UNIT-2

Fourier series

2.1 Definition,

2.2 Dirichlet’s conditions

2.3 Full range Fourier series,

2.4 Half range Fourier series,

2.5 Harmonic analysis,

2.6 Parseval’s identity and Applications to problems in Engineering.

Fourier series applications in engineering

- The **Fourier series** has many such **applications** in electrical **engineering**, vibration analysis, acoustics, optics, signal processing, image processing, quantum mechanics, econometrics, thin-walled shell theory, etc.

Fourier series

Applications

- Signal Processing
- Image processing
- Heat distribution mapping
- Wave simplification
- Light Simplication(Interference , Deffraction etc.)
- Radiation measurements etc.

What is Fourier Series?

The Fourier series allows us to model any arbitrary periodic signal or function $f(x)$ in the form $\frac{a_0}{2} + (a_1 cosx + a_2 cos2x + \cdots) + (b_1 sinx + b_2 sin2x + \cdots)$ the interval $[C, C + 2l]$ under some conditions called **Dirichlet's conditions** as given below:

(i) $f(x)$ is periodic with a period $2l$

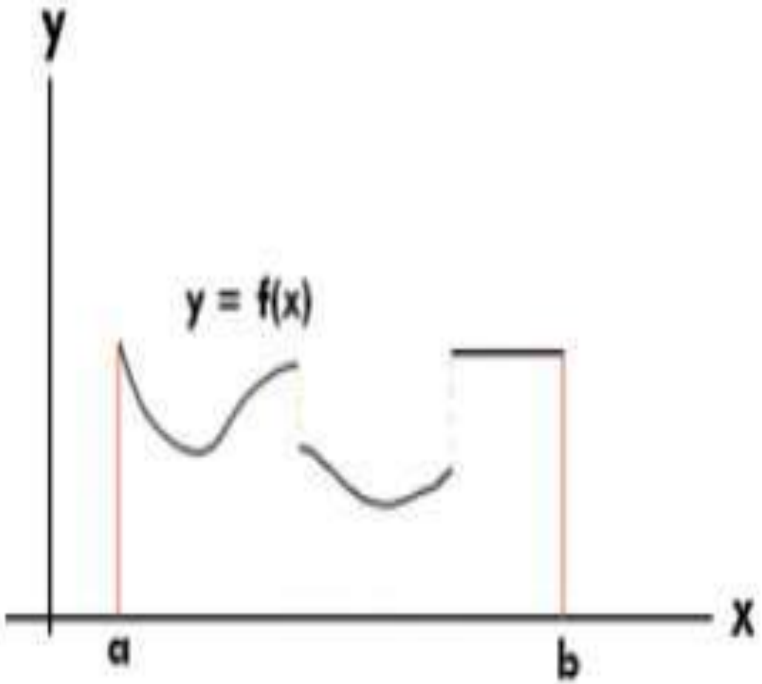
(ii) $f(x)$ and its integrals are finite and single valued in $[C, C + 2l]$

(iii) $f(x)$ is piecewise continuous* in the interval $[C, C + 2l]$

(iv) $f(x)$ has a finite no of maxima & minima in $[C, C + 2l]$

PIECEWISE CONTINUOUS FUNCTIONS

* A function $f(x)$ is said to be **piecewise continuous** in an interval $[a,b]$, if the interval can be subdivided into a finite number of intervals in each of which the function is continuous and has finite left and right hand limits i.e. it is bounded. In other words, a piecewise continuous function is a function that has a finite number of discontinuities and doesn't blow up to infinity anywhere in the given interval.



Periodic Functions

A function $f(x)$ is said to be periodic if there exists a positive number T such that $f(x+T) = f(x) \; \forall \, x \in R$.

Here T is the smallest positive real number such that $f(x+T) = f(x) \; \forall \, x \in R$ and is called the fundamental period of $f(x)$.

We know that $\sin x$, $\cos x$, $\sec x$, $\operatorname{cosec} x$ are periodic functions with period 2π whereas $\tan x$ and $\cot x$ are periodic with a period π. The functions $\sin nx$ and $\cos nx$ are periodic with period $\frac{2\pi}{n}$, while fundamental period of $\tan nx$ is $\frac{\pi}{n}$.

Fourier series

- If f(x) is periodic function of period 2π & it is defined in interval $c \leq x \leq c + 2\pi$ & satisfies Dirichlet's condition then f(x) can be represented by trigonometric series as
- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Where

- $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx,$
- $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\, dx,$
- $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\, dx$

Some useful results in computation of the Fourier series:

If m, n are non - zero integers then:

(i) $\int_c^{c+2\pi} sinnx\, dx = -\left[\frac{cosnx}{n}\right]_c^{c+2\pi} = 0$

(ii) $\int_c^{c+2\pi} cosnx\, dx = 0, n \neq 0$

(iii) $\int_c^{c+2\pi} sinmx.sinnx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

(iv) $\int_c^{c+2\pi} cosmx.cosnx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

(v) $\int_c^{c+2\pi} sinmx.cosnx\, dx = 0$

(vi) $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

(vii) $\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

(viii) $\sin n\pi = 0$

(ix) $\cos n\pi = (-1)^n$

(x) Integration by parts when first function vanishes after a finite number of differentiations:

If u and v are functions of x

$$\int u.v\,dx = uv_1 - u^{(1)}v_2 + u^{(2)}v_3 - u^{(3)}v_4 + \cdots$$

Here $u^{(n)}$ is derivative of $u^{(n-1)}$ and v_n is integral of v_{n-1}

For example

$$\int x^2.\sin nx\,dx = (x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)$$

$$= -x^2\cos x + 2x\sin x + 2\cos x$$

$$= -\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}$$

Formulae

1) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

2) $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$

3) $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

4) $-2 \sin A \sin B = \cos(A+B) - \cos(A-B)$

5) $\sin(n\pi) = 0, \cos(n\pi) = (-1)^n$

6) $\sin(2n\pi) = 0, \cos(2n\pi) = 1$

7) $\sin\left(\frac{(2n+1)\pi}{2}\right) = (-1)^n, \cos\left(\frac{(2n+1)\pi}{2}\right) = 0$

Problems on Fourier Series

1) Find the Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$.

Sol: We know that, the Fourier series of $f(x)$ defined in the interval $(0, 2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Here, $f(x) = x^2$

$$\text{Now, } a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}\left[(2\pi)^3 - 0\right] = \frac{8}{3}\pi^2$$

$$\Rightarrow \boxed{a_0 = \frac{8}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\cos nx}_{v}\,dx$$

$$= \frac{1}{\pi}\left[x^2\int \cos nx\,dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \cos nx\,dx\right)dx\right\}\right]$$

$$\boxed{\left[\because \int uv\,dx = u\int v\,dx - \left\{\int \frac{du}{dx}.\left(\int v\,dx\right)dx\right\}\right]}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \left\{\int 2x\left(\frac{\sin nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left\{\int \underbrace{x}_{u}\,\underbrace{\sin nx}_{v}\,dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \int 1.\frac{\cos nx}{n}dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\int \cos nx\,dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \qquad \left[\because \begin{matrix}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{matrix}\right]$$

$$\Longrightarrow \boxed{a_n = \frac{4}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\sin nx}_{v}\,dx$$

$$= \frac{1}{\pi}\left[x^2\int \sin nx\,dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \sin nx\,dx\right)dx\right\}\right]$$

$$\boxed{\left[\because \int uv\,dx = u\int v\,dx - \left\{\int \frac{du}{dx}.\left(\int v\,dx\right)dx\right\}\right]}$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - \left\{\int 2x\left(-\frac{\cos nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left\{\int \underbrace{x}_{u}\,\underbrace{\cos nx}_{v}\,dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \int 1.\frac{\sin nx}{n}dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\int \sin nx\,dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\frac{\cos nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n^2}x\sin nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$= -\frac{4\pi}{n} \quad \left[\because \begin{array}{l}\cos 2n\pi = 1\\ \sin 2n\pi = 0\end{array}\right]$$

$$\Rightarrow \boxed{b_n = -\frac{4\pi}{n}}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\therefore f(x) = x^2 = \frac{\frac{8\pi^2}{3}}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$\Rightarrow x^2 = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

This is the Fourier series for the function $f(x) = x^2$

Hence the result

Example 2 If $f(x+2\pi)=f(x)$, find the Fourier expansion $f(x)=x$ in the interval $[0,2\pi]$

Hence or otherwise prove that $\frac{1}{1}-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{\pi}{4}$

