Probability Theory Homework 6

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1 Problem 1

Problem. Find the PDF of $Y = 1 - X^3$, where X is the random variable distributed according to the Cauchy law, i. e. with the PDF

$$\phi(x) = \frac{1}{\pi(1+x^2)}$$

Solution. Let

$$Y = 1 - X^3.$$

Then

$$y = 1 - x^3 \iff x^3 = 1 - y \iff x = (1-y)^{1/3}.$$

Computing the CDF:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(1-X^3 \le y) = \mathbb{P}(X^3 \ge 1-y) = \mathbb{P}\Big(X \ge (1-y)^{1/3}\Big) = 1 - F_X\left((1-y)^{1/3}\right).$$

Differentiating:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -\frac{d}{dy}F_X\left((1-y)^{1/3}\right) = -f_X\left((1-y)^{1/3}\right)\frac{d}{dy}(1-y)^{1/3}.$$

$$\frac{d}{dy}(1-y)^{1/3} = -\frac{1}{3}(1-y)^{-2/3}, \qquad \left|\frac{d}{dy}(1-y)^{1/3}\right| = \frac{1}{3|1-y|^{2/3}}.$$

Hence

$$f_Y(y) = f_X\left((1-y)^{1/3}\right)\frac{1}{3|1-y|^{2/3}}.$$

Substituting f_X:

$$f_X\left((1-y)^{1/3}\right) = \frac{1}{\pi\left(1+\left((1-y)^{1/3}\right)^2\right)} = \frac{1}{\pi\left(1+|1-y|^{2/3}\right)}.$$

Therefore

$$f_Y(y) = \frac{1}{\pi\left(1+|1-y|^{2/3}\right)} \cdot \frac{1}{3|1-y|^{2/3}} = \boxed{\frac{1}{3\pi\,|1-y|^{2/3}\left(1+|1-y|^{2/3}\right)}}, \qquad y \in \mathbb{R}.$$

2 Problem 2

Problem. Find the expected value and the variance of the random variable $Y = 2 - 3\sin X$, given that the PDF of X is

$$\phi(x) = \frac{1}{2}\cos x \text{ for } x \in [-\pi/2, \pi/2]$$

Solution.
Letting $f_X(x) = \frac{1}{2}\cos x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Computing:

$$\mathbb{E}[Y] = \mathbb{E}[2-3\sin X] = \int_{-\pi/2}^{\pi/2}(2-3\sin x)f_X(x)\,dx = \int_{-\pi/2}^{\pi/2}(2-3\sin x)\frac{1}{2}\cos x\,dx$$

$$= \int_{-\pi/2}^{\pi/2}\cos x\,dx - \frac{3}{2}\int_{-\pi/2}^{\pi/2}\sin x\cos x\,dx = \Big[\sin x\Big]_{-\pi/2}^{\pi/2} - \frac{3}{2}\Big[\frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 2.$$

Computing:

$$\mathbb{E}[Y^2] = \mathbb{E}[(2-3\sin X)^2] = \int_{-\pi/2}^{\pi/2}(2-3\sin x)^2 f_X(x)\,dx = \int_{-\pi/2}^{\pi/2}(4-12\sin x+9\sin^2 x)\frac{1}{2}\cos x\,dx$$

$$= 2\int_{-\pi/2}^{\pi/2}\cos x\,dx - 6\int_{-\pi/2}^{\pi/2}\sin x\cos x\,dx + \frac{9}{2}\int_{-\pi/2}^{\pi/2}\sin^2 x\cos x\,dx.$$

$$2\int_{-\pi/2}^{\pi/2}\cos x\,dx = 2\Big[\sin x\Big]_{-\pi/2}^{\pi/2} = 4, \qquad -6\int_{-\pi/2}^{\pi/2}\sin x\cos x\,dx = -6\Big[\frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 0.$$

$$\frac{9}{2}\int_{-\pi/2}^{\pi/2}\sin^2 x\cos x\,dx = \frac{9}{2}\int_{-1}^{1}u^2\,du = \frac{9}{2}\Big[\frac{u^3}{3}\Big]_{-1}^{1} = 3.$$

$$\mathbb{E}[Y^2] = 4+0+3 = 7, \qquad \mathrm{Var}(Y) = \mathbb{E}[Y^2] - \big(\mathbb{E}[Y]\big)^2 = 7 - 2^2 = 3.$$

3 Problem 3

Problem. The random variable X is defined on the entire real axis with the probability density $\phi(x) = \frac{1}{2}e^{-|x|}$. Find the probability density of the random variable $Y = X^2$ and its mathematical expectation.

