

## EOQ Model

The first model we will present is called the economic order quantity (EOQ) model. This model is studied first owing to its simplicity. Simplicity and restrictive modeling assumptions usually go together, and the EOQ model is not an exception. However, the presence of these modeling assumptions does not mean that the model cannot be used in practice. There are many situations in which this model will produce good results. For example, these models have been effectively employed in automotive, pharmaceutical, and retail sectors of the economy for many years. Another advantage is that the model gives the optimal solution in closed form. This allows us to gain insights about the behavior of the inventory system. The closed-form solution is also easy to compute (compared to, for example, an iterative method of computation).

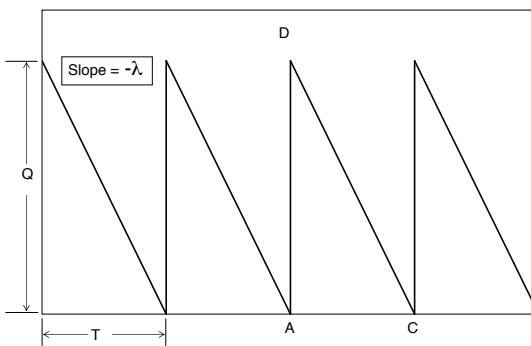
In this chapter, we will develop several models for a single-stage system in which we manage inventory of a single item. The purpose of these models is to determine how much to purchase (order quantity) and when to place the order (the reorder point). The common thread across these models is the assumption that demand occurs continuously at a constant and known rate. We begin with the simple model in which all demand is satisfied on time. In Section 2.2, we develop a model in which some of the demand could be backordered. In Section 2.3, we consider the EOQ model again; however, the unit purchasing cost depends on the order size. In the final section, we briefly discuss how to manage many item types when constraints exist that link the lot size decisions across items.

## 2.1 Model Development: Economic Order Quantity (EOQ) Model

We begin with a discussion of various assumptions underlying the model. This discussion is also used to present the notation.

1. Demand arrives continuously at a constant and known rate of  $\lambda$  units per year. Arrival of demand at a continuous rate implies that the optimal order quantity may be non-integer. The fractional nature of the optimal order quantity is not a significant problem so long as the order quantity is not very small; in practice, one simply rounds off the order quantity. Similarly, the assumption that demand arrives at a constant and known rate is rarely satisfied in practice. However, the model produces good results where demand is relatively stable over time.
2. Whenever an order is placed, a fixed cost  $K$  is incurred. Each unit of inventory costs  $\$I$  to stock per year per dollar invested in inventory. Therefore, if a unit's purchasing cost is  $C$ , it will cost  $I \cdot C$  to stock one unit of that item for a year.
3. The order arrives  $\tau$  years after the placement of the order. We assume that  $\tau$  is deterministic and known.
4. All the model parameters are unchanging over time.
5. The length of the planning horizon is infinite.
6. All the demand is satisfied on time.

Our goal is to determine the order quantity and the reorder interval. Since all the parameters are stationary over time, the order quantity, denoted by  $Q$ , also remains stationary. The reorder interval is related to when an order should be placed, since



**Fig. 2.1.** Change in inventory over time for the EOQ model.

a reorder interval is equal to the time between two successive epochs at which an order is placed and is called the cycle length. A cycle is the time between the placing of two successive orders. The question of when to place an order has a simple answer in this model. Since demand occurs at a deterministic and fixed rate and the order once placed arrives exactly  $\tau$  years later, we would want the order to

arrive exactly when the last unit is being sold. Thus the order should be placed  $\tau$  years before the depletion of inventory.

The first step in the development of the model is the construction of cost expressions. Since total demand per year is  $\lambda$ , the total purchasing cost for one year is  $C\lambda$ . Similarly, the number of orders placed per year is equal to  $\lambda/Q$ . Therefore, the total annual average cost of placing orders is  $K\lambda/Q$ . The derivation of the total holding cost per year is a bit more involved. We will begin by first computing the average inventory per cycle. Since each cycle is identical to any other cycle, the average inventory per year is the same as the average inventory per cycle. The holding cost is equal to the average inventory per year times the cost of holding one unit of inventory for one year. Using Figure 2.1, we find the average inventory per cycle is equal to:

$$\frac{\text{Area of triangle } ADC}{\text{Length of the cycle}} = \frac{\frac{1}{2}QT}{T} = \frac{Q}{2}.$$

The annual cost of holding inventory is thus equal to  $ICQ/2$ .

Adding the three types of costs together, we get the following objective function, which we want to minimize over  $Q$ :

$$\min_{Q \geq 0} Z(Q) = C\lambda + \frac{K\lambda}{Q} + \frac{ICQ}{2}. \quad (2.1)$$

Before we compute the optimal value of  $Q$ , let us take a step back and think about what the optimal solution should look like. First, the higher the value of the fixed cost  $K$ , the fewer the number of orders that should be placed every year. This means that the quantity ordered per order will be high. Second, if the holding cost rate is high, placing orders more frequently is economical since inventory will on average be lower. A higher frequency of order placement leads to lower amounts ordered per order. Therefore, our intuition tells us that the optimal order quantity should increase as the fixed ordering cost increases and decrease as the holding cost rate increases.

To compute the optimal order quantity, we take the first derivative of  $Z(Q)$  with respect to  $Q$  and set it equal to zero:

$$\begin{aligned} \frac{dZ}{dQ} &= 0 - K\frac{\lambda}{Q^2} + \frac{IC}{2} = 0 \\ \text{or } Q^* &= \sqrt{\frac{2K\lambda}{IC}}, \end{aligned} \quad (2.2)$$

where  $Q^*$  is the optimal order quantity. Note that the derivative of the purchasing cost  $C\lambda$  is zero since it is independent of  $Q$ . The following examples illustrate the computation of the optimal order quantity using (2.2).

In our first example, we assume an office supplies store sees a uniform demand rate of 10 boxes of pencils per week. Each box costs \$5. If the fixed cost of placing an

order is \$10 and the holding cost rate is .20 per year, let us determine the optimal order quantity using the EOQ model. Assume 52 weeks per year.

In this example  $K = 10$ ,  $I = .20$ ,  $C = 5$ , and the annual demand rate is  $\lambda = (10)(52) = 520$ . Substituting these values in (2.2), we get

$$Q^* = \sqrt{\frac{2(520)(10)}{(0.2)(5)}} = 101.98 \approx 102.$$

In our second example, suppose the regional distribution center (RDC) for a major auto manufacturer stocks approximately 20,000 service parts. The RDC fulfills demands of dozens of dealerships in the region. The RDC places orders with the national distribution center (NDC), which is also owned by the auto manufacturer. Given the huge size of these facilities, it is deemed impossible to coordinate the inventory management of the national and regional distribution centers. Accordingly, each RDC manages the inventory on its own regardless of the policies at the NDC.

We consider one part, a tail light, for a specific car model. The demand for this part is almost steady throughout the year at a rate of 100 units per week. The purchasing cost of the tail light paid by the RDC to the NDC is \$10 per unit. In addition, the RDC spends on average \$0.50 per unit in transportation. A breakdown of the different types of costs is as follows:

1. The RDC calculates its interest rate to be 15% per year.
2. The cost of maintaining the warehouse and its depreciation is \$100,000 per year, which is independent of the amount of inventory stored there. In addition, the costs of pilferage and misplacement of inventory are estimated to be 5 cents per dollar of average inventory stocked.
3. The annual cost of a computer-based order management system is \$50,000 and is not dependent on how often orders are placed.
4. The cost of invoice preparation, postage, time, etc. is estimated to be \$100 per order.
5. The cost of unloading every order that arrives is estimated to be \$10 per order.

