Probability Theory Homework 6

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1 Problem 1

Problem. Find the PDF of $Y = 1 - X^3$, where X is the random variable distributed according to the Cauchy law, i. e. with the PDF

$$\phi(x) = \frac{1}{\pi(1+x^2)}$$

Solution. Let's express x in terms of y:

$$y = 1 - x^3 \iff x^3 = 1 - y \iff x = (1-y)^{1/3}.$$

Now we'll find the CDF of Y and then differentiate to get the PDF:

$$F_Y(y) = \mathbb{P}(Y \le y) = \mathbb{P}(1-X^3 \le y) = \mathbb{P}(X^3 \ge 1-y) = \mathbb{P}\Big(X \ge (1-y)^{1/3}\Big) = 1-F_X\left((1-y)^{1/3}\right).$$

$$f_Y(y) = \frac{d}{dy}F_Y(y) = -\frac{d}{dy}F_X\left((1-y)^{1/3}\right) = -f_X\left((1-y)^{1/3}\right)\frac{d}{dy}(1-y)^{1/3}.$$

$$\frac{d}{dy}(1-y)^{1/3} = -\frac{1}{3}(1-y)^{-2/3}, \qquad \left|\frac{d}{dy}(1-y)^{1/3}\right| = \frac{1}{3|1-y|^{2/3}}.$$

So we get

$$f_Y(y) = f_X\left((1-y)^{1/3}\right)\frac{1}{3|1-y|^{2/3}}.$$

Now let's substitute the given PDF of X:

$$f_X\left((1-y)^{1/3}\right) = \frac{1}{\pi\left(1+\left((1-y)^{1/3}\right)^2\right)} = \frac{1}{\pi\left(1+|1-y|^{2/3}\right)}.$$

Finally:

$$f_Y(y) = \frac{1}{\pi\left(1+|1-y|^{2/3}\right)}\cdot\frac{1}{3|1-y|^{2/3}} = \boxed{\frac{1}{3\pi\,|1-y|^{2/3}\left(1+|1-y|^{2/3}\right)}}, \qquad y \in \mathbb{R}.$$

2 Problem 2

Problem. Find the expected value and the variance of the random variable $Y = 2 - 3\sin X$, given that the PDF of X is

$$\phi(x) = \frac{1}{2}\cos x \text{ for } x \in [-\pi/2, \pi/2]$$

Solution.
Let's find $\mathbb{E}[Y]$:

$$\mathbb{E}[Y] = \mathbb{E}[2-3\sin X] = \int_{-\pi/2}^{\pi/2}(2-3\sin x)f_X(x)\,dx = \int_{-\pi/2}^{\pi/2}(2-3\sin x)\frac{1}{2}\cos x\,dx$$

$$= \int_{-\pi/2}^{\pi/2}\cos x\,dx - \frac{3}{2}\int_{-\pi/2}^{\pi/2}\sin x\cos x\,dx = \Big[\sin x\Big]_{-\pi/2}^{\pi/2} - \frac{3}{2}\Big[\frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 2.$$

To find the variance, we need $\mathbb{E}[Y^2]$:

$$\mathbb{E}[Y^2] = \mathbb{E}[(2-3\sin X)^2] = \int_{-\pi/2}^{\pi/2}(2-3\sin x)^2 f_X(x)\,dx = \int_{-\pi/2}^{\pi/2}(4-12\sin x+9\sin^2 x)\frac{1}{2}\cos x\,dx$$

$$= 2\int_{-\pi/2}^{\pi/2}\cos x\,dx - 6\int_{-\pi/2}^{\pi/2}\sin x\cos x\,dx + \frac{9}{2}\int_{-\pi/2}^{\pi/2}\sin^2 x\cos x\,dx.$$

$$2\int_{-\pi/2}^{\pi/2}\cos x\,dx = 2\Big[\sin x\Big]_{-\pi/2}^{\pi/2} = 4, \qquad -6\int_{-\pi/2}^{\pi/2}\sin x\cos x\,dx = -6\Big[\frac{1}{2}\sin^2 x\Big]_{-\pi/2}^{\pi/2} = 0.$$

For the last integral, we will use substitution $u = \sin x$, so $du = \cos x\,dx$:

$$\frac{9}{2}\int_{-\pi/2}^{\pi/2}\sin^2 x\cos x\,dx = \frac{9}{2}\int_{-1}^{1}u^2\,du = \frac{9}{2}\Big[\frac{u^3}{3}\Big]_{-1}^{1} = 3.$$

So we have

$$\mathbb{E}[Y^2] = 4 + 0 + 3 = 7,$$

and therefore

$$\boxed{\mathbb{E}[Y] = 2, \qquad \operatorname{Var}(Y) = \mathbb{E}[Y^2] - \big(\mathbb{E}[Y]\big)^2 = 7 - 4 = 3.}$$

3 Problem 3

Problem. The random variable X is defined on the entire real axis with the probability density $\phi(x) = \frac{1}{2}e^{-|x|}$. Find the probability density of the random variable $Y = X^2$ and its mathematical expectation.