Solution. Let $Y = X^2$. Then $Y \geq 0$ and for $y > 0$ the equation $y = x^2$ has two solutions $x = \sqrt{y}$ and $x = -\sqrt{y}$. By the change-of-variables formula,

$$f_Y(y) = \sum_{x:\,x^2=y} f_X(x)\left|\frac{dx}{dy}\right| = f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}, \qquad y > 0.$$

Since $f_X(x) = \frac{1}{2}e^{-|x|}$, we have $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = \frac{1}{2}e^{-\sqrt{y}}$, hence

$$f_Y(y) = \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, \qquad y > 0, \qquad \text{and} \qquad f_Y(y) = 0,\ y < 0.$$

(The density has an integrable singularity at $y = 0$.)

For the expectation,

$$\mathbb{E}[Y] = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} e^{-|x|}\, dx = \int_{0}^{\infty} x^2 e^{-x}\, dx = 2.$$

4 Problem 4

Problem. Prove formally that if the correlation coefficient ρ_{XY} of two random variables X and Y is equal in absolute value to one, then there is a linear functional relationship between these random variables.

Remember how to prove that $Cov(X, Y) \leq \sigma_X \sigma_Y$.

Solution.

Assuming $\sigma_X > 0$ and $\sigma_Y > 0$.

Letting $\widetilde{X} = X - \mathbb{E}[X]$ and $\widetilde{Y} = Y - \mathbb{E}[Y]$.

Rewriting

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]}{\sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]}}.$$

Applying Cauchy–Schwarz

$$\left|\mathbb{E}[\widetilde{X}\widetilde{Y}]\right| \leq \sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]}.$$

Using $|\rho_{XY}| = 1$ and rewriting

$$|\rho_{XY}| = 1 \iff \left|\mathbb{E}[\widetilde{X}\widetilde{Y}]\right| = \sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]}.$$

Considering, for $t \in \mathbb{R}$,

$$0 \leq \mathbb{E}[(\widetilde{Y} - t\widetilde{X})^2] = \mathbb{E}[\widetilde{Y}^2] - 2t\,\mathbb{E}[\widetilde{X}\widetilde{Y}] + t^2\mathbb{E}[\widetilde{X}^2].$$

Minimizing in t by differentiating

$$\frac{d}{dt}\Big(\mathbb{E}[\widetilde{Y}^2] - 2t\,\mathbb{E}[\widetilde{X}\widetilde{Y}] + t^2\mathbb{E}[\widetilde{X}^2]\Big) = -2\mathbb{E}[\widetilde{X}\widetilde{Y}] + 2t\,\mathbb{E}[\widetilde{X}^2],$$

$$-2\mathbb{E}[\widetilde{X}\widetilde{Y}] + 2t\,\mathbb{E}[\widetilde{X}^2] = 0 \iff t = \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]}{\mathbb{E}[\widetilde{X}^2]}.$$

Substituting this t and computing

$$\mathbb{E}[(\widetilde{Y} - t\widetilde{X})^2] = \mathbb{E}[\widetilde{Y}^2] - \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]^2}{\mathbb{E}[\widetilde{X}^2]}.$$

Using

$$\left|\mathbb{E}[\widetilde{X}\widetilde{Y}]\right| = \sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]} \iff \mathbb{E}[\widetilde{X}\widetilde{Y}]^2 = \mathbb{E}[\widetilde{X}^2]\mathbb{E}[\widetilde{Y}^2],$$

obtaining

$$\mathbb{E}[(\widetilde{Y} - t\widetilde{X})^2] = \mathbb{E}[\widetilde{Y}^2] - \frac{\mathbb{E}[\widetilde{X}^2]\mathbb{E}[\widetilde{Y}^2]}{\mathbb{E}[\widetilde{X}^2]} = 0.$$

