HW - 2

1) a.

Group: $\{y_1, y_2\}$ → Nominal: Normal distribution

$\{y_3, y_4\}$ $\{y_5\}$ → Binary: Bernoulli

Training set $\{x_1 \dots x_6\}$

Priors: $P(N) = 3/6 = 1/2$ $P(P) = 3/6 = 1/2$

$\{y_3, y_4\}$:

N:

	y_3	y_4	
x_1	0	1	N
x_2	0	0	N
x_3	0	1	N

P:

	y_3	y_4	
x_4	1	0	P
x_5	0	0	P
x_6	0	0	P

$P(0,0|N) = 1/3$ $P(0,1|N) = 2/3$ $P(1,0|N) = 0$

$P(1,1|N) = 0$

$P(0,0|P) = 2/3$ $P(0,1|P) = 0$ $P(1,0|P) = 1/3$

$P(1,1|P) = 0$

$\{y_5\}$:

N:

	y_5	
x_1	1	N
x_2	0	N
x_3	1	N

	y_5	
x_4	1	P
x_5	1	P
x_6	1	P

$P(0|N) = 1/3$ $\quad P(1|N) = 2/3$

$P(0|P) = 0$ $\quad P(1|P) = 1$

$\{y_1, y_2\}$:

N:

	y_1	y_2	
x_1	0.52	0.8	N
x_2	0.53	0.92	N
x_3	0.42	0.42	N

P:

	y_1	y_2	
x_4	0.49	0.58	P
x_5	0.62	0.37	P
x_6	0.44	0.38	P

$\mu_{Ny_1} = 0{,}49$ $\quad \mu_{Ny_2} = 0{,}73$ $\quad \mu_N = \begin{pmatrix} 0{,}49 \\ 0{,}73 \end{pmatrix}$

$$\Sigma = \begin{bmatrix} Var(y_1) & Cov(y_1, y_2) \\ Cov(y_1, y_2) & Var(y_2) \end{bmatrix}$$ Para cada classe

$$Var(y_i) = \frac{1}{n-1} \sum_{k=1}^{n} (y_{i,k} - \bar{y}_i)^2$$

$$Cov(y_1, y_2) = \frac{1}{n-1} \sum_{k=1}^{n} (y_{1,k} - \bar{y}_1)(y_{2,k} - \bar{y}_2)$$

Classe N:

$$Var(y_1) = \frac{1}{2}\left[(0,52-0,49)^2 + (0,53-0,49)^2 + (0,42-0,49)^2\right]$$

$$= 0,0037$$

$$Var(y_2) = \frac{(0,8-0,73)^2 + (0,92-0,73)^2 + (0,48-0,73)^2}{2}$$

$$= 0,05175$$

$$Cov(y_1, y_2) = [(0,52-0,49)(0,8-0,73) + (0,53-0,49)$$
$$(0,92-0,73) + (0,42-0,49)(0,48-0,73)]/2$$

$$= 0,0136$$

$$\Sigma_N = \begin{pmatrix} 0,0037 & 0,0136 \\ 0,0136 & 0,05175 \end{pmatrix}$$

$$N\left(\mu = \begin{pmatrix} 0,49 \\ 0,73 \end{pmatrix}, \Sigma = \begin{pmatrix} 0,0037 & 0,00136 \\ 0,0136 & 0,05175 \end{pmatrix} \middle| N \right)$$

Classe P:

$\mu_P\, y_1 = 0,517$ $\quad \mu_P\, y_2 = 0,423$

$$\mu_P = \begin{pmatrix} 0,517 \\ 0,423 \end{pmatrix}$$

$$Var(y_1) = \frac{(0,49-0,517)^2 + (0,62-0,517)^2 + (0,44-0,517)^2}{2}$$

$$= 0,00863$$

$$Var(y_2) = \frac{(0,58-0,423)^2 + (0,31-0,423)^2 + (0,38-0,423)^2}{2}$$

$$= 0,01963$$

$$Cov(y_1,y_2) = [(0,49-0,517)(0,58-0,423) + (0,62-0,517)$$
$$(0,31-0,423) + (0,44-0,517)(0,38-0,423)]/2$$

$$= -0,0063$$

$$N\left(\mu = \begin{pmatrix} 0,517 \\ 0,423 \end{pmatrix}, \Sigma = \begin{pmatrix} 0,00863 & -0,0063 \\ -0,0063 & 0,01963 \end{pmatrix} \Bigg| P\right)$$

b) x_7:

$$\underline{p}(\underline{0,45}\ \underline{0,8}\ \underline{0\ 0\ 1}|N) = \underline{p}(0,45\ 0,8|N)\underline{p}(0\ 0|N)\underline{p}(1|N)$$

