Homework for Lecture # 02

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1. Solution:

(a)

$$P_b = \int_0^\infty Q(\sqrt{2x})\,\frac{1}{(L-1)!\,\bar{\gamma}_c^{\,L}}\,x^{L-1}e^{-\frac{x}{\bar{\gamma}_c}}\,dx$$

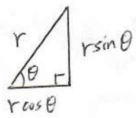
$$\boxed{Q(v) \triangleq \int_v^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt}$$

$$= \int_0^\infty \int_{\sqrt{2x}}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\,dt\,\frac{1}{(L-1)!\,\bar{\gamma}_c^{\,L}}\,x^{L-1}e^{-\frac{x}{\bar{\gamma}_c}}\,dx$$

$$= \int_{\sqrt{2x}}^\infty \int_0^\infty \frac{1}{\sqrt{2\pi}}\,\frac{1}{(L-1)!\,\bar{\gamma}_c^{\,L}}\,x^{L-1}e^{-\left(\frac{t^2}{2}+\frac{x}{\bar{\gamma}_c}\right)}\,dx\,dt$$

Let $r^2 = \frac{t^2}{2} + \frac{x}{\bar{\gamma}_c} = r^2\cos^2\theta + r^2\sin^2\theta \Rightarrow$ Set $\begin{cases} r^2\cos^2\theta = \frac{x}{\bar{\gamma}_c} \\ r^2\sin^2\theta = \frac{t^2}{2} \end{cases} \Rightarrow \begin{cases} r\cos\theta = \sqrt{\frac{x}{\bar{\gamma}_c}} \\ r\sin\theta = \sqrt{\frac{t^2}{2}} \end{cases}$

$re^{j\theta} = r\cos\theta + jr\sin\theta$



$$\therefore \begin{cases} x = \bar{\gamma}_c\, r^2\cos^2\theta \\ t = \sqrt{2}\, r\sin\theta \end{cases}$$

$$J_{xT} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial t}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial t}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 2r\bar{\gamma}_c\cos^2\theta & \sqrt{2}\sin\theta \\ -2\bar{\gamma}_c r^2\cos\theta\sin\theta & \sqrt{2}\,r\cos\theta \end{vmatrix}$$

$$= 2\sqrt{2}\,r^2\bar{\gamma}_c\cos^3\theta + 2\sqrt{2}\,r^2\bar{\gamma}_c\cos\theta\sin^2\theta$$

$$= 2\sqrt{2}\,r^2\bar{\gamma}_c\cos\theta\,(\cos^2\theta + \sin^2\theta)$$

$\boxed{1 = \cos^2\theta + \sin^2\theta}$

$$= 2\sqrt{2}\,r^2\bar{\gamma}_c\cos\theta$$

$$re^{j\theta} = r\cos\theta + jr\sin\theta = \sqrt{\frac{x}{\bar{\gamma}_c}} + j\sqrt{\frac{t^2}{2}} \Rightarrow \begin{cases} r^2 = \frac{x}{\bar{\gamma}_c} + \frac{t^2}{2} \\ \theta = \tan^{-1}\sqrt{\frac{\bar{\gamma}_c t^2}{2x}} \end{cases}$$

	下限	上限
x	0	∞
r	0	∞

	下限	上限
t	$\sqrt{2x}$	∞
θ	$\tan^{-1}\sqrt{\bar{\gamma}_c}$	$\frac{\pi}{2}$

$$t = \sqrt{2x} \Rightarrow \theta = \tan^{-1}\sqrt{\frac{\bar{\gamma}_c\,2x}{2x}} = \tan^{-1}\sqrt{\bar{\gamma}_c}$$

$$\Rightarrow P_b = \int_{\sqrt{2x}}^{\infty}\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{(L-1)!\,\bar{\gamma}_c^{\,L}}\, x^{L-1}\, e^{-\left(\frac{t^2}{2}+\frac{x}{\bar{\gamma}_c}\right)}\, dx\, dt$$