Solution: $f(x)=x$ is integrable and piecewise continuous in the interval $[0,2\pi]$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x)\approx\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n cosnx+\sum_{n=1}^{\infty}b_n sinnx \;......①$$

$$a_0=\frac{1}{\pi}\int_C^{C+2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}xdx=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}=2\pi$$

$$a_n=\frac{1}{\pi}\int_C^{C+2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x\cos nx\,dx$$

$$=\frac{1}{\pi}\left[(x)\left(\frac{sinnx}{n}\right)-(1)\left(\frac{-cosnx}{n^2}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[x\frac{sinnx}{n}+\frac{cosnx}{n^2}\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{cosnx}{n^2}\right]_0^{2\pi} \qquad \because sinnx=0 \text{ when } x=0 \text{ or } x=2\pi$$

$$=\frac{1}{\pi}\left[\frac{1}{n^2}-\frac{1}{n^2}\right]=0 \qquad \because \cos 2n\pi=1$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= -\frac{1}{\pi}\left[\frac{2\pi}{n}\right] = -\frac{2}{n} \qquad \because \sin nx = 0 \text{ when } x = 0 \text{ or } x = 2\pi \text{ and } \cos 2n\pi = 1$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \pi - 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

Putting $x = \frac{\pi}{2}$ on both sides

$$\frac{\pi}{2} = \pi - 2\left[\frac{1}{1} + 0 - \frac{1}{3} + 0 + \frac{1}{5} + \cdots\right]$$

$$\Rightarrow \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Fourier Series examples of neither even nor odd functions in (-l,l) period

Example 1

If $f(x+2\pi)=f(x)$, find the Fourier expansion $f(x)=e^{ax}$ in the interval $[-\pi,\pi]$

Solution: $f(x)=e^{ax}$ is integrable and piecewise continuous in the interval $[-\pi,\pi]$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x)\approx \frac{a_0}{2}+\sum_{n=1}^{\infty} a_n cosnx+\sum_{n=1}^{\infty} b_n sinnx \text{ ......①}$$

$$a_0=\frac{1}{\pi}\int_C^{C+2\pi} f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}dx$$

$$=\frac{1}{a\pi}[e^{ax}]_{-\pi}^{\pi}=\frac{1}{a\pi}[e^{a\pi}-e^{-a\pi}]=\frac{2}{a\pi}\sinh a\pi \quad \because \frac{e^{x}-e^{-x}}{2}=\sinh x$$

$$a_n=\frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\cos nx\,dx$$

$$=\frac{1}{\pi(a^2+n^2)}[e^{ax}\,(a\cos nx+n\sin nx)]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi(a^2+n^2)}[e^{a\pi}\,(a\cos n\pi+n\sin n\pi)-e^{-a\pi}\,(a\cos n\pi-n\sin n\pi)]$$

$$=\frac{a(-1)^n}{\pi(a^2+n^2)}[e^{a\pi}-e^{-a\pi}]=\frac{2a(-1)^n}{\pi(a^2+n^2)}\sinh a\pi$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{ax}\sin nx\,dx$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{ax}\,(a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[e^{a\pi}\,(a\sin n\pi - n\cos n\pi) - e^{-a\pi}\,(a\sin n\pi - n\cos n\pi)\right]$$

$$= \frac{-n(-1)^n}{\pi(a^2+n^2)}\left[e^{a\pi} - e^{-a\pi}\right] = \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)}\sinh a\pi$$

Substituting values of a_0, a_n, b_n in ①

$$f(x) \approx \frac{\sinh a\pi}{\pi}\left[\frac{1}{a} + 2a\left[-\frac{\cos x}{(a^2+1^2)} + \frac{\cos 2x}{(a^2+2^2)} - \frac{\cos 3x}{(a^2+3^2)} + \cdots\right] + 2\left[\frac{\sin x}{(a^2+1^2)} - \frac{2\sin 2x}{(a^2+2^2)} +\right.\right.$$

3sin3xa2+32−...

Example 2: If $f(x+2\pi) = f(x)$, find the Fourier series expansion of

$$f(x) = \begin{cases} 0, -\pi \le x \le 0 \\ x, \ 0 \le x \le \pi \end{cases}$$

Hence or otherwise prove that $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Solution: $f(x)$ is integrable and piecewise continuous in the interval $[-\pi, \pi]$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \ldots\ldots ①$$

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \text{ dx} + \int_0^{\pi} xdx\right] = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \ \cos nx \text{ dx} + \int_0^{\pi} x\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{sinnx}{n}\right) - (1)\left(\frac{-cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x\, sinnx}{n} + \frac{cosnx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad \because sinnx = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1] = \begin{cases} \frac{-2}{\pi n^2} & , n\ is\ odd \\ 0 & , n\ is\ even \end{cases}$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\ \sin nx\,\mathrm{dx} + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\frac{\pi(-1)^n}{n}\right] \qquad \because \frac{\sin nx}{n^2} = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= -\frac{1}{n}[(-1)^n] = \frac{(-1)^{n+1}}{n} = \begin{cases} \frac{1}{n} & ,n\ is\ odd \\ -\frac{1}{n} & ,n\ is\ even \end{cases}$$

Substituting values of a_0 , a_n , b_n in ①

$$f(x) \approx \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \frac{\sin 5x}{5} - \cdots\right]$$

Putting $x = \frac{\pi}{2}$ on both sides

$$\frac{\pi}{2} = \frac{\pi}{4} - 0 + \left[\frac{1}{1} - 0 - \frac{1}{3} - 0 + \frac{1}{5} - \cdots\right]$$

$$\Rightarrow \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Determination of Function Values at the Points of Discontinuity

A function satisfying Dirichlet's conditions may be expanded into Fourier series if it is discontinuous at a finite number of points.

Let the function be defined in (a, b) as

$$f(x) = \begin{cases} f_1(x), & a < x < x_o \\ f_2(x), & x_o < x < b \end{cases}$$

1. **To find $f(x)$ at $x = a$ or $x = b$ (End points discontinuity)**

   Since $f(a)$ and $f(b)$ are not defined in the interval (a, b)

   $$\therefore f(a) = f(b) = \frac{1}{2}[(RHL\ at\ x = a) + (LHL\ at\ x = b)]$$

   $$= \frac{1}{2}\left[\lim_{x \to a^+} f(x) + \lim_{x \to b^-} f(x)\right]$$

2. **To find $f(x)$ at $x = x_o$ (Mid point discontinuity)**

   Since $f(x_o)$ is not defined in the interval (a, b)

   $$\therefore f(x_o) = \frac{1}{2}[(LHL\ at\ x = x_o) + (RHL\ at\ x = x_o)]$$

   $$= \frac{1}{2}\left[\lim_{x \to x_o^-} f(x) + \lim_{x \to x_o^+} f(x)\right]$$

Example 1 If $f(x+2\pi) = f(x)$, find the Fourier series expansion of

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

Hence or otherwise prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

Solution: $f(x)$ is integrable and piecewise continuous in the interval $(-\pi, \pi)$.

$\therefore f(x)$ can be expanded into Fourier series given by:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx \;\ldots\ldots ①$$

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx \quad = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\, \mathrm{d}x + \int_0^{\pi} xdx\right] = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \;\cos nx\, \mathrm{d}x + \int_0^{\pi} x\cos nx\, dx\right]$$

$$= \frac{-\pi}{\pi}\left[\frac{sinnx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x)\left(\frac{sinnx}{n}\right) - (1)\left(\frac{-cosnx}{n^2}\right)\right]_0^{\pi}$$

$$= 0 + \frac{1}{\pi}\left[\frac{x\, sinnx}{n} + \frac{cosnx}{n^2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad \because sinnx = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1] = \begin{cases} \frac{-2}{\pi n^2} & , n\ is\ odd \\ 0 & , n\ is\ even \end{cases}$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\ \sin nx\,\mathrm{d}x + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{\pi}{\pi}\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \left[\frac{1}{n} - \frac{(-1)^n}{n}\right] - \frac{1}{\pi}\left[\frac{\pi(-1)^n}{n}\right] \qquad \because \frac{\sin nx}{n^2} = 0 \text{ when } x = 0 \text{ or } x = \pi$$

$$= \frac{1}{n}[1 - 2(-1)^n] = \begin{cases} \dfrac{3}{n} & , n \text{ is odd} \\ -\dfrac{1}{n} & , n \text{ is even} \end{cases}$$

Substituting values of a_0, a_n, b_n in ①

$$f(x) \approx -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \left[\frac{3\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right]$$

Putting $x = 0$ on both sides

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right] + 0 \ldots ②$$

Since $f(0)$ is not defined in the interval $(-\pi, \pi)$

$$\therefore f(0) = \frac{1}{2}[(LHL\ at\ x = 0) + (RHL\ at\ x = 0)]$$

$$= \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right]$$

$$= \frac{1}{2}\left[\lim_{h\to 0} f(0-h) + \lim_{h\to 0} f(0+h)\right]$$

$$= \frac{1}{2}[-\pi + 0] = -\frac{\pi}{2} \ \ldots ③$$

Using ③in ②, we get

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

Fourier Series for Arbitrary Period Length

Let $f(x)$ be a periodic function defined in the interval $[C, C+2l]$, then

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \;\ldots\ldots①$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$$

Note: If the interval length is 2π, putting $2l = 2\pi$ i.e. $= \pi$, then ① may be rewritten as $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cosnx + \sum_{n=1}^{\infty} b_n sinnx$, which is Fourier series expansion in the interval $[C, C+2\pi]$.

