Homework for Lecture # 03 &04

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1. Solution:

(a)

$$60\text{ mph} = 60 \times 1.61\text{ kmph} = 60 \times 1.61 \times \frac{1000\text{ m}}{3600\text{ s}} = 26.8\ \text{m/s}$$

$$\Rightarrow f_d^{max} = \frac{v}{c} \cdot f_c = \frac{26.8}{3\times10^8} \times 2 \times 10^9 = 178\text{ Hz} \ \#$$

$$\Rightarrow B_d = 2 f_d^{max} = 356\text{ Hz}$$

$$\Rightarrow T_c = \frac{1}{2B_d} = \frac{1}{2\times356} = 1.4\text{ ms} \ \#$$

(b)

$$h(t) = x(t) + i\,y(t)$$

① $X(0) \sim \mathcal{N}(\underset{\mu=0}{0}, \underset{\sigma^2=\frac{1}{2}}{\tfrac{1}{2}})$

$$\Rightarrow f_X(x(0)) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x(0)-\mu)^2}{2\sigma^2}\right]$$

$$\boxed{\begin{array}{l}\mu=0\\ \sigma^2=\frac{1}{2}\end{array}} \quad = \frac{1}{\sqrt{2\pi\cdot\frac{1}{2}}} \exp\left[-\frac{(x(0)-0)^2}{\cancel{2}\cdot\cancel{\frac{1}{2}}}\right]$$

$$= \frac{1}{\sqrt{\pi}} e^{-x^2(0)}$$

② $Y(0) \sim \mathcal{N}(0, \frac{1}{2})$

$$\Rightarrow f_Y(y(0)) = \frac{1}{\sqrt{\pi}} e^{-y^2(0)}$$

Assuming $x(0)$ and $y(0)$ are independent, we have the joint distribution that is

$$f_{X,Y}(x(0), y(0)) = f_X(x(0)) \cdot f_Y(y(0))$$

$$= \frac{1}{\sqrt{\pi}} e^{-x^2(0)} \cdot \frac{1}{\sqrt{\pi}} e^{-y^2(0)}$$

$$= \frac{1}{\pi} e^{-[x^2(0)+y^2(0)]}$$

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(c)

Employ the Jake's temporal correlation model

$$\boxed{\psi(\Delta t) = J_0(2\pi f_d^{max} \Delta t)}$$

(Bessel function of 0^{th} order)

The correlation of $h(0)$ and $h(0.5T_c)$

$\boxed{\Delta t = 0.5\,T_c}$

$$E\{h(0)h^*(0.5T_c)\} = J_0(2\pi f_d^{max}(0.5T_c))$$

$$\boxed{T_c = \frac{1}{4f_d^{max}} \Rightarrow f_d^{max} = \frac{1}{4T_c}}$$

$$= J_0\left(\cancel{2}\pi \times \frac{1}{4\cancel{T_c}} \times \frac{1}{\cancel{2}}\cancel{T_c}\right)$$

$$= J_0\left(\frac{\pi}{4}\right)$$

$$\approx 0.8516 \ \#$$

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2. Solution:

(1)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{V^H}$$

(NOT a valid SVD since $\sigma_1 = 1, \sigma_2 = \sqrt{5}, \sigma_3 = \sqrt{3}$, $\sigma_1 < \sigma_3 < \sigma_2$)

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{5} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{V^H} \quad \#$$

"valid" SVD

Check:

1. $U^H U = I$
2. $V^H V = I$
3. $\sigma_1 = \sqrt{5}, \sigma_2 = \sqrt{3}, \sigma_3 = 1$

   $\sigma_1 > \sigma_2 > \sigma_3$

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(2)

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\sqrt{5} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{5} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

"valid" SVD

$$= \underbrace{\begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{V^H} \#$$

Check:

1. $U^H U = I$
2. $V^H V = I$
3. $\sigma_1 = \sqrt{5}$, $\sigma_2 = \sqrt{3}$, $\sigma_1 = 1$

