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SCAN ME



Unit -V

MATRICES

➤ The Rank of a Matrix:

The matrix is said to be of rank 'r' if :

- i) there exist at least one non zero minor of the order r
- ii) every minor of order $(r + 1)$ is equal to zero.

The rank of the matrix A is denoted by $\rho(A) = r$

If a matrix has a non – zero minor of order r, then $\rho(A) \geq r$.

If a matrix has all minors of order $(r + 1)$ as zeroes, then $\rho(A) \leq r$.

If A is $m \times n$ matrix then $\rho(A) \leq \min(m, n)$

Finding the Rank by reducing the matrix to Echelon Form:

Ex. 1) Find the rank of A, where $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

Consider, $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

perform $R_2 - 3R_1$ and $R_3 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

Perform $R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A)$ = total number of non zero rows(at least one element is non zero) after reducing the matrix to echelon form.

Ex. 2) Find the rank of A, where $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Consider, $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ *Perform $R_2 \leftrightarrow R_1$* $\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

Perform $R_3 - 3R_1$ and $R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

Perform $R_3 - R_2$ and $R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

➤ System of Linear Algebraic Equations:

Consider a system of 'm' linear equations in 'n' unknowns $x_1, x_2, x_3, x_4, \dots, x_n$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \cdot &\quad \cdot \quad \quad \cdot \\ \cdot &\quad \cdot \quad \quad \cdot \\ \cdot &\quad \cdot \quad \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_n \end{aligned}$$

The above system of equations can be written in compact form by using matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} \quad \text{i.e. } AX = B$$

where, $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$

When the system $AX = B$ has a solution

i. e. set of values of $x_1, x_2 \dots x_n$ satisfy simultaneously all m equations then system is said to be **consistent**, otherwise the system is called **inconsistent**.

Augmented Matrix $[A, B]$:

$$[A, B] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \vdots & b_2 \\ & & \ddots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & \vdots & b_m \end{bmatrix}$$

Non – Homogeneous System of Equations:

For the system of equations $AX = B$ if **matrix B is not a null matrix** then the system $AX = B$ is known as non homogeneous system of equations.

Homogeneous System of Equations:

For the system of equations $AX = B$ if **matrix B is a null matrix** then the system $AX = B$ is known as homogeneous system of equations.

i. e. **$AX = 0$** (0 is null matrix).

System of Linear Equations

Non – Homogeneous System

$$A_{m \times n} X_{n \times 1} = B_{m \times 1} \text{ and } B \neq 0$$

Consistent if
 $\rho(A) = \rho[A, B] = r$

Unique Solution
if $r = n(\text{no. of variables})$

Inconsistent if
 $\rho(A) \neq \rho[A, B]$

No Solution

Infinite Solutions
if $r < n(\text{no. of variables})$

Homogeneous System

$$A_{m \times n} X_{n \times 1} = 0_{m \times 1} \text{ (Always Consistent)}$$

Unique Solution
trivial solution
if $\rho(A) = r = n(\text{no. of variables})$

Infinite Solutions
if $\rho(A) = r < n(\text{no. of variables})$

Ex. 1) Is the following system of equations consistent? If so solve it.

$$x + y + z = 3$$

$$x + 2y + 3z = 4$$

$$x + 4y + 9z = 6$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} \quad \text{i.e. } AX = B$$

Consider augmented matrix

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array} \right]$$

Perform $R_2 - R_1$ and $R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array} \right]$$

Perform $R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{array} \right]$$

$$\rho(A) = 3$$

$$\rho[A, B] = 3$$

$\therefore \rho(A) = \rho[A, B]$ **Other Subjects:** <https://www.studydrive.in/notes> given system is consistent.

$$\rho(A) = \rho[A, B] = 3 = \text{no. of variables}$$

\therefore The given system has unique solution

$$\text{by } R_3, \quad 2z = 0$$

$$\text{by } R_2, \quad y + 2z = 1$$

$$\therefore z = 0, y = 1, x = 2$$

$$\text{by } R_1, \quad x + y + z = 3$$

Hence the solution is, $x = 2, \quad y = 1, \quad z = 0$

Ex. 2) Is the following system of equations consistent? if so solve it.

