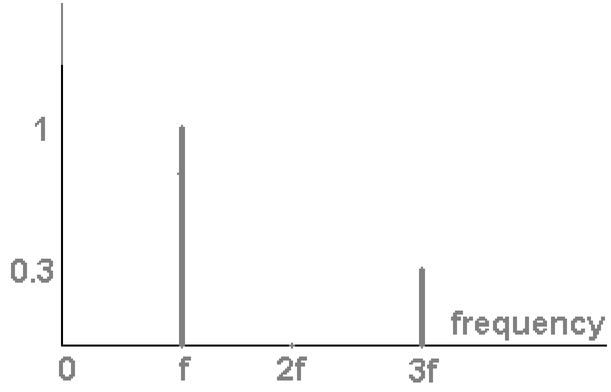


Frequency Spectra

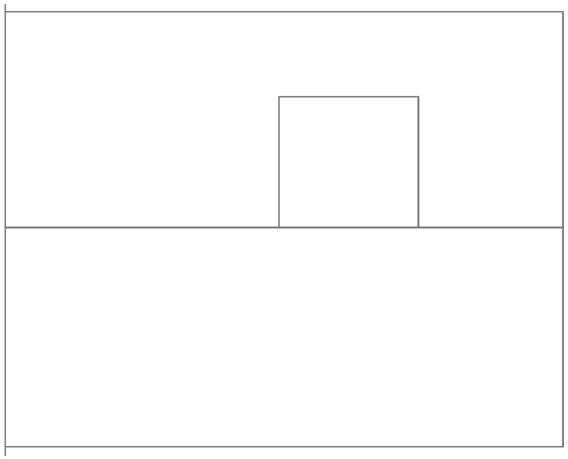
- example : $g(t) = \sin(2pf\,t) + (1/3)\sin(2p(3f)\,t)$



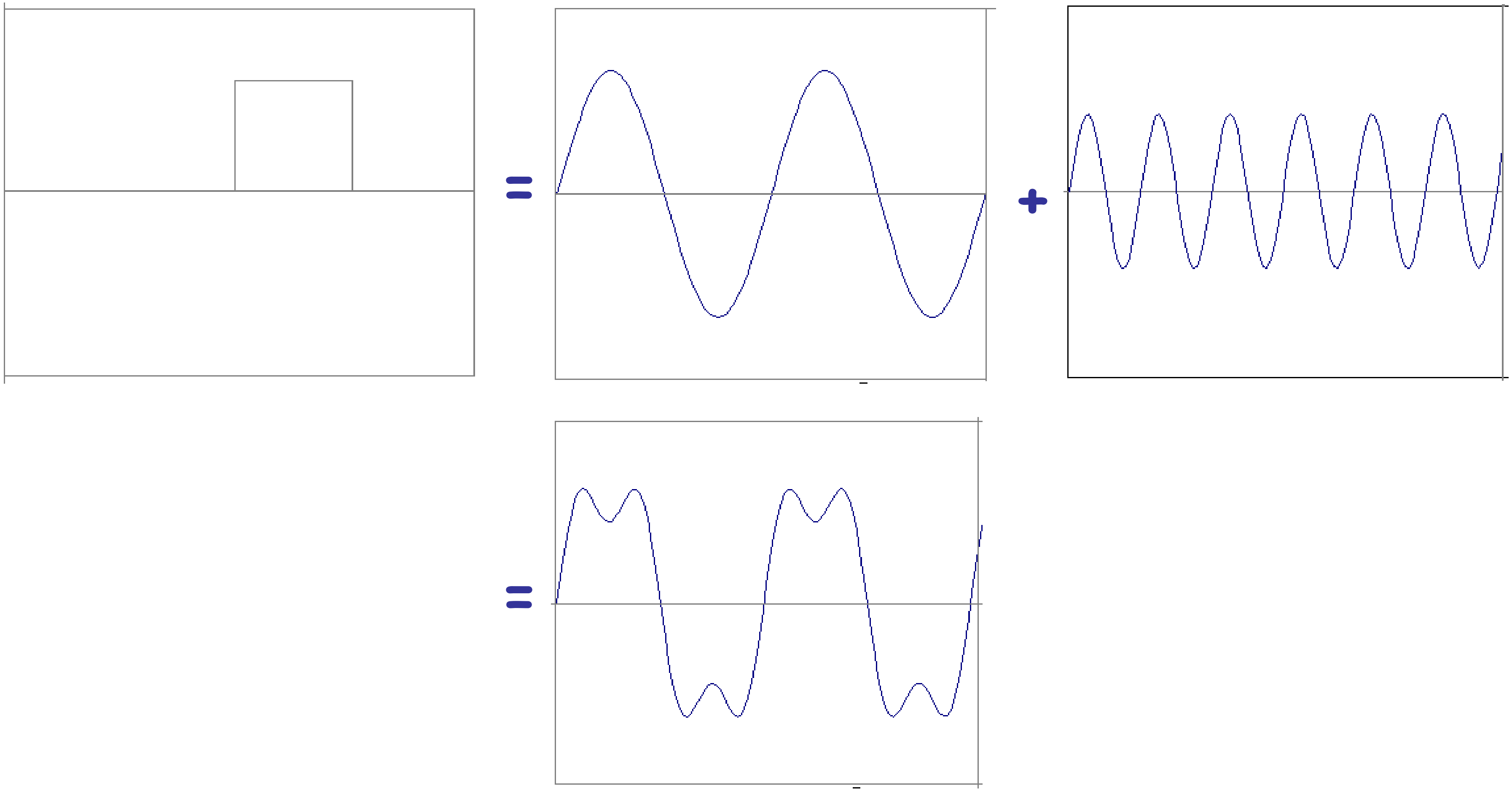


Frequency Spectra

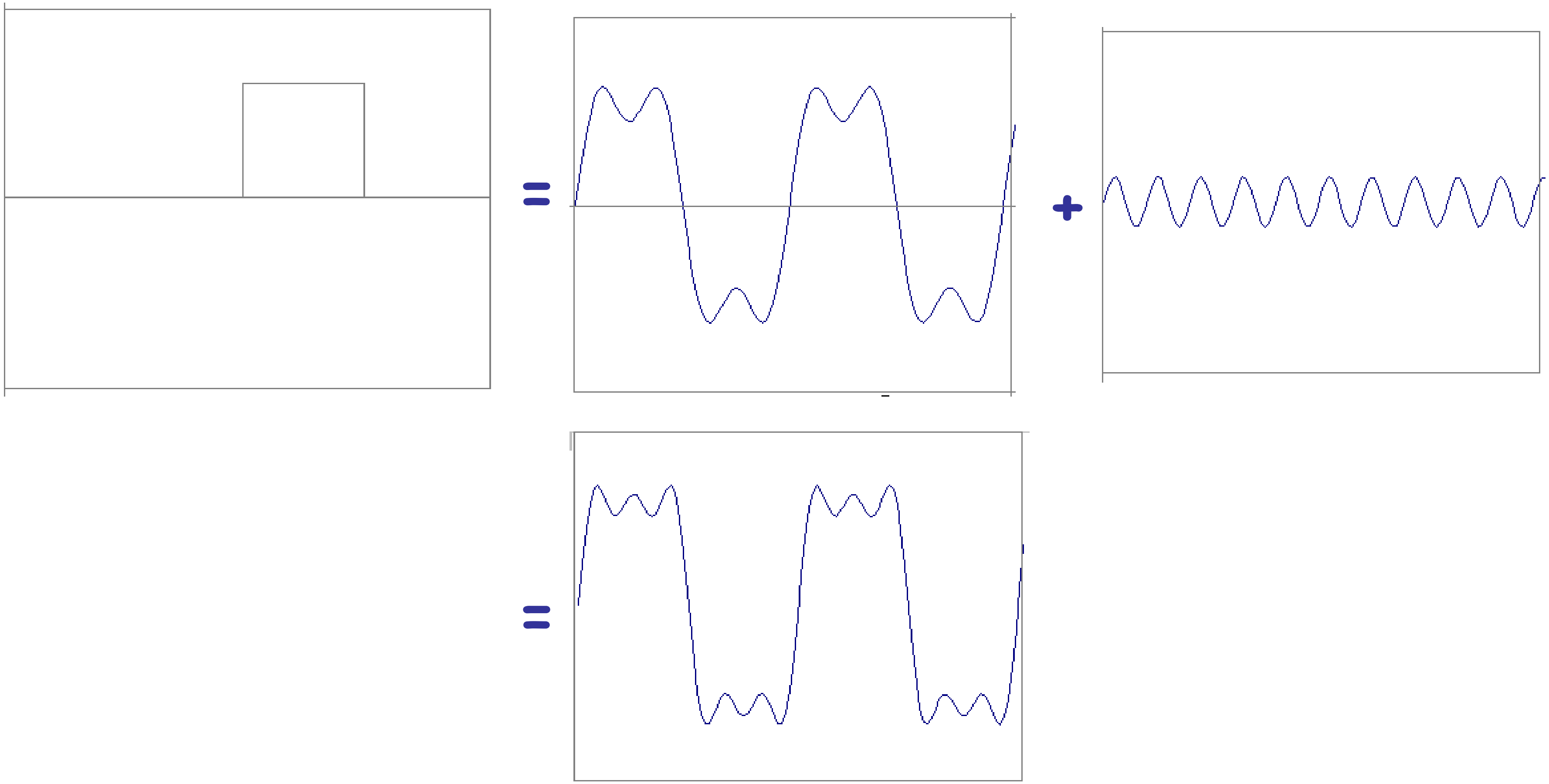
- Usually, frequency is more interesting than the phase