Also $a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)dx$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

Example 1 : If $f(x+10)=f(x)$, find the Fourier series expansion of the function

$$f(x)=\begin{cases}0, & -5\le x\le 0\\ 3, & 0\le x\le 5\end{cases}$$

Solution: Let $f(x)\approx\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

Here interval is $[-5,5]$, $\therefore 2l=10\ \Rightarrow l=5$

Putting $l=5$ in ①

$$f(x)\approx\frac{a_0}{2}+\sum a_n\cos\frac{n\pi x}{5}+\sum b_n\sin\frac{n\pi x}{5}\ \ldots\ldots ①$$

$$a_0=\frac{1}{l}\int_c^{c+2l}f(x)dx=\frac{1}{5}\int_{-5}^{5}f(x)dx=\frac{1}{5}\int_{-5}^{0}0dx+\frac{1}{5}\int_0^5 3dx=\frac{3}{5}[x]_0^5=3$$

$$a_n=\frac{1}{l}\int_c^{c+2l}f(x)\cos\frac{n\pi x}{l}dx$$

$$=\frac{1}{5}\int_{-5}^{5}f(x)\cos\frac{n\pi x}{5}dx$$

$$=\frac{1}{5}\int_{-5}^{0}0\cos\frac{n\pi x}{l}dx+\frac{1}{5}\int_0^5 3\cos\frac{n\pi x}{5}dx=0+\frac{3}{5}\left[\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right]_0^5=0$$

$$b_n=\frac{1}{l}\int_c^{c+2l}f(x)\sin\frac{n\pi x}{l}dx$$

$$=\frac{1}{5}\int_{-5}^{5}f(x)\sin\frac{n\pi x}{5}dx$$

$$= \frac{1}{5}\int_{-5}^{0} 0 \ \sin\frac{n\pi x}{l}dx + \frac{1}{5}\int_{0}^{5} 3\sin\frac{n\pi x}{5}dx$$

$$= 0 - \frac{3}{5}\left[\frac{5}{n\pi}\cos\frac{n\pi x}{5}\right]_0^5 = -\frac{3}{n\pi}[\cos n\pi - \cos 0]$$

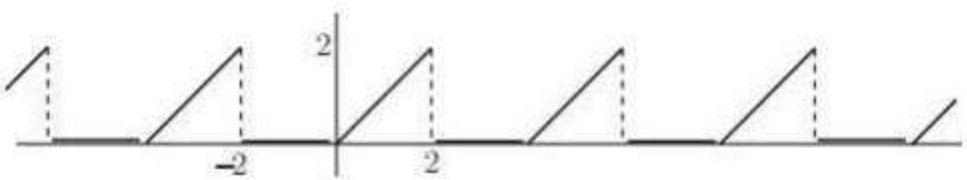
$$= -\frac{3}{n\pi}[(-1)^n - 1] = \begin{cases} \frac{6}{n\pi}, n \text{ is odd} \\ 0 \ , n \text{ is even} \end{cases}$$

Substituting values of a_0, a_n, b_n in ①

$$f(x) \approx \frac{3}{2} + \frac{6}{\pi}\left[\frac{\sin\frac{\pi x}{5}}{1} + \frac{\sin\frac{3\pi x}{5}}{3} + \frac{\sin\frac{5\pi x}{5}}{5} + \cdots\right]$$

$$\Rightarrow f(x) \approx \frac{3}{2} + \frac{6}{\pi}\left[\sin\frac{\pi x}{5} + \frac{1}{3}\sin\frac{3\pi x}{5} + \frac{1}{5}\sin \pi x + \cdots\right]$$

Example 2: Find the Fourier series expansion of the periodic function shown by the graph given below in the interval (-2,2)



Solution: From the graph $f(x) = \begin{cases} 0\,, & -2 < x < 0 \\ x\,, & 0 < x < 2 \end{cases}$

Clearly $f(x)$ is integrable and piecewise continuous in the interval $(-2,2)$

$\therefore f(x)$ can be expanded into Fourier series given by:

Let $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Here interval is $(-2,2)$, $\therefore 2l = 4 \Rightarrow l = 2$

Putting $l = 2$ in ①

$f(x) \approx \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{2} + \sum b_n \sin\frac{n\pi x}{2}$ ......①

$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} 0\,dx + \frac{1}{2}\int_0^2 x\,dx = \frac{1}{4}[x^2]_0^2 = 1$

$$a_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\,\cos\frac{n\pi x}{l}dx = \frac{1}{2}\int_{-2}^{2} f(x)\,\cos\frac{n\pi x}{5}dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0\,\cos\frac{n\pi x}{l}dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}dx$$

$$= 0 + \frac{1}{2}\left[(x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{2}{n^2\pi^2}\left[\cos\frac{n\pi x}{2}\right]_0^2 = \frac{2}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} \frac{-4}{n^2\pi^2}, & n\,is\,odd \\ 0\,, & n\,is\,even \end{cases}$$

$$b_n = \frac{1}{l}\int_{c}^{c+2l} f(x)\,\sin\frac{n\pi x}{l}dx = \frac{1}{2}\int_{-2}^{2} f(x)\,\sin\frac{n\pi x}{5}dx$$

$$= \frac{1}{2}\int_{-2}^{0} 0\,\sin\frac{n\pi x}{l}dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}dx$$

$$= 0 + \frac{1}{2}\left[(x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{-1}{n\pi}\left[x\cos\frac{n\pi x}{2}\right]_0^2 = \frac{-1}{n\pi}[2(-1)^n] = \frac{2(-1)^{n+1}}{n\pi} = \begin{cases} \frac{2}{n\pi}, & n \text{ is odd} \\ \frac{-2}{n\pi}, & n \text{ is even} \end{cases}$$

Substituting values of a_0, a_n, b_n in ①

$$f(x) \approx \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \frac{\cos\frac{5\pi x}{2}}{5^2} + \cdots\right] + \frac{2}{\pi}\left[\frac{\sin\frac{\pi x}{2}}{1} - \frac{\sin\frac{2\pi x}{2}}{2} + \frac{\sin\frac{3\pi x}{2}}{3} - \frac{\sin\frac{4\pi x}{2}}{4} + \cdots\right]$$

$$\Rightarrow f(x) \approx \frac{1}{2} - \frac{4}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{9}\cos\frac{3\pi x}{2} + \frac{1}{25}\cos\frac{5\pi x}{2} + \cdots\right] + \frac{2}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2}\ldots\right]$$

Fourier Series Expansion of Even Odd Functions

Computational procedure of Fourier series can be reduced to great extent, once a function is identified to be even or odd in an interval (-l,l)

Note: Properties of Even or Odd Function comply only if interval is $(-l, l)$ and any function in $(0, 2l)$ does not follow the properties of even/odd functions. For example for the function $f(x) = x^2$ in $(0, 2\pi)$, Fourier coefficients a_0 , a_n , b_n do not follow above given rules of even/odd functions.

Even Function

- A function f(x) is even if

1. Midpoint of interval is $x = 0$
2. $f(-x) = f(x)$
3. Graph of even function is symmetric about y-axis

- Examples

i. $f(x) = \cos x$
ii. $f(x) = x^2 \cos x$
iii. $f(x) = x \sin x$
iv. $f(x) = x^4 + x^2 + 5$

Odd function

- A function f(x) is said to be odd function if

1. Midpoint of interval is $x = 0$
2. $f(-x) = -f(x)$
3. Graph of even function is symmetric about opposite quadrant.

- Examples

i. $f(x) = \sin x$
ii. $f(x) = x^3 + 5x$
iii. $f(x) = x \cos x$
iv. 4) $f(x) = x^2 \sin x$

Product of two functions

- even $\times$ even $=$ even function
- even $\times$ odd $=$ odd function
- odd $\times$ even $=$ odd function
- odd $\times$ odd $=$ even function

Addition of two functions

- even $+$ even $=$ even function
- even $+$ odd $=$ cannot predict
- odd $+$ even $=$ cannot predict
- odd $+$ odd $=$ odd function

Integral

- If f(x) is even function then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$
- If f(x) is odd function then $\int_{-a}^{a} f(x)dx = 0.$

Fourier series expansion of even & odd function with arbitrary period

Fourier series expansion of Even Function

Let f(x) be an even function defined in the interval $-L \leq x \leq L$ .Then Fourier series expansion of f(x) is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

Where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

Fourier series expansion of even & odd function with arbitrary period

- **Fourier series expansion of odd Function**

- Let f(x) be an odd function defined in the interval $-L \leq x \leq L$ .Then Fourier series expansion of f(x) is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

Fourier series expansion of even function standard interval $-\pi \leq x \leq \pi$

- Let f(x) be an even function defined in the interval $-\pi \leq x \leq \pi$ .Then Fourier series expansion of f(x) is

- $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$

- where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx,$

- $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx.$

Fourier series expansion of odd function standard interval $-\pi \le x \le \pi$

- Let f(x) be an odd function defined in the interval $-\pi \le x \le \pi$ .Then Fourier series expansion of f(x) is
- $f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$
- $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Example 1 Obtain Fourier series expansion for the function $f(x) = x^3$ in the interval $(-\pi, \pi)$, if $f(x + 2\pi) = f(x)$

Solution: $f(x) = x^3$ is integrable and piecewise continuous in the interval $(-\pi, \pi)$ and also $f(x)$ is an odd function of x.