Let us determine the optimal order quantity. The first task is to determine the cost parameters. The holding cost rate  $I$  is equal to the interest rate (.15) plus the cost rate for pilferage and misplacement of inventory (.05). Therefore,  $I = .20$ . This rate applies to the value of the inventory when it arrives at the RDC. This value includes not only the purchasing cost (\$10) but also the value added through transportation (\$0.5). Therefore, the value of  $C$  is \$10.50. Finally, the fixed cost of order placement includes all costs that depend on the order frequency. Thus, it includes the order receiving cost (\$10) and the cost of invoice preparation, etc. (\$100), but not the cost of the order management system. Therefore,  $K = 110$ . We now substitute these parameters into (2.2) to get the optimal order quantity:

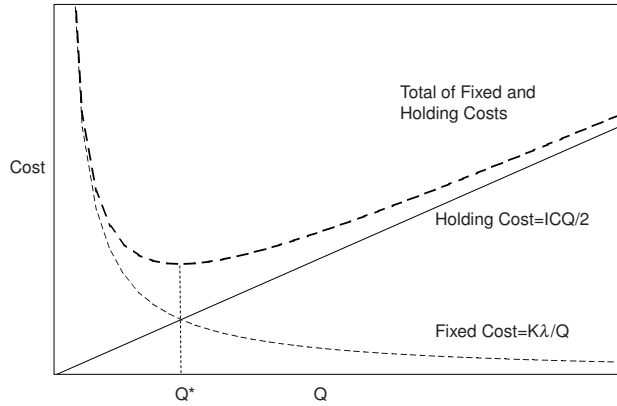
$$Q^* = \sqrt{\frac{2\lambda K}{IC}} = \sqrt{\frac{2(5200)(110)}{(0.2)(10.5)}} = 738.08 \text{ units.}$$

Next, let us determine whether or not the optimal EOQ solution matches our intuition. If the fixed cost  $K$  increases, the numerator of (2.2) increases and the optimal order quantity  $Q^*$  increases. Similarly, as the holding cost rate  $I$  increases, the denominator of (2.2) increases and the optimal order quantity  $Q^*$  decreases. Clearly, the solution fulfills our expectations.

To gain more insights, let us explore additional properties the optimal solution possesses. Figure 2.2 shows the plot of the average annual fixed order cost  $K\lambda/Q$  and the annual holding cost  $ICQ/2$  as functions of  $Q$ . The average annual fixed order cost decreases as  $Q$  increases because fewer orders are placed. On the other hand, the average annual holding cost increases as  $Q$  increases since units remain in inventory longer. Thus the order quantity affects the two types of costs in opposite ways. The annual fixed ordering cost is minimized by making  $Q$  as large as possible, but the holding cost is minimized by having  $Q$  as small as possible. The two curves intersect at  $Q = Q_1$ . By definition of  $Q_1$ ,

$$K \frac{\lambda}{Q_1} = \frac{ICQ_1}{2} \Rightarrow Q_1 = \sqrt{\frac{2K\lambda}{IC}}$$

and, in this case,  $Q_1 = Q^*$ . The exact balance of the holding and setup costs yields the optimal order quantity. In other words, the optimal solution is the *best compromise* between the two types of costs. (As we will see throughout this book, inventory models are based on finding the best compromise between opposing costs.) Since the annual holding cost  $ICQ^*/2$  and the fixed cost  $K\lambda/Q^*$  are equal in the optimal solution, the optimal average annual total cost is equal to



**Fig. 2.2.** Fixed and holding costs as functions of the order quantity.

$$\begin{aligned}
Z(Q^*) &= C\lambda + \frac{ICQ^*}{2} + K\frac{\lambda}{Q^*} = C\lambda + 2K\frac{\lambda}{Q^*} \\
&= C\lambda + 2K\frac{\lambda}{\sqrt{\frac{2\lambda K}{IC}}} = C\lambda + \sqrt{2\lambda KIC},
\end{aligned} \tag{2.3}$$

where we substitute for  $Q^*$  using (2.2).

Let us compute the optimal average annual cost for the office supplies example. Using (2.3), the optimal cost is equal to

$$C\lambda + \sqrt{2\lambda KIC} = (5)(520) + \sqrt{2(520)(10)(5)(0.2)} = \$2701.98.$$

The purchasing cost is equal to  $(\$5)(520) = \$2600$ , and the holding and order placement costs account for the remaining cost of \$101.98.

### 2.1.1 Robustness of the EOQ Model

In the real world, it is often difficult to estimate the model parameters accurately. The cost and demand parameter values used in models are at best an approximation to their actual values. The policy computed using the approximated parameters, henceforth referred to as approximated policy, cannot be optimal. The optimal policy cannot be computed without knowing the true values of the model's parameters. Clearly, if another policy is used, the realized cost will be greater than the cost of the true optimal policy. The following example illustrates this point.

Suppose in the office supplies example that the fixed cost of order placement is estimated to be \$4 and the holding cost rate is estimated to be .15. Let us calculate the alternative policy and the cost difference between employing this policy and the optimal policy. Recall that the average annual cost incurred when following the optimal policy is \$2701.98. To compute the alternative policy, we substitute the estimated parameter values into (2.2):

$$Q^* = \sqrt{\frac{2(520)(4)}{(0.15)(5)}} = 74.48.$$

The realized average annual cost if this policy is used when  $K = 10$  is

$$\begin{aligned}
Z(Q^*) &= C\lambda + \frac{K\lambda}{Q^*} + \frac{ICQ^*}{2} \\
&= (5)(520) + \frac{(10)(520)}{74.48} + \frac{(0.2)(5)(74.48)}{2} = \$2707.06.
\end{aligned}$$

Thus the cost difference between the alternative and optimal policies is  $\$2707.06 - \$2701.98 = \$5.08$ . Note that the cost of implementing the alternative policy is calculated using the actual cost parameters.

Let us now derive an upper bound on the realized average annual cost of using the approximate policy relative to the optimal cost. Suppose the *actual* order quantity is denoted by  $Q_a^*$ . This is the answer we would get from (2.2) if we could use the true cost and demand parameters. Let the true fixed cost and holding cost rate be denoted by  $K_a$  and  $I_a$ , respectively. We assume that the purchasing cost  $C$  and the demand rate  $\lambda$  have been estimated accurately. The estimates of the fixed cost and holding cost rate are denoted by  $K$  and  $I$ , respectively. The estimated order quantity is denoted by  $Q^*$ . Let  $Q^*/Q_a^* = \alpha$  or  $Q^* = \alpha Q_a^*$ . Thus

$$\begin{aligned} Q_a^* &= \sqrt{\frac{2\lambda K_a}{I_a C}} \\ Q^* &= \sqrt{\frac{2\lambda K}{I C}} = \alpha \sqrt{\frac{2\lambda K_a}{I_a C}} \\ \Rightarrow \alpha &= \sqrt{\left(\frac{K}{I}\right) \left(\frac{I_a}{K_a}\right)}. \end{aligned}$$

Since the purchasing cost  $C\lambda$  is not influenced by the order quantity, we do not include it in the comparison of costs. The true average annual operating cost (the sum of the holding and order placement costs) is equal to

$$Z(Q_a^*) = \sqrt{2K_a\lambda I_a C}.$$

The *actual* incurred average annual cost (sum of the holding and order placement costs) corresponding to the estimated order quantity  $Q^*$  is equal to

$$\begin{aligned} Z(Q^*) &= \frac{K_a\lambda}{Q^*} + \frac{I_a C Q^*}{2} \\ &= \frac{K_a\lambda}{\alpha Q_a^*} + \frac{I_a C (\alpha Q_a^*)}{2} \\ &= \frac{K_a\lambda}{\alpha \sqrt{\frac{2\lambda K_a}{I_a C}}} + \frac{I_a C \alpha \sqrt{\frac{2\lambda K_a}{I_a C}}}{2} \\ &= \frac{1}{2} \left( \alpha + \frac{1}{\alpha} \right) \sqrt{2K_a\lambda I_a C} = \frac{1}{2} \left( \alpha + \frac{1}{\alpha} \right) Z(Q_a^*). \end{aligned}$$

Therefore, if the estimated order quantity is  $\alpha$  times the optimal order quantity, the average annual cost corresponding to the estimated order quantity is  $\frac{1}{2}(\alpha + \frac{1}{\alpha})$  times the optimal cost. For example, if  $\alpha = 2$  (or  $\frac{1}{2}$ ), that is, the estimated order quantity is 100% greater (or 50% lower) than the optimal order quantity, then the cost corresponding to the estimated order quantity is 1.25 times the optimal cost. Similarly, when  $\alpha = 3$  (or  $\frac{1}{3}$ ), the cost corresponding to the estimated order quantity is approximately 1.67 times the optimal cost.

Two observations can be made. First, and importantly, even for significant inaccuracies in the order quantity, the cost increase is modest. As we showed, the cost increase is only 25% for a 100% increase in the estimated order quantity from the optimal order quantity. The moderate effect of inaccuracies in the cost parameters on the actual incurred average annual cost is very profound. Second, the cost increase is symmetric around  $\alpha = 1$  in a multiplicative sense. That is, the cost increase is the same for  $Q^*/Q_a^* = \alpha$  or  $Q^*/Q_a^* = \frac{1}{\alpha}$ . This observation will be useful in the discussion presented in the following chapter.