$$= \int_{\tan^{-1}\sqrt{\bar{\gamma}_c}}^{\frac{\pi}{2}}\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{(L-1)!\,\bar{\gamma}_c^{\,L}} \left(\bar{\gamma}_c r^2 \cos^2\theta\right)^{L-1} e^{-r^2} J_{XT}\, dr\, d\theta$$

$$\boxed{J_{XT} = 2\sqrt{2}\,\bar{\gamma}_c\, r^2 \cos\theta}$$

$$= \int_{\tan^{-1}\sqrt{\bar{\gamma}_c}}^{\frac{\pi}{2}}\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{(L-1)!\,\bar{\gamma}_c^{\,L}} \left(\bar{\gamma}_c r^2 \cos^2\theta\right)^{L-1} e^{-r^2} \left(2\sqrt{2}\,\bar{\gamma}_c r^2 \cos\theta\right) dr\, d\theta$$

$$= \frac{2}{\sqrt{\pi}\,(L-1)!} \int_{\tan^{-1}\sqrt{\bar{\gamma}_c}}^{\frac{\pi}{2}} \cos^{2L-1}\theta\, d\theta \int_0^{\infty} r^{2L} e^{-r^2}\, dr$$

(b)

$$\int_u^1 (1-v^2)^{L-1} dv = \int_{\sin^{-1}u}^{\frac{\pi}{2}} \left(1-\sin^2\theta\right)^{L-1} \cos\theta\, d\theta = \int_{\sin^{-1}u}^{\frac{\pi}{2}} \left(\cos^2\theta\right)^{L-1} \cos\theta\, d\theta$$

Let $v = \sin\theta$ $\quad (\theta = \sin^{-1} v)$

$\Longrightarrow dv = \cos\theta\, d\theta$

	下限	上限
v	u	1
θ	$\sin^{-1}u$	$\frac{\pi}{2}$

$1 = \cos^2\theta + \sin^2\theta$

$\Longrightarrow 1-\sin^2\theta = \cos^2\theta$

$$= \int_{\sin^{-1}u}^{\frac{\pi}{2}} \cos^{2L-1}\theta\, d\theta$$

$$\sin\theta = \sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}} = \sqrt{\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\sec^2\theta}}$$

$$\Longrightarrow \theta = \sin^{-1}\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}$$

$$\therefore \theta = \sin^{-1}\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}$$

$$= \sin^{-1} u$$

$$\Longrightarrow u = \sqrt{\frac{\bar{\gamma}_c}{1+\bar{\gamma}_c}} = \sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}} \Longrightarrow \bar{\gamma}_c = \tan^2\theta$$

$$\Longrightarrow \theta = \tan^{-1}\sqrt{\bar{\gamma}_c}$$

$$= \int_{\tan^{-1}\sqrt{\bar{\gamma}_c}}^{\frac{\pi}{2}} \cos^{2L-1}\theta\, d\theta$$

$$\boxed{\int_u^1 (1-v^2)^{L-1} dv = (1-u)^L \sum_{k=0}^{L-1} (1+u)^k\, 2^{L-1-k} \frac{(L-1)!\,(L-1+k)!}{k!\,(2L-1)!}}$$

$$= (1-u)^L \sum_{k=0}^{L-1} (1+u)^k\, 2^{L-1-k} \frac{(L-1)!\,(L-1+k)!}{k!\,(2L-1)!}$$