Concluding

$$\mathbb{E}[(\widetilde{Y} - t\widetilde{X})^2] = 0 \iff \widetilde{Y} - t\widetilde{X} = 0 \text{ a.s.} \iff Y - \mathbb{E}[Y] = t\,(X - \mathbb{E}[X]) \text{ a.s.}$$

Setting $c = t$ and computing

$$Y = cX + b \text{ a.s.}, \qquad b = \mathbb{E}[Y] - c\,\mathbb{E}[X].$$

5 Problem 5

Problem. The distribution surface (joint PDF) of the two-dimensional random variable (X, Y) is a right circular cone, the base of which is a circle centered at the origin with a unit radius. Outside this circle, the joint PDF of this two-dimensional random variable (X, Y) is zero. Find the joint PDF $f(x, y)$, the marginal PDFs and the conditional PDFs $f_x(y)$ and $f_y(x)$. Are the random variables X and Y dependent and/or correlated?

Solution.

Letting $r = \sqrt{x^2 + y^2}$. Writing

$$f_{X,Y}(x, y) = c(1 - r)\mathbf{1}_{\{r \le 1\}}.$$

Normalizing

$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x, y)\,dx\,dy = \int_0^{2\pi}\int_0^1 c(1-r)\,r\,dr\,d\theta = 2\pi c\int_0^1 (r-r^2)\,dr = 2\pi c\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi c}{3},$$

getting

$$c = \frac{3}{\pi}.$$

Therefore

$$f_{X,Y}(x, y) = \frac{3}{\pi}\Big(1 - \sqrt{x^2 + y^2}\Big)\mathbf{1}_{\{x^2+y^2 \le 1\}}.$$

Computing, for $|x| \le 1$ and $a = \sqrt{1 - x^2}$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dy = \int_{-a}^{a} \frac{3}{\pi}\Big(1-\sqrt{x^2 + y^2}\Big)\,dy = \frac{6}{\pi}\int_0^a \Big(1-\sqrt{x^2 + y^2}\Big)\,dy = \frac{6}{\pi}\left(a - \int_0^a \sqrt{x^2 + y^2}\right.$$

Using

$$\int \sqrt{x^2 + y^2}\,dy = \frac{1}{2}\Big(y\sqrt{x^2 + y^2} + x^2\ln\big(y + \sqrt{x^2 + y^2}\big)\Big) + C,$$

evaluating

$$\int_0^a \sqrt{x^2 + y^2}\,dy = \frac{1}{2}\Big(a\sqrt{x^2 + a^2} + x^2\ln(a+\sqrt{x^2 + a^2}) - x^2\ln|x|\Big) = \frac{1}{2}\Big(a + x^2\ln(1+a) - x^2\ln|x|\Big),$$

therefore, for $|x| \le 1$,

$$f_X(x) = \frac{6}{\pi}\left(a - \frac{1}{2}\Big(a + x^2\ln(1+a) - x^2\ln|x|\Big)\right) = \frac{3}{\pi}\left(a - x^2\ln\frac{1+a}{|x|}\right), \qquad a = \sqrt{1-x^2},$$

and

$$f_X(x) = 0, \qquad |x| > 1.$$

Taking limit at x=0

$$f_X(0) = \lim_{x\to 0}\frac{3}{\pi}\left(\sqrt{1-x^2} - x^2\ln\frac{1+\sqrt{1-x^2}}{|x|}\right) = \frac{3}{\pi}.$$

Computing analogously, for $|y| \le 1$ and $b = \sqrt{1-y^2}$,

$$f_Y(y) = \frac{3}{\pi}\left(b - y^2\ln\frac{1+b}{|y|}\right), \qquad b = \sqrt{1-y^2}, \qquad f_Y(y) = 0 \text{ for } |y| > 1.$$

Computing, for —x—¡1,

$$f_x(y) = f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{3}{\pi}\left(1-\sqrt{x^2+y^2}\right)\mathbf{1}_{\{x^2+y^2\le 1\}}}{\frac{3}{\pi}\left(\sqrt{1-x^2} - x^2\ln\frac{1+\sqrt{1-x^2}}{|x|}\right)} = \frac{1-\sqrt{x^2+y^2}}{\sqrt{1-x^2} - x^2\ln\frac{1+\sqrt{1-x^2}}{|x|}}\,\mathbf{1}_{\{|y|\le\sqrt{1-x^2}\}}$$

and $f_x(y) = 0$ otherwise. Computing, for $|y| < 1$,

$$f_y(x) = f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1-\sqrt{x^2+y^2}}{\sqrt{1-y^2} - y^2\ln\frac{1+\sqrt{1-y^2}}{|y|}}\,\mathbf{1}_{\{|x|\le\sqrt{1-y^2}\}},$$

and $f_y(x) = 0$ otherwise.