How do we estimate  $\alpha$ ? Clearly, if we could estimate  $\alpha$  precisely, then we could compute  $Q_a^*$  precisely as well and there would be no need to use the estimated order quantity. Since we cannot ascertain its value with certainty, perhaps we can estimate upper and lower bounds for  $\alpha$ . These bounds can give us bounds on the cost of using the estimated order quantity relative to the optimal cost. The following example illustrates this notion in more detail.

Suppose in the office supplies example that the retailer is confident that his *actual* fixed cost is at most 120% but no less than 80% of the estimated fixed cost. Similarly, he is sure that his actual holding cost rate is at most 110% but no less than 90% of the estimated holding cost rate. Let us determine the maximum deviation from the optimal cost by implementing a policy obtained on the basis of the estimated parameter values.

We are given that

$$0.8 \leq \frac{K_a}{K} \leq 1.2,$$

and that

$$0.9 \leq \frac{I_a}{I} \leq 1.1.$$

The cost increases when the estimated order quantity is either less than or more than the optimal order quantity. Our approach will involve computing the lower and upper bounds on  $\alpha$  and then computing the cost bounds corresponding to these values of  $\alpha$ . The maximum of these cost bounds will be the maximum possible deviation of the cost of using the estimated order quantity relative to the optimal cost.

Since  $\alpha = \sqrt{\left(\frac{K}{I}\right) \left(\frac{I_a}{K_a}\right)}$ , we use the upper bound on  $I_a/I$  and the lower bound on  $K_a/K$  to get an upper bound on  $\alpha$ . Thus,



$$\alpha \leq \sqrt{1.1 \times \frac{1}{0.8}} = 1.17.$$

The cost of using the estimated order quantity corresponding to  $\alpha = 1.17$  is  $\frac{1}{2}(1.17 + \frac{1}{1.17}) = 1.013$  times the optimal cost.

Similarly, to get a lower bound on  $\alpha$ , we use the lower bound on  $I_a/I$  and the upper bound on  $K_a/K$ . Thus,

$$\alpha \geq \sqrt{0.9 \times \frac{1}{1.2}} = 0.87.$$

The cost of using the estimated order quantity corresponding to  $\alpha = 0.87$  is  $\frac{1}{2}(0.87 + \frac{1}{0.87}) = 1.010$  times the optimal cost. The upper bound on the cost is the maximum of 1.01 and 1.013 times the optimal cost. Therefore, the cost of using the estimated order quantity is at most 1.3% higher than would be obtained if the optimal policy were implemented.

### 2.1.2 Reorder Point and Reorder Interval

In the EOQ model, the demand rate and lead time are known with certainty. Therefore, an order is placed such that the inventory arrives exactly when it is needed. This means that if the inventory is going to be depleted at time  $t$  and the lead time is  $\tau$ , then an order should be placed at time  $t - \tau$ . If we place the order before time  $t - \tau$ , then the order will arrive before time  $t$ . Clearly holding costs can be eliminated by having the order arrive at time  $t$ . On the other hand, delaying the placement of an order so that it arrives after time  $t$  is not permissible since a backorder will occur.

How should we determine the reorder point in terms of the inventory remaining on the shelf? There are two cases depending upon whether the lead time is less than or greater than the reorder interval, that is, whether  $\tau \leq T$  or  $\tau > T$ . We discuss the first case here; the details for the second case are left as an exercise. Since the on-hand inventory at the time an order arrives is zero, the inventory at time  $t - \tau$  should be equal to the total demand realized during the time interval  $(t - \tau, t]$ , which is equal to  $\lambda \tau$ . Therefore, the reorder point when  $\tau \leq T$  is equal to

$$r^* = \lambda \tau. \quad (2.4)$$

In other words, whenever the inventory drops to the level  $\lambda \tau$ , an order must be placed. Observe that  $r^*$  does not depend on the optimal order quantity.

On the other hand, when  $\tau > T$ , the reorder point is equal to

$$r^* = \lambda \tau_1, \quad (2.5)$$

where  $\tau_1$  is the remainder when  $\tau$  is divided by  $T$ . That is,  $\tau = mT + \tau_1$ , where  $m$  is a positive integer.

The time between the placement of two successive orders,  $T$ , is equal to the time between the receipt of two successive order deliveries, since the lead time is a known constant. Since orders are received when the inventory level is zero, the quantity received,  $Q^*$ , is consumed entirely at the demand rate  $\lambda$  by the time the next order is received. Therefore, if the optimal reorder interval is denoted by  $T^*$ ,  $Q^* = \lambda T^*$ . Hence

$$T^* = \frac{Q^*}{\lambda} = \sqrt{\frac{2K}{\lambda IC}}. \quad (2.6)$$

Suppose in our office supplies example that the lead time is equal to 2 weeks. In this case, the reorder interval  $T^*$  is equal to

$$T^* = \frac{Q^*}{\lambda} = \frac{101.98}{520} = 0.196 \text{ year} = 10.196 \text{ weeks}.$$

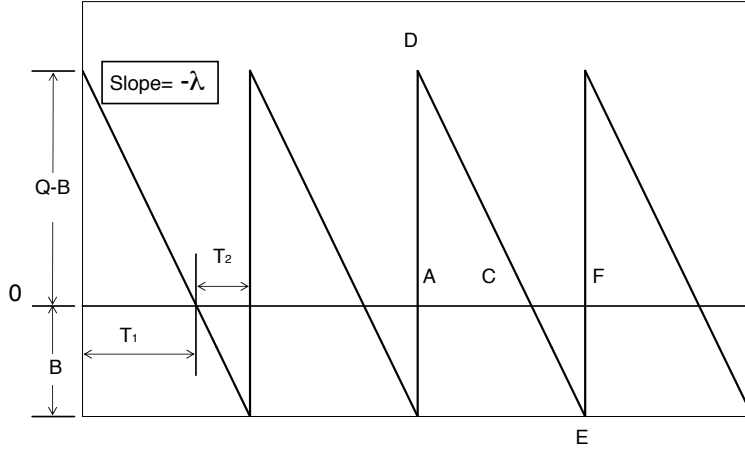
Since  $\tau = \frac{2}{52}$  years, which is less than the reorder interval, we can use (2.4) to find the reorder point. The reorder point  $r^*$  is equal to

$$r^* = \lambda \tau = (520)(2/52) = 20.$$

## 2.2 EOQ Model with Backordering Allowed

In this section we will relax one of the assumptions we have made about satisfying all demand on time. We will now allow some of the demand to be backordered, but there will be a cost penalty incurred. The rest of the modeling assumptions remain unaltered. As a result, the cost function now consists of four components: the purchasing cost, the fixed cost of order placement, the inventory holding cost, and the backlog penalty cost. The system dynamics are shown in Figure 2.3.

Each order cycle is comprised of two sub-cycles. The first sub-cycle (ADC) is  $T_1$  years long and is characterized by positive on-hand inventory which decreases at the demand rate of  $\lambda$ . The second sub-cycle (CEF) is  $T_2$  years long, during which demand is backordered. Hence there is no on-hand inventory during this time period. Since no demand is satisfied in this latter period, the backlog increases at the demand rate  $\lambda$ . The total length of a cycle is  $T = T_1 + T_2$ .



**Fig. 2.3.** Change in inventory over time for the EOQ model with backordering allowed.

There are two decisions to be made: How much to order whenever an order is placed, and how large the maximum backlog level should be in each cycle. The order quantity is denoted by  $Q$  as before, and we use  $B$  to denote the maximum amount of backlog allowed. When an order arrives, all the backordered demand is satisfied immediately. Thus, the remaining  $Q - B$  units of on-hand inventory satisfies demand in the first sub-cycle. Since this on-hand inventory decreases at rate  $\lambda$  and becomes zero in  $T_1$  years,

$$Q - B = \lambda T_1. \quad (2.7)$$

In the second sub-cycle, the number of backorders increases from 0 to  $B$  at rate  $\lambda$  over a period of length  $T_2$  years. Thus

$$B = \lambda T_2 \quad (2.8)$$

and

$$Q = \lambda(T_1 + T_2) = \lambda T. \quad (2.9)$$

The cost expressions for the purchasing cost and annual fixed ordering cost remain the same as for the EOQ model and are equal to  $C\lambda$  and  $K\lambda/Q$ , respectively. The expression for the average annual holding cost is different. We first compute the average inventory per cycle and then multiply the result by the holding cost  $IC$  to get the annual holding cost. The average inventory per cycle is equal to

$$\frac{\text{Area of triangle ADC}}{T} = \frac{\frac{(Q-B)T_1}{2}}{T}.$$

We next substitute for  $T_1$  and  $T$ . This results in an expression which is a function only of  $Q$  and  $B$ , our decision variables.

$$\text{Average Inventory per Cycle} = \frac{\frac{(Q-B)^2}{\lambda}}{2\frac{Q}{\lambda}} = \frac{(Q-B)^2}{2Q}.$$

The computation of the average annual backordering cost is similar. We let  $\pi$  be the cost of backordering a unit for one year. The first step is to compute the average number of backorders per cycle. Since all cycles are alike, this means that the average number of outstanding backorders per year is the same as the average per cycle. To get the average backorder cost per year, we multiply the average backorder quantities per year by the backorder cost rate. The average number of backorders per cycle is equal to the area of triangle CEF in Figure 2.3 divided by the length of the cycle  $T$ :

$$\frac{\text{Area of triangle CEF}}{T} = \frac{\frac{BT_2}{2}}{T} = \frac{\frac{B^2}{2\lambda}}{\frac{Q}{\lambda}} = \frac{B^2}{2Q}.$$

In the last equality, we have used the relationships  $T_2 = B/\lambda$  and  $T = Q/\lambda$ . Therefore, the average annual backorder cost is equal to  $\pi \frac{B^2}{2Q}$ .