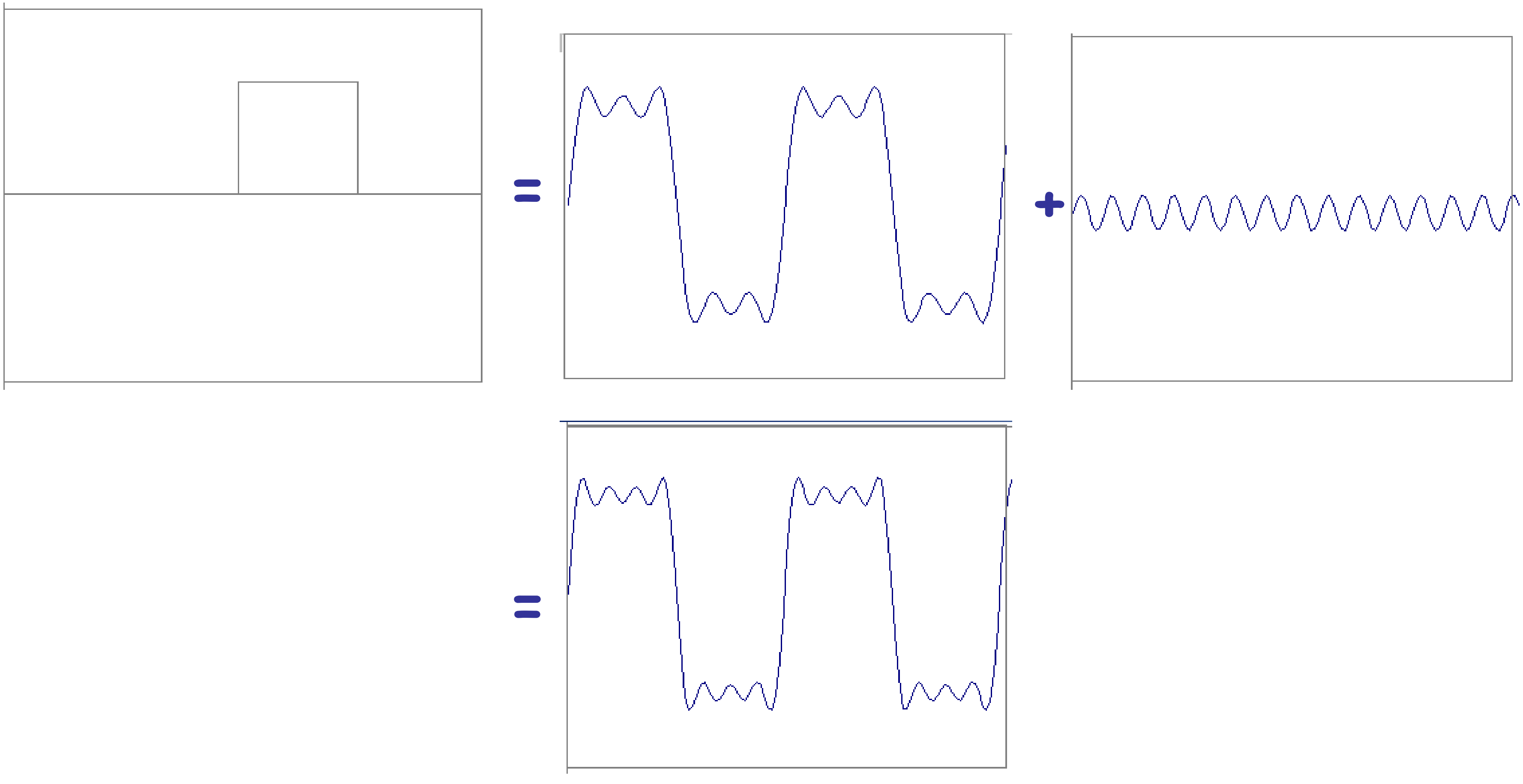
Frequency Spectra



Frequency Spectra



Frequency Spectra



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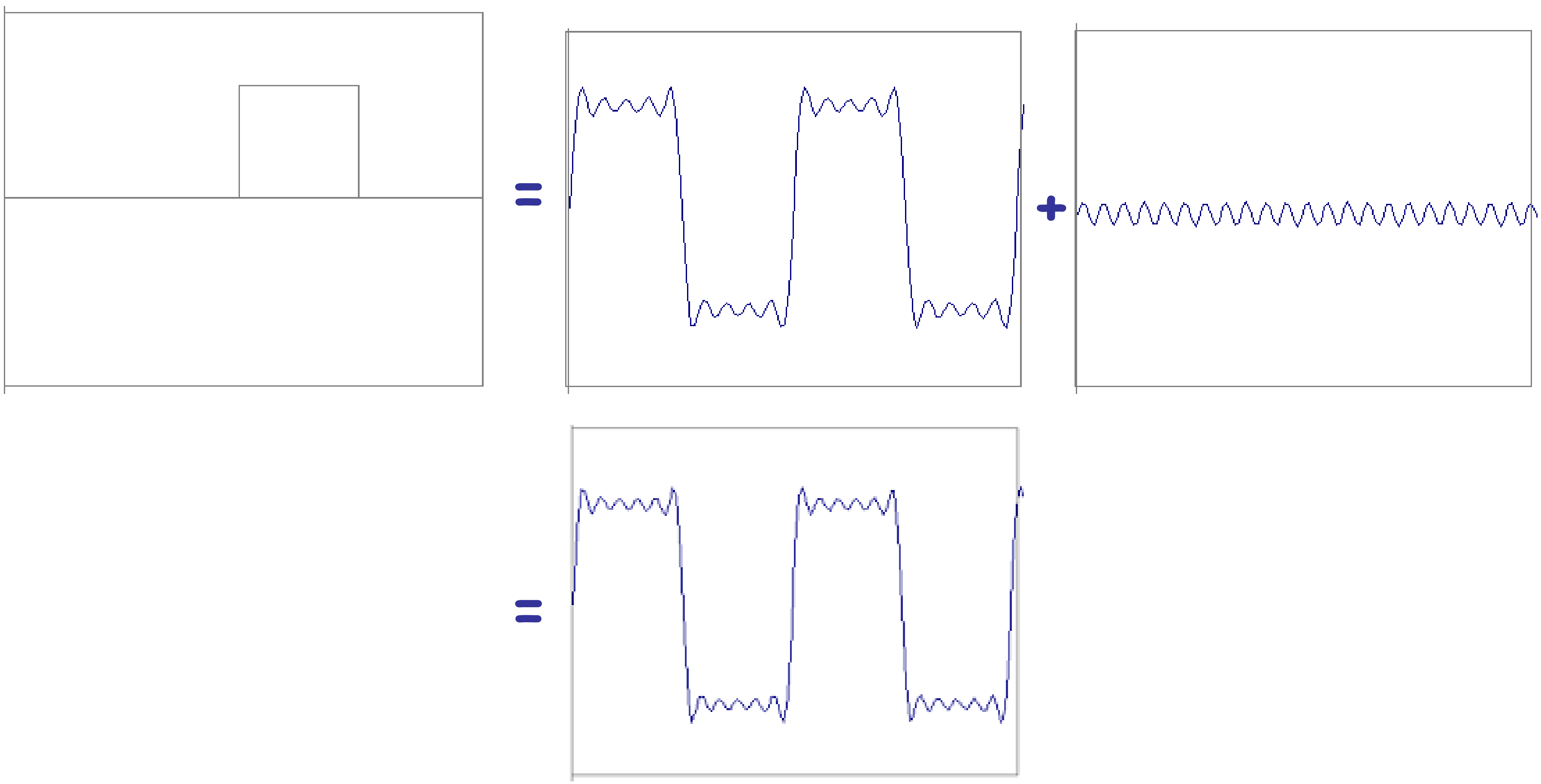
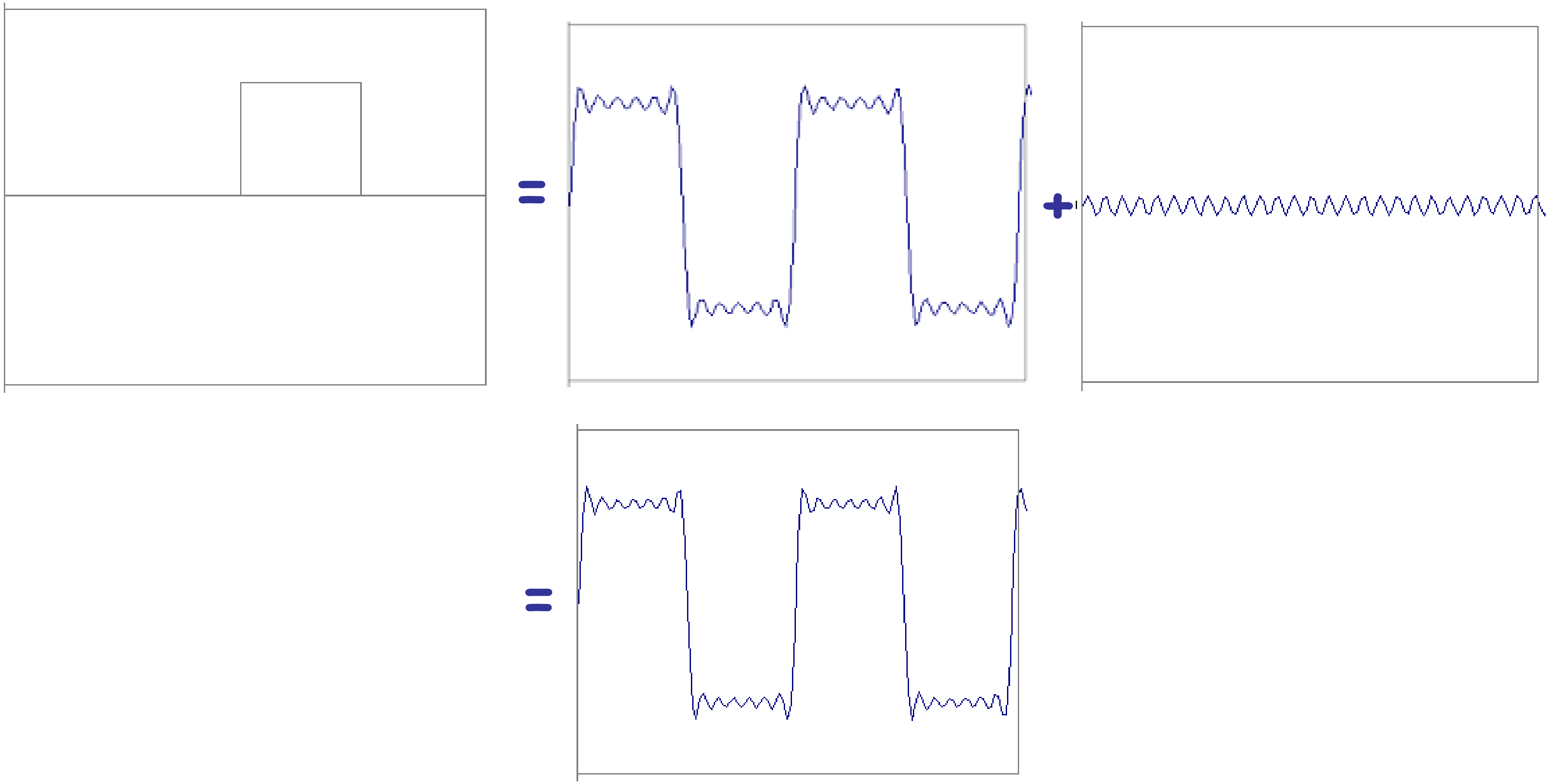
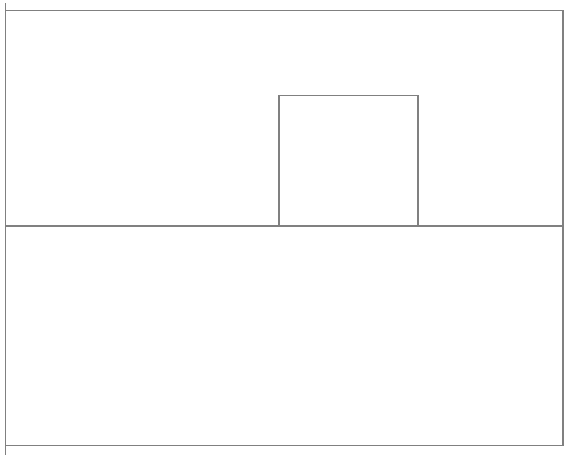


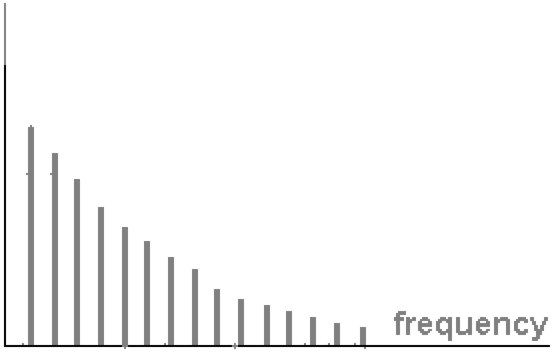
Frequency Spectra



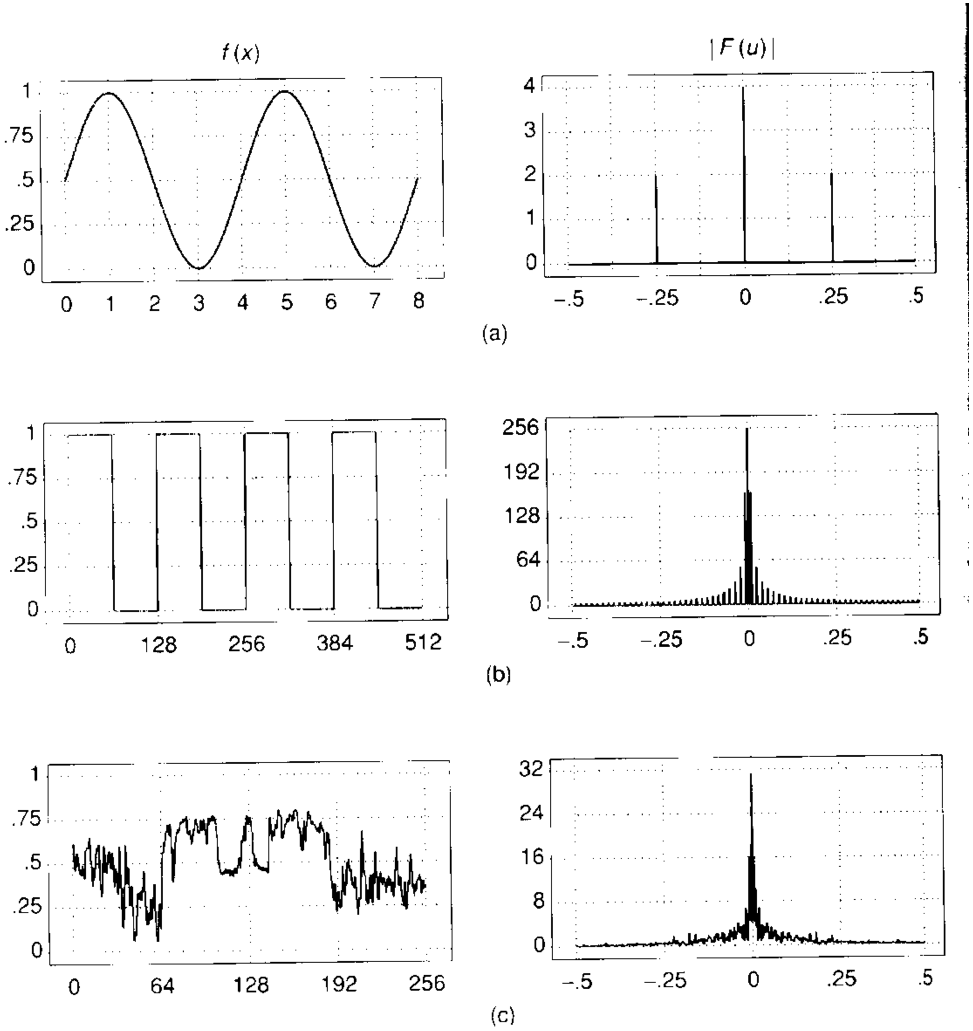
Frequency Spectra



$$= \quad \boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$



Frequency Spectra



Frequency Analysis

If a function f(x) can be expressed as a linear sum of scaled and shifted sinusoids:

$$f(x)=\sum_{\omega} F(\omega)e^{i2\pi\omega x}$$

it is possible to predict the system response to f(x):

$$g(x)=T\{f(x)\}=\sum_{\omega} H(\omega)F(\omega)e^{i2\pi\omega x}$$

The Fourier Transform:

It is possible to express any signal as a sum of shifted and scaled sinusoids at different frequencies.

$$\boxed{\begin{array}{l} f(x)=\sum_{\omega} F(\omega)e^{i2\pi\omega x} \\ \text{Or} \\ f(x)=\int_{\omega} F(\omega)e^{i2\pi\omega x}\,d\omega \end{array}}$$

The Fourier Transform

The inverse Fourier Transform composes a signal f(x) given F(ω):

$$\boxed{f(x) = \int_{\omega} F(\omega) e^{i 2\pi\omega x}\, d\omega}$$

The Fourier Transform finds the F(ω) given the signal f(x):

$$\boxed{F(\omega) = \int_{x} f(x) e^{-i 2\pi\omega x}\, dx}$$

F(ω) is the Fourier transform of f(x):

$$\tilde{F}\{f(x)\} = F(\omega)$$

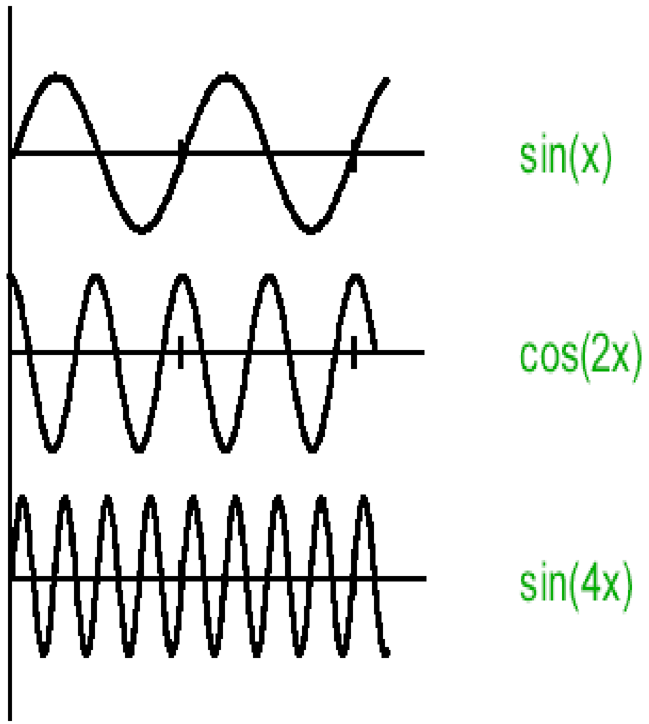
f(x) is the inverse Fourier transform of F(ω):

$$\tilde{F}^{-1}\{F(\omega)\} = f(x)$$

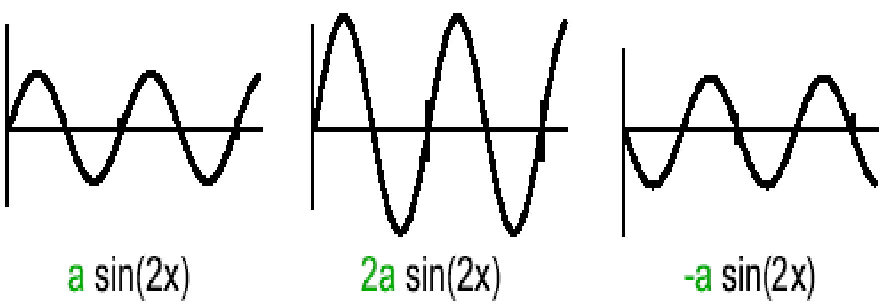
f(x) and F(ω) is a Fourier transform pair.

The Fourier Transform (cont.)

Basis Functions are sines and cosines



The transform coefficients determine the amplitude:



The Fourier Transform (cont.)

The Fourier transform $F(\omega)$ is a function over the complex numbers:

$$F(\omega) = R_\omega e^{i\theta_\omega}$$

- R_ω tells us how much of frequency ω is needed.
- θ_ω tells us the shift of the Sine wave with frequency ω.

Alternatively:

$$F(\omega) = a_\omega + ib_\omega$$

- a_ω tells us how much of cos with frequency ω is needed.
- b_ω tells us how much of sin with frequency ω is needed.

