Lezione 14-12

mercoledì 14 dicembre 2022 09:06

ESERCIZIO 1 $5\mathbb{Z} \subseteq \mathbb{Z}\times\mathbb{Z}$

$a \ 5\mathbb{Z} \ b \iff 5 \mid a-b \iff \exists k \in \mathbb{Z}$ t.c. $a-b=5k$

i) $\forall a \in \mathbb{Z}, \quad 5 \mid a-a$?

$5 \mid 0$ è vero perché $0 = 5 \cdot 0$

ii) $\forall a,b \in \mathbb{Z}$ t.c. $5 \mid (a-b) \Rightarrow 5 \mid (b-a)$

per ipotesi, $5 \mid (a-b) \Rightarrow \exists k$ t.c. $a-b=5k \Rightarrow$

$\Rightarrow \exists k$ t.c. $-(a-b) = -5k$

$\Rightarrow \exists k$ t.c. $b-a = 5(-k)$ tesi!

iii) $\forall a,b,c \in \mathbb{Z}$ t.c. $5 \mid (a-b)$ e $5 \mid (b-c) \overset{?}{\Rightarrow} 5 \mid (a-c)$

per ipotesi, $\exists k$ t.c. $(a-b) = 5k$

$\exists h$ t.c. $(b-c) = 5h$

$$a-c = (a-b)+(b-c) = 5k+5h = 5(k+h) \Rightarrow 5 \mid (a-c)$$

2) $\mathbb{Z}/_{5\mathbb{Z}} = \mathbb{Z}_5 = \{[0]_5, [1]_5, [2]_5, [3]_5, [4]_5\}$

3) $[0]_5 = \{x \in \mathbb{Z} \mid x \ 5\mathbb{Z} \ 0\} = \{x \in \mathbb{Z} \mid 5 \mid (x-0)\} =$

$= \{x \in \mathbb{Z} \mid 5 \mid x\} =$

$= \{$multipli di 5$\}$

ESERCIZIO 2

$$\begin{cases} x \equiv 6 \pmod 9 \\ x \equiv 2 \pmod{11} \end{cases}$$

$MCD(9, 11) = 1$

1) $x = 6 + k9$, $k \in \mathbb{Z}$

2) $6 + 9k \equiv 2 \pmod{11} \longrightarrow 9k \equiv -4 \pmod{11}$

3) $MCD(9,11) = 1$

$$\left.\begin{aligned} 11 &= 9 \cdot 1 + 2 \\ 9 &= 2 \cdot 4 + 1 \\ 2 &= 1 \cdot 2 + 0 \end{aligned}\right\} \quad 1 = 9 + (-4)2 =$$

$= 9 + (-4)(11 - 9) =$

$= 9 + (-4)11 + 4 \cdot 9 =$

$= 5 \cdot 9 + (-4)11$

$k \equiv -20 \pmod{11} \Rightarrow k \equiv 2 \pmod{11}$

3) $x = 6 + 2 \cdot 9 = 24$

4) $S = [24]_{99}$

ESERCIZIO 3

$(M_2(\mathbb{R}), +)$ è un gruppo abeliano

① $\forall A, B \in M_2(\mathbb{R})$, $A + B = B + A$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} e+a, & f+b \\ g+c, & h+d \end{pmatrix} =$$

$$= \begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = B + A$$

② $\forall A, B, C \in M_2(\mathbb{R})$, $(A+B)+C = A+(B+C)$

$$A = (a_{ij})_{ij}, \quad B = (b_{ij})_{ij}, \quad C = (c_{ij})_{ij}$$

$$(A+B)+C = (a_{ij}+b_{ij})_{ij} + (c_{ij})_{ij} = ((a_{ij}+b_{ij})+c_{ij})_{ij} =$$

$$= (a_{ij} + (b_{ij} + c_{ij}))_{ij} = A + (b_{ij}+c_{ij})_{ij} = A+(B+C)$$

③ $\forall A \in M_2(\mathbb{R})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, la matrice $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ è elemento neutro

$$\begin{pmatrix} a & b \\ c & c \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & c \end{pmatrix}$$

④ $\forall A \in M_2(\mathbb{R})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, la matrice $-A = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ è il suo simmetrico.

$\boxed{\text{ESERCIZIO 4}}$

$$\begin{cases} x - 3y + z = 0 \\ x + 2y = -1 \\ x - 3z = 2 \end{cases}$$

① CRAMER

$$\det \begin{pmatrix} 1 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -3 \end{pmatrix} = -6 + 0 + 0 - 2 + 0 - 9 = -17$$

$$\left(\begin{matrix} 1 & -3 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & -3 \end{matrix}\right) \begin{matrix} 1 & -3 \\ 1 & 2 \\ 1 & 0 \end{matrix} = -6 + 0 + 0 - 2 - 0 - (9) = -17$$

$$\det(A) = (-1)^{3+1} \cdot 1 \cdot \det\begin{pmatrix} -3 & 1 \\ 2 & 0 \end{pmatrix} + \cancel{(-1)^{3+2} \cdot 0 \cdot \det\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}} +$$

$$+ (-1)^{3+3} (-3) \det\begin{pmatrix} 1 & -3 \\ 1 & 2 \end{pmatrix} =$$

$$= -2 + (-3)(2+3) = -2 + (-15) = -17$$

$$x = \frac{\det\begin{pmatrix} 0 & -3 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}}{-17} = -\frac{5}{17}$$

$$y = \frac{\det\begin{pmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{pmatrix}}{-17} = -\frac{6}{17}$$

$$z = \frac{\det\begin{pmatrix} 1 & -3 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 2 \end{pmatrix}}{-17} = -\frac{13}{17}$$