Unit IV
Applications of Partial Differentiation

Jacobians

If $u = f(x, y)$and $v = g(x, y)$ then Jacobian of u, v w.r.t. x, y denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ and is defined as

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

The Jacobian of u, v, w w.r.t. x, y, z is given by

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

If J $= \frac{\partial(u,v)}{\partial(x,y)}$ then $\frac{\partial(x,y)}{\partial(u,v)}$ is denoted by J^* or J'.

If $J \neq 0$ then $JJ' = 1$

If $u = 2x^2 + 3y^2,\ v = 4x^2y^2$ then find

$$\frac{\partial(u,v)}{\partial(x,y)}$$

If $x = e^u \cos v$, $y = e^u \sin v$ then verify that

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

If $u = \frac{2yz}{x}$, $v = \frac{3xz}{y}$, $w = \frac{4xy}{z}$ then find

$$\frac{\partial(u, v, w)}{\partial(x, y, z)}$$

Jacobians

If $u = f(x, y)$and $v = g(x, y)$ then Jacobian of u, v w.r.t. x, y denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ and is defined as

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

The Jacobian of u, v, w w.r.t. x, y, z is given by

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

If J $= \dfrac{\partial(u,v)}{\partial(x,y)}$ then $\dfrac{\partial(x,y)}{\partial(u,v)}$ is denoted by J^* or J'.

If $J \neq 0$ then $JJ' = 1$

If $\quad u = x^2 + y^2 + z^2$,

$v = x + y + z$,

$w = xy + yz + zx$

then find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

If $x = 3r\sin\theta\cos\phi$,
$y = 2r\sin\theta\sin\phi$,
$z = r\cos\theta$

then show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = 6r^2\sin\theta$$

If

$$x = u^2 + v^2,$$
$$y = v^2 + w^2,$$
$$z = w^2 + u^2$$

then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

If

$$u = x + y + z,$$
$$u^2 v = y + z,$$
$$u^3 w = z$$

then evaluate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

Jacobian of Implicit Function

If u, v are functions of x, y and f, g be implicit functions of u, v, x, y such that $f(u, v, x, y) = 0$ and $g(u, v, x, y) = 0$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\frac{\partial(f,g)}{\partial(x,y)}}{\frac{\partial(f,g)}{\partial(u,v)}}$$

If $x^2 + y^2 + u^2 - v^2 = 0,\ uv + xy = 0$

then prove that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{x^2 - y^2}{u^2 + v^2}$$

$$\text{If } x+y+z=u^3+v^3+w^3,$$
$$x^2+y^2+z^2=u+v+w,$$
$$x^3+y^3+z^3=u^2+v^2+w^2$$

then show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

If $x + y = 2e^u \cos v$, $x - y = 2e^u \sin v$

then find $\dfrac{\partial(u,v)}{\partial(x,y)}$

If $u + v + w = x + y + z$

$$uv + vw + wu = x^2 + y^2 + z^2,$$

$$3uvw = x^3 + y^3 + z^3$$

then show that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = -\frac{(u - v)(v - w)(w - u)}{2(x - y)(y - z)(z - x)}$$

Partial derivative of Implicit functions by using Jacobian

If u, v are functions of x, y and f, g are implicit functions connecting u, v, x, y ,

i.e.,$f(u, v, x, y) = 0 = g(u, v, x, y)$ then

If $u = x + e^{-y}\sin x$, $v = y + e^{-y}\cos x$ then find $\frac{\partial y}{\partial u}$.

If $x = u + v^2,\ y = v + w^2,\ z = w + u^2$ then find $\frac{\partial u}{\partial x}$

If $x = u + v,\ y = v^2 + w^2,\ \ z = w^3 + u^3$ then show that

$$\frac{\partial u}{\partial x} = \frac{vw}{vw + u^2}$$

If $u = x + y + z,$

$$v = x^2 + y^2 + z^2,$$

$$w = x^3 + y^3 + z^3$$

then show that

$$\frac{\partial x}{\partial u} = \frac{yz}{(x-y)(x-z)}$$

Functional Dependence

Let $u = f(x, y)$ and $v = g(x, y)$ be given differentiable functions, then u and v are said to be functionally dependent if either u is a function of v or v is a function of u.

Two functions u and v are functionally dependent if their Jacobian is zero, i.e.,

$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

If $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ then u and v are said to be functionally independent.

Examine for functional dependence of $u = \frac{x-y}{x+y}$, $v = \frac{x+y}{x}$. If functionally dependent find relation between them.

Examine for functional dependence of

$$u = x + y + z,$$
$$v = x - y + z,$$
$$w = x^2 + y^2 + z^2 + 2xz$$

If functionally dependent find relation between them.

Examine for functional dependence of

$$u = y + z,$$
$$v = x + 2z^2,$$
$$w = x - 4yz - 2y^2$$

If functionally dependent find relation between them.

Examine for functional dependence of

$$u = e^x \sin y\,,$$
$$v = e^x \cos y$$

If functionally dependent find relation between them.