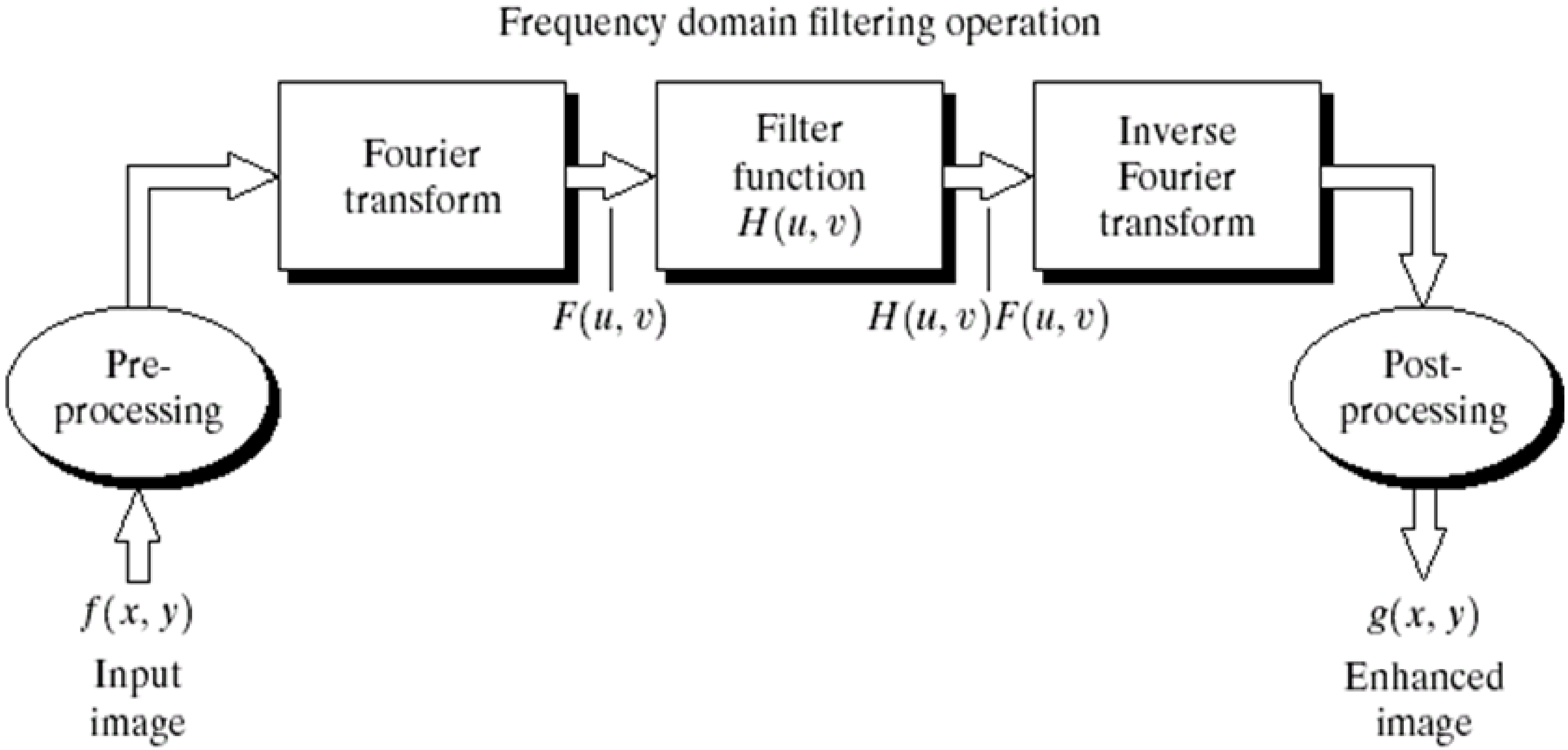
The Fourier Transform (cont.)

$$F(\omega) = R_{\omega} e^{i\theta_{\omega}}$$

$$F(\omega) = a_{\omega} + ib_{\omega}$$

- R_{ω} - is the amplitude of $F(\omega)$.
- θ_{ω} - is the phase of $F(\omega)$.
- $|R_{\omega}|^2 = F^*(\omega)\, F(\omega)$ - is the power spectrum of $F(\omega)$.
- If a signal f(x) has a lot of fine details $F(\omega)$ will be high for high ω.
- If the signal f(x) is "smooth" $F(\omega)$ will be low for high ω.

Frequency Domain Filtering



2D Fourier Transform

Given a continuous real function $f(x,y)$,
it's fourier transform $F(u,v)$ is defined as:

$$\boxed{\tilde{F}\{f(x,y)\} = F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-2\pi i(ux+vy)}dxdy}$$

The Inverse Fourier Transform:

$$\boxed{\tilde{F}^{-1}\{F(u,v)\} = f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{2\pi i(ux+vy)}dudv}$$

$$F(u,v) = a(u,v) + ib(u,v) = |F(u,v)|e^{i\phi(u,v)}$$

Phase = $\phi(u,v) = tg^{-1}(b(u,v)/a(u,v))$

Spectrum (Amplitude) = $|F(u,v)| = \sqrt{a^2(u,v) + b^2(u,v)}$

Power Spectrum = $|F(u,v)|^2 = a^2(u,v) + b^2(u,v)$

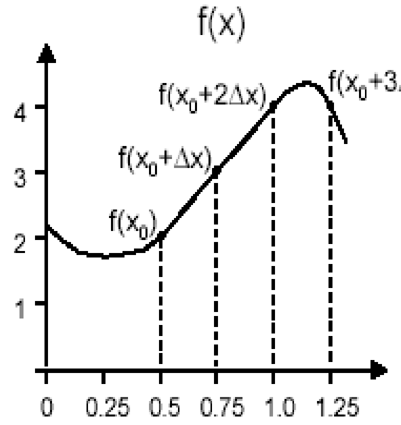
Discrete Fourier Transform

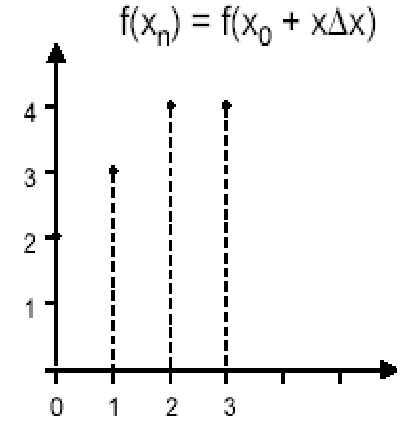
Move from $f(x)$ $(x \in R)$ to $f(x)$ $(x \in Z)$
by sampling at equal intervals.

$f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots, f(x_0+[n-1]\Delta x),$

Given N samples at equal intervals, we redefine f as:

$f(x) = f(x_0+x\Delta x) \quad x = 0, 1, 2, \ldots, N-1$





Discrete Fourier Transform (cont.)

The Discrete Fourier Transform (DFT) is defined as:

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{\frac{-2\pi iux}{N}} \qquad u = 0, 1, 2, \ldots, N\text{-}1$$

The Inverse Discrete Fourier Transform (IDFT) is defined as:

$$f(x) = \sum_{u=0}^{N-1} F(u) e^{\frac{2\pi iux}{N}} \qquad x = 0, 1, 2, \ldots, N\text{-}1$$

Discrete Fourier Transform (cont.)

Discrete Fourier Transform - 2D

Image $f(x,y)$ $x = 0,1,\ldots,N-1$ $y = 0,1,\ldots,M-1$

The Discrete Fourier Transform (DFT) is defined as:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x,y)e^{-2\pi i(\frac{ux}{N}+\frac{vy}{M})} \quad u = 0, 1, 2, \ldots, N-1 \quad v = 0, 1, 2, \ldots, M-1$$

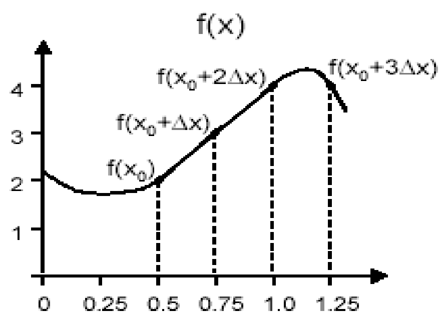
The Inverse Discrete Fourier Transform (IDFT) is defined as:

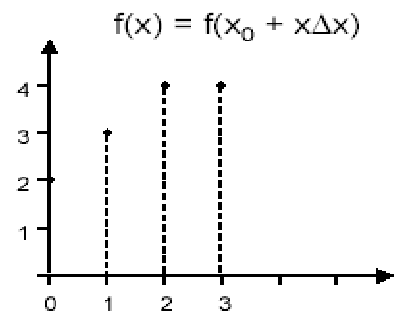
$$f(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{M-1} F(u,v)e^{2\pi i(\frac{ux}{N}+\frac{vy}{M})} \quad x = 0, 1, 2, \ldots, N-1 \quad y = 0, 1, 2, \ldots, M-1$$

Discrete Fourier Transform (cont.)

The Discrete Fourier Transform (DFT) is defined as:

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x) e^{\frac{-2\pi i u x}{N}} \qquad u = 0, 1, 2, \dots, N-1$$





$$F(0) = 1/4 \sum_{x=0}^{3} f(x)\, e^{\frac{-2\pi j 0 x}{4}} = 1/4 \sum_{x=0}^{3} f(x)\, 1$$

$$= 1/4(f(0) + f(1) + f(2) + f(3)) = 1/4(2+3+4+4) = 3.25$$

$$F(1) = 1/4 \sum_{x=0}^{3} f(x)\, e^{\frac{-2\pi i x}{4}} = 1/4\, [2e^{0}+3e^{-i\pi/2}+4e^{-\pi i} +4e^{-i3\pi/2}] = \frac{1}{4}\, [-2+i]$$

$$F(2) = 1/4 \sum_{x=0}^{3} f(x)\, e^{\frac{-4\pi i x}{4}} = 1/4\, [2e^{0}+3e^{-i\pi}+4e^{-2\pi i}\ 4e^{-3\pi i}] = \frac{-1}{4}\, [-1\text{-}0i] = \frac{-1}{4}$$

$$F(3) = 1/4 \sum_{x=0}^{3} f(x)\, e^{\frac{-6\pi i x}{4}} = 1/4\, [2e^{0}+3e^{-i3\pi/2}+4e^{-3\pi i}\ 4e^{-i9\pi/2}] = \frac{1}{4}\, [-2-i]$$

Fourier Spectrum:

$$|F(0)| = 3.25$$

$$|F(1)| = [(-1/2)^2 + (1/4)^2]^{0.5} = \sqrt{5}\,/4$$

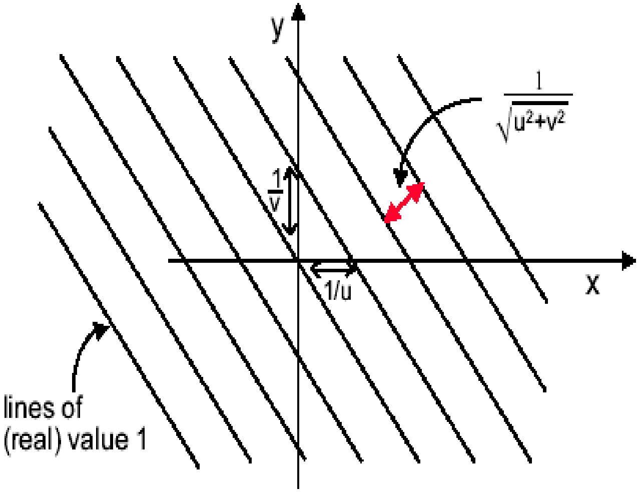
$$|F(2)| = [(-1/4)^2 + (0)^2]^{0.5} = 1/4$$

$$|F(3)| = [(-1/2)^2 + (-1/4)^2]^{0.5} = \sqrt{5}\,/4$$

Discrete Fourier Transform (cont.)

Fourier Wave Functions - 2D

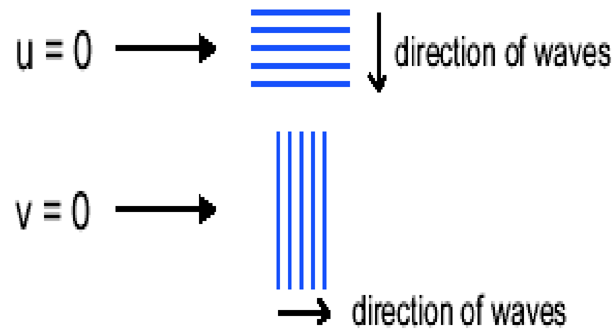
$F(u,v)$ is the coefficient of the sine wave $e^{2\pi i(ux+vy)}$



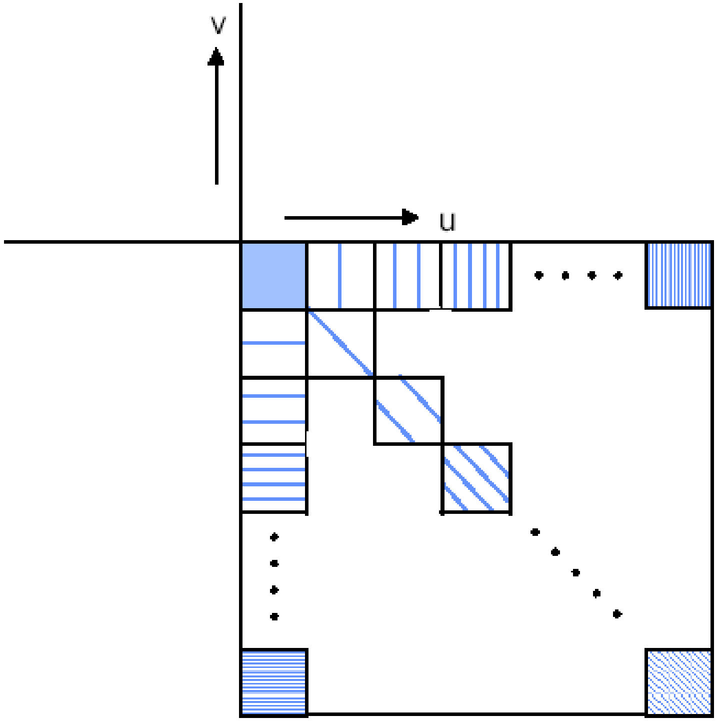
$$e^{2\pi i(ux+vy)} = \cos(2\pi(ux+vy)) + i\sin(2\pi(ux+vy))$$

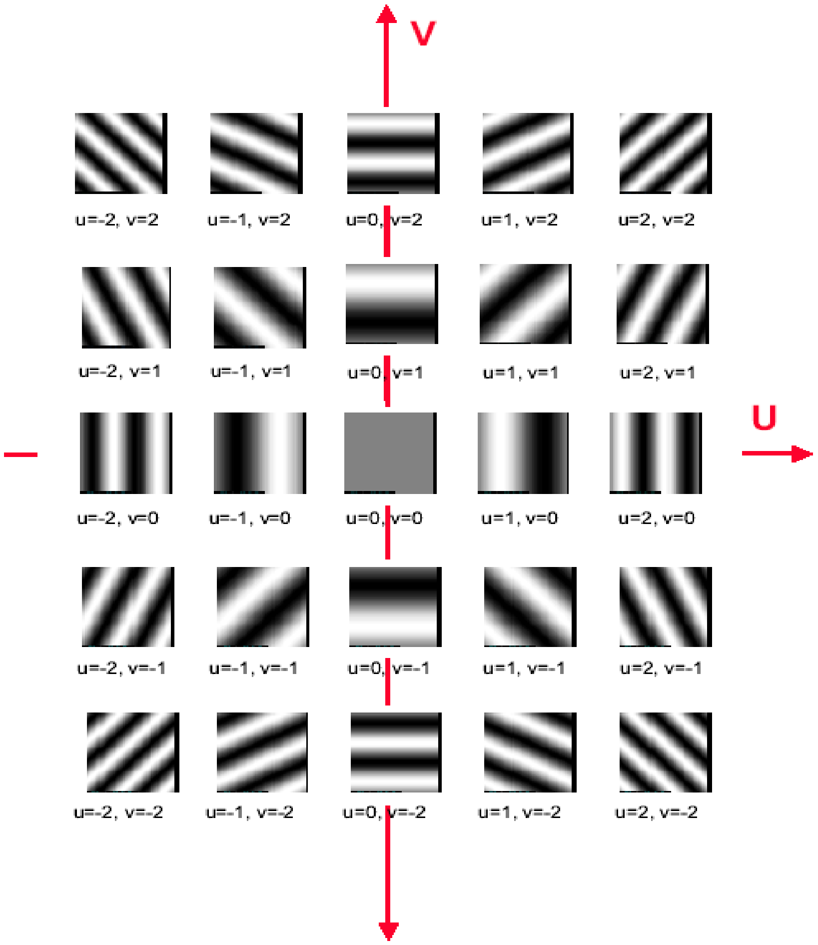
The ratio $\frac{u}{v}$ determines the Direction.

The size of u,v determines the Frequency.

