Probability Theory Homework 5

Gregory Matsnev

December 23, 2025

1 Problem 1

Problem. Find the PDF of $Y = 1 - X^3$, where X is the random variable distributed according to the Cauchy law, i. e. with the PDF

$$\phi(x) = \frac{1}{\pi(1+x^2)}$$

Solution. Let

$$Y = 1 - X^3.$$

Then

$$y = 1 - x^3 \iff x^3 = 1 - y \iff x = (1-y)^{1/3}.$$

Computing the CDF:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(1-X^3 \le y) = \mathbb{P}(X^3 \ge 1-y) = \mathbb{P}\Big(X \ge (1-y)^{1/3}\Big) = 1-F_X\left((1-y)^{1/3}\right).$$

Differentiating:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -\frac{d}{dy}F_X\left((1-y)^{1/3}\right) = -f_X\left((1-y)^{1/3}\right)\frac{d}{dy}(1-y)^{1/3}.$$

$$\frac{d}{dy}(1-y)^{1/3} = -\frac{1}{3}(1-y)^{-2/3}, \qquad \left|\frac{d}{dy}(1-y)^{1/3}\right| = \frac{1}{3|1-y|^{2/3}}.$$

Hence

$$f_Y(y) = f_X\left((1-y)^{1/3}\right)\frac{1}{3|1-y|^{2/3}}.$$

Substituting f_X:

$$f_X\left((1-y)^{1/3}\right) = \frac{1}{\pi\left(1+\left((1-y)^{1/3}\right)^2\right)} = \frac{1}{\pi\left(1+|1-y|^{2/3}\right)}.$$

Therefore

$$f_Y(y) = \frac{1}{\pi\left(1+|1-y|^{2/3}\right)}\cdot\frac{1}{3|1-y|^{2/3}} = \boxed{\frac{1}{3\pi\,|1-y|^{2/3}\left(1+|1-y|^{2/3}\right)}}, \qquad y \in \mathbb{R}.$$

2 Problem 2

Problem. Find the expected value and the variance of the random variable $Y = 2 - 3\sin X$, given that the PDF of X is

$$\phi(x) = \frac{1}{2}\cos x \text{ for } x \in [-\pi/2, \pi/2]$$

Solution.
Letting $f_X(x) = \frac{1}{2}\cos x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Computing:

$$\mathbb{E}[Y] = \mathbb{E}[2-3\sin X] = \int_{-\pi/2}^{\pi/2} (2-3\sin x) f_X(x)\,dx = \int_{-\pi/2}^{\pi/2} (2-3\sin x)\frac{1}{2}\cos x\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos x\,dx - \frac{3}{2}\int_{-\pi/2}^{\pi/2} \sin x \cos x\,dx = \Big[\sin x\Big]_{-\pi/2}^{\pi/2} - \frac{3}{2}\Big[\frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 2.$$

Computing:

$$\mathbb{E}[Y^2] = \mathbb{E}[(2-3\sin X)^2] = \int_{-\pi/2}^{\pi/2} (2-3\sin x)^2 f_X(x)\,dx = \int_{-\pi/2}^{\pi/2} (4-12\sin x+9\sin^2 x)\frac{1}{2}\cos x\,dx$$

$$= 2\int_{-\pi/2}^{\pi/2} \cos x\,dx - 6\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx + \frac{9}{2}\int_{-\pi/2}^{\pi/2} \sin^2 x\cos x\,dx.$$

$$2\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\Big[\sin x\Big]_{-\pi/2}^{\pi/2} = 4, \qquad -6\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx = -6\Big[\frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 0.$$

$$\frac{9}{2}\int_{-\pi/2}^{\pi/2} \sin^2 x\cos x\,dx = \frac{9}{2}\int_{-1}^{1} u^2\,du = \frac{9}{2}\Big[\frac{u^3}{3}\Big]_{-1}^{1} = 3.$$

$$\mathbb{E}[Y^2] = 4 + 0 + 3 = 7, \qquad \mathrm{Var}(Y) = \mathbb{E}[Y^2] - \big(\mathbb{E}[Y]\big)^2 = 7 - 2^2 = 3.$$

3 Problem 3

Problem. The random variable X is defined on the entire real axis with the probability density $\phi(x) = \frac{1}{2}e^{-|x|}$. Find the probability density of the random variable $Y = X^2$ and its mathematical expectation.

Solution.

4 Problem 4

Problem. Prove formally that if the correlation coefficient ρ_{XY} of two random variables X and Y is equal in absolute value to one, then there is a linear functional relationship between these random variables.

Remember how to prove that $Cov(X, Y) \leq \sigma_X \sigma_Y$.

Solution.

5 Problem 5

Problem. The distribution surface (joint PDF) of the two-dimensional random variable (X, Y) is a right circular cone, the base of which is a circle centered at the origin with a unit radius. Outside this circle, the joint PDF of this two-dimensional random variable (X, Y) is zero. Find the joint PDF $f(x, y)$, the marginal PDFs and the conditional PDFs $f_x(y)$ and $f_y(x)$. Are the random variables X and Y dependent and/or correlated?

Solution.

6 Problem 6

Problem. Let X and Y be continuous random variables with a (spherically symmetric) joint PDF of the form $f(x, y) = g(x^2 + y^2)$ for some function g. Let (R, θ) be the polar coordinates of (X, Y), so that $R^2 = X^2 + Y^2$ is the squared distance from the origin and θ is the angle $\in [0, 2\pi)$, with $X = R\cos\theta$, $Y = R\sin\theta$.

a) Prove that R and θ are independent and explain intuitively why this result makes sense;

b) What is the joint PDF of (R, θ) if (X, Y) is Uniform on the unit disk, i. e. $x^2 + y^2 \leq 1$? If X, Y are i. i. d. $N(0, 1)$?

Solution.