   $\sigma_1 > \sigma_2 > \sigma_3$

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(3)

$$\mathbb{A}=\begin{bmatrix}1&2\\1&2\end{bmatrix}$$

$$\mathbb{A}=\mathbb{U}\Sigma\mathbb{V}^T$$

$$\mathbb{A}^T\mathbb{A}=\mathbb{V}\Sigma^T\mathbb{U}^T\mathbb{U}\Sigma\mathbb{V}^T=\mathbb{V}\Sigma^T\Sigma\mathbb{V}^T$$

$$\mathbb{A}\mathbb{A}^T=\mathbb{U}\Sigma\mathbb{V}^T\mathbb{V}\Sigma^T\mathbb{U}^T=\mathbb{U}\Sigma\Sigma^T\mathbb{U}^T$$

$$\mathbb{A}^T\mathbb{A}=\begin{bmatrix}1&1\\2&2\end{bmatrix}\begin{bmatrix}1&2\\1&2\end{bmatrix}=\begin{bmatrix}2&4\\4&8\end{bmatrix}$$

$$\left|\begin{bmatrix}2-\lambda&4\\4&8-\lambda\end{bmatrix}\right|=(2-\lambda)(8-\lambda)-16=0$$

$$\Rightarrow \lambda^2-10\lambda=0\Rightarrow\lambda(\lambda-10)=0$$

$$\therefore \lambda_1=10,\ \lambda_2=0$$

① $\lambda_1=10$

$$\begin{bmatrix}2&4\\4&8\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=10\begin{bmatrix}1\\2\end{bmatrix}$$

$$\Rightarrow \underline{v}_1=\begin{bmatrix}\frac{1}{\sqrt5}\\\frac{2}{\sqrt5}\end{bmatrix}$$

② $\lambda_2=0$

$$\begin{bmatrix}2&4\\4&8\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix}=0\begin{bmatrix}-2\\1\end{bmatrix}$$

$$\Rightarrow \underline{v}_2=\begin{bmatrix}-\frac{2}{\sqrt5}\\\frac{1}{\sqrt5}\end{bmatrix}$$

$$\mathbb{A}\mathbb{A}^T=\begin{bmatrix}1&2\\1&2\end{bmatrix}\begin{bmatrix}1&1\\2&2\end{bmatrix}=\begin{bmatrix}5&5\\5&5\end{bmatrix}$$

$$\left|\begin{bmatrix}5-\lambda&5\\5&5-\lambda\end{bmatrix}\right|=(5-\lambda)^2-25=0$$

$$\Rightarrow \lambda^2-10\lambda=0\Rightarrow\lambda(\lambda-10)=0$$

$$\therefore \lambda_1=10,\ \lambda_2=0$$

$\lambda_1=10,\ \lambda_2=0$

$$\Rightarrow \Sigma=\begin{bmatrix}\sqrt{10}&0\\0&0\end{bmatrix}$$

① $\lambda_1=10$

$$\begin{bmatrix}5&5\\5&5\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=10\begin{bmatrix}1\\1\end{bmatrix}$$

$$\Rightarrow \underline{u}_1=\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}$$

② $\lambda_2=0$

$$\begin{bmatrix}5&5\\5&5\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=0\begin{bmatrix}-1\\1\end{bmatrix}$$

$$\Rightarrow \underline{u}_2=\begin{bmatrix}-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}$$

$$\therefore \mathbb{U}=\begin{bmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}\qquad \mathbb{V}=\begin{bmatrix}\frac{1}{\sqrt5}&-\frac{2}{\sqrt5}\\\frac{2}{\sqrt5}&\frac{1}{\sqrt5}\end{bmatrix}$$

$$\begin{bmatrix}1&2\\1&2\end{bmatrix}=\underbrace{\begin{bmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}}_{\mathbb{U}}\underbrace{\begin{bmatrix}\sqrt{10}&0\\0&0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\\-\frac{2}{\sqrt5}&\frac{1}{\sqrt5}\end{bmatrix}}_{\mathbb{V}^T}\ \#$$