(c)

$$P_b = \int_0^\infty Q(\sqrt{2x}) \frac{1}{(L-1)!\,\bar{\gamma}_c^{\,L}} x^{L-1} e^{-\frac{x}{\bar{\gamma}_c}} dx$$

$$= \frac{2}{\sqrt{\pi}\,(L-1)!} \int_{\tan^{-1}\sqrt{\bar{\gamma}_c}}^{\frac{\pi}{2}} \cos^{2L-1}\theta\, d\theta \int_0^\infty r^{2L} e^{-r^2} dr$$

$$\boxed{\begin{aligned} &\int_{\tan^{-1}\sqrt{\bar{\gamma}_c}}^{\frac{\pi}{2}} \cos^{2L-1}\theta\, d\theta = (1-u)^L \sum_{k=0}^{L-1} (1+u)^k 2^{L-1-k} \frac{(L-1)!\,(L-1+k)!}{k!\,(2L-1)!} \\ &\int_0^\infty r^{2L} e^{-r^2} dr = \frac{(2L-1)!!}{2^{L+1}} \sqrt{\pi} \end{aligned}}$$

$$= \frac{\cancel{2}}{\cancel{\sqrt{\pi}\,(L-1)!}} (1-u)^L \sum_{k=0}^{L-1} (1+u)^k 2^{L-1-k} \frac{\cancel{(L-1)!}\,(L-1+k)!}{k!\,(2L-1)!} \frac{(2L-1)!!}{2^{L+1}} \cancel{\sqrt{\pi}}$$

$$= (1-u)^L \sum_{k=0}^{L-1} (1+u)^k\, 2^{L-1-k} \frac{(L-1+k)!\,(2L-1)!!}{k!\,(2L-1)!\,2^L}$$

Double factorial:

$$n!! = \begin{cases} n\cdot(n-2)\cdots 5\cdot 3\cdot 1, & n \text{ is odd.} \\ n\cdot(n-2)\cdots 6\cdot 4\cdot 2, & n \text{ is even.} \end{cases} \quad (n \text{ is a positive integer})$$

$\because$ $2L-1$ is odd $\Longrightarrow$ $(2L-1)!! = (2L-1)\cdot(2L-3)\cdots 5\cdot 3\cdot 1$

$$\therefore \frac{(2L-1)!!}{(2L-1)!} = \frac{\cancel{(2L-1)}\cancel{(2L-3)}\cdots \cancel{5}\cdot\cancel{3}\cdot\cancel{1}}{\cancel{(2L-1)}(2L-2)\cdots \cancel{5}\cdot 4\cdot\cancel{3}\cdot 2\cdot\cancel{1}} = \frac{1}{(2L-2)(2L-4)\cdots 4\cdot 2}$$

$$= \frac{1}{2^{L-1}\cdot(L-1)!}$$

$$= (1-u)^L \sum_{k=0}^{L-1} (1+u)^k\, 2^{\cancel{L-1}-k} \frac{(L-1+k)!}{k!\,2^L} \cdot \frac{1}{\cancel{2^{L-1}}(L-1)!}$$

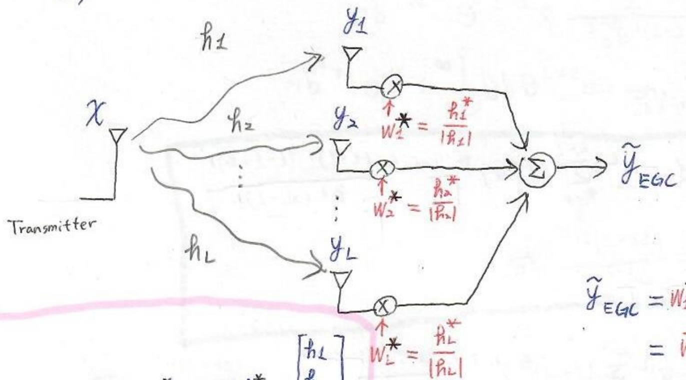
$$= \left(\frac{1-u}{2}\right)^L \sum_{k=0}^{L-1} \left(\frac{1+u}{2}\right)^k \frac{(L+k-1)!}{k!\,(L-1)!}$$

$$\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \Longrightarrow \binom{L+k-1}{k} = \frac{(L+k-1)!}{k!\,(L-1)!}}$$

$$= \left(\frac{1-u}{2}\right)^L \sum_{k=0}^{L-1} \binom{L+k-1}{k} \left(\frac{1+u}{2}\right)^k$$

