Probability Theory Homework 6

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December 23, 2025

1 Problem 1

Problem. Find the PDF of $Y = 1 - X^3$, where X is the random variable distributed according to the Cauchy law, i. e. with the PDF

$$\phi(x) = \frac{1}{\pi(1+x^2)}$$

Solution. Let

$$Y = 1 - X^3.$$

Then

$$y = 1 - x^3 \iff x^3 = 1 - y \iff x = (1-y)^{1/3}.$$

Computing the CDF:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(1-X^3 \le y) = \mathbb{P}(X^3 \ge 1-y) = \mathbb{P}\Big(X \ge (1-y)^{1/3}\Big) = 1-F_X\left((1-y)^{1/3}\right).$$

Differentiating:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -\frac{d}{dy}F_X\left((1-y)^{1/3}\right) = -f_X\left((1-y)^{1/3}\right)\frac{d}{dy}(1-y)^{1/3}.$$

$$\frac{d}{dy}(1-y)^{1/3} = -\frac{1}{3}(1-y)^{-2/3}, \qquad \left|\frac{d}{dy}(1-y)^{1/3}\right| = \frac{1}{3|1-y|^{2/3}}.$$

Hence

$$f_Y(y) = f_X\left((1-y)^{1/3}\right)\frac{1}{3|1-y|^{2/3}}.$$

Substituting f_X:

$$f_X\left((1-y)^{1/3}\right) = \frac{1}{\pi\left(1+\left((1-y)^{1/3}\right)^2\right)} = \frac{1}{\pi\left(1+|1-y|^{2/3}\right)}.$$

Therefore

$$f_Y(y) = \frac{1}{\pi\left(1+|1-y|^{2/3}\right)}\cdot\frac{1}{3|1-y|^{2/3}} = \boxed{\frac{1}{3\pi\,|1-y|^{2/3}\left(1+|1-y|^{2/3}\right)}}, \qquad y \in \mathbb{R}.$$

2 Problem 2

Problem. Find the expected value and the variance of the random variable $Y = 2 - 3\sin X$, given that the PDF of X is

$$\phi(x) = \frac{1}{2}\cos x \text{ for } x \in [-\pi/2, \pi/2]$$

Solution.
Letting $f_X(x) = \frac{1}{2}\cos x$ for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Computing:

$$\mathbb{E}[Y] = \mathbb{E}[2-3\sin X] = \int_{-\pi/2}^{\pi/2} (2-3\sin x) f_X(x)\,dx = \int_{-\pi/2}^{\pi/2} (2-3\sin x)\frac{1}{2}\cos x\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos x\,dx - \frac{3}{2}\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx = \Big[\sin x\Big]_{-\pi/2}^{\pi/2} - \frac{3}{2}\Big[\frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 2.$$

Computing:

$$\mathbb{E}[Y^2] = \mathbb{E}[(2-3\sin X)^2] = \int_{-\pi/2}^{\pi/2} (2-3\sin x)^2 f_X(x)\,dx = \int_{-\pi/2}^{\pi/2} (4-12\sin x+9\sin^2 x)\frac{1}{2}\cos x\,dx$$

$$= 2\int_{-\pi/2}^{\pi/2} \cos x\,dx - 6\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx + \frac{9}{2}\int_{-\pi/2}^{\pi/2} \sin^2 x\cos x\,dx.$$

$$2\int_{-\pi/2}^{\pi/2} \cos x\,dx = 2\Big[\sin x\Big]_{-\pi/2}^{\pi/2} = 4, \qquad -6\int_{-\pi/2}^{\pi/2} \sin x\cos x\,dx = -6\Big[\frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 0.$$

$$\frac{9}{2}\int_{-\pi/2}^{\pi/2} \sin^2 x\cos x\,dx = \frac{9}{2}\int_{-1}^{1} u^2\,du = \frac{9}{2}\Big[\frac{u^3}{3}\Big]_{-1}^{1} = 3.$$

$$\mathbb{E}[Y^2] = 4+0+3 = 7, \qquad \mathrm{Var}(Y) = \mathbb{E}[Y^2] - \big(\mathbb{E}[Y]\big)^2 = 7 - 2^2 = 3.$$

3 Problem 3

Problem. The random variable X is defined on the entire real axis with the probability density $\phi(x) = \frac{1}{2}e^{-|x|}$. Find the probability density of the random variable $Y = X^2$ and its mathematical expectation.

Solution. Let $Y = X^2$. Then $Y \geq 0$ and for $y > 0$ the equation $y = x^2$ has two solutions $x = \sqrt{y}$ and $x = -\sqrt{y}$. By the change-of-variables formula,

$$f_Y(y) = \sum_{x:\,x^2=y} f_X(x)\left|\frac{dx}{dy}\right| = f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\cdot\frac{1}{2\sqrt{y}}, \qquad y > 0.$$

Since $f_X(x) = \frac{1}{2}e^{-|x|}$, we have $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = \frac{1}{2}e^{-\sqrt{y}}$, hence

$$f_Y(y) = \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, \qquad y > 0, \qquad \text{and} \qquad f_Y(y) = 0,\ y < 0.$$

(The density has an integrable singularity at $y = 0$.)

