

Probability Theory Homework 5

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1 Problem 1

Problem. Find all moments of the normal distribution using MGF.

Solution. Let $X \sim \mathcal{N}(\mu, \sigma^2)$. Its MGF is

$$M_X(t) = \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right).$$

The n -th raw moment is $m_n = M_X^{(n)}(0)$. Differentiating the exponential gives

$$\mathbb{E} X^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)! k! 2^k} \mu^{n-2k} \sigma^{2k},$$

so all odd centered moments vanish and even centered moments are $\mathbb{E}(X-\mu)^{2k} = \frac{(2k)!}{2^k k!} \sigma^{2k}$.

2 Problem 2

Problem. For independent and identically distributed $X, Y \sim \mathcal{N}(0, 1)$ find $\mathbb{E}(|X - Y|)$. Use MGF of independent normal distributions.

Solution. The difference $Z = X - Y$ is normal with mean 0 and variance $1 + 1 = 2$, hence $Z \sim \mathcal{N}(0, 2)$ and $M_Z(t) = \exp(t^2)$. For a centered normal $N(0, \sigma^2)$, $\mathbb{E}|Z| = \sigma \sqrt{2/\pi}$. Here $\sigma = \sqrt{2}$, so

$$\mathbb{E}|X - Y| = \sqrt{2} \sqrt{\frac{2}{\pi}} = \frac{2}{\sqrt{\pi}}.$$

3 Problem 3

Problem. Let $X, Y \sim \text{Expo}(1)$ be independent and identically distributed random variables. Find the correlation between $\max(X, Y)$ and $\min(X, Y)$.

Solution. Let $U = \min(X, Y)$ and $D = \max(X, Y) - \min(X, Y)$. For exponentials, $U \sim \text{Expo}(2)$ and $D \sim \text{Expo}(1)$, and U and D are independent by the memoryless property. Then $\max(X, Y) = U + D$.

$$\mathbb{E}U = \frac{1}{2}, \quad \text{Var}(U) = \frac{1}{4}, \quad \mathbb{E}D = 1, \quad \text{Var}(D) = 1.$$

Thus $\mathbb{E}\max = \frac{3}{2}$ and $\text{Var}(\max) = \text{Var}(U) + \text{Var}(D) = \frac{5}{4}$. Since D is independent of U , $\text{Cov}(\max, \min) = \text{Cov}(U + D, U) = \text{Var}(U) = \frac{1}{4}$. Therefore

$$\rho(\max, \min) = \frac{\text{Cov}(\max, \min)}{\sqrt{\text{Var}(\max) \text{Var}(\min)}} = \frac{\frac{1}{4}}{\sqrt{(\frac{5}{4})(\frac{1}{4})}} = \frac{1}{\sqrt{5}}.$$

4 Problem 4

Problem. Consider the Log-Normal distribution $Y \sim LN(\mu, \sigma^2)$, where $Y = e^X$ and $X \sim \mathcal{N}(\mu, \sigma^2)$. Check that the MGF of the Log-Normal distribution doesn't exist. Despite this, obtain all moments of the Log-Normal, using the MGF of the Normal.

Solution. The putative MGF of Y is $M_Y(t) = \mathbb{E}(e^{tY}) = \mathbb{E}(e^{te^X})$. For any $t > 0$ the integrand grows like $\exp(te^x)$ as $x \rightarrow \infty$, which dominates the Gaussian decay and makes the integral diverge; no open neighborhood of 0 yields a finite MGF, so it does not exist.

Nevertheless, all moments of Y exist. For $k > 0$,

$$\mathbb{E}Y^k = \mathbb{E}e^{kX} = M_X(k) = \exp(k\mu + \frac{1}{2}k^2\sigma^2).$$

Thus every raw moment of the Log-Normal has the above closed form even though its MGF diverges.

5 Problem 5

Problem. The distribution function of a continuous random variable X , distributed according to the Cauchy law, is $F(x) = A + B \arctan \frac{x}{a}$ for $a > 0$. Find the constants A and B , the PDF, the probability $\mathbb{P}(-a \leq X \leq a)$. What are the mathematical expectation and variance of this random variable?

Solution. As $x \rightarrow -\infty$, $\arctan(x/a) \rightarrow -\frac{\pi}{2}$, so $0 = \lim_{x \rightarrow -\infty} F(x) = A - \frac{\pi}{2}B$. As $x \rightarrow \infty$, $1 = \lim_{x \rightarrow \infty} F(x) = A + \frac{\pi}{2}B$. Solving gives $A = \frac{1}{2}$ and $B = \frac{1}{\pi}$, yielding

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{x}{a}.$$

Differentiation gives the PDF

$$f(x) = \frac{1}{\pi} \cdot \frac{1/a}{1 + (x/a)^2} = \frac{a}{\pi(a^2 + x^2)}.$$

The probability $\mathbb{P}(-a \leq X \leq a) = F(a) - F(-a) = \frac{1}{\pi}(\arctan 1 - \arctan(-1)) = \frac{1}{2}$. The mean and variance of the Cauchy distribution do not exist (they diverge).

6 Problem 6

Problem. Find the skewness $A = \mu_3/\sigma^3$ and kurtosis $E = \mu_4/\sigma^4 - 3$ of a random variable distributed according to the Laplace law with a probability density function $\phi(x) = \frac{1}{2}e^{-|x|}$.

Solution. This is a $\text{Laplace}(0, 1)$ density. It is symmetric, so $\mu_3 = 0$ and skewness $A = 0$. The MGF is $M(t) = (1 - t^2)^{-1}$ for $|t| < 1$, implying raw (and centered) moments $\mathbb{E}X^{2k} = (2k)!$ and $\mathbb{E}X^{2k+1} = 0$. In particular,

$$\mu_2 = \mathbb{E}X^2 = 2, \quad \mu_4 = \mathbb{E}X^4 = 24.$$

Hence $\sigma^2 = 2$, so $\sigma^4 = 4$, and

$$E = \frac{\mu_4}{\sigma^4} - 3 = \frac{24}{4} - 3 = 3.$$

Thus skewness is 0 and excess kurtosis is 3.