Checking independence

$$f_{X,Y}(0,0) = \frac{3}{\pi}, \qquad f_X(0) = \frac{3}{\pi}, \qquad f_Y(0) = \frac{3}{\pi}, \qquad f_X(0)f_Y(0) = \frac{9}{\pi^2} \ne \frac{3}{\pi},$$

getting X,Y depending.

Computing

$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} x f_{X,Y}(x,y)\,dx\,dy = \iint_{\mathbb{R}^2}(-x)f_{X,Y}(-x,y)\,dx\,dy = -\mathbb{E}[X] \Longrightarrow \mathbb{E}[X] = 0,$$

$$\mathbb{E}[Y] = \iint_{\mathbb{R}^2} y f_{X,Y}(x,y)\,dx\,dy = \iint_{\mathbb{R}^2}(-y)f_{X,Y}(x,-y)\,dx\,dy = -\mathbb{E}[Y] \Longrightarrow \mathbb{E}[Y] = 0,$$

$$\mathbb{E}[XY] = \iint_{\mathbb{R}^2} xy f_{X,Y}(x,y)\,dx\,dy = \iint_{\mathbb{R}^2}(-x)y f_{X,Y}(-x,y)\,dx\,dy = -\mathbb{E}[XY] \Longrightarrow \mathbb{E}[XY] = 0,$$

therefore

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0, \qquad \rho_{XY} = 0.$$

$$
\boxed{\begin{aligned}
&f_{X,Y}(x,y) = \frac{3}{\pi}\left(1-\sqrt{x^2+y^2}\right)\mathbf{1}_{\{x^2+y^2\le 1\}},\\
&f_X(x) = \frac{3}{\pi}\left(\sqrt{1-x^2} - x^2\ln\frac{1+\sqrt{1-x^2}}{|x|}\right)\mathbf{1}_{\{|x|\le 1\}},\\
&f_Y(y) = \frac{3}{\pi}\left(\sqrt{1-y^2} - y^2\ln\frac{1+\sqrt{1-y^2}}{|y|}\right)\mathbf{1}_{\{|y|\le 1\}},\\
&f_x(y) = \frac{1-\sqrt{x^2+y^2}}{\sqrt{1-x^2} - x^2\ln\frac{1+\sqrt{1-x^2}}{|x|}}\mathbf{1}_{\{|x|<1,\ |y|\le\sqrt{1-x^2}\}},\\
&f_y(x) = \frac{1-\sqrt{x^2+y^2}}{\sqrt{1-y^2} - y^2\ln\frac{1+\sqrt{1-y^2}}{|y|}}\mathbf{1}_{\{|y|<1,\ |x|\le\sqrt{1-y^2}\}},\\
&X,Y \text{ depending},\qquad \mathrm{Cov}(X,Y)=0,\qquad \rho_{XY}=0.
\end{aligned}}
$$

6 Problem 6

Problem. Let X and Y be continuous random variables with a (spherically symmetric) joint PDF of the form $f(x,y) = g(x^2+y^2)$ for some function g. Let (R,θ) be the polar coordinates of (X,Y), so that $R^2 = X^2+Y^2$ is the squared distance from the origin and θ is the angle $\in [0,2\pi)$, with $X = R\cos\theta$, $Y = R\sin\theta$.

a) Prove that R and θ are independent and explain intuitively why this result makes sense;

b) What is the joint PDF of (R,θ) if (X,Y) is Uniform on the unit disk, i. e. $x^2+y^2\le 1$? If X,Y are i. i. d. $N(0,1)$?

Solution.

