

JOIN



Telegram
@PuneEngineers

For more Subjects

<https://www.studymedia.in/fe/notes>



SCAN ME



CLICK HERE
@PuneEngineers



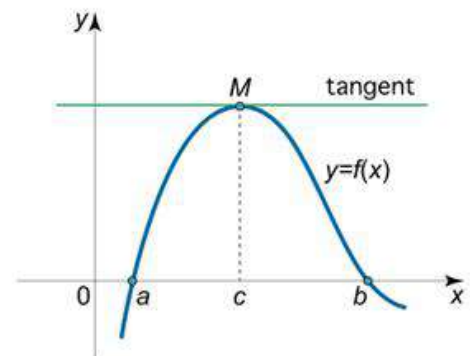
Unit I : Differential Calculus

Rolle's Theorem:

Let $f(x)$ be a function defined in $[a, b]$

- i) function $f(x)$ is continuous on the closed interval $[a, b]$
- ii) differentiable on the open interval (a, b)
- iii) $f(a) = f(b)$

then there exists at least one point $x = c$ in the open interval (a, b) such that $f'(c) = 0$.



Geometric interpretation

There is a point c on the interval (a, b) where the tangent to the graph of the function is horizontal.

Lagrange's Mean Value Theorem:

Let $f(x)$ be a function defined in $[a, b]$

- i) Function $f(x)$ is continuous on a closed interval $[a, b]$
- ii) Function $f(x)$ differentiable on the open interval (a, b)

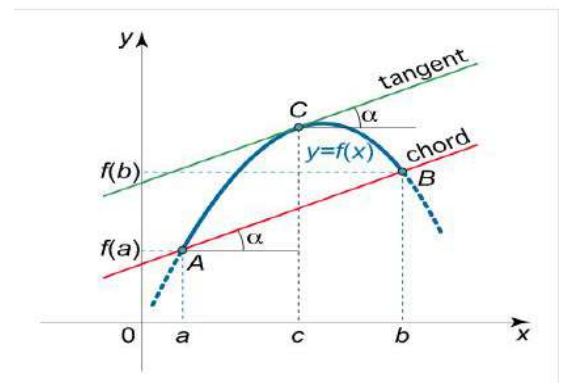
then there is at least one point $x = c$ on this interval (a, b) , such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometric Interpretation

The chord passing through the points of the graph corresponding to the ends of the segment a and b has the slope equal to $k =$

$$\tan \alpha = \frac{f(b) - f(a)}{b - a}$$



Then there is a point $x = c$ inside the interval $[a, b]$ where the tangent to the graph is parallel to the chord.



Cauchy's Mean Value Theorem:

Let $f(x)$ be a function defined in $[a, b]$

- i) Function $f(x)$ and $g(x)$ is continuous on a closed interval $[a, b]$
- ii) Function $f(x)$ and $g(x)$ differentiable on the open interval (a, b)
- iii) $g'(x) \neq 0$ for all value of x in (a, b)

then there is at least one point $x = c$ on this interval (a, b) , such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Example 1: Verify Rolle's Mean Value theorem for $f(x) = x^2 - 5x + 4$ in $[1, 4]$

Solution : $f(x) = x^2 - 5x + 4$

As $f(x) = x^2 - 5x + 4$ is a polynomial

Every polynomial is continuous and differentiable everywhere

$\therefore f(x) = x^2 - 5x + 4$ is continuous in $[1, 4]$ and differentiable in $(1, 4)$

$$f(x) = x^2 - 5x + 4$$

$$f(a) = f(1) = 1^2 - 5(1) + 4 = 1 - 5 + 4 = 0$$

$$f(b) = f(4) = 4^2 - 5(4) + 4 = 16 - 20 + 4 = 0$$

$$f(a) = f(b)$$

All condition of Rolle's Mean Value theorem satisfied.

then there exist at least one $c \in (1, 4)$ such that $f'(c) = 0$

$$f(x) = x^2 - 5x + 4$$

$$f'(x) = 2x - 5$$

$$\text{put } x = c$$

$$f'(c) = 2c - 5$$

$$\therefore f'(c) = 0 \Rightarrow 2c - 5 = 0 \quad \therefore 2c = 5 \quad \therefore c = \frac{5}{2} = 2.5$$

$\therefore c = 2.5 \in (1, 4)$ hence Lagrange's Mean Value theorem verified.



Example 2: Verify Rolle's Mean Value theorem for

$$f(x) = e^x(\sin x - \cos x) \text{ in } \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$$

Solution : $f(x) = e^x(\sin x - \cos x)$ As $f(x)$ is a combination of exponential and sine, cosine functions

Exponential and Sine, Cosine function are continuous and differentiable

 $\therefore f(x) = e^x(\sin x - \cos x)$ is continuous in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and differentiable in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$$f(x) = e^x(\sin x - \cos x) =$$

$$f(a) = f\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\sin \frac{\pi}{4} - \cos \frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

$$f(b) = f\left(\frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(\sin \frac{5\pi}{4} - \cos \frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)\right) = 0$$

$$f(a) = f(b)$$

All condition of Rolle's Mean Value theorem satisfied.

then there exist at least one $c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that $f'(c) = 0$

$$f(x) = e^x(\sin x - \cos x)$$

$$f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x)$$

$$f'(x) = 2e^x \sin x$$

put $x = c$

$$f'(c) = 2e^c \sin c$$

$$\therefore f'(c) = 0 \Rightarrow 2e^c \sin c = 0$$

$$\therefore \sin c = 0 \quad \therefore c = \sin^{-1} 0 \quad \therefore c = n\pi \quad n = 0, 1, 2, 3 \dots$$

$$c = 0, \pi, 2\pi, 3\pi, 4\pi \dots$$

 $\therefore c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ hence Rolle's Mean Value theorem verified.

Example 3: Verify Lagrange's Mean Value theorem for

$$f(x) = (x - 1)(x - 2)(x - 3) \text{ in } [0, 3]$$

Solution : $f(x) = (x - 1)(x - 2)(x - 3)$

$$f(x) = x^3 - 6x^2 + 11x - 6$$

As $f(x) = (x - 1)(x - 2)(x - 3)$ is a polynomial

Every polynomial is continuous and differentiable everywhere

$\therefore f(x) = (x - 1)(x - 2)(x - 3)$ is continuous in $[0, 3]$ and differentiable in $(0, 3)$

All condition of Lagrange's Mean Value theorem satisfied.

then there exist at least one $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f(x) = x^3 - 6x^2 + 11x - 6$$

$$f(a) = f(0) = (0)^3 - 6(0)^2 + 11(0) - 6 = -6$$

$$f(b) = f(3) = (3)^3 - 6(3)^2 + 11(3) - 6 = 27 - 54 + 33 - 6 = 0$$

$$f'(x) = 3x^2 - 12x + 11$$

put $x = c$

$$f'(c) = 3c^2 - 12c + 11$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\therefore 3c^2 - 12c + 11 = \frac{0 - (-6)}{3 - 0} = \frac{6}{3}$$

$$\therefore 3c^2 - 12c + 11 = 2$$

$$\therefore 3c^2 - 12c + 11 - 2 = 0$$

$$\therefore 3c^2 - 12c + 9 = 0$$

$$\therefore c^2 - 4c + 3 = 0$$

$$\therefore c^2 - 3c - c + 3 = 0$$

$$\therefore c(c - 3) - 1(c - 3) = 0$$

$$\therefore (c - 3)(c - 1) = 0 \quad \therefore c = 3, 1$$

$\therefore c = 1 \in (0, 3)$ hence Lagrange's Mean Value theorem verified.



