Revised algorithm description section

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1 ISA framework, formal description and implementation

In this work, we propose an approximate quantum state preparation framework, which we call "Iterative Sparse Approximation" (ISA). Let the target state be $|x_0\rangle$. Then the ISA framework transforms $|x_0\rangle$ to $|x_1\rangle$, then $|x_1\rangle$ to $|x_2\rangle$, and so on, until it reaches some state $|x_m\rangle$, such that

$$|\langle 0|x_0\rangle|^2 < |\langle 0|x_1\rangle|^2 < ... < |\langle 0|x_m\rangle|^2$$

And

$$|\langle 0|x_m\rangle|^2 \geq 1 - \epsilon$$

Each transformation from $|x_i\rangle$ to $|x_{i+1}\rangle$ generates a state closer to $|0\rangle$; these steps are called "refinements." The sequence of gates used in each refinement step are combined to create the state preparation sequence for $|x_0\rangle$.

To refine $|x\rangle$, the ISA framework selects an index subset $K \subset \{0, , , 2^n - 1\}$ and constructs the sparse, usually non-normalized approximation

$$|x\rangle \approx |x'\rangle = \sum_{k \in K} \langle k|x\rangle$$

If $|x\rangle = \sum_{i=0}^{2^n-1} c_i |i\rangle$, then $|x'\rangle = \sum_{k \in K} c_k |k\rangle$, showing that $|x'\rangle$ has at most $|K|$ non-zero elements and is therefore sparse.

Next, the sparse state preparation method for $|x'\rangle$ is adapted towards the refinement of $|x\rangle$. Because $|x'\rangle$ approximates $|x\rangle$, the state preparation sequence for $|x'\rangle$ approximately prepares $|x\rangle$. However, $|x'\rangle$ and $|x\rangle$ are not the same state, motivating the need for an adaptation step. Index subset selection K ("Select"), constructing and preparing $|x'\rangle$ ("Prepare"), and adapting the state preparation sequence for $|x'\rangle$ to refine $|x\rangle$ ("Adapt") constitute the central steps in the ISA framework, as shown in Algorithm 1.

Algorithm 1 ISA Framework

Target state $|x\rangle$
$i \leftarrow 0$
output $\leftarrow []$
$|x_0\rangle \leftarrow |x\rangle$
while $|\langle 0|x_i\rangle|^2 < 1 - \epsilon$ **do**
    $K \leftarrow \text{Select}(|x_i\rangle)$
    $|x_i'\rangle \leftarrow \sum_{k \in K} \langle k|x_i\rangle$
    $g \leftarrow \text{Prepare}(|x_i'\rangle)$
    $g' \leftarrow \text{Adapt}(g, |x_i\rangle)$
    $|x_{i+1}\rangle \leftarrow g' |x_i\rangle$
    output $+= g'$
    $i \leftarrow i + 1$
end while
return output

In the remainder of this section, we describe our specific implementation of the ISA framework for linear nearest neighbor architectures. In section x.1, we describe the two types of index subsets K considered for the Select step, as well as the procedure for choosing one. In section x.2, we describe our implementation of the Prepare step for each index subset type. In section x.3, describe our implementation of the Adapt step. Finally, in section x.4, we describe an additional step in our implementation, called "Refinement by RZ-RY," in which the starting state $|x_0\rangle$ is first refined to $|x_1\rangle$ using RZ and RY gates before applying the ISA framework.

1.1 Index subsets and implementing Select

Type 1 index subsets are those of the form:

$$K(b) = \{0, b\}$$

For some b between 1 and $2^n - 1$ inclusive.

Type 2 index subsets take one of two different forms:

$$K^{(+)}(a, p) = \{a * 2^{p+2} + i * 2^p | 0 \leq i \leq 3\} \cup \{i * 2^p | 0 \leq i \leq 3\}$$

$$K^{(-)}(a, p) = \{a + i * 2^p | 0 \leq i \leq 3\} \cup \{i * 2^p | 0 \leq i \leq 3\}$$

For some a and p. For the $K^{(+)}$ case, $0 \leq a \leq 2^{n-p-2} - 1$; in the $K^{(-)}$ case, $0 \leq a \leq 2^p - 1$. In both cases, $0 \leq p \leq n - 2$. When written in binary, the indices in $K^{(+)}$ subsets are those where the left-most $n - p - 2$ bits are either a or zeroes, the right-most p bits are zeroes, and the middle two bits can be any length-2 bitstring, and the indices in $K^{(-)}$ subsets are those where the left-most $n - p - 2$ bits are zeroes, the right-most p bits are either a or zeroes, and the middle two bits can be any length-2 bitstring.

When refining a state $|x\rangle$, each subset K corresponds to a non-normalized sparse approximation $|x'(K)\rangle$. Let CX_count(K) be the number of CX gates required to prepare $|x'(K)\rangle$. Then, at the Select step, we choose the index subset K that maximizes the fidelity increase ratio:

$$\text{fidelity increase ratio}(K) = \frac{\langle x'(K)|x'(K)\rangle - |\langle 0|x\rangle|^2}{\text{CX\_count}(K) + 1}$$

This quantity is named "fidelity increase ratio" because if $|y(K)\rangle$ is the state after applying the state preparation sequence of $|x'(K)\rangle$ to $|x\rangle$, then $|\langle 0|y(K)\rangle|^2 = \langle x'(K)|x'(K)\rangle$, corresponding to a fidelity increase of $|\langle 0|y(K)\rangle|^2 - |\langle 0|x\rangle|^2 = \langle x'(K)|x'(K)\rangle - |\langle 0|x\rangle|^2$. This improvement is divided by the number of CX gates required to achieve it to compute the fidelity increase per CX gate; $+1$ is added to the denominator to prevent division by zero. Thus, at each step, K is chosen to maximize the projected fidelity increase per CX gate applied.