Solution.

Since $Y \geq 0$ and for any $y > 0$ the equation $y = x^2$ has two solutions $x = \pm\sqrt{y}$, we can use the change-of-variables formula:

$$f_Y(y) = \sum_{x:\, x^2=y} f_X(x) \left|\frac{dx}{dy}\right| = f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}, \qquad y > 0.$$

Because of the symmetry of f_X, we have $f_X(\sqrt{y}) = f_X(-\sqrt{y}) = \frac{1}{2}e^{-\sqrt{y}}$. Therefore:

$$\boxed{f_Y(y) = \begin{cases} \frac{e^{-\sqrt{y}}}{2\sqrt{y}}, & y > 0, \\ 0, & y \leq 0. \end{cases}}$$

The density has an integrable singularity at $y = 0$.

For the expectation:

$$\mathbb{E}[Y] = \mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{2}e^{-|x|}\, dx = \int_0^{\infty} x^2 e^{-x}\, dx = \boxed{2}.$$

4 Problem 4

Problem. Prove formally that if the correlation coefficient ρ_{XY} of two random variables X and Y is equal in absolute value to one, then there is a linear functional relationship between these random variables.

Remember how to prove that $Cov(X, Y) \leq \sigma_X \sigma_Y$.

Solution.

Let's assume $\sigma_X > 0$ and $\sigma_Y > 0$ (otherwise the correlation isn't defined). Lets denote the centered variables as $\widetilde{X} = X - \mathbb{E}[X]$ and $\widetilde{Y} = Y - \mathbb{E}[Y]$.

The correlation coefficient can be written as

$$\rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]}{\sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]}}.$$

We know from the Cauchy–Schwarz inequality that

$$\left|\mathbb{E}[\widetilde{X}\widetilde{Y}]\right| \leq \sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]}.$$

Now, if $|\rho_{XY}| = 1$, this means we have equality in Cauchy–Schwarz:

$$|\rho_{XY}| = 1 \iff \left|\mathbb{E}[\widetilde{X}\widetilde{Y}]\right| = \sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]}.$$

Lets consider the quadratic form for any $t \in \mathbb{R}$:

$$0 \leq \mathbb{E}[(\widetilde{Y} - t\widetilde{X})^2] = \mathbb{E}[\widetilde{Y}^2] - 2t\,\mathbb{E}[\widetilde{X}\widetilde{Y}] + t^2\mathbb{E}[\widetilde{X}^2].$$

To find the minimum, let's differentiate with respect to t:

$$\frac{d}{dt}\Big(\mathbb{E}[\widetilde{Y}^2] - 2t\,\mathbb{E}[\widetilde{X}\widetilde{Y}] + t^2\mathbb{E}[\widetilde{X}^2]\Big) = -2\mathbb{E}[\widetilde{X}\widetilde{Y}] + 2t\,\mathbb{E}[\widetilde{X}^2],$$

Setting this to zero gives us

$$t = \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]}{\mathbb{E}[\widetilde{X}^2]}.$$

Plugging this back in:

$$\mathbb{E}[(\widetilde{Y} - t\widetilde{X})^2] = \mathbb{E}[\widetilde{Y}^2] - \frac{\mathbb{E}[\widetilde{X}\widetilde{Y}]^2}{\mathbb{E}[\widetilde{X}^2]}.$$

But from our condition $|\rho_{XY}| = 1$, we have

$$\left|\mathbb{E}[\widetilde{X}\widetilde{Y}]\right| = \sqrt{\mathbb{E}[\widetilde{X}^2]}\sqrt{\mathbb{E}[\widetilde{Y}^2]} \iff \mathbb{E}[\widetilde{X}\widetilde{Y}]^2 = \mathbb{E}[\widetilde{X}^2]\mathbb{E}[\widetilde{Y}^2],$$

which means

$$\mathbb{E}[(\widetilde{Y} - t\widetilde{X})^2] = \mathbb{E}[\widetilde{Y}^2] - \frac{\mathbb{E}[\widetilde{X}^2]\mathbb{E}[\widetilde{Y}^2]}{\mathbb{E}[\widetilde{X}^2]} = 0.$$