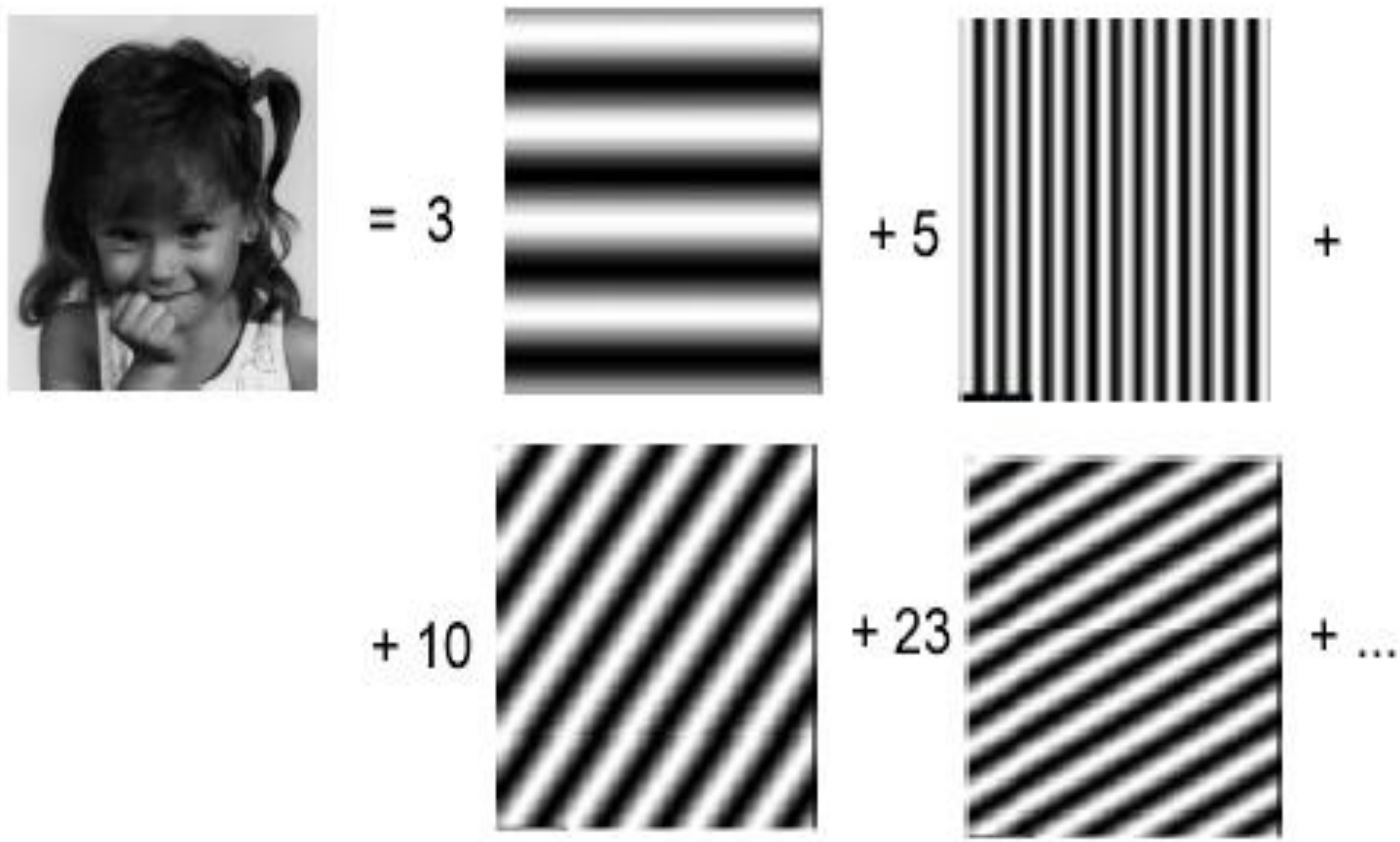


Discrete Fourier Transform (cont.)





Discrete Fourier Transform (cont.)



Discrete Fourier Transform (cont.)

Properties of The Fourier Transform

Distributive (addition)

$$\tilde{F}[f_1(x,y) + f_2(x,y)] = \tilde{F}[f_1(x,y)] + \tilde{F}[f_2(x,y)]$$

Linearity

$$\tilde{F}[a\, f(x,y)] = a\, \tilde{F}[f(x,y)]$$

$$a\, f(x,y) \longleftrightarrow a\, F(u,v)$$

Cyclic

$$F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N)$$

$$F(x,y) = F(x+N,y+N)$$

Symmetric if f(x) is real:

$$F(u,v) = F^*(-u,-v)$$

thus:

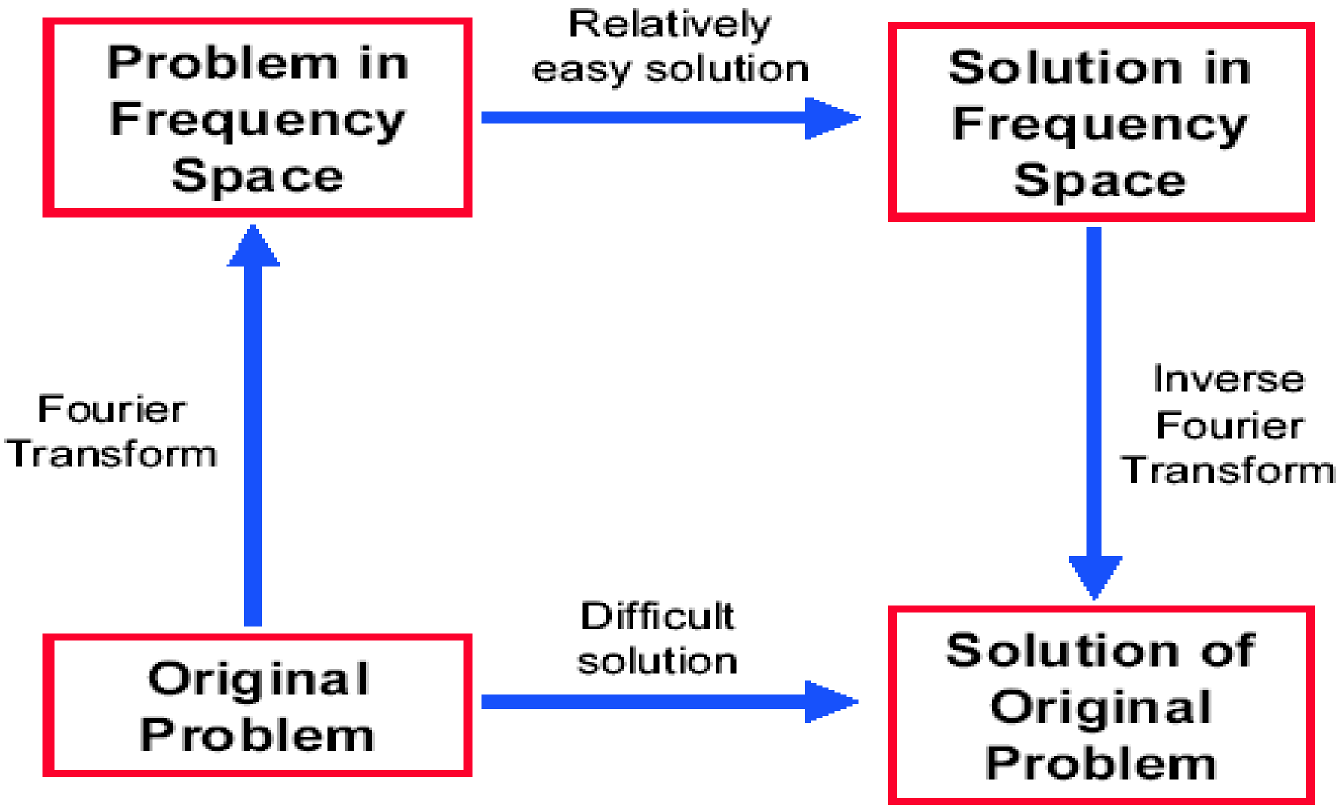
$$|F(u,v)| = |F(-u,-v)|$$

Fourier Spectrum is symmetric

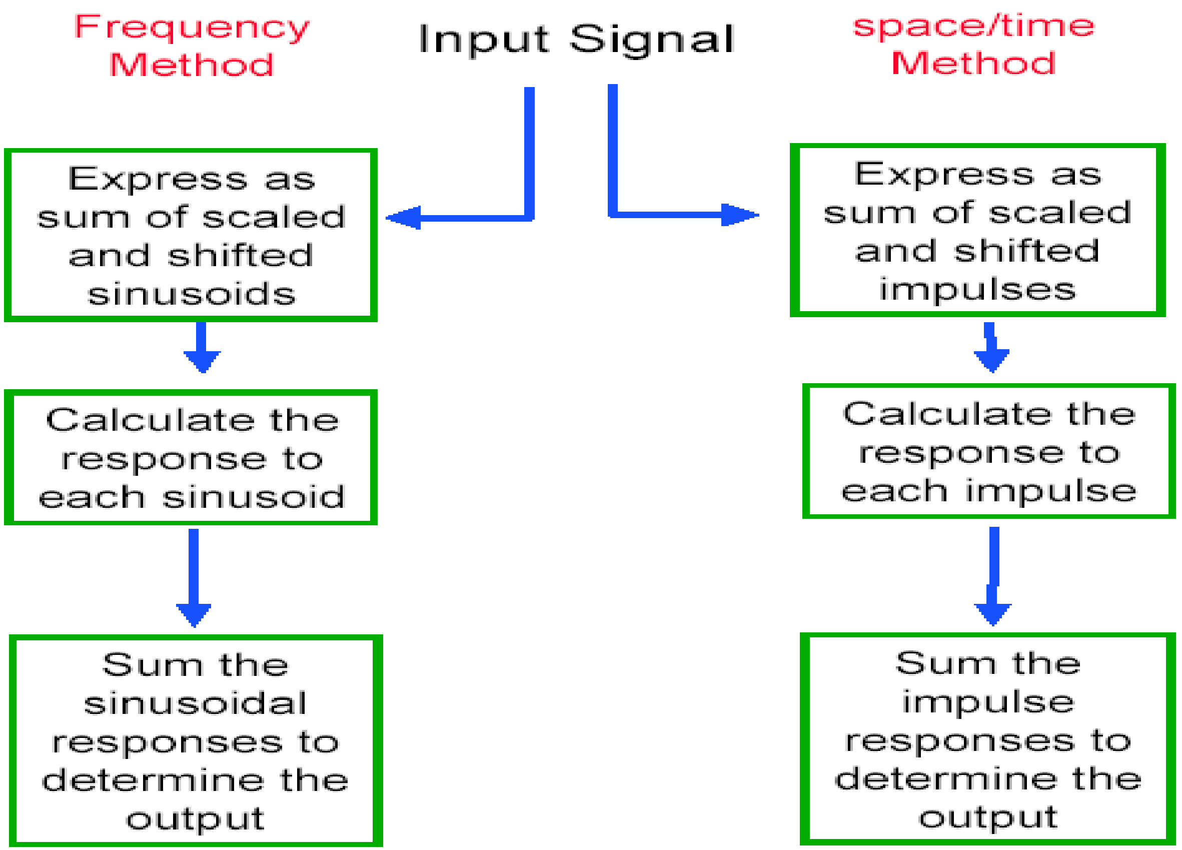
DC (Average)

$$F(0,0) = \frac{1}{N}\frac{1}{M}\sum_{x=0}^{N-1}\sum_{y=0}^{M-1} f(x)\, e^0$$

Spatial vs Frequency Domain



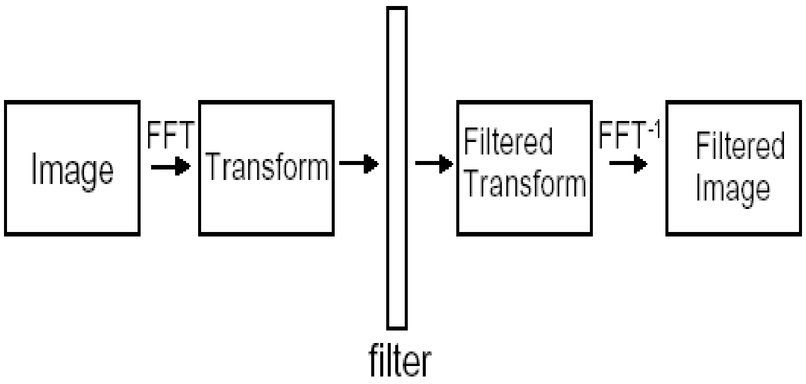
Spatial vs Frequency Domain (cont.)

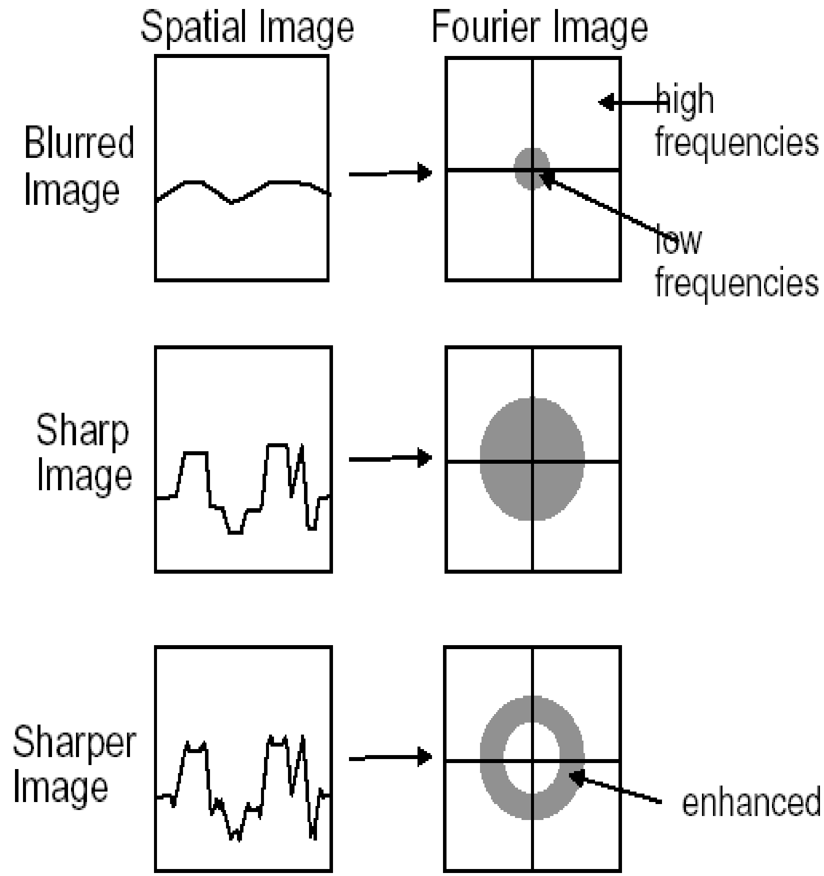


$$G(\omega) = F(\omega)H(\omega)$$

$$g(x) = f(x) * h(x)$$

Spatial vs Frequency Domain (cont.)





