Entanglement Entropy in Two-dimensional Conformal Field Theory

Entropia splątania w dwuwymiarowej konforemnej teorii pola

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Abstract

The study has three major purposes: to define the Quantum Mechanics and the 2-dimensional Conformal Field Theory, to derive the entanglement entropy for a given set of systems in the frame of the 2-dimensional Conformal Field Theory and show its application to 1-dimensional statistical systems considering an example of quantum XX model.

In the first chapter of the work there is given a background of historical and current researches treating the entanglement. In the second chapter there is given an introduction to the Quantum Mechanics with an emphasis on quantum entanglement, next the picture is generalised to the Quantum Field Theory formalism. The third chapter defines the Conformal Field Theory with a special considerations of a 2-dimensional case is defined. The fourth chapter shows methods of deriving the entanglement entropy in the frame of 2-dimensional Conformal Field Theory for three cases: an interval at zero temperature, an interval at non-zero temperature and an interval on a circle at zero temperature. In the fifth chapter there is considered the quantum 1-dimensional XX model as an example of 1-dimensional statistical systems. Its ground state is derived what enables one to calculate the entanglement entropy in such a system and show that at its critical point it agrees with the theoretical prediction.

The thesis shows entanglement entropy in a perspective of the 2-dimensional Conformal Field Theory and that is applicable to 1-dimensional statistical models at their critical point.

Streszczenie

Poniższa praca ma trzy główne cele: zdefiniować mechanikę kwantową i dwuwymiarową konforemną teorię pola, wyprowadzić entropię splątania dla danego zbioru układów w ramach dwuwymiarowej konforemnej teorii pola i pokazać jej zastosowanie do jednowymiarowych układów statystycznych biorąc jako przykład kwantowy model XX.

W pierwszym rozdziale pracy dane jest tło historycznych i aktualnych badań traktujących o splątaniu. W drugim rozdziale dane jest wprowadzenie do mechaniki kwantowej z naciskiem na kwantowe splątanie, dalej ten obraz uogólniony jest do formalizmu kwantowej teorii pola. Trzeci rozdział definiuje konforemną teorię pola ze szczególnym uwzględnieniem przypadku dwuwymiarowego. Czwarty rozdział pokazuje metody wyprowadzenia entropii splątania w ramach dwuwymiarowej konforemnej teorii pola dla trzech przypadków: odcinka w zerowej temperaturze, odcinka w niezerowej temperaturze i odcinka na okręgu w zerowej temperaturze. W rozdziale piątym rozważany jest kwantowy jednowymiarowy model XX jako przykład jednowymiarowych układów statystycznych. Wyprowadzony jest jego stan podstawowy, co pozwala na wyliczenie entropii splątania w takim układzie i pokazanie, że w punkcie krytycznym zgadza się ona z przewidywaniem teoretycznym.

Praca ta pokazuje entropię splątania w perspektywie dwuwymiarowej konforemnej teorii pola oraz jej stosowalność do jednowymiarowych modeli statystycznych w punkcie krytycznym.

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Chapter 1

Motivation

In 1935 when Quantum Mechanics was not yet a fully established theory A. Einstein, B. Podolsky and N. Rosen published a paper "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?". In the paper they gave a doubt about completeness of the quantum-mechanical description given in form of a wave function. They pointed out *"This feature implies the existence of global states of composite system which cannot be written as a product of the states of individual subsystems"* [1]. They show that it is possible to generate a physical situation in which measurement results do not obey Heisenberg Principle. But in fact what they encountered was quantum entanglement. Which can be a surprising phenomena for somebody who is not familiar with the Quantum Mechanics. The doubt was called **EPR** Paradox. It was answered by Bell inequalities describing possible results of such a non-classical behavior. So far every experiment confirms them proving quantum mechanics to be complete.

During many years of research on entanglement many ways of its description arose. The most universal one is entanglement entropy. It is used in many different physical theories, such as: *Condensed Matter Physics and Statistical Models, Quantum Field Thoery, Black Holes, AdS/CFT correspondence, Entropic Gravity, Minimal Surfaces in Quantum Gravity, Classical and Spectral Geometry on Cones* [2].

This work's main subject is entanglement entropy in 2-dimensional Conformal Field Theory. Further there is discussed correspondence between 2**dCFT** and 1-dimensional thermodynamical systems at their critical point. As an example **XY Model** is presented with calculations of entanglement entropy.

As a possible extension of this work could be taken AdS_3/CFT_2 correspondence. It identifies physical properties of interval on a real line in 2**dCFT** and black hole in anti de Sitter 3-dimensional gravity [3]. This work began research on theory of quantum gravity using

methods on Quantum Information Theory[1]. This is also interesting in perspective of recent research on so-called $ER = EPR$. It identifies entanglement of black holes with Einstein-Rosen bridge between them. Einstein-Rosen bridge is a solution of General Relativity granting connection between distant points in space-time causing non-trivial topology. Proper description of such problem could shed light on yet to be solved puzzles as information paradox and foundation of quantum nature of our world. As Susskind writes in [4] $ER = EPR$ may change our view on interpretations of quantum mechanics.

Independently of the outcomes of such researches the entanglement is a very important phenomena which needs to be taken in construction of a new potential fundamental theory.

[1] https://www.simonsfoundation.org/mathematics-and-physical-science/it-from-qubit-simons-collaboration-on-quantum-fields-gravity-and-information/

Chapter 2

Introduction to Quantum Theory

This chapter is devoted to a definition of entanglement and ways of its description and measurement. Then the description is generalized to Quantum Field Theory picture and from that perspective Quantum Thermodynamics is considered.

2.1 Quantum Mechanics

Quantum Mechanics is defined by following axioms [5]:

1. "**States**: A state is a complete description of a physical system. In quantum mechanics, a state is a ray in *Hilbert space*.

2. **Observables**: An observable is a property of a physical system that can be measured. In quantum mechanics, an observable is a *self-adjoint operator*.

3. **Measurement**: A measurement is a process in which information about states of a physical system is acquired by an observer. In quantum mechanics, the measurement of an observable $\boldsymbol{A}$ prepares an eigenstate of $\boldsymbol{A}$, and the observer learns the value of the corresponding eigenvalue. If quantum state just prior to the measurement is $|\psi\rangle_n$, then the outcome a_n is obtained with *a priori probability*."

$$P(a_n) = \| \Pi_n |\psi\rangle \|^2 = \langle \psi | \Pi_n | \psi \rangle. \tag{2.1}$$

Where Π_n is an orthogonal projection onto a space of eigenvectors with eigenvalue a_n. A measurement performed immediately after the first one yields the same result with probability one.

4. "**Dynamics**: Dynamics describes how a state evolves over time. In quantum mechanics, the time evolution of a closed system is described by a *unitary operator*.

5. **Composite systems**: If the Hilbert space of system A is $\mathscr{H}_A$ and the Hilbert space of system B is $\mathscr{H}_B$, then the Hilbert space of the composite systems AB is the tensor product $\mathscr{H}_A \otimes \mathscr{H}_B$. If system A is prepared in the state $|\psi\rangle_A$ and system B is prepared in the state $|\phi\rangle_B$, then the composite system's state is the product $|\psi\rangle_A \otimes |\phi\rangle_B$."

Those axioms show us a way to represent a quantum system. But due to them many other properties of the quantum description arise. In example due to the fact that $\hat{A}$ is self-adjoint one can take orthonormal base such that

$$\hat{A}(t)|i\rangle = a_i|i\rangle, \quad \langle i|j\rangle = \delta_{ij},$$
$$|i\rangle \in \mathscr{H}, \quad (|i\rangle)^\dagger = \langle i| \in \mathscr{H}^*.$$

By taking an arbitrary state $|\psi\rangle$ it is possible to represent it in the mentioned base

$$|\psi\rangle = \mathbf{1}|\psi\rangle = \sum_i |i\rangle \otimes \langle i|\psi\rangle = \sum_i c_i|i\rangle.$$

The coefficients $c_i \equiv \langle i|\psi\rangle$ have probabilistic interpretation of

$$\mathbf{1} = \| \, |\psi\rangle \, \|^2 = \langle\psi|\psi\rangle = \sum_i \langle\psi|i\rangle\langle i|\psi\rangle = \sum_i c_i^* c_i = \sum_i |c_i|^2.$$

Where $|c_i|^2$ are probabilities of each basis state. The same can be done for operator with continuous spectrum by implementing integration instead of sums.

Density Operator formulation

Instead of considering state vector formalism one can introduce **Denisty Operator**.

Definition: Density Operator(DO)
Density operator is operator $\hat{\rho} : \mathscr{H} \to \mathscr{H}$ defined as:

$$\hat{\rho} = |\psi\rangle \otimes \langle\psi| \tag{2.2}$$

It posses following properties:

- Hermiticity: $\hat{\rho}^\dagger = \hat{\rho}$,

- Positive definiteness: $\forall |\psi\rangle \in \mathscr{H} \quad \frac{\langle\psi|\hat{\rho}|\psi\rangle}{\langle\psi|\psi\rangle} \geq 0$,
- Probabilistic interpretation: $\mathrm{tr}\hat{\rho} = 1$.

Where trace of operator $\hat{A}$ is defined as follows

$$\mathrm{tr}\hat{A} \equiv \sum_i \langle i|\hat{A}|i\rangle \tag{2.3}$$

Expectation value of operator $\hat{A}$ can be expressed as

$$\langle\hat{A}\rangle = \mathrm{tr}(\hat{\rho}\hat{A}) \tag{2.4}$$

The **DO** formulation of **QM** is especially useful in description of systems consisting of subsystems. Let's simplify the considerations by narrowing them to systems containing 2 subsystems A and B. Space describing the whole system is tensor product of all Hilbert spaces corresponding the subsystems.

$$\mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$$

Elements of such space are tensor products of elements of the subspaces. Taking two collections of basis vectors for the subsystems

$$\mathscr{H}_A = \mathrm{span}\{|i\rangle_A\}, \qquad \mathscr{H}_B = \mathrm{span}\{|j\rangle_B\}$$
$$\mathscr{H}_{AB} = \mathrm{span}\{|i,j\rangle_{AB} = |i\rangle_A \otimes |j\rangle_B\}.$$

The particular handiness of **DO** formalism in **QM** lays in construction of **Reduced Density Operator**.

Definition: Reduced Density Operator(RDO)
Reduced Density Operator of subsystem A is operator
$\hat{\rho}_A : \mathscr{H}_A \to \mathscr{H}_A$ defined as:

$$\hat{\rho}_A = \mathrm{tr}_B\hat{\rho} = \sum_{\{|j\rangle_B\}} {}_B\langle j|\hat{\rho}|j\rangle_B, \tag{2.5}$$

fulfilling following conditions:

- Hermiticity: $\hat{\rho}_A^\dagger = \hat{\rho}_A$,
- Positive definiteness: $\forall |\psi\rangle \in \mathscr{H}_A \quad \frac{\langle\psi|\hat{\rho}_A|\psi\rangle}{\langle\psi|\psi\rangle} > 0$,

- Probability interpretation: $\mathrm{tr}\hat{\rho}_A = 1$.

2.1.1 Quantum entanglement

First discussed in 1935 in the paper [1] by Einstein, Rosen and Podolski. Their main concern expressed in the work was that it is possible to construct a system consisting of two parts (i.e. particles like electron) that cannot be described by a product of two vector states corresponding to the two states respectively as in

$$|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\psi\rangle_B \tag{2.6}$$

Physical result of such phenomena is that measurements of some physical observables like spin, momentum or polarization for the particles are correlated [5]. Any state that cannot be decomposed as in (2.6) are called entangled. To distinguish between entangled and not-entangled states one can use Schmidt theorem which will be introduced in this subsection. But for a general case one can introduce following definitions:

Definition: Pure state
Subsystem **A** is in pure state if its **RDO** is idempotent ($\hat{\rho}_A^2 = \hat{\rho}_A$).

And thanks to this definition one can define mixed state as:

Definition: Mixed state
Subsystem **A** is in mixed state if it is not pure state.
Its **RDO** obeys ($\hat{\rho}_A^2 \neq \hat{\rho}_A$). Alternatively $\mathrm{tr}\hat{\rho}_A^2 < 1$.

In order to describe such a phenomena as entanglement one needs a way to quantitatively measure it. To do that one can introduce entanglement entropy:

Definition: von Neuman Entropy
For subsystem **A** described by $\hat{\rho}_A$ on $\mathscr{H}_A$ entanglement entropy of the system with its surrounding is defined as follows

$$S_A = -\mathrm{tr}\{\hat{\rho}_A \log \hat{\rho}_A\}.$$

Due to the fact that **RDO** can be given in basis of its eigenvectors as $\hat{\rho}_A = \sum_i p_i |i\rangle_A \otimes {}_A\langle i|$ one can express the entropy in terms of probabilities

$$S_A = -\sum_i p_i \log p_i. \tag{2.7}$$

Facts:

- Due to that p_i are the same for both system the entropy is equal for both systems $S_A = S_B = S$.
- Maximal entropy represents system in which all probabilities p_i of all states are equal.

Schmidt decomposition

To represent the **RDO** in a form that can be used to read the probabilities p_i one can use Schmidt Theorem [5]. It enables one to choose such a basis in which the quantum state of the whole system can be expressed by eigenvalues of **RDO** of one subsystem.

Theorem: Schmidt Decomposition
For system described by $|\psi\rangle_{AB} \in \mathscr{H}_{AB} = \mathscr{H}_A \otimes \mathscr{H}_B$ one can choose basis in $\mathscr{H}_A$ such that it diagonalizes the **RDO** $\hat{\rho}_A$

$$\hat{\rho}_A = \sum_i p_i |i\rangle_A \otimes {}_A\langle i|, \qquad \sum_i p_i = 1.$$

Then the system can be described by

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |i\rangle_A \otimes |i\rangle_B.$$

This form of the theorem is valid for non-degenerated non-zero spectrum for the **RDO**.

Definition: Schmidt number
Number of non-zero probabilities in Schmidt decomposition:

$$Sch\#(\hat{\rho}_A) \equiv \#\{p_i > 0, i \in (1,2,\ldots,dim\mathscr{H}_A)\}.$$

And by the Schmidt number one can distinguish pure and entangled states:

- state $|\psi\rangle_A$ is pure if $Sch\#(\hat{\rho}_A) = 1$,

- state $|\psi\rangle_A$ is entangled if $Sch\#(\hat{\rho}_A) > 1$.

By the theorem one can conclude that for separable state $|\psi\rangle_{AB}$ the entropy vanishes as there is only one non-vanishing probability $p = 1$ which vanishes the expression (2.7).

2.2 Functional Formulation of Quantum Mechanics

The quantization of the classical theory can be performed in two complementary ways: the canonical and the functional one. Operator approach is well defined mathematically, but it comes with some limitations in applications. On the other hand, functional approach can be applied to wider group of physical systems, but it is not mathematically rigorous. Nevertheless, as closer inspection reveals, the statistical interpretation of QM is more transparent within this approach as it will be seen below. Despite its ill definitness the functional approach turned out to be equivalent to the canonical one. It has proved useful in many cases in the intuitive description of quantum physics (e.g., path integral approach to hydrogen atom). To define path integral formulation one can start form Shroedinger equation[1] defining time evolution of a state

$$i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle = \hat{\mathscr{H}}|\Psi(t)\rangle.$$

Definition: Evolution operator
Operator $\hat{U}(t,t_0)$ which evolves state from time t_0 to t the time dependent state vector can be represented as

$$|\Psi(t)\rangle = \hat{U}(t,t_0)|\Psi(t_0)\rangle, \quad \hat{U}(t=t_0,t_0) = \mathbf{1}.$$

By substitution in Shroedinger's equation one gets

$$i\hbar\partial_t\hat{U}(t,t_0) = \hat{\mathscr{H}}\hat{U}(t,t_0).$$

What leads to expression on evolution operator

$$\hat{U}(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt\hat{\mathscr{H}}\right]$$

[1] Time-independent Hamiltonian is considered for simplicity of derivation.

