$$E(g(X)) = \begin{cases} \sum_i g(x_i) p_i \\ \int_{-\infty}^{\infty} g(y) f_X(y)\, dy \end{cases}$$

① $D^2(X) = E(X^2) - E^2(X)$

$$E(X) = \int_0^1 cxe^x dx = \left[cxe^x\right]_0^1 - c\int_0^1 e^x dx = c(e - 0 - e + 1) = c$$

$f(x) = ce^x, \quad x \in (0,1)$

$$E(X^2) = \int_0^1 cx^2e^x dx = \left[cx^2e^x\right]_0^1 - \int_0^1 2cxe^x dx = ce - 2c\left[\left[xe^x\right]_0^1 - \int_0^1 e^x dx\right] =$$

$$= ce - 2c\left[e - 0 - e + 1\right] = ce - 2c$$

$$D^2(X) = ce - 2c - c^2 = c(e-2) - c^2$$

$$1 = \int_0^1 ce^x dx = \left[ce^x\right]_0^1 = c(e-1) \Rightarrow c = \frac{1}{e-1}$$

$$D^2(X) = \frac{e-2}{e-1} - \frac{1}{(e-1)^2} = \frac{(e-2)(e-1) - 1}{(e-1)^2} = \frac{e^2 - 3e + 2 - 1}{(e-1)^2}$$

Liniowość:

② $D^2(aX+b) \overset{?}{=} a^2 D^2(X)$

$$E(aX+b) = aE(X) + b$$

$$E^2(aX+b) = (E(aX+b))^2 = (aE(X)+b)^2 = a^2E^2(X) + 2ab\,E(X) + b^2$$

$$D^2(X) = E(X^2) - E^2(X)$$

$$E((aX+b)^2) = E(a^2X^2 + 2abX + b^2) = a^2E(X^2) + 2ab\,E(X) + b^2$$

$$D^2(aX+b) = a^2E(X^2) + \cancel{2abE(X)} + \cancel{b^2} - a^2E^2(X) - \cancel{2abE(X)} - \cancel{b^2}$$

$$= a^2(E(X^2) - E^2(X)) = a^2 D^2(X)$$

③ a) $X: \{p_0 = p,\ p_1 = 1-p\}$ 0-1-kowy $\quad E(X) = 0\cdot p + 1(1-p) = 1-p \overset{p=\frac{1}{2}}{=} \frac{1}{2}$

$E(X^2) = 0^2\cdot p + 1^2(1-p) = 1-p \quad D^2(X) = 1-p-1+p = 0$

b) dwumianowy (Bernoulliego)

$$\phi_X(t) = \sum_{k=0}^{n} e^{itk} \cancel{\binom{n}{k}} p P(k) = \sum_{k=0}^{n} e^{itk}\binom{n}{k}p^k(1-p)^{n-k} \overset{\text{Dwumian Newtona}}{=} (pe^{it}+(1-p))^n$$

$$P(X=k) = \binom{n}{k}p^k(1-p)^{n-k}$$

$$E(X) = \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}k \qquad \phi_X^{(n)}(t=0) = i^n E(X^n)$$

$$\phi_X'(t=0) = \left[n(pe^{it}+(1-p))^{n-1}pie^{it}\right]_{t=0} = npi = iE(X) \Rightarrow E(X) = np$$

$$\cancel{\phi_X''(t=0) = \left[npi(n-1)(pe^{it}+(1-p))^{n-2}pi\right]_{t=0} = (n(n-1)p^2i^2(p+1-p))^{n-2} =}$$

$$\cancel{= n(n-1)p^2i^2 = i^2E(X^2) \Rightarrow E(X^2) = n(n-1)p^2}$$

$$\cancel{D^2(X) = E(X^2) - E^2(X) = n(n-1)p^2 - n^2p^2 = p^2(n^2-n-n^2) = -np^2}$$

$$\phi_X''(t=0) = \left[npie^{it}(n-1)(pe^{it}+(1-p))^{n-1}pie^{it} + n(pe^{it}+(1-p))^{n-1}pi^2e^{it}\right]_{t=0} =$$

$$= n(n-1)p^2i^2(p+1-p)^{n-1} + npi^2(p+1-p)^{n-1} =$$

$$= p^2i^2n(n-1) + i^2 np \Rightarrow E(X^2) = p^2n(n-1) + pn$$

$$D^2(X) = p^2n(n-1) + pn - n^2p^2 = \cancel{p^2n^2} - p^2n + pn - \cancel{n^2p^2} = pn(1-p)$$

