Probability Theory Homework 5

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December 22, 2025

1 Problem 1

Problem. Find the PDF of $Y = 1 - X^3$, where X is the random variable distributed according to the Cauchy law, i. e. with the PDF

$$\phi(x) = \frac{1}{\pi(1 + x^2)}$$

Solution.

2 Problem 2

Problem. Find the expected value and the variance of the random variable $Y = 2 - 3\sin X$, given that the PDF of X is

$$\phi(x) = \frac{1}{2}\cos x \text{ for } x \in [-\pi/2, \pi/2]$$

Solution.

3 Problem 3

Problem. The random variable X is defined on the entire real axis with the probability density $\phi(x) = \frac{1}{2}e^{-|x|}$. Find the probability density of the random variable $Y = X^2$ and its mathematical expectation.

Solution.

4 Problem 4

Problem. Prove formally that if the correlation coefficient ρ_{XY} of two random variables X and Y is equal in absolute value to one, then there is a linear functional relationship between these random variables.

Remember how to prove that $Cov(X, Y) \leq \sigma_X \sigma_Y$.

Solution.

5 Problem 5

Problem. The distribution surface (joint PDF) of the two-dimensional random variable (X, Y) is a right circular cone, the base of which is a circle centered at the origin with a unit radius. Outside this circle, the joint PDF of this two-dimensional random variable (X, Y) is zero. Find the joint PDF $f(x, y)$, the marginal PDFs and the conditional PDFs $f_x(y)$ and $f_y(x)$. Are the random variables X and Y dependent and/or correlated?

Solution.

6 Problem 6

Problem. Let X and Y be continuous random variables with a (spherically symmetric) joint PDF of the form $f(x, y) = g(x^2 + y^2)$ for some function g. Let (R, θ) be the polar coordinates of (X, Y), so that $R^2 = X^2 + Y^2$ is the squared distance from the origin and θ is the angle $\in [0, 2\pi)$, with $X = R\cos\theta$, $Y = R\sin\theta$.

a) Prove that R and θ are independent and explain intuitively why this result makes sense;

b) What is the joint PDF of (R, θ) if (X, Y) is Uniform on the unit disk, i. e. $x^2 + y^2 \leq 1$? If X, Y are i. i. d. $N(0, 1)$?

Solution.