$\therefore a_0 = a_n = 0$, $f(x)$ can be expanded into Fourier series given by:

$f(x) \approx \sum_{n=1}^{\infty} b_n sinnx$ ......①

$$b_n = \frac{2}{\pi} \int_0^c f(x) \sin nx \, dx = \frac{2}{\pi} \int_0^{\pi} x^3 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[(x^3)\left(\frac{-\cos nx}{n}\right) - (3x^2)\left(\frac{-\sin nx}{n^2}\right) + (6x)\left(\frac{\cos nx}{n^3}\right) - (6)\left(\frac{\sin nx}{n^4}\right)\right]_o^{\pi}$$

$$= \frac{2}{\pi}\left[(x^3)\left(\frac{-\cos nx}{n}\right) + (6x)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} \quad \because sinnx = 0 \text{ when } x = \pi \text{ or } 0$$

$$= \frac{2}{\pi}\left[(\pi^3)\left(\frac{-\cos n\pi}{n}\right) + (6\pi)\left(\frac{\cos n\pi}{n^3}\right)\right]$$

$$= \frac{2}{\pi}\left[(\pi^3)\left(\frac{-(-1)^n}{n}\right) + (6\pi)\left(\frac{(-1)^n}{n^3}\right)\right]$$

$$= 2(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore f(x) \approx 2\left[-\left(\frac{-\pi^2}{1} + \frac{6}{1^3}\right)\right] sinx + \left(\frac{-\pi^2}{2} + \frac{6}{2^3}\right) sin2x - \left(\frac{-\pi^2}{3} + \frac{6}{3^3}\right) \sin 3x + \cdots$$

Example 2

If $f(x+2\pi) = f(x)$, obtain Fourier series expansion for the function given by $f(x) = |x|$ in the interval $(-\pi, \pi)$

Hence or otherwise prove that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Solution: $f(-x) = |-x| = |x| = f(x)$

$f(-x) = f(x) \therefore f(x)$ is even function of x.

Rewriting $f(x)$ as $|x| = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$

Being even function of , $b_n = 0$,

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \ldots\ldots (1)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}[\pi^2] = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx$$

$$= \frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}[\cos n\pi]_0^{\pi} = \frac{2}{\pi n^2}[(-1)^n - 1] = \begin{cases} \frac{-4}{\pi n^2}, n \text{ is odd} \\ 0, n \text{ is even} \end{cases}$$

Substituting values of a_0 and a_n in (1)

$$f(x) \approx \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2}\right]$$

Putting $x = 0$ on both sides

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Example 3

Expand the function $f(x) = x^2$ as Fourier series in $[-\pi, \pi]$.

Hence deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$

f(x) be an even function defined in the interval $-\pi \leq x \leq \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

Here, $f(x) = x^2$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{2}{\pi}\int_0^{\pi} x^2\, dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$\Rightarrow \boxed{a_0 = \frac{2\pi^2}{3}}$$

Again, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx \qquad \left[\because f(x) \text{ is even} \Rightarrow \int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x \cos nx}{n^2} - \frac{2x \sin nx}{n^3}\right]_0^{\pi} = \frac{4}{n^2}(-1)^n$$

$$\Rightarrow \boxed{a_n = \frac{4}{n^2}(-1)^n}$$

$$f(x) = x^2 = \frac{\left(\frac{2\pi^2}{3}\right)}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + 4\left(-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \dots\right)$$

$$f(x) = x^2 = \frac{\left(\frac{2\pi^2}{3}\right)}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + 4\left(-\cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \dots\right)$$

Deduction: Put $x = \pi$ in the above equation, we get

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\left(-\cos \pi + \frac{\cos 2\pi}{2^2} - \frac{\cos 3\pi}{3^2} + \frac{\cos 4\pi}{4^2} - \dots\right)$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right)$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right)$$

$$\Rightarrow \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

Hence the Result

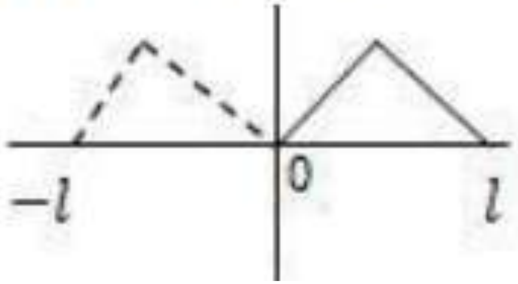
Half Range Fourier Series in the Interval (0,l)

If it is required to expand $f(x)$ in $(0, l)$, it is immaterial what the function may be outside the range $0 < x < l$, we are free to choose the function in $(-l, 0)$.

Half Range Cosine Series

To develop into Cosine series, we extend $f(x)$ in $(-l, 0)$ by reflecting it in y – axis as shown in adjoining figure, so that $f(-x) = f(x)$, function becomes even function and $b_n = 0$

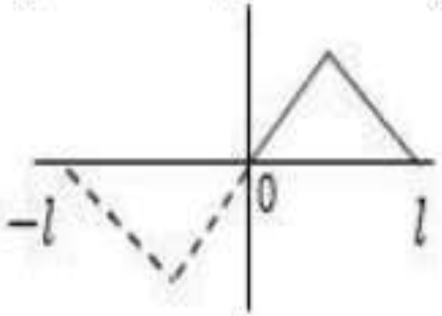
$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$



Half Range Sine Series

To develop into Sine series, we extend $f(x)$ in $(-l, 0)$, by reflecting it in origin, so that $f(-x) = -f(x)$, function becomes odd function and $a_0 = a_n = 0$

$$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$



Half rang expansion

Half rang expansion is used when period of function is 2L (or 2π) but function is defined only in half period $0 \le x \le L$ (or $0 \le x \le \pi$)

Half rang cosine expansion

Let f(x) be a periodic function of period 2L defined in $0 \le x \le L$ then Half rang cosine expansion of f(x) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx,$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

Half rang expansion

Half rang expansion is used when period of function is 2L (or 2π) but function is defined only in half period $0 \le x \le L$ (or $0 \le x \le \pi$)

Half rang sine expansion

Let f(x) be a periodic function of period 2L defined in $0 \le x \le L$ then Half rang sine expansion of f(x) is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Example 1

Expand $f(x) = x,\ 0 < x < 2$ in a half-range (a) Sine Series. (b) Cosine Series.

(a) Sine Series: (L=2)

$$
\begin{aligned}
b_n &= \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi}{\ell}t\,dt \\
&= \int_0^2 t\sin\frac{n\pi}{2}t\,dt \\
&= -\left.\frac{t\cos\frac{n\pi}{2}t}{\left(\frac{n\pi}{2}\right)}\right|_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi}{2}t\,dt \\
&= -\frac{4}{n\pi}\cos(n\pi) + \left(\frac{2}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}t\right)\Big|_0^2 \\
&= -\frac{4}{n\pi}(-1)^n
\end{aligned}
$$

Therefore

$$f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}t\right).$$

$$f(1) = 1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi}{2}\right)$$

therefore $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

(b) *Cosine Series: (L=2)*

$$a_0 = \frac{2}{2}\int_0^2 t\,dt = \frac{t^2}{2}\Big|_0^2 = 2$$

$$a_n = \int_0^2 t\cos\frac{n\pi}{2}t\,dt = \left(\frac{2}{n\pi}\right)t\sin\frac{n\pi}{2}t\Big|_0^2 - \left(\frac{2}{n\pi}\right)\int_0^2 \sin\frac{n\pi}{2}t\,dt$$

$$= +\left(\frac{2}{n\pi}\right)^2\cos\frac{n\pi}{2}t\Big|_0^2 = \frac{4}{n^2\pi^2}\{\cos n\pi - 1\}$$

Therefore

$$f(t) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos\frac{n\pi}{2}t$$

$$= 1 - \frac{8}{\pi^2}\sum_{n=0}^{\infty}\cos\frac{(2n+1)}{2}\pi t/(2n+1)^2.$$

The cosine series converges faster than Sine Series.