$p(0,45\ 0,8|N)$:

$$|\Sigma_N| = 0,0037 \times 0,05175 - (0,0136)^2 = 6,515 \times 10^{-6}$$

$$(y - \mu_N) = \begin{pmatrix} -0,04 \\ 0,07 \end{pmatrix} \quad \Sigma_N^{-1} = \frac{1}{6,515 \times 10^{-6}} \begin{pmatrix} 0,05175 & -0,0316 \\ -0,0316 & 0,0037 \end{pmatrix}$$

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$$= \begin{pmatrix} 7943,21 & -2087,49 \\ -2087,49 & 567,92 \end{pmatrix}$$

CamScanner

$$(-0{,}04 \quad 0{,}07)\begin{pmatrix} 7943{,}21 & -2087{,}49 \\ -2087{,}49 & 567{,}92 \end{pmatrix}\begin{pmatrix} -0{,}04 \\ 0{,}07 \end{pmatrix}$$

$$= 27{,}1819$$

$$P(0{,}45 \quad 0{,}8 \mid N) = \frac{e^{-1/2\,(27{,}1819)}}{2\pi\,(6{,}515 \times 10^{-4})^{1/2}} = 7{,}805 \times 10^{-5}$$

$$p(x_7 \mid N) = 7{,}805 \times 10^{-5} \times 1/3 \times 2/3 \approx 1{,}73 \times 10^{-5}$$

$$P(N \mid x_7) = \frac{1{,}73 \times 10^{-5} \times 0{,}5}{P(x_7)} = \frac{8{,}65 \times 10^{-6}}{P(x_7)}$$

$$P(x_7 \mid P) = P(0{,}45 \quad 0{,}8 \quad 0 \quad 0 \quad 1 \quad 1 \mid P)$$

$$P(0{,}45 \quad 0{,}8 \mid P)$$

$$(Y - \mu_P) = \begin{pmatrix} -0{,}067 \\ 0{,}377 \end{pmatrix}$$

$$|\Sigma_P| = 0{,}00863 \times 0{,}01963 - (0{,}0063)^2$$
$$= 1{,}297 \times 10^{-4}$$

$$\Sigma_P^{-1} = \frac{1}{1{,}297 \times 10^{-4}}\begin{pmatrix} 0{,}01963 & 0{,}0063 \\ 0{,}0063 & 0{,}00863 \end{pmatrix}$$

$$= \begin{pmatrix} 151{,}35 & 48{,}57 \\ 48{,}57 & 66{,}54 \end{pmatrix}$$

$$(-0{,}067 \;\; 0{,}377)\begin{pmatrix} 151{,}35 & 48{,}57 \\ 48{,}57 & 66{,}54 \end{pmatrix}\begin{pmatrix} -0{,}067 \\ 0{,}377 \end{pmatrix}$$

$$= 7{,}6830$$

$$P(0{,}45 \;\; 0{,}8 \mid P) = \frac{e^{-\frac{1}{2}(7{,}683)}}{2\pi \times (1{,}297\times10^{-4})^{1/2}} \approx 0{,}3$$

$$P(x_7 \mid P) = 0{,}3 \times \frac{2}{3} \times 1 = 0{,}2$$

$$P(P \mid x_7) = \frac{0{,}1}{P(x_7)} \rightarrow x_7 \text{ classifies as } P$$

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x_8: $P(x_8 \mid N) \rightarrow$

$$(Y - \mu_N) = \begin{pmatrix} 0{,}01 \\ -0{,}03 \end{pmatrix}$$

$$(0{,}01 \;\; -0{,}03)\begin{pmatrix} 7943{,}21 & -2087{,}49 \\ -2087{,}49 & 567{,}92 \end{pmatrix}\begin{pmatrix} 0{,}01 \\ -0{,}03 \end{pmatrix}$$

$$= 2{,}558$$

$$\frac{e^{-\frac{1}{2}(2{,}558)}}{2\pi(6{,}515\times10^{-4})^{0{,}5}} \approx 17{,}354$$

$$P(x_8 \mid N) = 17{,}354 \times \frac{2}{3} \times \frac{2}{3} = 7{,}713$$

$$P(N \mid x_8) = \frac{7{,}713 \times 0{,}5}{P(x_8)} = \frac{3{,}8565}{P(x_8)}$$

$P(x_8|P)$:

$$(y - \bar{y}/_p) = \begin{pmatrix} -0,017 \\ 0,277 \end{pmatrix}$$

$$(-0,017 \quad 0,277)\begin{pmatrix} 151,35 & 48,57 \\ 48,57 & 66,54 \end{pmatrix}\begin{pmatrix} -0,017 \\ 0,277 \end{pmatrix}$$

$$= 4,6919$$

$$\frac{e^{-\frac{1}{2}(4,6919)}}{2\pi(1,297 \times 10^{-4})^{0.5}} \simeq 1,338 \rightarrow P(x_8|P) = 1 \times 0 \times 1,338 = 0$$