Maxima and Minima

Let $z = f(x, y)$ be a function of two independent variables x, y, then f is said to have a maximum at point (a, b) if $f(a, b) \geq f(x, y) \; \forall \; x, y$ and minimum at point (a, b) if $f(a, b) \leq f(x, y) \; \forall \; x, y$.

A minimum or maximum value of a function is called extreme value.

A point (a, b) is said to be stationary point of $f(x, y)$ if $f_x(a, b) = 0 = f_y(a, b)$.

Every extreme value of a function is attended at stationary point.

Method of finding maxima and minima of a function

Let $z = f(x, y)$. Find f_x , f_y and equate them to zero. Solving these equations simultaneously for x, y gives the stationary points.

Consider $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$.

Find the values of r, s, t at each stationary point.

(a) If $\boldsymbol{rt - s^2 > 0}$ and $\boldsymbol{r < 0}$ at (a, b) then f is maximum at (a, b) and $f_{max} = f(a, b)$.

(b) If $\boldsymbol{rt - s^2 > 0}$ and $\boldsymbol{r > 0}$ at (a, b) then f is minimum at (a, b) and $f_{min} = f(a, b)$.

(c) If $\boldsymbol{rt - s^2 < 0}$ then f take neither maximum nor minimum at (a, b) and such points are called saddle points.

Discuss the maxima and minima of the function

$$3x^2 - y^2 + x^3$$

Discuss the maxima and minima of the function

$$x^3 + y^3 - 3axy$$

Discuss the maxima and minima of the function

$$x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

Discuss the maxima and minima of the function

$$x^3y^2(12 - x - y),\ x, y > 0$$

Method of finding maxima and minima of a function

Let $z = f(x, y)$. Find f_x , f_y and equate them to zero. Solving these equations simultaneously for x, y gives the stationary points.

Consider $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$.

Find the values of r, s, t at each stationary point.

(a) If $\boldsymbol{rt - s^2 > 0}$ and $\boldsymbol{r < 0}$ at (a, b) then f is maximum at (a, b) and $f_{max} = f(a, b)$.

(b) If $\boldsymbol{rt - s^2 > 0}$ and $\boldsymbol{r > 0}$ at (a, b) then f is minimum at (a, b) and $f_{min} = f(a, b)$.

(c) If $\boldsymbol{rt - s^2 < 0}$ then f take neither maximum nor minimum at (a, b) and such points are called saddle points.

Discuss the maxima and minima of the function

$$x^2 + y^2 + 6x + 12$$

Lagrange's method of undermined multipliers

Let $u = f(x, y, z)$ be the given function whose extreme values to be determined under the condition $\phi(x, y, z) = 0$.

Construct $F = u + \lambda\phi$, where λ is a non-zero constant called as Lagrange's multiplier.

Equate $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ to zero.

Solve these equation to get the stationary points (x, y, z).

If $u = \frac{x^2}{a^3} + \frac{y^2}{b^3} + \frac{z^2}{c^3}$, where $x + y + z = 1$ then prove that the stationary point of u is given by $x = \frac{a^3}{a^3+b^3+c^3}$, $y = \frac{b^3}{a^3+b^3+c^3}$, $z = \frac{c^3}{a^3+b^3+c^3}$

Find the maximum value of xy^3/z^2 if $x + y + z = 2$.

Find the maximum and minimum distance of a point $(3,4,12)$ from the sphere $x^2 + y^2 + z^2 = 1$.

A space probe in the shape of ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earths' atmosphere and its surface begins to heat. After one hour, the temperature at point (x, y, z) on the surface of the probe is $T(x, y, z) = 8x^2 + 4yz - 16z + 600,\ x \neq 0$. Find the hottest point on probe's surface.

Error and Approximations

Let $z = f(x, y)$ and $dx,\ dy$ be small changes in x and y respectively then the corresponding change in f is given by

$$df = f(x + dx, y + dy) - f(x, y)$$
$$= \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

dx, dy, df is called as actual errors in x, y, f resp.

$\frac{dx}{x}, \frac{dy}{y}, \frac{df}{f}$ is called as relative errors in x, y, f resp.

$\frac{100dx}{x}, \frac{100dy}{y}, \frac{100df}{f}$ is called as % errors in x, y, f resp.

If the kinetic energy, $k = mv^2/2$, find the approximately the change in K.E. as m changes from 49 to 49.5 and v changes from 1600 to 1590.

The focal length of a mirror is found from $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find the percentage error in f if u and v are both in error by 2% each.

The period of a simple pendulum with small oscillations is $T = 2\pi\sqrt{l/g}$. If T is calculated using $l = 8\ ft, g = 40\ ft/\sec^2$, find % error in calculating T if the % errors in calculating l and g are $0.05\,ft$ and $0.01\,ft/\sec^2$ respectively.

In calculating volume of right circular cylinder, errors of 2% and 1% are found in measuring height and base radius respectively. Find the percentage error in calculating volume of the cylinder.

Find the percentage error in the area of an ellipse, when the errors of 2% and 3% are made in measuring its major and minor axes respectively.

The area of ΔABC is calculated using the formula $\Delta = \frac{1}{2} ab\, sin\, C$. Error of 2%, 3%, 4% are made in measuring a, b, C resp. If the corrected value of C is 30°, find % error in the calculating value of Δ.