For time independent Hamiltonian this expression take the form

$$\hat{U}(t,t_0) = \exp\left[-\frac{i}{\hbar}(t-t_0)\hat{\mathscr{H}}\right]$$

Considering one-dimensional space and position operator $\hat{x}$ with its eigenvectors $\hat{x}|x\rangle = x|x\rangle$ the state takes form

$$|\Psi(t)\rangle = \mathbf{1}|\Psi(t)\rangle = \int_{\mathbb{R}} dx|x\rangle\langle x|\Psi(t)\rangle = \int_{\mathbb{R}} dx\Psi(x,t)|x\rangle, \quad \Psi(x,t) \equiv \langle x|\Psi(t)\rangle$$

Where identity operator in position representation was used.
One can express the $\Psi(x,t)$ by evolution operator as

$$\begin{aligned}\Psi(x,t) &= \langle x|\Psi(t)\rangle = \langle x|\hat{U}(t,t_0)|\Psi(t_0)\rangle = \langle x|\hat{U}(t,t_0)\mathbf{1}|\Psi(t_0)\rangle \\ &= \int_{\mathbb{R}} dx'\langle x|\hat{U}(t,t_0)|x'\rangle\langle x'|\Psi(t_0)\rangle = \int_{\mathbb{R}} dx'\langle x|\hat{U}(t,t_0)|x'\rangle\Psi(x',t_0)\end{aligned}$$

The integral kernel in last expression is called propagator. $\langle x|\hat{U}(t,t_0)|x'\rangle$ propagates state from point x' to x. This idea can be brought further. To do so one can consider $\delta t \ll 1$ implemented into

$$\hat{U}(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} dt\hat{\mathscr{H}}\right] \approx \exp\left[-i\hat{\mathscr{H}}\delta t\right].$$

Taking general Hamiltonian $\hat{\mathscr{H}} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, $\hbar = 1$ and using Backer-Haussdorf formula $\exp\left(\delta t(\hat{X}+\hat{Y})\right) = \exp(\delta t\hat{X})\exp(\delta t\hat{Y})\exp\left(-\delta t^2\left[\hat{X},\hat{Y}\right]\right)\ldots$ the evolution operator transforms into

$$\hat{U}(t,t_0) \approx \exp\left(-i\frac{\hat{p}^2}{2m}\delta t\right)\exp\left(-iV(\hat{x})\delta t\right)\exp(\mathscr{O}(\delta t^2)).$$

Using the fact that $\langle x|p\rangle = \exp(ipx)$ matrix elements of the propagator $\langle x'|\hat{U}(\delta t)|x\rangle$ take the form

$$\begin{aligned}\langle x'|\hat{U}(\delta t)|x\rangle &\approx \langle x'|\exp\left(-i\frac{\hat{p}^2}{2m}\delta t\right)\exp\left(-iV(\hat{x})\delta t\right)|x\rangle \\ &= \int\frac{dp}{2\pi}\langle x'|\exp\left(-i\frac{\hat{p}^2}{2m}\delta t\right)|p\rangle\langle p|\exp\left(-iV(\hat{x})\delta t\right)|x\rangle \\ &= \int\frac{dp}{2\pi}\exp\left[-i\delta t\left(\frac{p^2}{2m}-p\frac{x'-x}{\delta t}+V(x)\right)\right] \\ &= \sqrt{\frac{m}{2\pi i\delta t}}\exp\left(iS\left[x',x,\delta t\right]\right).\end{aligned}$$

Considering $\delta t = \frac{\Delta t}{N}$, $\Delta t = t_f - t_i$ where t_f is final time and t_i is initial time. For sufficiently large number of intervals $N \gg 1$ the propagator transforms into

$$\begin{aligned}\langle x_f|\hat{U}(t_f,t_i)|x_i\rangle = \lim_{N\to\infty}\left(\frac{mN}{2\pi i\Delta t}\right)^{\frac{N}{2}}\int\prod_{j=1}^{N-1}dx_j\langle x_f|\hat{U}(\frac{\Delta t}{N})|x_{N-1}\rangle \\ \times\ldots\langle x_1|\hat{U}(\frac{\Delta t}{N})|x_i\rangle\end{aligned}$$

Where under the integral there are propagators between points on a trajectory. And the expression is integral over any possible trajectory. To visualize better the above equation one can construct new integration set and integral measure

$$\langle x_f|\hat{U}(t_f,t_i)|x_i\rangle = \int_{\mathscr{F}(x_i,x_f)}\mathscr{D}x\exp\left(iS[x]\right),\quad \mathscr{D}x\equiv\prod_{t\in[t_i,t_f]}dx(t) \tag{2.8}$$

Where $\mathscr{D}x$ is topological product over all time points. And the set $\mathscr{F}(x_i,x_f)$ is defined as

$$\mathscr{F}(x_i,x_f) = \left\{x:\left[t_i,t_f\right]\to\mathbb{R}, x(t)\in C^1, x_i = x(t_i), x_f = x(t_f)\right\}.$$

And $S[x]$ is action given by Lagrangian $L(\dot{x}(t),x(t))$ with potential $V(x(t))$:

$$S[x,t_2,t_1]\equiv\int_{t_1}^{t_2}dtL(\dot{x}(t),x(t)),\quad L(\dot{x}(t),x(t)) = \frac{1}{2}g_{ij}\dot{x}^i(t)\dot{x}^j(t) - V(x(t)).$$

Expression (2.8) can be interpreted as integrating action over any possible trajectory between points x_i and x_f. And $\mathscr{F}(x_i,x_f)$ is set of the mentioned trajectories.

Generalization to n-dimensions of expression (2.8) takes form

$$\langle \mathbf{x}_f | \hat{U}(t_f,t_i) | \mathbf{x}_i \rangle = \int_{\mathscr{F}(\mathbf{x}_i,\mathbf{x}_f)} \mathscr{D}\mathbf{x} \exp\left(iS[\mathbf{x}]\right), \quad \mathscr{D}\mathbf{x} \equiv \prod_{\mu=1}^{n} \prod_{t\in[t_i,t_f]} dx^{\mu}(t)$$

$$\mathbf{x} = (x^1,\ldots,x^n) \in \mathbb{R}^n, \quad |\mathbf{x}\rangle = |x^1\rangle \otimes \cdots \otimes |x^n\rangle \in \bigotimes_{i=1}^{n} \mathscr{H}_i$$

Where now $\mathscr{D}$ now is topological product over any time point and all spatial dimensions.

For further considerations in this work there is one really crucial object witch enables one to derive **DO**.

Definition: n-point correlation function
Function defined for n-points as:

$$\langle x(t_1)\ldots x(t_n)\rangle \equiv \langle 0 | \mathscr{T}\left(x(t_1)\ldots x(t_n)\right) | 0 \rangle.$$

Ground state $|0\rangle$ was used and time ordering operator defined as

$$\mathscr{T}\left(x(t_1)\ldots x(t_n)\right) \equiv x(t_1)\ldots x(t_n), \quad \text{for} \quad t_1 < t_2 < \cdots < t_n.$$

Generating functional

The functional formulation of **QM** offers the efficient way to obtain the correlation functions.

Definition: Generating functional

$$Z[J] = \int_{\mathscr{F}} \mathscr{D}x \exp\left(iS[x] - i\int dt x(t)J(t)\right).$$

Where the original action was modified by adding **source term** known as **Schwinger source** $J(t)$. Using the generating functional one can obtain n-point correlation function by taking n-th functional derivative of $Z[J]$ and setting $J = 0$ what gives

$$\langle x(t_1)\ldots x(t_n)\rangle = i^n \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta J(t_1)\ldots\delta J(t_n)}\Big|_{J=0}.$$

In n-dimensional case the Schwinger source and the correlation function become n-dimensional vectors. The source term can be expressed as a scalar product of vectors $\mathbf{x}(t)\cdot\mathbf{J}(t) = \sum_{i=1}^{n} x^i(t)J_i(t)$.

2.3 Quantum Thermodynamics

The **Density Operator** given by the equation in section 5.7 is valid for the $T = 0$ case. But for finite temperatures one can define alternative operator [6] which enables one to work in the frame of Quantum Thermodynamics.

Definition: Thermal Density Operator

$$\hat{\rho} \equiv Z^{-1}\exp -\beta\hat{\mathscr{H}}, \quad \beta \equiv \frac{1}{k_B T}. \tag{2.9}$$

The coefficient Z^{-1} appears, so the new operator fulfills conditions given in section 2.1. From the Classical Statistical Mechanics one can remind that every possible configuration from the phase space Ω in counted into the partition function Z with a Boltzman coefficient $\exp(-\beta\hat{\mathscr{H}})$ with $\beta = \frac{1}{k_b T}$. Using that one can define partition function as sum over all energy states or equivalently as trace in (2.3) of recently defined thermal operator

$$Z = \sum_i \exp -\beta\varepsilon_i = \sum_i \langle\,\varepsilon_i\,|\exp -\beta\hat{\mathscr{H}}|\,\varepsilon_i\,\rangle = \mathrm{tr}\exp -\beta\hat{\mathscr{H}}. \tag{2.10}$$

One can define thermal average of an operator $\hat{A}$ similarly to definition in (2.4)

$$\langle\hat{A}\rangle = \sum_n \langle n|Z^{-1}\exp\left(-\beta\hat{H}\right)\hat{A}|\,n\,\rangle = \mathrm{tr}(\hat{\rho}\hat{A}). \tag{2.11}$$

In the above formula one can observe similarity to the previously derived propagator in (2.8). Due to that one can use the Lagrangian formalism and express the **Thermal Density Operator** as a functional integral. One can consider a kernel given as

$$\hat{\rho}(x_f, x_i) = \langle\,x_f\,|\exp -\beta\hat{H}|\,x_i\,\rangle. \tag{2.12}$$

Functional integral formalism can be modified for this expression by the **Wick's rotation**. Namely time t is substituted by $-i\tau$, where τ is from $[0,\beta]$ (real). And action $S[x(t)]$ is changed to euclidean action $S_E[x(\tau)]$. The density operator in position space is now given as

a 2-point correlation function:

$$\hat{\rho}(x_f,x_i)=\int_{\mathscr{F}(x_i,x_f)}\mathscr{D}x\exp\left(-S_E[x]\right),\quad \mathscr{D}x\equiv\prod_{\tau\in[0,\beta]}dx(\tau). \tag{2.13}$$

Partition function is integration of the **DO** over all degree of freedom in the system - in this case it is over $x\in\mathbb{R}$. What results in integrating over all possible trajectories with periodic boundary conditions which are represented by set $\mathscr{C}(\beta)$.

$$Z(\beta)=\int_{\mathbb{R}}dx\hat{\rho}(x,x)=\int_{\mathbb{R}}dx\int_{\mathscr{F}(x,x)}\mathscr{D}x\exp\left(-S_E[x]\right)=\int_{\mathscr{C}(\beta)}\mathscr{D}x\exp\left(-S_E[x]\right).$$

Concluding, the partition function is defined as

$$\begin{aligned}Z(\beta)&=\int_{\mathscr{C}(\beta)}\mathscr{D}x\exp\left(-S_E[x]\right),\\ \mathscr{C}(\beta)&\equiv\prod_{x\in\mathbb{R}}\mathscr{F}(x,x),\quad \mathscr{D}x\equiv\prod_{\tau\in[0,\beta]}dx(\tau),\\ \mathscr{F}(x,x)&=\left\{x:[0,\beta]\to\mathbb{R},x(\tau)\in C^1,x(0)=x(\beta)\right\}.\end{aligned} \tag{2.14}$$

And by using the identity operator in position representation the thermal average in functional approach takes form

$$\begin{aligned}\langle\hat{A}\rangle=&Z^{-1}\int_{\mathbb{R}}dx\langle x|\hat{\rho}\hat{A}|x\rangle=Z^{-1}\int_{\mathbb{R}^2}dxdy\langle x|\hat{\rho}|y\rangle\langle y|\hat{A}|x\rangle\\ =&Z^{-1}\int_{\mathbb{R}^2}dxdy\int\limits_{(x,0)}^{(y,\beta)}\mathscr{D}x\langle y|\hat{A}|x\rangle\exp-S_E[x]\\ =&Z^{-1}\int_{\mathbb{R}}dxdy\int\limits_{(x,0)}^{(y,\beta)}\mathscr{D}x\hat{A}(x)\delta(x-y)\exp-S_E[x]\\ =&Z^{-1}\int_{\mathscr{C}(\beta)}\mathscr{D}x\hat{A}(x(0))\exp-S_E[x].\end{aligned}$$

Where relation $\langle y|\hat{A}|x\rangle=\hat{A}(x)\delta(x-y)$ was assumed.

It is possible to describe bipartite system in such picture. Subsystem A is defined as interval on $\mathbb{R}$ and the other subsystem $B=\mathbb{R}\setminus A$. **Reduced Density Operator** takes form

$$\begin{aligned}
\hat{\rho}_A(y,y') =& Z^{-1} \int_B dx\, {}_B\langle x|\exp -\beta\hat{H}|x\rangle_B \\
=& Z^{-1} \int_B dx dx_1 dx_2 \int_A dy_1 dy_2\, {}_B\langle x|x_1\rangle_B |y_1\rangle_{A\,A}\langle y_1|_B\langle x_1| \\
& \exp -\beta\hat{H}|y_2\rangle_a |x_2\rangle_{B\,A}\langle y_2|_B\langle x_2|x\rangle_B \\
=& \int_A dy_1 dy_2 \left(Z^{-1} \int_B {}_{AB}\langle y_1, x|\exp -\beta\hat{H}|y_2, x\rangle_{AB} \right) |y_1\rangle_{A\,A}\langle y_2|
\end{aligned}$$

$$\begin{aligned}
=& \int_A dy_1 dy_2 a(y_1,y_2) |y_1\rangle_{A\,A}\langle y_2| \\
=& Z^{-1} \int_B dx \int_{\mathscr{F}(x,x)\times\mathscr{F}(y,y')} \mathscr{D}x \exp -S_E[x].
\end{aligned}$$

Where the integration in last line is over sum of two sets. One represents trajectories with boundary conditions and the other trajectories between points y and y'. And its final form is

$$\hat{\rho}_A(y,y') = Z^{-1} \int_{\mathscr{C}_B\times\mathscr{F}(y,y')} \mathscr{D}x \exp -S_E[x], \quad y,y' \in A, \quad \mathscr{C}_B = \prod_{x\in B} C_x. \tag{2.15}$$

Here the region of integration can be interpreted as all trajectories that are periodic on B and are not periodic on A.

2.4 Generalization to QFT

To study field theory one needs to consider theory with continuum degrees of freedom. Instead of trajectories one should consider field configurations interpolating between some initial and final configurations. The transition from n dimensional theory to the field theory formally can be formally described as

$$\mathbf{x}(\tau) \to \phi(\tau,\mathbf{x}), \quad \bigotimes_{i=1}^{n} |x^i\rangle \to |\phi\rangle \in \bigotimes_{x\in\mathbb{R}} \mathscr{H}_x.$$

The $\bigotimes_{x\in\mathbb{R}} \mathscr{H}_x$ is formal topological tensor product over all points in $\mathbb{R}$.

Generalizing the **Density Operator** form to field formalism

$$\hat{\rho}(\phi_2,\phi_1) = \int_{\mathscr{F}(\phi_1,\phi_2)} \mathscr{D}\phi \exp iS[x], \quad \mathscr{D}\phi = \prod_{x\in\mathbb{R}} \prod_{t\in[t_1,t_2]} d\phi(x,t),$$
$$\mathscr{F}(\phi_1,\phi_2) = \{\phi : [t_1,t_2]\times\mathbb{R}\to\mathbb{R}, \forall x\in\mathbb{R}\, \phi_1(x) = \phi(x,t_1), \phi_2(x) = \phi(x,t_2)\}.$$

Where now $\mathscr{F}(\phi_1,\phi_2)$ is space of field configurations interpolating between fields ϕ_1 and ϕ_2. Transition to the thermal field theory can be obtained by the **Wick's rotation** and taking $\tau\in[0,\beta]$.

$$\hat{\rho}(\phi_2,\phi_1) = \int_{\mathscr{F}(\phi_1,\phi_2)} \mathscr{D}\phi \exp -S_E[x], \quad \mathscr{D}\phi = \prod_{x\in\mathbb{R}} \prod_{\tau\in[0,\beta]} d\phi(x,\tau).$$

The new integration measure represents topological product over all points of $\mathbb{R}$ and all time points from interval $\tau\in[0,\beta]$. Partition function is again obtained by implementing boundary conditions

$$Z(\beta) = \int_{\mathscr{S}(\beta)} \mathscr{D}\phi \exp -S_E[x], \quad \mathscr{D}\phi = \prod_{x\in\mathbb{R}} \prod_{\tau\in[0,\beta]} d\phi(x,\tau),$$
$$\mathscr{S}(\beta) = \{\phi : [0,\beta]\times\mathbb{R}\to\mathbb{R}, \forall x\in\mathbb{R}\, \phi(x,0) = \phi(x,\beta)\}.$$

The space $\mathscr{S}(\beta)$ can be interpreted as an infinite strip with a circumference β glued along the x direction into an infinite cylinder.

