Probability Theory Homework 3

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1 Problem 1

Problem. Let $X \sim \mathrm{Unif}(-1, 3)$. Find the probability density function (PDF) of $Y = X^2$.

Solution. The density of X is $f_X(x) = \frac{1}{4}$ on $[-1, 3]$. For $y > 0$, the equation $x^2 = y$ has roots $x = \pm\sqrt{y}$. Both lie in $[-1, 3]$ when $0 < y \leq 1$; only $+\sqrt{y}$ lies in the support when $1 < y \leq 9$. By the change-of-variable formula,

$$f_Y(y) = \sum_{x:\, x^2 = y} \frac{f_X(x)}{|2x|} = \begin{cases} \dfrac{1}{4\sqrt{y}}, & 0 < y < 1, \\ \dfrac{1}{8\sqrt{y}}, & 1 < y < 9, \\ 0, & \text{otherwise.} \end{cases}$$

(The value at the single point $y = 1$ is immaterial for a density; setting $f_Y(1) = \frac{1}{4}$ keeps the first branch continuous.)

2 Problem 2

Problem. Let $X \sim \mathrm{Bin}(n, p)$. Find the probability mass function (PMF) of $Y = e^X$.

Solution. The random variable Y takes the discrete values e^k for $k = 0, 1, \ldots, n$. Since $Y = e^X$ is a bijection on the support of X,

$$\Pr(Y = e^k) = \Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

3 Problem 3

Problem. Let X be the number of failures in i.i.d. Bernoulli trials with success probability p, when sampling continues until r successes have been observed. Find the PMF of X, and recall the name of this distribution.

Solution. To stop with exactly x failures, the sample must consist of $x+r-1$ trials containing $r-1$ successes and x failures, followed by a final success. The number of ways to position the first $r-1$ successes is $\binom{x+r-1}{r-1}$. Therefore

$$\Pr(X = x) = \binom{x+r-1}{r-1}(1-p)^x p^r, \qquad x = 0, 1, 2, \ldots$$

This is the (Pascal) negative binomial distribution with parameters r and p.

4 Problem 4

Problem. An airline sells 110 tickets for a flight that has 100 seats. Each ticketed passenger shows up independently with probability 0.9. Find the probability that everyone who shows up can be seated.

Solution. Let S be the number of passengers who show up. Then $S \sim \text{Bin}(110, 0.9)$, and seats suffice precisely when $S \leq 100$. Hence

$$\Pr(S \leq 100) = \sum_{k=0}^{100} \binom{110}{k}(0.9)^k(0.1)^{110-k} \approx 0.6710.$$

(Equivalently, one may compute $1 - \sum_{k=101}^{110} \binom{110}{k}(0.9)^k(0.1)^{110-k}$.)

5 Problem 5

Problem. Assume $X \sim \text{HGeom}(w, b, n)$, with total population $N = w + b$. If $N \to \infty$ while the success proportion $p = \frac{w}{N}$ stays fixed, prove that the PMF of X converges to that of $\text{Bin}(n, p)$.

Solution. For $k = 0, 1, \ldots, n$,

$$\Pr(X = k) = \frac{\binom{w}{k}\binom{b}{n-k}}{\binom{N}{n}}.$$

Writing the binomial coefficients with falling factorials $(a)_m = a(a-1)\cdots(a-m+1)$ yields

$$\Pr(X = k) = \binom{n}{k}\left[\prod_{i=0}^{k-1} \frac{w-i}{N-i}\right]\left[\prod_{j=0}^{n-k-1} \frac{b-j}{N-k-j}\right].$$

As $N \to \infty$ with $w/N \to p$ (hence $b/N \to 1-p$), every factor satisfies $\frac{w-i}{N-i} \to p$ and $\frac{b-j}{N-k-j} \to 1-p$. Taking the product yields

$$\Pr(X = k) \longrightarrow \binom{n}{k}p^k(1-p)^{n-k},$$

which is exactly the PMF of $\text{Bin}(n, p)$.

6 Problem 6

Problem. Let X and Y be independent random variables with distribution functions F_X and F_Y, respectively. Find the cumulative distribution functions (CDFs) of $Z_1 = \max(X, Y)$ and $Z_2 = \min(X, Y)$.

Solution. For any real z,

$$\begin{aligned} F_{Z_1}(z) &= \Pr(\max(X, Y) \le z) \\ &= \Pr(X \le z,\, Y \le z) \\ &= F_X(z)F_Y(z), \end{aligned}$$

because the events are independent. Similarly,

$$\begin{aligned} F_{Z_2}(z) &= \Pr(\min(X, Y) \le z) \\ &= 1 - \Pr(X > z,\, Y > z) \\ &= 1 - (1 - F_X(z))(1 - F_Y(z)). \end{aligned}$$