$$x + y + z = 6$$

$$x - y + 2z = 5$$

$$3x + y + z = 8$$

$$2x - 2y + 3z = 7$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

Consider augmented matrix,

$$[A, B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{array} \right]$$

Perform, $R_2 - R_1$, $R_3 - 3R_1$ and $R_4 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{array} \right]$$

Perform $R_3 - R_2$ and $R_4 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -1 & -3 \end{array} \right]$$

Perform $R_4 - \frac{1}{3}R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rho(A) = 3$$

$$\rho[A, B] = 3$$

$$\rho(A) = \rho[A, B] = 3 = \text{no. of variables}$$

\therefore given system is consistent and has unique solution

$$\text{by } R_3, \quad -3z = -9 \quad \therefore z = 3$$

$$\text{by } R_2, \quad -2y + z = -1 \quad \therefore -2y + 3 = -1, \quad y = 2$$

$$\text{by } R_1, \quad x + y + z = 6 \quad \therefore x + 2 + 3 = 6, \quad x = 1$$

Hence the solution is $x = 1, \quad y = 2, \quad z = 3$

Ex. 3) By considering the ranks of relevant matrices, examine for consistency the system of equations:

$$2x - y - z = 2$$

$$x + 2y + z = 2$$

$$4x - 7y - 5z = 2$$

And solve them if consistent.

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ 4 & -7 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \quad i. e. \quad AX = B$$

Consider augmented matrix,

$$[A, B] = \left[\begin{array}{ccc|c} 2 & -1 & -1 & 2 \\ 1 & 2 & 1 & 2 \\ 4 & -7 & -5 & 2 \end{array} \right]$$

perform $R_2 \leftrightarrow R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 2 & -1 & -1 & 2 \\ 4 & -7 & -5 & 2 \end{array} \right]$$

perform $R_2 - 2R_1$ and $R_3 - 4R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & -15 & -9 & -6 \end{array} \right]$$

Perform $R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & -5 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rho(A) = 2$$

$$\rho[A, B] = 2$$

$$\therefore \rho(A) = \rho[A, B]$$

\therefore given system is consistent

$$\rho(A) = \rho[A, B] = 2 < \text{no. of variables.}$$

\therefore system has infinite number of solutions.

by R_2 , $-5y - 3z = -2$

$$y = \frac{2 - 3z}{5}$$

by R_1 , $x + 2y + z = 2$

$$x = 2 - 2y - z, \quad x = 2 - \frac{2 - 3z}{5} - z \quad x = \frac{6 + z}{5}$$

Let $z = t$

Hence the solution set is $x = \frac{6 + t}{5}, \quad y = \frac{2 - 3t}{5}, \quad z = t$

Ex. 4) Examine for consistency and if consistent then solve it

$$2x - 3y + 5z = 1$$

$$3x + y - z = 2$$

$$x + 4y - 6z = 1$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 1 & -1 \\ 1 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Consider augmented matrix,

$$[A, B] = \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{array} \right]$$

Perform $R_3 \leftrightarrow R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 3 & 1 & -1 & 2 \\ 2 & -3 & 5 & 1 \end{array} \right]$$

Perform, $R_2 - 3R_1$ and $R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & -11 & 17 & -1 \end{array} \right]$$

Perform $R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & -6 & 1 \\ 0 & -11 & 17 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rho(A) = 2 \qquad \rho[A, B] = 2$$

$$\therefore \rho(A) = \rho[A, B]$$

$\therefore \rho(A) = \rho[A, B] = 2 < \text{no. of variables}$

\therefore system is consistent and system has infinite number of solutions.

by R_2 , $-11y + 17z = -1$

$$\therefore y = \frac{1 + 17z}{11}$$

by R_1 , $x + 4y - 6z = 1$

$$x = 1 - 4\left(\frac{1 + 17z}{11}\right) + 6z$$

$$\therefore x = \frac{7 - 2z}{11}$$

Let, $z = t$

Hence the solution set is, $x = \frac{7 - 2t}{11}$, $y = \frac{1 + 17t}{11}$, $z = t$

Ex. 5) Examine for consistency and if consistent then solve it

$$x + 2y + 3z = 0$$

$$4x + 5y + 6z = 2$$

$$7x + 8y + 9z = 3$$

Solution:

System is inconsistent.

