4 Quadric Surfaces

A frequently used class of objects are the *quadric surfaces*, which are described with second-degree equations (quadratics). They include spheres, ellipsoids, tori, paraboloids, and hyperboloids. Quadric surfaces, particularly spheres and ellipsoids, are common elements of graphics scenes, and routines for generating these surfaces are often available in graphics packages. Also, quadric surfaces can be produced with rational spline representations.

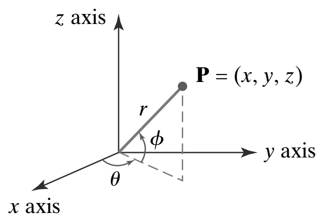

FIGURE 2
Parametric coordinate position (r, θ, ϕ) on the surface of a sphere with radius r.

Sphere

In Cartesian coordinates, a spherical surface with radius r centered on the coordinate origin is defined as the set of points (x, y, z) that satisfy the equation

$$x^2 + y^2 + z^2 = r^2 \tag{1}$$

We can also describe the spherical surface in parametric form, using latitude and longitude angles (Figure 2):

$$\begin{aligned} x &= r\cos\phi\cos\theta, & -\pi/2 \le \phi \le \pi/2 \\ y &= r\cos\phi\sin\theta, & -\pi \le \theta \le \pi \\ z &= r\sin\phi \end{aligned} \tag{2}$$

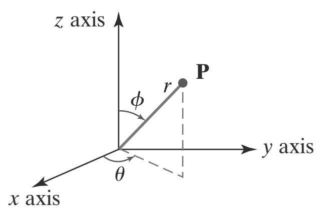

FIGURE 3
Spherical coordinate parameters (r, θ, ϕ), using colatitude for angle ϕ.

The parametric representation in Equations 2 provides a symmetric range for the angular parameters θ and ϕ. Alternatively, we could write the parametric equations using standard spherical coordinates, where angle ϕ is specified as the colatitude (Figure 3). Then, ϕ is defined over the range $0 \le \phi \le \pi$, and θ is often taken in the range $0 \le \theta \le 2\pi$. We could also set up the representation using parameters u and v defined over the range from 0 to 1 by substituting $\phi = \pi u$ and $\theta = 2\pi v$.

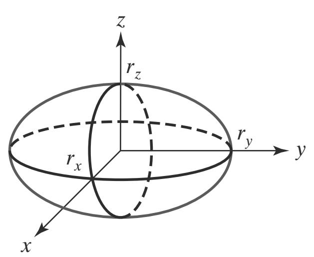

FIGURE 4
An ellipsoid with radii r_x, r_y, and r_z, centered on the coordinate origin.

Ellipsoid

An ellipsoidal surface can be described as an extension of a spherical surface where the radii in three mutually perpendicular directions can have different values (Figure 4). The Cartesian representation for points over the surface of an ellipsoid centered on the origin is

$$\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 = 1 \tag{3}$$

And a parametric representation for the ellipsoid in terms of the latitude angle ϕ and the longitude angle θ in Figure 2 is

$$\begin{aligned} x &= r_x\cos\phi\cos\theta, & -\pi/2 \le \phi \le \pi/2 \\ y &= r_y\cos\phi\sin\theta, & -\pi \le \theta \le \pi \\ z &= r_z\sin\phi \end{aligned} \tag{4}$$

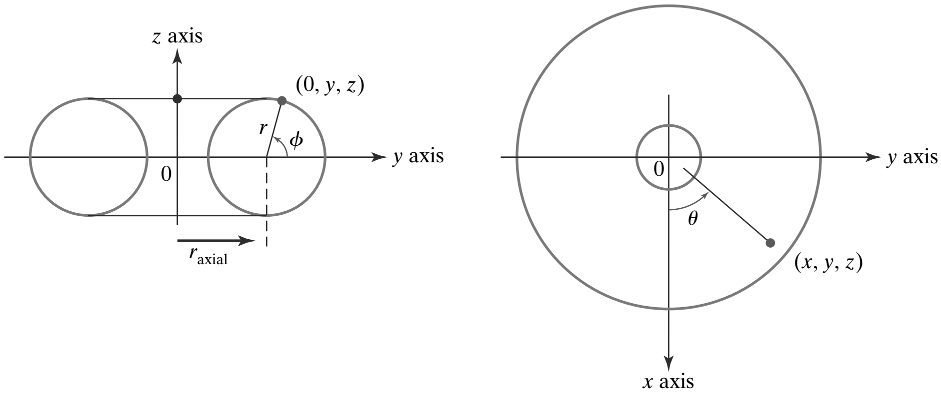

FIGURE 5
A torus, centered on the coordinate origin, with a circular cross-section and with the torus axis along the z axis.

Torus

A doughnut-shaped object is called a *torus* or *anchor ring*. Most often it is described as the surface generated by rotating a circle or an ellipse about a coplanar axis line that is external to the conic. The defining parameters for a torus are then the distance of the conic center from the rotation axis and the dimensions of the conic. A torus generated by the rotation of a circle with radius r in the yz plane about the z axis is shown in Figure 5. With the circle center on the y axis, the axial radius, r_{axial}, of the resulting torus is equal to the distance along the y axis to the circle center from the z axis (the rotation axis); and the cross-sectional radius of the torus is the radius of the generating circle.

The equation for the cross-sectional circle shown in the side view of Figure 5 is

$$(y - r_{\text{axial}})^2 + z^2 = r^2$$

Rotating this circle about the z axis produces the torus whose surface positions are described with the Cartesian equation

$$\left(\sqrt{x^2 + y^2} - r_{\text{axial}}\right)^2 + z^2 = r^2 \tag{5}$$

The corresponding parametric equations for the torus with a circular cross-section are

$$\begin{aligned} x &= (r_{\text{axial}} + r\cos\phi)\cos\theta, \qquad & -\pi \le \phi \le \pi \\ y &= (r_{\text{axial}} + r\cos\phi)\sin\theta, \qquad & -\pi \le \theta \le \pi \\ z &= r\sin\phi \end{aligned} \tag{6}$$

We could also generate a torus by rotating an ellipse, instead of a circle, about the z axis. For an ellipse in the yz plane with semimajor and semiminor axes denoted as r_y and r_z, we can write the ellipse equation as

$$\left(\frac{y - r_{\text{axial}}}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 = 1$$

where r_{axial} is the distance along the y axis from the rotation z axis to the ellipse center. This generates a torus that can be described with the Cartesian equation

$$\left(\frac{\sqrt{x^2 + y^2} - r_{\text{axial}}}{r_y}\right) + \left(\frac{z}{r_z}\right)^2 = 1 \tag{7}$$

The corresponding parametric representation for the torus with an elliptical cross-section is

$$
\begin{aligned}
x &= (r_{\text{axial}} + r_y \cos\phi)\cos\theta, \qquad & -\pi \le \phi \le \pi \\
y &= (r_{\text{axial}} + r_y \cos\phi)\sin\theta, \qquad & -\pi \le \theta \le \pi \\
z &= r_z \sin\phi
\end{aligned}
\tag{8}
$$

Other variations on the preceding torus equations are possible. For example, we could generate a torus surface by rotating either a circle or an ellipse along an elliptical path around the rotation axis.