

# Probability Theory Homework 5

Gregory Matsnev

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## 1 Problem 1

**Problem.** Find all moments of the normal distribution using MGF.

**Solution.** Let  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Its MGF is

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

The  $n$ -th raw moment is  $m_n = M_X^{(n)}(0)$ . Differentiating the exponential gives

$$\mathbb{E} X^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)! k! 2^k} \mu^{n-2k} \sigma^{2k},$$

so all odd centered moments vanish and even centered moments are  $\mathbb{E}(X-\mu)^{2k} = \frac{(2k)!}{2^k k!} \sigma^{2k}$ .

## 2 Problem 2

**Problem.** For independent and identically distributed  $X, Y \sim \mathcal{N}(0, 1)$  find  $\mathbb{E}(|X - Y|)$ . Use MGF of independent normal distributions.

**Solution.** The difference  $Z = X - Y$  is normal with mean 0 and variance  $1 + 1 = 2$ , hence  $Z \sim \mathcal{N}(0, 2)$  and  $M_Z(t) = \exp(t^2)$ . For a centered normal  $N(0, \sigma^2)$ ,  $\mathbb{E}|Z| = \sigma\sqrt{2/\pi}$ . Here  $\sigma = \sqrt{2}$ , so

$$\mathbb{E}|X - Y| = \sqrt{2} \sqrt{\frac{2}{\pi}} = \frac{2}{\sqrt{\pi}}.$$

## 3 Problem 3

**Problem.** Let  $X, Y \sim \text{Expo}(1)$  be independent and identically distributed random variables. Find the correlation between  $\max(X, Y)$  and  $\min(X, Y)$ .

**Solution.** Let  $U = \min(X, Y)$  and  $D = \max(X, Y) - \min(X, Y)$ . For exponentials,  $U \sim \text{Expo}(2)$  and  $D \sim \text{Expo}(1)$ , and  $U$  and  $D$  are independent by the memoryless property. Then  $\max(X, Y) = U + D$ .

$$\mathbb{E}U = \frac{1}{2}, \quad \text{Var}(U) = \frac{1}{4}, \quad \mathbb{E}D = 1, \quad \text{Var}(D) = 1.$$

Thus  $\mathbb{E} \max = \frac{3}{2}$  and  $\text{Var}(\max) = \text{Var}(U) + \text{Var}(D) = \frac{5}{4}$ . Since  $D$  is independent of  $U$ ,  $\text{Cov}(\max, \min) = \text{Cov}(U + D, U) = \text{Var}(U) = \frac{1}{4}$ . Therefore

$$\rho(\max, \min) = \frac{\text{Cov}(\max, \min)}{\sqrt{\text{Var}(\max) \text{Var}(\min)}} = \frac{\frac{1}{4}}{\sqrt{(\frac{5}{4})(\frac{1}{4})}} = \frac{1}{\sqrt{5}}.$$

## 4 Problem 4

**Problem.** Consider the Log-Normal distribution  $Y \sim LN(\mu, \sigma^2)$ , where  $Y = e^X$  and  $X \sim \mathcal{N}(\mu, \sigma^2)$ . Check that the MGF of the Log-Normal distribution doesn't exist. Despite this, obtain all moments of the Log-Normal, using the MGF of the Normal.

**Solution.** The putative MGF of  $Y$  is  $M_Y(t) = \mathbb{E}(e^{tY}) = \mathbb{E}(e^{te^X})$ . For any  $t > 0$  the integrand grows like  $\exp(te^x)$  as  $x \rightarrow \infty$ , which dominates the Gaussian decay and makes the integral diverge; no open neighborhood of 0 yields a finite MGF, so it does not exist.

Nevertheless, all moments of  $Y$  exist. For  $k > 0$ ,

$$\mathbb{E}Y^k = \mathbb{E}e^{kX} = M_X(k) = \exp\left(k\mu + \frac{1}{2}k^2\sigma^2\right).$$

Thus every raw moment of the Log-Normal has the above closed form even though its MGF diverges.

## 5 Problem 5

**Problem.** The distribution function of a continuous random variable  $X$ , distributed according to the Cauchy law, is  $F(x) = A + B \arctan \frac{x}{a}$  for  $a > 0$ . Find the constants  $A$  and  $B$ , the PDF, the probability  $\mathbb{P}(-a \leq X \leq a)$ . What are the mathematical expectation and variance of this random variable?

**Solution.** As  $x \rightarrow -\infty$ ,  $\arctan(x/a) \rightarrow -\frac{\pi}{2}$ , so  $0 = \lim_{x \rightarrow -\infty} F(x) = A - \frac{\pi}{2}B$ . As  $x \rightarrow \infty$ ,  $1 = \lim_{x \rightarrow \infty} F(x) = A + \frac{\pi}{2}B$ . Solving gives  $A = \frac{1}{2}$  and  $B = \frac{1}{\pi}$ , yielding

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{a}.$$

Differentiation gives the PDF

$$f(x) = \frac{1}{\pi} \cdot \frac{1/a}{1 + (x/a)^2} = \frac{a}{\pi(a^2 + x^2)}.$$

The probability  $\mathbb{P}(-a \leq X \leq a) = F(a) - F(-a) = \frac{1}{\pi}(\arctan 1 - \arctan(-1)) = \frac{1}{2}$ . The mean and variance of the Cauchy distribution do not exist (they diverge).

## 6 Problem 6

**Problem.** Find the skewness  $A = \mu_3/\sigma^3$  and kurtosis  $E = \mu_4/\sigma^4 - 3$  of a random variable distributed according to the Laplace law with a probability density function  $\phi(x) = \frac{1}{2}e^{-|x|}$ .

**Solution.** This is a Laplace(0, 1) density. It is symmetric, so  $\mu_3 = 0$  and skewness  $A = 0$ . The MGF is  $M(t) = (1 - t^2)^{-1}$  for  $|t| < 1$ , implying raw (and centered) moments  $\mathbb{E}X^{2k} = (2k)!$  and  $\mathbb{E}X^{2k+1} = 0$ . In particular,

$$\mu_2 = \mathbb{E}X^2 = 2, \quad \mu_4 = \mathbb{E}X^4 = 24.$$

Hence  $\sigma^2 = 2$ , so  $\sigma^4 = 4$ , and

$$E = \frac{\mu_4}{\sigma^4} - 3 = \frac{24}{4} - 3 = 3.$$

Thus skewness is 0 and excess kurtosis is 3.