c) rozkład Poissona $\quad P_X(k) = e^{-\lambda}\frac{\lambda^k}{k!} \quad \lambda > 0 \quad k \in \mathbb{N}\cup\{0\}$

$$E(X) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}k = \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{k'=0}^{\infty}\frac{\lambda^{k'}}{k'!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

$$E(X^2) = \sum_{k=0}^{\infty}k^2e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}k = e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}(k-1+1) =$$

$$= e^{-\lambda}\lambda\sum_{k'=0}^{\infty}\frac{\lambda^{k'}}{k'!}(k'+1) = \cancel{\lambda + e^{-\lambda}\lambda^2\sum_{k'=0}^{\infty}\frac{\lambda^{k'-1}}{(k'-1)!} - \lambda \sum + \lambda e^{-\lambda}\sum_{k''=-1}^{\infty}\frac{\lambda^{k''}}{k''!}}$$

$$= \lambda + e^{-\lambda}\lambda\frac{\lambda^0}{0!}\cdot 0 + e^{-\lambda}\lambda\sum_{k'=1}^{\infty}\frac{\lambda^{k'}}{k'!}k' = \lambda + \lambda^2e^{-\lambda}\sum_{k'=1}^{\infty}\frac{\lambda^{k'-1}}{(k'-1)!} = \lambda + \lambda^2e^{-\lambda}e^{\lambda} = \lambda + \lambda^2$$

$$D^2(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

③ d) geometryczny $P(X=k)=p(1-p)^{k-1}, \; p\in(0,1), \; k\in\mathbb{N}, k>0$

$$E(X)=\sum_{k=1}^{\infty} kp(1-p)^{k-1} = p\sum_{k=1}^{\infty} k(1-p)^{k-1} \qquad \phi(t)=\sum_{k=1}^{\infty} e^{itk}p(1-p)^{k-1}$$

$$\phi'(t)=\sum_{k=1}^{\infty} ike^{itk}p(1-p)^{k-1} \qquad \phi'(0)=i\sum_{k=1}^{\infty} kp(1-p)^{k-1}$$

$$E(X)=p\sum_{k=1}^{\infty} k(1-p)^{k-1} = p\frac{1}{p^2}=\frac{1}{p}$$

$$E(X^2)=\sum_{k=1}^{\infty} k^2p(1-p)^{k-1} = \frac{p^2-3p+2}{(1-p)p^3}$$

$$D^2(X)=E(X^2)-E^2(X)=\frac{p^2-3p+2}{(1-p)p^2}-\frac{1}{p^2}=\frac{p^2-3p+2-1+p}{(1-p)p^2} \cancel{=\frac{2p^2-4p+2}{(1-p)p^3}}$$

$$\cancel{2\frac{p^2-2p+1}{(1-p)p^3}=2\frac{(p-1)^2}{(1-p)p^3}=2\frac{p-1}{p^3}}=\frac{p^2-2p+1}{(1-p)p^2}=\frac{(1-p)^2}{(1-p)p^2}=$$

$$=\frac{1-p}{p^2}$$

e) wykładniczy $f_X(y)=\begin{cases}\lambda e^{-\lambda y}, & y\geq 0\\ 0, & y<0\end{cases}$

$$E(X)=\int_0^{+\infty} y\lambda e^{-\lambda y}dy=\lambda\left[-\frac{1}{\lambda}ye^{-\lambda y}\right]_0^{\infty}-\lambda\left(-\frac{1}{\lambda}\right)\int e^{-\lambda y}dy=-\lambda\left[-\frac{1}{\lambda}e^{-\lambda y}\right]_0^{+\infty}=$$

$$=\frac{1}{\lambda}$$

$$E(X^2)=\int_0^{+\infty} y^2\lambda e^{-\lambda y}dy=\lambda\left[-\frac{1}{\lambda}y^2e^{-\lambda y}\right]_0^{+\infty}+2\int_0 ye^{-\lambda y}dy=$$

$$=2\left[-\frac{1}{\lambda}ye^{-\lambda y}\right]_0^{+\infty}+\frac{2}{\lambda}\int_0^{+\infty}e^{-\lambda y}dy=+\frac{2}{\lambda^2}$$

$$D^2(X)=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$$

f) normalny $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$z = \frac{(x-\mu)^2}{2\sigma^2}$

$dz = \frac{\cancel{2}(x-\mu)}{\cancel{2}\,\sigma^2} dx$

$$E(X) = \int_{-\infty}^{+\infty} \frac{x}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx =$$