For the expectation,

$$\mathbb{E}[Y] = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2} e^{-|x|}\, dx = \int_0^{\infty} x^2 e^{-x}\, dx = 2.$$

4 Problem 4

Problem. Prove formally that if the correlation coefficient ρ_{XY} of two random variables X and Y is equal in absolute value to one, then there is a linear functional relationship between these random variables.

Remember how to prove that $Cov(X,Y) \le \sigma_X \sigma_Y$.

Solution. Assume $\sigma_X > 0$ and $\sigma_Y > 0$ (otherwise one of the variables is a.s. constant and the claim is trivial). Let $\widetilde{X} = X - \mathbb{E}[X]$ and $\widetilde{Y} = Y - \mathbb{E}[Y]$. Then

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]}{\sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]}}.$$

By the Cauchy–Schwarz inequality,

$$\left|\mathbb{E}[\widetilde{X}\widetilde{Y}]\right| \le \sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]},$$

and equality holds if and only if $\widetilde{Y} = c\,\widetilde{X}$ a.s. for some constant c.

If $|\rho_{XY}| = 1$, then the Cauchy–Schwarz inequality is an equality, hence there exists a constant c such that

$$Y - \mathbb{E}[Y] = c\,(X - \mathbb{E}[X]) \quad \text{a.s.}$$

Equivalently,

$$Y = cX + b \quad \text{a.s.}, \qquad b = \mathbb{E}[Y] - c\,\mathbb{E}[X],$$

which is a linear functional relationship between X and Y.

5 Problem 5

Problem. The distribution surface (joint PDF) of the two-dimensional random variable (X,Y) is a right circular cone, the base of which is a circle centered at the origin with a unit radius. Outside this circle, the joint PDF of this two-dimensional random variable (X,Y) is zero. Find the joint PDF $f(x,y)$, the marginal PDFs and the conditional PDFs $f_x(y)$ and $f_y(x)$. Are the random variables X and Y dependent and/or correlated?

Solution. Let $r = \sqrt{x^2+y^2}$. A natural "cone" density over the unit disk is linear in r and vanishes at the boundary $r = 1$, hence

$$f_{X,Y}(x,y) = \begin{cases} c(1-r), & r \le 1, \\ 0, & r > 1, \end{cases} \qquad r = \sqrt{x^2+y^2},$$

with a constant c determined by normalization. In polar coordinates (r,θ), $dx\,dy = r\,dr\,d\theta$, so

$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \int_0^{2\pi}\int_0^1 c(1-r)\,r\,dr\,d\theta = 2\pi c\int_0^1 (r-r^2)\,dr = 2\pi c\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\pi c}{3},$$

thus $c = \frac{3}{\pi}$. Therefore

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{3}{\pi}\Big(1-\sqrt{x^2+y^2}\Big), & x^2+y^2 \le 1,\\ 0, & x^2+y^2 > 1.\end{cases}$$

Marginals. By symmetry, $f_X = f_Y$. For $|x| < 1$ we integrate over $y \in \big[-\sqrt{1-x^2}, \sqrt{1-x^2}\big]$:

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{\pi}\left(1-\sqrt{x^2+y^2}\right)\,dy.$$

Let $a = \sqrt{1-x^2}$. Using evenness in y,

$$f_X(x) = \frac{6}{\pi}\int_0^a \left(1-\sqrt{x^2+y^2}\right)\,dy = \frac{6}{\pi}\left(a - \int_0^a \sqrt{x^2+y^2}\,dy\right).$$

Recall

$$\int \sqrt{x^2+y^2}\,dy = \frac{y}{2}\sqrt{x^2+y^2} + \frac{x^2}{2}\ln\Big(y+\sqrt{x^2+y^2}\Big) + C.$$

Since $\sqrt{x^2+a^2} = 1$, we get

$$\int_0^a \sqrt{x^2+y^2}\,dy = \frac{a}{2}\cdot 1 + \frac{x^2}{2}\ln(a+1) - \frac{x^2}{2}\ln|x|.$$

Hence for $|x| < 1$,

$$f_X(x) = \frac{6}{\pi}\left(a - \frac{a}{2} - \frac{x^2}{2}\ln\frac{a+1}{|x|}\right) = \frac{3}{\pi}\left(a - x^2\ln\frac{1+a}{|x|}\right), \qquad a = \sqrt{1-x^2}.$$

