Homework Problems for Lecture # 02

Assigned Date: October 31, 2022

1. Suppose that BPSK modulation is used with L-branch diversity and coherent maximal ratio combining. Assume uncorrelated diversity branches. The probability of bit error for a Rayleigh fading channel is

$$P_b = \int_0^{\infty} \mathcal{Q}\left(\sqrt{2x}\right) \frac{1}{(L-1)!\overline{\gamma}_c^L} x^{L-1} e^{-x/\overline{\gamma}_c} \mathrm{d}x, \tag{1}$$

where $\overline{\gamma}_c$ is the branch bit energy-to-noise ratio.

(a) (1%) Show that P_b can be reduced to

$$P_b = \frac{2}{\sqrt{\pi}(L-1)!} \int_{\arctan\sqrt{\overline{\gamma}_c}}^{\pi/2} \cos^{2L-1}\theta \mathrm{d}\theta \int_0^{\infty} r^{2L} e^{-r^2} \mathrm{d}r \tag{2}$$

by interchanging the order of integrals in (1) and using the polar coordinates.

(b) (1%) Use the following equation

$$\int_u^1 \left(1-v^2\right)^{L-1} \mathrm{d}v = (1-u)^L \sum_{k=0}^{L-1} (1+u)^k \, 2^{L-1-k} \frac{(L-1)!(L-1+k)!}{k!(2L-1)!} \tag{3}$$

to show that

$$\int_{\arctan\sqrt{\overline{\gamma}_c}}^{\pi/2} \cos^{2L-1}\theta \mathrm{d}\theta = (1-u)^L \sum_{k=0}^{L-1} (1+u)^k \, 2^{L-1-k} \frac{(L-1)!(L-1+k)!}{k!(2L-1)!}, \tag{4}$$

where $u = \sqrt{\frac{\overline{\gamma}_c}{1+\overline{\gamma}_c}}$.

(c) (2%) Use (a), (b), and the following integral:

$$P_b = \int_0^{\infty} r^{2L} e^{-r^2} \mathrm{d}r = \frac{(2L-1)!!}{2^{L+1}} \sqrt{\pi} \tag{5}$$

to show

$$P_b = k \left(\frac{1-u}{2}\right)^L \sum_{k=0}^{L-1} \binom{L+k-1}{k} \left(\frac{1+u}{2}\right)^k \tag{6}$$

Note: The double factorial of a positive integer n is a generalization of the usual factorial $n!$ defined by

$$n!! = \begin{cases} n \cdot (n-2) \cdots 5 \cdot 3 \cdot 1, & n \text{ is odd,} \\ n \cdot (n-2) \cdots 6 \cdot 4 \cdot 2, & n \text{ is even.} \end{cases} \tag{7}$$

2. In class, we mention that the diversity order of MRC with L independent receiver antennas is L. Consider a *slightly different* beamforming vector given as below

$$\mathbf{w}_e = \begin{bmatrix} \frac{h_1}{|h_1|} \\ \frac{h_2}{|h_2|} \\ \vdots \\ \frac{h_L}{|h_L|} \end{bmatrix} \tag{8}$$

   (a) (1%) The above beamforming vector is termed as the *Equal Gain Combiner*. Can you explain why?

   (b) (1%) Given the expression for $\tilde{\mathbf{w}}_e$, the above beamformer normalized to have unit-norm

   (c) (2%) Employing the high SNR argument, derive the diversity order of the above system when the receiver beamformer is $\tilde{\mathbf{w}}_e$.

3. **Matlab Assignment:** Perform simulation and plot the error probability performance, i.e., bit error rate vs E_b/N_0 (dB) of a BPSK communication system over Rayleigh Fading and AWGN channel. In your figure, you should show 2 curves: (i) The results by simulation. (ii) The results by analysis, i.e., average BER $= \frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{2+\text{SNR}}}\right)$ (2%)