Ex. 5) Investigate for what values of λ and μ , the system of simultaneous equations:

$$2x - y + 3z = 2$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 2 \\ 5 & -1 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ \mu \end{bmatrix}$$

Consider augmented matrix

$$[A, B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 5 & -1 & \lambda & \mu \end{array} \right]$$

Perform $R_2 \leftrightarrow R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 3 & 2 \\ 5 & -1 & \lambda & \mu \end{array} \right]$$

Perform $R_2 - 2R_1$ and $R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & -6 & \lambda - 10 & \mu - 10 \end{array} \right]$$

perform $R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & \lambda - 8 & \mu - 6 \end{array} \right]$$

(i) if $\rho(A) \neq \rho[A, B]$ then system has no solution

\therefore Let, $\lambda = 8$, then $\rho(A) = 2$

Let, $\mu \neq 6$, then $\rho[A, B] = 3$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & \lambda - 8 & \mu - 6 \end{array} \right]$$

(ii) If $\rho(A) = \rho[A, B] = \text{no. of variables}$, then system has unique solution

\therefore let, $\lambda \neq 8$ and μ may have any value then
 $\rho(A) = \rho[A, B] = 3$

(iii) if $\rho(A) = \rho[A, B] < \text{no of variables}$, then system has infinite no. of solutions.

\therefore Let, $\lambda = 8$ and $\mu = 6$ then
 $\rho(A) = \rho[A, B] = 2 < \text{no. of variables}$

Ex. 6) Investigate for what value of k the equations

$$x + y + z = 1$$

$$2x + y + 4z = k$$

$$4x + y + 10z = k^2$$

have infinite number of solutions ? Hence find solutions.

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ k^2 \end{bmatrix}$$

Consider augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & k \\ 4 & 1 & 10 & k^2 \end{array} \right]$$

perform $R_2 - 2R_1$ and $R_3 - 4R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & -3 & 6 & k^2-4 \end{array} \right]$$

perform $R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & k^2-3k+2 \end{array} \right]$$

$$\rho(A) = 2$$

if $\rho(A) = \rho[A, B] < \text{number of variables}$, then system has infinite number of solutions

\therefore consider $k^2 - 3k + 2 = 0$ then $\rho(A) = \rho[A, B] = 2 < \text{number of variables}$

\therefore we get, $k = 1$ or $k = 2$

For, $k = 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & k^2-3k+2 \end{array} \right]$$

we get, $-y + 2z = -1$

$$\therefore y = 1 + 2z$$

$$x + y + z = 1$$

$$\therefore x = 1 - (1 + 2z) - z$$

$$\therefore x = -3z$$

Let, $z = t$

\therefore Solution set is $x = -3t, \quad y = 1 + 2t, \quad z = t$

For, $k = 2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

we get,

$$-y + 2z = 0$$

$$\therefore y = 2z$$

$$x + y + z = 1$$

$$\therefore x = 1 - (2z) - z$$

$$\therefore x = 1 - 3z$$

Let $z = t$

\therefore Solution set is $x = 1 - 3t, \quad y = 2t, \quad z = t$

Ex. 7) Examine for non – trivial solution, the following set of equations and solve

$$x - 2y + z + 3w = 0$$

$$3x + y + z + w = 0$$

$$2x - 4y + 2z + 6w = 0$$

$$4x - y + 2z + 4w = 0$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & 1 & 1 & 1 \\ 2 & -4 & 2 & 6 \\ 4 & -1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider coefficient matrix

$$A = \begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & 1 & 1 & 1 \\ 2 & -4 & 2 & 6 \\ 4 & -1 & 2 & 4 \end{bmatrix}$$

perform $R_2 - 3R_1$, $R_3 - 2R_1$ and $R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & -2 & -8 \end{bmatrix}$$

perform $R_4 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, $\rho(A) = 2 < \text{number of variables}$