Example 4: Verify Lagrange's Mean Value theorem for $f(x) = \log x$ in $[1, e]$

Solution : $f(x) = \log x$

As $f(x) = \log x$ is a logarithmic function

Every logarithmic function is continuous and differentiable in its domain

$\therefore f(x) = \log x$ is continuous in $[1, e]$ and differentiable in $(1, e)$

All condition of Lagrange's Mean Value theorem satisfied.

then there exist at least one $c \in (1, e)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$f(x) = \log x$$

$$f(a) = f(1) = \log 1 = 0$$

$$f(b) = f(e) = \log e = 1$$

$$f'(x) = \frac{1}{x} \quad \text{put } x = c \quad f'(c) = \frac{1}{c}$$

$$\frac{1}{c} = \frac{1-0}{e-1} \quad \therefore c = e - 1$$

$\therefore c = e - 1 \in (1, e)$ hence Lagrange's Mean Value theorem verified.

Example 5: Verify Cauchy's Mean value theorem for

$$f(x) = x^3 \text{ and } g(x) = x^4 \text{ in } [0, 2]$$

Solution : $f(x) = x^3$ and $g(x) = x^4$

As $f(x) = x^3$ and $g(x) = x^4$ are polynomials

Every polynomial is continuous and differentiable everywhere

$\therefore f(x) = x^3$ and $g(x) = x^4$ is continuous in $[0, 2]$ and differentiable in $(0, 2)$

$$\text{As } g(x) = x^4 \quad g'(x) = 4x^3 \neq 0 \text{ for } x \text{ in } (0, 2)$$

All condition of Cauchy's Mean Value theorem satisfied.



then there exist at least point $c \in (0, 2)$ such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\text{As } f(x) = x^3 \text{ and } g(x) = x^4$$

$$f(b) = f(2) = 2^3 = 8 \quad \text{and} \quad g(b) = g(2) = 2^4 = 16$$

$$f(a) = f(0) = 0 \quad \text{and} \quad g(a) = g(0) = 0$$

$$f'(x) = 3x^2 \quad \text{and} \quad g'(x) = 4x^3$$

$$\text{Put } x = c$$

$$f'(c) = 3c^2 \quad \text{and} \quad g'(c) = 4c^3$$

$$\frac{3c^2}{4c^3} = \frac{8 - 0}{16 - 0}$$

$$\frac{3}{4c} = \frac{1}{2} \quad \therefore 4c = 6 \quad \therefore c = \frac{6}{4} \quad \therefore c = \frac{3}{2}$$

$\therefore c = \frac{3}{2} \in (0, 2)$ hence Cauchy's Mean value theorem verified.



Expansion of Function

Taylor's Theorem:

Statement : Let $f(a + h)$ be a function of h which can be expanded in powers of h and let the expansion be differentiable term by term any number of times w.r.t. h

$$\text{then } f(a + h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots$$

Expansion of $f(x + h)$ in power of ' h '

$$f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots + \frac{h^n}{n!} f^n(x) + \dots$$

Expansion of $f(x + h)$ in power of ' x '

$$f(x + h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \dots + \frac{x^n}{n!} f^n(h) + \dots$$

Expansion $f(x)$ in power of $(x - a) = 0$ or about $x = a$

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \frac{(x - a)^3}{3!} f'''(a) + \dots + \frac{(x - a)^n}{n!} f^n(a) + \dots$$



Example 1: Using Taylor's theorem express

$$(x - 2)^4 - 3(x - 2)^3 + 4(x - 2)^2 + 5 \text{ in powers of } x$$

Solution: Let $f(x + h) = (x - 2)^4 - 3(x - 2)^3 + 4(x - 2)^2 + 5$

$$\text{Here } x + h = x - 2 \quad \therefore h = -2$$

By using Taylor's theorem, **Expansion of $f(x + h)$ in power of x**

$$f(x + h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \frac{x^4}{4!} f^{iv}(h) + \dots \dots \dots$$

$$f(x - 2) = f(-2) + x f'(-2) + \frac{x^2}{2!} f''(-2) + \frac{x^3}{3!} f'''(-2) + \frac{x^4}{4!} f^{iv}(-2) + \dots \dots \quad (A)$$

$$\therefore f(x) = x^4 - 3x^3 + 4x^2 + 5$$

$$f(x) = x^4 - 3x^3 + 4x^2 + 5 \quad \therefore f(h) = f(-2) = 61$$

$$f'(x) = 4x^3 - 9x^2 + 8x \quad \therefore f'(h) = f'(-2) = -84$$

$$f''(x) = 12x^2 - 18x + 8 \quad \therefore f''(-2) = f''(-2) = 92$$

$$f'''(x) = 24x - 18 \quad \therefore f'''(-2) = f'''(-2) = -66$$

$$f^{iv}(x) = 24 \quad \therefore f^{iv}(-2) = f^{iv}(-2) = 24$$

$$f^v(x) = 0 \quad \therefore f^v(-2) = f^v(-2) = 0$$

Equation (A) becomes

$$f(x - 2) = 61 + x(-84) + \frac{x^2}{2} (92) + \frac{x^3}{6} (-66) + \frac{x^4}{24} (24)$$

$$f(x - 2) = 61 - 84x + 46x^2 - 11x^3 + x^4$$



Example 2: Using Taylor's theorem express

$$(x + 2)^4 + 3(x + 2)^3 + (x + 2) + 7 \quad \text{in powers of } x$$

Solution: Let $f(x + h) = (x + 2)^4 + 3(x + 2)^3 + (x + 2) + 7$

$$\text{Here } x + h = x + 2 \quad \therefore h = 2$$

By using Taylor's theorem, **Expansion of $f(x + h)$ in power of x**

$$f(x + h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \frac{x^4}{4!} f^{iv}(h) + \dots \dots \dots$$

$$f(x - 2) = f(2) + x f'(2) + \frac{x^2}{2!} f''(2) + \frac{x^3}{3!} f'''(2) + \frac{x^4}{4!} f^{iv}(2) + \dots \dots \quad (A)$$

$$\therefore f(x) = x^4 + 3x^3 + x + 7$$

$$f(x) = x^4 + 3x^3 + x + 7 \quad \therefore f(2) = 49$$

$$f'(x) = 4x^3 + 9x^2 + 1 \quad \therefore f'(2) = 69$$

$$f''(x) = 12x^2 + 18x \quad \therefore f''(2) = 84$$

$$f'''(x) = 24x + 18 \quad \therefore f'''(2) = 66$$

$$f^{iv}(x) = 24 \quad \therefore f^{iv}(2) = 24$$

$$f^v(x) = 0 \quad \therefore f^v(2) = 0$$

Equation (A) becomes

$$f(x + 2) = 49 + x(69) + \frac{x^2}{2} (84) + \frac{x^3}{6} (66) + \frac{x^4}{24} (24) + 0$$

$$f(x + 2) = 49 + 69x + 42x^2 + 11x^3 + x^4$$



Example 3: Using Taylor's theorem express

$$49 + 69x + 42x^2 + 11x^3 + x^4 \text{ in powers of } (x + 2)$$

Solution: Let $f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$

$$\text{Here } x - a = x + 2 \quad \therefore -a = 2 \quad \therefore a = -2$$

By using Taylor's theorem, **Expansion f(x) in power of (x - a) or about x = a**

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \frac{(x - a)^3}{3!} f'''(a) + \dots + \frac{(x - a)^n}{n!} f^n(a) \dots$$

$$f(x) = f(-2) + (x + 2)f'(-2) + \frac{(x + 2)^2}{2} f''(-2) + \frac{(x + 2)^3}{6} f'''(-2) + \frac{(x + 2)^4}{24} f^{iv}(-2) \dots (A)$$

$$\therefore f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4$$

$$f(x) = 49 + 69x + 42x^2 + 11x^3 + x^4 \quad \therefore f(-2) = 7$$

$$f'(x) = 69 + 84x + 33x^2 + 4x^3 \quad \therefore f'(-2) = 1$$

$$f''(x) = 84 + 66x + 12x^2 \quad \therefore f''(-2) = 0$$

$$f'''(x) = 66 + 24x \quad \therefore f'''(-2) = 18$$

$$f^{iv}(x) = 24 \quad \therefore f^{iv}(-2) = 24$$

$$f^v(x) = 0 \quad \therefore f^v(-2) = 0$$

Equation (A) becomes

$$f(x) = 7 + (x + 2)1 + \frac{(x + 2)^2}{2} (0) + \frac{(x + 2)^3}{6} 18 + \frac{(x + 2)^4}{24} 24$$

$$f(x) = 7 + (x + 2) + 3(x + 2)^3 + (x + 2)^4$$



Example 4: Using Taylor's theorem express $x^3 + 7x^2 + x - 6$ in powers of $(x - 3)$

