(a) discrete-time system model (Equivalent Baseband Model)

$$r(t) = x(t-\tau_0) + n(t), \qquad \tau_0 < t < \tau_0 + T$$

$$= d\cdot\sqrt{\frac{2E_b}{T}}\cos(w_c t - w_c\tau_0) + n(t), \qquad \tau_0 < t < \tau_0 + T$$

$$= d\cdot\sqrt{\frac{2E_b}{T}}\cos(w_c t + \theta) + n(t), \qquad \tau_0 < t < \tau_0 + T$$

denote $\theta \triangleq -w_c\tau_0$

$$z = \int_{\tau_0}^{\tau_0+T} r(t)\cdot 2\cos(w_c t+\theta)\,dt$$

$$= \int_{\tau_0}^{\tau_0+T}\left[d\cdot\sqrt{\frac{2E_b}{T}}\cos(w_c t+\theta) + n(t)\right]\cdot 2\cos(w_c t+\theta)\,dt$$

$$= \int_{\tau_0}^{\tau_0+T}\left[d\cdot\sqrt{\frac{2E_b}{T}}\cos(w_c t+\theta)\right]\cdot 2\cos(w_c t+\theta)\,dt + \underline{\int_{\tau_0}^{\tau_0+T} n(t)\cdot 2\cos(w_c t+\theta)\,dt}$$

$$= \int_{\tau_0}^{\tau_0+T} d\cdot\sqrt{\frac{2E_b}{T}}\cdot 2\cos^2(w_c t+\theta)\,dt + \underbrace{w}_{\text{noise (random variable)}}$$

$$\boxed{2\cos^2(w_c t+\theta) = 1 + \cos(2w_c t + 2\theta)}$$

$$= d\cdot\sqrt{\frac{2E_b}{T}}\cdot\left(\int_{\tau_0}^{\tau_0+T} 1\,dt + \int_{\tau_0}^{\tau_0+T}\cos(2w_c t+2\theta)\,dt\right) + w$$

$$= d\cdot\sqrt{\frac{2E_b}{T}}\cdot T + w$$

$$= \underbrace{d}_{+1 \text{ or } -1}\cdot\sqrt{2E_b T} + w$$

$$\therefore\ z = s + w$$

$$s = \begin{cases} \sqrt{2E_b T}, & \text{if } d = +1 \text{ is transmitted} \\ -\sqrt{2E_b T}, & \text{if } d = -1 \text{ is transmitted} \end{cases}$$

(b) Specify the distribution of the random variable W

- The noise component W is Gaussian distribution because $n(t)$ is WSS White Gaussian process.

$$W = \int_{T_0}^{T_0+T} n(t)\cdot 2\cos(w_c t+\theta)\,dt \quad \Longrightarrow \quad W \sim N(0, \sigma^2)$$

$$E[W] = E\left[\int_{T_0}^{T_0+T} n(t)\cdot 2\cos(w_c t+\theta)\,dt\right]$$

$$= \int_{T_0}^{T_0+T} E[n(t)]\cdot 2\cos(w_c t+\theta)\,dt$$

$$= 0$$

$$var[W] = \sigma^2 = E[W^2]$$

$$= E\left[\left(\int_{T_0}^{T_0+T} n(s)\cdot 2\cos(w_c s+\theta)\,ds\right)\left(\int_{T_0}^{T_0+T} n(t)\cdot 2\cos(w_c t+\theta)\,dt\right)\right]$$

$$= \int_{T_0}^{T_0+T}\int_{T_0}^{T_0+T} E[n(s)n(t)]\cdot 2\cos(w_c s+\theta)\cdot 2\cos(w_c t+\theta)\,ds\,dt$$

$$\boxed{E[n(s)n(t)] = \underbrace{\frac{N_0}{2}}_{\text{two-sided PSD}}\ \underbrace{\delta(s-t)}_{\text{Dirac delta function}}}$$

$$= \int_{T_0}^{T_0+T}\int_{T_0}^{T_0+T} \frac{N_0}{2}\,\delta(s-t)\cdot 2\cos(w_c s+\theta)\cdot 2\cos(w_c t+\theta)\,ds\,dt$$

$$\boxed{\text{Sifting property: } \int_{-\infty}^{\infty} x(t)\cdot\delta(t-t_0)\,dt = x(t_0)}$$

$$= \int_{T_0}^{T_0+T} \frac{N_0}{2}\cdot 2\cdot 2\cos^2(w_c t+\theta)\,dt$$

$$\boxed{2\cos^2(w_c t+\theta) = 1+\cos(2w_c t+2\theta)}$$

$$= N_0\left(\int_{T_0}^{T_0+T} 1\,dt + \int_{T_0}^{T_0+T}\cos(2w_c t+2\theta)\,dt\right)$$

$$= N_0 T$$

$\therefore$ r.v $W \sim N(0, N_0 T)$

$$f_W(w) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{w^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi N_0 T}} e^{-\frac{w^2}{2N_0 T}}$$