Check:

① $\mathbb{U}^T\mathbb{U}=\mathbb{I}$

② $\mathbb{V}^T\mathbb{V}=\mathbb{I}$

③ $\sigma_1=\sqrt{10},\ \sigma_2=0,\ \sigma_1>\sigma_2$

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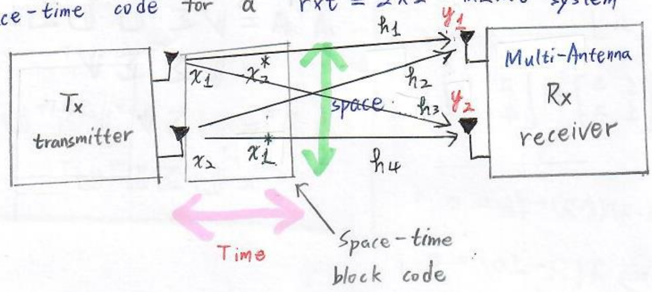
3. Solution:

※ The space-time code for a "$r \times t = 2 \times 2$" MIMO system



① the first transmit instant	$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$	$\underbrace{y_1(1)}_{\text{time index}} = [h_1 \ \ h_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n_1(1)$ $y_2(1) = [h_3 \ \ h_4]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n_2(1)$
② the second transmit instant	$\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix}$	$y_1(2) = [h_1 \ \ h_2]\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + n_1(2)$ $y_2(2) = [h_3 \ \ h_4]\begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix} + n_2(2)$

※ Consider the conjugates of $y_1(2)$ and $y_2(2)$ at the receiver, the above equations can be simplified as

$$y_1^*(2) = [h_1^* \ \ h_2^*]\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + n_1^*(2) = [-h_1^* \ \ h_2^*]\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + n_1^*(2)$$

$$= [h_2^* \ \ -h_1^*]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n_1^*(2)$$

$$y_2^*(2) = [h_3^* \ \ h_4^*]\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} + n_2^*(2) = [-h_3^* \ \ h_4^*]\begin{bmatrix} x_2 \\ x_1 \end{bmatrix} + n_2^*(2)$$

$$= [h_4^* \ \ -h_3^*]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n_2^*(2)$$

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※ The combined system model for the first and second time instants in the space-time code

$$\underbrace{\begin{bmatrix} y_1(1) \\ y_1^*(2) \\ y_2(1) \\ y_2^*(2) \end{bmatrix}}_{\underline{y}} = \underbrace{\begin{bmatrix} \overset{\underline{c}_1}{h_1} & \overset{\underline{c}_2}{h_2} \\ h_2^* & -h_1^* \\ h_3 & h_4 \\ h_4^* & -h_3^* \end{bmatrix}}_{\mathbb{H}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} n_1(1) \\ n_1^*(2) \\ n_2(1) \\ n_2^*(2) \end{bmatrix}}_{\underline{n}}$$

$$\underline{c}_1^H \underline{c}_2 = [h_1^* \ \ h_2 \ \ h_3^* \ \ h_4] \begin{bmatrix} h_2 \\ -h_1^* \\ h_4 \\ -h_3^* \end{bmatrix} = h_1^* h_2 - h_2 h_1^* + h_3^* h_4 - h_4 h_3^* = 0$$

$\Longrightarrow$ $\underline{c}_1$ and $\underline{c}_2$ are orthogonal

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※ Consider now beamforming using the vector $\underline{w}_1$ defined in terms of $\underline{c}_1$ as

$$\underline{w}_1 = \frac{1}{\|\underline{c}_1\|} \underline{c}_1 = \frac{1}{\|\underline{h}\|} \begin{bmatrix} h_1 \\ h_2^* \\ h_3 \\ h_4^* \end{bmatrix}$$

