

Probability Theory Homework 6

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1 Problem 1

Problem. Find the PDF of $Y = 1 - X^3$, where X is the random variable distributed according to the Cauchy law, i. e. with the PDF

$$\phi(x) = \frac{1}{\pi(1+x^2)}$$

Solution. We have $Y = 1 - X^3$, so we need to find the PDF of Y using the transformation method. First, let's express x in terms of y :

$$y = 1 - x^3 \iff x^3 = 1 - y \iff x = (1 - y)^{1/3}.$$

Now I'll find the CDF of Y and then differentiate to get the PDF:

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(1 - X^3 \leq y) = \mathbb{P}(X^3 \geq 1 - y) = \mathbb{P}\left(X \geq (1 - y)^{1/3}\right) = 1 - F_X((1 - y)^{1/3}).$$

To get the PDF, we differentiate with respect to y :

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -\frac{d}{dy}F_X((1 - y)^{1/3}) = -f_X((1 - y)^{1/3}) \frac{d}{dy}(1 - y)^{1/3}.$$

The derivative of $(1 - y)^{1/3}$ is

$$\frac{d}{dy}(1 - y)^{1/3} = -\frac{1}{3}(1 - y)^{-2/3}, \quad \left| \frac{d}{dy}(1 - y)^{1/3} \right| = \frac{1}{3|1 - y|^{2/3}}.$$

So we get

$$f_Y(y) = f_X((1 - y)^{1/3}) \frac{1}{3|1 - y|^{2/3}}.$$

Now let's substitute the given PDF of X :

$$f_X((1 - y)^{1/3}) = \frac{1}{\pi \left(1 + ((1 - y)^{1/3})^2\right)} = \frac{1}{\pi (1 + |1 - y|^{2/3})}.$$

Finally, putting it all together:

$$f_Y(y) = \frac{1}{\pi (1 + |1 - y|^{2/3})} \cdot \frac{1}{3|1 - y|^{2/3}} = \boxed{\frac{1}{3\pi |1 - y|^{2/3} (1 + |1 - y|^{2/3})}}, \quad y \in \mathbb{R}.$$

2 Problem 2

Problem. Find the expected value and the variance of the random variable $Y = 2 - 3 \sin X$, given that the PDF of X is

$$\phi(x) = \frac{1}{2} \cos x \text{ for } x \in [-\pi/2, \pi/2]$$

Solution.

We're given that $f_X(x) = \frac{1}{2} \cos x$ for $x \in [-\pi/2, \pi/2]$ and $Y = 2 - 3 \sin X$. Let's find $\mathbb{E}[Y]$ first:

$$\begin{aligned} \mathbb{E}[Y] &= \mathbb{E}[2 - 3 \sin X] = \int_{-\pi/2}^{\pi/2} (2 - 3 \sin x) f_X(x) dx = \int_{-\pi/2}^{\pi/2} (2 - 3 \sin x) \frac{1}{2} \cos x dx \\ &= \int_{-\pi/2}^{\pi/2} \cos x dx - \frac{3}{2} \int_{-\pi/2}^{\pi/2} \sin x \cos x dx = \left[\sin x \right]_{-\pi/2}^{\pi/2} - \frac{3}{2} \left[\frac{1}{2} \sin^2 x \right]_{-\pi/2}^{\pi/2} = 2. \end{aligned}$$

To find the variance, we need $\mathbb{E}[Y^2]$:

$$\begin{aligned} \mathbb{E}[Y^2] &= \mathbb{E}[(2 - 3 \sin X)^2] = \int_{-\pi/2}^{\pi/2} (2 - 3 \sin x)^2 f_X(x) dx = \int_{-\pi/2}^{\pi/2} (4 - 12 \sin x + 9 \sin^2 x) \frac{1}{2} \cos x dx \\ &= 2 \int_{-\pi/2}^{\pi/2} \cos x dx - 6 \int_{-\pi/2}^{\pi/2} \sin x \cos x dx + \frac{9}{2} \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x dx. \end{aligned}$$

Let me compute each integral separately:

$$2 \int_{-\pi/2}^{\pi/2} \cos x dx = 2 \left[\sin x \right]_{-\pi/2}^{\pi/2} = 4, \quad -6 \int_{-\pi/2}^{\pi/2} \sin x \cos x dx = -6 \left[\frac{1}{2} \sin^2 x \right]_{-\pi/2}^{\pi/2} = 0.$$

For the last integral, I'll use substitution $u = \sin x$, so $du = \cos x dx$:

$$\frac{9}{2} \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x dx = \frac{9}{2} \int_{-1}^1 u^2 du = \frac{9}{2} \left[\frac{u^3}{3} \right]_{-1}^1 = 3.$$

So we have

$$\mathbb{E}[Y^2] = 4 + 0 + 3 = 7,$$

and therefore

$$\boxed{\mathbb{E}[Y] = 2, \quad \text{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 7 - 4 = 3.}$$

3 Problem 3

Problem. The random variable X is defined on the entire real axis with the probability density $\phi(x) = \frac{1}{2} e^{-|x|}$. Find the probability density of the random variable $Y = X^2$ and its mathematical expectation.