Since the expectation of a non-negative random variable is zero, the variable itself must be zero almost surely:

$$\mathbb{E}[(\widetilde{Y} - t\widetilde{X})^2] = 0 \iff \widetilde{Y} - t\widetilde{X} = 0 \text{ a.s.} \iff Y - \mathbb{E}[Y] = t\,(X - \mathbb{E}[X]) \text{ a.s.}$$

So if we set $c = t$ and $b = \mathbb{E}[Y] - c\,\mathbb{E}[X]$, we get the linear relationship:

$$Y = cX + b \text{ a.s.}$$

5 Problem 5

Problem. The distribution surface (joint PDF) of the two-dimensional random variable (X, Y) is a right circular cone, the base of which is a circle centered at the origin with a unit radius. Outside this circle, the joint PDF of this two-dimensional random variable (X, Y) is zero. Find the joint PDF $f(x, y)$, the marginal PDFs and the conditional PDFs $f_x(y)$ and $f_y(x)$. Are the random variables X and Y dependent and/or correlated?

Solution.

Converting to polar coordinates:

$$1 = \iint_{\mathbb{R}^2} f_{X,Y}(x, y)\,dx\,dy = \int_0^{2\pi}\int_0^1 c(1-r)\,r\,dr\,d\theta = 2\pi c\int_0^1 (r-r^2)\,dr = 2\pi c\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi c}{3},$$

so we have $c = \frac{3}{\pi}$. Therefore:

$$f_{X,Y}(x, y) = \frac{3}{\pi}\left(1 - \sqrt{x^2 + y^2}\right)\mathbf{1}_{\{x^2+y^2\leq 1\}}.$$

Now let's find the marginal PDFs. For $|x| \leq 1$, we will denote $a = \sqrt{1 - x^2}$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dy = \int_{-a}^{a} \frac{3}{\pi}\left(1-\sqrt{x^2 + y^2}\right)dy = \frac{6}{\pi}\int_0^a \left(1-\sqrt{x^2 + y^2}\right)dy = \frac{6}{\pi}\left(a - \int_0^a \sqrt{x^2 + y^2}\right.$$

To evaluate this integral, we will use the antiderivative formula:

$$\int \sqrt{x^2+y^2}\,dy = \frac{1}{2}\Big(y\sqrt{x^2+y^2} + x^2 \ln\big(y+\sqrt{x^2+y^2}\big)\Big) + C,$$

so we get

$$\int_0^a \sqrt{x^2+y^2}\,dy = \frac{1}{2}\Big(a\sqrt{x^2+a^2}+x^2\ln(a+\sqrt{x^2+a^2})-x^2\ln|x|\Big) = \frac{1}{2}\Big(a+x^2\ln(1+a)-x^2\ln|x|\Big),$$

where $\sqrt{x^2+a^2} = \sqrt{x^2+1-x^2} = 1$. Therefore, for $|x| \le 1$:

$$f_X(x) = \frac{6}{\pi}\left(a - \frac{1}{2}\Big(a + x^2\ln(1+a) - x^2\ln|x|\Big)\right) = \frac{3}{\pi}\left(a - x^2 \ln\frac{1+a}{|x|}\right), \qquad a = \sqrt{1-x^2},$$

and $f_X(x) = 0$ for $|x| > 1$. At $x = 0$ we need to take a limit:

$$f_X(0) = \lim_{x\to 0} \frac{3}{\pi}\left(\sqrt{1-x^2} - x^2\ln\frac{1+\sqrt{1-x^2}}{|x|}\right) = \frac{3}{\pi}.$$

By symmetry, for $|y| \le 1$ with $b = \sqrt{1-y^2}$:

$$f_Y(y) = \frac{3}{\pi}\left(b - y^2\ln\frac{1+b}{|y|}\right), \qquad b = \sqrt{1-y^2}, \qquad f_Y(y) = 0 \text{ for } |y| > 1.$$