We now combine all the cost components and express the average annual cost of managing inventory as

$$Z(B, Q) = C\lambda + K\frac{\lambda}{Q} + IC\frac{(Q-B)^2}{2Q} + \pi\frac{B^2}{2Q}. \quad (2.10)$$

Before we obtain the optimal solution, let us anticipate what properties we expect the optimal solution to possess. As before, if the fixed order cost  $K$  increases, fewer orders will be placed, which will increase the order quantity. An increase in the holding cost rate should drive the order quantity to lower values. The effect of the backorder cost on the maximum possible number of units backordered should be as follows: the higher the backorder cost, the lower the maximum desirable number of backorders. We will state additional insights after deriving the optimal solution.

To obtain the optimal solution, we take the first partial derivatives of  $Z(B, Q)$  in (2.10) with respect to  $Q$  and  $B$  and set them equal to zero. This yields two simultaneous equations in  $Q$  and  $B$ :

$$\frac{\partial Z(B, Q)}{\partial Q} = -K \frac{\lambda}{Q^2} + IC \frac{(Q-B)}{Q} - IC \frac{(Q-B)^2}{2Q^2} - \pi \frac{B^2}{2Q^2}, \quad (2.11)$$

$$\frac{\partial Z(B, Q)}{\partial B} = -IC \frac{(Q-B)}{Q} + \pi \frac{B}{Q}. \quad (2.12)$$

Equation (2.12) appears simpler, so let us set it to zero first. This results in

$$B = Q \frac{IC}{IC + \pi}. \quad (2.13)$$

Substituting for  $B$  in (2.11) results in

$$\begin{aligned} \frac{\partial Z(B, Q)}{\partial Q} &= -K \frac{\lambda}{Q^2} + IC \frac{\pi}{IC + \pi} - IC \frac{(\frac{\pi}{IC + \pi})^2}{2} - \pi \frac{(\frac{IC}{IC + \pi})^2}{2} \\ &= -K \frac{\lambda}{Q^2} + IC \frac{\pi}{IC + \pi} - IC \frac{\pi}{2(IC + \pi)} \\ &= -K \frac{\lambda}{Q^2} + IC \frac{\pi}{2(IC + \pi)} \\ \Rightarrow Q^* &= \sqrt{\frac{2K\lambda(IC + \pi)}{IC\pi}} = \sqrt{\frac{(IC + \pi)}{\pi}} \sqrt{\frac{2K\lambda}{IC}} = Q_E \sqrt{\frac{IC + \pi}{\pi}}, \end{aligned} \quad (2.14)$$

where  $Q_E$  is the optimal solution to the EOQ model. Also,

$$\begin{aligned} B^* &= Q^* \frac{IC}{IC + \pi} = \sqrt{\frac{2K\lambda IC}{(IC + \pi)\pi}}, \\ T^* &= \frac{Q^*}{\lambda} = \sqrt{\frac{2K(IC + \pi)}{\lambda IC \pi}}. \end{aligned} \quad (2.15)$$

Again, let us return to our office supplies example. Assume now that the pencils sold by the office-supplies retailer are somewhat exotic and that they are not available anywhere else in the town. This means that the customers wait if the retailer runs out of the pencil boxes. Sensing this, the retailer now wants to allow backordering of demand when determining the inventory replenishment policy. Suppose that the cost to backorder a unit per year is \$10. Let us determine the optimal order quantity and the maximum quantity of backorders that will be permitted to accumulate. Since  $IC = (0.2)(5) = 1$ ,  $\pi = 10$ , and  $Q_E = 101.98$ , we can use (2.14) to determine

$$Q^* = Q_E \sqrt{\frac{IC + \pi}{\pi}} = (101.98) \sqrt{\frac{1 + 10}{10}} = 106.96.$$

Next, we use (2.15) to determine the maximum level of backlog

$$B^* = Q^* \frac{IC}{IC + \pi} = (106.96) \frac{1}{1 + 10} = 9.72.$$

Several observations can now be made.

1. Our pre-derivation intuition holds. As  $K$  increases,  $Q^*$  increases, implying a decrease in the number of orders placed per year. As the holding cost rate  $I$  increases,  $\frac{IC + \pi}{IC} = 1 + \frac{\pi}{IC}$  decreases, and  $Q^*$  decreases. Finally, as  $\pi$  increases, the denominator of (2.15) increases, and  $B^*$  decreases.
2. The maximum number of backorders per cycle,  $B^*$ , cannot be more than the order quantity per cycle  $Q^*$  since  $\frac{B^*}{Q^*} = \frac{IC}{IC + \pi} \leq 1$ .
3.  $Q^*$  is never smaller than  $Q_E$ , the optimal order quantity for the EOQ model without backordering. Immediately after the arrival of an order, part of the order is used to fulfill the backordered demand. This saves the holding cost on that part of the received order and allows the placement of a bigger order. As  $\pi$  increases, backordering becomes more expensive and  $B^*$  decreases. As a result, the component of  $Q^*$  that is used to satisfy the backlog decreases and  $Q^*$  comes closer to  $Q_E$ .
4. As the holding cost rate  $I$  increases,  $\frac{IC}{IC + \pi}$  increases and  $B^*$  increases.

To improve our understanding even further, let us compare the fixed cost to the sum of the holding and backordering costs at  $Q = Q^*$  and  $B = B^*$ .

From (2.15),  $\frac{B^*}{Q^*} = \frac{IC}{IC + \pi}$ , which means that

$$\frac{Q^* - B^*}{Q^*} = 1 - \frac{B^*}{Q^*} = 1 - \frac{IC}{IC + \pi} = \frac{\pi}{IC + \pi}.$$

Now, the holding cost at  $Q = Q^*$  and  $B = B^*$  is equal to

$$\frac{IC(Q^* - B^*)^2}{2Q^*} = \frac{ICQ^*}{2} \left( \frac{Q^* - B^*}{Q^*} \right)^2 = \frac{ICQ^*}{2} \left( \frac{\pi}{IC + \pi} \right)^2.$$

Similarly, the backordering cost at  $Q = Q^*$  and  $B = B^*$  is equal to

$$\frac{\pi B^{*2}}{2Q^*} = \frac{\pi}{2} \left( \frac{IC}{IC + \pi} \right)^2 Q^*.$$

Therefore, the sum of the holding and backordering costs is equal to

$$\frac{Q^*}{2} \left( \frac{\pi^2 IC + (IC)^2 \pi}{(IC + \pi)^2} \right) = \frac{Q^*}{2} \left( \frac{IC \pi}{IC + \pi} \right),$$

which, after substitution of  $Q^*$ , is equal to  $\sqrt{\frac{\lambda K I C \pi}{2(IC + \pi)}}$ .

On the other hand, the fixed cost of order placement at  $Q = Q^*$  is equal to

$$\frac{\lambda K}{Q^*} = \sqrt{\frac{\lambda K I C \pi}{2(IC + \pi)}},$$

where we have substituted for  $Q^*$ . The expressions for the average annual fixed cost and the sum of the holding and backordering costs are equal. Once again we note that in an optimal solution the costs are balanced.