Solution. We need to find the PDF of $Y = X^2$ where X has PDF $f_X(x) = \frac{1}{2}e^{-|x|}$.

Since $Y \geq 0$ and for any $y > 0$ the equation $y = x^2$ has two solutions $x = \pm\sqrt{y}$, we can use the change-of-variables formula:

$$f_Y(y) = \sum_{x: x^2=y} f_X(x) \left| \frac{dx}{dy} \right| = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}, \quad y > 0.$$

Notice that because of the symmetry of f_X (it only depends on $|x|$), we have $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = \frac{1}{2}e^{-\sqrt{y}}$. Therefore:

$$f_Y(y) = \begin{cases} \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, & y > 0, \\ 0, & y \leq 0. \end{cases}$$

(Note: there's an integrable singularity at $y = 0$, but that's okay for a PDF.)

Now for the expectation:

$$\mathbb{E}[Y] = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2}e^{-|x|} dx = \int_0^{\infty} x^2 e^{-x} dx = \boxed{2}.$$

4 Problem 4

Problem. Prove formally that if the correlation coefficient ρ_{XY} of two random variables X and Y is equal in absolute value to one, then there is a linear functional relationship between these random variables.

Remember how to prove that $\text{Cov}(X, Y) \leq \sigma_X \sigma_Y$.

Solution.

Let's assume $\sigma_X > 0$ and $\sigma_Y > 0$ (otherwise the correlation isn't defined). I'll denote the centered variables as $\tilde{X} = X - \mathbb{E}[X]$ and $\tilde{Y} = Y - \mathbb{E}[Y]$.

The correlation coefficient can be written as

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{E}[\tilde{X}\tilde{Y}]}{\sqrt{\mathbb{E}[\tilde{X}^2]}\sqrt{\mathbb{E}[\tilde{Y}^2]}}.$$

We know from the Cauchy–Schwarz inequality that

$$|\mathbb{E}[\tilde{X}\tilde{Y}]| \leq \sqrt{\mathbb{E}[\tilde{X}^2]}\sqrt{\mathbb{E}[\tilde{Y}^2]}.$$

Now, if $|\rho_{XY}| = 1$, this means we have equality in Cauchy–Schwarz:

$$|\rho_{XY}| = 1 \iff |\mathbb{E}[\tilde{X}\tilde{Y}]| = \sqrt{\mathbb{E}[\tilde{X}^2]}\sqrt{\mathbb{E}[\tilde{Y}^2]}.$$

The key insight is to consider the quadratic form for any $t \in \mathbb{R}$:

$$0 \leq \mathbb{E}[(\tilde{Y} - t\tilde{X})^2] = \mathbb{E}[\tilde{Y}^2] - 2t \mathbb{E}[\tilde{X}\tilde{Y}] + t^2 \mathbb{E}[\tilde{X}^2].$$

To find the minimum, let's differentiate with respect to t :

$$\frac{d}{dt} \left(\mathbb{E}[\tilde{Y}^2] - 2t \mathbb{E}[\tilde{X}\tilde{Y}] + t^2 \mathbb{E}[\tilde{X}^2] \right) = -2\mathbb{E}[\tilde{X}\tilde{Y}] + 2t \mathbb{E}[\tilde{X}^2],$$

Setting this to zero gives us

$$t = \frac{\mathbb{E}[\tilde{X}\tilde{Y}]}{\mathbb{E}[\tilde{X}^2]}.$$

Plugging this back in:

$$\mathbb{E}[(\tilde{Y} - t\tilde{X})^2] = \mathbb{E}[\tilde{Y}^2] - \frac{\mathbb{E}[\tilde{X}\tilde{Y}]^2}{\mathbb{E}[\tilde{X}^2]}.$$