One can also define the **Thermal QFT Reduced Density Operator**. To do so one takes periodic boundary conditions for the whole system excluding the considered subsystem.

$$\hat{\rho}_A(\phi_2,\phi_1) = \int\limits_{\mathscr{S}(\beta)\setminus\mathscr{A}(\phi_2,\phi_1)} \mathscr{D}\phi \exp -S_E[x], \quad \mathscr{D}\phi = \prod_{x\in\mathbb{R}} \prod_{\tau\in[0,\beta]} d\phi(x,\tau),$$
$$\mathscr{S}(\beta)\setminus\mathscr{A}(\phi_2,\phi_1) = \{\phi : [0,\beta]\times\mathbb{R}\to\mathbb{R}, \forall x\in\mathbb{R}\setminus A :\ \phi(x,0) = \phi(x,\beta)\}.$$

The geometrical interpretation of such a space is an infinite strip glued along the x direction with exclusion of the considered subsystem A.

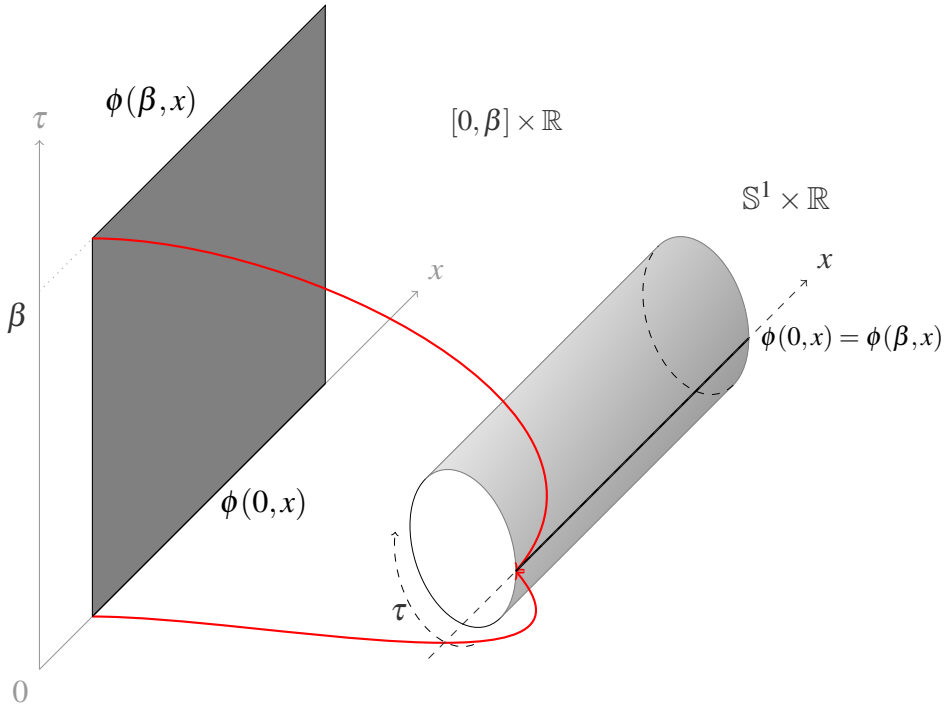

Fig. 2.1 Geometrical interpretation of the partition function.

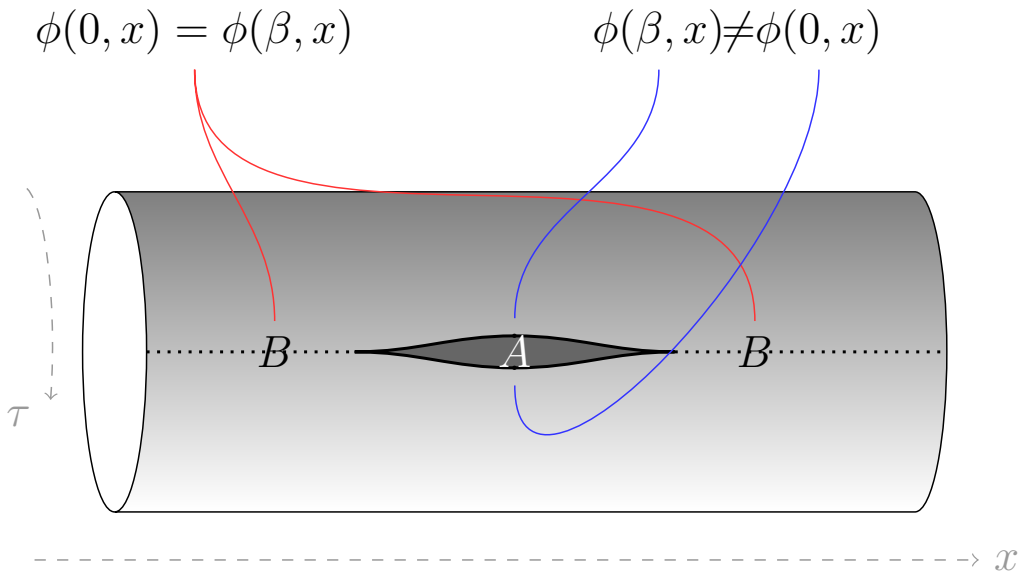

Fig. 2.2 Geometrical interpretation of **RDO**.

Chapter 3

Introduction to Conformal Field Theory

In this chapter foundations of Conformal Field Theory are given with a special emphasis on 2-dimensional case.

Conformal transformations

Let's consider flat space of dimension d. In simple words conformal transformations preserve locally angles between lines. But for more precision one can consider a following definition [7].

Definition: Conformal Transformation
Considering a differentiable mapping $\varphi : U \subset M \to V \subset M'$, where M and M' are manifolds. The conformal map φ obeys condition

$$g'_{\rho\sigma}(\varphi(x))\frac{\partial\varphi^{\rho}(x)}{\partial x^{\mu}}\frac{\partial\varphi^{\sigma}(x)}{\partial x^{\nu}} = \Lambda(x)g_{\mu\nu}(x). \tag{3.1}$$

Einstein summation convention is assumed. Λ is called a conformal scale factor.

In this work it is assumed that $M = M'$ what results in fact that $g' = g$. Considered metric is taken as $\eta_{\mu\nu} = \mathrm{diag}(-1,1,\ldots,1,\ldots)$. And the condition of conformal transformation takes form

$$\eta_{\rho\sigma}\frac{\partial\varphi^{\rho}(x)}{\partial x^{\mu}}\frac{\partial\varphi^{\sigma}(x)}{\partial x^{\nu}} = \Lambda(x)\eta_{\mu\nu}.$$

Conformal invariance

To find the explicit form of the conformal transformation it is convenient to derive Killing's equation. One can take a family of infinitesimal coordinate transformations dependent on a

parameter ε which can be expressed as

$$x'^{\rho} = x^{\rho} + \varepsilon^{\rho}(x) + \mathscr{O}(\varepsilon^2).$$

Substituting this into the metric transformation and remembering the index lowering rule $\varepsilon_{\mu} = \eta_{\mu\nu}\varepsilon^{\nu}$ and notation $x' = \varphi(x)$ one gets

$$\begin{aligned}\eta_{\rho\sigma}\frac{\partial x'^{\rho}}{\partial x^{\mu}}\frac{\partial x'^{\sigma}}{\partial x^{\nu}} &= \eta_{\rho\sigma}\left(\delta^{\rho}_{\mu} + \frac{\partial\varepsilon^{\rho}}{\partial x^{\mu}} + \mathscr{O}(\varepsilon^2)\right)\left(\delta^{\sigma}_{\nu} + \frac{\partial\varepsilon^{\sigma}}{\partial x^{\nu}} + \mathscr{O}(\varepsilon^2)\right)\\ &= \eta_{\mu\nu} + \eta_{\mu\sigma}\frac{\partial\varepsilon^{\sigma}}{\partial x^{\nu}} + \eta_{\rho\nu}\frac{\partial\varepsilon^{\rho}}{\partial x^{\mu}} + \mathscr{O}(\varepsilon^2)\\ &= \eta_{\mu\nu} + \frac{\partial\varepsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial\varepsilon_{\nu}}{\partial x^{\mu}} + \mathscr{O}(\varepsilon^2)\end{aligned}$$

Neglecting terms higher than ε one gets an equation

$$\eta_{\mu\nu} + \frac{\partial\varepsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial\varepsilon_{\nu}}{\partial x^{\mu}} = \Lambda(x)\eta_{\mu\nu}. \tag{3.2}$$

By introducing notation as $K(x) = \Lambda(x) - 1$ a following equation appears

$$\frac{\partial\varepsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial\varepsilon_{\nu}}{\partial x^{\mu}} = K(x)\eta_{\mu\nu}.$$

Taking trace of such an expression one can find the form of the $K(x)$ function

$$\begin{aligned}\eta^{\mu\nu}\left(\frac{\partial\varepsilon_{\mu}}{\partial x^{\nu}} + \frac{\partial\varepsilon_{\nu}}{\partial x^{\mu}}\right) &= K(x)\eta^{\mu\nu}\eta_{\mu\nu},\\ K(x) &= \frac{2}{d}\frac{\partial\varepsilon_{\nu}}{\partial x_{\nu}}.\end{aligned}$$

Substituting this to the former equation with notation $\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$ and $\partial\cdot\varepsilon = \partial^{\nu}\varepsilon_{\nu}$ one gets the Killing's equation

$$\partial_{\nu}\varepsilon_{\mu} + \partial_{\mu}\varepsilon_{\nu} = \frac{2}{d}\partial\cdot\varepsilon\eta_{\mu\nu}. \tag{3.3}$$

And now one can give the explicit form of the scale factor

$$\Lambda = 1 + \frac{2}{d}\partial\cdot\varepsilon + \mathscr{O}(\varepsilon^2).$$

Further insights

Now the Killing's equation can be used to find the explicit form of the conformal transformations. One can derivate the equation (3.3)

$$\partial^\nu \left(\partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu\right) = \partial^\nu \frac{2}{d} \partial \cdot \varepsilon \eta_{\mu\nu}.$$

Introducing a notation $\Box = \partial^\rho \partial_\rho$ the equation above transforms into

$$\Box \varepsilon_\mu + \partial_\mu \partial \cdot \varepsilon = \partial_\mu \frac{2}{d} \partial \cdot \varepsilon.$$

Taking an additional derivative ∂_ν one gets

$$\partial_\nu \Box \varepsilon_\mu + \partial_\nu \partial_\mu \partial \cdot \varepsilon = \partial_\nu \partial_\mu \frac{2}{d} \partial \cdot \varepsilon.$$

Swapping μ with ν in the above equation and adding them result

$$\Box \left(\frac{2}{d} \partial \cdot \varepsilon \eta_{\mu\nu}\right) + 2\partial_\nu \partial_\mu \partial \cdot \varepsilon = \frac{4}{d} \partial_\nu \partial_\mu \partial \cdot \varepsilon,$$
$$\left(\eta_{\mu\nu} \Box + (d-2) \partial_\nu \partial_\mu\right) \partial \cdot \varepsilon = 0.$$

By contraction with $\eta_{\mu\nu}$ one gets

$$(d-1) \Box \partial \cdot \varepsilon = 0. \tag{3.4}$$

This equation implies that the $\partial \cdot \varepsilon$ is at most linear in x:

$$\partial \cdot \varepsilon(x) = A + B_\mu x^\mu.$$

Where A and B_μ are constant. This impose that ε can be at most quadratic in x:

$$\varepsilon(x)_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho. \tag{3.5}$$

Where to $a, b_{\mu\nu}, c_{\mu\nu\rho} \ll 1$ and $c_{\mu\nu\rho} = c_{\mu\rho\nu}$. Following insights can be taken from such an equation:

- Any conditions for the term a are not given by equation (3.3). Transformation $x'^\mu = x^\mu + a^\mu$ corresponds to infinitesimal translations and the momentum operator $P_\mu = -i\partial_\mu$ is its generator.

- To study the linear term one can insert into (3.3) the form of (3.5). It results in

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{d}\eta^{\rho\sigma} b_{\rho\sigma}\eta_{\mu\nu}.$$

Such a form suggest that $b_{\mu\nu}$ should be divided into symmetric and antisymmetric parts as follows

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}.$$

The antisymmetric term $m_{\mu\nu} = -m_{\nu\mu}$ describes infinitesimal rotations $x'^{\mu} = (\delta^{\mu}_{\nu} + m^{\mu}_{\nu})x^{\nu}$ with the angular momentum operator $L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$ as its generator. The symmetric term corresponds to scale transformations $x'^{\mu} = (1+\alpha)x^{\mu}$ with the generator being $D = -ix^{\mu}\partial_{\mu}$.

- For further insights a second formula can be derived by taking a derivative ∂_{ρ} of the equation (3.3) and permuting all three indices

$$\begin{aligned}
\partial_{\rho}\partial_{\nu}\varepsilon_{\mu} + \partial_{\rho}\partial_{\mu}\varepsilon_{\nu} &= \frac{2}{d}\partial_{\rho}\partial\cdot\varepsilon\eta_{\mu\nu},\\
\partial_{\mu}\partial_{\rho}\varepsilon_{\nu} + \partial_{\nu}\partial_{\rho}\varepsilon_{\mu} &= \frac{2}{d}\partial_{\nu}\partial\cdot\varepsilon\eta_{\rho\mu},\\
\partial_{\nu}\partial_{\mu}\varepsilon_{\rho} + \partial_{\mu}\partial_{\nu}\varepsilon_{\rho} &= \frac{2}{d}\partial_{\mu}\partial\cdot\varepsilon\eta_{\nu\rho}.
\end{aligned}$$

Now by adding the second and the third line and subtracting the first line one gets

$$2\partial_{\mu}\partial_{\nu}\varepsilon_{\rho} = \frac{2}{d}\left(\eta_{\rho\mu}\partial_{\nu} + \eta_{\nu\rho}\partial_{\mu} - \eta_{\mu\nu}\partial_{\rho}\right)\partial\cdot\varepsilon. \tag{3.6}$$

And this makes it possible to study the quadratic term. One can insert (3.5) into (3.6) and get

$$\partial\cdot\varepsilon = b^{\mu}_{\mu} + 2c^{\mu}_{\mu\rho}x^{\rho} \rightarrow \partial_{\nu}(\partial\cdot\varepsilon) = 2c^{\mu}_{\mu\nu},$$

what leads to the form of the $c_{\mu\nu\rho}$ tensor

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_{\nu} + \eta_{\mu\nu}b_{\rho} - \eta_{\nu\rho}b_{\mu}, \qquad b_{\mu} = \frac{1}{d}c^{\rho}_{\rho\mu}.$$

And this gives an another transformation called Special Conformal Transformation (**SCT**) with its infinitesimal form

$$x'^{\mu} = x^{\mu} + 2(x \cdot b)x^{\mu} - (x \cdot x)b^{\mu}. \tag{3.7}$$

The SCT's generator takes form $K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - (x \cdot x)\partial_{\mu})$.

All of the above insights can be concluded in a following table.

Transformation		Generator
Translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
Dilation	$x'^{\mu} = \alpha x^{\mu}$	$D = -ix^{\mu}\partial_{\mu}$
Rotation	$x'^{\mu} = M^{\mu}_{\nu}x^{\nu}$	$L_{\mu\nu} = i(x\mu\partial_{\nu} - x_{\nu}\partial_{\mu})$
SCT	$x'^{\mu} = \frac{x^{\mu} - (x \cdot x)b^{\mu}}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)}$	$K_{\mu} = -i(2x\mu x^{\nu}\partial_{\nu} - (x \cdot x)\partial_{\mu})$

Table 3.1 All possible transformations within **CFT** [7].

Special Conformal Transformation

The **SCT** can be expressed also as

$$\frac{x'^{\mu}}{x' \cdot x'} = \frac{x^{\mu}}{x \cdot x} - b^{\mu}.$$

According to this the **SCT** can be understood as an inversion of x, translation by b and one more inversion. But such a construction reveals that the **SCT** is not well defined on the $\mathbb{R}^{d,0}$ or the $\mathbb{R}^{d-1,1}$ [7]. It requires more sophisticated constructions including additional points which will be discussed later.