Solution:

$$\text{Let } f(x) = x^3 + 7x^2 + x - 6$$

$$\text{Here } x - a = x - 3 \quad \therefore a = 3$$

By using Taylor's theorem, **Expansion f(x) in power of (x - a) or about x = a**

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \frac{(x - a)^3}{3!} f'''(a) + \dots + \frac{(x - a)^n}{n!} f^n(a) \dots$$

$$f(x) = f(3) + (x - 3)f'(3) + \frac{(x - 3)^2}{2!} f''(3) + \frac{(x - 3)^3}{3!} f'''(3) + \frac{(x - 3)^4}{4!} f^{iv}(3) + \dots \quad (A)$$

$$\therefore f(x) = x^3 + 7x^2 + x - 6$$

$$f(x) = x^3 + 7x^2 + x - 6 \quad \therefore f(3) = 87$$

$$f'(x) = 3x^2 + 14x + 1 \quad \therefore f'(3) = 70$$

$$f''(x) = 6x + 14 \quad \therefore f''(3) = 32$$

$$f'''(x) = 6 \quad \therefore f'''(3) = 6$$

$$f^{iv}(x) = 0 \quad \therefore f^{iv}(3) = 0$$

Equation (A) becomes

$$f(x) = 87 + (x - 3)70 + \frac{(x - 3)^2}{2} 32 + \frac{(x - 3)^3}{6} 6$$

$$f(x) = 87 + 70(x - 3) + 16(x - 3)^2 + (x - 3)^3$$

Example 5: Using Taylor's theorem express $(x - 1)^4 - 3(x - 1)^3 + 4(x - 1)^2 + 5$ in powers of x.

Example 6: Using Taylor's theorem express $2(x - 2)^3 + 19(x - 2)^2 + 53(x - 2) + 40$ in powers of x.

Example 7: Using Taylor's theorem express $3x^3 - 2x^2 + x - 6$ in powers of $x - 2$

Example 8: Using Taylor's theorem express $1 + 2x + 3x^2 + 4x^3$ in powers of $x + 1$



Maclaurin's Theorem:

Statement: Let $f(x)$ be a function of x which can be expanded in ascending powers and let the expansion be differentiable term by term any number of times then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

Note:

1) If $y = f(x)$ then $f(0) = (y)_0$, $f'(0) = (y_1)_0$, $f''(0) = (y_2)_0$... $f^n(0) = (y_n)_0$

Maclaurin's Theorem stated as

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!} (y_2)_0 + \frac{x^3}{3!} (y_3)_0 + \dots + \frac{x^n}{n!} (y_n)_0 + \dots$$

2) The $(n + 1)^{th}$ term of expansion $\frac{x^n}{n!} f^n(0)$ is called general term.

$$3) \cosh x = \frac{e^x + e^{-x}}{2} \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$4) \frac{d}{dx}(\cosh x) = \sinh x$$

$$5) \frac{d}{dx}(\sinh x) = \cosh x$$

$$6) \int \sinh x \, dx = \cosh x + c$$

$$7) \int \cosh x \, dx = \sinh x + c$$



Example : Expansion of e^x **Solution :** Let $f(x) = e^x$

by Maclaurin's theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = e^x \quad \therefore f(0) = e^0 = 1$$

$$f'(x) = e^x \quad \therefore f'(0) = e^0 = 1$$

$$f''(x) = e^x \quad \therefore f''(0) = e^0 = 1$$

$$f'''(x) = e^x \quad \therefore f'''(0) = e^0 = 1$$

$$f^{iv}(x) = e^x \quad \therefore f^{iv}(0) = e^0 = 1$$

$$f^v(x) = e^x \quad \therefore f^v(0) = e^0 = 1$$

.....

$$f^n(x) = e^x \quad \therefore f^n(0) = e^0 = 1$$

by Maclaurin's Theorem:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = 1 + x(1) + \frac{x^2}{2!} 1 + \frac{x^3}{3!} 1 + \dots + \frac{x^n}{n!} 1 + \dots$$

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots + \frac{x^n}{n!} + \dots$$



Example : Expansion of e^{-x} **Solution :** Let $f(x) = e^{-x}$

by Maclaurin's Theorem:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = e^{-x} \quad \therefore f(0) = e^0 = 1$$

$$f'(x) = -e^{-x} \quad \therefore f'(0) = -e^0 = -1$$

$$f''(x) = e^{-x} \quad \therefore f''(0) = e^0 = 1$$

$$f'''(x) = -e^{-x} \quad \therefore f'''(0) = -e^0 = -1$$

$$f^{iv}(x) = e^{-x} \quad \therefore f^{iv}(0) = e^0 = 1$$

$$f^v(x) = -e^{-x} \quad \therefore f^v(0) = -e^0 = -1$$

.....

$$f^n(x) = (-1)^n e^{-x} \quad \therefore f^n(0) = (-1)^n e^0 = (-1)^n$$

by Maclaurin's Theorem:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = 1 + x(-1) + \frac{x^2}{2!} 1 + \frac{x^3}{3!} (-1) + \dots + \frac{x^n}{n!} (-1)^n + \dots$$

$$f(x) = e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^n}{n!} + \dots$$



Example : Expansion of $\sin x$ **Solution :** Let $f(x) = \sin x$

by Maclaurin's Theorem:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = \sin x \quad \therefore f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \quad \therefore f'(0) = \cos 0 = 1$$

$$f''(x) = -\sin x \quad \therefore f''(0) = -\sin 0 = 0$$

$$f'''(x) = -\cos x \quad \therefore f'''(0) = -\cos 0 = -1$$

$$f^{iv}(x) = \sin x \quad \therefore f^{iv}(0) = \sin 0 = 0$$

$$f^v(x) = \cos x \quad \therefore f^v(0) = \cos 0 = 1$$

by Maclaurin's Theorem:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) \dots$$

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$



Example : Expansion of $\sinh x$ **Solution :** Let $f(x) = \sinh x$

by Maclaurin's Theorem:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = \sinh x \quad \therefore f(0) = \sin 0 = 0$$

$$f'(x) = \cosh x \quad \therefore f'(0) = \cos 0 = 1$$

$$f''(x) = \sinh x \quad \therefore f''(0) = \sin 0 = 0$$

$$f'''(x) = \cosh x \quad \therefore f'''(0) = \cos 0 = 1$$

$$f^{iv}(x) = \sinh x \quad \therefore f^{iv}(0) = \sin x = 0$$

$$f^v(x) = \cosh x \quad \therefore f^v(0) = \cos 0 = 1$$

by Maclaurin's Theorem:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

$$f(x) = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) \dots$$

$$f(x) = \sin x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} - \dots$$

Standard Expansions:

$$1) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

$$2) e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \dots$$

$$3) \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$4) \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \dots$$

$$5) \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$6) \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$



Engineering Mathematics I

$$7) \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \dots$$

$$8) \quad \tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \dots$$

$$9) \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

$$10) \quad \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$$

$$11) \quad (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \dots$$

$$12) \quad \frac{1}{(1+x)} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$$

$$13) \quad \frac{1}{(1-x)} = (1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

$$14) \quad \sin^{-1} x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \frac{3}{4} \frac{x^5}{5} + \frac{1}{2} \frac{3}{4} \frac{5}{6} \frac{x^7}{7} + \dots$$

$$15) \quad \sinh^{-1} x = x - \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \frac{3}{4} \frac{x^5}{5} - \frac{1}{2} \frac{3}{4} \frac{5}{6} \frac{x^7}{7} + \dots$$

$$16) \quad \cos^{-1} x = \frac{\pi}{2} - \left[x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \frac{3}{4} \frac{x^5}{5} + \frac{1}{2} \frac{3}{4} \frac{5}{6} \frac{x^7}{7} + \dots \right]$$

$$17) \quad \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$18) \quad \tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$