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$$(\mu = E[W] = 0,\ \sigma^2 = var[W] = N_0 T)$$

(c) Determine SNR of the decision variable z

$$z = s + w = \underbrace{d}_{+1 \text{ or } -1} \cdot \sqrt{2E_bT} + w$$

$$E[|s|^2] = \left(|d| \cdot \sqrt{2E_bT}\right)^2 = 2E_bT$$

$$E[|w|^2] = \sigma^2 = N_0T$$

$$\therefore \ SNR_z \equiv \frac{E[|s|^2]}{E[|w|^2]} = \frac{2E_bT}{N_0T} = \frac{2E_b}{N_0}$$

(d) theoretical derivation of BER

- transmitted signal

$$s \in \{\underbrace{-\sqrt{2E_bT}}_{s_0}, \underbrace{\sqrt{2E_bT}}_{s_1}\}$$

$$\Longrightarrow s = \begin{cases} \sqrt{2E_bT}, & \text{if } d = +1 \text{ is transmitted} \\ -\sqrt{2E_bT}, & \text{if } d = -1 \text{ is transmitted} \end{cases}$$

- additive Gaussian noise

$$w \sim N(0, \sigma^2), \quad \sigma^2 = N_0T$$

- probability of the transmitted bits

$$P(d=+1) = P(s=s_1) = P_1$$
$$P(d=-1) = P(s=s_0) = 1 - P_1$$

$$\therefore \ z = s + w = \begin{cases} N(s_1, \sigma^2), & s = s_1 \\ N(s_0, \sigma^2), & s = s_0 \end{cases}$$

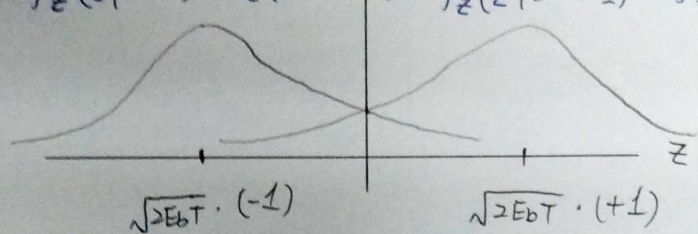
$$(\sigma^2 = N_0T)$$

- The maximum a posteriori probability (MAP) criterion

$$P(d=+1)\cdot f_z(z|d=+1) \underset{s_0}{\overset{s_1}{\gtrless}} P(d=-1) f_z(z|d=-1)$$

$$\Rightarrow P(s=s_1)\cdot f_z(z|s=s_1) \underset{s_0}{\overset{s_1}{\gtrless}} P(s=s_0) f_z(z|s=s_0)$$

$f_z(z|s=s_0)=f_z(z|d=-1)$ $\quad f_z(z|s=s_1)=f_z(z|d=+1)$



$$\Rightarrow P_1 \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(z-s_1)^2}{2\sigma^2}} \underset{s_0}{\overset{s_1}{\gtrless}} (1-P_1)\cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(z-s_0)^2}{2\sigma^2}}$$

$$\Rightarrow \frac{\frac{1}{\sqrt{2\pi}\,\sigma}\cdot e^{-\frac{(z-s_1)^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi}\,\sigma}\cdot e^{-\frac{(z-s_0)^2}{2\sigma^2}}} \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1-P_1}{P_1}$$

$$\Rightarrow e^{\frac{(z-s_0)^2-(z-s_1)^2}{2\sigma^2}} \underset{s_0}{\overset{s_1}{\gtrless}} \frac{1-P_1}{P_1}$$

$$\Rightarrow \frac{2(s_1-s_0)z+(s_0^2-s_1^2)}{2\sigma^2} \underset{s_0}{\overset{s_1}{\gtrless}} \ln\left(\frac{1-P_1}{P_1}\right)$$