But from our condition $|\rho_{XY}| = 1$, we have

$$|\mathbb{E}[\tilde{X}\tilde{Y}]| = \sqrt{\mathbb{E}[\tilde{X}^2]} \sqrt{\mathbb{E}[\tilde{Y}^2]} \iff \mathbb{E}[\tilde{X}\tilde{Y}]^2 = \mathbb{E}[\tilde{X}^2]\mathbb{E}[\tilde{Y}^2],$$

which means

$$\mathbb{E}[(\tilde{Y} - t\tilde{X})^2] = \mathbb{E}[\tilde{Y}^2] - \frac{\mathbb{E}[\tilde{X}^2]\mathbb{E}[\tilde{Y}^2]}{\mathbb{E}[\tilde{X}^2]} = 0.$$

Since the expectation of a non-negative random variable is zero, the variable itself must be zero almost surely:

$$\mathbb{E}[(\tilde{Y} - t\tilde{X})^2] = 0 \iff \tilde{Y} - t\tilde{X} = 0 \text{ a.s.} \iff Y - \mathbb{E}[Y] = t(X - \mathbb{E}[X]) \text{ a.s.}$$

So if we set $c = t$ and $b = \mathbb{E}[Y] - c\mathbb{E}[X]$, we get the linear relationship:

$$Y = cX + b \text{ a.s.}$$

5 Problem 5

Problem. The distribution surface (joint PDF) of the two-dimensional random variable (X, Y) is a right circular cone, the base of which is a circle centered at the origin with a unit radius. Outside this circle, the joint PDF of this two-dimensional random variable (X, Y) is zero. Find the joint PDF $f(x, y)$, the marginal PDFs and the conditional PDFs $f_x(y)$ and $f_y(x)$. Are the random variables X and Y dependent and/or correlated?

Solution.

So we have a cone with base as a unit circle. The PDF must have the form $f_{X,Y}(x, y) = c(1-r)\mathbf{1}_{\{r \leq 1\}}$, where $r = \sqrt{x^2 + y^2}$ and c is some constant. Let's find c from the normalization condition.

Converting to polar coordinates:

$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x, y) dx dy = \int_0^{2\pi} \int_0^1 c(1-r) r dr d\theta = 2\pi c \int_0^1 (r - r^2) dr = 2\pi c \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi c}{3},$$

so we have $c = \frac{3}{\pi}$. Therefore:

$$f_{X,Y}(x,y) = \frac{3}{\pi} \left(1 - \sqrt{x^2 + y^2}\right) \mathbf{1}_{\{x^2 + y^2 \leq 1\}}.$$

Now let's find the marginal PDFs. For $|x| \leq 1$, I'll denote $a = \sqrt{1 - x^2}$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \int_{-a}^a \frac{3}{\pi} \left(1 - \sqrt{x^2 + y^2}\right) dy = \frac{6}{\pi} \int_0^a \left(1 - \sqrt{x^2 + y^2}\right) dy = \frac{6}{\pi} \left(a - \int_0^a \sqrt{x^2 + y^2} dy\right).$$

To evaluate this integral, I'll use the antiderivative formula:

$$\int \sqrt{x^2 + y^2} dy = \frac{1}{2} \left(y \sqrt{x^2 + y^2} + x^2 \ln(y + \sqrt{x^2 + y^2}) \right) + C,$$

so we get

$$\int_0^a \sqrt{x^2 + y^2} dy = \frac{1}{2} \left(a \sqrt{x^2 + a^2} + x^2 \ln(a + \sqrt{x^2 + a^2}) - x^2 \ln|x| \right) = \frac{1}{2} \left(a + x^2 \ln(1+a) - x^2 \ln|x| \right),$$

where in the last step I used $\sqrt{x^2 + a^2} = \sqrt{x^2 + 1 - x^2} = 1$. Therefore, for $|x| \leq 1$:

$$f_X(x) = \frac{6}{\pi} \left(a - \frac{1}{2} \left(a + x^2 \ln(1+a) - x^2 \ln|x| \right) \right) = \frac{3}{\pi} \left(a - x^2 \ln \frac{1+a}{|x|} \right), \quad a = \sqrt{1 - x^2},$$

and $f_X(x) = 0$ for $|x| > 1$. At $x = 0$ we need to take a limit:

$$f_X(0) = \lim_{x \rightarrow 0} \frac{3}{\pi} \left(\sqrt{1 - x^2} - x^2 \ln \frac{1 + \sqrt{1 - x^2}}{|x|} \right) = \frac{3}{\pi}.$$

By symmetry, for $|y| \leq 1$ with $b = \sqrt{1 - y^2}$:

$$f_Y(y) = \frac{3}{\pi} \left(b - y^2 \ln \frac{1+b}{|y|} \right), \quad b = \sqrt{1 - y^2}, \quad f_Y(y) = 0 \text{ for } |y| > 1.$$