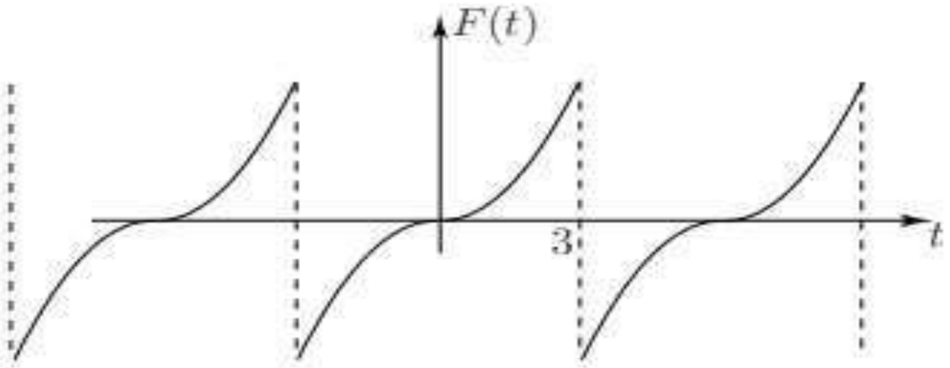
$$f(2) = 2 = 1 + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}, \qquad \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Example 2

Obtain a half range Fourier Sine Series to represent the function

$$f(t) = t^2 \qquad 0 < t < 3.$$

We first extend $f(t)$ as an odd periodic function $F(t)$ of **period 6**: $f(t) = -t^2, \quad -3 < t < 0$



We now evaluate the Fourier Series of F(t) by standard techniques but take advantage of the symmetry and put an = 0, n = 0, 1, 2,....

$$b_n = \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} F(t)\sin\left(\frac{2n\pi t}{P}\right)\,\mathrm{d}t,$$

we put P = 6

since the integrand is even (a product of 2 odd functions), we can write

$$
\begin{aligned}
b_n &= \frac{2}{3}\int_0^3 F(t)\sin\left(\frac{2n\pi t}{6}\right)dt \\
&= \frac{2}{3}\int_0^3 t^2 \sin\left(\frac{n\pi t}{3}\right)dt.
\end{aligned}
$$

(Note that we always carry out integration over the originally defined range of the function, in this case 0 <t< 3.) We now have to integrate by parts (twice!)

$$
\begin{aligned}
b_n &= \frac{2}{3}\left\{\left[-\frac{3t^2}{n\pi}\cos\left(\frac{n\pi t}{3}\right)\right]_0^3 + 2\left(\frac{3}{n\pi}\right)\int_0^3 t\cos\left(\frac{n\pi t}{3}\right)dt\right\} \\
&= \frac{2}{3}\left\{-\frac{27}{n\pi}\cos n\pi + \frac{6}{n\pi}\left[\frac{3}{n\pi}t\sin\frac{n\pi t}{3}\right]_0^3 - \left(\frac{6}{n\pi}\right)\left(\frac{3}{n\pi}\right)\int_0^3 \sin\left(\frac{n\pi t}{3}\right)dt\right\} \\
&= \frac{2}{3}\left\{-\frac{27}{n\pi}\cos n\pi - \frac{18}{n^2\pi^2}\left[-\frac{3}{n\pi}\cos\left(\frac{n\pi t}{3}\right)\right]_0^3\right\} \\
&= \frac{2}{3}\left\{-\frac{27}{n\pi}\cos n\pi + \frac{54}{n^3\pi^3}(\cos n\pi - 1)\right\}
\end{aligned}
$$

$$
\text{i.e.} \qquad b_n = \begin{cases} -\frac{18}{n\pi} & n = 2, 4, 6, \ldots \\ \frac{18}{n\pi} - \frac{72}{n^3\pi^3} & n = 1, 3, 5, \ldots \end{cases}
$$

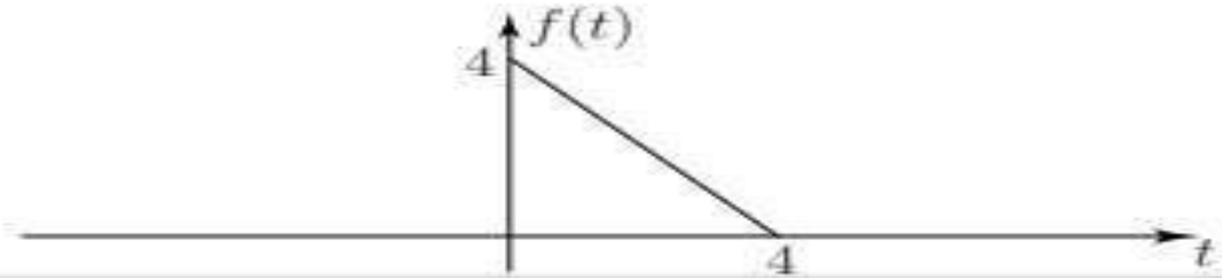
So the required Fourier Sine Series is

$$
F(t) = 18\left(\frac{1}{\pi} - \frac{4}{\pi^3}\right)\sin\left(\frac{\pi t}{3}\right) - \frac{18}{2\pi}\sin\left(\frac{2\pi t}{3}\right) + 18\left(\frac{1}{3\pi} - \frac{4}{27\pi^3}\right)\sin(\pi t) - \ldots
$$

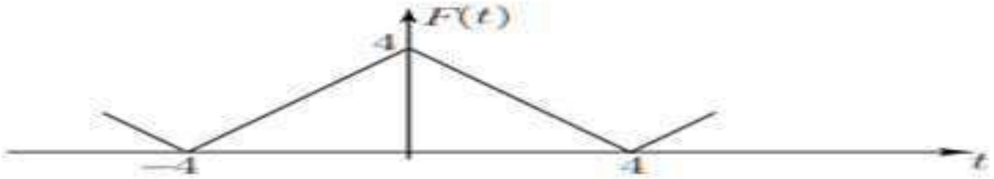
Example 3

Obtain a half-range Fourier Cosine Series to represent the function

$$f(t) = 4 - t \qquad 0 < t < 4.$$



First complete the definition to obtain an even periodic function F(t) of period 8. Sketch F(t).



We have with $P = 8$

$$a_n = \frac{2}{8}\int_{-4}^{4} F(t)\cos\left(\frac{2n\pi t}{8}\right) dt$$

Utilising the fact that the integrand here is even we get

$$a_n = \frac{1}{2}\int_{0}^{4} (4-t)\cos\left(\frac{n\pi t}{4}\right) dt$$

Using integration by parts we obtain for $n = 1, 2, 3, \ldots$

$$\begin{aligned} a_n &= \frac{1}{2}\left\{\left[(4-t)\frac{4}{n\pi}\sin\left(\frac{n\pi t}{4}\right)\right]_0^4 + \frac{4}{n\pi}\int_0^4 \sin\left(\frac{n\pi t}{4}\right) dt\right\} \\ &= \frac{1}{2}\left(\frac{4}{n\pi}\right)\left(\frac{4}{n\pi}\right)\left[-\cos\left(\frac{n\pi t}{4}\right)\right]_0^4 \\ &= \frac{8}{n^2\pi^2}\left[-\cos(n\pi) + 1\right] \end{aligned}$$

$$\text{i.e.} \qquad a_n = \begin{cases} 0 & n = 2, 4, 6, \ldots \\ \frac{16}{n^2\pi^2} & n = 1, 3, 5, \ldots \end{cases}$$

Also $a_0 = \frac{1}{2}\int_0^4 (4-t)dt = 4$. So the constant term is $\frac{a_0}{2} = 2$.

Now write down the required Fourier Series

$$2 + \frac{16}{\pi^2}\left\{\cos\left(\frac{\pi t}{4}\right) + \frac{1}{9}\cos\left(\frac{3\pi t}{4}\right) + \frac{1}{25}\cos\left(\frac{5\pi t}{4}\right) + \ldots\right\}$$

Example 4 : Obtain half range Fourier Cosine series for

$f(x) = 2x - 1$ in the interval $(0,1)$.