\therefore System has infinite number of solutions

Other Subjects: <https://www.studymedia.in/fe/notes>

by R_2 , $7y - 2z - 8w = 0$

by R_1 , $x - 2y + z + 3w = 0$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 7 & -2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore y = \frac{2z + 8w}{7} \quad \therefore x = 2y - z - 3w \quad \therefore x = 2\left(\frac{2z + 8w}{7}\right) - z - 3w$$

$$\therefore x = \frac{-3z - 5w}{7}$$

Let $z = t$ and $w = s$

\therefore **Solution set is** $x = \frac{-3t - 5s}{7}, \quad y = \frac{2t + 8s}{7}, \quad z = t, \quad w = s$

Ex. 8) For different value of k , discuss the nature of solution of the following eqs

$$x + 2y - z = 0$$

$$3x + (k + 7)y - 3z = 0$$

$$2x + 4y + (k - 3)z = 0.$$

Solution: Given system of equations in matrix form can be written as

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & k+7 & -3 \\ 2 & 4 & k-3 \end{bmatrix}$$

perform $R_2 - 3R_1$ and $R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & k+1 & 0 \\ 0 & 0 & k-1 \end{bmatrix}$$

If $k \neq \pm 1$, then $\rho(A) = 3 = \text{number of variables}$

\therefore system has unique solution which is trivial solution if $k \neq \pm 1$

i. e. $x = 0, y = 0$ and $z = 0$

If $k = \pm 1$, then $\rho(A) = 2 < \text{number of variables}$

\therefore System has infinite number solutions if $k = \pm 1$

➤ LINEAR DEPENDENT AND INDEPEDENT VECTORS:

Vector:

We define n dimentional vector as an ordered set of n elements $x_1, x_2, x_3, \dots, x_n$ and is denoted by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}, X = [x_1 \quad x_2 \quad x_3 \quad \cdot \quad \cdot \quad x_n]. \text{ The element } x_1, x_2, x_3, \dots, x_n \text{ are called componant of } x.$$

Linear Dependent Vectors:

Let $X_1, X_2, X_3, \dots, X_n$ be n vectors of the same order

If ther exist n scalars $c_1, c_2, c_3, \dots, c_n$, **not all zero**

such that $c_1X_1 + c_2X_2 + c_3X_3 + \dots + c_nX_n = 0$

then these n vectors are called as linearly dependent.

Linear Independent Vectors:

Let $X_1, X_2, X_3, \dots, X_n$ be n vectors of the same order

These n vectors are said to be linearly independent if

$c_1X_1 + c_2X_2 + c_3X_3 + \dots + c_nX_n = 0$, then

$c_1 = c_2 = c_3 = \dots = c_n = 0$

Other Subjects: <https://www.studymedia.in/fe/notes>

Ex. 1) Examine for linear dependence or independence of vectors

$(2, -1, 3, 2)$, $(1, 3, 4, 2)$ and $(3, -5, 2, 2)$. Find the relation between them if dependent.

Solution: Let, given vectors be $X_1 = (2, -1, 3, 2)$, $X_2 = (1, 3, 4, 2)$ and $X_3 = (3, -5, 2, 2)$

Consider linear combination $c_1X_1 + c_2X_2 + c_3X_3 = 0$

$$c_1(2, -1, 3, 2) + c_2(1, 3, 4, 2) + c_3(3, -5, 2, 2) = (0, 0, 0, 0)$$

$$i.e., \quad 2c_1 + c_2 + 3c_3 = 0$$

$$-c_1 + 3c_2 - 5c_3 = 0$$

$$3c_1 + 4c_2 + 2c_3 = 0$$

$$2c_1 + 2c_2 + 2c_3 = 0$$

In matrix form,
$$\begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Let, } A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 3 & -5 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \quad \begin{matrix} \text{perform } R_{12} \\ \sim \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} \end{matrix}$$

Perform, $R_2 + 2R_1, R_3 + 3R_1, R_4 + 2R_1$

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 0 & 7 & -7 \\ 0 & 13 & -13 \\ 0 & 8 & -8 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 2 & 1 & 3 \\ 3 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

perform, $R_3 - \frac{13}{7}R_2$ and $R_4 - \frac{8}{7}R_2$

$$\sim \begin{bmatrix} -1 & 3 & -5 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(A) = 2 < \text{no of variables}$$