One can now employ this as a receive beamformer to derive the processed symbol as

$$\underline{w}_1^H \underline{y} = \left[\frac{h_1^*}{\|\underline{h}\|} \ \ \frac{h_2}{\|\underline{h}\|} \ \ \frac{h_3^*}{\|\underline{h}\|} \ \ \frac{h_4}{\|\underline{h}\|} \right] \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \\ h_3 & h_4 \\ h_4^* & -h_3^* \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underline{w}_1^H \underline{n}$$

$$= [\, \|\underline{h}\| \ \ 0 \,] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \tilde{n}_1$$

$$= \|\underline{h}\| \, x_1 + \tilde{n}_1 \Longrightarrow SNR = \frac{\|\underline{h}\|^2}{\sigma_n^2} E\{|x_1|^2\} = \frac{\|\underline{h}\|^2}{\sigma_n^2} P_1$$

$$= \frac{1}{2} \frac{\|\underline{h}\|^2}{\sigma_n^2} \cdot P$$

$$P_1 + P_2 = P \Rightarrow P_1 = P_2 = \frac{P}{2}$$

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Similarly, to decode x_2, the beamformer $\underline{w}_2$ is given as

$$\underline{w}_2 = \frac{1}{\|\underline{c}_2\|}\underline{c}_2 = \frac{1}{\|\underline{h}\|}\begin{bmatrix} h_2 \\ -h_1^* \\ h_4 \\ -h_3^* \end{bmatrix} \Longrightarrow SNR = \frac{\|\underline{h}\|^2}{\sigma_n^2} E\{|x_2|^2\} = \frac{\|\underline{h}\|^2}{\sigma_n^2} P$$

$$= \frac{1}{2}\frac{\|\underline{h}\|^2}{\sigma_n^2} P$$

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4. Solution:

(a)

$$\mathbb{H}_{zf} = (\mathbb{H}^H \mathbb{H})^{-1} \mathbb{H}^H$$

① $\mathbb{H}^H \mathbb{H} = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 7 \\ 7 & 11 \end{bmatrix}$

② $(\mathbb{H}^H \mathbb{H})^{-1} = \dfrac{1}{11\times 11 - 7\times 7} \begin{bmatrix} 11 & -7 \\ -7 & 11 \end{bmatrix} = \dfrac{1}{72} \begin{bmatrix} 11 & -7 \\ -7 & 11 \end{bmatrix}$

③ $\mathbb{H}_{zf} = (\mathbb{H}^H \mathbb{H})^{-1} \mathbb{H}^H = \dfrac{1}{72} \begin{bmatrix} 11 & -7 \\ -7 & 11 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -0.14 & 0.36 & 0.06 \\ 0.36 & -0.14 & 0.06 \end{bmatrix}$ #

(b)

$$P_t = 13\,\text{dB} = 10^{\frac{13}{10}} = 20$$

$$\sigma_n^2 = 0\,\text{dB} = 10^{\frac{0}{10}} = 1$$

$$\mathbb{W} = \boxed{\mathbb{R}_{yy}^{-1}} \boxed{\mathbb{R}_{yx}} = \boxed{(P_t\, \mathbb{H}\mathbb{H}^H + \sigma_n^2 \mathbb{I})^{-1}} \boxed{P_t\, \mathbb{H}} = P_t (P_t\, \mathbb{H}\mathbb{H}^H + \sigma_n^2 \mathbb{I})^{-1} \mathbb{H}$$

① $\mathbb{H}\mathbb{H}^H = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 4 \\ 6 & 10 & 4 \\ 4 & 4 & 2 \end{bmatrix}$

② $P_t\, \mathbb{H}\mathbb{H}^H + \sigma_n^2 \mathbb{I} = 20 \begin{bmatrix} 10 & 6 & 4 \\ 6 & 10 & 4 \\ 4 & 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 201 & 120 & 80 \\ 120 & 201 & 80 \\ 80 & 80 & 41 \end{bmatrix}$