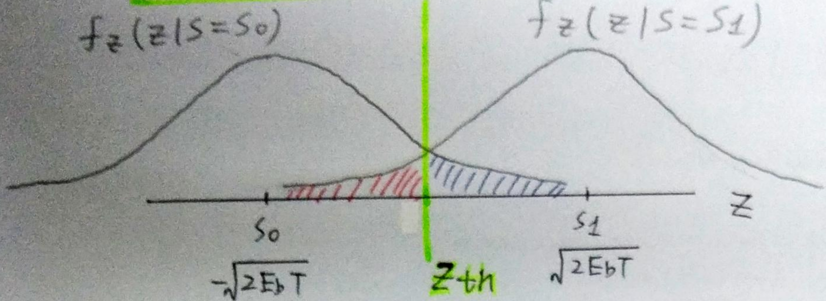
$$\Rightarrow 2(s_1-s_0)z \underset{s_0}{\overset{s_1}{\gtrless}} 2\sigma^2\ln\left(\frac{1-P_1}{P_1}\right)+(s_1^2-s_0^2)$$

$$\Rightarrow z \underset{s_0}{\overset{s_1}{\gtrless}} \frac{2\sigma^2\ln\left(\frac{1-P_1}{P_1}\right)+(s_1^2-s_0^2)}{2(s_1-s_0)}$$

$$\boxed{s_1^2-s_0^2 = 2E_bT-2E_bT=0}$$

$$\Rightarrow z \underset{s_0}{\overset{s_1}{\gtrless}} \frac{\sigma^2}{s_1-s_0}\ln\left(\frac{1-P_1}{P_1}\right) = \underbrace{z_{th}}_{\text{threshold}}$$

$$Z_{th} = \frac{\sigma^2}{S_1 - S_0} \ln\left(\frac{1-P_1}{P_1}\right) = \frac{N_0 T}{2\sqrt{2E_b T}} \ln\left(\frac{1-P_1}{P_1}\right)$$



Error probability of binary anti-podal signaling

$$P_e = P(S=S_1)\cdot\int_{-\infty}^{Z_{th}} f_z(z|S=S_1)\,dz + P(S=S_0)\cdot\int_{Z_{th}}^{\infty} f_z(z|S=S_0)\,dz$$

$$= P_1 \cdot P_r\{\mathcal{N}(\underbrace{\sqrt{2E_bT}}_{\mu=E[w]}, \underbrace{N_0T}_{\sigma^2}) < Z_{th}\} + (1-P_1)\cdot P_r\{\mathcal{N}(-\sqrt{2E_bT}, N_0T) > Z_{th}\}$$

$$= P_1 \cdot P_r\{\mathcal{N}(\sqrt{2E_bT}, N_0T) < Z_{th}\} + (1-P_1)\cdot P_r\{\mathcal{N}(\sqrt{2E_bT}, N_0T) < -Z_{th}\}$$

$$\boxed{P_r\{\mathcal{N}(\mu,\sigma^2) < r\} = \int_{-\infty}^{r} f_x(x)\,dx = \int_{-\infty}^{r} \frac{1}{\sqrt{2\pi}\,\sigma}\cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \underbrace{Q\left(\frac{\mu - x}{\sigma}\right)}_{Q\text{ function}}}$$

$$= P_1 \cdot Q\left(\frac{\sqrt{2E_bT} - Z_{th}}{\sqrt{N_0T}}\right) + (1-P_1)\cdot Q\left(\frac{\sqrt{2E_bT} + Z_{th}}{\sqrt{N_0T}}\right)$$

$$= BER_{MAP}$$

- The maximum likelihood (ML) criterion

$$f_z(z|d=+1) \underset{S_0}{\overset{S_1}{\gtrless}} f_z(z|d=-1)$$

$$\Longrightarrow f_z(z|S=S_1) \underset{S_0}{\overset{S_1}{\gtrless}} f_z(z|S=S_0)$$

The MAP criterion = ML criterion when the a priori probabilities are all equal. $P(S=S_1) = P(S=S_0) = 0.5$

When $P(d=+1) = P(d=-1) = \frac{1}{2} = 0.5$,

$$Z_{th} = \frac{N_0 T}{2\sqrt{2E_b T}} \cdot \ln\left(\frac{0.5}{0.5}\right) = \frac{N_0 T}{2\sqrt{2E_b T}} \ln(1) = 0$$

$$P_e = P_1 \cdot Q\left(\frac{\sqrt{2E_b T} - Z_{th}}{\sqrt{N_0 T}}\right) + (1-P_1) \cdot Q\left(\frac{\sqrt{2E_b T} + Z_{th}}{\sqrt{N_0 T}}\right)$$

$$= \frac{1}{2} \cdot Q\left(\frac{\sqrt{2E_b T}}{\sqrt{N_0 T}}\right) + \frac{1}{2} \cdot Q\left(\frac{\sqrt{2E_b T}}{\sqrt{N_0 T}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b T}{N_0 T}}\right)$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$\boxed{SNR_Z \equiv \frac{E[|S|^2]}{E[|W|^2]} = \frac{2E_b}{N_0}}$$

$$= Q\left(\sqrt{SNR_Z}\right)$$

$$\therefore BER_{ML} = Q\left(\sqrt{SNR_Z}\right)$$