The conditional PDFs are just the ratio of joint to marginal. For $|x| < 1$:

$$f_x(y) = f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{3}{\pi} \left(1 - \sqrt{x^2 + y^2}\right) \mathbf{1}_{\{x^2 + y^2 \leq 1\}}}{\frac{3}{\pi} \left(\sqrt{1 - x^2} - x^2 \ln \frac{1 + \sqrt{1 - x^2}}{|x|}\right)} = \frac{1 - \sqrt{x^2 + y^2}}{\sqrt{1 - x^2} - x^2 \ln \frac{1 + \sqrt{1 - x^2}}{|x|}} \mathbf{1}_{\{|y| \leq \sqrt{1 - x^2}\}},$$

and $f_x(y) = 0$ otherwise. Similarly, for $|y| < 1$:

$$f_y(x) = f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1 - \sqrt{x^2 + y^2}}{\sqrt{1 - y^2} - y^2 \ln \frac{1 + \sqrt{1 - y^2}}{|y|}} \mathbf{1}_{\{|x| \leq \sqrt{1 - y^2}\}},$$

and $f_y(x) = 0$ otherwise.

Now let's check if X and Y are independent. At the origin:

$$f_{X,Y}(0,0) = \frac{3}{\pi}, \quad f_X(0) = \frac{3}{\pi}, \quad f_Y(0) = \frac{3}{\pi}, \quad f_X(0)f_Y(0) = \frac{9}{\pi^2} \neq \frac{3}{\pi},$$

so X and Y are dependent.

What about correlation? By symmetry:

$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} xf_{X,Y}(x,y) dx dy = \iint_{\mathbb{R}^2} (-x)f_{X,Y}(-x,y) dx dy = -\mathbb{E}[X] \implies \mathbb{E}[X] = 0,$$

$$\mathbb{E}[Y] = \iint_{\mathbb{R}^2} yf_{X,Y}(x,y) dx dy = \iint_{\mathbb{R}^2} (-y)f_{X,Y}(x,-y) dx dy = -\mathbb{E}[Y] \implies \mathbb{E}[Y] = 0,$$

$$\mathbb{E}[XY] = \iint_{\mathbb{R}^2} xyf_{X,Y}(x,y) dx dy = \iint_{\mathbb{R}^2} (-x)yf_{X,Y}(-x,y) dx dy = -\mathbb{E}[XY] \implies \mathbb{E}[XY] = 0,$$

So we have

$$\text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0, \quad \rho_{XY} = 0.$$

So to summarize:

$$f_{X,Y}(x,y) = \frac{3}{\pi} \left(1 - \sqrt{x^2 + y^2}\right) \mathbf{1}_{\{x^2 + y^2 \leq 1\}},$$

$$f_X(x) = \frac{3}{\pi} \left(\sqrt{1-x^2} - x^2 \ln \frac{1+\sqrt{1-x^2}}{|x|} \right) \mathbf{1}_{\{|x| \leq 1\}},$$

$$f_Y(y) = \frac{3}{\pi} \left(\sqrt{1-y^2} - y^2 \ln \frac{1+\sqrt{1-y^2}}{|y|} \right) \mathbf{1}_{\{|y| \leq 1\}},$$

$$f_x(y) = \frac{1 - \sqrt{x^2 + y^2}}{\sqrt{1-x^2} - x^2 \ln \frac{1+\sqrt{1-x^2}}{|x|}} \mathbf{1}_{\{|x| < 1, |y| \leq \sqrt{1-x^2}\}},$$

$$f_y(x) = \frac{1 - \sqrt{x^2 + y^2}}{\sqrt{1-y^2} - y^2 \ln \frac{1+\sqrt{1-y^2}}{|y|}} \mathbf{1}_{\{|y| < 1, |x| \leq \sqrt{1-y^2}\}},$$

X, Y dependent, $\text{Cov}(X, Y) = 0, \quad \rho_{XY} = 0.$

6 Problem 6

Problem. Let X and Y be continuous random variables with a (spherically symmetric) joint PDF of the form $f(x,y) = g(x^2 + y^2)$ for some function g . Let (R, θ) be the polar coordinates of (X, Y) , so that $R^2 = X^2 + Y^2$ is the squared distance from the origin and θ is the angle $\in [0, 2\pi)$, with $X = R \cos \theta$, $Y = R \sin \theta$.

a) Prove that R and θ are independent and explain intuitively why this result makes sense;

b) What is the joint PDF of (R, θ) if (X, Y) is Uniform on the unit disk, i. e. $x^2 + y^2 \leq 1$? If X, Y are i. i. d. $N(0, 1)$?