3.1 Conformal Group

All previous considerations lead to the formulation of such a construction in terms of the group theory.

Definition: Conformal group
Group consisting of globally defined invertible and finite conformal transformations (conformal diffeomorphism) [7].

Definition: Conformal algebra
Lie algebra corresponding to the conformal group [7].

$d \geq 3$ Conformal Field Theory

Although this work is not using the formalism needed for higher dimensions than two it is valuable to show the construction of the conformal group in such theories. By studying the transformations given in the table 3.1 one can deduce the dimension of the algebra by counting number of generators. We have one dilatation generator, d translation generators and d **SCT** generators. Rotation generators are antisymmetric in indices what gives that there are $d(d-1)/2$ generators. That gives that the dimension of the algebra is $N = \frac{(d+2)(d+1)}{2}$. This leads to a definition of the group elements as follows

$$J_{\mu,\nu} = L_{\mu\nu}, \qquad J_{-1,\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right),$$
$$J_{-1,0} = D, \qquad J_{0,\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right).$$

The generators $J_{m,n}$ with $m,n = -1,0,1,\ldots,d-1$ obey following commutation rules

$$[J_{m,n}, J_{r,s}] = i\left(\eta_{ms}J_{nr} + \eta_{nr}J_{ms} - \eta_{mr}J_{ns} - \eta_{ns}J_{mr}\right) \tag{3.8}$$

If η_{mn} describes $\mathbb{R}^{d,0}$ meaning $\eta_{mn} = \mathrm{diag}(-1,1,\ldots,1)$ then the commutation rules correspond to the Lie algebra $\mathbb{SO}(d+1,1)$. And if the metric describes $\mathbb{R}^{d-1,1}$ meaning $\eta_{mn} = \mathrm{diag}(-1,-1,\ldots,1)$ then the commutation rules correspond to the Lie algebra $\mathbb{SO}(d,2)$ [7].

$d = 2$ Conformal Field Theory

For $d = 2$ in the Euclidean space the equation (3.3) reduces to two conditions

$$\partial_0\varepsilon_0 = \partial_1\varepsilon_1, \qquad \partial_0\varepsilon_1 = -\partial_1\varepsilon_0. \tag{3.9}$$

Those conditions should bring to mind the Cauchy-Rieman conditions what lets introduction of complex coordinates

$$z = x^0 + ix^1, \qquad \varepsilon = \varepsilon^0 + i\varepsilon^1, \qquad \partial_z = \frac{1}{2}(\partial_0 - i\partial_1),$$
$$\bar{z} = x^0 - ix^1, \qquad \bar{\varepsilon} = \varepsilon^0 - i\varepsilon^1, \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + i\partial_1). \tag{3.10}$$

Every function obeying (3.9) is holomorphic in some open set. Then function $f(z) = z + \varepsilon(z)$ is as well holomorphic. What gives that such functions are infinitesimal 2-dimensional

conformal transformations. Then the metric changes as

$$ds^2 = dzd\bar{z} \to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dzd\bar{z},$$

where the $\left|\frac{\partial f}{\partial z}\right|^2$ is the previously defined conformal scale factor.

Witt Algebra

In fact the $\varepsilon(z)$ can be not holomorphic, but only meromorphic with isolated singularities meaning depend only on z, but posses certain singularities. Knowing that one can perform a Laurent expansion of $\varepsilon(z)$ around e.g. $z = \bar{z} = 0$. Then the infinitesimal conformal transformations takes form

$$\begin{aligned} z' &= z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n(-z^{n+1}), \\ \bar{z}' &= \bar{z} + \varepsilon(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\varepsilon}_n(-\bar{z}^{n+1}) \end{aligned}$$

Parameters ε_n and $\bar{\varepsilon}_n$ are constant. Generators for such transformation with fixed n are

$$l_n = -z^{n+1}\partial_z, \qquad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{3.11}$$

Due to the fact that $n \in \mathbb{Z}$ the number of the transformations is infinite. It is exclusive only for the two-dimensional Conformal Field Theory. Such generators obey following commutation rules [7]

$$\begin{aligned} [l_m, l_n] &= (m-n)l_{m+n}, \\ \left[\bar{l}_m, \bar{l}_n\right] &= (m-n)\bar{l}_{m+n}, \\ \left[l_m, \bar{l}_n\right] &= 0. \end{aligned} \tag{3.12}$$

It have to be pointed out that there exist two copies of the Witt algebra. And they commute with each other and can be identified with coordinates as in the equation (3.11) what effectively gives the $\mathbb{C}^2$.

Global Conformal Transformations

By considering only the Witt algebra corresponding to $[l_n]$ on $\mathbb{R}^2 \approx \mathbb{C}$ one can see that the generators are not well defined everywhere. To avoid such problems one can study the algebra on a Riemann sphere $S^2 \approx \mathbb{C} \cup \{\infty\}$ which is a conformal compactification of the $\mathbb{R}^2$.

Even though one introduces the algebra on the S^2 some problems appears. Considering $z = 0$ it is true that

$$l_n = -z^{n+1}\partial_z \quad \text{are non} - \text{singular only for } n \geq -1.$$

The other case of ill-defines is $z = \infty$. One can change variables as $z = -\frac{1}{w}$ what results in $\partial_z = w^2\partial_w$. Considering $w \to 0$ one gets that

$$l_n = -(-\frac{1}{w})^{n-1}\partial_w \quad \text{are non} - \text{singular only for } n \leq 1.$$

Globally defined conformal transformations on the Riemann sphere $S^2 = \mathbb{C} \cup \infty$ are generated by $\{l_{-1}, l_0, l_1\}$ [7].

2-dimensional conformal group

Knowing the global conformal transformations one can derive form of the conformal group for $d = 2$. Following [7] one can point out that:

- The generator l_{-1} corresponds to translations $z \to z + b$.
- The generator $l_0 = -z\partial_z$ gives transformations $z \to az$, $a \in \mathbb{C}$. To understand meaning of such an operator one can introduce the polar form of complex coordinates $z = r\exp(i\phi)$. It leads to

  $$l_0 = -\frac{1}{2}r\partial_r + \frac{i}{2}\partial_\phi, \quad \bar{l}_0 = -\frac{1}{2}r\partial_r - \frac{i}{2}\partial_\phi.$$

  Taking linear combinations of such operators

  $$l_0 + \bar{l}_0 = -r\partial_r \quad i\left(l_0 - \bar{l}_0\right) = -\partial_\phi, \tag{3.13}$$

  one gets generators for dilations and rotations respectively.
- The last generator l_1 gives **SCT** namely transformation $w = -\frac{1}{z}$. Infinitesimal form of $z \to \frac{z}{cz+1}$ what corresponds to $w' = w - c$.

Concluding, the three generators result in a general group element of the global conformal transformations given as:

$$z' = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C}. \tag{3.14}$$

Invertibility of such a transformation requires the coefficients to obey $ad-bc=1$ condition. Furthermore the expression is invariant under a simultaneous sign change of the coefficients $(a,b,c,d) \to (-a,-b,-c,-d)$. This can be concluded by a following statement [7]:

The global conformal group of the Riemann sphere $S^2 = \mathbb{C} \cup \{\infty\}$ is Moebius group $SL(2,\mathbb{C})/\mathbb{Z}_2$.

Virasoro Algebra

In order for the conformal group to be the symmetry of the quantum field theory in two dimensions the Witt algebra need to be extended by a central extension. Following [7] the $\tilde{g} = g \oplus \mathbb{C}$ extension of the Lie algebra g by $\mathbb{C}$ is given by following commutation rules

$$\begin{aligned} &[\tilde{x},\tilde{y}]_{\tilde{g}} = [x,y]_g + cp(x,y), && \tilde{x},\tilde{y} \in \tilde{g}, \\ &[\tilde{x},c]_{\tilde{g}} = 0, && x,y \in g, \\ &[c,c]_{\tilde{g}} = 0, && c \in \mathbb{C}. \end{aligned}$$

Where the $p(x,y)$ is a bilinear form such that the $p: g \times g \to \mathbb{C}$. The commutation relations for such an extension is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}. \tag{3.15}$$

Where the c is called a central charge. The derivation of the $p(m,n) = \frac{1}{12}(m^3-m)\delta_{m+n,0}$ is given in [7] on pg. 15-16.

Primary Fields

One needs to remember the correspondence between coordinates z and $\bar{z}$ and the two commuting parts of the Virasoro algebra - L_n and $\bar{L}_n$. Taking such coordinations as in (3.10) is equivalent to an identification $\mathbb{R}^2 \approx \mathbb{C}$. Such coordinates can be treated as two independent complex variables. All this results in a statement

$$\phi(x^0,x^1) \quad \rightarrow \quad \phi(z,\bar{z}), \quad (x^0,x^1) \in \mathbb{R}^2, \quad (z,\bar{z}) \in \mathbb{C}^2.$$

But due to their definitions eventually the $\bar{z}$ have to be identified with the complex conjugation of the z.

Definition: Chiral(Anti-chiral) field
Field ϕ depending only on z ($\bar{z}$) is called chiral (anti-chiral), sometimes also holomorphic (anti-holomorphic).

Definition: Conformal Dimension
If field $\phi(z,\bar{z})$ transforms under scaling as follows

$$\phi'(z,\bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}).$$

it is said that it has conformal dimensions $(h,\bar{h})$.

Definition: Primary field
If field $\phi(z,\bar{z})$ transforms under a conformal transformation $f(z)$ as follows

$$\phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z})) \tag{3.16}$$

it is called a **primary field** with conformal dimensions $(h,\bar{h})$.

Definition: Quasi-Primary field
If field $\phi(z,\bar{z})$ transforms only under a transformation $f(z) \in SL(2,\mathbb{C})/\mathbb{Z}_2$ as follows

$$\phi'(z,\bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z),\bar{f}(\bar{z}))$$

it is called a **quasi-primary field** with conformal dimensions $(h,\bar{h})$.

Every primary filed is quasi-primary, but it is not always true in reverse.

Radial quantization

One can consider compactification of the flat 2-dimensional euclidean space on a circle with a radius R which for simplicity can be taken unitary $R = 1$. The compactification is taken along the direction x^1 which is chosen to be a spatial dimension. Coordinates for such a compactification are taken as

$$w = x^0 + ix^1, \quad w \equiv w + 2\pi i. \tag{3.17}$$

Such a transformation gives a infinite cylinder with radius $R = 1$.

2dCFT realization

To use such a construction in the previously developed theory one needs to map it from the cylinder to the complex plane. To do that one can use a following mapping

$$z = \exp(w) = \exp(x^0)\exp(ix^1), \quad z : \mathbb{R} \times S^1 \to \mathbb{C}. \tag{3.18}$$

Time translations $x'^0 = x^0 + a$ become dilations $z' = \exp(a)z$ and spatial translations $x'^1 = x^1 + b$ become rotations $z' = \exp(ib)z$. And the generators for such transformations are the Hamiltonian for dilations and the momentum operator for rotations:

$$H = L_0 + \bar{L}_0, \quad P = i(L_0 - \bar{L}_0). \tag{3.19}$$

Asympotic states

Considering a field $\phi(z,\bar{z})$ with conformal dimensions $(h,\bar{h})$ one can expand it in the Laurent manner around $z = \bar{z} = 0$:

$$\phi(z,\bar{z}) = \sum_{n,\bar{m}\in\mathbb{Z}} z^{-n-h}\bar{z}^{-\bar{m}-\bar{h}}\phi_{n,\bar{m}}. \tag{3.20}$$

To perform the quantization of such a field one needs to treat the mods as operators just as in Fourier expansion of $\phi(x^0,x^1)$ [7].

Due to the fact that the $x^0 = -\infty$ is mapped to $z = \bar{z} = 0$ the *in*-state is given as

$$|h,\bar{h}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle. \tag{3.21}$$

But expecting such an expression to be non-singular at $z = 0$ one needs to constrain the mods as

$$\phi_{n,\bar{m}}|0\rangle = 0, \quad n > -h, \quad \bar{m} > -\bar{h}. \tag{3.22}$$

Concluding such results the *in*-state take form:

$$|h,\bar{h}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle = \phi_{-h,-\bar{h}}|0\rangle, \quad n > -h, \quad \bar{m} > -\bar{h}. \tag{3.23}$$

Form of the two-point function

Employing the previously described global conformal $SL(2,\mathbb{C})/\mathbb{Z}_2$ symmetry to deduce the form of the two-point function. Obviously such a function has to a function of two points:

$$\langle \phi_1(z)\phi_2(w)\rangle = g(z,w).$$

As the function has to be invariant under translations ($f(z) = z + a$) generated by L_{-1} it has to be a function $g(z,w) = g(z-w)$. Further the function hast to be invariant under dilations ($f(z) = \lambda z$) generated by L_0 what gives

$$\langle \phi_1(z)\phi_2(w)\rangle \to \langle \lambda^{h_1}\phi_1(\lambda z)\lambda^{h_2}\phi_2(\lambda w)\rangle = \lambda^{h_1+h_2} g(\lambda(z-w)) \neq g(z-w).$$

This leads to a conclusion that the function has to be of the following form:

$$g(z-w) = \frac{A}{(z-w)^{h_1+h_2}}.$$

Where the A is a structure constant. Further symmetry to be considered is invariance under transformations $f(z) = -\frac{1}{z}$ generated L_1. This leads to

$$\begin{aligned}\langle\phi_1(z)\phi_2(w)\rangle \rightarrow &\langle\frac{1}{z^{2h_1}}\frac{1}{z^{2h_2}}\phi_1(-\frac{1}{z})\phi_2(-\frac{1}{w})\rangle \\ &= \frac{1}{z^{2h_1}w^{2h_2}}\frac{A}{(-\frac{1}{z}+\frac{1}{w})^{h_1+h_2}} \\ &\neq \frac{A}{(z-w)^{h_1+h_2}}\end{aligned}$$

The only possibility for the above functions to be equal is to take $h_1 = h_2 = h$. These insights can be concluded as

Due to the global conformal $SL(2,\mathbb{C})/\mathbb{Z}_2$ symmetry the two-point function for chiral fields takes form:

$$\langle\phi_i(z)\phi_j(w)\rangle = \frac{A\delta_{ij}}{(z-w)^{2h}} \tag{3.24}$$

Local Ward Identities

The Ward identities are manifestations of the symmetry on correlations functions in the Quantum Field Theory. For conformal symmetry in 2-dimensions the Local Ward identities according to [7] take the form

$$\langle T(z)\phi_1\dots\phi_n\rangle = \sum_i^n \left(\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\partial_i\right)\langle\phi_1\dots\phi_n\rangle. \tag{3.25}$$

3.1.1 Locality problem

An important problem arises when operator products are considered as in 2-point functions. It happens for such a products:

Definition: Non-locality
Field is non-local when there are no such h and $\bar{h}$ that obey the following condition:

$$(z-w)^h(\bar{z}-\bar{w})^{\bar{h}} = |z-w|^{2\bar{h}}(z-w)^{h-\bar{h}}, \quad h-\bar{h}\in\mathbb{Z}. \tag{3.26}$$

Fact of non-existence of such weights leads to existence of such "correlation functions which have brunch cuts in some of their variables, and one would encounter integrals like $\oint dz\sqrt{z}$ around 0. Since the integrand is not periodic around the origin, the integral depends on the choice of the beginning and ending of the interval, and is thus not defined" according to [8]. What results in some conceptual problems in case of statistical applications.

3.2 Energy-Momentum Tensor

The main object defining every Field Theory is the action of the theory given by the Lagrangian. But in case od **2dCFT** all what is required is a 3-point function which is given by the Global Ward Identities up to a structure constant which can be derived thanks to bootstrap method without the knowledge of the action. But in this work all needed information will be given by transformation of **E-M** tensor.