And $f_X(x) = 0$ for $|x| \ge 1$. (At $x = 0$ the formula is understood by continuity and gives $f_X(0) = \frac{3}{\pi}$.) By symmetry,

$$f_Y(y) = \frac{3}{\pi}\left(\sqrt{1-y^2} - y^2\ln\frac{1+\sqrt{1-y^2}}{|y|}\right)\mathbf{1}_{\{|y|<1\}}.$$

Conditional densities. For $|x| < 1$ and $|y| \le \sqrt{1-x^2}$,

$$f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{3}{\pi}\left(1-\sqrt{x^2+y^2}\right)}{f_X(x)},$$

and $f_{Y|X=x}(y) = 0$ otherwise. Similarly, for $|y| < 1$ and $|x| \le \sqrt{1-y^2}$,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{3}{\pi}\left(1 - \sqrt{x^2+y^2}\right)}{f_Y(y)},$$

and 0 otherwise.

Dependence and correlation. X and Y are *dependent* since the support is not a rectangle: if $|X|$ is close to 1, then necessarily $|Y|$ must be small (because $X^2 + Y^2 \le 1$).

They are *uncorrelated*: by symmetry $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, and

$$\mathbb{E}[XY] = \iint xy\, f_{X,Y}(x,y)\, dx\, dy = 0$$

because the integrand is odd in x (or in y) while the domain and $f_{X,Y}$ are even. Hence $\mathrm{Cov}(X,Y) = 0$ and the correlation coefficient is 0.

6 Problem 6

Problem. Let X and Y be continuous random variables with a (spherically symmetric) joint PDF of the form $f(x,y) = g(x^2+y^2)$ for some function g. Let (R,θ) be the polar coordinates of (X,Y), so that $R^2 = X^2 + Y^2$ is the squared distance from the origin and θ is the angle $\in [0, 2\pi)$, with $X = R\cos\theta$, $Y = R\sin\theta$.

a) Prove that R and θ are independent and explain intuitively why this result makes sense;

b) What is the joint PDF of (R,θ) if (X,Y) is Uniform on the unit disk, i. e. $x^2+y^2 \le 1$? If X, Y are i. i. d. $N(0,1)$?

Solution. a) Independence of R and θ. Use the change of variables $(x,y) = (r\cos\theta, r\sin\theta)$ with Jacobian $|J| = r$. Then for $r \ge 0$ and $\theta \in [0, 2\pi)$,

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(r\cos\theta, r\sin\theta)\, r = g(r^2)\, r.$$

This expression does not depend on θ. The marginal density of Θ is therefore constant:

$$f_\Theta(\theta) = \int_0^\infty g(r^2)\, r\, dr.$$

Since $f_{R,\Theta}$ must integrate to 1,

$$1 = \int_0^{2\pi}\int_0^\infty g(r^2)\, r\, dr\, d\theta = 2\pi \int_0^\infty g(r^2)\, r\, dr,$$

so $\int_0^\infty g(r^2)\, r\, dr = \frac{1}{2\pi}$ and hence

$$f_\Theta(\theta) = \frac{1}{2\pi}, \qquad \theta \in [0, 2\pi),$$

i.e. Θ is uniform on $[0, 2\pi)$. Also

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta)\, d\theta = 2\pi r g(r^2), \qquad r \geq 0.$$

Thus

$$f_{R,\Theta}(r,\theta) = \underbrace{(2\pi r g(r^2))}_{f_R(r)} \cdot \underbrace{\frac{1}{2\pi}}_{f_\Theta(\theta)},$$

so R and Θ are independent.

Intuition. Spherical symmetry means the distribution is invariant under rotations, so the angle cannot favor any direction (hence uniform), while the radius controls how far from the origin we are; these two pieces of information do not interact.

b) Two examples.

Uniform on the unit disk. If (X, Y) is uniform on $\{x^2 + y^2 \leq 1\}$, then

$$f_{X,Y}(x,y) = \frac{1}{\pi}\mathbf{1}_{\{x^2+y^2\leq 1\}}.$$

Hence for $0 \leq r \leq 1$ and $\theta \in [0, 2\pi)$,

$$f_{R,\Theta}(r,\theta) = \frac{1}{\pi}\, r,$$

and 0 otherwise. In particular $f_\Theta(\theta) = \frac{1}{2\pi}$ and $f_R(r) = 2r$ for $0 \leq r \leq 1$.

X, Y *i.i.d.* $N(0, 1)$. Then

$$f_{X,Y}(x,y) = \frac{1}{2\pi}\exp\left(-\frac{x^2+y^2}{2}\right) = g(x^2+y^2), \quad g(u) = \frac{1}{2\pi}e^{-u/2}.$$

So for $r \geq 0$ and $\theta \in [0, 2\pi)$,

$$f_{R,\Theta}(r,\theta) = \frac{1}{2\pi}e^{-r^2/2}\, r,$$

and again $\Theta \sim \mathrm{Unif}[0, 2\pi)$ while R has the Rayleigh density $f_R(r) = re^{-r^2/2}$ for $r \geq 0$.