Find $[(3.82)^2 + 2(2.1)^3]^{1/5}$ by using the theory of approximations.

Method of finding maxima and minima of a function

Let $z = f(x,y)$. Find f_x , f_y and equate them to zero. Solving these equations simultaneously for x, y gives the stationary points.

Consider $r = f_{xx}$, $s = f_{xy}$, $t = f_{yy}$.

Find the values of $r,\ s,\ t$ at each stationary point.

(a) If $rt - s^2 > 0$ and $r < 0$ at (a,b) then f is maximum at (a,b) and $f_{max} = f(a,b)$.

(b) If $rt - s^2 > 0$ and $r > 0$ at (a,b) then f is minimum at (a,b) and $f_{min} = f(a,b)$.

(c) If $rt - s^2 < 0$ then f take neither maximum nor minimum at (a,b) and such points are called saddle points.

Lagrange's method of undermined multipliers.

In many practical and theoretical problems it is required to find the extreme values of a function of several variable, where the variables are connected by some given relation. Such problems can be solved by using Lagrange's multiplier's method.

Let $u = f(x,y,z)$ be the given function whose extreme values to be determined under the condition $\phi(x,y,z) = 0$.

Construct $F = u + \lambda\phi$, where λ is a non-zero constant called as Lagrange's multiplier.

Equate $\frac{\partial F}{\partial x}$,$\frac{\partial F}{\partial y}$,$\frac{\partial F}{\partial z}$ to zero.

Solve these equation to get the stationary points (x,y,z).

1) Discuss the maxima and minima of the function.

i) $x^2 + 3xy^2 - 15x^2 - 15y^2 + 72x$

ii) $x^3 + y^3 - 3axy$

iii) $3x^2 - y^2 + x^3$

iv) $x^3 + xy^2 - 12x^2 - 2y^2 + 21x$

v) $x^3y^2(12 - x - y),\ x, y > 0.$

vi) $xy(a - x - y)$

vii) $x^2 + y^2 + 6x + 12$

viii) $2(x^2 - y^2) - x^4 + y^4$

ix) $3xy - x^2y - xy^2$

x) $x^3 + 3xy^2 - 3x^2 - 3y^2 - 2$

2) Find the point on the surface $z^2 = xy + 1$ nearest to the origin by using Lagrange's method.

3) A rectangular box open at the top is to have a volume 108 m^3. What must be the dimensions so that the total surface area is minimum?

4) Find the maximum value of xy^3/z^2 if $x + y + z = 2$.

5) Find stationary values of $u = x + y + z$ if $xy + yz + xz = 3a^2$.

6) Use Lagrange's method to find the maximum value of $x^2y^3z^4$, such that $2x + 3y + 4z = 9$.

7) Find the maximum and minimum distance of a point (3,4,12) from the sphere $x^2 + y^2 + z^2 = 1$.

8) The sum of three positive numbers is $'a'$. Determine maximum value of their product.

9) A space probe in the shape of ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earths' atmosphere and its surface begins to heat. After one hour, the temperature at point (x, y, z) on the surface of the probe is $T(x, y, z) = 8x^2 + 4yz - 16z + 600,\ x \neq 0$. Find the hottest point on probe's surface.

10) Divide 120 in to three parts so that the sum of their product taken tow at a time shall be maximum.

11) If $u = \frac{x^2}{a^3} + \frac{y^2}{b^3} + \frac{z^2}{c^3}$, where $x + y + z = 1$ then prove that the stationary value of u is given by $x = \frac{a^3}{a^3+b^3+c^3}$, $y = \frac{b^3}{a^3+b^3+c^3}$, $z = \frac{c^3}{a^3+b^3+c^3}$

12) Using Lagrange's method, divide 24 into three parts such that the continued product of the first, square of the second and cube of the third may be maximum.

Lagrange's method of undermined multipliers

Let $u = f(x, y, z)$ be the given function whose extreme values to be determined under the condition $\phi(x, y, z) = 0$.

Construct $F = u + \lambda\phi$, where λ is a non-zero constant called as Lagrange's multiplier.

Equate $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ to zero.

Solve these equation to get the stationary points (x, y, z).

If $u = \frac{x^2}{a^3} + \frac{y^2}{b^3} + \frac{z^2}{c^3}$, where $x + y + z = 1$ then prove that the stationary point of u is given by $x = \frac{a^3}{a^3+b^3+c^3}$, $y = \frac{b^3}{a^3+b^3+c^3}$, $z = \frac{c^3}{a^3+b^3+c^3}$

Find the maximum value of xy^3/z^2 if $x + y + z = 2$.

Find the maximum and minimum distance of a point $(3,4,12)$ from the sphere $x^2 + y^2 + z^2 = 1$.

A space probe in the shape of ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earths' atmosphere and its surface begins to heat. After one hour, the temperature at point (x, y, z) on the surface of the probe is $T(x, y, z) = 8x^2 + 4yz - 16z + 600,\ x \neq 0$. Find the hottest point on probe's surface.