③ $(P_t\, \mathbb{H}\mathbb{H}^H + \sigma_n^2 \mathbb{I})^{-1} = \begin{bmatrix} \frac{1841}{29241} & \frac{1480}{29241} & \frac{-80}{361} \\ \frac{1480}{29241} & \frac{1841}{29241} & \frac{-80}{361} \\ \frac{-80}{361} & \frac{-80}{361} & \frac{321}{361} \end{bmatrix}$

④ $P_t (P_t\, \mathbb{H}\mathbb{H}^H + \sigma_n^2 \mathbb{I})^{-1} \mathbb{H} = 20 \begin{bmatrix} \frac{1841}{29241} & \frac{1480}{29241} & \frac{-80}{361} \\ \frac{1480}{29241} & \frac{1841}{29241} & \frac{-80}{361} \\ \frac{-80}{361} & \frac{-80}{361} & \frac{321}{361} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -0.14 & 0.36 \\ 0.36 & -0.14 \\ 0.06 & 0.06 \end{bmatrix} \quad \#$$

$$\mathbb{W}^H = \begin{bmatrix} -0.14 & 0.36 & 0.06 \\ 0.36 & -0.14 & 0.06 \end{bmatrix}$$

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5. Solution:

(a) Choose the constant α that minimizes the mean square error $(E\{|x-\hat{x}|^2\})$

$$
\begin{aligned}
E\{|x-\hat{x}|^2\} &= E\{|x-\alpha\underline{c}^*\underline{y}|^2\} = E\{|x-\alpha\underline{c}^*\underline{h}x-\alpha\underline{c}^*\underline{z}|^2\} \\
&= E\{(x-\alpha\underline{c}^*\underline{h}x-\alpha\underline{c}^*\underline{z})(x-\alpha\underline{c}^*\underline{h}x-\alpha\underline{c}^*\underline{z})^*\} \\
&= E\{(x-\alpha\underline{c}^*\underline{h}x-\alpha\underline{c}^*\underline{z})(x^*-x^*\underline{h}^*\underline{c}\alpha^*-\underline{z}^*\underline{c}\alpha^*)\} \\
&\triangleq f(\alpha)
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \frac{\partial f(\alpha)}{\partial \alpha^*} &= E\{(x-\alpha\underline{c}^*\underline{h}x-\alpha\underline{c}^*\underline{z})(0-x^*\underline{h}^*\underline{c}-\underline{z}^*\underline{c})\} \\
&= -E\{|x|^2\underline{h}^*\underline{c}\} - \underbrace{E\{x\underline{z}^*\underline{c}\}}_{0} + E\{\alpha\underline{c}^*\underline{h}|x|^2\underline{h}^*\underline{c}\} \\
&\quad + \underbrace{E\{\alpha\underline{c}^*\underline{h}x\underline{z}^*\underline{c}\}}_{0} + \underbrace{E\{\alpha\underline{c}^*\underline{z}x^*\underline{h}^*\underline{c}\}}_{0} + E\{\alpha\underline{c}^*\underline{z}\underline{z}^*\underline{c}\} \\
&= -E\{|x|^2\}\underline{h}^*\underline{c} + \alpha E\{|x|^2\}|\underline{c}^*\underline{h}|^2 + \alpha\underline{c}^*\underbrace{E\{\underline{z}\underline{z}^*\}}_{\mathbb{K}_z}\underline{c} \\
&= -E\{|x|^2\}\underline{h}^*\underline{c} + \alpha\left(E\{|x|^2\}|\underline{c}^*\underline{h}|^2 + \underline{c}^*\mathbb{K}_z\underline{c}\right) = 0
\end{aligned}
$$

The noise $\underline{z}$ and the data symbol x are assumed to be uncorrelated.