Solution: To develop $f(x) = 2x - 1$ into Cosine series, extending $f(x)$ in $(-1, 0)$ by reflecting it in y – axis, so that $f(-x) = f(x)$, function becomes even function and $b_n = 0$

$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \ldots\ldots (1)$$

Here $2l = 2 \quad \therefore \ l = 1$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = 2\int_0^1 (2x-1)dx = 0$$

$$a_n = 2\int_0^1 (2x-1)\cos n\pi x \, dx$$

$$= 2\left[(2x-1)\left(\frac{\sin n\pi x}{n\pi}\right) - (2)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}[\cos n\pi - \cos 0]$$

$$= \frac{4}{n^2\pi^2}[(-1)^n - 1] = \begin{cases} \frac{-8}{n^2\pi^2} & , n \text{ is odd} \\ 0 & , n \text{ is even} \end{cases}$$

Substituting values of a_0, a_n in (1)

$$f(x) \approx -\frac{8}{\pi^2}\left[\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right]$$

Assignment

Obtain the half-range Fourier series specified for each of the following functions:

1. $f(t) = 1 \quad 0 \le t \le \pi$ (sine series)
2. $f(t) = t \quad 0 \le t \le 1$ (sine series)
3. (a) $f(t) = e^{2t} \quad 0 \le t \le 1$ (cosine series)

   (b) $f(t) = e^{2t} \quad 0 \le t \le \pi$ (sine series)
4. (a) $f(t) = \sin t \quad 0 \le t \le \pi$ (cosine series)

   (b) $f(t) = \sin t \quad 0 \le t \le \pi$ (sine series)

Answers

1. $\frac{4}{\pi}\left\{\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right\}$
2. $\frac{2}{\pi}\{\sin \pi t - \frac{1}{2}\sin 2\pi t + \frac{1}{3}\sin 3\pi t - \cdots\}$
3. (a) $\frac{e^2-1}{2} + \sum_{n=1}^{\infty}\frac{4}{4+n^2\pi^2}[e^2\cos(n\pi) - 1]\cos n\pi t$

   (b) $\sum_{n=1}^{\infty}\frac{2n\pi}{4+n^2\pi^2}[1 - e^2\cos(n\pi)]\sin n\pi t$
4. (a) $\frac{2}{\pi} + \sum_{n=2}^{\infty}\frac{1}{\pi}\left[\frac{1}{1-n}(1-\cos(1-n)\pi) + \frac{1}{1+n}(1-\cos(1+n)\pi)\right]\cos nt$

   (b) $\sin t$ itself (!)

Practical Harmonic Analysis

In many engineering and scientific problems, $f(x)$ is not given directly, rather set of discrete values of function are given in the form $(x_i, y_i), i = 1, 2, 3, \ldots, m$ where x_i's are equispaced. The process of obtaining $f(x)$ in terms of Fourier series from given set of values (x_i, y_i), is known as practical harmonic analysis.

In a given interval $(0, 2l)$, $f(x)$ is represented in terms of harmonics as shown below:

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$$

Where $n = 1, 2, 3$ give 1^{st}, 2^{nd} and 3^{rd} harmonics respectively.

$\therefore \left(a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l} \right)$ is the first harmonic

$\left(a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l} \right)$ is the second harmonic

$\left(a_3 \cos\frac{\pi x}{l} + b_3 \sin\frac{\pi x}{l} \right)$ is the third harmonic

$\vdots$

$\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l} \right)$ is the n^{th} harmonic

Fourier coefficient a_0 is computed using the relation

2 [Mean value of y in the interval $(0,2l)$]

$\therefore a_0 = \frac{2}{m}\sum_{i=1}^{m} y_i$, where m denotes number of observations

Similarly a_n and b_n can be found out using the relations

$$a_n = 2\left[\text{Mean value of } y\cos\frac{n\pi x}{l} \text{ in the interval } (0,2l)\right] = \frac{2}{m}\sum_{i=1}^{m} y_i\cos\frac{n\pi x_i}{l}$$

$$b_n = 2\left[\text{Mean value of } y\sin\frac{n\pi x}{l} \text{ in the interval } (0,2l)\right] = \frac{2}{m}\sum_{i=1}^{m} y_i\sin\frac{n\pi x_i}{l}$$

Also when interval length is 2π, putting $2l = 2\pi$ i.e. $l = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$a_0 = \frac{2}{m}\sum_{i=1}^{m} y_i\,, \quad a_n = \frac{2}{m}\sum_{i=1}^{m} y_i\cos nx_i\,, \quad b_n = \frac{2}{m}\sum_{i=1}^{m} y_i\sin nx_i$$

- The amplitude of first harmonic is given by $\sqrt{a_1^2 + b_1^2}$ and similarly amplitudes of second and third harmonics are given by $\sqrt{a_2^2 + b_2^2}$ and $\sqrt{a_3^2 + b_3^2}$ respectively.
- For $f(x)$ in discrete form, values of Fourier coefficients a_0, a_n and b_n have been computed using trapezoidal rule for definite integration.

Harmonic Analysis

We know that $\int_{x_0}^{x_1} ydx = h\sum y$

1. **Fourier series**

Suppose $y = f(x)$ be a periodic function of period period $2L$ defined in $0 \le x \le 2L$ then Fourier series expansion of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + bn \ \sin\left(\frac{n\pi x}{L}\right)$$

where $\boxed{a_0 = \frac{2}{m}\sum y_i}$, $\boxed{a_n = \frac{2}{m}\sum y_i \cos\left(\frac{n\pi x}{L}\right)}$, $\boxed{b_n = \frac{2}{m}\sum y_i \sin\left(\frac{n\pi x}{L}\right)}$

Where m is number of divisions of interval $[0,2L]$

Half rang Fourier cosine expansion

Suppose $y = f(x)$ be a periodic function of period $2L$ defined in $0 \leq x \leq 2L$ then half range Fourier cosine expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where

$$a_0 = \frac{2}{m}\sum y_i$$ &

$$a_n = \frac{2}{m}\sum y_i \cos\left(\frac{n\pi x}{L}\right) ,$$

Half rang Fourier sine expansion

Suppose $y = f(x)$ be a periodic function of period $2L$ defined in $0 \leq x \leq 2L$ then half range Fourier sine expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \, sin\left(\frac{n\pi x}{L}\right)$$

Where

$$b_n = \frac{2}{m} \sum y_i \sin\left(\frac{n\pi x}{L}\right) ,$$

Q1:The following values of ‘y ’ give the displacement of a machine part for the rotation x of a flywheel. Express ‘y ’ in Fourier series up to third harmonic.

x	0^0	60^0	120^0	180^0	240^0	300^0	360^0
y	1.98	2.15	2.77	-0.22	-0.31	1.43	1.98

Solution: Here number of observations (m) are 6, period length is 2π $\because [y]_{0^0} \equiv [y]_{360^0}$

$$\text{Let } f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\therefore y \approx \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) \ldots ①$$

$$a_0 = \frac{2}{m}\sum_{i=1}^{m} y_i\,,\ a_n = \frac{2}{m}\sum_{i=1}^{m} y_i \cos nx_i\,,\ b_n = \frac{2}{m}\sum_{i=1}^{m} y_i \sin nx_i$$

$$a_0 = \frac{2}{m}\sum_{i=1}^{m} y_i\,,\quad a_n = \frac{2}{m}\sum_{i=1}^{m} y_i \cos nx_i\,,\quad b_n = \frac{2}{m}\sum_{i=1}^{m} y_i \sin nx_i$$

x_i	y_i	$\cos x_i$	$\sin x_i$	$\cos 2x_i$	$\sin 2x_i$	$\cos 3x_i$	$\sin 3x_i$
0^0	19.8	1.0	0	1.0	0	1.0	0
60^0	2.15	0.5	0.866	-0.5	0.866	-1.0	0
120^0	2.77	-0.5	0.866	-0.5	-0.866	1.0	0
180^0	-0.22	-1	0	1.0	0	-1.0	0
240^0	-0.31	-0.5	-0.866	-0.5	0.866	1.0	0
300^0	1.43	0.5	-0.866	-0.5	-0.866	-1.0	0

$$a_0 = \frac{2}{6}\sum_{i=1}^{6} y_i = \frac{2}{6}[1.98 + 2.15 + 2.77 - 0.22 - 0.31 + 1.4] = 2.6$$

$$a_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos x_i = \frac{2}{6}[(1.98)(1) + (2.15)(0.5) + \cdots + (1.43)(0.5)] = 0.92$$

$$b_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin x_i = \frac{2}{6}[(1.98)(0) + (2.15)(0.866) + \cdots + (1.43)(-0.866)] = 1.097$$

$$a_2 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos 2x_i = \frac{2}{6}[(1.98)(1) + (2.15)(-0.5) + \cdots + (1.43)(-0.5)] = -0.42$$

$$b_2 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin 2x_i = \frac{2}{6}[(1.98)(0) + (2.15)(0.866) + \cdots + (1.43)(-0.866)] = -0.681$$

$$a_3 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos 3x_i = \frac{2}{6}[(1.98)(1) + (2.15)(-1) + \cdots + (1.43)(-1)] = 0.36$$

$$b_3 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin 3x_i = \frac{2}{6}[(1.98)(0) + (2.15)(0) + \cdots + (1.43)(0)] = 0$$

Substituting values of a_0, a_n, b_n in ① where $n = 1,2,3$

$$\boxed{y \approx 1.3 + (0.92\cos x + 1.097\sin x) - (0.42\cos 2x + 0.681\sin 2x) + 0.36\cos 3x + \cdots}$$

Q2)Experimental values of y corresponding to x are tabulated below:

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{8\pi}{6}$	$\frac{9\pi}{6}$	$\frac{10\pi}{6}$	$\frac{11\pi}{6}$	2π
y	298	356	373	337	254	155	80	51	60	93	147	221	298

Express y in Fourier series up to second harmonic.