2. Solution:

(a)



$$\tilde{y}_{EGC} = w_1^* y_1 + w_2^* y_2 + \dots + w_L^* y_L = \bar{W}_e^H \bar{y} = \bar{W}_e^H (\bar{h} x + \bar{n}) = \underbrace{\bar{W}_e^H \bar{h} x}_{\text{signal component}} + \underbrace{\bar{W}_e^H \bar{n}}_{\text{noise component}}$$

$$\bar{W}_e^H \bar{h} = \left[\frac{h_1^*}{|h_1|} \ \frac{h_2^*}{|h_2|} \ \cdots \ \frac{h_L^*}{|h_L|} \right] \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{bmatrix}$$

$$= \frac{|h_1|^2}{|h_1|} + \frac{|h_2|^2}{|h_2|} + \dots + \frac{|h_L|^2}{|h_L|} = |h_1| + |h_2| + \dots + |h_L|$$

① signal component

$$\bar{W}_e^H \bar{h} x = |h_1| x + |h_2| x + \dots + |h_L| x$$

$$\begin{cases} y_1 = h_1 x + n_1 \\ y_2 = h_2 x + n_2 \\ \vdots \\ y_L = h_L x + n_L \end{cases}$$

Observe that

$$\bar{W}_e = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_L \end{bmatrix} = \begin{bmatrix} \frac{h_1}{|h_1|} \\ \frac{h_2}{|h_2|} \\ \vdots \\ \frac{h_L}{|h_L|} \end{bmatrix}$$

$$|w_1| = |w_2| = \dots = |w_L| = 1$$

Hence, the beamforming vector is termed as the Equal Gain Combiner. ▨

(b)

$$\|\bar{W}_e\|^2 = \bar{W}_e^H \bar{W}_e = \left[\frac{h_1^*}{|h_1|} \ \frac{h_2^*}{|h_2|} \ \cdots \ \frac{h_L^*}{|h_L|} \right] \begin{bmatrix} \frac{h_1}{|h_1|} \\ \frac{h_2}{|h_2|} \\ \vdots \\ \frac{h_L}{|h_L|} \end{bmatrix}$$

$$= \frac{|h_1|^2}{|h_1|^2} + \frac{|h_2|^2}{|h_2|^2} + \dots + \frac{|h_L|^2}{|h_L|^2}$$

$$= 1 + 1 + \dots + 1$$

$$= L$$

Set $\tilde{W_e} = c\,\bar{W_e}$, where c is a constant.

$\|\tilde{W_e}\|^2 = c^2 \|\bar{W_e}\|^2 = c^2 L = 1 \Longrightarrow c = \frac{1}{\sqrt{L}}$

$\tilde{W_e}$ has unit-form; i.e., $\|\tilde{W_e}\| = 1$.

$\therefore$ The normalized beamformer

$$\tilde{W_e} = c\,\bar{W_e} = \frac{1}{\sqrt{L}} \begin{bmatrix} \frac{h_1}{|h_1|} \\ \frac{h_2}{|h_2|} \\ \vdots \\ \frac{h_L}{|h_L|} \end{bmatrix}$$

(c)

$$\boxed{\tilde{y}_{EGC} = \tilde{W_e}^H \bar{y} = \tilde{W_e}^H(\bar{h}x+\bar{n}) = \tilde{W_e}^H\bar{h}\,x + \tilde{W_e}^H\bar{h}\,n}$$

$\boxed{SNR = \frac{P}{\sigma^2}}$

SNR after EGC, $SNR_e = \frac{P|\tilde{W_e}^H\bar{h}|^2}{\sigma^2\|\tilde{W_e}\|^2} = \frac{1}{L}\left(\sum_{i=0}^{L-1}|h_i|\right)^2 SNR$