$$
\Longrightarrow E\{|x|^2\}\underline{h}^*\underline{c} = \alpha\left(E\{|x|^2\}|\underline{c}^*\underline{h}|^2 + \underline{c}^*\mathbb{K}_z\underline{c}\right)
$$

$$
\therefore \alpha = \frac{E\{|x|^2\}\underline{h}^*\underline{c}}{E\{|x|^2\}|\underline{c}^*\underline{h}|^2+\underline{c}^*\mathbb{K}_z\underline{c}}
= \frac{E\{|x|^2\}|\underline{c}^*\underline{h}|^2}{E\{|x|^2\}|\underline{c}^*\underline{h}|^2+\underline{c}^*\mathbb{K}_z\underline{c}} \cdot \frac{\underline{h}^*\underline{c}}{|\underline{h}^*\underline{c}|^2} \quad \#
$$

$\boxed{|\underline{c}^*\underline{h}|^2 = |\underline{h}^*\underline{c}|^2}$

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(b) Calculate the MMSE of the above linear estimate

$$E\{|x-\hat{x}|^2\} = E\{|x-\alpha \underline{c}^* \underline{h}\, x - \alpha\, \underline{c}^* \underline{z}|^2\}$$

$$= E\{(x-\alpha \underline{c}^* \underline{h}\, x - \alpha \underline{c}^* \underline{z})(x-\alpha \underline{c}^* \underline{h}\, x - \alpha \underline{c}^* \underline{z})^*\}$$

$$= E\{(x-\alpha \underline{c}^* \underline{h}\, x - \alpha \underline{c}^* \underline{z})(x^* - x^* \underline{h}^* \underline{c}\,\alpha^* - \underline{z}^* \underline{c}\,\alpha^*)\}$$

The noise $\underline{z}$ and the data symbol x are assumed to be uncorrelated

$$= E\{|x|^2\} - E\{\alpha \underline{c}^* \underline{h} |x|^2\} - \cancel{E\{\alpha \underline{c}^* \underline{z}\, x^*\}}^{\,0} -$$

$$E\{|x|^2 \underline{h}^* \underline{c}\,\alpha^*\} + E\{\alpha \underline{c}^* \underline{h} |x|^2 \underline{h}^* \underline{c}\,\alpha^*\} + \cancel{E\{\alpha \underline{c}^* \underline{z}\, x^* \underline{h}^* \underline{c}\,\alpha^*\}}^{\,0}$$

$$- \cancel{E\{x\, \underline{z}^* \underline{c}\,\alpha^*\}}^{\,0} + \cancel{E\{\alpha \underline{c}^* \underline{h}\, x\, \underline{z}^* \underline{c}\,\alpha^*\}}^{\,0} + E\{\alpha \underline{c}^* \underline{z}\,\underline{z}^* \underline{c}\,\alpha^*\}$$

$$= E\{|x|^2\} - E\{|x|^2\}\alpha \underline{c}^* \underline{h} - E\{|x|^2\} \underline{h}^* \underline{c}\,\alpha^*$$

$$+ |\alpha|^2 E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + |\alpha|^2 \underline{c}^* \underbrace{E\{\underline{z}\,\underline{z}^*\}}_{\mathbb{K}_z} \underline{c}$$

$$= E\{|x|^2\}\left[1 - \alpha \underline{c}^* \underline{h} - \alpha^* \underline{h}^* \underline{c}\right] + |\alpha|^2 \left[E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}\right]$$

Using the value of α obtained in (a)

$$\alpha = \frac{E\{|x|^2\} \underline{h}^* \underline{c}}{E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}}$$

$$= E\{|x|^2\}\left[1 - \frac{E\{|x|^2\} \underline{h}^* \underline{c}\, \underline{c}^* \underline{h}}{E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}} - \frac{E\{|x|^2\} |\underline{h}^* \underline{c}|^2}{E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}}\right]$$

$$+ \frac{\left[E\{|x|^2\} \underline{h}^* \underline{c}\right]^2}{E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}} = \text{MMSE}$$

$$\Rightarrow \text{MMSE} = \frac{(E\{|x|^2\})^2}{E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}} \left[\cancel{|\underline{c}^* \underline{h}|^2} + \frac{\underline{c}^* \mathbb{K}_z \underline{c}}{E\{|x|^2\}} - \cancel{|\underline{c}^* \underline{h}|^2} - \cancel{|\underline{c}^* \underline{h}|^2} + \cancel{|\underline{h}^* \underline{c}|^2}\right]$$