\therefore system has infinite number of solutions

by R_2 , $7c_2 - 7c_3 = 0$

$$c_2 = c_3$$

by R_1 , $-c_1 + 3c_2 - 5c_3 = 0$

$$c_1 = 3c_2 - 5c_3$$

$$\therefore c_1 = -2c_3$$

Let, $c_3 = t$

$$\therefore c_1 = -2t, \quad c_2 = t, \quad c_3 = t$$

we can find c_1, c_2 and c_3 are not all zero

\therefore the given vectors X_1, X_2 and X_3 are linearly dependents.

Relation: $c_1X_1 + c_2X_2 + c_3X_3 = 0$

$$-2tX_1 + tX_2 + tX_3 = 0 \quad \text{Let, } t = 1$$

we get, $X_2 + X_3 = 2X_1$

Ex. 2) Examine for linear dependence or independence of vectors

$X_1 = (2, 2, 7, -1), X_2 = (3, -1, 2, 4), X_3 = (1, 1, 3, 1)$. Find relation if dependent.

Solution: Consider linear combination

$$c_1X_1 + c_2X_2 + c_3X_3 = 0$$

$$c_1(2, 2, 7, -1) + c_2(3, -1, 2, 4) + c_3(1, 1, 3, 1) = (0, 0, 0, 0)$$

$$2c_1 + 3c_2 + c_3 = 0$$

$$2c_1 - c_2 + c_3 = 0$$

$$7c_1 + 2c_2 + 3c_3 = 0$$

$$-c_1 + 4c_2 + c_3 = 0$$

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Let, } A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix}$$

$$\text{perform, } R_{14} \quad \sim \begin{bmatrix} -1 & 4 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 1 \\ 2 & -1 & 1 \\ 7 & 2 & 3 \\ -1 & 4 & 1 \end{bmatrix}$$

perform $R_2 + 2R_1, R_3 + 7R_1$ and $R_4 + 2R_1$

$$\sim \begin{bmatrix} -1 & 4 & 1 \\ 0 & 7 & 3 \\ 0 & 30 & 10 \\ 0 & 11 & 3 \end{bmatrix}$$

perform, $R_3 - \frac{30}{7}R_2, R_4 - \frac{11}{7}R_2$

$$\sim \begin{bmatrix} -1 & 4 & 1 \\ 0 & 7 & 3 \\ 0 & 0 & -\frac{20}{7} \\ 0 & 0 & -\frac{12}{7} \end{bmatrix}$$

perform $R_4 - \frac{12}{20}R_3$

$$\sim \begin{bmatrix} -1 & 4 & 1 \\ 0 & 7 & 3 \\ 0 & 0 & -\frac{20}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3 = \text{number of variables.}$

\therefore system has unique solution

\therefore we get, trivial solution

i. e. $c_1 = 0, \quad c_2 = 0, \quad c_3 = 0$

\therefore vectors X_1, X_2 and X_3 are linearly independent.

Ex. 3) Examine for linear dependence or independenace of vectors
 $X_1 = (1, -1, 1), X_2 = (2, 1, 1), X_3 = (3, 0, 2)$. Find relation if dependent.

Ex. 4) Examine for linear dependence or independenace of vectors
 $X_1 = (1, 2, 3), X_2 = (2, -2, 6)$. *Find relation if dependent.*

Ex. 5) Examine for linear dependence or independenace of vectors
 $X_1 = (3, 1, -4)^T, X_2 = (2, 2, -3)^T, X_3 = (0, -4, 1)^T$ *Find relation if dependent.*

➤ Linear transformation:

The relation $Y = AX$ where A is square matrix expresses $y_1, y_2, y_3, \dots y_n$ in terms of $x_1, x_2, x_3, \dots x_n$ is called linear transformation.

' A ' is called matrix of transformation. $|A|$ is known as modulus of transformation.

If $|A| = 0$, then linear transformation $Y = AX$ is called singular transformation.

If $|A| \neq 0$, then linear transformation $Y = AX$ is called non – singular transformation.

