

Probability Theory Homework 1

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1 Problem 1

Problem. Sixteen volleyball teams are randomly divided into two groups of eight. What is the probability that the two best teams are placed in different groups?

Solution. Let's place the best team into an arbitrary group. Among the remaining 15 positions, 7 are in the same group and 8 are in the other group. Therefore, the probability that the second-best team is in the other group is

$$\frac{8}{15}$$

2 Problem 2

Problem. Of the 20 clocks received for repair, 8 require a general cleaning of the mechanism. What is the probability that among 3 clocks taken at random simultaneously, at least two require a general cleaning of the mechanism?

Solution. Let X be the number of clocks among the three that require cleaning. Then X has a hypergeometric distribution with population size 20, number of "successes" 8, and draws 3. We need

$$\begin{aligned}\mathbb{P}(X \geq 2) &= \mathbb{P}(X = 2) + \mathbb{P}(X = 3) \\ &= \frac{\binom{8}{2}\binom{12}{1} + \binom{8}{3}\binom{12}{0}}{\binom{20}{3}} \\ &= \frac{28 \cdot 12 + 56 \cdot 1}{1140} = \frac{392}{1140} = \frac{98}{285}.\end{aligned}$$

3 Problem 3

Problem. There are 20 people at a chess club on a certain day. They each find opponents and start playing. How many possibilities are there for how they are matched up, assuming that in each game it does matter who has the white pieces (in a chess game, one player has the white pieces and the other player has the black pieces)?

Solution. We are counting the number of ways to partition 20 labeled people into 10 ordered pairs (assigning white/black in each game). Arrange the 20 people in order ($20!$ ways) and group them into consecutive pairs (1st plays with 2nd, 3rd plays with 4th, etc.). Let's take the first in each pair to be White and the second to be Black (in standard pair division problem we do not care about the order within the pairs, so we also dividing by 2 in factor of number of pairs, but in this case we do care). We should also take into account pairs permutations by dividing by a factor of $10!$. The final formula for the number of possible matchings would be

$$\frac{20!}{10!} = 670,442,572,800.$$

4 Problem 4

Problem. What is the probability that a point thrown at random into a circle will land inside the square inscribed in it?

Solution. The point is uniformly distributed over the area of the circle, so the probability equals the ratio of the area of the inscribed square to the area of the circle. If the circle has radius R , then the square's diagonal is $2R$, hence its side length is $s = \sqrt{2}R$. Therefore, the square's area is $s^2 = 2R^2$, while the circle's area is πR^2 . Thus,

$$\mathbb{P} = \frac{\text{area of square}}{\text{area of circle}} = \frac{2R^2}{\pi R^2} = \frac{2}{\pi}.$$

5 Problem 5

Problem. Seven passengers were sold 7 tickets in a compartment carriage (9 compartments with 4 seats each). Find the probabilities that the passengers got into:

- a) two compartments;
- b) seven compartments;
- c) three compartments. Be careful with the counting!

Solution. Treat a ticket as specifying a particular seat. Equivalently, we choose uniformly at random a 7-seat subset from the 36 seats in the carriage. The sample space size is $\binom{36}{7}$.

a) Two compartments. To place 7 seats across two compartments with at most 4 per compartment, the occupancies must be 3 and 4. Choose the two compartments $\binom{9}{2}$, choose which one has 4 seats (2 ways), and then choose the

seats inside: $\binom{4}{4}$ and $\binom{4}{3}$. Therefore,

$$\begin{aligned}\mathbb{P}(\text{two compartments}) &= \frac{\binom{9}{2} \cdot 2 \cdot \binom{4}{4} \binom{4}{3}}{\binom{36}{7}} \\ &= \frac{36 \cdot 8}{8,347,680} = \frac{288}{8,347,680}.\end{aligned}$$

b) Seven compartments. Then every chosen compartment contains exactly 1 seat; choose the compartments $\binom{9}{7}$ and the seats $\binom{4}{1}^7 = 4^7$:

$$\begin{aligned}\mathbb{P}(\text{seven compartments}) &= \frac{\binom{9}{7} \cdot 4^7}{\binom{36}{7}} \\ &= \frac{36 \cdot 16384}{8347680} = \frac{589824}{8347680}.\end{aligned}$$

c) Three compartments. The only feasible seat-count patterns are 4–2–1, 3–3–1, and 3–2–2. Choose the three compartments $\binom{9}{3}$, assign which compartment gets which count (3! = 6 ways for 4–2–1; 3 ways for 3–3–1; 3 ways for 3–2–2), and then choose seats inside each used compartment (using $\binom{4}{r}$ for r seats). Therefore,

$$\begin{aligned}\mathbb{P}(\text{three compartments}) &= \frac{\binom{9}{3} \left(6 \binom{4}{4} \binom{4}{2} \binom{4}{1} + 3 \binom{4}{3} \binom{4}{3} \binom{4}{1} + 3 \binom{4}{3} \binom{4}{2} \binom{4}{2} \right)}{\binom{36}{7}} \\ &= \frac{84 \left(6 \cdot 1 \cdot 6 \cdot 4 + 3 \cdot 4 \cdot 4 \cdot 4 + 3 \cdot 4 \cdot 6 \cdot 6 \right)}{8,347,680} \\ &= \frac{84(144 + 192 + 432)}{8,347,680} = \frac{84 \cdot 768}{8,347,680} = \frac{64,512}{8,347,680}.\end{aligned}$$

6 Problem 6

Problem. There are two gold-bearing regions, each divided into four plots. According to forecasts, the first region contains three gold-bearing plots, and the second contains two, but it is not known which ones specifically. A region is chosen at random, and then one plot is randomly purchased within it; this plot turns out to be gold-bearing.

What is the probability of a second successful purchase under the same conditions (i.e., choosing a random plot in the same region)?

Solution. Let R_1 denote the region with 3 gold plots out of 4 and R_2 the region with 2 gold plots out of 4. With priors $\mathbb{P}(R_1) = \mathbb{P}(R_2) = \frac{1}{2}$ and likelihoods $\mathbb{P}(\text{success} \mid R_1) = \frac{3}{4}$, $\mathbb{P}(\text{success} \mid R_2) = \frac{1}{2}$, Bayes' formula after the

first success gives

$$\mathbb{P}(R_1 \mid \text{success}) = \frac{\frac{1}{2} \cdot \frac{3}{4}}{\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{3}{5},$$
$$\mathbb{P}(R_2 \mid \text{success}) = \frac{2}{5}.$$

For the second purchase in the same region without replacement, the success probabilities are $\frac{2}{3}$ in R_1 and $\frac{1}{3}$ in R_2 . Therefore,

$$\mathbb{P}(\text{second success} \mid \text{first success}) = \frac{3}{5} \cdot \frac{2}{3} + \frac{2}{5} \cdot \frac{1}{3} = \frac{8}{15}.$$