The conditional PDFs are just the ratio of joint to marginal. For $|x| < 1$:

$$f_x(y) = f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{3}{\pi}\left(1-\sqrt{x^2+y^2}\right)\mathbf{1}_{\{x^2+y^2\le 1\}}}{\frac{3}{\pi}\left(\sqrt{1-x^2} - x^2\ln\frac{1+\sqrt{1-x^2}}{|x|}\right)} = \frac{1-\sqrt{x^2+y^2}}{\sqrt{1-x^2}-x^2\ln\frac{1+\sqrt{1-x^2}}{|x|}}\,\mathbf{1}_{\{|y|\le\sqrt{1-x^2}\}}$$

and $f_x(y) = 0$ otherwise. Similarly, for $|y| < 1$:

$$f_y(x) = f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1-\sqrt{x^2+y^2}}{\sqrt{1-y^2}-y^2\ln\frac{1+\sqrt{1-y^2}}{|y|}}\,\mathbf{1}_{\{|x|\le\sqrt{1-y^2}\}},$$

and $f_y(x) = 0$ otherwise.

Now let's check if X and Y are independent. At the origin:

$$f_{X,Y}(0,0) = \frac{3}{\pi}, \qquad f_X(0) = \frac{3}{\pi}, \qquad f_Y(0) = \frac{3}{\pi}, \qquad f_X(0)f_Y(0) = \frac{9}{\pi^2} \ne \frac{3}{\pi},$$

so X and Y are dependent.

What about correlation? By symmetry:

$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} x f_{X,Y}(x,y)\,dx\,dy = \iint_{\mathbb{R}^2} (-x) f_{X,Y}(-x,y)\,dx\,dy = -\mathbb{E}[X] \Longrightarrow \mathbb{E}[X] = 0,$$

$$\mathbb{E}[Y] = \iint_{\mathbb{R}^2} y f_{X,Y}(x,y)\,dx\,dy = \iint_{\mathbb{R}^2} (-y) f_{X,Y}(x,-y)\,dx\,dy = -\mathbb{E}[Y] \Longrightarrow \mathbb{E}[Y] = 0,$$

$$\mathbb{E}[XY] = \iint_{\mathbb{R}^2} xy f_{X,Y}(x,y)\,dx\,dy = \iint_{\mathbb{R}^2} (-x)y f_{X,Y}(-x,y)\,dx\,dy = -\mathbb{E}[XY] \Longrightarrow \mathbb{E}[XY] = 0,$$

So we have

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0, \qquad \rho_{XY} = 0.$$

So to summarize:

$$\boxed{\begin{aligned}
&f_{X,Y}(x,y) = \frac{3}{\pi}\left(1 - \sqrt{x^2+y^2}\right)\mathbf{1}_{\{x^2+y^2\le 1\}},\\
&f_X(x) = \frac{3}{\pi}\left(\sqrt{1-x^2} - x^2 \ln\frac{1+\sqrt{1-x^2}}{|x|}\right)\mathbf{1}_{\{|x|\le 1\}},\\
&f_Y(y) = \frac{3}{\pi}\left(\sqrt{1-y^2} - y^2 \ln\frac{1+\sqrt{1-y^2}}{|y|}\right)\mathbf{1}_{\{|y|\le 1\}},\\
&f_x(y) = \frac{1 - \sqrt{x^2+y^2}}{\sqrt{1-x^2} - x^2 \ln\frac{1+\sqrt{1-x^2}}{|x|}}\mathbf{1}_{\{|x|<1,\ |y|\le\sqrt{1-x^2}\}},\\
&f_y(x) = \frac{1 - \sqrt{x^2+y^2}}{\sqrt{1-y^2} - y^2 \ln\frac{1+\sqrt{1-y^2}}{|y|}}\mathbf{1}_{\{|y|<1,\ |x|\le\sqrt{1-y^2}\}},\\
&X, Y \text{ dependent}, \qquad \mathrm{Cov}(X,Y) = 0, \qquad \rho_{XY} = 0.
\end{aligned}}$$

6 Problem 6

Problem. Let X and Y be continuous random variables with a (spherically symmetric) joint PDF of the form $f(x,y) = g(x^2+y^2)$ for some function g. Let (R,θ) be the polar coordinates of (X,Y), so that $R^2 = X^2+Y^2$ is the squared distance from the origin and θ is the angle $\in [0,2\pi)$, with $X = R\cos\theta$, $Y = R\sin\theta$.

a) Prove that R and θ are independent and explain intuitively why this result makes sense;

b) What is the joint PDF of (R,θ) if (X,Y) is Uniform on the unit disk, i. e. $x^2+y^2 \le 1$? If X,Y are i. i. d. $N(0,1)$?

Solution.

Part (a): Let's convert to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$ where $r \ge 0$ and $\theta \in [0,2\pi)$.