### 2.2.1 The Optimal Cost

Since the optimal average annual cost is equal to the purchasing cost plus two times the average annual fixed cost,

$$\begin{aligned} Z(B^*, Q^*) &= C\lambda + K \frac{\lambda}{Q^*} + IC \frac{(Q^* - B^*)^2}{2Q^*} + \pi \frac{(B^*)^2}{2Q^*} = C\lambda + 2K \frac{\lambda}{Q^*} \\ &= C\lambda + \sqrt{\frac{2K\lambda IC\pi}{IC + \pi}}. \end{aligned}$$

Let us compute the cost of following the optimal policy in our backordering example. Recall that  $K = 10$ ,  $\lambda = 520$ ,  $I = .20$ ,  $C = \$5$ , and  $\pi = 10$ . Thus the optimal cost is

$$Z(B^*, Q^*) = C\lambda + \sqrt{\frac{2K\lambda IC\pi}{IC + \pi}} = (5)(520) + \sqrt{\frac{2(10)(520)(0.2)(5)(10)}{(5)(0.2) + 10}} = 2697.23.$$

The purchasing cost is equal to \$2600 and the remaining cost of \$97.23 arises from inventory/backlog management. Notice that the optimal cost is lower than the cost obtained without backordering.

## 2.3 Quantity Discount Model

In this section, we will study inventory management when the unit purchasing cost decreases with the order quantity  $Q$ . In other words, a discount is given by the seller if the buyer purchases a large number of units. Our objective is to determine the optimal

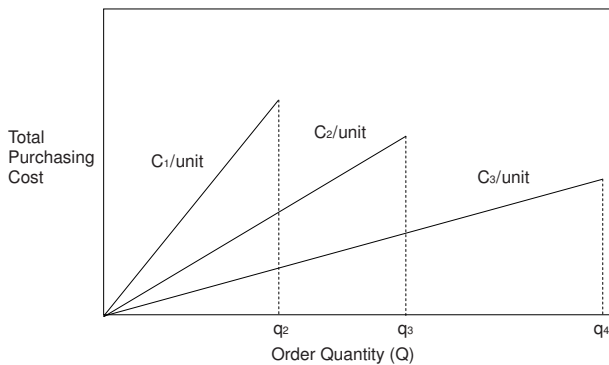
ordering policy for the buyer in the presence of such incentives. The remainder of the model is the same as stated in Section 2.1.

We will discuss two types of quantity discount contracts: all units discounts and incremental quantity discounts. An example of all units discounts is as follows.

**Table 2.1.** Example data for all units discount.

Quantity Purchased	Per-Unit Price
0–100	\$5.00
101–250	\$4.50
251 and higher	\$4.00

If the buyer purchases 100 or fewer units, he pays \$5 for each unit purchased. If he purchases at least 101 but no more than 250 units, he pays \$4.50 for each unit purchased. Thus the cost of purchasing 150 units in a single order is  $(150)(4.5) = \$675$ . Similarly, his purchasing cost comes down to only \$4.00 per unit if he purchases at least 251 units.



**Fig. 2.4.** Total purchasing cost for all units discount.

Therefore, in the all units discounts model, as the order quantity increases, the unit purchasing cost decreases for *every* unit purchased. Figure 2.4 shows the total purchasing cost function for the all units discounts.

The incremental quantity discount is an alternative type of discount. Let us consider an example. Here the unit purchasing price is \$1.00 for every unit up to 200 units. If

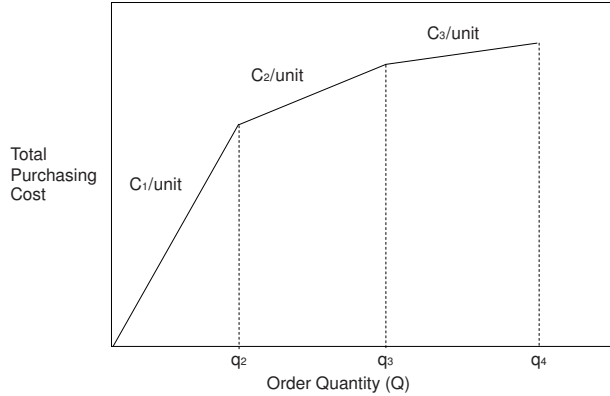
**Table 2.2.** Example data for incremental quantity discount.

Quantity Purchased	Per-Unit Price
0–200	\$1.00
201–500	\$0.98
501 and higher	\$0.95

the order quantity is between 201 and 500, the unit purchasing price drops to \$0.98 but



only for units numbered 201 through 500. The total purchasing cost of 300 units will be  $(200)(1) + (300 - 200)(0.98) = \$298$ . Similarly, if at least 501 units are purchased, the unit purchasing price decreases to \$0.95 for units numbered 501, 502, etc. Therefore, in the incremental quantity discounts case, the unit purchasing cost decreases only for units beyond a certain threshold and not for every unit as in the all units discounts case. Figure 2.5 illustrates the total purchasing cost function.



**Fig. 2.5.** Total purchasing cost for incremental quantity discount.

Why are these discounts offered by suppliers? The reason is to make the customers purchase more per order. As we learned before, large orders result in high inventory holding costs because an average unit spends a longer time in the system before it gets sold. Thus the seller's price discount subsidizes the buyer's inventory holding cost.

We will begin with an analysis of the all units discounts contract. We want to find the order quantity that minimizes the average annual sum of the purchasing, holding, and fixed ordering costs.

### 2.3.1 All Units Discount

Similarly to the EOQ model, we assume that backordering is not allowed. Let  $m$  be the number of discount possibilities. In the example in Table 2.1, there are three discount levels, so  $m = 3$ . Let  $q_1 = 0, q_2, q_3, \dots, q_j, q_{j+1}, \dots, q_m$  be the order quantities at which the purchasing cost changes. Even though we gave an example in which units were discrete, for simplicity in our analysis we will assume from now on that units are infinitely divisible. The unit purchasing cost is the same for all  $Q$  in  $[q_j, q_{j+1})$ ; let the corresponding unit purchasing cost be denoted by  $C_j$ . Thus, the  $j$ th lowest unit purchasing cost is denoted by  $C_j$ ;  $C_m$  is the lowest possible purchasing cost with  $C_1$  being the highest unit purchasing cost. Going back to the example in Table 2.1,  $q_1 = 0, q_2 = 101, q_3 = 251, C_1 = \$5, C_2 = \$4.50$ , and  $C_3 = \$4$ .

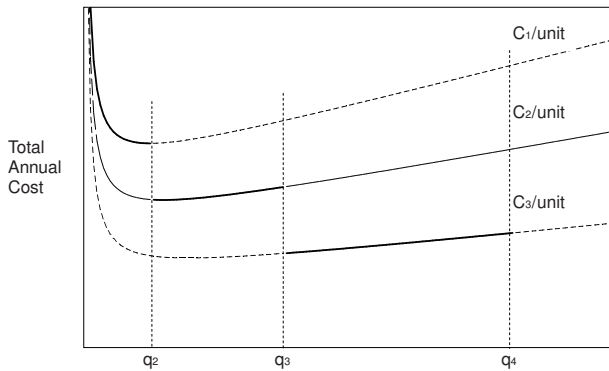
The expression for the average annual cost is similar to (2.1) with one difference. The purchasing cost now depends on the value of  $Q$ . We rewrite the average annual cost to take this difference into account:

$$Z_j(Q) = C_j\lambda + K\frac{\lambda}{Q} + \frac{IC_jQ}{2}, \quad q_j \leq Q < q_{j+1}. \quad (2.16)$$

Thus we have a family of cost functions indexed by the subscript  $j$ . The  $j$ th cost function is defined for only those values of  $Q$  that lie in  $[q_j, q_{j+1})$ . These functions are shown in Figure 2.6. The solid portion of each curve corresponds to the interval in which it is defined. The average annual cost function is thus a combination of these solid portions.

Two observations can be made. First, the average annual cost function is not continuous. It is segmented such that each segment is defined over a discount interval  $[q_j, q_{j+1})$ . The segmented nature of the average annual cost function makes solving the problem slightly more difficult since we now cannot simply take a derivative and set it to zero to find the optimal solution.

It is the second observation that paves the way for an approach to find the optimal solution. Different curves in Figure 2.6 are arranged in an order of decreasing unit purchasing cost. The curve at the top corresponds to the highest per-unit purchasing cost  $C_1$ , and the lowest curve corresponds to the lowest per-unit purchasing cost  $C_m$ .



**Fig. 2.6.** Total cost for all units discount.

Also, the curves do not cross each other. To obtain the optimal solution, we will start from the bottom-most curve which is defined for  $Q$  in  $[q_m, \infty)$ , and compute the lowest possible cost in this interval. Then, in the second iteration, we will consider the second-lowest curve and check to see if we could do better in terms of the cost.

The algorithm continues as long as the cost keeps decreasing. The steps of the algorithm are outlined in the following subsection.