Low Pass Filter

spatial domain → frequency domain

$f(x,y) \longrightarrow F(u,v)$

$G(u,v) = F(u,v) \cdot H(u,v)$ ← filter

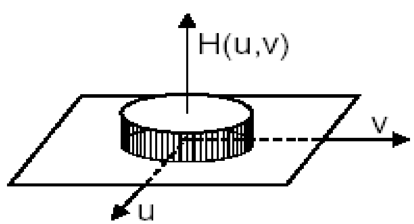
$g(x,y) \longleftarrow G(u,v)$

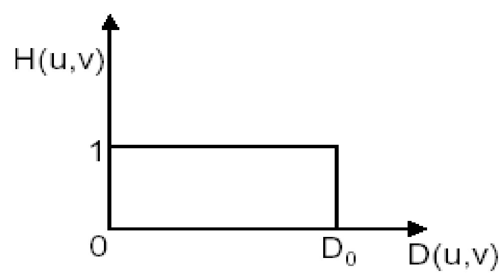
H(u,v) - Ideal Filter

$$H(u,v) = \begin{cases} 1 & D(u,v) \leq D_0 \\ 0 & D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \sqrt{u^2 + v^2}$$

D_0 = cut off frequency





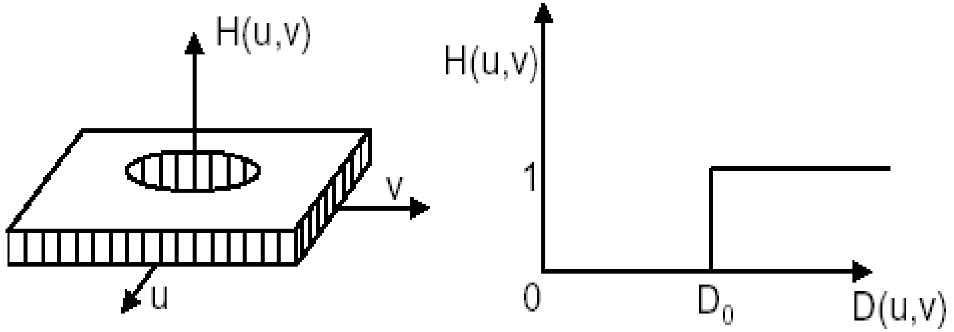
Sharpening (High Pass) Filter

H(u,v) - Ideal Filter

$$H(u,v) = \begin{cases} 0 & D(u,v) \le D_0 \\ 1 & D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \sqrt{u^2 + v^2}$$

D_0 = cut off frequency



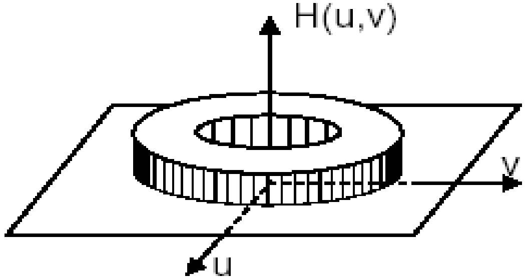
Band Pass Filter

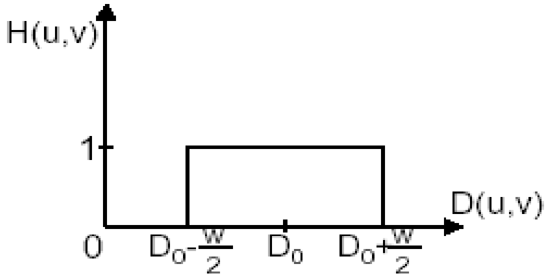
$$H(u,v) = \begin{cases} 0 & D(u,v) \leq D_0 - \frac{w}{2} \\ 1 & D_0 - \frac{w}{2} \leq D(u,v) \leq D_0 + \frac{w}{2} \\ 0 & D(u,v) > D_0 + \frac{w}{2} \end{cases}$$

$$D(u,v) = \sqrt{u^2 + v^2}$$

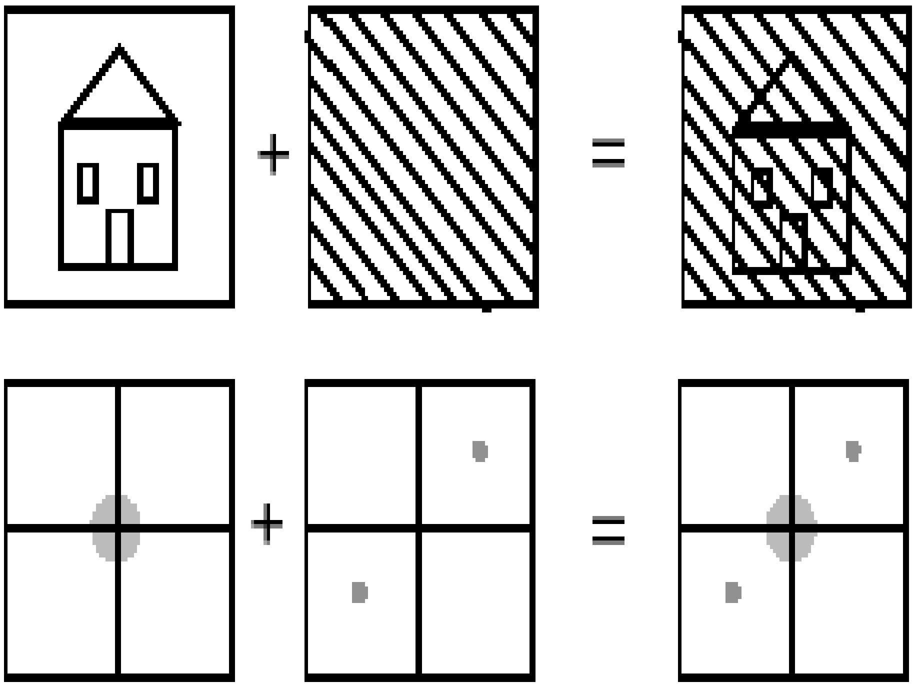
D_0 = cut off frequency

w = band width

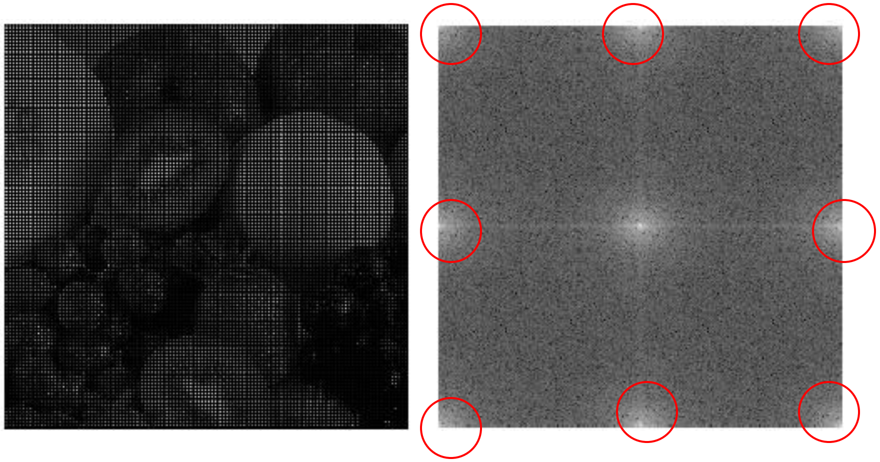


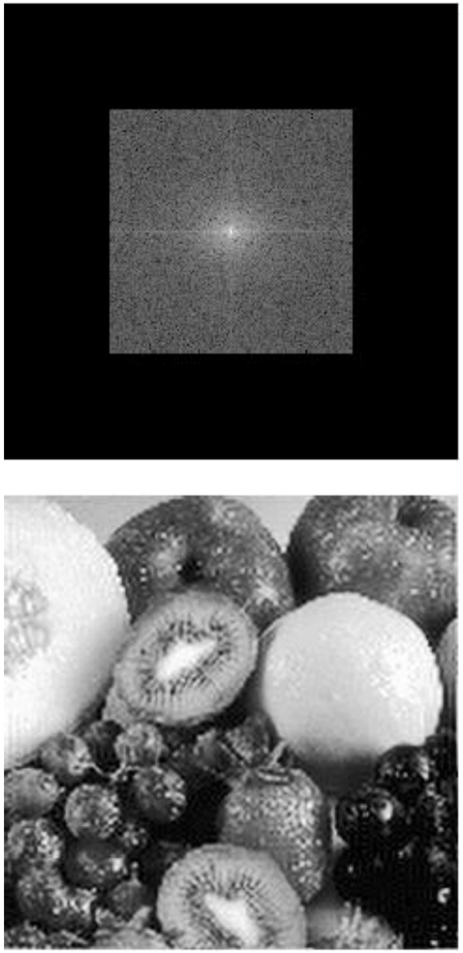


Filtering Example (cont.)



Filtering Example

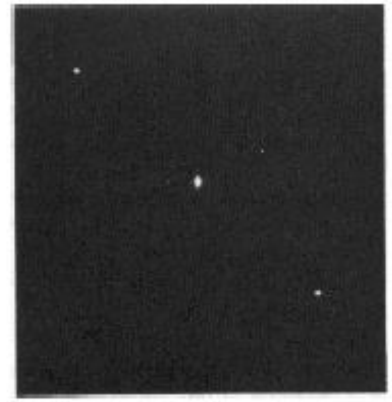


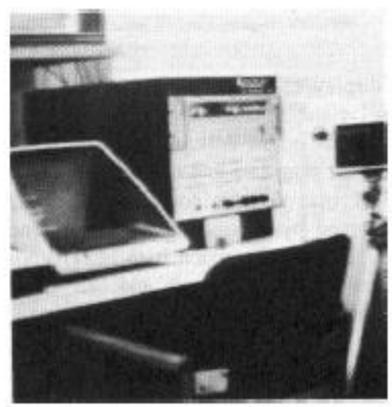


Filtering Example (cont.)

Original Noisy image

Fourier Spectrum



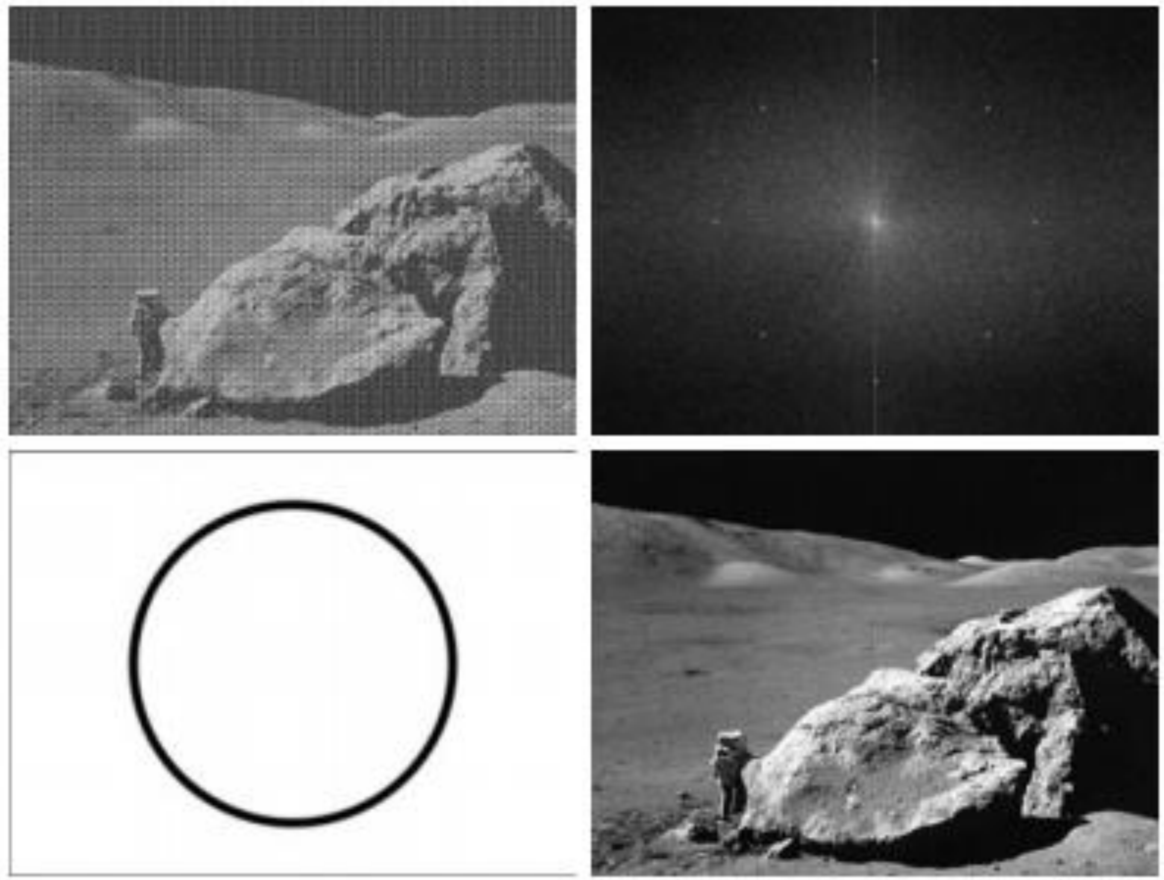


Band Reject Filter

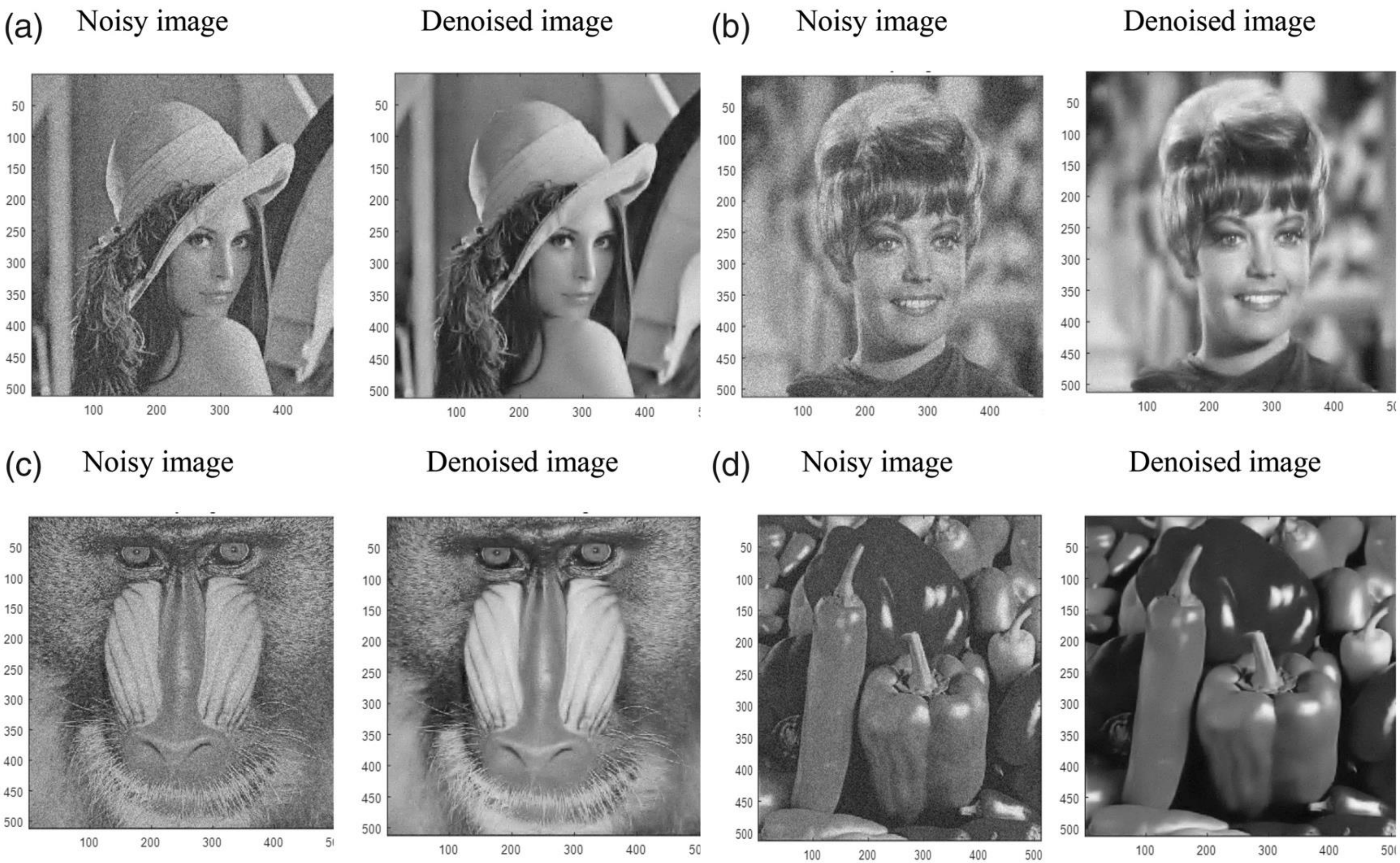
Filtering Example (cont.)

a b
c d

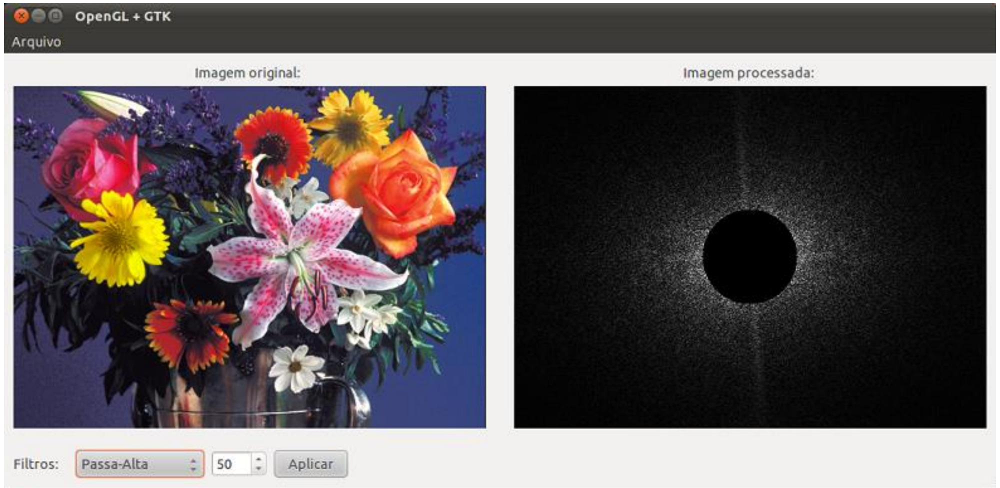
FIGURE 2.40
(a) Image corrupted by sinusoidal interference. (b) Magnitude of the Fourier transform showing the bursts of energy responsible for the interference. (c) Mask used to eliminate the energy bursts. (d) Result of computing the inverse of the modified Fourier transform. (Original image courtesy of NASA.)