① $\tilde{W_e}^H\bar{h} = \frac{1}{\sqrt{L}}\left[\frac{h_1^*}{|h_1|}\ \frac{h_2^*}{|h_2|}\ \cdots\ \frac{h_L^*}{|h_L|}\right]\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_L \end{bmatrix} = \frac{1}{\sqrt{L}}(|h_1|+|h_2|+\ldots+|h_L|) = \frac{1}{\sqrt{L}}\sum_{i=1}^{L}|h_i|$

$\Longrightarrow |\tilde{W_e}^H\bar{h}|^2 = \frac{1}{L}\left(\sum_{i=0}^{L}|h_i|\right)^2$

② $\|\tilde{W_e}\|^2 = L \cdot \frac{1}{L} = 1$

The poor error probability performance of the wireless communication arises from the deep fade. $\Longrightarrow$ BER $\propto P_{DF}$ deep fade

$\therefore$ Deep fade occurs, when $\boxed{\text{received power}} < \boxed{\text{noise power}}$

$$\boxed{\frac{1}{L}\left(\sum_{i=0}^{L}|h_i|\right)^2 P} < \boxed{\sigma^2}$$

$$\Longrightarrow \frac{1}{L}\left(\sum_{i=0}^{L}|h_i|\right)^2 < \frac{\sigma^2}{P} = \frac{1}{SNR}$$

$$\Longrightarrow \left(\sum_{i=1}^{L}|h_i|\right)^2 < \frac{L}{SNR} \quad \Longrightarrow \sum_{i=1}^{L}|h_i|^2 < \sqrt{\frac{L}{SNR}}$$

$\because$ ① $|h_i| \neq |h_j| \ \forall\, i\neq j$, ② $|h_i| \geq 0 \ \forall\, i=1,2,\ldots,L$

There fore,

$$\sum_{i=1}^{L}|h_i| < \sqrt{\frac{L}{SNR}} \Longrightarrow |h_1|, |h_2|, \ldots, |h_L| < \sqrt{\frac{L}{SNR}}$$

$h_1, h_2, \ldots, h_L$: I.I.D. Rayleigh fading channel coefficient with $\underbrace{E\{|h_i|^2\}=1}_{\text{average power}}$

$$\therefore P_r(\text{deep fade}) = P_r\left(\sum_{i=1}^{L}|h_i| < \sqrt{\frac{L}{SNR}}\right) = \prod_{i=1}^{L} P_r\left(|h_i| < \sqrt{\frac{L}{SNR}}\right)$$

$$= \prod_{i=1}^{L} P_r\left(a < \sqrt{\frac{L}{SNR}}\right)$$

$$\approx \left(\frac{L}{SNR}\right)^L$$

$$\Longrightarrow \boxed{Pe(SNR) \approx \left(\frac{L}{SNR}\right)^L}$$

at high SNR

$$P_r\left(a < \sqrt{\frac{L}{SNR}}\right)$$

$$= \int_0^{\sqrt{\frac{L}{SNR}}} f_A(a)\, da$$

$\boxed{f_A(a) = 2ae^{-a^2} \text{ Rayleigh density}}$

$$= \int_0^{\sqrt{\frac{L}{SNR}}} 2ae^{-a^2}\, da$$

$$= e^{-a^2}\Big|_0^{\sqrt{\frac{L}{SNR}}} = 1 - e^{-\frac{L}{SNR}}$$

$\frac{L}{SNR}$ = "x"

$e^{-x} \approx 1 - x$ if $x \approx 0$

$x \triangleq \frac{L}{SNR}$

$x \approx 0$ 是指 $\boxed{\text{high SNR}}$

$\therefore$ We derive the diversity :

$$d = -\lim_{SNR\to\infty} \frac{\log Pe(SNR)}{\log SNR} = -\lim_{SNR\to\infty} \frac{\log\left(\frac{L}{SNR}\right)^L}{\log SNR}$$