Letting $x = r\cos\theta$, $y = r\sin\theta$, $r\ge 0$, $\theta\in[0,2\pi)$. Computing the Jacobian,

$$
\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r, \qquad dx\,dy = r\,dr\,d\theta.
$$

Substituting into the change of variables formula,

$$
f_{R,\Theta}(r,\theta) = f_{X,Y}(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = g\big((r\cos\theta)^2 + (r\sin\theta)^2\big)\ r = g(r^2)\,r, \qquad r\ge 0,\ \theta\in[0,2\pi).
$$

Computing the normalization integral,

$$
1 = \int_0^{2\pi}\int_0^\infty f_{R,\Theta}(r,\theta)\,dr\,d\theta = \int_0^{2\pi}\int_0^\infty g(r^2)\,r\,dr\,d\theta = 2\pi\int_0^\infty g(r^2)\,r\,dr,
$$

$$
\int_0^\infty g(r^2)\,r\,dr = \frac{1}{2\pi}.
$$

Computing the marginal of Θ,

$$f_\Theta(\theta) = \int_0^\infty f_{R,\Theta}(r,\theta)\,dr = \int_0^\infty g(r^2)\,r\,dr = \frac{1}{2\pi}, \qquad \theta \in [0, 2\pi).$$

Computing the marginal of R,

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta)\,d\theta = \int_0^{2\pi} g(r^2)\,r\,d\theta = 2\pi r g(r^2), \qquad r \geq 0.$$

Factoring,

$$f_R(r) f_\Theta(\theta) = \big(2\pi r g(r^2)\big)\left(\frac{1}{2\pi}\right) = g(r^2)\,r = f_{R,\Theta}(r,\theta),$$

getting R and Θ independent. Interpreting $f(x,y) = g(x^2+y^2)$ as rotation-invariant, getting Θ uniform and R radial.

Substituting the uniform-on-disk density,

$$f_{X,Y}(x,y) = \frac{1}{\pi}\mathbf{1}_{\{x^2+y^2\leq 1\}} = \frac{1}{\pi}\mathbf{1}_{\{0\leq r\leq 1\}},$$

$$f_{R,\Theta}(r,\theta) = \frac{r}{\pi}\mathbf{1}_{\{0\leq r\leq 1\}}\mathbf{1}_{\{0\leq\theta<2\pi\}}.$$

Integrating for marginals,

$$f_\Theta(\theta) = \int_0^1 \frac{r}{\pi}\,dr = \frac{1}{2\pi}, \qquad 0 \leq \theta < 2\pi,$$

$$f_R(r) = \int_0^{2\pi} \frac{r}{\pi}\,d\theta = 2r, \qquad 0 \leq r \leq 1.$$

Substituting the i.i.d. $N(0,1)$ density,

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\exp\!\left(-\frac{x^2+y^2}{2}\right) = \frac{1}{2\pi}\exp\!\left(-\frac{r^2}{2}\right),$$

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi}\exp\!\left(-\frac{r^2}{2}\right)\mathbf{1}_{\{r\geq 0\}}\mathbf{1}_{\{0\leq\theta<2\pi\}}.$$

Integrating for marginals,

$$f_\Theta(\theta) = \int_0^\infty \frac{r}{2\pi}e^{-r^2/2}\,dr = \frac{1}{2\pi}, \qquad 0 \leq \theta < 2\pi,$$

$$f_R(r) = \int_0^{2\pi} \frac{r}{2\pi}e^{-r^2/2}\,d\theta = re^{-r^2/2}, \qquad r \geq 0.$$

$$\boxed{\begin{aligned} &f_{R,\Theta}(r,\theta) = g(r^2)\,r, \qquad r\geq 0,\ 0\leq\theta<2\pi,\\ &f_{R,\Theta}(r,\theta) = \frac{r}{\pi}\mathbf{1}_{\{0\leq r\leq 1\}}\mathbf{1}_{\{0\leq\theta<2\pi\}},\\ &f_{R,\Theta}(r,\theta) = \frac{r}{2\pi}\exp\!\left(-\frac{r^2}{2}\right)\mathbf{1}_{\{r\geq 0\}}\mathbf{1}_{\{0\leq\theta<2\pi\}}. \end{aligned}}$$