Examples: Expand of $e^x \cos x$ in ascending powers of x upto a term in x^4

Solution: Let $f(x) = e^x \cos x$

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$f(x) = e^x \cos x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right)$$

$$e^x \cos x = 1 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right) + x \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right) + \frac{x^2}{2!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right) + \frac{x^3}{3!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right) + \frac{x^4}{4!} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\right)$$

$$e^x \cos x = 1 \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right) + x \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right) + \frac{x^2}{2} \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right) + \frac{x^3}{6} \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right) + \frac{x^4}{24} \left(1 - \frac{x^2}{2} + \frac{x^4}{24} \dots\right)$$

$$e^x \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + x - \frac{x^3}{2} + \frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$

OR

Examples: Expand of $e^x \cos x$ in ascending powers of x upto a term in x^4

Solution: Let $f(x) = e^x \cos x$

by Maclaurin's Theorem

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

$$f(x) = e^x \cos x \quad \therefore f(0) = e^0 \cos 0 = 1$$

$$f'(x) = e^x \cos x - e^x \sin x \quad \therefore f'(0) = e^0 \cos 0 - e^0 \sin 0 = 1$$

$$f''(x) = e^x \cos x - e^x \sin x - e^x \cos x - e^x \sin x = -2e^x \sin x \quad \therefore f''(0) = 0$$



$$f'''(x) = -2e^x \sin x - 2e^x \cos x \quad \therefore f'''(0) = -2$$

$$f''(x) = -2e^x \sin x - 2e^x \cos x + 2e^x \sin x - 2e^x \cos x \quad \therefore f''(0) = -4$$

by Maclaurin's Theorem:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(4)}(0) + \dots$$

$$f(x) = 1 + x(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(-2) + \frac{x^4}{24}(-4)$$

$$f(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$

Examples : Expand $\sqrt{1 + \sin x}$ in ascending powers of x upto a term in x^6

Solution : Let $f(x) = \sqrt{1 + \sin x}$

$$f(x) = \sqrt{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) + 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}$$

$$f(x) = \sqrt{\left(\sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)\right)^2} \quad \because (a+b)^2 = a^2 + 2ab + b^2$$

$$f(x) = \sqrt{1 + \sin x} = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)$$

We know that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Put $x = \frac{x}{2}$ in above expansion

$$\sin\left(\frac{x}{2}\right) = \frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 - \dots$$

$$\cos\left(\frac{x}{2}\right) = 1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 - \dots$$

$$f(x) = \sqrt{1 + \sin x} = \sin\left(\frac{x}{2}\right) + \cos\left(\frac{x}{2}\right)$$

$$f(x) = \frac{x}{2} - \frac{1}{3!}\left(\frac{x}{2}\right)^3 + \frac{1}{5!}\left(\frac{x}{2}\right)^5 - \dots + 1 - \frac{1}{2!}\left(\frac{x}{2}\right)^2 + \frac{1}{4!}\left(\frac{x}{2}\right)^4 - \frac{1}{6!}\left(\frac{x}{2}\right)^6 + \dots$$

$$f(x) = \frac{x}{2} - \frac{1}{6} * \frac{x^3}{8} + \frac{1}{120} * \frac{x^5}{32} - \dots + 1 - \frac{1}{2} * \frac{x^2}{4} + \frac{1}{24} * \frac{x^4}{16} - \frac{1}{720} * \frac{x^6}{64} + \dots$$

$$f(x) = 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384} + \frac{x^5}{3840} - \frac{x^6}{46080}$$



Type: Expansions of functions by using substitution:-

Ex.1 - Expand $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in ascending powers of x**Solution:**

$$\text{Let } f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$

$$\text{Put } x = \tan\theta \longrightarrow (\because \theta = \tan^{-1} x)$$

$$\therefore f(x) = \sin^{-1}\left(\frac{2 \tan \theta}{1 + \tan^2 \theta}\right)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$= 2\theta$$

$$= 2(\tan^{-1} x) \quad (\text{If } x = \tan\theta \text{ then } \theta = \tan^{-1} x)$$

$$\text{We know that } \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \dots \dots$$

$$\therefore \sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2 \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \dots \dots \right]$$

Ex.2- Prove that $\sec^{-1}\left[\frac{1}{1-2x^2}\right] = 2 \left[x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \frac{3}{4} \frac{x^5}{5} \dots \dots \right]$ **Solution:**

$$\text{Let } f(x) = \sec^{-1}\left[\frac{1}{1-2x^2}\right]$$

$$\text{Put } x = \sin\theta \longrightarrow (\because \theta = \sin^{-1} x)$$

$$\therefore f(x) = \sec^{-1}\left[\frac{1}{1-2\sin^2\theta}\right]$$

$$= \sec^{-1}\left[\frac{1}{\cos 2\theta}\right]$$

$$= \sec^{-1}[\sec 2\theta]$$

$$= 2\theta \quad (\text{If } x = \sin\theta \text{ then } \theta = \sin^{-1} x)$$

$$= 2 \sin^{-1} x$$

$$\sec^{-1}\left[\frac{1}{1-2x^2}\right] = 2 \left[x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \frac{3}{4} \frac{x^5}{5} \dots \dots \right] \quad \text{Hence Proved.}$$



Ex.3 – Expand $\cos^{-1}(4x^3 - 3x)$ in ascending powers of x .

Solution:

$$\text{Let } f(x) = \cos^{-1}(4x^3 - 3x)$$

$$\text{Put } x = \cos\theta$$

$$f(x) = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$$

$$= \cos^{-1}(\cos 3\theta)$$

$$= 3\theta$$

$$= 3 \cos^{-1} x \quad (\text{If } x = \cos\theta \text{ then } \theta = \cos^{-1} x)$$

$$= 3 \left[\frac{\pi}{2} - \left(x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \frac{3}{4} \frac{x^5}{5} + \frac{1}{2} \frac{3}{4} \frac{5}{6} \frac{x^7}{7} + \dots \right) \right]$$

$$= 3 \frac{\pi}{2} - 3 \left[x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \frac{3}{4} \frac{x^5}{5} + \frac{1}{2} \frac{3}{4} \frac{5}{6} \frac{x^7}{7} + \dots \right]$$

Ex.4- Prove that $\cot^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right) = \frac{\pi}{2} - 3 \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right)$

Ex.5- Prove that $\sin^{-1}(3x - 4x^3) = 3 \left(x + \frac{x^3}{6} + \frac{3x^5}{40} + \dots \right)$



Indeterminate Forms

Definition: Let $f(x)$ and $g(x)$ be any two function of x such that $f(a) = 0$ and $g(a) = 0$

then the ratio $\frac{f(x)}{g(x)}$ is said to be the indeterminate form $\frac{0}{0}$ at $x = a$

There are several Indeterminate Forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0 , ∞^0 , 1^∞

True Value (Limit):

The limiting value of an indeterminate form is called its true value.

Type I : Indeterminate Form $\frac{0}{0}$ (L'Hospital Rule)

Let $f(x)$ and $g(x)$ be any two function of x such that $f(a) = 0$ and $g(a) = 0$

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

Type II : Indeterminate Form $\frac{\infty}{\infty}$

If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ in $\frac{\infty}{\infty}$ form then reduces to $\frac{0}{0}$ by

$\frac{f(x)}{g(x)} = \frac{1/f(x)}{1/g(x)}$ and L'Hospital Rule is applicable.

L'Hospital Rule is applied to the $\frac{\infty}{\infty}$ form Thus $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

Type III : Indeterminate Form $0 \times \infty$

If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = \infty$ then $\lim_{x \rightarrow a} f(x) \cdot g(x)$ takes $0 \times \infty$

$f(x) \cdot g(x) = \frac{f(x)}{\frac{1}{g(x)}}$ or $\frac{g(x)}{\frac{1}{f(x)}}$ and the limit reduces to either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form

and L'Hospital Rule is applicable.