$\boxed{\text{Using L'Hopital's rule}}$

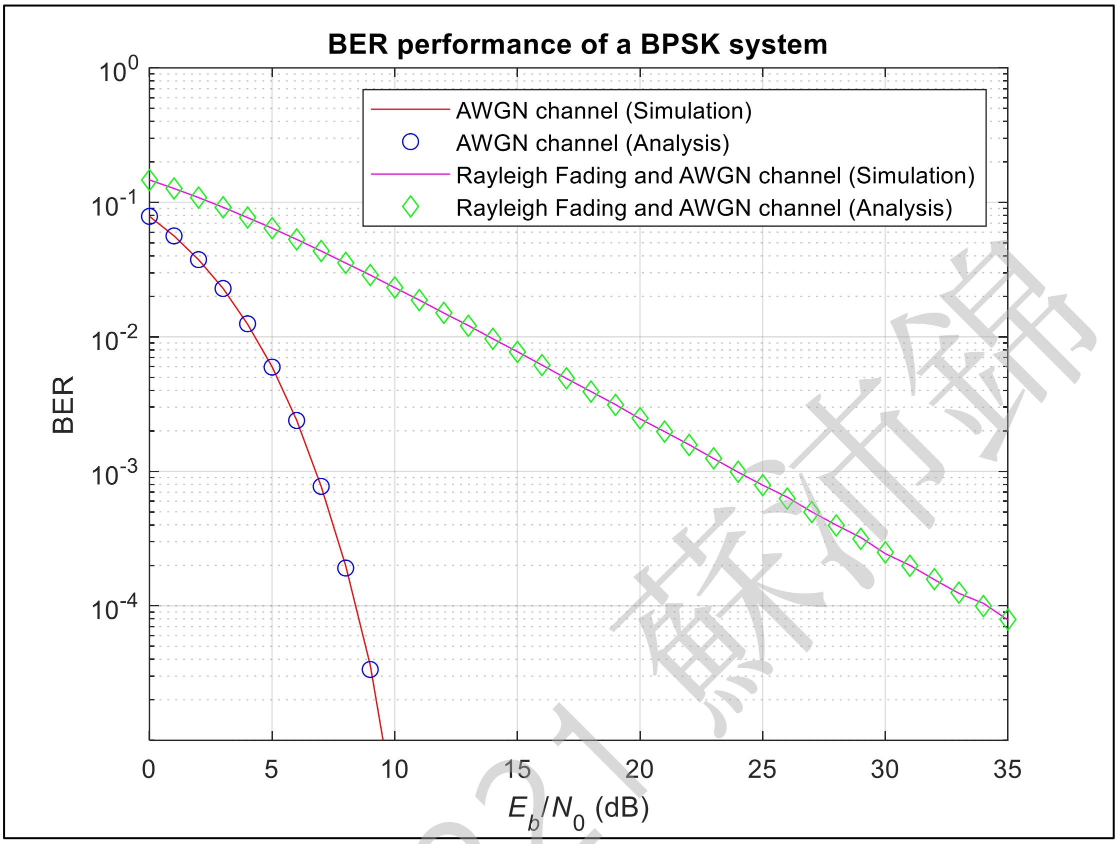
$$= -\lim_{SNR\to\infty} \frac{\cancel{L}\left(-\frac{L}{\cancel{SNR^2}}\right)\frac{\cancel{SNR}}{\cancel{L}}}{\frac{1}{\cancel{SNR}}}$$

$$= \lim_{SNR\to\infty} L = \text{"}L\text{"}$$

$\Longrightarrow$ The diversity of EGC is "L".