For a non – singular transformation $Y = AX$, A^{-1} exist and we can write the **inverse transformation**,

$$X = A^{-1}Y$$

If a transformation from $(x_1, x_2, \dots x_n)$ to (y_1, y_2, \dots, y_n) is given by $Y = AX$ and another transformation from (y_1, y_2, \dots, y_n) to (z_1, z_2, \dots, z_n) is given by $Z = BY$, then transformation from $(x_1, x_2, \dots x_n)$ to (z_1, z_2, \dots, z_n) is given by

$$Z = BY = B(AX) = (BA)X$$

$$\text{i. e. } Z = (BA)X.$$

Ex. 1) Given the transformation $Y = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Find the co – ordinates (x_1, x_2, x_3) in X corresponding to $(1, 2, -1)$ in Y.

Solution: $Y = AX, \quad i.e. AX = Y$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

This is non – homogeneous system of equations

perform R_{12}

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 0 & -2 & -1 \end{bmatrix}$$

perform $R_2 - 2R_1$ and $R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & -1 & -4 & -3 \end{bmatrix}$$

perform $R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\rho(A) = 3 = \rho[A, B] = \text{number of variables}$$

System has unique solution.

$$\text{by } R_3, \quad -x_3 = 0$$

$$x_3 = 0$$

$$\text{by } R_2, \quad -x_2 - 3x_3 = -3$$

$$x_2 = 3$$

$$\text{by } R_1, \quad x_1 + x_2 + 2x_3 = 2$$

$$x_1 = -1$$

Other Subject: <https://www.studymedia.in/fe/notes/>
Hence $(-1, 3, 0)$ corresponds to $(1, 2, -1)$ in Y.

Orthogonal Matrix:

A square matrix A is said to be orthogonal if $AA' = A'A = I$

If A is orthogonal then $A^{-1} = A'$

Ex. 1) Show that $A = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$ is orthogonal matrix.

Solution:

$$\begin{aligned} \text{Consider, } AA' &= \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta & 0 & -\sin \theta \cos \theta + \cos \theta \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta \cos \theta + \cos \theta \sin \theta & 0 & \sin^2 \theta + \cos^2 \theta \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$AA' = I, \quad \therefore A$ is orthogonal.

Other Subjects: <https://www.studymedia.in/fe/notes>

Ex. 2) Determine the values of a, b, c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

Solution: If A is orthogonal then it requires $AA' = I$

$$\begin{aligned} AA' &= \begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix} \begin{bmatrix} 0 & a & a \\ 2b & b & -b \\ c & -c & c \end{bmatrix} \\ &= \begin{bmatrix} 4b^2 + c^2 & 2b^2 - c^2 & -2b^2 + c^2 \\ 2b^2 - c^2 & a^2 + b^2 + c^2 & a^2 - b^2 - c^2 \\ -2b^2 + c^2 & a^2 - b^2 - c^2 & a^2 + b^2 + c^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

we get, $4b^2 + c^2 = 1, \quad 2b^2 - c^2 = 0, \quad a^2 + b^2 + c^2 = 1, \quad a^2 - b^2 - c^2 = 0$

we get, $a = \pm \frac{1}{\sqrt{2}}, \quad b = \pm \frac{1}{\sqrt{6}}, \quad c = \pm \frac{1}{\sqrt{3}}$

Ex. 3) Verify whether the matrix A is orthogonal or not. If so find A^{-1} .

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Other Subjects: <https://www.studymedia.in/fe/notes>

Solution:

Consider,

$$\begin{aligned} AA' &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + 0 + \frac{1}{2} & 0 + 0 + 0 & \frac{1}{2} + 0 - \frac{1}{2} \\ 0 + 0 + 0 & 0 + 1 + 0 & 0 + 0 + 0 \\ \frac{1}{2} + 0 - \frac{1}{2} & 0 + 0 + 0 & \frac{1}{2} + 0 + \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\therefore AA' = I \quad \therefore A \text{ is Orthogonal Matrix}$$

As matrix A is orthogonal, $A^{-1} = A'$

$$\therefore A^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

➤ Applications of System of Linear Equations:

Ex. 1) Find the currents in the circuit shown in figure.