$$\boxed{|\underline{c}^* \underline{h}|^2 = |\underline{h}^* \underline{c}|^2}$$

$$= \frac{(E\{|x|^2\})^{\cancel{2}}\, \underline{c}^* \mathbb{K}_z \underline{c}}{E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}} \cdot \frac{1}{\cancel{E\{|x|^2\}}}$$

$$= \frac{E\{|x|^2\}\, \underline{c}^* \mathbb{K}_z \underline{c}}{E\{|x|^2\} |\underline{c}^* \underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}}$$

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$$\Rightarrow \frac{E\{|x|^2\}}{\text{MMSE}} = \frac{E\{|x|^2\}|\underline{c}^*\underline{h}|^2 + \underline{c}^* \mathbb{K}_z \underline{c}}{\underline{c}^* \mathbb{K}_z \underline{c}}$$

$$= 1 + \frac{E\{|x|^2\}|\underline{c}^*\underline{h}|^2}{\underline{c}^* \mathbb{K}_z \underline{c}}$$

Consider the following estimate of x from $\underline{y}$

$$\hat{x} := \alpha \underline{c}^* \underline{y} = \underbrace{\alpha \underline{c}^* \underline{h} x}_{\substack{\text{signal} \\ \text{component}}} + \underbrace{\alpha \underline{c}^* \underline{z}}_{\substack{\text{noise} \\ \text{component}}} \quad \Rightarrow \text{SNR} = \frac{\alpha^2 E\{|x|^2\}|\underline{c}^*\underline{h}|^2}{\alpha^2 \underline{c}^* \mathbb{K}_z \underline{c}}$$

$$= \frac{E\{|x|^2\}|\underline{c}^*\underline{h}|^2}{\underline{c}^* \mathbb{K}_z \underline{c}}$$

$$\therefore \frac{E\{|x|^2\}}{\text{MMSE}} = 1 + \frac{E\{|x|^2\}|\underline{c}^*\underline{h}|^2}{\underline{c}^* \mathbb{K}_z \underline{c}}$$

$$= 1 + \text{SNR} \quad \#$$

■

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6. Solution:

(a) $\mathbb{H}_{zf} = (\mathbb{H}^H \mathbb{H})^{-1} \mathbb{H}^H$

① $\mathbb{H}^H \mathbb{H} = \begin{bmatrix} 1 & 0.2 \\ -0.8 & -1 \end{bmatrix} \begin{bmatrix} 1 & -0.8 \\ 0.2 & -1 \end{bmatrix} = \begin{bmatrix} 1.04 & -1 \\ -1 & 1.64 \end{bmatrix}$

② $(\mathbb{H}^H \mathbb{H})^{-1} = \dfrac{1}{1.04 \times 1.64 - 1 \times 1} \begin{bmatrix} 1.64 & 1 \\ 1 & 1.04 \end{bmatrix} = \dfrac{1}{0.7056} \begin{bmatrix} 1.64 & 1 \\ 1 & 1.04 \end{bmatrix}$

③ $\mathbb{H}_{zf} = (\mathbb{H}^H \mathbb{H})^{-1} \mathbb{H}^H = \dfrac{1}{0.7056} \begin{bmatrix} 1.64 & 1 \\ 1 & 1.04 \end{bmatrix} \begin{bmatrix} 1 & 0.2 \\ -0.8 & -1 \end{bmatrix} = \begin{bmatrix} 1.19 & -0.95 \\ 0.24 & -1.19 \end{bmatrix}$ #

(b) $\underline{x}_{zf} = (\mathbb{H}^H \mathbb{H})^{-1} \mathbb{H}^H \underline{y}$

$$= \begin{bmatrix} 1.19 & -0.95 \\ 0.24 & -1.19 \end{bmatrix} \begin{bmatrix} -1.5 \\ 0.27 \end{bmatrix} = \begin{bmatrix} -2.04 \\ -0.68 \end{bmatrix} \quad \#$$