Current conservation

The Noether's theorem is often used in the Field Theory. It states that for every continuous symmetry there exist a conserved current meaning $\exists\, j_\mu : \partial^\mu j_\mu = 0$. For the conformal symmetry with transformations $x'^\mu = x^\mu + \varepsilon^\mu(x)$ the current takes form [7]

$$j_\mu = T_{\mu\nu}\varepsilon^\nu. \tag{3.27}$$

Where the object $T_{\mu\nu}$ is an energy-momentum tensor. It is symmetric.
Using the Noether's theorem results in

$$\begin{aligned} 0 = \partial^\mu j_\mu &= \partial^\mu(T_{\mu\nu}\varepsilon^\nu) = (\partial^\mu T_{\mu\nu})\varepsilon^\nu + T_{\mu\nu}(\partial^\mu\varepsilon^\nu) \\ &= \frac{1}{2}T_{\mu\nu}(\partial^\mu\varepsilon^\nu + \partial^\nu\varepsilon^\mu) = \frac{1}{2}T_{\mu\nu}\eta^{\mu\nu}(\partial\cdot\varepsilon)\frac{2}{d} = \frac{1}{d}T^\mu_\mu(\partial\cdot\varepsilon). \end{aligned}$$

Where the equation (3.3) and the symmetry of the energy-momentum tensor were used. Due to the fact that the $\varepsilon(z)$ can be any infinitesimal transformation then the $\partial\cdot\varepsilon$ is not constrained. What results in a statement

The energy-momentum tensor in **CFT** is traceless - $T^\mu_\mu = 0$.

Energy-momentum Tensor for $d = 2$

In 2-dimensional euclidean space one can express the energy-momentum tensor in complex coordinates by changing the coordinates as $T_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}T_{\alpha\beta}$ and taking $x^0 = \frac{1}{2}(z+\bar{z})$ and $x^1 = \frac{1}{2i}(z-\bar{z})$ one gets that

$$T_{zz} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}), \quad T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}),$$
$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = 0.$$

What by the tracelessness of the $T_{\mu\nu}$ and metric being $\mathrm{diag}(1,1)$ gives results

$$T_{zz} = \frac{1}{2}(T_{00} - iT_{10}), \quad T_{\bar{z}\bar{z}} = \frac{1}{2}(T_{00} + iT_{10}). \tag{3.28}$$

For the translation ε is constant and one can revisit the Noether's theorem and see that

$$0 = \partial^\mu j_\mu = (\partial^\mu T_{\mu\nu})\varepsilon^\nu \Rightarrow \partial^\mu T_{\mu\nu} = 0.$$

And now using that fact one can show that

$$\partial_0 T_{00} + \partial_1 T_{10} = 0, \quad \partial_0 T_{01} + \partial_1 T_{11} = 0.$$

What can be used to show the chirality of the z component of the tensor:

$$\begin{aligned}\partial_{\bar{z}} T_{zz} =& \frac{1}{4}(\partial_0 + i\partial_1)(T_{00} - iT_{10}) = \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 T_{00} + -i\partial_0 T_{10}) \\ =& \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} - i(\partial_1 T_{11} + \partial_0 T_{01})) = 0\end{aligned}$$

Where was used fact that $T^\mu_\mu = 0$.
That can be concluded in a following way:

The non-zero components of the energy-momentum tensor are chiral and anti-chiral conformal fields

$$T_{zz}(z,\bar{z}) = T(z), \quad T_{\bar{z}\bar{z}}(z,\bar{z}) = \bar{T}(\bar{z}).$$

Transformation Rule for Energy-Momentum Tensor

As given in [9] the transformation rule of the energy-moment tensor under conformal transformations $w = f(z)$ takes form

$$T(w) = \left(\frac{dw}{dz}\right)^{-2} \left[T(z) - \frac{c}{12}\{w,z\}\right]. \tag{3.29}$$

Where the object $\{w,z\}$ is called a Schwarz derivative and is defined as

$$\{w,z\} \equiv \frac{\partial_z^3 w}{\partial_z w} - \frac{3}{2}\frac{(\partial_z^2 w)}{(\partial_z w)^2}. \tag{3.30}$$

Chapter 4

Entanglement Entropy in 2dCFT

In this chapter basing on previously presented **QM**, **QFT** and **2dCFT** further methods enabling entanglement entropy calculations are presented. Then using these methods entanglement entropy is computed for following three cases:

1. an interval on real line at $T = 0$,
2. an interval on real line at $T > 0$,
3. an interval on circle at $T = 0$.

4.1 Alternative definition of Entropy

Calculations of entropy may be an uneasy task. And because of that there are many other definitions of entropy beside so far introduced von Neuman entropy. One of them is Renyi entropy defined as follows

$$S^{(n)}_{A,Renyi} \equiv -\frac{1}{n-1}\log \mathrm{tr}\{\hat{\rho}^n_A\}. \tag{4.1}$$

Using the fact that $\mathrm{tr}\hat{\rho}_A = 1$ and by changing the integer parameter to complex continuous parameter and its limit going to 1 one can define the entanglement entropy via so-called **Replica trick**:

$$S_{A,replica} = \lim_{n\to 1} -\frac{\log \mathrm{tr}\hat{\rho}^n_A - \log \mathrm{tr}\hat{\rho}_a}{n-1} = -\frac{\partial}{\partial n}\mathrm{tr}\hat{\rho}^n_A\Big|_{n=1}. \tag{4.2}$$

Replicas are n cylinders cut along a given interval and glued cyclically along the intervals $A_i = [a_i, b_i]$. Such construction results in a n-sheeted Riemann surface which is shown in

the Figure 4.1. It is an example of an orbifold - a common construction in **CFT** enabling generalizing the theory to another fields by different topological conditions [8].

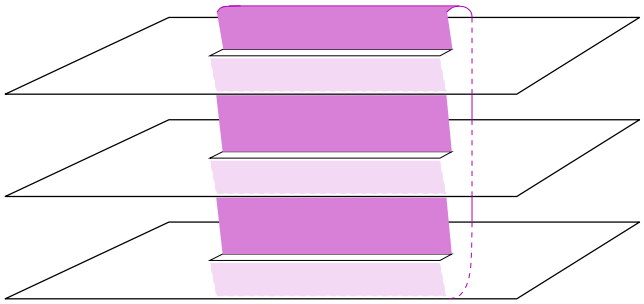

Fig. 4.1 Replicas - cylinders cut along an interval and glued cyclically. In the picture there is presented case of infinite radius what physically corresponds to $T = 0$.

According to Gauss-Bonnet Theorem such a surface has vanishing curvature with exclusion of a finite set of points being ends of the considered interval $A = [a,b]$. As a result one obtains theory which is not local. It depends on these points where the curvature does not vanish. To solve the problem of non-locality following Cardy in [9] one can consider n pieces of complex planes $\mathbb{C}$ instead of the Riemann surface $\mathscr{R}_n$ and move the problem of cuts to target space in form of Twist fields. But to fulfil the picture one needs to introduce fields defined on the boundary points which mimic the non-vanishing curvature in these points. The partition function on $\mathscr{R}_n$ can be defined as

$$Z_{\mathscr{R}_n} = \int_{\mathscr{F}(\mathscr{R}_n)} \mathscr{D}\phi \exp\left[-\int_{\mathscr{R}_n} dx d\tau \mathscr{L}[\phi](x,\tau)\right],$$

where the $\mathscr{L}[\phi](x,\tau)$ is Lagrangian density depending on fields which are non-local due to the branch cuts. To fix the problem of the non-locality one can move it to the target space. What one gets is Lagrangian defined on n sheets of complex planes $\mathbb{C}$ with the boundary conditions encoded into the path integral formalism. Thus, the partition function can be now expressed as

$$Z_{\mathscr{R}_n} = \int_{\mathscr{C}(a,b)} \prod_{i=1}^{n} \mathscr{D}\phi_i \exp\left[-\sum_{i=1}^{n} \int_{\mathbb{C}} dz d\bar{z} \mathscr{L}[\phi_i](z,\bar{z})\right], \quad z = x + i\tau \in \mathbb{C}.$$

Where one integrates over the collection of field configurations $\mathscr{C}(a,b)$ is defined in way to preserve the mechanism connecting different sheets:

$$\mathscr{C}(a,b) = \left\{ \phi_i : [0,\beta] \times \mathbb{R} \to \mathbb{R}, \forall\, x \in [a,b]\ \phi_i(x,0^+) = \phi_{i+1}(x,0^-), i = 1,\ldots,n \right\}.$$

The 0^+ and 0^- correspond to 0 and β respectively and periodicity of the construction is imposed as $i+n \equiv i$. The Lagrangian can be decomposed into parts describing separated sheets:

$$\mathscr{L}^{(n)}[\phi_1,\ldots,\phi_n](z,\bar{z}) = \mathscr{L}[\phi_1](z,\bar{z}) + \cdots + \mathscr{L}[\phi_n](z,\bar{z}).$$

By such a construction one obtains n sheets which are dependent on each other only in terms of the local fields defined on the boundary points $z = (a,0)$ and $z = (b,0)$. Such fields are called **Twist fields**. In **QFT** they signal existence of a global symmetry σ in the system. In this case the symmetry σ is a symmetry of exchange of the sheets.

$$\mathscr{L}[\sigma\phi_i](z,\bar{z}) = \mathscr{L}[\phi_{i+1}](z,\bar{z}).$$

The **Twist fields** can be defined as follows

$$\begin{aligned} &\mathscr{T}_n \equiv \mathscr{T}_\sigma, \quad \sigma : i \to i+1 \bmod n, \\ &\tilde{\mathscr{T}}_n \equiv \mathscr{T}_{\sigma^{-1}}, \quad \sigma^{-1} : i \to i-1 \bmod n. \end{aligned} \tag{4.3}$$

It can be observed that partition function for the Riemann surface $\mathscr{R}_n$ with the cut interval $A = [a,b]$ is given as

$$Z_{\mathscr{R}_n} \propto \langle \mathscr{T}_n(a,0)\tilde{\mathscr{T}}_n(b,0)\rangle, \tag{4.4}$$

as the consecutive copies are connected through $\tau = 0$ on the interval A by the $\tilde{\mathscr{T}}_n(b,0)$ and on the $B = \mathbb{C}\setminus A$ copies for $\tau = 0$ are connected to themselves as the $\tilde{\mathscr{T}}_n(b,0)$ and $\mathscr{T}_n(a,0)$ cancel each other. Generalizing this result the correlation functions in the system described by $\mathscr{L}$ on $\mathscr{R}_n$ can be expressed as the

$$\langle O(x,\tau;\text{sheet } i)\ldots\rangle_{\mathscr{L},\mathscr{R}_n} = \frac{\langle \mathscr{T}_n(a,0)\tilde{\mathscr{T}}_n(b,0)O_i(z,\bar{z})\ldots\rangle_{\mathscr{L}^{(n)},\mathbb{C}}}{\langle \mathscr{T}_n(a,0)\tilde{\mathscr{T}}_n(b,0)\rangle_{\mathscr{L}^{(n)},\mathbb{C}}}. \tag{4.5}$$

And the expression needed for entropy calculations according to (4.2) can be expressed as a following path integral

$$\mathrm{tr}\hat{\rho}_A^n = Z^{-n} \int\limits_{\mathscr{R}_n} \mathscr{D}\phi \mathrm{e}^{-S_E[\phi]} = \frac{Z_{\mathscr{R}_n}}{Z^n}. \tag{4.6}$$

According to the equation (4.5) one can express the expectation value of the energy-momentum tensor on n replicas as

$$\sum_{i=1}^{n} \langle T(x,\tau,i)\rangle_{\mathscr{L},\mathscr{R}_n} = \frac{\langle T_n(w)\mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}}{\langle \mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}}. \tag{4.7}$$

Where T_n is the energy-momentum tensor of the whole system, which means it is a sum of $T(x,\tau,i)$ - energy-momentum tensors on i-th replica. Due to the symmetry of all the single-leaf tensors expectation values are the same, then the sum becomes just a factor n

$$n\langle T(x,\tau)\rangle_{\mathscr{L},\mathscr{R}_n} = \frac{\langle T_n(w)\mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}}{\langle \mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}}. \tag{4.8}$$

To calculate such an expectation value on the replica one can first express the energy-momentum tensor via the transformation rule (3.29). To simplify calculations one can show that the transformation rule of the **E-M** tensor (3.29) can be transformed to a less complicated form:

$$T(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z,w\}. \tag{4.9}$$

Due to that the mapping $\frac{dw}{dz}$ is a diffeomorphism one can freely write down a following equality:

$$\left(\frac{dw}{dz}\right)^{-2} = \left(\frac{dz}{dw}\right)^2.$$

Then what is left to be proved is a following statement

$$\{z,w\} = -\left(\frac{dw}{dz}\right)^{-2}\{w,z\}.$$

To prove it two mappings $u = u(w)$ and $w = w(z)$ can be can considered. Then according to the transformation rule (3.29) the $T''(u)$ can be written down as

$$T''(u) = \left(\frac{du}{dw}\right)^{-2}\left[T'(w) - \frac{c}{12}\{u,w\}\right].$$

Transforming further the $T'(w)$:

$$T''(u) = \left(\frac{du}{dw}\right)^{-2}\left[\left(\frac{dw}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{w,z\}\right] - \frac{c}{12}\{u,w\}\right] \tag{4.10}$$

$$= \left(\frac{du}{dw}\right)^{-2}\left(\frac{dw}{dz}\right)^{-2} T(z) \tag{4.11}$$

$$- \frac{c}{12}\left(\frac{du}{dw}\right)^{-2}\left(\frac{dw}{dz}\right)^{-2}\{w,z\} - \frac{c}{12}\left(\frac{du}{dw}\right)^{-2}\{u,w\} \tag{4.12}$$

$$= \left(\frac{du}{dz}\right)^{-2} T(z) - \frac{c}{12}\left(\frac{du}{dz}\right)^{-2}\{w,z\} - \frac{c}{12}\left(\frac{du}{dw}\right)^{-2}\{u,w\}. \tag{4.13}$$

Due to the fact that the u is a function of z one can construct relation between $T(z)$ and $T''(u)$:

$$T''(u) = \left(\frac{du}{dz}\right)^{-2}\left[T(z) - \frac{c}{12}\{u,z\}\right]. \tag{4.14}$$

By subtraction of the (4.14) from (4.13) one gets:

$$0 = -\frac{c}{12}\left(\frac{du}{dz}\right)^{-2}\{w,z\} - \frac{c}{12}\left(\frac{du}{dw}\right)^{-2}\{u,w\} + \left(\frac{du}{dz}\right)^{-2}\frac{c}{12}\{u,z\}.$$

Taking $u(z) = z$ one gets that $\{u,z\}$ vanishes trivially and result is

$$\{z,w\} = -\left(\frac{dw}{dz}\right)^{-2}\{w,z\}. \tag{4.15}$$

According to (4.9) to evaluate expectation value one needs two terms. One of them is **E-M** tensor one flat space. To calculate expectation value of energy-momentum tensor one can use the Laurent expansion of the **E-M** tensor for modes as in [7]:

$$T(z) = \sum_{m\in\mathbb{Z}} z^{-m-2} L_m. \tag{4.16}$$

The expectation value on complex plane for vacuum states $|0\rangle$ can be calculated as follows:

$$\begin{aligned}\langle T(z)\rangle_{\mathbb{C}} &= \langle 0|T(z)|0\rangle \\ &= \langle 0|\sum_{m\in\mathbb{Z}} z^{-m-2}L_m|0\rangle = \sum_{m\in\mathbb{Z}} z^{-m-2}\langle 0|L_m|0\rangle \\ &= \sum_{m\geq -1} z^{-m-2}\langle 0|L_m|0\rangle + \sum_{m<-1} z^{-m-2}(L_{-m}|0\rangle)^{\dagger}|0\rangle\end{aligned}$$

Where identity $L_n^{\dagger} = L_{-n}$ was used for generators of Virasoro algebra.
Knowing that action of generators of this algebra for indices $m \geq -1$ on vacuum states vanishes one sees that the expectation value of the energy-momentum tensor on complex plane vanishes

$$\langle T(z)\rangle_{\mathbb{C}} = 0. \tag{4.17}$$

The second term needed for calculation of the expectation value of the **E-M** tensor is the term with Schwartz derivative defined in equation (3.30). Conformal transformation from replicas to complex flat space is given by

$$f : \mathscr{R}_n \to \mathbb{C}, \qquad \mathbb{C} \ni z = f(w) = \left(\frac{w-a}{w-b}\right)^{\frac{1}{n}}, \quad w \in \mathscr{R}_n.$$