3. Matlab Assignment

My Matlab figure:



My Matlab codes:

```
% Problem # 03 (Matlab Assignment)
% Perform simulation and plot the bit error rate (BER) vs Eb/N0 (dB)
% of a BPSK communication system over Rayleigh fading and AWGN channel.
% Written by P.-J. Su 2022/11/08
% Run: To compile and run this computer program using MATLAB.


% Clear Command Window
clc
% Remove all variables from the current workspace,
% releasing them from system memory.
clear
% Close all figures whose handles are visible.
close all
% Start stopwatch timer.
```

```
tic

num_bits = 10^7;    % Number of data bit
Eb = 1;     % Energy per bit
EbOverN0_dB = 0:1:35;     % Eb/N0 in dB
EbOverN0 = 10.^(EbOverN0_dB/10);     % Eb/N0
num_of_bit_error_AWGN = zeros(size(EbOverN0_dB));    % Number of bit error (AWGN)
num_of_bit_error_Rayleigh = zeros(size(EbOverN0_dB));    % Number of bit error (Rayleigh)
BER_theo_AWGN = zeros(size(EbOverN0_dB));
BER_theo_Rayleigh = zeros(size(EbOverN0_dB));

% Mapping to the signal constellation
BPSK = [-sqrt(Eb), sqrt(Eb)];

for Idx = 1: length(EbOverN0_dB)
    tx_bit = ceil(2.*rand(1, num_bits))-1;  %% Transmitted data bit
    tx_sym = BPSK(tx_bit(1:num_bits)+1);  %% Transmitted symbol

    % AWGN channel
    sigma = sqrt(Eb/(2*EbOverN0(Idx)));
    n = randn(1, length(tx_sym))*sigma;

    % Rayleigh Fading channel
    x = randn(1, length(tx_sym))*sqrt(0.5);
    y = randn(1, length(tx_sym))*sqrt(0.5);
    h = sqrt(x.*x + y.*y);

    % The received signal at the detector (AWGN channel)
    rx_sym_AWGN = tx_sym + n;

    rx_bit_AWGN = zeros(size(tx_bit)); %% Reset rx bit decisions
    rx_bit_AWGN(rx_sym_AWGN > 0) = 1;   %% Decision of bit '1'
    err_pat_AWGN = xor(tx_bit, rx_bit_AWGN); %% Compare tx and rx bits

    % The received signal at the detector (Rayleigh)
    rx_sym_Rayleigh = h.*tx_sym + n;

    rx_bit_Rayleigh = zeros(size(tx_bit)); %% Reset rx bit decisions
```

```
    rx_bit_Rayleigh(rx_sym_Rayleigh > 0) = 1;   %% Decision of bit '1'
    err_pat_Rayleigh = xor(tx_bit, rx_bit_Rayleigh); %% Compare tx and rx bits

    % Error counting
    num_of_bit_error_AWGN(Idx) = num_of_bit_error_AWGN(Idx) + sum(err_pat_AWGN);
    num_of_bit_error_Rayleigh(Idx) = num_of_bit_error_Rayleigh(Idx) + sum(err_pat_Rayleigh);

    % Error probability calculation (Analysis)
    BER_theo_AWGN(Idx) = qfunc(sqrt(2*EbOverN0(Idx)));
    SNR = 2*EbOverN0(Idx);
    BER_theo_Rayleigh(Idx) = 0.5*(1-sqrt(SNR/(2+SNR)));
end

% Error probability calculation (Simulation)
BER_sim_AWGN = num_of_bit_error_AWGN/num_bits;
BER_sim_Rayleigh = num_of_bit_error_Rayleigh/num_bits;

% Plot the bit error rate (BER) curves
semilogy(EbOverN0_dB, BER_sim_AWGN, '-r');
hold on
semilogy(EbOverN0_dB, BER_theo_AWGN,'bo');
hold on
semilogy(EbOverN0_dB, BER_sim_Rayleigh, '-m');
hold on
semilogy(EbOverN0_dB, BER_theo_Rayleigh,'gd');
axis([0 35 1e-5 1]);
title('BER performance of a BPSK system');
legend('AWGN channel (Simulation)','AWGN channel (Analysis)', ...
      'Rayleigh Fading and AWGN channel (Simulation)', ...
      'Rayleigh Fading and AWGN channel (Analysis)');
xlabel('\itE_{b}\rm/\itN\rm_{0} (dB)');
ylabel('BER');
grid on
```