1.2 Implementing Prepare

Type 1 and Type 2 index subsets give rise to Type 1 and Type 2 sparse quantum states. Our preparation method for these states is an extension of [Gleinig and Hoefler] for limited connectivity architectures.

Type 1 index subsets give rise to quantum states of the form

$$|x'(b)\rangle = c_0 |0\rangle + c_b |b\rangle$$

For some $b > 0$ and complex amplitudes c_0 and c_b. To prepare such states, a sequence of CX gates, $g_1 g_2 ... g_l$ is used to move $|b\rangle$ to a sequence of other basis states, $|b_1\rangle, |b_2\rangle, ..., |b_l\rangle$ such that

$$g_i |b_{i-1}\rangle = |b_i\rangle \quad \text{for all} \quad 1 \leq i \leq l, b_0 = b$$

$$b_l = 2^j \quad \text{for some } j$$

CX gates leave $|0\rangle$ unchanged, therefore, this sequence of CX gates transforms $|x'(b)\rangle$ to a new state:

$$|x''\rangle = c_0 |0\rangle + c_b |2^j\rangle$$

Which is a one-qubit state, and can be prepared using an RZ and RY gate [ref]. Let $CXD(b) = l$ be the minimum number of CX gates required to prepare $|x'\rangle$; the formula for $CXD(b)$ is presented in the Appendix. Then, each CX gate g_i can be computed by selecting a CX gate such that $CXD(b_i) = CXD(b_{i-1}) - 1$.

The Type 2 index subsets $K^{(+)}(a, p)$ give rise to Type 2 quantum states of the form:

$$|x'^{(+)}(a, p)\rangle = (|a\rangle \otimes |\psi_1\rangle + |0\rangle \otimes |\psi_2\rangle) \otimes |0_p\rangle$$

. Where $|\psi_1\rangle$ and $|\psi_2\rangle$ are non-normalized, two-qubit states and $|0_p\rangle$ represents all zeroes in the lowest p qubits. To prepare such states, a sequence of CX gates, $g_1 g_2 ... g_m$ are applied to the upper $n - p - 2$ qubits to transform $|a\rangle$ to a sequence of other basis states, $|a_1\rangle, |a_2\rangle, ..., |a_m\rangle$ such that

$$g_i |a_{i-1}\rangle = |a_i\rangle \quad \text{for all} \quad 1 \le i \le m, |a_0\rangle = |a\rangle$$

$$|a_m\rangle = |1\rangle$$

Again, CX gates leave $|0\rangle$ unchanged, so this sequence transforms $|x'^{(+)}(a, p)\rangle$ to

$$|x''^{(+)}\rangle = (|1\rangle \otimes |\psi_1\rangle + |0\rangle \otimes |\psi_2\rangle) \otimes |0_p\rangle$$

The result is a three-qubit state, which can be prepared exactly using three CX gates [ref]. We give our construction for linear nearest neighbor architectures in the Appendix. In addition, let $CXD^{(+)}(p) = m$ be the minimum number of CX gates required to transform $|a\rangle$ to $|1\rangle$; the formula for $CXD^{(+)}$ is given in the Appendix.

In the special case where $a = 0$, the index subset $K^{(+)}(a, p)$ gives rise to a two qubit state:

$$|x'\rangle = |0_{n-p-2}\rangle \otimes |\psi_0\rangle \otimes |0_p\rangle$$

Which can be prepared using one CX gate [ref].

The Type 2 index subsets $K^{(-)}(a, p)$ give rise to quantum states of the form

$$|x'^{(-)}(a, p)\rangle = |0_{n-p-2}\rangle \otimes (|\psi_1\rangle \otimes |a\rangle + \otimes |\psi_2\rangle \otimes |0\rangle)$$

Where $|\psi_1\rangle$, $|\psi_2\rangle$, and $|0_{n-p-2}\rangle$ are defined as in the $K^{(+)}(a, p)$ case. Note that these states are the same as those in [ref equation], but with the qubits in reverse order. Thus, the same state preparation method applies to this case.

1.3 Implementing Adapt

If K is a Type 1 index subset, then the corresponding sparse approximation takes the form

$$|x'\rangle = c_0 |0\rangle + c_b |b\rangle$$

. And the state preparation method involves applying CX gates to move $|b\rangle$ to $|2^i\rangle$, before applying a one-qubit state preparation procedure. To adapt this to refining $|x\rangle$, we re-imagine the process of moving $|b\rangle$ using CX gates as an iterative merging procedure using CX, RZ, and RY gates. If b and b' are two length-n bitstrings that differ in only the ith bit and both b and b' have a 1 at position j for $|i - j| = 1$, then the amplitude at $|b\rangle$ can be merged into the amplitude at $|b'\rangle$ using the following procedure:

[subroutine description]

Granted, this procedure will generally shuffle the amplitudes of many other bases, but it leaves the amplitude at $|0\rangle$ unchanged (up to a phase). Then, the procedure for adapting $|x'\rangle$ for refining $|x\rangle$ is:

[Refinement by ISA, Type 1]

At each iteration, the algorithm constructs a list of all possible CX gates that modify $|b\rangle$. For each propective CX gate, the projected fidelity increase ratio is updated, and the maximal fidelity increase ratio is selected.

1.4 Refinement by RZ-RY

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Our specific implementation of the ISA framework is summarized in Algorithm 2. [pseudo-code for specific ISA implementation].