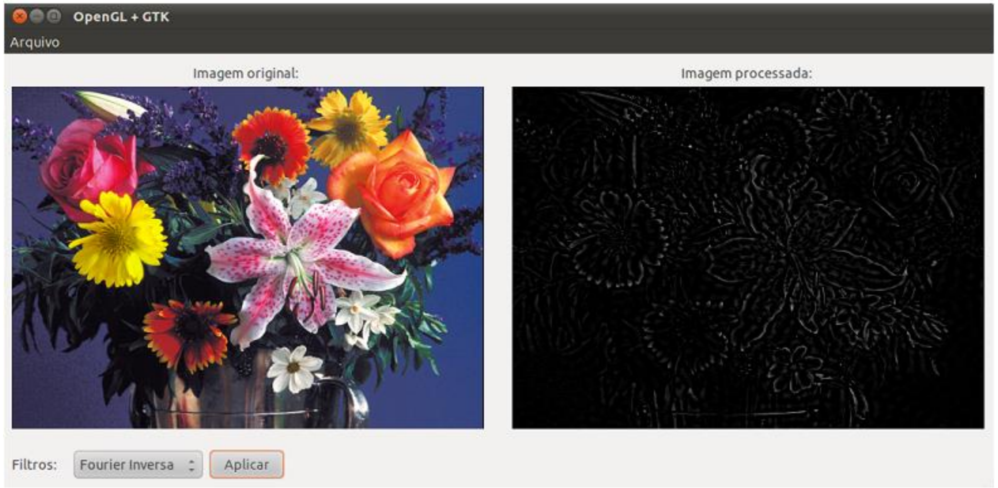


Filtering Example (cont.)



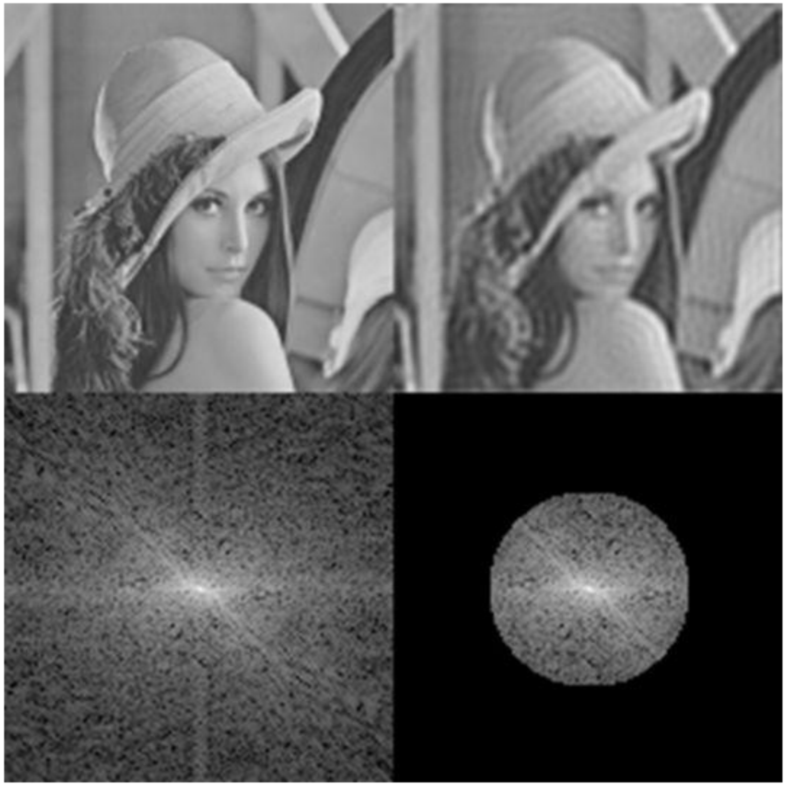
Filtering Example (cont.)





Filtering Example (cont.)

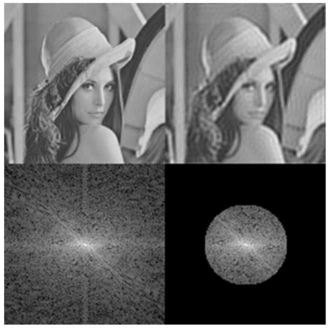
Trasformata di Fourier



Compressione di immagini.

Filtering Example (cont.)

Trasformata di Fourier



Compressione di immagini.

Edge Detection

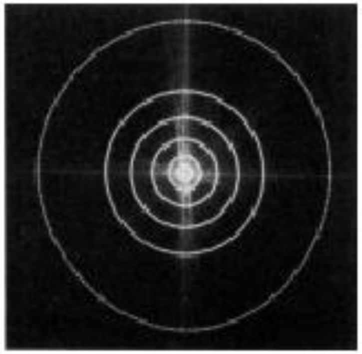
Frequency Bands

- Percentage of image power enclosed in circles (small to large):

90, 95, 98, 99, 99.5, 99.9

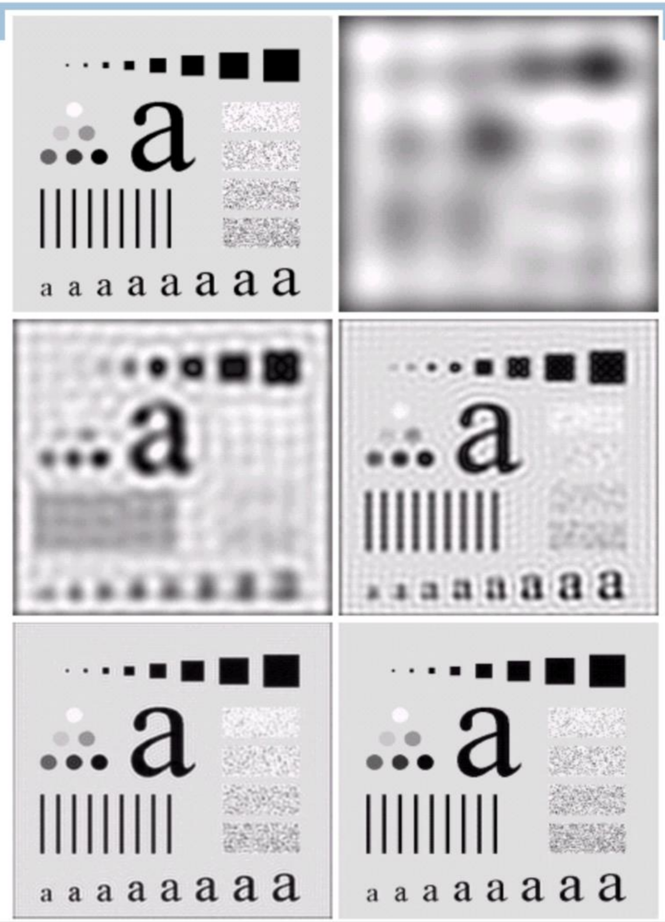
Image

Fourier Spectrum



Frequency Bands

Original image



Result of filtering with ideal low pass filter of radius 5

Result of filtering with ideal low pass filter of radius 15

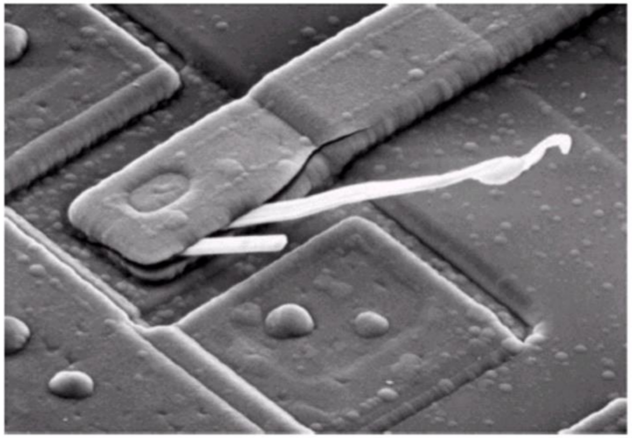
Result of filtering with ideal low pass filter of radius 30

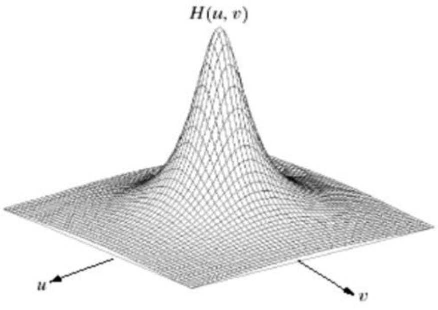
Result of filtering with ideal low pass filter of radius 80

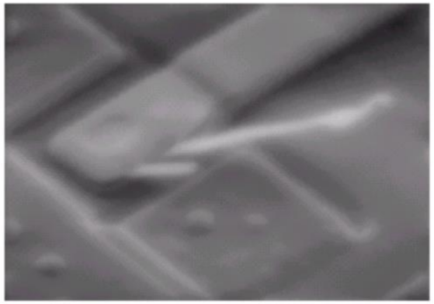
Result of filtering with ideal low pass filter of radius 230

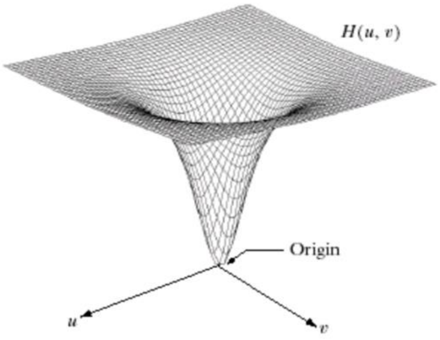
Frequency Bands

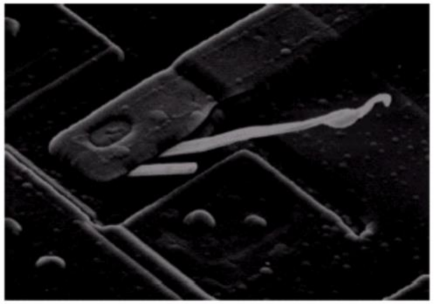
Low Pass Filter











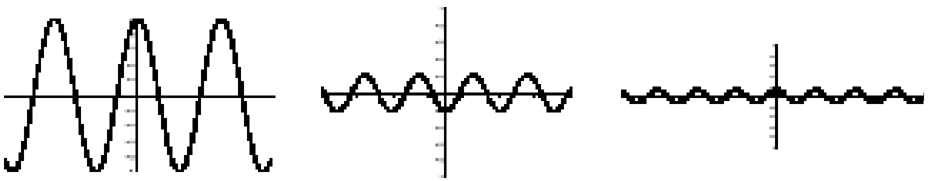
High Pass Filter

Fourier Transform

- Computing Time
  - $O(n^2)$
- Fast Fourier Transform (FFT)
  - $O(n\,log\,n)$

Fourier Transform

What is FT ?



FT decomposes a function into a weighted sum of sinusoidal functions
=> We can reconstruct the original function:

$$p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Fp(\omega) e^{i\omega t} d\omega$$

FT maps a function to its frequencies

Fourier Transform of p

$i^2 = -1$

$$Fp(\omega) = \int_{-\infty}^{\infty} p(t) e^{-i\omega t} dt$$

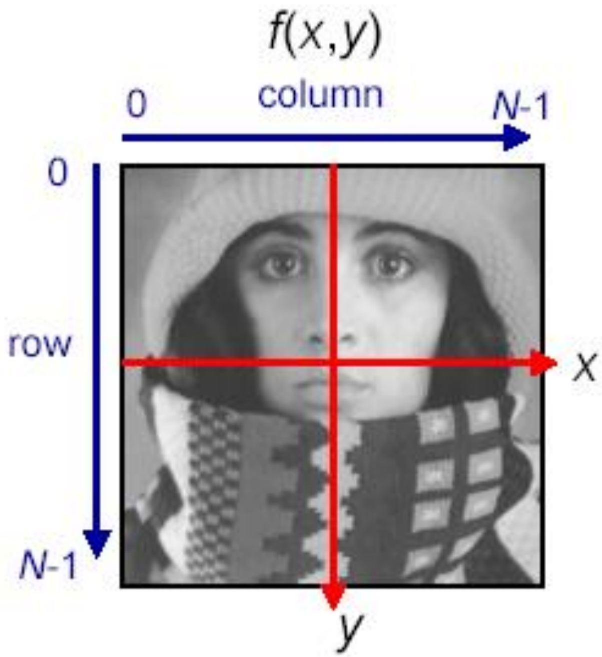
$e^{-i\omega t} = \cos(\omega t) - i \sin(\omega t)$

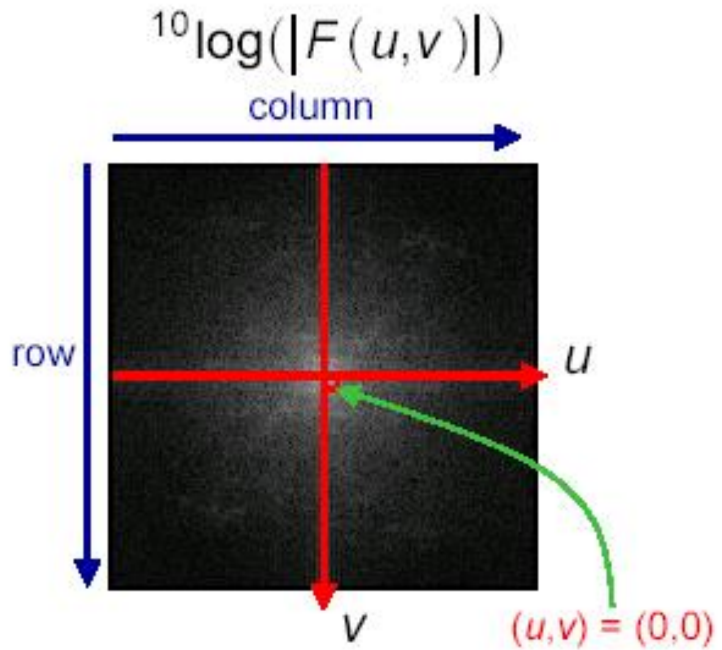
Angular frequency

Continuous function

Fourier Transform (cont.)