Solution.

Part (a): Let's convert to polar coordinates: $x = r \cos \theta$, $y = r \sin \theta$ where $r \geq 0$ and $\theta \in [0, 2\pi]$.

The Jacobian of the transformation is

$$\left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r, \quad dx dy = r dr d\theta.$$

Using the change of variables formula:

$$f_{R, \Theta}(r, \theta) = f_{X, Y}(r \cos \theta, r \sin \theta) \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| = g((r \cos \theta)^2 + (r \sin \theta)^2) r = g(r^2) r, \quad r \geq 0, \theta \in [0, 2\pi].$$

Notice that the joint PDF doesn't depend on θ at all! Let me verify this leads to independence. The normalization gives:

$$1 = \int_0^{2\pi} \int_0^\infty f_{R, \Theta}(r, \theta) dr d\theta = \int_0^{2\pi} \int_0^\infty g(r^2) r dr d\theta = 2\pi \int_0^\infty g(r^2) r dr,$$

so

$$\int_0^\infty g(r^2) r dr = \frac{1}{2\pi}.$$

The marginal of Θ is:

$$f_\Theta(\theta) = \int_0^\infty f_{R, \Theta}(r, \theta) dr = \int_0^\infty g(r^2) r dr = \frac{1}{2\pi}, \quad \theta \in [0, 2\pi].$$

The marginal of R is:

$$f_R(r) = \int_0^{2\pi} f_{R, \Theta}(r, \theta) d\theta = \int_0^{2\pi} g(r^2) r d\theta = 2\pi r g(r^2), \quad r \geq 0.$$

We can check:

$$f_R(r) f_\Theta(\theta) = (2\pi r g(r^2)) \left(\frac{1}{2\pi} \right) = g(r^2) r = f_{R, \Theta}(r, \theta),$$

so R and Θ are independent! This makes intuitive sense: a spherically symmetric distribution is rotationally invariant, so the angle should be uniformly distributed and independent of the distance from the origin.

Part (b): Let's apply this to the two specific cases.

Uniform on the unit disk: Here $f_{X, Y}(x, y) = \frac{1}{\pi} \mathbf{1}_{\{x^2 + y^2 \leq 1\}}$, which has the form $g(x^2 + y^2)$ with $g(u) = \frac{1}{\pi} \mathbf{1}_{\{u \leq 1\}}$. So:

$$f_{R, \Theta}(r, \theta) = \frac{r}{\pi} \mathbf{1}_{\{0 \leq r \leq 1\}} \mathbf{1}_{\{0 \leq \theta < 2\pi\}}.$$

The marginals are:

$$f_{\Theta}(\theta) = \int_0^1 \frac{r}{\pi} dr = \frac{1}{2\pi}, \quad 0 \leq \theta < 2\pi,$$

$$f_R(r) = \int_0^{2\pi} \frac{r}{\pi} d\theta = 2r, \quad 0 \leq r \leq 1.$$

i.i.d. $N(0, 1)$: When X, Y are independent standard normals, we have

$$f_{X,Y}(x, y) = \frac{1}{2\pi} \exp\left(-\frac{x^2 + y^2}{2}\right) = \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right),$$

so $g(u) = \frac{1}{2\pi} e^{-u/2}$. This gives:

$$f_{R,\Theta}(r, \theta) = \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) \mathbf{1}_{\{r \geq 0\}} \mathbf{1}_{\{0 \leq \theta < 2\pi\}}.$$

The marginals are:

$$f_{\Theta}(\theta) = \int_0^{\infty} \frac{r}{2\pi} e^{-r^2/2} dr = \frac{1}{2\pi}, \quad 0 \leq \theta < 2\pi,$$

$$f_R(r) = \int_0^{2\pi} \frac{r}{2\pi} e^{-r^2/2} d\theta = r e^{-r^2/2}, \quad r \geq 0.$$

(This is the Rayleigh distribution!)

$f_{R,\Theta}(r, \theta) = g(r^2) r, \quad r \geq 0, 0 \leq \theta < 2\pi,$ $f_{R,\Theta}(r, \theta) = \frac{r}{\pi} \mathbf{1}_{\{0 \leq r \leq 1\}} \mathbf{1}_{\{0 \leq \theta < 2\pi\}},$ $f_{R,\Theta}(r, \theta) = \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) \mathbf{1}_{\{r \geq 0\}} \mathbf{1}_{\{0 \leq \theta < 2\pi\}}.$