Solution:

Applying KCL at the nodes P, Q, and R

$$I_1 + I_3 = I_2$$

$$I_4 = I_1 + I_5$$

$$I_2 + I_5 = I_3 + I_4$$

Applying KVL to three loops

$$I_1 + 2I_2 = 2 \quad 2I_2 + 2I_3 = 4 \quad 5I_5 = 10$$

from last equation $I_5 = 2$

we get,

$$I_1 - I_2 + I_3 = 0$$

$$-I_1 + I_4 = 2$$

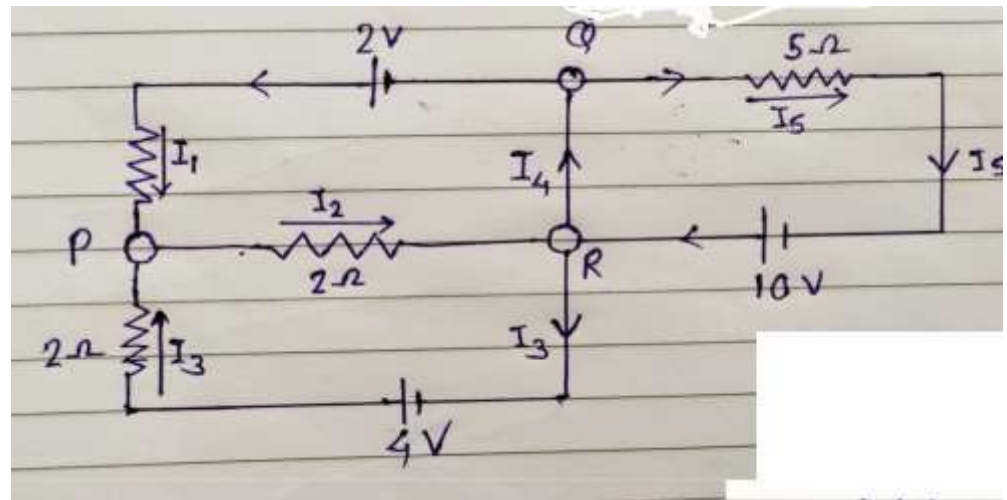
$$-I_2 + I_3 + I_4 = 2$$

$$I_1 + 2I_2 = 2$$

$$I_2 + I_3 = 2$$

In matrix form,

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$



Consider Augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 & 2 \\ 1 & 2 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 2 \end{array} \right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 2 \\ 0 & 0 & -2 & 3 & 6 \\ 0 & 0 & 0 & 4 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\text{by } R_4, \quad I_4 = \frac{5}{2}$$

$$\text{by } R_3, \quad -2I_3 + 3I_4 = 6$$

$$\text{by } R_2, \quad -I_2 + I_3 + I_4 = 2$$

$$\text{by } R_1, \quad I_1 - I_2 + I_3 = 0$$

$$\begin{aligned} &\text{we get, } I_4 = \frac{5}{2} \\ I_3 &= \frac{3}{4}, \quad I_2 = \frac{5}{4}, \quad I_1 = \frac{1}{2} \end{aligned}$$

Ex. 2) Solve the following traffic problem

Solution:

At junction A, $x_1 + x_4 = 300 + 200$

At junction B, $x_3 + x_4 = 500 + 300$

At junction C, $x_2 + x_3 = 600 + 500$

At junction D, $x_1 + x_2 = 400 + 400$

The system of equation is,

$$\begin{aligned}x_1 + x_4 &= 500 \\x_3 + x_4 &= 800 \\x_2 + x_3 &= 1100 \\x_1 + x_2 &= 800\end{aligned}$$