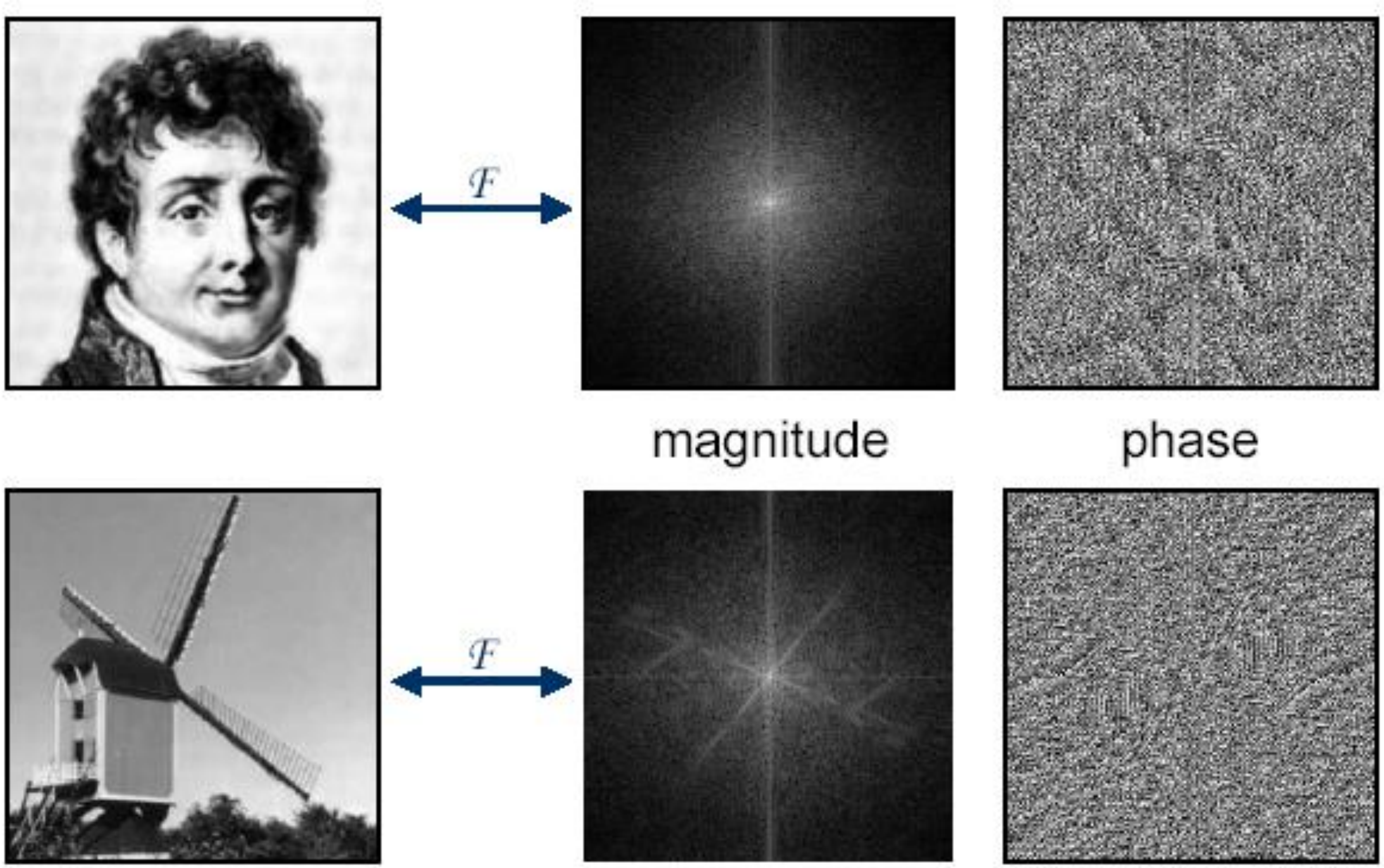
- $F(u,v)$ is the complex amplitude of the eigenfunction $\exp(j\,(2\pi/N)(ux+vy))$
  Note that $\mathbf{exp}(j\,(2\pi/N)(ux+vy)) = \mathbf{cos}((2\pi/N)(ux+vy)) + j\,\mathbf{sin}((2\pi/N)(ux+vy))$
- Standard display is the logarithm of the magnitude: $\log(|F(u,v|)$





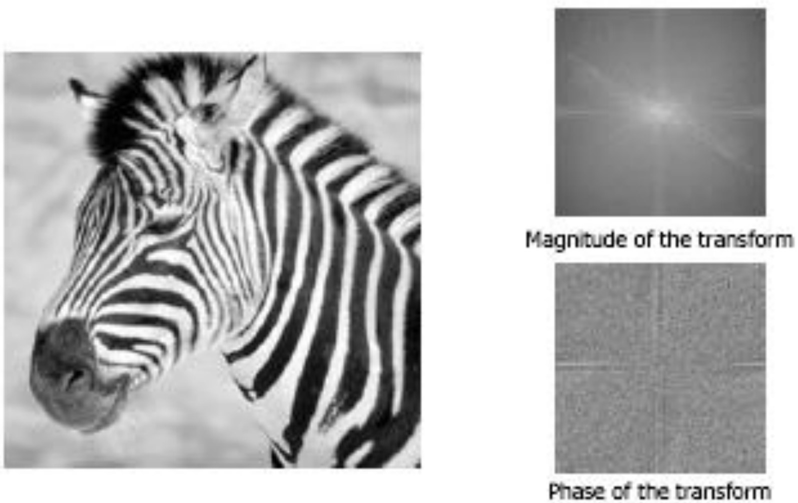
Fourier spectrum

Fourier Transform (cont.)

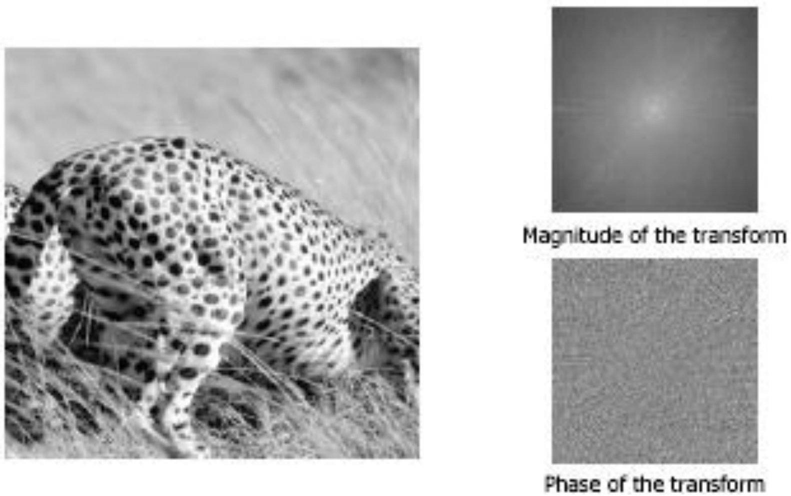


Fourier Transform (cont.)

PHASE AND MAGNITUDE

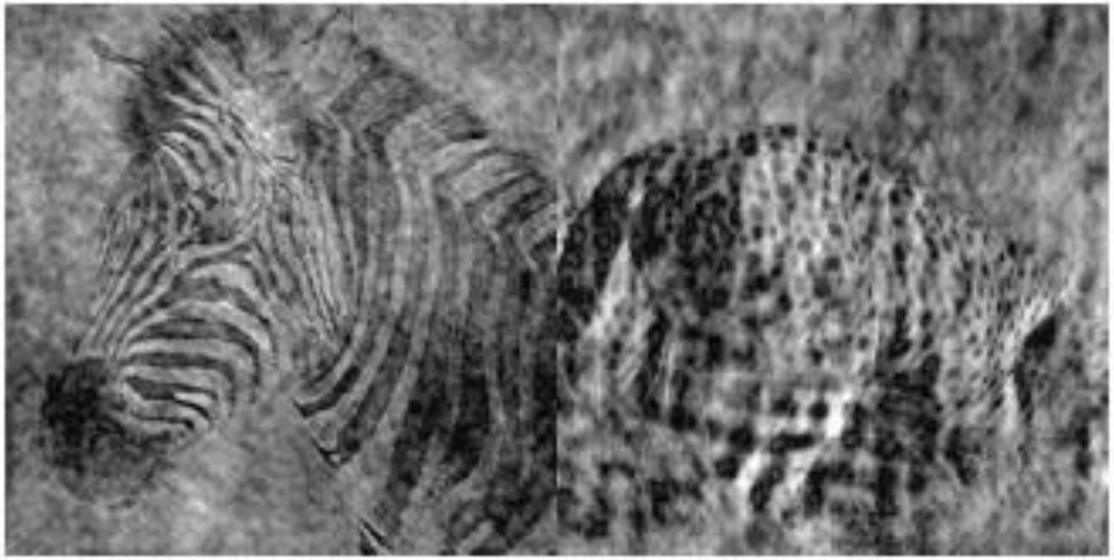


PHASE AND MAGNITUDE



Fourier Transform (cont.)

SWITCHING
PHASE AND MAGNITUDE



- Zebra phase
- Cheetah magnitude

- Cheetah phase
- Zebra magnitude

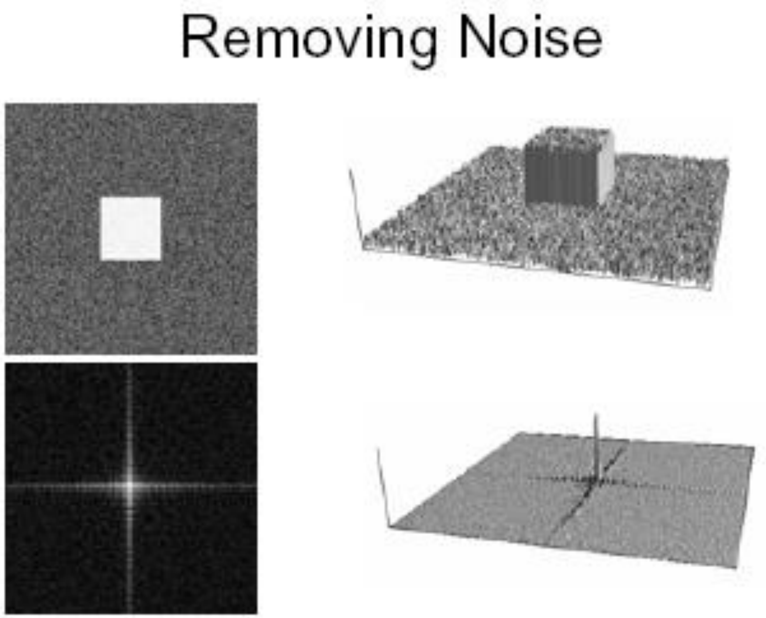
Fourier Transform (cont.)

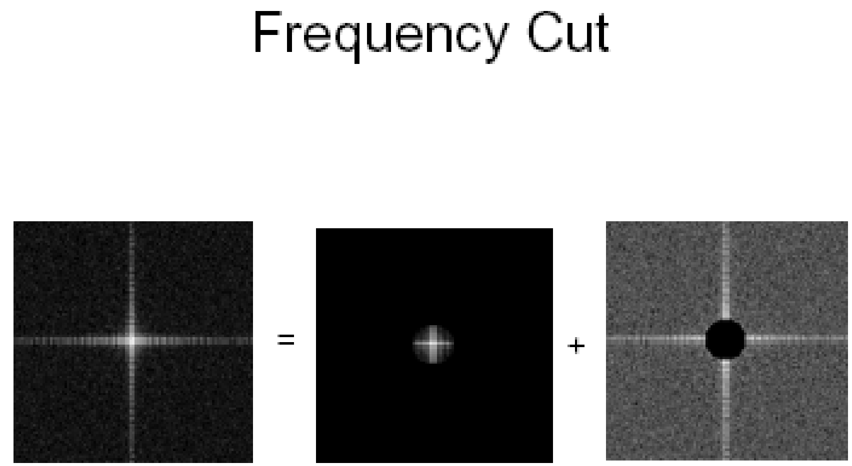
FT is Shift Invariant

After shifting:

- Magnitude stay constant
- Phase changes

Fourier Transform (cont.)





Linear Systems and Responses

	Spatial Domain	Frequency Domain
Input	f	F
Output	g	G
Impulse Response	h	
Freq. Response		H
Relationship	g=f*h	G=FH

The Convolution Theorem

$g = f * h$	$g = f\,h$
implies	implies
$G = F\,H$	$G = F * H$

Convolution in one domain is multiplication in the other and vice versa

The Convolution Theorem (cont.)

$$\boxed{\tilde{F}\{f(x) * g(x)\} = \tilde{F}\{f(x)\}\tilde{F}\{g(x)\}}$$

and likewise

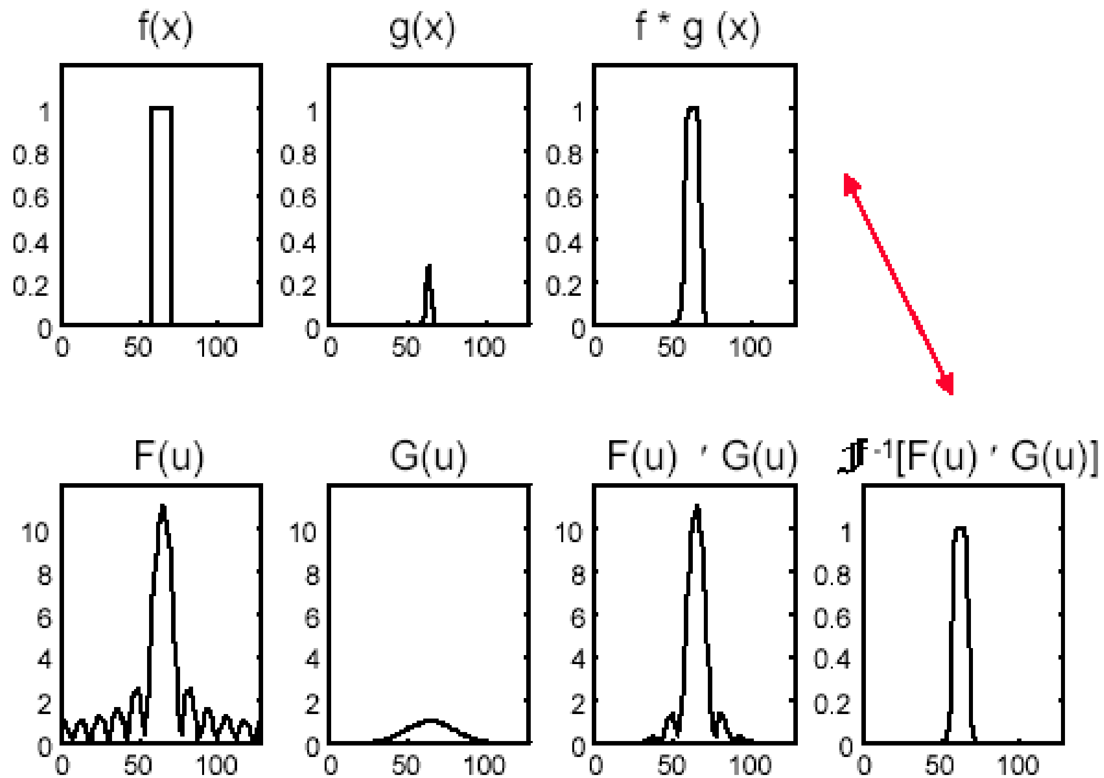
$$\boxed{\tilde{F}\{f(x)g(x)\} = \tilde{F}\{f(x)\} * \tilde{F}\{g(x)\}}$$

f(x,y) * g(x,y) ⟷ F(u,v) ' G(u,v)