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{a x^2}{2} + \cancel{bx}} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}$$

$$\cancel{= \int_{-\infty}^{+\infty} \frac{\sigma}{2\sigma\sqrt{2\pi}} \exp\left(-\frac{z}{2\sigma}\right) dz + \int_{-\infty}^{+\infty} \frac{\mu}{2\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx =}$$

$y = x - \mu \Rightarrow dy = dx$

do poprawki

$$\cancel{= \int_{-\infty}^{+\infty} \frac{2\sigma}{\sqrt{2\pi}} \left[-\exp(-z)\right]_{0}^{+\infty}} + \frac{\mu}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2\sigma^2}\right) dy =$$

$$= \frac{2\sigma}{\sqrt{2\pi}} + \frac{\mu}{\sigma\sqrt{2\pi}} \boxed{\sqrt{2\pi\sigma^2}} = \frac{2\sigma}{\sqrt{2\pi}} + \mu$$

$$\int_{-\infty}^{+\infty} \frac{x}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = \int_{-\infty}^{0} \frac{x}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx + \int_{0}^{+\infty} \frac{x}{\sigma\sqrt{2\pi}} \exp\left(\frac{(x-\mu)^2}{2\sigma^2}\right) dx$$

$= \int$

$E(X) = \mu \quad D^2(X) = \sigma^2$

(4) Rozwiązaniem jest rozkład dwumianowy (Bernoulliego)

$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k} \qquad n_1=3 \quad n_2=5$$

$k\in\{0,1,\dots,n\}$

$$\sum_{i=2}^{n_1}\binom{n_1}{i}p^i(1-p)^{n_1-i} > \sum_{j=3}^{n_2}\binom{n_2}{j}p^j(1-p)^{n_2-j}$$

$$\binom{3}{2}p^2(1-p)+\binom{3}{3}p^3 > \binom{5}{3}p^3(1-p)^2+\binom{5}{4}p^4(1-p)+\binom{5}{5}p^5$$

$$\frac{3\cdot\cancel{2!}}{\cancel{2!}1!}p^2(1-p)+p^3 > \frac{\cancel{3!}\,4\cdot5}{\cancel{3!}\,2!}p^3(1-p)^2+\frac{\cancel{4!}\,5}{\cancel{4!}1!}p^4(1-p)+p^5$$

$$3p^2-3p^3+p^3 > 10p^3+10p^5-20p^4+5p^4-5p^5+p^5$$

$$0 > 6p^5-15p^4+12p^3-3p^2 \quad \Big|\Big| :p^2 \quad ,\ p>0$$

$$0 > 6p^3-15p^2+12p-3 \quad \Big|\Big| :3$$

$$0 > 2p^3-5p^2+4p-1 = 2p^2(p-1)-3p(p-1)+(p-1) =$$

$$= (p-1)(2p^2-3p+1) = 2(p-1)\left(p^2-\frac{3}{2}p+\frac{1}{2}\right)$$

$$= 2(p-1)(p-1)\left(p-\frac{1}{2}\right) = 2(p-1)^2\left(p-\frac{1}{2}\right)$$

$$0 > 2(p-1)^2\left(p-\frac{1}{2}\right)$$

Ale $p\in\langle 0,1\rangle$, więc $p\in\left(0,\frac{1}{2}\right)$

(5) $f(x,y)=2e^{-(x+2y)}$, $0<x<\infty$, $0<y<\infty$

$$P\{X<Y\} = \int_0^\infty\int_0^y 2e^{-(x+2y)}\,dx\,dy = \int_0^\infty\int_0^y 2e^{-x}e^{-2y}\,dx\,dy$$

$$= \int_0^\infty 2e^{-2y}\left[-e^{-x}\right]_0^y = -\int_0^\infty 2e^{-2y}\left(e^{-y}-1\right)dy =$$

$$= -2\int_0^\infty \left(e^{-3y}-e^{-2y}\right)dy = -2\left[-\frac{1}{3}e^{-3y}\right]_0^\infty + 2\left[-\frac{1}{2}e^{-2y}\right]_0^\infty =$$

$$= -\frac{2}{3}\cdot 1 + 2\cdot\frac{1}{2} = 1-\frac{2}{3} = \frac{1}{3}$$

(6) Rozkład ~~geome~~ Bernoulliego

$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$, $p=0.6$

$P=\sum_{k=5}^{12}\binom{k}{5}p^5(1-p)^{k-5}$ = Numorzę nie, ale czy to jest poprawne

(7.)