### 2.3.2 An Algorithm to Determine the Optimal Order Quantity for the All Units Discount Case

Step 1: Set  $j = m$ . Compute the optimal EOQ for the  $m$ th cost curve, which we denote by  $Q_m^*$ :

$$Q_m^* = \sqrt{\frac{2K\lambda}{IC_m}}.$$

Step 2: Is  $Q_m^* \geq q_m$ ? If yes,  $Q_m^*$  is the optimal order quantity and we are done. If not, the minimum cost occurs at  $Q = q_m$  owing to the convexity of the cost function. Since the minimum point  $Q_m^* < q_m$ , the cost function for the  $m$ th discount level is increasing on the right of  $q_m$ . Consequently, among all the feasible order quantities, that is, for order quantities greater than or equal to  $q_m$ , the minimum cost occurs at  $Q = q_m$ .

Compute the cost corresponding to  $Q = q_m$ . Let this cost be denoted by  $Z_{min}$  and  $Q_{min} = q_m$  and go to Step 3.

Step 3: Set  $j = j - 1$ . Compute the optimal EOQ for the  $j$ th cost curve:

$$Q_j^* = \sqrt{\frac{2K\lambda}{IC_j}}.$$

Step 4: Is  $Q_j^*$  in  $[q_j, q_{j+1})$ ? If yes, compute  $Z(Q_j^*)$  and compare with  $Z_{min}$ . If  $Z(Q_j^*) < Z_{min}$ ,  $Q_j^*$  is the optimal order quantity. Otherwise,  $Q_{min}$  is the optimal order quantity. In either case, we are done.

Otherwise, if  $Q_j^*$  is not in  $[q_j, q_{j+1})$ , then the minimum cost for the  $j$ th curve occurs at  $Q = q_j$  owing to the convexity of the cost function. Compute this cost  $Z(q_j)$ . If  $Z(q_j) < Z_{min}$ , then set  $Q_{min} = q_j$  and  $Z_{min} = Z(q_j)$ . If  $j \geq 2$ , go to Step 3; otherwise, stop.

We now demonstrate the execution of the above algorithm with the following example.

Let us revisit the office supplies company once again. Suppose now that the retailer gets an all units discount of 5% per box of pencils if he purchases at least 110 pencil boxes in a single order. The deal is further sweetened if the retailer purchases at least 150 boxes in which case he gets a 10% discount. Should the retailer change the order quantity?

We apply the algorithm to compute the optimal order quantity. There are three discount categories, so  $m = 3$ . Also,  $C_1 = 5$ ,  $C_2 = 5(1 - 0.05) = 4.75$ ,  $C_3 = 5(1 - 0.1) = 4.50$ ,  $q_1 = 0$ ,  $q_2 = 110$ , and  $q_3 = 150$ .

**Iteration 1:** Initialize  $j = 3$ .

Step 1: We compute  $Q_3^* = \sqrt{\frac{2(520)(10)}{(0.2)(4.50)}} = 107.5$ .

Step 2: Since  $Q_3^*$  is not greater than  $q_3 = 150$ , the minimum cost occurs at  $Q = 150$ . This cost is equal to

$$Z_{min} = (4.50)(520) + \frac{(10)(520)}{150} + \frac{(0.2)(4.50)(150)}{2} = \$2442.17.$$

Step 3: Now, we set  $j = 2$ . We compute  $Q_2^* = \sqrt{\frac{2(520)(10)}{(0.2)(4.75)}} = 104.63$ .

Step 4: Once again,  $Q_2^*$  is not feasible since it does not lie in the interval  $[110, 150)$ . The minimum feasible cost thus occurs at  $Q = q_2 = 110$  and is equal to

$$Z(q_2) = (4.75)(520) + \frac{(10)(520)}{110} + \frac{(0.2)(4.75)(110)}{2} = \$2569.52,$$

which is higher than  $Z_{min}$  and so the value of  $Z_{min}$  remains unchanged. Now we proceed to Iteration 2.

**Iteration 2:**

Step 3: Now, we set  $j = 1$ . Recall that the value of  $Q_1^*$  is equal to 101.98.

Step 4:  $Q_1^*$  is feasible since it lies in the interval  $[0, 110)$ . The corresponding cost is equal to \$2701.98, which is also higher than  $Z_{min}$ .

Thus the optimal solution is to order 150 units and the corresponding average annual cost of purchasing and managing inventory is equal to \$2442.17.

Observe that the algorithm stops as soon as we find a discount for which  $Q_j^*$  is feasible. This does not mean that the optimal solution is equal to  $Q_j^*$ ; the optimal solution can still correspond to the  $(j + 1)$ st or higher-indexed discount. That is, the optimal solution cannot be less than  $Q_j^*$ .

### 2.3.3 Incremental Quantity Discounts

The incremental quantity discount case differs from the all units discount case. In this situation, as the quantity per order increases, the unit purchasing cost declines incrementally on additional units purchased as opposed to on all the units purchased. Let  $q_1 = 0, q_2, \dots, q_j, q_{j+1}, \dots, q_m$  be the order quantities at which the unit purchasing cost changes. The number of discount levels is  $m$ . In the example, we assumed that the

units are discrete; for analysis we will assume that the units are infinitely divisible and the purchasing quantity can assume any real value. The unit purchasing cost is the same for all values of  $Q$  in  $[q_j, q_{j+1})$ , and we denote this cost by  $C_j$ . By definition,  $C_1 > C_2 > \dots > C_j > C_{j+1} > \dots > C_m$ . In the above example,  $m = 3$ ,  $q_2 = 201$ ,  $q_3 = 501$  ( $q_1$  by definition is 0),  $C_1 = \$1$ ,  $C_2 = \$0.98$ ,  $C_3 = \$0.95$ .

Our goal is to determine the optimal number of units to be ordered. We first write an expression for the average annual purchasing cost if  $Q$  units are ordered. Let  $Q$  be in the  $j$ th discount interval, that is,  $Q$  lies between  $q_j$  and  $q_{j+1}$ . The purchasing cost for  $Q$  in this interval is equal to

$$C(Q) = C_1(q_2 - q_1) + C_2(q_3 - q_2) + \dots + C_{j-1}(q_j - q_{j-1}) + C_j(Q - q_j).$$

Note that only the last term on the right-hand side of the above expression depends on  $Q$ . Let  $R_j$  be used to denote the sum of the terms that are independent of  $Q$ . That is,

$$R_j = C_1(q_2 - q_1) + C_2(q_3 - q_2) + \dots + C_{j-1}(q_j - q_{j-1}), \quad j \geq 2,$$

with  $R_1 = 0$ . Therefore,

$$C(Q) = R_j + C_j(Q - q_j).$$

Now the average unit purchasing cost for  $Q$  units is equal to  $\frac{C(Q)}{Q}$ , which is equal to

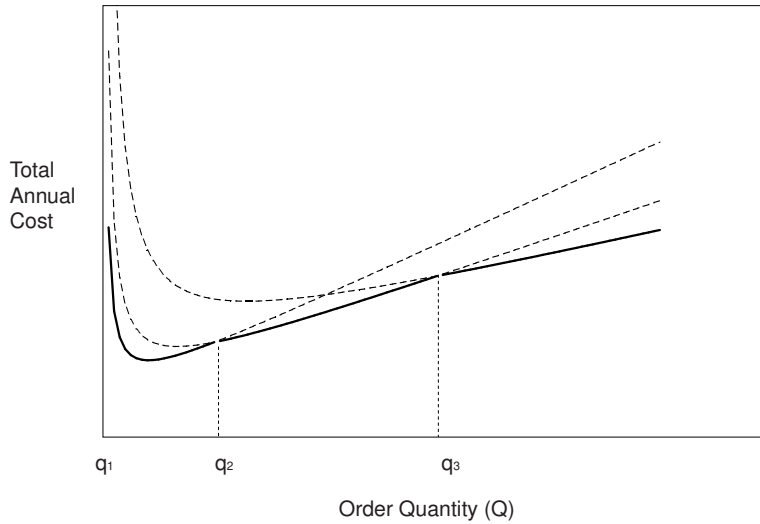
$$\frac{C(Q)}{Q} = \frac{R_j}{Q} + C_j - C_j \frac{q_j}{Q}.$$

The average annual purchasing cost when each order consists of  $Q$  units is equal to  $\frac{C(Q)}{Q} \lambda$ . The fixed order cost and holding cost terms remain similar to those in the basic EOQ model except that the average annual holding cost per unit is now equal to  $I \frac{C(Q)}{Q}$ . This is not surprising since the holding cost per unit depends on the *cost* of each unit, which is a function of the order size given the cost structure here. The average annual cost of managing inventory is thus equal to

$$\begin{aligned} Z(Q) &= \frac{C(Q)}{Q} \lambda + K \frac{\lambda}{Q} + \frac{I \frac{C(Q)}{Q} Q}{2} \\ &= \left( \frac{R_j}{Q} + C_j - C_j \frac{q_j}{Q} \right) \lambda + K \frac{\lambda}{Q} + \frac{I(R_j + C_j(Q - q_j))}{2}. \end{aligned}$$

Realigning terms yields

$$Z(Q) = C_j \lambda + (R_j - C_j q_j + K) \frac{\lambda}{Q} + \frac{I C_j Q}{2} + \frac{I(R_j - C_j q_j)}{2},$$



**Fig. 2.7.** Total cost for incremental quantity discount case.

which is valid for  $q_j \leq Q < q_{j+1}$ .