The Jacobian of the transformation is

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r, \qquad dx\,dy = r\,dr\,d\theta.$$

Using the change of variables formula:

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = g\big((r\cos\theta)^2 + (r\sin\theta)^2\big)\ r = g(r^2)\,r, \qquad r\ge 0,\ \theta\in[0,2\pi).$$

Notice that the joint PDF doesn't depend on θ at all. Let's verify this leads to independence. The normalization gives:

$$1 = \int_0^{2\pi} \int_0^{\infty} f_{R,\Theta}(r,\theta)\, dr\, d\theta = \int_0^{2\pi} \int_0^{\infty} g(r^2)\, r\, dr\, d\theta = 2\pi \int_0^{\infty} g(r^2)\, r\, dr,$$

so

$$\int_0^{\infty} g(r^2)\, r\, dr = \frac{1}{2\pi}.$$

The marginal of Θ is:

$$f_{\Theta}(\theta) = \int_0^{\infty} f_{R,\Theta}(r,\theta)\, dr = \int_0^{\infty} g(r^2)\, r\, dr = \frac{1}{2\pi}, \qquad \theta \in [0, 2\pi).$$

The marginal of R is:

$$f_R(r) = \int_0^{2\pi} f_{R,\Theta}(r,\theta)\, d\theta = \int_0^{2\pi} g(r^2)\, r\, d\theta = 2\pi r g(r^2), \qquad r \geq 0.$$

We can check:

$$f_R(r) f_{\Theta}(\theta) = \big(2\pi r g(r^2)\big) \left(\frac{1}{2\pi}\right) = g(r^2)\, r = f_{R,\Theta}(r,\theta),$$

so R and Θ are independent. Since the distribution is spherically symmetric, it is rotationally invariant, which means the angle is uniformly distributed and independent of the distance from the origin.

Part (b): Let's apply this to the two specific cases.

Uniform on the unit disk: Here $f_{X,Y}(x,y) = \frac{1}{\pi} \mathbf{1}_{\{x^2+y^2 \leq 1\}}$, which has the form $g(x^2+y^2)$ with $g(u) = \frac{1}{\pi}\mathbf{1}_{\{u \leq 1\}}$. So:

$$f_{R,\Theta}(r,\theta) = \frac{r}{\pi} \mathbf{1}_{\{0 \leq r \leq 1\}} \mathbf{1}_{\{0 \leq \theta < 2\pi\}}.$$

The marginals are:

$$f_{\Theta}(\theta) = \int_0^1 \frac{r}{\pi}\, dr = \frac{1}{2\pi}, \qquad 0 \leq \theta < 2\pi,$$

$$f_R(r) = \int_0^{2\pi} \frac{r}{\pi}\, d\theta = 2r, \qquad 0 \leq r \leq 1.$$

i.i.d. $N(0,1)$: When X, Y are independent standard normals, we have

$$f_{X,Y}(x,y) = \frac{1}{2\pi} \exp\left(-\frac{x^2+y^2}{2}\right) = \frac{1}{2\pi} \exp\left(-\frac{r^2}{2}\right),$$

so $g(u) = \frac{1}{2\pi} e^{-u/2}$. This gives:

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi} \exp\left(-\frac{r^2}{2}\right) \mathbf{1}_{\{r \geq 0\}} \mathbf{1}_{\{0 \leq \theta < 2\pi\}}.$$

The marginals are:

$$f_\Theta(\theta) = \int_0^\infty \frac{r}{2\pi} e^{-r^2/2}\, dr = \frac{1}{2\pi}, \qquad 0 \le \theta < 2\pi,$$

$$f_R(r) = \int_0^{2\pi} \frac{r}{2\pi} e^{-r^2/2}\, d\theta = re^{-r^2/2}, \qquad r \ge 0.$$

$$\boxed{\begin{aligned} f_{R,\Theta}(r,\theta) &= g(r^2)\, r, \qquad r \ge 0,\ 0 \le \theta < 2\pi, \\ f_{R,\Theta}(r,\theta) &= \frac{r}{\pi}\mathbf{1}_{\{0\le r\le 1\}}\mathbf{1}_{\{0\le\theta<2\pi\}}, \\ f_{R,\Theta}(r,\theta) &= \frac{r}{2\pi}\exp\!\left(-\frac{r^2}{2}\right)\mathbf{1}_{\{r\ge 0\}}\mathbf{1}_{\{0\le\theta<2\pi\}}. \end{aligned}}$$