$\Longrightarrow$ The decoded transmitted symbols on each transmit antenna is

$$\hat{\underline{x}} = \begin{bmatrix} -\sqrt{P} \\ -\sqrt{P} \end{bmatrix} = \begin{bmatrix} -\sqrt{0.5} \\ -\sqrt{0.5} \end{bmatrix} \quad \#$$

$$\boxed{P = -3\,dB = 10^{\frac{-3}{10}} = 0.5}$$

(c) The vector transmit constellation corresponding to BPSK

$\underline{x}_1 = \begin{bmatrix} \sqrt{P} \\ \sqrt{P} \end{bmatrix} = \begin{bmatrix} \sqrt{0.5} \\ \sqrt{0.5} \end{bmatrix}$, $\underline{x}_2 = \begin{bmatrix} -\sqrt{P} \\ \sqrt{P} \end{bmatrix} = \begin{bmatrix} -\sqrt{0.5} \\ \sqrt{0.5} \end{bmatrix}$, $\underline{x}_3 = \begin{bmatrix} \sqrt{P} \\ -\sqrt{P} \end{bmatrix} = \begin{bmatrix} \sqrt{0.5} \\ -\sqrt{0.5} \end{bmatrix}$ and

$\underline{x}_4 = \begin{bmatrix} -\sqrt{P} \\ -\sqrt{P} \end{bmatrix} = \begin{bmatrix} -\sqrt{0.5} \\ -\sqrt{0.5} \end{bmatrix}$ $\quad \boxed{P = 0.5}$ #

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(d)

Employ the optimal ML decoder

$\Longrightarrow$ Select the signal $\underline{x}_i$ if

$$\|\underline{y} - \mathbb{H}\underline{x}_i\|^2 \leq \|\underline{y} - \mathbb{H}\underline{x}_j\|^2 \quad \forall\, i \neq j$$

where $i = 1, 2, 3, 4$

$\therefore$

① $$\|\underline{y} - \mathbb{H}\underline{x}_1\|^2 = \left\| \begin{bmatrix} -1.5 \\ 0.23 \end{bmatrix} - \begin{bmatrix} 1 & -0.8 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{0.5} \\ \sqrt{0.5} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} -1.64 \\ 0.8 \end{bmatrix} \right\|^2 = 3.33$$

② $$\|\underline{y} - \mathbb{H}\underline{x}_2\|^2 = \left\| \begin{bmatrix} -1.5 \\ 0.23 \end{bmatrix} - \begin{bmatrix} 1 & -0.8 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{0.5} \\ \sqrt{0.5} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} -0.23 \\ 1.08 \end{bmatrix} \right\|^2 = 1.21$$

③ $$\|\underline{y} - \mathbb{H}\underline{x}_3\|^2 = \left\| \begin{bmatrix} -1.5 \\ 0.23 \end{bmatrix} - \begin{bmatrix} 1 & -0.8 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{0.5} \\ -\sqrt{0.5} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} -2.77 \\ -0.62 \end{bmatrix} \right\|^2 = 8.07$$

④ $$\|\underline{y} - \mathbb{H}\underline{x}_4\|^2 = \left\| \begin{bmatrix} -1.5 \\ 0.23 \end{bmatrix} - \begin{bmatrix} 1 & -0.8 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} -\sqrt{0.5} \\ -\sqrt{0.5} \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} -1.36 \\ -0.34 \end{bmatrix} \right\|^2 = 1.96$$

Therefore,

$$\|\underline{y} - \mathbb{H}\underline{x}_2\|^2 \leq \|\underline{y} - \mathbb{H}\underline{x}_j\|^2 \quad \text{for } j = 1, 3, 4$$

$\Longrightarrow$ The optical ML detector selects the signal $\underline{x}_2 = \begin{bmatrix} -\sqrt{0.5} \\ \sqrt{0.5} \end{bmatrix}$ as the decoded transmitted symbol.

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