Schwartz derivative of this transformation takes form

$$\{z,w\} = \frac{n^2-1}{2n^2}\frac{(a-b)^2}{(w-a)^2(w-b)^2}. \tag{4.18}$$

Concluding results from the transformation rule (4.9), vanishing expectation value of **E-M** tensor on flat space in (4.17) and Schwartz derivative term (4.18) one can get the expectation value of the **E-M** tensor on one replica:

$$\begin{aligned}\langle T(x,\tau,i)\rangle_{\mathscr{L},\mathscr{R}_n} &= \langle\left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z,w\}\rangle_{\mathbb{C}} \\ &= \langle\left(\frac{dz}{dw}\right)^2 T(z)\rangle_{\mathbb{C}} + \frac{c}{12}\langle\{z,w\}\rangle_{\mathbb{C}} \\ &= \frac{c}{12}\langle\{z,w\}\rangle_{\mathbb{C}} = \frac{c}{12}\{z,w\}_{\mathbb{C}}\langle 0|0\rangle_{\mathbb{C}} \\ &= \frac{c}{12}\{z,w\}.\end{aligned}$$

Including the Schwartz derivative term's explicit form as in (4.18) one gets final result

$$\langle T(x,\tau,i)\rangle_{\mathscr{L},\mathscr{R}_n} = \frac{c(n^2-1)}{24n^2}\frac{(a-b)^2}{(w-a)^2(w-b)^2}. \tag{4.19}$$

What gives one side of the equality (4.8) and one gets

$$\frac{\langle T_n(w)\mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}}{\langle \mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}} = \frac{c(n^2-1)}{24n}\frac{(a-b)^2}{(w-a)^2(w-b)^2}. \tag{4.20}$$

To express the second side of the equality one can use Ward identities. As in section **??** due to the Global Ward identities the 2-point correlation function takes a following general form:

$$\langle \phi_1\phi_2\rangle = A(z_1-z_2)^{-2h}(\bar{z}_1-\bar{z}_2)^{-2\bar{h}}. \tag{4.21}$$

To relate to the previously derived results one can point out that the $\langle \mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}$ is a two-point correlation function of the **Twist fields** and takes explicit form

$$\langle \mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}} = A(a-b)^{-2(h+\bar{h})} \tag{4.22}$$

The second expression which appears in the missing side of the equality (4.8) can be expressed by the Local Ward identities which according to the equation (3.25) take form

$$\langle T(w)\phi_1\dots\phi_n\rangle = \sum_i^n \left(\frac{h_i}{(w-z_i)^2}+\frac{1}{w-z_i}\partial_i\right)\langle\phi_1\dots\phi_n\rangle.$$

For the considered 2-point correlation function of the **Twist fields** as in (4.22) the Local Ward identities carry out

$$\begin{aligned}\langle T_n(w)\mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}} =& \left(\frac{h}{(w-a)^2}+\frac{1}{w-a}\partial_a\right)A(a-b)^{-2(h+\bar{h})}\\ &+\left(\frac{h}{(w-b)^2}+\frac{1}{w-b}\partial_b\right)A(a-b)^{-2(h+\bar{h})}\\ =&hA(a-b)^{-2(h+\bar{h})}\left(\frac{1}{(w-a)^2}+\frac{1}{(w-b)^2}\right)\\ &-2hA(a-b)^{-2(h+\bar{h})-1}\left(\frac{1}{w-a}-\frac{1}{w-b}\right)\end{aligned}$$

Collecting results from the (4.20) and the (4.22) one gets the following equation

$$\begin{aligned}\frac{c(n^2-1)}{24n}\frac{(u-v)^2}{(w-u)^2(w-v)^2}A(a-b)^{-2(h+\bar{h})} \\ =&hA(a-b)^{-2(h+\bar{h})}\left(\frac{1}{(w-a)^2}+\frac{1}{(w-b)^2}\right) \\ &-2hA(a-b)^{-2(h+\bar{h})-1}\left(\frac{1}{w-a}-\frac{1}{w-b}\right).\end{aligned}$$

By simplifying and solving for h one gets the conformal weight for the chiral field. Also due to the symmetry of the equations for chiral and anti-chiral parts one gets that the weights are equal.

$$h=\bar{h}=\frac{c(n^2-1)}{24n} \tag{4.23}$$

And finally the 2-point correlation function of the **Twist fields** takes form

$$\langle\mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}=A(a-b)^{-\frac{c(n^2-1)}{6n}}. \tag{4.24}$$

4.2 Entropy Calculation

According to the entropy definition in the equation (4.2) one needs to calculate trace of the **Reduced Density Operator** as in the (4.6). What is equivalent to deriving the 2-point correlation function of the **Twist fields** what was done in last section. Then by collecting those results and taking a proper conformal transformations one can finally calculate the entropy in framework of the 2**dCFT**.

4.2.1 Entropy for interval at $T=0$

Taking an expression as in the (4.6) and using form of the 2-point function as in the (4.24) one gets

$$\mathrm{tr}\{\rho_A^n\}=\frac{Z_{\mathscr{R}_n}}{Z^n}=\frac{\langle\mathscr{T}(a,a)\bar{\mathscr{T}}(b,b)\rangle_{\mathscr{L}_n,\mathbb{C}}}{Z^n}=\frac{A}{Z^n}(a-b)^{-\frac{c(n^2-1)}{6n}}.$$

Defining a function $\alpha(n) = \frac{A}{Z^n}$ the entropy can be calculated as

$$\begin{aligned} S_A =& -\frac{d}{dn}\left(\frac{A}{Z^n}(a-b)^{-\frac{c(n^2-1)}{6n}}\right)\Big|_{n=1} \\ =& -\left[\alpha(n)\left(-\frac{c(n^2+1)}{6n^2}\right)(a-b)^{-\frac{c(n^2-1)}{6n}}\ln(a-b) + (a-b)^{-\frac{c(n^2-1)}{6n}}\frac{d\alpha}{dn}\right]_{n=1} \end{aligned}$$

Resulting in final form

$$S_A = \frac{c}{3}\ln(a-b) - \frac{d\alpha}{dn}\Big|_{n=1}. \tag{4.25}$$

Where the $\alpha(n)$ can be defined in such a way to match normalization condition $\alpha(1) = 1$. The term $\frac{d\alpha}{dn}\,|_{n=1}$ is model dependent.

4.2.2 Entropy for interval at finite temperature

Considering finite temperatures β becomes finite as well so the sheets become cylinders of radius $\frac{\beta}{2\pi}$. According to the definition of the Primary Fields in the section 3.1 the previously derived correlation function (4.24) transforms as

$$\langle \mathscr{T}'(u,u)\bar{\mathscr{T}}'(v,v)\rangle_{\mathbb{R}\times S^1} = \left|\frac{dw}{dz}\Big|_a\right|^{2h}\left|\frac{dw}{dz}\Big|_b\right|^{2h}\langle \mathscr{T}(g(a),g(a))\bar{\mathscr{T}}(g(b),g(b))\rangle_{\mathbb{C}}. \tag{4.26}$$

The transformation from Riemann Sphere to a cylinder of radius $\frac{\beta}{2\pi}$ is defined as

$$\begin{aligned} g:&\quad \mathbb{C}\cup\{\infty\} \to \mathbb{R}\times S^1 \\ w =& g^{-1}(z) = \exp\left(\frac{\beta}{2\pi}z\right). \end{aligned}$$

Taking that $u = g(a)$, $v = g(b)$ one can derive

$$\begin{aligned}\langle \mathscr{T}'(u,u)\bar{\mathscr{T}}'(v,v)\rangle_{\mathbb{R}\times S^1} =& A\left(\frac{2\pi}{\beta}\right)^{4h}\exp\left(\frac{4h\pi}{\beta}(u+v)\right)\left(\exp(\frac{2\pi}{\beta}u)-\exp(\frac{2\pi}{\beta}v)\right)^{-4h}\\ =& A\left(\frac{2\pi}{\beta}\right)^{4h}\exp\left(\frac{4h\pi}{\beta}(u+v)-\frac{4h\pi}{\beta}(u+v)\right)\\ &\times\quad\left(2\frac{\exp\left(\frac{\pi}{\beta}(u-v)\right)-\exp\left(-\frac{\pi}{\beta}(u-v)\right)}{2}\right)^{-4h}\\ =& A\left(\frac{2\pi}{\beta}\right)^{4h}\left(2\mathrm{sh}\left(\frac{\pi}{\beta}(u-v)\right)\right)^{-4h}\\ =& A\left(\frac{\beta}{\pi}\mathrm{sh}\left(\frac{\pi}{\beta}(u-v)\right)\right)^{-4h}.\end{aligned}$$

Introducing again $h = \frac{c(n^2-1)}{24n}$ the transformed correlation function takes form:

$$\langle \mathscr{T}'(u,u)\bar{\mathscr{T}}'(v,v)\rangle_{\mathbb{R}\times S^1} = A\left(\frac{\beta}{\pi}\mathrm{sh}\left(\frac{\pi}{\beta}(u-v)\right)\right)^{-\frac{c(n^2-1)}{6n}}. \tag{4.27}$$

Again defining the function $\alpha(n) = \frac{A}{Z^n}$ with the normalization condition $\alpha(1) = 1$ and performing same calculations as in the previous section one gets

$$\begin{aligned}S_A =& -\frac{d}{dn}\left(\alpha(n)\left(\frac{\beta}{\pi}\mathrm{sh}\left(\frac{\pi}{\beta}(u-v)\right)\right)^{-\frac{c(n^2-1)}{6n}}\right)\Big|_{n=1}\\ =& -\frac{d\alpha}{dn}\Big|_{n=1}+\frac{c}{3}\ln\left(\frac{\beta}{\pi}\mathrm{sh}\left(\frac{\pi}{\beta}(u-v)\right)\right).\end{aligned}$$

The entanglement entropy for interval at non-zero temperature is given by

$$S_A = -\frac{d\alpha}{dn}\Big|_{n=1}+\frac{c}{3}\ln\left(\frac{\beta}{\pi}\mathrm{sh}\left(\frac{\pi}{\beta}(u-v)\right)\right). \tag{4.28}$$

4.2.3 Entropy for interval on a circle at $T = 0$

To consider interval on a circle at $T = 0$ one have to consider again space which is compactified along one dimension and infinite along the other. But in this case the spatial dimension becomes compactified and the time one is infinite. To achieve the rotation of the space considered in previous subsection is needed. Such a rotation in complex becomes just a

simple substitution

$$\beta \to iL.$$

Performing such calculations

$$\begin{aligned} S_A &= -\frac{d\alpha}{dn}\Big|_{n=1} + \frac{c}{3}\ln\left(\frac{\beta}{\pi}\mathrm{sh}\left(\frac{\pi}{\beta}(u-v)\right)\right) \\ &= -\frac{d\alpha}{dn}\Big|_{n=1} + \frac{c}{3}\ln\left(\frac{iL}{\pi}\left(-i\sin\left(\frac{\pi}{L}(u-v)\right)\right)\right). \end{aligned}$$

End finally the entanglement entropy for interval on a circle at $T = 0$ is given by

$$S_A = -\frac{d\alpha}{dn}\Big|_{n=1} + \frac{c}{3}\ln\left(\frac{L}{\pi}\sin\left(\frac{\pi}{L}(u-v)\right)\right). \tag{4.29}$$

4.2.4 Conclusions and physical context

In this chapter means needed to derive the entanglement entropy in 2**dCFT** were presented and the entanglement entropy for three different cases was calculated. The entropy is entropy of entanglement of 1-dimensional systems in their critical point of quantum phase transition. To demonstrate validity of this statement in next chapter there will be derived entanglement entropy for 1-dimensional **XX model** which is an example of a system where quantum phase transition can be observed.

Chapter 5

Correspondence with Spin Models

In this chapter it is shown that the results from **2dCFT** can be applied to statistical models. **Spin Models** are defined and their relation with actual physical systems is discussed. Isotropic **XY model** is considered and its correlation function is derived and then by numerical computations entanglement entropy is calculated for $T = 0$ case.

5.1 Spin Models

Spin models are constructed on a fixed d-dimensional lattice (usually square or cubic) with a lattice constant a where in every nod there is a D-dimensional spin placed. Both the spatial dimension d and the spin dimension D numbers can take value from 1 to 3. In dependence of the D number we introduce the following terminology:

- Ising model - $D = 1$
- XY model - $D = 2$
- Heisenberg model - $D = 3$

All of them can be considered in any d-dimensional case. But only for $d = 1$ and $d = 2$ some exact results can be derived [10].

5.1.1 Spin model Hamiltonian

Despite the dimensional differences all the models can be described by a Hamiltonian of a following form:

$$\mathscr{H} = \mu_B \sum_i \mathbf{B} \cdot \mathbf{g} \cdot \mathbf{S}_i + \sum_i D(\mathbf{S}_i) + \sum_i \mathbf{F}_{(i-j)}(\mathbf{S}_i, \mathbf{S}_j). \tag{5.1}$$

Where S_i are spin operators for all spins in the lattice. **B** is homogeneous magnetic field. μ_B is Bohr magneton. **g** is a tensor which in case of free ions is reduced to the Lande g-factor. The $D(\mathbf{S}_i)$ function represents anisotropic interactions of spins with surroundings. And the function $\mathbf{F}_{(i-j)}(\mathbf{S}_i, \mathbf{S}_j)$ describes interactions among spins [10].

5.1.2 Physical realizations

Spin models seem to be extremely simplified version of the reality. That impression can rise a question whether they can be really used in description of reality. This question can be addressed from two perspectives - physical systems and results given by the models.

The regular lattice of objects with some physical properties should bring to mind crystals and their composition. And in frame of **QM** used in the Solid State Physics it can be actually shown that ions (i.e. Iron group ions and rare earth elements' ions) can be described by effective spin. It is possible due to the smallness of the corrections from non-symmetrical interactions at least in sufficiently low temperatures.

On the other hand results given by such models in research of ferromagnetic and anti-ferromagnetic materials show their effectiveness.

5.2 Entanglement entropy for 1d XY model

To show correspondence between **2dCFT** and statistical models isotropic **XY model** is defined and considered. To do so one can consider a **d=**1-dimensional chain of **D=**2-dimensional spins in homogeneous magnetic field **B** oriented orthogonally to the spin-interaction plane. The spin model Hamiltonian (5.1) reduces to:

$$\mathscr{H}_{XY} = \sum_{i=1}^{N} \left[J(1+\gamma) S_i^x S_{i+1}^x + J(1-\gamma) S_i^y S_{i+1}^y - 2B S_i^z \right]. \tag{5.2}$$

Where the γ is an anisotropy parameter of the system. And J is a coupling constant of the interaction. Further one can consider a special case of the general model: isotropic **XY model** ($\gamma = 0$) which for simplicity is called **XX model**. Periodic boundary conditions $S_{N+i} = S_i$ are introduced. Now the Hamiltonian takes form:

$$\mathscr{H}_{XY} = \sum_{i=1}^{N} \left[JS_i^x S_{i+1}^x + JS_i^y S_{i+1}^y - 2BS_i^z \right]. \tag{5.3}$$

Where the spin operators are constructed from Pauli matrices:

$$S_i^a = \frac{1}{2}\sigma_i^a, \quad a = x,y,z.$$

Operators for a given nod i obviously obey Pauli matrices algebra (up to a factor), but operators for different nods $i \neq j$ commute. The description can be simplified to a single commutation rule:

$$\left[S_i^a, S_j^b\right] = i\delta_{ij}\varepsilon_{abc}S_i^c.$$

In the case of two different states, either the spin is down $|\downarrow\rangle_i$ or it is up $|\uparrow\rangle_i$. For N particles the system is given by Hilbert space of dimension 2^N. In this picture one can introduce lowering and rising operators. They can be thought as operators flipping spins between $|\uparrow\rangle$ and $|\downarrow\rangle$ states.

$$\begin{aligned} S^+ &= S^x + iS^y, & S^- &= S^x - iS^y, \\ S^x &= \frac{S^+ + S^-}{2}, & S^y &= \frac{S^+ - S^-}{2i}, \\ S_i^+|\downarrow\rangle_i &= |\uparrow\rangle_i, & S_i^-|\uparrow\rangle_i &= |\downarrow\rangle_i, \\ S^z &= S^+S^- - \frac{1}{2}, & \left(S^+\right)^2 &= \left(S^-\right)^2 = 0 \end{aligned}$$

Unfortunately the lowering and rising operators obey mixed bosonic-fermionic algebra:

- $i = j$ case:

$$\begin{aligned} \left\{S_i^-, S_i^+\right\} &= \left(S_i^x - iS_i^y\right)\left(S_i^x + iS_i^y\right) + \left(S_i^x + iS_i^y\right)\left(S_i^x - iS_i^y\right) \\ &= (S_i^x)^2 - iS_i^y S_i^x + iS_i^x S_i^y + (S_i^y)^2 \\ &\quad + (S_i^x)^2 + iS_i^y S_i^x - iS_i^x S_i^y + (S_i^y)^2 \\ &= 2(S_i^x)^2 + 2(S_i^y)^2 = 4 \cdot \frac{1}{4} = 1. \end{aligned}$$

What gives anti-commutation rules characteristic for fermions:

$$\left\{S_i^-, S_i^+\right\} = 1, \qquad \left\{S_i^+, S_i^+\right\} = \left\{S_i^-, S_i^-\right\} = 0.$$

- $i \neq j$ case:

$$\begin{aligned}\left[S_i^-, S_j^+\right] &= \left(S_i^x - iS_i^y\right)\left(S_j^x + iS_j^y\right) - \left(S_j^x + iS_j^y\right)\left(S_i^x - iS_i^y\right) \\ &= S_i^x S_j^x - iS_i^y S_j^x + iS_i^x S_j^y + S_i^y S_j^x \\ &\quad - S_j^x S_i^x - iS_j^y S_i^x + iS_j^x S_i^y - S_j^x S_i^y \\ &= \left[S_i^x, S_j^x\right] + \left[S_i^y, S_j^y\right] + i\left[S_i^x, S_j^y\right] + i\left[S_i^y, S_j^x\right] = 0.\end{aligned}$$

What results in bosonic-like commutation rules:

$$\left[S_i^+, S_j^-\right] = \left[S_i^-, S_j^-\right] = \left[S_i^+, S_j^+\right] = 0$$

The Hamiltonian can be expressed with these operators:

$$\mathscr{H}_{XY} = \frac{J}{2}\sum_{i=1}^{N}\left[S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+\right] - B\sum_{i=1}^{N}\left(2S_i^+ S_i^- - 1\right). \tag{5.4}$$

To gather further insight into the model one can construct field-theory-like interpretation. The different possible states of one nod in system can be treated as existence of quasi-particles. Due to the fact that spin can be risen only once, it can be said that these quasi-particles avoid themselves - they behave as fermions. What leads to the Jordan-Wigner transformation - translation of spin variables into fermion variables:

$$\begin{aligned} S_j^+ &= c_j \exp\left(-i\pi\sum_{k=1}^{j-1} c_k^\dagger c_k\right), & S_1^+ &= c_1, \\ S_j^- &= \exp\left(i\pi\sum_{k=1}^{j-1} c_k^\dagger c_k\right) c_j^\dagger, & S_1^- &= c_1^\dagger. \end{aligned}$$

Defining an excitation number operator for such quasi-particles the previous definition can be simplified:

$$n_k = c_k^\dagger c_k \Rightarrow \exp\left(-i\pi \sum_{k=1}^{j-1} c_k^\dagger c_k\right) = \exp\left(-i\pi \sum_{k=1}^{j-1} n_k\right).$$

And due to that the sum goes up to $j-1$ ($j \leq N$) index the operators obey commutation relation:

$$\left[c_j^{(\dagger)}, \exp\left(-i\pi \sum_{k=1}^{j-1} n_k\right)\right] = 0.$$

To simplify further considerations following relations for functions of $c_i^\dagger c_i$ can be proved:

$$\begin{aligned} c_i^\dagger f(c_i^\dagger c_i) &= c_i^\dagger f(0), & c_i f(c_i^\dagger c_i) &= c_i f(1), \\ f(c_i^\dagger c_i) c_i^\dagger &= f(1) c_i^\dagger, & f(c_i^\dagger c_i) c_i &= f(0) c_i. \end{aligned}$$

To prove two of them one can expand them around 0. Also identities $\left(c_i^\dagger c_i\right)^n = c_i^\dagger c_i$ and $\left(c_i^{(\dagger)}\right)^2 = 0$ are used.

$$\begin{aligned} c_i^\dagger f(c_i^\dagger c_i) =& c_i^\dagger \left(f(0) + \sum_{n\geq 1} \frac{f^{(n)}(0)}{n!} \left(c_i^\dagger c_i\right)^n\right) \\ =& c_i^\dagger f(0) + c_i^\dagger c_i^\dagger c_i \left(\sum_{n\geq 1} \frac{f^{(n)}(0)}{n!}\right) = c_i^\dagger f(0), \end{aligned}$$

$$\begin{aligned} f(c_i^\dagger c_i) c_i =& \left(f(0) + \sum_{n\geq 1} \frac{f^{(n)}(0)}{n!} \left(c_i^\dagger c_i\right)^n\right) c_i \\ =& f(0) c_i + c_i^\dagger c_i c_i \left(\sum_{n\geq 1} \frac{f^{(n)}(0)}{n!}\right) = f(0) c_i. \end{aligned}$$

To prove second pair of the identities one can expand them around 1:

$$\begin{aligned} f(c_i^\dagger c_i)c_i^\dagger =& f(1-c_i c_i^\dagger)c_i^\dagger \\ =& \left(f(1) + \sum_{n\geq 1} \frac{(-1)^n f^{(n)}(1)}{n!} \left(c_i c_i^\dagger\right)^n \right) c_i^\dagger \\ =& f(1)c_i^\dagger + c_i c_i^\dagger c_i^\dagger \left(\sum_{n\geq 1} \frac{(-1)^n f^{(n)}(1)}{n!} \right) = f(1)c_i^\dagger, \end{aligned}$$

$$\begin{aligned} c_i f(c_i^\dagger c_i) =& c_i f(1-c_i c_i^\dagger) \\ =& c_i \left(f(1) + \sum_{n\geq 1} \frac{(-1)^n f^{(n)}(1)}{n!} \left(c_i c_i^\dagger\right)^n \right) \\ =& c_i f(1) + c_i c_i c_i^\dagger \left(\sum_{n\geq 1} \frac{(-1)^n f^{(n)}(1)}{n!} \right) = c_i f(1). \end{aligned}$$

The $c_i^{(\dagger)}$ operators obviously obey the fermionic algebra. It can be checked, let $i < j$:

$$\begin{aligned} 0 = \left[S_i^-, S_j^+\right] =& \left[\exp\left(i\pi \sum_{k=1}^{i-1} c_k^\dagger c_k \right) c_i^\dagger, c_j \exp\left(-i\pi \sum_{k=1}^{j-1} n_k \right) \right] \\ =& c_i^\dagger \exp\left(-i\pi \sum_{k=i}^{j-1} n_k \right) c_j - c_j \exp\left(-i\pi \sum_{k=i}^{j-1} n_k \right) c_i^\dagger \\ =& \exp\left(-i\pi \sum_{k=i+1}^{j-1} n_k \right) \left(c_i^\dagger \exp(-i\pi n_i) c_j - c_j \exp(-i\pi n_i) c_i^\dagger \right) \\ =& \exp\left(-i\pi \sum_{k=i+1}^{j-1} n_k \right) \left(c_i^\dagger c_j - c_j \exp(-i\pi) c_i^\dagger \right) \\ =& \exp\left(-i\pi \sum_{k=i+1}^{j-1} n_k \right) \left\{ c_j, c_i^\dagger \right\}. \end{aligned}$$

And for $i = j$ case:

$$\begin{aligned}
1 =& \left\{S_i^-, S_i^+\right\} \\
=& \left\{\exp\left(i\pi \sum_{k=1}^{i-1} c_k^\dagger c_k\right) c_i^\dagger, c_i \exp\left(-i\pi \sum_{k=1}^{j-1} n_k\right)\right\} \\
=& c_i^\dagger \exp\left(i\pi \sum_{k=1}^{i-1} (n_k - n_k)\right) c_i + c_i \exp\left(-i\pi \sum_{k=1}^{i-1} (n_k - n_k)\right) c_i^\dagger \\
=& c_i^\dagger c_i + c_i c_i^\dagger = \left\{c_i^\dagger, c_i\right\} = \left\{c_i, c_i^\dagger\right\}.
\end{aligned}$$

Thus the anti-commutation rules can be given

$$\left\{c_i, c_j\right\} = \left\{c_i^\dagger, c_j^\dagger\right\} = 0, \qquad \left\{c_i, c_j^\dagger\right\} = \delta_{ij}.$$

Using such defined operators one can express all the operators existing in the Hamiltonian in terms of fermion variables:

$$\begin{aligned}
S_i^+ S_{i+1}^- =& c_i \exp\left(-i\pi \sum_{k=1}^{i-1} c_k^\dagger c_k\right) \exp\left(i\pi \sum_{k=1}^{i} c_k^\dagger c_k\right) c_{i+1}^\dagger \\
=& c_i \exp\left(i\pi c_i^\dagger c_i\right) c_{i+1}^\dagger = c_i \exp(i\pi)\, c_{i+1}^\dagger \\
=& - c_i c_{i+1}^\dagger = c_{i+1}^\dagger c_i,
\end{aligned}$$

$$\begin{aligned}
S_i^- S_{i+1}^+ =& \exp\left(i\pi \sum_{k=1}^{i-1} c_k^\dagger c_k\right) c_i^\dagger c_{i+1} \exp\left(-i\pi \sum_{k=1}^{i} c_k^\dagger c_k\right) \\
=& c_i^\dagger \exp\left(i\pi c_i^\dagger c_i\right) c_{i+1} = c_i^\dagger \exp(i\pi \cdot 0)\, c_{i+1} \\
=& c_i^\dagger c_{i+1}.
\end{aligned}$$

And for the boundary conditions $S_{N+i}^\pm = S_i^\pm$ one can find relation between operators $c_{N+1}^{(\dagger)}$ and $c_1^{(\dagger)}$

$$\begin{aligned}
S_{N+1}^+ =& c_{N+1} \exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right), \\
S_{N+1}^+ =& S_1^+ = c_1, \\
c_1 =& c_{N+1} \exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right).
\end{aligned}$$

And for the S^- operators situation is analogical:

$$\begin{aligned} S^-_{N+1} &= \exp\left(i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) c_{N+1}^\dagger, \\ S^-_{N+1} &= S^-_1 = c_1^\dagger, \\ c_1^\dagger &= \exp\left(i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) c_{N+1}^\dagger. \end{aligned}$$

What results in relations as follows

$$\begin{aligned} c_{N+1} &= c_1 \exp\left(i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right), \\ c_{N+1}^\dagger &= \exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) c_1^\dagger. \end{aligned}$$

Thanks to this boundary terms can be calculated as

$$\begin{aligned} S_N^+ S_{N+1}^- &= c_N \exp\left(-i\pi \sum_{k=1}^{N-1} c_k^\dagger c_k\right) \exp\left(i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) c_{N+1}^\dagger \\ &= c_N \exp\left(-i\pi \sum_{k=1}^{N-1} c_k^\dagger c_k\right) \exp\left(i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) \exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) c_1^\dagger \\ &= c_N \exp\left(-i\pi \sum_{k=1}^{N-1} c_k^\dagger c_k\right) c_1^\dagger = \exp\left(-i\pi \sum_{k=1}^{N-1} c_k^\dagger c_k\right) c_N c_1^\dagger \\ &= \exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) \exp\left(i\pi c_N^\dagger c_N\right) c_N c_1^\dagger \\ &= \exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) c_N c_1^\dagger = -\exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) c_1^\dagger c_N. \end{aligned}$$

And the second boundary term is

$$\begin{aligned}
S_N^- S_{N+1}^+ =& \exp\left(i\pi \sum_{k=1}^{N-1} c_k^\dagger c_k\right) c_N^\dagger c_{N+1} \exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) \\
=& \exp\left(i\pi \sum_{k=1}^{N-1} c_k^\dagger c_k\right) c_N^\dagger c_1 \exp\left(i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) \exp\left(-i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) \\
=& \exp\left(i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) \exp\left(-i\pi c_N^\dagger c_N\right) c_N^\dagger c_1 \\
=& -\exp\left(i\pi \sum_{k=1}^{N} c_k^\dagger c_k\right) c_N^\dagger c_1.
\end{aligned}$$

Rewriting the Hamiltonian with the new expressions one gets

$$\begin{aligned}
\mathscr{H}_{XY} =& \frac{J}{2} \sum_{i=1}^{N-1} \left[c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}\right] \\
& - \frac{J}{2} \exp\left(i\pi \sum_{k=1}^{N} n_k\right) \left(c_1^+ c_N + c_N^+ c_1\right) + 2B \sum_{i=1}^{N} c_i^\dagger c_i - NB
\end{aligned} \tag{5.5}$$

Defining number of quasi-particles in the system as $M = \sum_{k=1}^{N} n_k$ the coefficient can be simplified $\exp\left(i\pi \sum_{k=1}^{N} n_k\right) = (-1)^M$. Second term of the Hamiltonian corresponds to the boundary conditions which are periodic (anti-periodic) for M being even (odd). This is called "c-cyclic" boundary conditions.

Further one can try to diagonalize the Hamiltonian. Due to translational symmetry one can perform Fourier transformation of the fermionic variables as

$$c_q = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp\left(-iqk\right) c_k, \quad q = \frac{2\pi n}{N}.$$

To make sure that the boundary conditions are absorbed by the Hamiltonian one requires that the boundary conditions are restricted to be periodic in the momentum space:

$$\begin{aligned}
-(-1)^M c_1^+ c_N &= c_{N+1}^+ c_N, \\
\exp\left(i\pi\left(1+M\right)\right) c_1^+ c_N &= c_{N+1}^+ c_N, \\
\sum_{q,q'} \exp\left(i\pi(M+1) + iq - iq'N\right) c_q^\dagger c_{q'} &= \sum_{q,q'} \exp\left(iq(N+1) - iq'N\right) c_q^\dagger c_{q'}.
\end{aligned}$$

Due to the fact that q lives on a circle the condition takes form:

$$\pi(M+1) = qN + 2k\pi, \quad k \in \mathbb{Z}.$$

What gives following formula for parameter q

$$q = \frac{\pi(M+1-2k)}{N}, \quad k \in \mathbb{Z}.$$

It can be shown that the fermionic algebra holds for such operators:

$$\begin{aligned}
\left\{c_p, c_q^\dagger\right\} =& \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp(-ipk)\, c_k \frac{1}{\sqrt{N}} \sum_{k'=1}^{N} c_{k'}^\dagger \exp\left(ipk'\right) \\
&+ \frac{1}{\sqrt{N}} \sum_{k'=1}^{N} c_{k'}^\dagger \exp\left(iqk'\right) \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp(-iqk)\, c_k \\
=& \frac{1}{N} \sum_{k,k'=1}^{N} \exp\left(iqk' - ipk)\right) \left(c_k c_{k'}^\dagger + c_{k'}^\dagger c_k\right) \\
=& \frac{1}{N} \sum_{k,k'=1}^{N} \exp\left(iqk' - ipk\right) \left\{c_k, c_{k'}^\dagger\right\} \\
=& \frac{1}{N} \sum_{k,k'=1}^{N} \exp\left(iqk' - ipk\right) \delta_{kk'} \\
=& \frac{1}{N} \sum_{k=1}^{N} \exp\left(ik(q-p)\right) = \delta_{pq}.
\end{aligned}$$

Concluding the anti-commutation rules

$$\left\{c_p, c_q\right\} = \left\{c_p^\dagger, c_q^\dagger\right\} = 0, \qquad \left\{c_p, c_q^\dagger\right\} = \delta_{pq}.$$

One can express terms existing in the Hamiltonian in terms of those operators. To simplify calculations it is convenient to start from the following form of Hamiltonian and show that it is equivalent to the former one

$$\mathscr{H}_{XY} = -NB + \sum_q \left(2B + J\cos q\right) c_q^\dagger c_q. \tag{5.6}$$

The $-NB$ term already exists in former form of Hamiltonian. Then it does not require to be proved. The field interaction term can be expressed as

$$\begin{aligned}2B\sum_q c_q^\dagger c_q =& \frac{2B}{N}\sum_q \sum_{k,k'=1}^{N} \exp\left(iq(k-k')\right) c_k^\dagger c_{k'} \\ =& 2B\sum_{k,k'=1}^{N} \delta_{kk'} c_k^\dagger c_{k'} = 2B\sum_{k=1}^{N} c_k^\dagger c_k.\end{aligned}$$