Type IV : Indeterminate Form $\infty - \infty$

If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$ then $\lim_{x \rightarrow a} [f(x) - g(x)]$ takes $\infty - \infty$

simplify the expression $f(x) - g(x)$ and the limit reduces to either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form

and L'Hospital Rule is applicable.



Note:

- 1) If $\lim_{x \rightarrow a} \frac{f_1(x)}{f_2(x)} - \frac{g_1(x)}{g_2(x)}$ in $\infty - \infty$ form then $\lim_{x \rightarrow a} \frac{f_1(x)g_2(x) - g_1(x)f_2(x)}{f_2(x)g_2(x)}$ is in $\frac{0}{0}$
- 2) If $f'(x), f''(x) \dots f^{n-1}(x)$ and $g'(x), g''(x) \dots g^{n-1}(x)$ all are zero,
But $f^n(x)$ and $g^n(x)$ are not both zero then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f^n(x)}{g^n(x)}$
- 3) Use of L'Hospital Rule : Differentiate numerator and denominator separately and then put $x = a$. If this reduces to indeterminate form then apply the rule again.
- 4) If logarithmic term is present in $0 * \infty$ form then keep logarithmic term in numerator

Type V : Indeterminate Form $0^0, \infty^0, 1^\infty$

- 1) If $\lim_{x \rightarrow a} f(x) = 0$ and $\lim_{x \rightarrow a} g(x) = 0$ then $\lim_{x \rightarrow a} \{f(x)\}^{g(x)}$ takes 0^0
- 2) If $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = 0$ then $\lim_{x \rightarrow a} \{f(x)\}^{g(x)}$ takes ∞^0
- 3) If $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \infty$ then $\lim_{x \rightarrow a} \{f(x)\}^{g(x)}$ takes 1^∞

If the true value of limit is denoted by L

$$\text{Then } L = \lim_{x \rightarrow a} \{f(x)\}^{g(x)}$$

Taking Log on both sides

$$\log L = \log \lim_{x \rightarrow a} \{f(x)\}^{g(x)}$$

$$\log L = \lim_{x \rightarrow a} g(x) \log f(x)$$

limit can be determined by $0 \times \infty$ form, true value is b

$$\log L = b$$

$$L = e^b$$

Note: $e^\infty = \infty$ $e^{-\infty} = 0$



Standard Limits:

$$1) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$3) \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$6) \lim_{x \rightarrow 0} (1 + x)^{1/x} = e$$

$$2) \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1$$

$$4) \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$$

$$7) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$3) \lim_{x \rightarrow 0} \frac{\sinh x}{x} = 1$$

$$5) \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$8) \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a$$

Example 1) Evaluate $\lim_{x \rightarrow 0} \frac{xe^x - \log(1 + x)}{x^2}$

Solutoin: Let $L = \lim_{x \rightarrow 0} \frac{xe^x - \log(1 + x)}{x^2}$ (1)

x → 0 put x = 0

$$L = \frac{(0)e^0 - \log(1 + 0)}{0^2} = \frac{0(1) - \log 1}{0} = \frac{0 - 0}{0} = \frac{0}{0} \quad \therefore \frac{0}{0} \text{ form}$$

$$\text{From (1) } L = \lim_{x \rightarrow 0} \frac{xe^x + e^x - \frac{1}{(1+x)}}{2} \dots\dots\dots (2)$$

x → 0 put x = 0

$$L = \frac{(0)e^0 + e^0 - \frac{1}{1+0}}{2(0)} = \frac{0(1) + 1 - \frac{1}{1-0}}{2(0)} = \frac{0}{0} \quad \therefore \frac{0}{0} \text{ form}$$

$$\text{From (2) } L = \lim_{x \rightarrow 0} \frac{xe^x + e^x + e^x - \frac{-1}{(1+x)^2}}{2}$$

$$L = \lim_{x \rightarrow 0} \frac{xe^x + e^x + e^x + \frac{1}{(1+x)^2}}{2} \dots\dots\dots (3)$$

x → 0 put x = 0

$$L = \frac{(0)e^0 + e^0 + e^0 + \frac{1}{(1+0)^2}}{2}$$

$$L = \frac{0 + 1 + 1 + 1}{2} = \frac{3}{2}$$



Example 2) Evaluate $\lim_{x \rightarrow \frac{1}{2}} \frac{\cos^2(\pi x)}{e^{2x} - 2xe}$

Solutoin: Let $L = \lim_{x \rightarrow \frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2xe} \dots \dots \dots (1)$

$$x \rightarrow \frac{1}{2} \text{ put } x = \frac{1}{2}$$

$$L = \frac{\cos^2 \pi(1/2)}{e^{2(1/2)} - 2(1/2)e} = \frac{0}{e - e} = \frac{0}{0}$$

From (1) $L = \lim_{x \rightarrow \frac{1}{2}} \frac{2 \cos \pi x (-\sin \pi x) \pi}{2e^{2x} - 2e} \quad \because \sin 2x = 2 \sin x \cos x$

$$L = \lim_{x \rightarrow \frac{1}{2}} \frac{-\pi \sin(2\pi x)}{2e^{2x} - 2e} \dots \dots \dots (2)$$

$$x \rightarrow \frac{1}{2} \text{ put } x = \frac{1}{2}$$

$$L = \frac{-\pi \sin(2\pi/2)}{2e^{2(1/2)} - 2e} = \frac{-\pi(0)}{2e - 2e} = \frac{0}{0}$$

$$L = \lim_{x \rightarrow \frac{1}{2}} \frac{-\pi \cos(2\pi x) (2\pi)}{4e^{2x} - 0} \dots \dots \dots (3)$$

$$x \rightarrow \frac{1}{2} \text{ put } x = \frac{1}{2}$$

$$L = \frac{-2\pi^2 \cos(2\pi/2)}{4e^{2(1/2)}}$$

$$L = \frac{-2\pi^2 (-1)}{4e}$$

$$L = \frac{2\pi^2}{4e} = \frac{\pi^2}{2e}$$



Example 3) Evaluate $\lim_{x \rightarrow 0} \frac{\log \sin 2x}{\log \sin x}$

Solutoin: Let $L = \lim_{x \rightarrow 0} \frac{\log \sin 2x}{\log \sin x}$ (1)

x → 0 put x = 0

$$L = \frac{\log \sin 2(0)}{\log \sin (0)} = \frac{\log 0}{\log 0} = \frac{\infty}{\infty} \text{ from}$$

$$\text{From (1) } L = \lim_{x \rightarrow 0} \frac{\frac{1}{\sin 2x} \cos 2x (2)}{\frac{1}{\sin x} \cos x}$$

$$\because \frac{d}{dx} \log x = \frac{1}{x}$$

$$L = \lim_{x \rightarrow 0} \frac{2 \cot 2x}{\cot x}$$

$$\because \frac{\cos x}{\sin x} = \cot x \quad \frac{1}{\cot x} = \tan x$$

$$L = \lim_{x \rightarrow 0} \frac{\frac{2}{\tan 2x}}{\frac{1}{\tan x}} = \lim_{x \rightarrow 0} \frac{2 \tan x}{\tan 2x} \dots \dots \dots (2)$$

x → 0 put x = 0

$$L = \frac{2 \tan 0}{\tan 20} = \frac{0}{0} \text{ from}$$

$$L = \lim_{x \rightarrow 0} \frac{2 \sec^2 x}{\sec^2 2x (2)} = \frac{2 \sec^2 0}{2 \sec^2 2(0)} = \frac{1}{1} = 1$$