$P(P|x_8) = 0$

x_8 classifies as N

2) $K=3$

i	(y_3, y_4, y_5)	3NN	Pred	True	
x_1	(0 1 1)	$x_8\, x_3\, x_9$	N	N	✓
x_2	(0 0 0)	$x_5\, x_6\, x_7$	P	N	X
x_3	(0 1 1)	$x_1\, x_8\, x_5$	N	N	✓
x_4	(1 0 1)	$x_5\, x_6\, x_7$	P	P	✓
x_5	(0 0 1)	$x_6\, x_7\, x_4$	P	P	✓
x_6	(0 0 1)	$x_5\, x_7\, x_4$	P	P	✓
x_7	(0 0 1)	$x_5\, x_6\, x_8$	P	P	✓
x_8	(0 1 1)	$x_1\, x_3\, x_7$	N	N	✓

Hamming:

$d(x_1, x_2) = 2$ $d(x_1, x_3) = 0$ $d(x_1, x_4) = 2$
$d(x_1, x_5) = 1$ $d(x_1, x_6) = 1$ $d(x_1, x_7) = 1$
$d(x_1, x_8) = 0$

$d(x_2\, x_1) = 2$ $d(x_2\, x_3) = 2$ $d(x_2\, x_4) = 2$
$d(x_2\, x_5) = 1$ $d(x_2\, x_6) = 1$ $d(x_2\, x_7) = 1$
$d(x_2\, x_8) = 2$

$d(x_3\, x_1) = 0$ $d(x_3, x_2) = 2$ $d(x_3\, x_4) = 2$
$d(x_3\, x_5) = 1$ $d(x_3\, x_6) = 1$ $d(x_3\, x_7) = 1$
$d(x_3\, x_8) = 0$

$d(x_4\, x_1) = 2$ $d(x_4\, x_2) = 2$ $d(x_4\, x_3) = 2$
$d(x_4\, x_3) = 1$ $d(x_4\, x_6) = 1$ $d(x_4\, x_7) = 1$
$d(x_4\, x_8) = 2$

$d(x_5\, x_1) = 1$ $d(x_5\, x_2) = 1$ $d(x_5\, x_3) = 1$
$d(x_5\, x_4) = 1$ $d(x_5\, x_6) = 0$ $d(x_5\, x_7) = 0$
$d(x_5\, x_8) = 1$

x_6 e x_7 se comportam igual

$d(x_8\, x_1) = 0$ $d(x_8\, x_2) = 2$ $d(x_8\, x_3) = 0$
$d(x_8\, x_4) = 2$ $d(x_8\, x_5) = 1$ $d(x_8\, x_6) = 1$
$d(x_8\, x_7) = 1$

Accuracy $= \frac{7}{8} \approx 0{,}875 \rightarrow 87{,}5\%$

3) a) $P(\theta=0)=p$, $P(X=n|\theta=0)=P(X=n|\theta=1)$

$\varepsilon_{bayes}=P(\theta\neq\theta_{bayes})$

MAP:

$\theta_{Bayes}(n)=\arg\max \underbrace{P(\theta=c|X=n)}_{\text{Posterior}}$

Teorema de Bayes

$$P(\theta=0|X=n)=\frac{P(X=n|\theta=0)P(\theta=0)}{P(X=n)}$$

$$P(X=n)=P(X=n|\theta=0)P(\theta=0)+P(X=n|\theta=1)P(\theta=1)$$
$$=P(X=n|\theta=0)(\cancel{p+1-p})=P(X=n|\theta=0)$$

$P(\theta=0|X=n)=P(\theta=0)=p$

Como $p\in[1/2,1]$, $p\geq 1-p$

$\theta_{Bayes}=P(\theta=0|X=n)=p$ $\quad\varepsilon_{Bayes}\to P(\theta\neq 0|X=n)=1-p$

b. $P(\theta=0|X=n)=P(\theta=0)=p$

$$P(\theta=1|X=n)=\frac{\cancel{P(X=n|\theta=0)}P(\theta=1)}{\cancel{P(X=n|\theta=0)}}=P(\theta=1)$$

com $P(\theta=1)=1-p$

$2\times P(\theta=0|X=n)\times P(\theta=1|X=n)=2p(1-p)$

c. $\varepsilon_{bayes} = 1 - p \quad 1 - \varepsilon_{bayes} = p$

Pela alínea anterior: $\varepsilon_{1NN} = 2p(1-p)$

$\varepsilon_{1NN} = 2 \times \varepsilon_{bayes} \times (1 - \varepsilon_{bayes})$