Figure 2.7 shows the  $Z(Q)$  function. Once again, we have a family of curves, each of which is valid for a given interval. The valid portion is shown using a solid line. Thus the curve with the solid line constitutes the  $Z(Q)$  function. Unlike the all units discount cost function,  $Z(Q)$  is a continuous function. It turns out that the optimal solution  $Q^*$  cannot be equal to any one of  $\{q_1, q_2, \dots, q_m, q_{m+1}\}$ . We use this fact to construct an algorithm to determine the optimal order quantity.

### 2.3.4 An Algorithm to Determine the Optimal Order Quantity for the Incremental Quantity Discount Case

The algorithm for this case is as follows.

Step 1: Compute the order quantity that minimizes  $Z_j(Q)$  for each  $j$ , which is denoted by  $Q_j^*$  and is obtained by setting  $\frac{dZ_j(Q)}{dQ} = 0$ :

$$\begin{aligned} \frac{dZ_j(Q)}{dQ} &= -(R_j - C_j q_j + K) \frac{\lambda}{Q^2} + \frac{IC_j}{2} \\ \Rightarrow Q_j^* &= \sqrt{\frac{2(R_j - C_j q_j + K)\lambda}{IC_j}}. \end{aligned}$$

This step gives us a total of  $m$  possible order quantities.

Step 2: In this step we check which one of these potential values for  $Q^*$  is feasible, that is,  $q_j \leq Q_j^* < q_{j+1}$ . Disregard the ones that do not satisfy this inequality.

Step 3: Calculate the cost  $Z_j(Q_j^*)$  corresponding to each remaining  $Q_j^*$ . The order quantity  $Q_j^*$  that produces the least cost is the optimal order quantity.

Let us now illustrate the execution of this algorithm. Suppose in the preceding example the retailer is offered an incremental quantity discount. Let us determine the optimal order quantity.

Step 1:  $R_1 = 0$ ,  $R_2 = C_1(q_2 - q_1) = (5)(110 - 0) = 550$ , and  $R_3 = C_1(q_2 - q_1) + C_2(q_3 - q_2) = R_2 + (4.75)(150 - 110) = 740$ . We compute

$$\begin{aligned} Q_1^* &= \sqrt{\frac{2(R_1 - C_1q_1 + K)\lambda}{IC_1}} = \sqrt{\frac{2(0 - 0 + 10)(520)}{(0.2)(5)}} = 101.98, \\ Q_2^* &= \sqrt{\frac{2(R_2 - C_2q_2 + K)\lambda}{IC_2}} = \sqrt{\frac{2(550 - (4.75)(110) + 10)(520)}{(0.2)(4.75)}} = 202.61, \\ Q_3^* &= \sqrt{\frac{2(R_3 - C_3q_3 + K)\lambda}{IC_3}} = \sqrt{\frac{2(740 - (4.5)(150) + 10)(520)}{(0.2)(4.5)}} = 294.39. \end{aligned}$$

Step 2: We disregard  $Q_2^*$  since it does not lie in the interval  $[110, 150)$ .  $Q_1^*$  and  $Q_3^*$  are feasible.

Step 3: We now compute the costs corresponding to  $Q_1^*$  and  $Q_3^*$ :

$$\begin{aligned} Z(Q_1^*) &= C_1\lambda + (R_1 - C_1q_1 + K)\frac{\lambda}{Q_1^*} + \frac{IC_1Q_1^*}{2} + \frac{I(R_1 - C_1q_1)}{2} \\ &= (5)(520) + (0 - 0 + 10)\frac{520}{101.98} + \frac{(0.2)(5)(101.98)}{2} + \frac{(0.2)(0 - 0)}{2} \\ &= 2701.98, \\ Z(Q_3^*) &= C_3\lambda + (R_3 - C_3q_3 + K)\frac{\lambda}{Q_3^*} + \frac{IC_3Q_3^*}{2} + \frac{I(R_3 - C_3q_3)}{2} \\ &= (4.5)(520) + (740 - (4.5)(150) + 10)\frac{520}{294.39} + \frac{(0.2)(4.5)(294.39)}{2} \\ &\quad + \frac{(0.2)(740 - (4.5)(150))}{2} \\ &= 2611.45. \end{aligned}$$

Thus, the optimal solution is to order 294.39 units and the corresponding average annual cost of purchasing and managing inventory is equal to \$2611.45.

## 2.4 Lot Sizing When Constraints Exist

In the preceding portions of this chapter, we focused on determining the optimal ordering policy for a single item. In many if not most real-world situations, decisions are not made for each item independently. There may be limitations on space to store items in warehouses; there may be constraints on the number of orders that can be received per year; there may be monetary limitations on the value of inventories that are stocked. Each of these situations requires stocking decisions to be made jointly among the many items managed at a location. We will illustrate how lot sizing decisions can be made for a group of items.

Holding costs are often set to limit the amount of space or investment consumed as a result of the lot sizing decisions. Rather than assuming a holding cost rate is used to calculate the lot sizes, suppose a constraint is placed on the average amount of money invested in inventory. Thus a budget constraint is imposed that limits investment across items. Let  $Q_i$  be the procurement lot size for item  $i$  and  $C_i$  the per-unit purchasing cost for item  $i$ ,  $i = 1, \dots, n$ , where  $n$  is the number of items being managed. The sum

$$\sum_{i=1}^n C_i \frac{Q_i}{2}$$

measures the average amount invested in inventory over time. Let  $b$  be the maximum amount that can be invested in inventory on average. Furthermore, suppose our goal is to minimize the average annual total fixed procurement cost over all item types while adhering to the budget constraint. Let  $\lambda_i$  and  $K_i$  represent the average annual demand rate and fixed order cost for item  $i$ , respectively. Recall that  $\lambda_i K_i / Q_i$  measures the average annual fixed order cost incurred for item  $i$  given  $Q_i$  is the lot size for item  $i$ . Define  $F_i(Q_i) = \lambda_i K_i / Q_i$ .

This procurement problem can be stated as follows:

$$\text{minimize } \sum_{i=1}^n \frac{\lambda_i K_i}{Q_i} = \sum_{i=1}^n F_i(Q_i)$$

subject to

$$\begin{aligned} \sum_{i=1}^n C_i \frac{Q_i}{2} &\leq b, \\ Q_i &\geq 0. \end{aligned}$$



We will obtain the solution for this problem by constructing the corresponding Karush–Kuhn–Tucker conditions. There are four such conditions. The parameter  $\theta$  is the Lagrange multiplier associated with the budget constraint.

$$\begin{aligned}
 \text{(i)} \quad & \frac{dF_i(Q_i)}{dQ_i} + \frac{\theta C_i}{2} \geq 0 \\
 \text{(ii)} \quad & \theta \left( \sum_{i=1}^n C_i \frac{Q_i}{2} - b \right) = 0 \\
 \text{(iii)} \quad & Q_i \left( \frac{dF_i(Q_i)}{dQ_i} + \frac{\theta C_i}{2} \right) = 0 \\
 \text{(iv)} \quad & \begin{cases} \sum_{i=1}^n C_i \frac{Q_i}{2} \leq b \\ Q_i \geq 0. \end{cases}
 \end{aligned}$$

The function  $F_i(Q_i)$  is a strictly decreasing function for all  $i$ . Hence if there were no limit on investment in inventory,  $Q_i^* = \infty$ . Thus in an optimal solution to the problem the budget constraint must hold as a strict equality. That is,

$$\sum_{i=1}^n C_i \frac{Q_i}{2} = b$$

in any optimal solution.