OR

Example 3) Evaluate $\lim_{x \rightarrow 0} \frac{\log \sin 2x}{\log \sin x}$

Solutoin: Let $L = \lim_{x \rightarrow 0} \frac{\log \sin 2x}{\log \sin x}$ (1)

x → 0 put x = 0

$$L = \frac{\log \sin 2(0)}{\log \sin (0)} = \frac{\log 0}{\log 0} \quad \frac{\infty}{\infty} \text{ form}$$

$$L = \lim_{x \rightarrow 0} \frac{\frac{1}{\sin 2x} \cos 2x (2)}{\frac{1}{\sin x} \cos x}$$



$$L = \lim_{x \rightarrow 0} \frac{2 \cos 2x \sin x}{\sin 2x \cos x}$$

$$L = \lim_{x \rightarrow 0} \frac{2 \cos 2x \sin x}{2 \sin x \cos x \cos x}$$

$$\because \sin 2x = 2 \sin x \cos x$$

$$L = \lim_{x \rightarrow 0} \frac{\cos 2x}{\cos^2 x}$$

$$\because \cos 2x = \cos^2 x - \sin^2 x$$

$$L = \lim_{x \rightarrow 0} \frac{\cos^2 x - \sin^2 x}{\cos^2 x}$$

$$L = \lim_{x \rightarrow 0} \frac{\cos^2 x}{\cos^2 x} - \frac{\sin^2 x}{\cos^2 x}$$

$$L = \lim_{x \rightarrow 0} 1 - \tan^2 x$$

x → 0 put x = 0

$$L = 1 - \tan^2 0 \quad L = 1 - 0 = 1$$

Example 4) Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{\log(1+x) - x}$

Solutoin: Let $L = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{\log(1+x) - x} \dots \dots \dots (1)$

x → 0 put x = 0

$$L = \frac{e^0 - 1 - 0}{\log(1+0) - 0} = \frac{1 - 1}{\log(1)} \quad \frac{0}{0} \text{ form}$$

From (1) $L = \lim_{x \rightarrow 0} \frac{e^x - 0 - 1}{\frac{1}{1+x} - 1} \dots \dots \dots (2)$

$$L = \frac{e^0 - 0 - 1}{\frac{1}{1+0} - 1} \quad \frac{0}{0} \text{ form}$$

From (2) $L = \lim_{x \rightarrow 0} \frac{e^x}{\frac{-1}{(1+x)^2}} \dots \dots \dots (3)$

x → 0 put x = 0

$$L = \frac{e^0}{\frac{-1}{(1+0)^2}} = -1$$



Example 5) Find a and b if $\lim_{x \rightarrow 0} \frac{x(1 + a \cos x) - b \sin x}{x^3} = 1$

Solutoin: Let $L = \lim_{x \rightarrow 0} \frac{x(1 + a \cos x) - b \sin x}{x^3} \dots \dots \dots (1)$

x → 0 put x = 0

$$L = \frac{(0)(1 + a \cos 0) - b \sin 0}{0^3} = \frac{0}{0} \text{ form}$$

From (1) $L = \lim_{x \rightarrow 0} \frac{x(0 - a \sin x) + (1 + a \cos x) - b \cos x}{3x^2}$

$$L = \lim_{x \rightarrow 0} \frac{-a x \sin x + 1 + a \cos x - b \cos x}{3x^2} \dots \dots \dots (2)$$

x → 0 put x = 0

$$L = \frac{0(0 - a \sin 0) + (1 + a \cos 0) - b \cos 0}{3(0)^2} = \frac{(1 + a) - b}{0} = \infty$$

but limit is finite i.e. 1 $1 \neq \frac{(1 + a) - b}{0} = \infty$

∴ For Finite limit it should be $\frac{0}{0}$ from $\frac{(1 + a) - b}{0} = \frac{0}{0}$ from

$$\therefore (1 + a) - b = 0 \quad \therefore a - b = -1 \dots \dots \dots (A)$$

From (2) $L = \lim_{x \rightarrow 0} \frac{-a x \cos x - a \sin x + 0 - a \sin x + b \sin x}{6x} \dots \dots \dots (3)$

x → 0 put x = 0

$$L = \frac{-a(0) \cos(0) - a \sin(0) - a \sin(0) + b \sin(0)}{6(0)} = \frac{0}{0}$$

From (3) $L = \lim_{x \rightarrow 0} \frac{a x \sin x - a \cos x - a \cos x - a \cos x + b \cos x}{6} \dots \dots \dots (4)$

x → 0 put x = 0

$$L = \frac{a(0) \sin(0) - a \cos(0) - a \cos(0) - a \cos(0) + b \cos(0)}{6}$$

$$L = \frac{0 - a - a - a + b}{6}$$

$$L = \frac{-3a + b}{6} \text{ but limit value } L = 1 \text{ is given}$$



but limit is finite 1 $\therefore 1 = \frac{-3a + b}{6}$

$\therefore -3a + b = 6 \dots \dots \dots (B)$

$\therefore a - b = -1 \dots \dots \dots (A)$

Solving (A) and (B) $a - b = -1 \quad -3a + b = 6$

Add (A) and (B) $-2a = 5 \quad \therefore a = -\frac{5}{2}$

Put $a = -\frac{5}{2}$ in (A) $\therefore -\frac{5}{2} - b = 1 \quad \therefore b = -\frac{3}{2}$

$\therefore a = -\frac{5}{2} \quad b = -\frac{3}{2}$

Example 6) Find a and b if $\lim_{x \rightarrow 0} [x^{-3} \sin x + ax^{-2} + b] = 0$

Solutoin: Let $L = \lim_{x \rightarrow 0} [x^{-3} \sin x + ax^{-2} + b]$

$L = \lim_{x \rightarrow 0} \left[\frac{\sin x}{x^3} + \frac{a}{x^2} + b \right]$

$L = \lim_{x \rightarrow 0} \left[\frac{\sin x + ax + bx^3}{x^3} \right] \dots \dots \dots (1)$

x \rightarrow 0 put x = 0

$L = \frac{\sin(0) + a(0) + b(0)^3}{(0)^3} = \frac{0}{0} \text{ form}$

From (1) $L = \lim_{x \rightarrow 0} \left[\frac{\cos x + a + 3x^2b}{3x^2} \right] \dots \dots \dots (2)$

$L = \frac{\cos(0) + a + 3x(0)^2b}{3(0)^2} = \frac{1 + a}{0} = \infty$

but limit is finite $L = 0$

\therefore For Finite limit it should be $\frac{0}{0} \quad \therefore \frac{1 + a}{0} = \frac{0}{0}$

$\therefore 1 + a = 0 \quad \therefore a = -1$



$$\text{From (2)} \quad L = \lim_{x \rightarrow 0} \left[\frac{-\sin x + 6xb}{6x} \right] \dots \dots \dots (3)$$

x → 0 put x = 0

$$L = \frac{-\sin(0) + 6(0)b}{6(0)} \quad \frac{0}{0} \text{ form}$$

x → 0 put x = 0

$$\text{From (3)} \quad L = \lim_{x \rightarrow 0} \left[\frac{-\cos x + 6b}{6} \right] \dots \dots \dots (4)$$

$$L = \frac{-\cos(0) + 6b}{6}$$

$$L = \frac{-1 + 6b}{6}$$

$$\text{but limit is finite } 0 \quad \therefore \frac{-1 + 6b}{6} = 0$$

$$\therefore -1 + 6b = 0 \quad b = \frac{1}{6}$$

$$\therefore a = -1 \quad b = \frac{1}{6}$$

Example 7) Find a and b if $\lim_{x \rightarrow 0} \frac{a \cos x - a + bx^2}{x^4} = \frac{1}{12}$

Solutoin: Let $L = \lim_{x \rightarrow 0} \frac{a \cos x - a + bx^2}{x^4} \dots \dots \dots (1)$

x → 0 put x = 0

$$L = \frac{a \cos(0) - a + b(0)^2}{(0)^4} = \frac{a - a}{0} \quad \frac{0}{0} \text{ form}$$

$$\text{From (1)} \quad L = \lim_{x \rightarrow 0} \frac{-a \sin x + 2xb}{4x^3} \dots \dots \dots (2)$$

x → 0 put x = 0

$$L = \frac{-a \sin(0) + 2(0)b}{4(0)^3} \quad \frac{0}{0} \text{ form}$$

$$\text{From (2)} \quad L = \lim_{x \rightarrow 0} \left[\frac{-a \cos x + 2b}{12x^2} \right] \dots \dots \dots (3)$$