Solution: Here number of observations (m) are 12, period length is $2\pi \because [y]_0 \equiv [y]_{2\pi}$

$$\text{Let } y \approx \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \cdots \text{①}$$

$$a_0 = \frac{2}{m}\sum_{i=1}^{m} y_i\ ,\ \ a_n = \frac{2}{m}\sum_{i=1}^{m} y_i \cos nx_i\ ,\ \ b_n = \frac{2}{m}\sum_{i=1}^{m} y_i \sin nx_i$$

x_i	y_i	$\cos x_i$	$\sin x_i$	$\cos 2x_i$	$\sin 2x_i$
0	298	1	0	1	0
$\frac{\pi}{6}$	356	0.866	0.5	0.5	0.866
$\frac{2\pi}{6}$	373	0.5	0.866	-0.5	0.866
$\frac{3\pi}{6}$	337	0	1	-1	0
$\frac{4\pi}{6}$	254	-0.5	0.866	-0.5	-0.866
$\frac{5\pi}{6}$	155	-0.866	0.5	0.5	-0.866
π	80	-1	0	1	1
$\frac{7\pi}{6}$	51	-0.866	-0.5	0.5	0.866
$\frac{8\pi}{6}$	60	-0.5	-0.866	-0.5	0.866
$\frac{9\pi}{6}$	93	0	-1	-1	0
$\frac{10\pi}{6}$	147	0.5	-0.866	-0.5	-0.866
$\frac{11\pi}{6}$	221	0.866	-0.5	0.5	-0.866

$$a_0 = \frac{2}{12}\sum_{i=1}^{12} y_i = \frac{1}{6}[298 + 356 + \cdots + 221] = 404.17$$

$$a_1 = \frac{2}{12}\sum_{i=1}^{12} y_i \cos x_i = \frac{1}{6}[(298)(1) + (356)(0.866) + \cdots + (221)(0.866)] = 107.048$$

$$b_1 = \frac{2}{12}\sum_{i=1}^{12} y_i \sin x_i = \frac{1}{6}[(298)(0) + (356)(0.5) + \cdots + (221)(-0.5)] = 121.203$$

$$a_2 = \frac{2}{12}\sum_{i=1}^{12} y_i \cos 2x_i = \frac{1}{6}[(298)(1) + (356)(0.5) + \cdots + (221)(0.5)] = -13$$

$$b_2 = \frac{2}{12}\sum_{i=1}^{12} y_i \sin 2x_i = \frac{1}{6}[(298)(0) + (356)(0.866) + \cdots + (221)(-0.866)] = 9.093$$

Substituting values of a_0, a_1, b_1, a_2, b_2 in ①

$$y \approx 202.09 + (107.048\cos x + 121.203\sin x) + (-13\cos 2x + 9.093\sin 2x) + \cdots$$

Q3)The following table connects values of x and y for a statistical input:

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Express y in Fourier series up to first harmonic. Also find amplitude of the first harmonic.

Solution: Here $m = 6$, Also putting $2l = 6 \Rightarrow l = 3 \because [y]_{x=0} \equiv [y]_{x=6}$, if y is periodic

$$\therefore y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{3} + b_n \sin\frac{n\pi x}{3}\right)$$

$$\Rightarrow y \approx \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{3} + b_1 \sin\frac{\pi x}{3}\right) + \cdots \qquad \ldots\ldots①$$

$$a_0 = \frac{2}{6}\sum_{i=1}^{6} y_i\ ,\ a_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos\frac{\pi x_i}{3}\ ,\ b_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin\frac{\pi x_i}{3}$$

x_i	y_i	$\cos\frac{\pi x_i}{3}$	$\sin\frac{\pi x_i}{3}$
0	9	1	0
1	18	0.5	0.866
2	24	-0.5	0.866
3	28	-1	0
4	26	-0.5	-0.866
5	20	0.5	-0.866

$$a_0 = \frac{2}{6}\sum_{i=1}^{6} y_i = \frac{1}{3}[9 + 18 + 24 + 28 + 26 + 20] = 41.67$$

$$a_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \cos\frac{\pi x_i}{3} = \frac{1}{3}[(9)(1) + (18)(0.5) + \cdots + (20)(0.5)] = -8.33$$

$$b_1 = \frac{2}{6}\sum_{i=1}^{6} y_i \sin\frac{\pi x_i}{3} = \frac{1}{3}[(9)(0) + (18)(0.866) + \cdots + (20)(-0.866)] = -1.15$$

Substituting values of a_0, a_1, b_1 in ①

$$\Rightarrow y \approx 20.835 - \left(8.33\cos\frac{\pi x}{3} + 1.15\sin\frac{\pi x}{3}\right) + \cdots$$

The amplitude of first harmonic is given by $\sqrt{(-8.33)^2 + (-1.15)^2} = 8.41$

Q4)The following table gives the variation of a periodic current over a period ' T'

Time(t) Sec	0	T/6	T/3	T/2	2T/3	5T/6	T
Current(A) Amp	1.98	1.30	1.05	1.3	-0.88	-0.25	1.98

Show that there is a direct current part of 0.75 amp in the variable current. Also obtain the amplitude of the first harmonic.

Solution: Let $A \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{l}$

Here $m = 6$, Also $2l = T \Rightarrow l = \frac{T}{2}$

$\therefore A \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}$

$\Rightarrow A \approx \frac{a_0}{2} + a_1 \cos\frac{2\pi t}{T} + b_1 \sin\frac{2\pi t}{T}$ for the first harmonic…①

$a_0 = \frac{2}{m}\sum A,\ a_1 = \frac{2}{m}\sum A\cos\frac{2\pi t}{T},\ b_1 = \frac{2}{m}\sum A\sin\frac{2\pi t}{T}$

Time(t) sec	Current(A) amp	$\cos\frac{2\pi t}{T}$	$\sin\frac{2\pi t}{T}$
0	1.98	1	0
T/6	1.3	0.5	0.866
T/3	1.05	-0.5	0.866
T/2	1.3	-1	0
2T/3	-0.88	-0.5	-0.866
5T/6	-0.25	0.5	-0.866

$$a_0 = \frac{2}{6}\sum A = \frac{1}{3}[1.98 + 1.3 + 1.05 + 1.3 - 0.88 - 0.25] = 1.5$$

$$a_1 = \frac{2}{6}\sum A\cos\frac{2\pi t}{T} = \frac{1}{3}[(1.98)(1) + (1.3)(0.5) + \cdots + (-0.25)(0.5)] = 0.373$$

$$b_1 = \frac{2}{6}\sum A\sin\frac{2\pi t}{T} = \frac{1}{3}[(1.98)(0) + (1.3)(0.866) + \cdots + (-0.25)(-0.866)] = 1.005$$

Substituting values of a_0 , a_1 , b_1 in ①

$$\therefore A \approx 0.75 + 0.373\cos\frac{2\pi t}{T} + 1.005\sin\frac{2\pi t}{T}$$

Here $\frac{a_0}{2}$ represents the direct current part and the amplitude of the first harmonic is given by $\sqrt{a_1^2 + b_1^2}$

$\therefore$ A has a direct current part of 0.75 amp

The amplitude of first harmonic is given by $\sqrt{(0.373)^2 + (1.005)^2} = \sqrt{1.1491} = 1.072$

Harmonic Analysis for Half Range Series

If it is required to express $f(x)$ given in discrete form $(x_i, y_i), i = 1, 2, 3, \dots, m$, taken in the interval $(0, l)$ into half range sine or cosine series, we extend $f(x)$ in $(-l, 0)$ to make it odd or even respectively.

Sine Series

To develop $f(x)$ into sine series, extend $f(x)$ in the interval $(-l, 0)$ by reflecting in origin, so that $f(-x) = -f(x)$, function becomes odd function and $a_0 = a_n = 0$

$$\therefore f(x) \approx \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = 2\left[\text{Mean value of } y \sin\frac{n\pi x}{l} \text{ in the interval } (0, l)\right] = \frac{2}{m}\sum_{i=1}^{m} y_i \sin\frac{n\pi x_i}{l}$$

Note: To express $f(x)$ into sine series, y_1 must be zero, otherwise it cannot be reflected in origin.