Observe that  $Q_i > 0$  in any optimal solution. As a consequence of (iii),

$$\frac{dF_i(Q_i)}{dQ_i} + \frac{\theta C_i}{2} = 0.$$

This observation results in

$$Q_i^* = \sqrt{\frac{2\lambda_i K_i}{\theta C_i}}.$$

Since  $\sum_{i=1}^n C_i \frac{Q_i}{2} = b$ ,

$$\sum_{i=1}^n \frac{C_i}{2} \sqrt{\frac{2\lambda_i K_i}{\theta C_i}} = \frac{1}{\sqrt{\theta}} \sum_{i=1}^n \sqrt{\frac{\lambda_i C_i K_i}{2}} = b$$

and

$$\theta = \left[ \frac{1}{b} \sum_{i=1}^n \sqrt{\frac{\lambda_i C_i K_i}{2}} \right]^2.$$

Observe that the expression for  $Q_i^*$  is the same as the EOQ solution with  $I$  equal to  $\theta$ . Thus  $\theta$  is the imputed holding cost rate given the budget limitation,  $b$ . Note also how  $\theta$  is related to the value of  $b$ :  $\theta$  is proportional to the square of the inverse of  $b$ .

Let us examine an example problem. Suppose  $n = 2$  and  $b = 7000$ . Additional data are given in the following table.

Item	1	2
$\lambda_i$	1000	500
$C_i$	20	100
$K_i$	50	75

The optimal solution is

$$Q_1^* = 238.38$$

$$Q_2^* = 92.324$$

with

$$\theta^* = .0879895.$$

Thus as the budget level is increased the average annual cost of placing orders decreases by the value of  $\theta^*$ .

Many other constrained multi-item lot sizing problems can be formulated and solved. Some examples are given in the exercises.

## 2.5 Exercises

**2.1.** Derive an expression for the reorder point in the EOQ model when the lead time is greater than the reorder interval.

**2.2.** Show that the optimal order quantity for the incremental quantity discount model cannot be equal to any one of  $\{q_1, q_2, \dots, q_m\}$ .

**2.3.** The demand for toilet paper in a convenience store occurs at the rate of 100 packets per month. Each packet contains a dozen rolls and costs \$8. Assuming the order placement cost to be \$20 per order and the holding cost rate to be .25, determine the optimal order quantity, reorder interval, and optimal cost using the EOQ model. Take one month to be the equivalent of four weeks.

**2.4.** Suppose now that backordering of demand is allowed in the situation described in the previous question. The cost of backordering is \$10/packet/year. Use the EOQ model with backordering to determine the optimal order quantity, maximum backlog allowed, and optimal average annual cost. Also, compute the lengths of the portions of a cycle with positive on-hand inventory and backorders, respectively.

**2.5.** Compare the optimal average annual costs obtained in the last two exercises. Assume that the relationship between the optimal costs of the EOQ model and the EOQ model with backordering holds true in general with the same holding cost, fixed ordering cost, and demand rate (for example, you may find that the optimal cost of the EOQ model with backordering is lower than the optimal cost of the EOQ model.) Can you think of an intuitive explanation for your conclusion?

**2.6.** If the lead time in Exercise 2.3 is 1 week, determine the reorder point.

**2.7.** The demand for leather laptop cases in an electronics store is fairly regular throughout the year at 200 cases per year. Each leather case costs \$50. Assuming the fixed order placement cost to be \$5/order and holding cost rate to be .30, determine the optimal order quantity using the EOQ model. Compute the reorder point assuming the lead time to be 4 weeks.

**2.8.** Suppose in Exercise 2.3 the convenience store receives a discount of 5% per packet on all units purchased if the order quantity is at least 200 and 7.5% per packet if the order quantity is at least 250. Would your answer change? If so, what is the new recommended purchase quantity?

**2.9.** Solve the last exercise assuming the discount type to be an incremental quantity discount.

**2.10.** We illustrated that in the all units discount case, the cost curves for different discount levels are in increasing order of per-unit cost. That is, a curve with a higher per-unit cost always lies above another curve with lower per-unit cost for *any* order quantity. Establish this observation mathematically by comparing the total average annual costs for the two levels of discounts with purchasing costs  $C_1$  and  $C_2$  such that  $C_1 < C_2$ .

**2.11.** The purchasing agent of Doodaldoo Dog Food company can buy horsemeat from one source for \$0.06 per pound for the first 1000 pounds and \$0.058 for each additional pound. The company requires 50,000 pounds per year. The cost of placing an order is \$1.00. The inventory holding cost rate is  $I = .25$  per year. Compute the optimal purchase quantity and the total average annual cost.

**2.12.** Derive an expression for the reorder point in the EOQ model with backordering when the lead time is either greater than or less than the length of the reorder interval.

**2.13.** Show in the EOQ model with no backordering that it is optimal to place an order such that it arrives when the on-hand inventory is zero. Specifically, show that the cost of any other policy such that the order arrives when the on-hand inventory is positive is greater than the cost of the policy in which the order arrives when the on-hand inventory is zero.

**2.14.** For the all units discount case, let  $j^*$  be the largest discount index such that  $Q_j^* = \sqrt{\frac{2\lambda K}{IC_j}} \in [q_j, q_{j+1})$ . Prove that the optimal order quantity cannot be less than  $Q_j^*$ .

**2.15.** Consider the basic EOQ model. Suppose the order is delivered in boxes each of which holds  $A$  units. Thus, the order quantity can only be a multiple of  $A$ . Using the convexity of the cost function in  $Q$ , compute the optimal order quantity.

**2.16.** Recall that in the algorithm for the all units discount model we stop as soon as we identify a discount for which the optimal EOQ,  $Q_j^*$ , is feasible. An alternative stopping criterion is as follows. In Step 3, compute  $Z(Q_j^*)$  and stop if  $Z(Q_j^*) \geq Z_{min}$ . While this stopping criterion involves additional computation, it may eliminate some iterations and save computational effort overall.

Prove that an algorithm that utilizes the above stopping criterion generates an optimal solution. That is, show that if  $j$  is the largest discount index for which  $Z(Q_j^*) \geq Z_{min}$ , then the optimal order quantity cannot be less than  $Q_j^*$ .

**2.17.** The basic EOQ model permitted the value of  $Q^*$  to be any positive real number. Suppose we restrict  $Q$  to be integer-valued. Develop an approach for finding  $Q^*$  in this case.

**2.18.** Llenroc Automotive produces power steering units in production lots. These units go into Jaguar's 4.2 liter engine models and have a demand rate estimated to be 20,000 units per year. Llenroc makes a variety of power steering units in addition to the one used in this model of Jaguars. Switching from producing one power steering model type to another incurs a cost of \$400. The cost of carrying a power steering unit in stock for a year is estimated to be \$35. While the goal is to minimize the average annual cost of changeovers and carrying inventory, management has decided that no more than 10 setups per year are permissible for the pump in question.

Develop a mathematical model and a solution approach for determining the optimal lot size given the constraint on production. What is the solution if 20 setups were possible?

**2.19.** Llenroc Electronics stocks  $n$  items. These items are ordered from a group of suppliers. The cost of placing an order for items of type  $i$  is  $K_i$ . The unit purchase cost for an item of type  $i$  is  $C_i$ . The holding cost factor  $I$  is the same for all item types. Suppose a maximum of  $b$  orders can be processed in total in a year. Derive formulas that can be used to determine the optimal procurement quantities for each item so that the average annual cost of holding inventories is minimized subject to a constraint on the total number of orders that can be processed per year. Also, construct formulas for each item when the objective is to minimize the sum of the average annual fixed ordering costs and the average annual holding costs subject to the same constraint.

**2.20.** The procurement department of Llenroc Electronics purchases the component types from a particular supplier. The fixed cost of placing an order for any of three types of components is \$100. The inventory holding cost factor,  $I$ , is .20. Other relevant data are presented in the following table.

Item	1	2	3
Annual Demand Rate	10,000	30,000	20,000
Unit Cost	350	125	210
Space Req'm't $\left(\frac{\text{ft}^2}{\text{unit}}\right)$	10	15	12

The CFO has placed a \$75,000 limit on the average value of inventory carried in stock. Furthermore, there is a limit on the amount of space available to store inventory of 10,000  $\text{ft}^2$ .

Find the optimal order quantity for each item type so as minimize the total average annual fixed ordering and holding costs subject to the constraints.



<http://www.springer.com/978-0-387-24492-1>

Principles of Inventory Management

When You Are Down to Four, Order More

Muckstadt, J.A.; Sapro, A.

2010, XVIII, 339 p., Hardcover

ISBN: 978-0-387-24492-1