$x \rightarrow 0$ put $x = 0$

$$L = \frac{-a \cos(0) + 2b}{12(0)^2} = \frac{-a + 2b}{0} \neq \frac{1}{12}$$

but limit is finite $\frac{1}{12}$

\therefore For Finite limit it should be $\frac{0}{0}$

$$\therefore -a + 2b = 0 \dots\dots\dots (A)$$

From (3) $L = \lim_{x \rightarrow 0} \left[\frac{a \sin x}{24x} \right] \dots\dots\dots (3)$

$$L = \frac{a}{24} \lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right]$$

$$\frac{1}{12} = \frac{a}{24} (1)$$

$$a = 2$$

$$\therefore -a + 2b = 0 \quad \therefore -2 + 2b = 0 \quad b = 1$$

$$a = 2 \text{ and } b = 1$$

Example 8) Find a and b if $\lim_{x \rightarrow 0} \frac{a \sin^2 x + b \log \cos x}{x^4} = -\frac{1}{2}$

Example 9) If $\lim_{x \rightarrow 0} \frac{\sin 2x + p \sin x}{x^3}$ is finite then find the value of p

and hence find the value of limit



Example 10) Evaluate $\lim_{x \rightarrow 0} \sin x \log x$

Solution: Let $L = \lim_{x \rightarrow 0} \sin x \log x \dots \dots \dots (1)$

$x \rightarrow 0$ put $x = 0$

$$L = \sin 0 \log 0 \quad \mathbf{0 \times \infty \text{ form}}$$

$$L = \lim_{x \rightarrow 0} \frac{\log x}{\frac{1}{\sin x}} = \lim_{x \rightarrow 0} \frac{\log x}{\operatorname{cosec} x} \dots \dots \dots (2)$$

$x \rightarrow 0$ put $x = 0$

$$L = \frac{\log 0}{\operatorname{cosec} 0} \quad \mathbf{\frac{\infty}{\infty} \text{ form}}$$

From (2)
$$L = \lim_{x \rightarrow 0} \frac{1/x}{-\operatorname{cosec} x \cot x}$$

$$L = \lim_{x \rightarrow 0} \frac{\sin x \tan x}{-x}$$

$$L = \lim_{x \rightarrow 0} \frac{\sin x \tan x}{-x} \quad \mathbf{\frac{0}{0} \text{ form}} \dots \dots \dots (3)$$

From (3)
$$L = \lim_{x \rightarrow 0} \frac{\sin x \sec^2 x + \tan x \cos x}{-1} \dots \dots \dots (4)$$

$x \rightarrow 0$ put $x = 0$

$$L = \frac{\sin(0) \sec^2(0) + \tan(0) \cos(0)}{-1}$$

$$L = 0$$



Example 11) Evaluate $\lim_{x \rightarrow \pi/2} (1 - \sin x) \tan x$

Solution: Let $L = \lim_{x \rightarrow \pi/2} (1 - \sin x) \tan x \dots \dots \dots (1)$

$$x \rightarrow \pi/2 \text{ put } x = \pi/2$$

$$L = (1 - \sin \pi/2) \tan \pi/2 \quad 0 \times \infty \text{ form}$$

$$\text{From (1)} \quad L = \lim_{x \rightarrow \pi/2} \frac{(1 - \sin x)}{\frac{1}{\tan x}} = \lim_{x \rightarrow \pi/2} \frac{(1 - \sin x)}{\cot x} \dots \dots \dots (2)$$

$$x \rightarrow \pi/2 \text{ put } x = \pi/2$$

$$L = \frac{(1 - \sin \pi/2)}{\cot \pi/2} \quad \frac{0}{0} \text{ form}$$

$$\text{From (2)} \quad L = \lim_{x \rightarrow \pi/2} \frac{-\cos x}{-\operatorname{cosec}^2 x}$$

$$x \rightarrow \pi/2 \text{ put } x = \pi/2$$

$$L = \frac{-\cos \pi/2}{-\operatorname{cosec}^2 \pi/2}$$

$$L = \frac{0}{1^2}$$

$$L = 0$$

Example 12) Evaluate $\lim_{x \rightarrow 0} \left[\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x} + 1)} \right]$

Solution : Let $L = \lim_{x \rightarrow 0} \left[\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x} + 1)} \right] \dots \dots \dots (1)$

$$x \rightarrow 0 \text{ put } x = 0$$

$$L = \frac{\pi}{4(0)} - \frac{\pi}{2(0)(e^{\pi 0} + 1)} \quad \infty - \infty \text{ form}$$

$$\text{From (1)} \quad L = \lim_{x \rightarrow 0} \left[\frac{2x\pi(e^{\pi x} + 1) - 4x\pi}{8x^2(e^{\pi x} + 1)} \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{\pi(e^{\pi x} + 1) - 2\pi}{4x(e^{\pi x} + 1)} \right]$$



$$L = \lim_{x \rightarrow 0} \left[\frac{\pi e^{\pi x} + \pi - 2\pi}{4x(e^{\pi x} + 1)} \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{\pi e^{\pi x} - \pi}{4x(e^{\pi x} + 1)} \right] \dots \dots \dots (2)$$

x → 0 put x = 0

$$L = \frac{\pi e^{0x} - \pi}{4(0)(e^{\pi 0} + 1)} = \frac{\pi - \pi}{0} \quad \frac{0}{0} \text{ form}$$

From (2)
$$L = \lim_{x \rightarrow 0} \left[\frac{\pi^2 e^{\pi x}}{4x(\pi e^{\pi x} + 0) + 4(\pi e^{\pi x} + 1)} \right]$$

$$L = \frac{\pi^2 e^{\pi 0}}{4(0)(\pi e^0 + 0) + 4(e^0 + 1)}$$

$$L = \frac{\pi^2}{4(1 + 1)}$$

$$L = \frac{\pi^2}{8}$$

Example 13) Evaluate $\lim_{x \rightarrow 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1 + x) \right]$

Solution:

Let
$$L = \lim_{x \rightarrow 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1 + x) \right] \dots \dots \dots (1)$$

x → 0 put x = 0

$$L = \frac{1}{0} - \frac{1}{0} \log(1 + 0) \quad \infty - \frac{0}{0} \text{ form}$$

From (1)
$$L = \lim_{x \rightarrow 0} \left[\frac{x}{x^2} - \frac{1}{x^2} \log(1 + x) \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{x * x^2 - x^2 \log(1 + x)}{x^2 x^2} \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{x^2(x - \log(1 + x))}{x^2 x^2} \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{x - \log(1 + x)}{x^2} \right] \quad \frac{0}{0} \text{ form} \dots \dots \dots (1)$$



$$\text{From (1)} \quad L = \lim_{x \rightarrow 0} \left[\frac{1 - \frac{1}{1+x}}{2x} \right] \dots \dots \dots (2)$$

x → 0 put x = 0

$$L = \frac{1 - \frac{1}{1+0}}{2(0)} \quad \frac{0}{0} \text{ form}$$

$$\text{From (2)} \quad L = \lim_{x \rightarrow 0} \left[\frac{0 - \frac{-1}{(1+x)^2}}{2} \right]$$

x → 0 put x = 0

$$L = \frac{\frac{1}{(1+0)^2}}{2}$$

$$L = \frac{1}{2}$$

Example 14) Evaluate $\lim_{x \rightarrow 1} \left[\frac{1}{x-1} - \frac{2}{x^2-1} \right]$