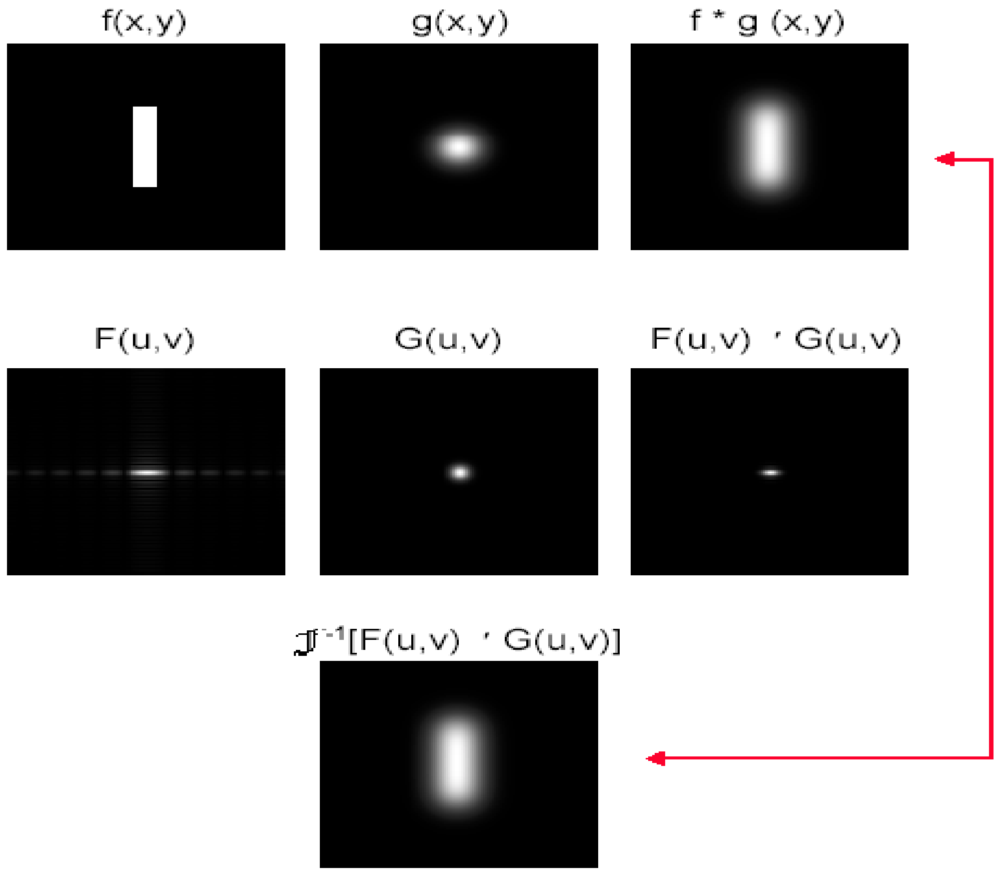
f(x,y) ' g(x,y) ⟷ F(u,v) * G(u,v)

Convolution in one domain is multiplication in the other and vice versa

The Convolution Theorem (cont.)

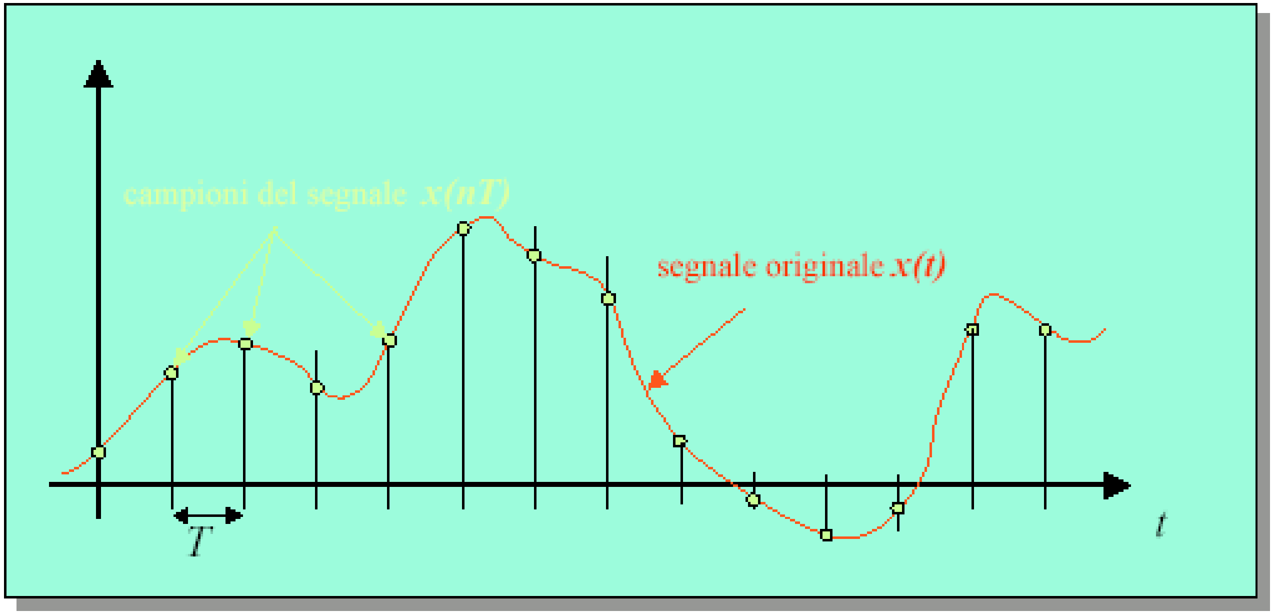


The Convolution Theorem (cont.)



Campionamento

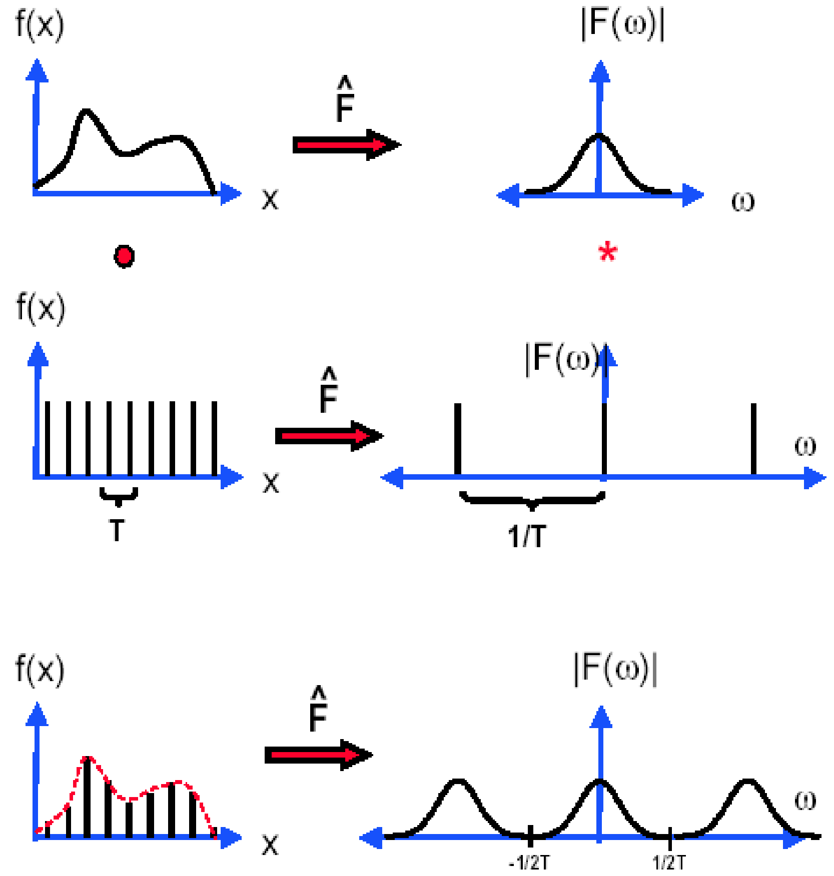
Campionare i segnali (discretizzare nel tempo)



- T e' detto **periodo (o passo) di campionamento**
- $f_c = 1/T$ e' detta **frequenza di campionamento**

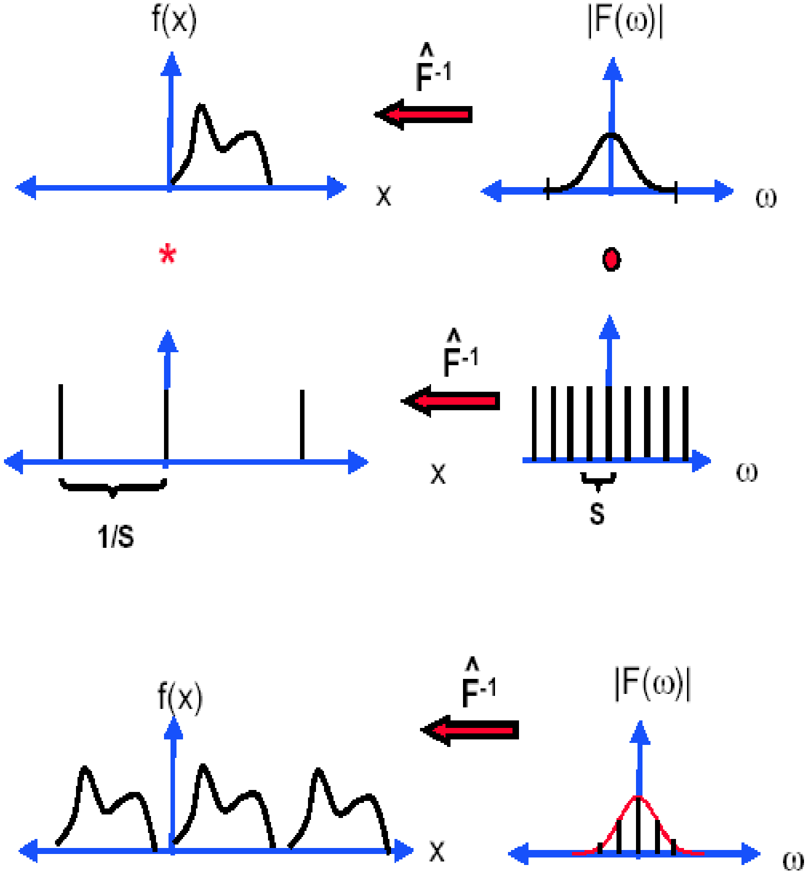
Sampling Image

- Sampling a function f(x) with impulse train of cycle T produces replicas in the frequency domain with cycle 1/T

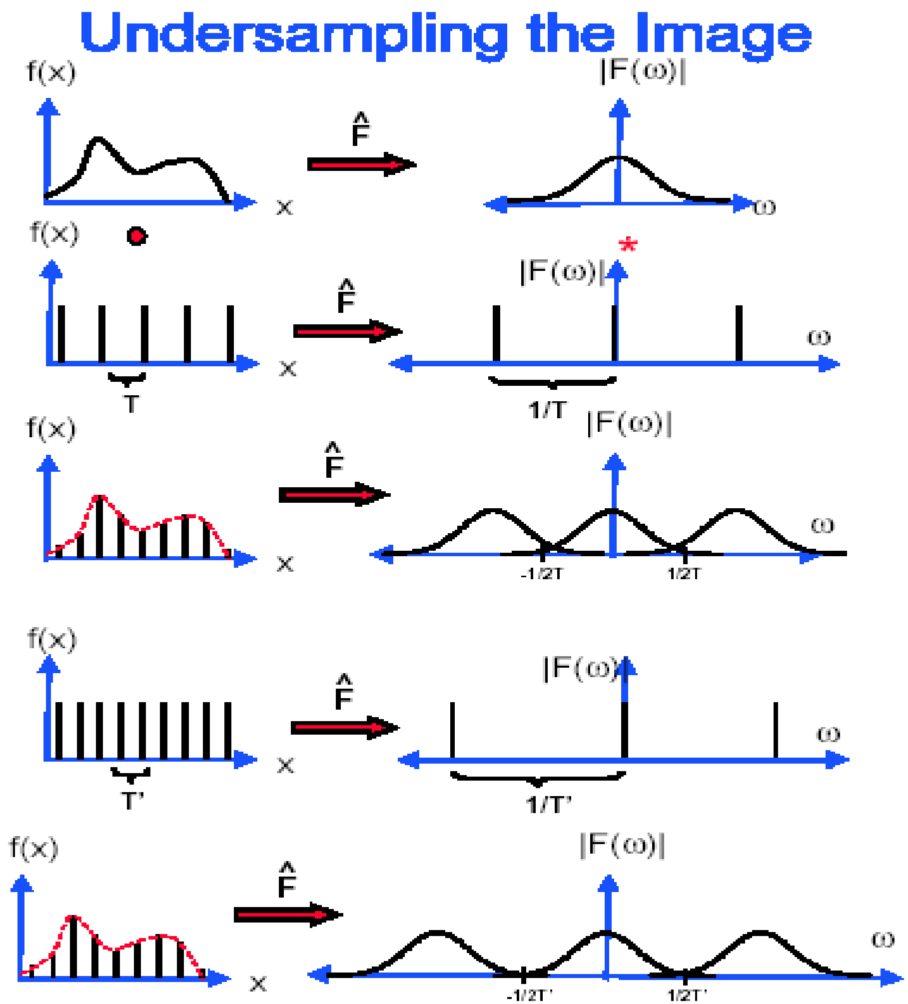


Sampling Image (cont.)

- Sampling a function F(ω) with impulse train of cycle S produces replicas in the image domain with cycle 1/S



Sampling Image (cont.)



Nyquist Theorem (Sampling Theorem)

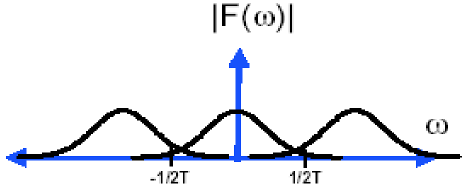
- If the maximal frequency of f(x) is ω_{max} , it is clear from the above replicas that ω_{max} should be smaller that 1/2T.

- Alternatively:

$$\frac{1}{T} > 2\omega_{\max}$$

- **<u>Nyquist Theorem</u>**: If the maximal frequency of f(x) is ω_{max} the sampling rate should be larger than $2\omega_{max}$ in order to fully reconstruct f(x) from its samples.

- If the sampling rate is smaller than $2\omega_{max}$ overlapping replicas produce **aliasing**.



Nyquist Theorem
(Sampling Theorem)

Dalle proprieta' della trasformata di Fourier (moltiplicazione per esponenziali complessi, oppure convoluzione delle trasformate) e' immediato verificare che il segnale campionato ha come trasformata di Fourier la ripetizione periodica della trasformata *X(f)* del segnale continuo *x(t)*, con periodo pari alla frequenza di campionamento $f_c=1/T$, moltiplicata per $f_c=1/T$

$$X_c(f) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(f - k/T)$$

Se le infinite repliche traslate in frequenza di *X(f)* non si sovrappongono e' facile estrarre mediante filtraggio *X (f)* da $X_c(f)$, cioe' riottenere *x(t)* da $x_c(t)$. Ovviamente cio' richiede una conoscenza a priori della banda *B* occupata da *x(t)*. Tale valore e' disponibile, o facilmente misurabile, nella grande maggioranza dei casi di interesse pratico.

Nyquist Theorem
(Sampling Theorem)

Teorema del campionamento

Se e' noto a priori che il segnale tempo continuo $x(t)$ non contiene frequenze maggiori di $f_c / 2$ e inferiori a $-f_c / 2$, esiste un legame univoco tra il segnale continuo nel tempo e i suoi campioni $x(nT)$.

Se un segnale $x(t)$ e' campionato con frequenza di campionamento f_c almeno doppia della massima frequenza contenuta e' perfettamente ricostruibile (*le repliche in frequenza sono disgiunte*). Altrimenti le repliche sono sovrapposte e vi sono frequenze alle quali non e' possibile distinguere tra repliche diverse.

$$x(t) \xrightarrow{TF} X(f)$$

$$x(t)\exp(j2\pi f_c t) \xrightarrow{TF} X(f - kf_c)$$