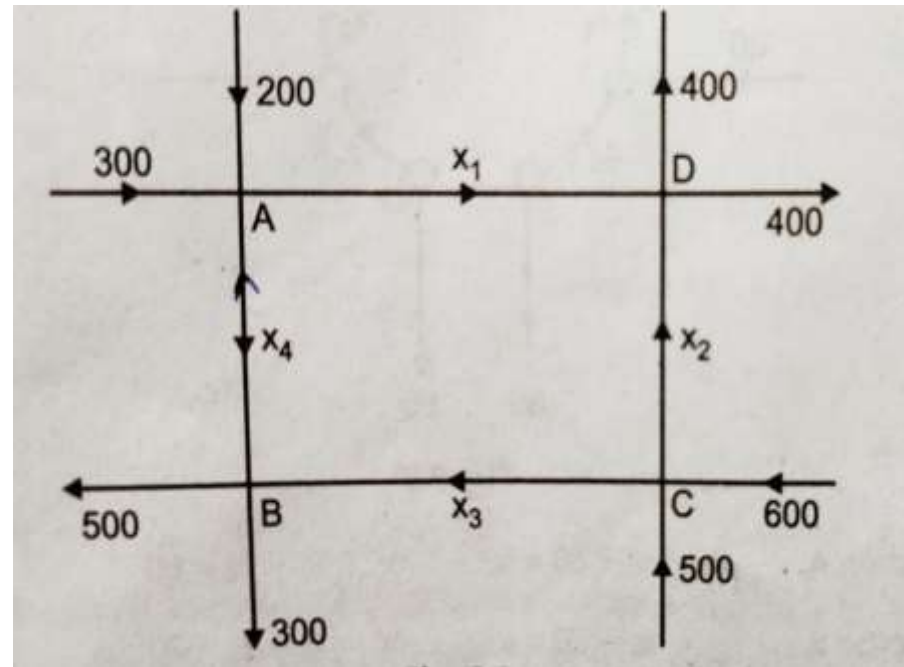
$$[A, B] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & 0 & 1 & 1 & 800 \\ 0 & 1 & 1 & 0 & 1100 \\ 1 & 1 & 0 & 0 & 800 \end{array} \right]$$

Matrix form is,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 500 \\ 800 \\ 1100 \\ 800 \end{bmatrix}$$

i. e. $AX = B$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$



Here $\rho(A) = 3 = \rho[A, B] < \text{number of variables}$.

system has infinite number of solutions

by R_3 , $x_3 + x_4 = 800$

Let, $x_4 = t$

by R_2 , $x_2 - x_4 = 300$

by R_1 , $x_1 + x_4 = 500$

$$x_3 = 800 - t$$

$$x_2 = 300 + t$$

$$x_1 = 500 - t$$

Take $t = 100$

we get, $x_1 = 400$, $x_2 = 400$, $x_3 = 700$, $x_4 = 100$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 500 \\ 0 & 1 & 0 & -1 & 300 \\ 0 & 0 & 1 & 1 & 800 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Ex. 3) Find the currents in the various branches of the following network, Shown in figure.

Solution:

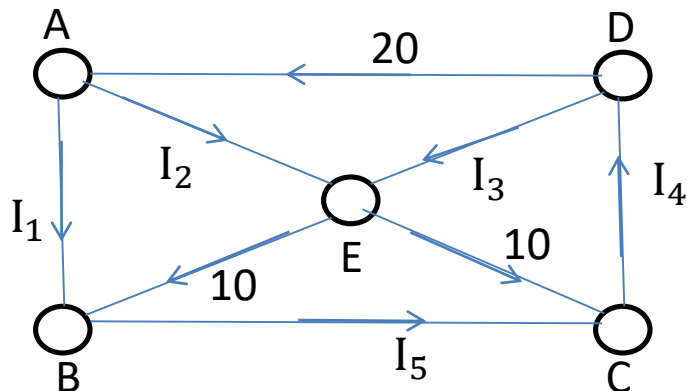
At node A, $20 = I_1 + I_2$

At node B, $10 + I_1 = I_5$

At node C, $10 + I_5 = I_4$

At node D, $I_4 = I_3 + 20$

At node E, $I_2 + I_3 = 10 + 10$



i. e.

In Matrix Form,

$$I_1 + I_2 = 20$$

$$I_1 - I_5 = -10$$

$$I_4 - I_5 = 10$$

$$I_3 - I_4 = -20$$

$$I_2 + I_3 = 20$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \\ 10 \\ -20 \\ 20 \end{bmatrix}$$