Solution : Let $L = \lim_{x \rightarrow 1} \left[\frac{1}{x-1} - \frac{2}{x^2-1} \right] \dots \dots \dots (1)$

x → 1 put x = 1

$$L = \frac{1}{1-1} - \frac{2}{1^2-1} = \frac{1}{0} - \frac{2}{0} \quad \infty - \infty \text{ form}$$

$$\text{From (1)} \quad L = \lim_{x \rightarrow 0} \left[\frac{(x^2-1) - 2(x-1)}{(x-1)(x^2-1)} \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{(x-1)(x+1) - 2(x-1)}{(x-1)(x^2-1)} \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{(x+1) - 2}{(x^2-1)} \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{x-1}{(x^2-1)} \right]$$



$$L = \lim_{x \rightarrow 0} \left[\frac{x-1}{(x-1)(x+1)} \right]$$

$$L = \lim_{x \rightarrow 0} \left[\frac{1}{(x+1)} \right]$$

x → 1 put x = 1

$$L = \frac{1}{1+1}$$

$$L = \frac{1}{2}$$

Example 15) Evaluate $\lim_{x \rightarrow 0} \{\sin x\}^{\tan x}$

Solution: Let $L = \lim_{x \rightarrow 0} \{\sin x\}^{\tan x} \dots \dots \dots (1)$

x → 0 put x = 0

$$L = \{\sin 0\}^{\tan 0} \quad \text{0}^0 \text{ form}$$

Taking log on both sides

$$\log L = \log \lim_{x \rightarrow 0} \{\sin x\}^{\tan x}$$

$$\log L = \lim_{x \rightarrow 0} \log \{\sin x\}^{\tan x} \quad \because \log a^n = n * \log a$$

$$\log L = \lim_{x \rightarrow 0} \tan x \log \sin x \quad \dots \dots \dots (2)$$

x → 0 put x = 0

$$\log L = \tan 0 \log \sin 0 = \tan 0 \log 0 \quad \text{0} \times \infty \text{ form}$$

From (2) $\log L = \lim_{x \rightarrow 0} \frac{\log \sin x}{1/\tan x}$

$$\log L = \lim_{x \rightarrow 0} \frac{\log \sin x}{\cot x} \quad \because \frac{1}{\tan x} = \cot x \quad \dots \dots \dots (3)$$

x → 0 put x = 0

$$\log L = \frac{\log \sin 0}{\cot 0} \quad \frac{\infty}{\infty} \text{ form}$$

From (3) $\log L = \lim_{x \rightarrow 0} \frac{\left(\frac{1}{\sin x}\right) \frac{d}{dx} \sin x}{-\operatorname{cosec}^2 x} = \frac{\frac{\cos x}{\sin x}}{-\operatorname{cosec}^2 x}$



$$\log L = \lim_{x \rightarrow 0} \frac{\cot x}{-\operatorname{cosec}^2 x} \quad \frac{\infty}{\infty} \text{ form} \quad \dots \dots \dots (4)$$

From (4) $\log L = \lim_{x \rightarrow 0} \frac{-\operatorname{cosec}^2 x}{-2 \operatorname{cosec}^1 x \operatorname{cosec} x \cot x} \quad \because x^n = nx^{n-1} \frac{d}{dx}(x)$

$$\log L = \lim_{x \rightarrow 0} \frac{1}{2 \cot x} = \lim_{x \rightarrow 0} \frac{1}{2} \tan x$$

$x \rightarrow 0$ put $x = 0$

$$\log L = \frac{1}{2} \tan 0$$

$$\log L = 0$$

$$e^{\log L} = e^0 \quad L = e^0 = 1$$

Example 16) Evaluate $\lim_{x \rightarrow 0} \{\cot x\}^{\sin x}$

Solution: Let $L = \lim_{x \rightarrow 0} \{\cot x\}^{\sin x}$

$x \rightarrow 0$ put $x = 0$

$$L = \lim_{x \rightarrow 0} \{\cot 0\}^{\sin 0} \quad \infty^0 \text{ form} \quad \dots \dots \dots (1)$$

Taking log on both sides

$$\log L = \log \lim_{x \rightarrow 0} \{\cot x\}^{\sin x}$$

$$\log L = \lim_{x \rightarrow 0} \log \{\cot x\}^{\sin x}$$

$$\log L = \lim_{x \rightarrow 0} \sin x \log \cot x \quad \dots \dots \dots (2)$$

$x \rightarrow 0$ put $x = 0$

$$\log L = \sin 0 \log \cot 0 \quad 0 \times \infty \text{ form}$$

From (2) $\log L = \lim_{x \rightarrow 0} \frac{\log \cot x}{1/\sin x}$

$$\log L = \lim_{x \rightarrow 0} \frac{\log \cot x}{\operatorname{cosec} x} \quad \because \frac{1}{\sin x} = \operatorname{cosec} x \quad \dots \dots \dots (3)$$

$x \rightarrow 0$ put $x = 0$

$$\log L = \frac{\log \cot 0}{\operatorname{cosec} 0} \quad \frac{\infty}{\infty} \text{ form}$$



$$\text{From (3)} \quad \log L = \lim_{x \rightarrow 0} \frac{-\operatorname{cosec}^2 x / \cot x}{-\operatorname{cosec} x \cot x}$$

$$\log L = \lim_{x \rightarrow 0} \frac{\operatorname{cosec} x}{\cot^2 x} \quad \frac{\infty}{\infty} \text{ form} \dots\dots\dots (4)$$

$$\text{From (4)} \quad \log L = \lim_{x \rightarrow 0} \frac{\tan^2 x}{\sin x} \quad \text{taking reciprocal} \quad \frac{0}{0} \text{ form}$$

$$\log L = \lim_{x \rightarrow 0} \frac{2 \tan x \sec^2 x}{\cos x} \quad \because x^n = nx^{n-1} \frac{d}{dx}(x)$$

$x \rightarrow 0$ put $x = 0$

$$\log L = \frac{2 \tan 0 \sec^2 0}{\cos 0} = \frac{0}{1}$$

$$\log L = 0$$

$$L = e^0 = 1$$

Example 17) Evaluate $\lim_{x \rightarrow 0} \left[\frac{a^x + b^x}{2} \right]^{1/x}$

Solution: Let $L = \lim_{x \rightarrow 0} \left[\frac{a^x + b^x}{2} \right]^{1/x}$

$x \rightarrow 0$ put $x = 0$

$$L = \left[\frac{a^0 + b^0}{2} \right]^{1/0} = \left[\frac{1+1}{2} \right]^\infty \quad 1^\infty \text{ form} \dots\dots\dots (1)$$

Taking log on both sides

$$\log L = \log \lim_{x \rightarrow 0} \left[\frac{a^x + b^x}{2} \right]^{1/x}$$

$$\log L = \lim_{x \rightarrow 0} \log \left[\frac{a^x + b^x}{2} \right]^{1/x}$$

$$\log L = \lim_{x \rightarrow 0} \frac{\log \left[\frac{a^x + b^x}{2} \right]}{x}$$

$$\because \log a^n = n \log a \dots\dots (2)$$

$x \rightarrow 0$ put $x = 0$

$$\log L = \log \left[\frac{a^0 + b^0}{2} \right] / 0 \quad \frac{0}{0} \text{ form}$$

$$\text{From (2)} \quad \log L = \lim_{x \rightarrow 0} \left[\frac{1}{\frac{a^x + b^x}{2}} \cdot \frac{(a^x \log a + b^x \log b)}{2} \right] / 1$$



$$\log L = \lim_{x \rightarrow 0} \frac{2}{a^x + b^x} \frac{(a^x \log a + b^x \log b)}{2}$$

$$\log L = \lim_{x \rightarrow 0} \frac{a^x \log a + b^x \log b}{a^x + b^x}$$

x → 0 put x = 0

$$\log L = \frac{a^0 \log a + b^0 \log b}{a^0 + b^0}$$

$$\log L = \frac{\log a + \log b}{1 + 1}$$

$$\log L = \frac{\log(ab)}{2}$$

$$\therefore \log a + \log b = \log ab$$

$$\log L = \frac{1}{2} \log ab$$

$$\log L = \log(ab)^{\frac{1}{2}}$$

$$L = (ab)^{\frac{1}{2}}$$