Harmonic Analysis for Half Range Series

Cosine Series

To develop $f(x)$ into cosine series, extend $f(x)$ in the interval $(-l, 0)$ by reflecting in y-axis, so that $f(-x) = f(x)$, function becomes even function and $b_n = 0$

$$\therefore f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$a_0 = 2$ [Mean value of y in the interval $(0, l)$]

$$= \frac{2}{m}\left[\frac{y_1 + y_m}{2} + y_2 + y_3 + \cdots + y_{m-1}\right] \quad \ldots\ldots \text{ using trapezoidal rule}$$

$$a_n = 2\left[\text{Mean value of } y \cos\frac{n\pi x}{l} \text{ in the interval } (0, l)\right]$$

$$= \frac{2}{m}\left[\frac{y_1 \cos\frac{n\pi x}{l} + y_m \cos\frac{n\pi x}{l}}{2} + y_2 \cos\frac{n\pi x}{l} + y_3 \cos\frac{n\pi x}{l} + \cdots + y_{m-1} \cos\frac{n\pi x}{l}\right]$$

$$n = 1,2,3 \ldots$$

Example 1:The turning moment ' M' units of a crank shaft of a steam engine are given for a series of values of the crank angle 'θ ' in degrees. Obtain first three terms of sine series to represent . Also verify the value from obtained function at $\theta=60^0$

θ	0^0	30^0	60^0	90^0	120^0	150^0
M	0	5224	8097	7850	5499	2656

Solution: Assuming M periodic, to represent into sine series (half range series), extending M in the interval $(-180^0, 0)$ by reflecting in origin, so that $M(-\theta) = -M(\theta)$, function becomes odd function and $a_0 = a_n = 0$

$$\therefore M \approx \sum_{n=1}^{\infty} b_n \sin \frac{n\pi\theta}{l}$$

Here $m = 6$, Also $2l = 2\pi \Rightarrow l = \pi$

$$\Rightarrow M \approx \sum_{n=1}^{\infty} b_n \sin n\theta$$

$$\Rightarrow \mathrm{M} \approx b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \cdots$$

$$b_n = \frac{2}{6} \sum M \sin n\theta \ , n = 1,2,3 \ldots$$

θ	M	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$
0^0	0	0	0	0
30^0	5224	0.5	0.866	1
60^0	8097	0.866	0.866	0
90^0	7850	1	0	-1
120^0	5499	0.866	-0.866	0
150^0	2656	0.5	-0.866	1

θ	M	$\sin\theta$	$\sin 2\theta$	$\sin 3\theta$
0^0	0	0	0	0
30^0	5224	0.5	0.866	1
60^0	8097	0.866	0.866	0
90^0	7850	1	0	-1
120^0	5499	0.866	-0.866	0
150^0	2656	0.5	-0.866	1

$$b_1 = \frac{2}{6}\sum M\sin\theta = \frac{1}{3}[(0)(0) + (5224)(0.5) + \cdots + (2656)(0.5)] = 7850$$

$$b_2 = \frac{2}{6}\sum M\sin 2\theta = \frac{1}{3}[(0)(0) + (5224)(0.866) + \cdots + (2656)(-0.866)] = 1500$$

$$b_3 = \frac{2}{6}\sum M\sin 3\theta = \frac{1}{3}[(0)(0) + (5224)(1) + \cdots + (2656)(1)] = 0$$

$$\therefore \text{M} \approx 7850\sin\theta + 1500\sin 2\theta + 0 + \cdots$$

When $\theta = 60^0$, $\text{M} \approx 7850\sin 60^0 + 1500\sin 120^0 + 0 + \cdots$

$$\approx 7850(0.866) + 1500(0.866) + \cdots$$

$$\approx 8097.1$$

Example 2 : Obtain half range Fourier cosine series for the data given below:

x	0	1	2	3	4	5
y	4	8	11	15	12	7

Also check value of y at $x = 2$ from the obtained cosine series.

Solution: Assuming y periodic, to represent it into half range cosine series, extending y in the interval $(-6,0)$ by reflecting it in y-axis, so that $y(-x) = y(x)$, function becomes even function and $b_n = 0$

$$\therefore y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Here $m = 6$, Also $2l = 12 \Rightarrow l = 6$

$$\Rightarrow y \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{6}$$

$$\Rightarrow y \approx \frac{a_0}{2} + a_1 \cos\frac{\pi x}{6} + a_2 \cos\frac{2\pi x}{6} + a_3 \cos\frac{3\pi x}{6} + \cdots$$

$$a_0 = \frac{2}{m}\left[\frac{y_1 + y_m}{2} + y_2 + y_3 + \cdots + y_{m-1}\right]$$

$$a_n = \frac{2}{m}\left[\frac{y_1 \cos\frac{n\pi x}{l} + y_m \cos\frac{n\pi x}{l}}{2} + y_2 \cos\frac{n\pi x}{l} + y_3 \cos\frac{n\pi x}{l} + \cdots + y_{m-1} \cos\frac{n\pi x}{l}\right]$$

$n = 1,2,3 \ldots$

x_i	y_i	$\cos\frac{\pi x}{6}$	$\cos\frac{2\pi x}{6}$	$\cos\frac{3\pi x}{6}$	$y\cos\frac{\pi x}{6}$	$y\cos\frac{2\pi x}{6}$	$y\cos\frac{3\pi x}{6}$
0	4	1	1	1	4	4	4
1	8	0.866	0.5	0	0.6928	4	0
2	11	0.5	-0.5	-1	5.5	-5.5	-11
3	15	0	-1	0	0	-15	0
4	12	-0.5	-0.5	1	-6	-6	12
5	7	-0.866	0.5	0	-6.062	3.5	0

$$a_0 = \frac{2}{6}\left[\frac{4+7}{2} + 8 + 11 + 15 + 12\right] = \frac{1}{3}[51.5] = 17.2$$

$$a_1 = \frac{2}{6}\left[\frac{4-6.062}{2} + 0.6928 + 5.5 + 0 - 6\right] = \frac{1}{3}[-0.8382] = -0.2794$$

$$a_2 = \frac{2}{6}\left[\frac{4+3.5}{2} + 4 - 5.5 - 15 - 6\right] = \frac{1}{3}[-18.75] = -6.25$$

$$a_3 = \frac{2}{6}\left[\frac{4+0}{2} + 0 - 11 + 0 + 12\right] = \frac{1}{3}[3] = 1$$

$$\therefore y \approx 8.6 - 0.2794\cos\frac{\pi x}{6} - 6.25\cos\frac{2\pi x}{6} + \cos\frac{3\pi x}{6} + \cdots$$

When $x = 2$, $y \approx 8.6 - 0.2794\cos\frac{2\pi}{6} - 6.25\cos\frac{4\pi}{6} + \cos\frac{6\pi}{6} + \cdots$

$$\approx 8.6 - 0.2794(0.5) - 6.25(-0.5) - 1 + \cdots$$

$$\approx 10.5853$$

Parseval's identity

For a full Fourier Series on $[-L, L]$ Parseval's Theorem assumes the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\frac{1}{L}\int_{-L}^{L} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$ $0 < x < L$. *Then* $\boxed{\frac{2}{L}\int_{0}^{L} [f(x)]^2\, dx = \sum_{n=1}^{\infty} b_n^2.}$

Parseval's identity

For Fourier Sine Components:

$$\frac{2}{L}\int_{0}^{L}(f(x))^2\,dx=\sum_{n=1}^{\infty}b_n^2.$$

$$Let\; f(x)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)\; 0<x<L.\;\; Then\; \boxed{\frac{2}{L}\int_{0}^{L}[f(x)]^2\,dx=\sum_{n=1}^{\infty}b_n^2.}$$

Example 1: Consider $f(x) = x^2 \quad -\pi < x < \pi.$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

Solution 1:

n	1	2	3	4
$\cos\left(\frac{n\pi}{2}\right)$	0	-1	0	1

Let

$$x = \frac{\pi}{2} \quad \Rightarrow \quad \frac{\pi^2}{4} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi}{2}\right)$$

$$-\frac{\pi^2}{12} = 4\sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)^2}$$

Therefore

$$\frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}.$$

By Parseval's Formula:

$$\frac{2}{\pi}\int_0^{\pi} x^4\,dx = 2\left(\frac{\pi^2}{3}\right)^2 + 16\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{2}{\pi}\left.\frac{x^5}{5}\right|_0^{\pi} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{9-5}{45} = \frac{4}{45} = \frac{8}{90}$$

$$\frac{1}{90}$$

Therefore

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4} = \zeta(4).$$