The spin-spin interaction term can be transformed as

$$\begin{aligned}\sum_q \cos q c_q^\dagger c_q =& \sum_q \frac{1}{2}\left(\exp(iq)+\exp(-iq)\right) c_q^\dagger c_q \\ =& \frac{1}{2N}\sum_q \sum_{k,k'=1}^{N} \left(\exp(iq(k-k'+1))c_k^\dagger c_{k'} + \exp(iq(k-k'-1))c_k^\dagger c_{k'}\right) \\ =& \frac{1}{2}\sum_{k,k'=1}^{N} \left(\delta_{k+1,k'} c_k^\dagger c_{k'} + \delta_{k,k'+1} c_k^\dagger c_{k'}\right) \\ =& \frac{1}{2}\sum_{k=1}^{N} \left(c_k^\dagger c_{k+1} + c_{k+1}^\dagger c_k\right).\end{aligned}$$

It can be shown that M is a constant of motion [11].

$$\begin{aligned}\left[\mathscr{H}_{XY}, \hat{M}\right] =& \left(-NB + \sum_q \left(2B + J\cos q\right) c_q^\dagger c_q\right)\left(\sum_{q'} c_{q'}^\dagger c_{q'}\right) \\ & - \left(\sum_{q'} c_{q'}^\dagger c_{q'}\right)\left(-NB + \sum_q \left(2B + J\cos q\right) c_q^\dagger c_q\right) \\ =& (NB - NB)\left(\sum_{q'} c_{q'}^\dagger c_{q'}\right) + 2B\left(\sum_q c_q^\dagger c_q \sum_{q'} c_{q'}^\dagger c_{q'} - \sum_{q'} c_{q'}^\dagger c_{q'} \sum_q c_q^\dagger c_q\right) \\ & + J\left(\sum_q \cos(q) c_q^\dagger c_q \sum_{q'} c_{q'}^\dagger c_{q'} - \sum_{q'} c_{q'}^\dagger c_{q'} \sum_q \cos(q) c_q^\dagger c_q\right)\end{aligned}$$

Using the anti-commutation rules it can be shown that $\hat{M}$ commutes with $\sum_q \cos(q) c_q^\dagger c_q$

$$\begin{aligned}
\sum_q \cos(q) c_q^\dagger c_q \sum_{q'} c_{q'}^\dagger c_{q'} &= \sum_{q,q'} \cos(q) c_q^\dagger c_q c_{q'}^\dagger c_{q'} \\
&= -\sum_{q,q'} \cos(q) c_q^\dagger c_{q'}^\dagger c_q c_{q'} \\
&= (-1)^2 \sum_{q,q'} \cos(q) c_q^\dagger c_{q'}^\dagger c_{q'} c_q \\
&= (-1)^3 \sum_{q,q'} c_{q'}^\dagger \cos(q) c_q^\dagger c_{q'} c_q \\
&= (-1)^4 \sum_{q,q'} c_{q'}^\dagger c_{q'} \cos(q) c_q^\dagger c_q \\
&= \sum_{q'} c_{q'}^\dagger c_{q'} \sum_q \cos(q) c_q^\dagger c_q .
\end{aligned}$$

Hence, the the commutator $\left[\mathscr{H}_{XY}, \hat{M}\right]$ is proved to vanish and M is in fact the constant of motion.

The Hamiltonian (5.6) can be transformed to more intuitive form by using fact that $\sum_{k=1}^{N} \cos q_k = 0$ as follows

$$\begin{aligned}
\mathscr{H}_{XY} &= -NB + \sum_q (2B + J\cos q)\, c_q^\dagger c_q \\
&= 2B \sum_q \left(c_q^\dagger c_q - \frac{1}{2} \right) + J \sum_q \cos q \left(c_q^\dagger c_q - \frac{1}{2} \right) \\
&= \sum_q \cos q \left(c_q^\dagger c_q - \frac{1}{2} \right) (2B + J\cos q) \\
&= \sum_q \cos q \left(c_q^\dagger c_q - \frac{1}{2} \right) \varepsilon_q
\end{aligned}$$

Where energy for each momentum was denoted as $\varepsilon_q \equiv 2B + J\cos q$. This representation of Hamiltonian is similar for Hamiltonian of harmonic oscillators with energy levels ε_q.

5.2.1 Density operator

According to the methods developed in 2.1 one need to derive vector state of the system to define **DO** and calculate entanglement entropy. In case of $T = 0$ the system is described by a **ground state**. To construct it one needs a vacuum state which is defined as follows

$$\forall_q \quad c_q|0\rangle = 0.$$

In case of M quasi-particles in the system any state can be represented as

$$|\Psi\rangle = \prod_{\{q\}} c_q^\dagger |0\rangle.$$

The **ground state** will be a state of the mentioned form with the set $\{q\}$ minimizing energy given by action of the Hamiltonian on the state. Due to the fact that M is constant of motion of the system the only non-constant part of Hamiltonian is the one with $\cos(q)$. To find the **ground state** it is needed to find such values of q the maximizes $\cos(q)$ function. Cosine function has its maximum in 0, so one needs to construct set $\{q\}$ with M elements being as close to 0 as possible. The set need to be symmetric around 0. The effect of those considerations is given by:

$$\{q\} = \{q = \frac{\pi}{N}(M+1-2l),\, l = 1,\ldots,M\}.$$

But it can be shown that it is symmetric around zero:

$$q(1) = \frac{\pi}{N}(M-1), \qquad\qquad q(M) = -\frac{\pi}{N}(M-1).$$

And the ground state is given by:

$$|GS\rangle = \prod_{l=1}^{M} c^\dagger_{\pi(M+1-2l)/N}|0\rangle.$$

To derive entanglement entropy one needs the **DO** and further the **RDO** of the chosen subsystem. As shown in (5.7) density operator for a system described by state $|\Psi\rangle$ is defined as follows:

$$\hat{\rho}_\Psi = |\Psi\rangle \otimes \langle\Psi|.$$

For the **ground state** the density operator in the momentum space takes form

$$\begin{aligned}\hat{\rho}_{GS} &= |GS\rangle \otimes \langle GS| \\ &= \prod_{l=1}^{M} c^{\dagger}_{\pi(M+1-2l)/N}|0\rangle \otimes \left(\prod_{l'=1}^{M} c^{\dagger}_{\pi(M+1-2l')/N}|0\rangle\right)^{\dagger} \\ &= \prod_{l=1}^{M} c^{\dagger}_{\pi(M+1-2l)/N}|0\rangle \otimes \langle 0| \prod_{l'=1}^{M} c_{-\pi(M+1-2l')/N}.\end{aligned}$$

In order to have control over the division of the system into subsystems one needs representation in lattice space.

$$\begin{aligned}\hat{\rho}_{GS} &= \prod_{l=1}^{M} c^{\dagger}_{\pi(M+1-2l)/N}|0\rangle \otimes \langle 0| \prod_{l'=1}^{M} c_{-\pi(M+1-2l')/N} \\ &= \prod_{l=1}^{M} \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \exp\left(\frac{i\pi(M+1-2l)k}{N}\right) c^{\dagger}_{k}|0\rangle \otimes \langle 0| \prod_{l'=1}^{M} \frac{1}{\sqrt{N}} \sum_{k'=1}^{N} \exp\left(\frac{i\pi(M+1-2l')k'}{N}\right) c_{k'} \\ &= \frac{1}{N^M} \prod_{l,l'=1}^{M} \sum_{k,k'=1}^{N} \exp\left(\frac{i\pi\left((M+1-2l)k+(M+1-2l')k'\right)}{N}\right) c^{\dagger}_{k}|0\rangle \otimes \langle 0|c_{k'}\end{aligned}$$

Operators of creation acting on the vacuum state produce

$$c_{k_i}|0\rangle = |k_i\rangle, \quad k_i = 1,\ldots,N.$$

What means that the basis of this space is given by a following set of vectors:

$$\mathscr{H} = \bigotimes_{M=0}^{N} \mathscr{H}^M = \bigotimes_{M=0}^{N} \operatorname{span}\left\{|k_1,\ldots,k_M\rangle : 1 \le k_1 < \cdots < k_M \le N\right\}.$$

5.2.2 Final form of RDO

The considered subsystem A consists of L nods. Basis states can be decomposed as:

$$|0\rangle = |0\rangle_A \otimes |0\rangle_B.$$

As was already shown the **RDO** is obtained according to the equation (2.5) by trace over the surroundings of the given subsystem. What means in this case sum over a following set

of states:

$$\mathscr{H}_B = \bigotimes_{M=0}^{N-L} \operatorname{span}\{|\,k_1,\ldots,k_M\,\rangle_B : L < k_1 < \cdots < k_M \leq N\}\,.$$

The dimension of such a space is $\dim \mathscr{H}_B = \sum_{k=0}^{M} \begin{pmatrix} N-L \\ k \end{pmatrix}$. And the **RDO** takes form:

$$\begin{aligned}\hat{\rho}_A^M =& \operatorname{tr}_B\left(\hat{\rho}^M\right)\\ =& {}_B\langle 0|\hat{\rho}^M|0\rangle_B + \sum_{L<k_M\leq M} {}_B\langle k_M|\hat{\rho}^M|k_M\rangle_B\\ &+ \sum_{L<k_{M-1}<k_M\leq M} {}_B\langle k_{M-1},k_M|\hat{\rho}^M|k_{M-1},k_M\rangle_B\\ &+ \sum_{L<k_1<\cdots<k_M\leq M} {}_B\langle k_1,\ldots,k_M|\hat{\rho}^M|k_1,\ldots,k_M\rangle_B\end{aligned}$$

What results in final form:

$$\hat{\rho}_A^M = \sum_{\substack{1\leq k_1<\cdots<k_M\leq L\\ 1\leq k_1'<\cdots<k_M'\leq L}} N_{1\ldots M|k_1\ldots k_m} N^*_{1\ldots M|k_1'\ldots k_m'} |\,k_1,\ldots,k_M\,\rangle_{A\,A}\langle\,k_1',\ldots,k_M'\,| \tag{5.7}$$

$$+ \sum_{\substack{1\leq k_1<\cdots<k_{M-1}\leq L\\ 1\leq k_1'<\cdots<k_{M-1}'\leq L}} \left(\sum_{L<k_M\leq N} N_{1\ldots M|k_1\ldots k_{M-1}k_M} N^*_{1\ldots M|k_1'\ldots k_{M-1}'k_M}\right) |\,k_1,\ldots,k_{M-1}\,\rangle_{A\,A}\langle\,k_1',\ldots,k_{M-1}'\,|$$

(5.8)

$$\vdots \tag{5.9}$$

$$+ \sum_{\substack{1\leq k_1\leq L\\ 1\leq k_1'\leq L}} \left(\sum_{L<k_2<\ldots k_M\leq N} N_{1\ldots M|k_1k_2\ldots k_M} N^*_{1\ldots M|k_1'k_2\ldots k_M}\right) |\,k_1\,\rangle_{A\,A}\langle\,k_1'\,| \tag{5.10}$$

$$+ \sum_{L<k_1<\cdots<k_M\leq M} |N_{1\ldots M|k_1\ldots k_m}|^2 |\,0\,\rangle_{A\,A}\langle\,0\,|. \tag{5.11}$$

Where coefficient function is defined as

$$N_{1\ldots M|k_1\ldots k_{M-1}k_M} \equiv \frac{2^{\frac{L(L-1)}{2}}}{N^{\frac{L}{2}}} \mathrm{e}^{i\frac{\pi}{N}\left[(2-M)\sum\limits_i k_i + \sum\limits_{i<l}(k_i-k_j+N)\right]} \prod_{1\leq i<j\leq L} \sin\left(\frac{\pi}{N}(k_i-k_j)\right).$$

Approach to computations of the above results was taken in form of code in Mathematica which can be found in the appendix. Results have not be obtained yet due to limited time, but further research will be held.

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Appendix A

Code for computation of Entanglement Entropy for XY Model

In this appendix there is shown code used to compute Entanglement Entropy for quantum 1-dimensional XX Model by using form of Reduced Density Matrix as in chapter (5.11).

Description of the code

Code of the algorithm was written in a modular way. Following elements can be distinguished:

1. Lines $1-7$: Functions generating set of indices used in dynamically generated sums
2. Lines $9-19$: Function generating basis vectors corresponding to particles existing in given list of nods.
3. Lines $21-24$: Function generating coefficients N_M.
4. Lines $26-29$: Generating external product of two basis vectors corresponding to two lists of given nods.
5. Lines $31-34$: Function checking whether given list of numbers is increasing.
6. Lines $36-48$: Functions generating the **RDO**.
7. Lines $50-52$: Function calculating von Neuman entropy for a given set of parameters.

Code of the algorithm

```
GenSingIndList[MM_, A_, B_, it_, NN_] :=
        Table[{j[ii + it NN], If[ii < 2, A, j[ii + it NN - 1] +
           1], B}, {ii, 1, MM}]

GenIndList[NN_, M_, L_, MM_] :=
        Join[GenSingIndList[MM, 1, L, 0, NN],
        GenSingIndList[MM, 1, L, 1, NN],
        GenSingIndList[M - MM, L + 1, NN, 2, NN]]

PartVector[NN_, k_] :=
        SparseArray[{1, 2^NN (1 - 2^(-k))} -> 1, {1, 2^NN}]

PartVectors[NN_, inds_] :=
        Table[If[i > 1, If[i < Length[inds],
        PartVector[inds[[i]] - inds[[i - 1]], inds[[i]] - inds[[
           i - 1]]],
        PartVector[NN - inds[[i - 1]], inds[[i]] - inds[[i -
           1]]]],
        PartVector[inds[[1]], inds[[1]]]], {i, 1, Length[inds]}]

BasisVector[NN_, inds_] :=
        If[Length[inds] > 1, KroneckerProduct[##] & @@
           PartVectors[NN, inds], PartVector[NN, inds[[1]]]]

NCoeff[NN_, M_, L_, ks_] :=
        2^((L (L - 1))/2) NN^(-L/2) Exp[I Pi/ NN ((2 - M) Sum[ks
           [[i]], {i, 1, Length[ks]}] +
        Sum[ks[[i]] - ks[[j]] + NN, {j, 1, Length[ks]}, {i, 1,
        j - 1}])] Product[ Sin[Pi/NN (ks[[i]] - ks[[j]])], {j,
           2, Length[ks]}, {i, 1, j - 1}]

KMatrix[L_, ksl_, ksp_] :=
        If[Length[ksl] == 0 && Length[ksp] == 0,
        SparseArray[{2^L, 2^L} -> 1, {2^L, 2^L}],
        Transpose[BasisVector[L, ksl]].BasisVector[L, ksp]]

Grow[list_] :=
```

```
        If[Length[list] > 1, For[i = 1; temp = True, i < Length[
           list], i++, If[i == Length[list] - 1,
        Return[temp = temp && (list[[i]] < list[[i + 1]])],
        temp = temp && (list[[i]] < list[[i + 1]])]], Return[
           True]]

FRDO[ind_] :=
        If[Grow[ind[[5 ;; ind[[4]] + 4]]] &&
        Grow[ind[[ind[[4]] + 5 ;; 2 ind[[4]] + 4]]] &&
        Grow[ind[[2 ind[[4]] + 5 ;; Length[ind]]]],
        Conjugate[ NCoeff[ind[[1]], ind[[2]], ind[[3]],
        Join[ind[[5 ;; ind[[4]] + 4]],
        ind[[2 ind[[4]] + 5 ;; Length[ind]]]]] NCoeff[ind[[1]],
        ind[[2]], ind[[3]], Join[ind[[ind[[4]] + 5 ;; 2 ind[[4]]
            + 4]], ind[[2 ind[[4]] + 5 ;; Length[ind]]]]]
           KMatrix[ind[[3]], ind[[5 ;; ind[[4]] + 4]], ind[[ind
           [[4]] + 5 ;; 2 ind[[4]] + 4]]],0]

RDO[NN_, M_, L_] :=
        Sum[Sum[FRDO @@ {Join[{NN, M, L, m}, First /@ {##}]},
           ##] & @@ GenIndList[NN, M, L, m], {m, 1, If[M>L,L,M
           ]}]

vNE[NN_, M_, L_] :=
        Total[If[Abs[#] > 0, -Abs[#] Log[Abs[#]], 0] & /@
        Eigenvalues[N[RDO[NN, M, L]]]]
```