Ex.3) Expand $(1+x)^x$ *in a series* upto the term containing x^4

Sol. Let y = $(1+x)^x$

Taking log on both sides,

Logy = x log (1+x)

Using standard expansion for log (1+x)

Logy = x $\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right]$

= $\left[x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\cdots\right]$

= z (say)

y = e^z =1+ $z+\frac{z^2}{2!}+\frac{z^3}{3!}$ +……

= 1 +$\left[x^2-\frac{x^3}{2}+\frac{x^4}{3}-\frac{x^5}{4}+\cdots\right]+\frac{1}{2}\left[x^2-\frac{x^3}{2}+\frac{x^4}{3}\right]^2+\frac{1}{6}\left[x^2-\frac{x^3}{2}\right]^3$ +…

Neglecting the higher powers of x we get ,

= 1 + $x^2-\frac{x^3}{2}+\frac{5}{6}x^4-\frac{3x^5}{4}+\cdots$

Ex.3)Expand $(1+x)^{\frac{1}{x}}$ upto the term containing x^2

Ans. e$\left[1-\frac{x}{2}+\frac{11}{24}x^2+\cdots.\right]$

Ex.5) Expand $\sqrt{1+sinx}$ upto x^6.

Sol. Let y =$\sqrt{1+sinx}$

$$= \sqrt{\left(sin^2\frac{x}{2}+cos^2\frac{x}{2}\right)+2sin\frac{x}{2}\ cos\frac{x}{2}}$$

$$= \sqrt{(sin\frac{x}{2}+cos\frac{x}{2})^2}$$

$$= sin\frac{x}{2}+cos\frac{x}{2}$$

$$=\left[\left(\frac{x}{2}\right)-\frac{1}{3!}\left(\frac{x}{2}\right)^3+\frac{1}{5!}\left(\frac{x}{2}\right)^5-\cdots\right]+\left[1-\frac{1}{2!}\left(\frac{x}{2}\right)^2+\frac{1}{4!}\left(\frac{x}{2}\right)^4-\cdots\right]$$

$$=1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}+\frac{x^4}{384}+\frac{x^5}{3840}-\ldots\ldots.$$

Ex.6) Show that $e^{e^x} = e\left(1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \cdots\right)$

Sol.

By exponential series,

$(a+b)^2 = a^2 + 2ab + b^2$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\ldots = 1 + y$

Where $y = x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots$

$\therefore\ e^{e^x} = e^{1+y} = e.\ e^y$

$= e\left(1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} \ldots\ldots\right)$

$= e\left[1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) + \frac{1}{2}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + ..\right)^2 + \frac{1}{6}\left(x + \frac{x^2}{2} + \frac{x^3}{6} + ..\right)^3 + \frac{1}{24}(x + ..)^4 ..\right]$

$= e\left[1 + \left(x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right) + \frac{x^2}{2}\left(1 + \frac{x}{2} + \frac{x^2}{6} + ..\right)^2 + \frac{x^3}{6}\left(1 + \frac{x}{2} + \frac{x^2}{6} + ..\right)^3 + \frac{x^4}{24}(1 + ..)^4 ..\right]$

$= e\left[1 + x + \left(\frac{1}{2} + \frac{1}{2}\right)x^2 + \left(\frac{1}{6} + \frac{1}{2} + \frac{1}{6}\right)x^3 + \left(\frac{1}{24} + \frac{7}{24} + \frac{1}{4} + \frac{1}{24}\right)x^4 + \cdots\right]$

$e^{e^x} = e\left[1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \cdots\right]$

Ex.7) Expand log $\left(\mathbf{1} + \boldsymbol{x} + \boldsymbol{x}^{\mathbf{2}} + \boldsymbol{x}^{\mathbf{3}}\right)$ upto a term in $\boldsymbol{x}^{\mathbf{8}}$.

Sol. Let f(x) = log $(1 + x + x^2 + x^3)$

$$= \log\left[\frac{(1+x+x^2+x^3)(1-x)}{(1-x)}\right]$$

$$= \log\left[\frac{1-x^4}{1-x}\right]$$

$$= \log(1 - x^4) - \log(1 - x)$$

$$= \left[-x^4 - \frac{x^8}{2}\right] - \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \frac{x^8}{8} - \cdots\right]$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} \ldots$$

Ex .7) Prove that log $(1 + x + x^2 + x^3 + x^4) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} - \frac{4}{5}x^5 + \ldots\ldots$

Ex. 8) Expand 40 + $53(x-2)+19(x-2)^2+2(x-2)^3$ in ascending powers at x.

Sol. $-6+x+7x^2+2x^3$

Ex. 8) Using Taylor's theorem express $7 + (x+2)^2 + 3(x+2)^3 + (x+2)^4$ in ascending powers at x.

Sol. : let $f(x+h) = 7 + (x+h)^2 + 3(x+h)^3 + (x+h)^4$

$$f(x) = 7 + x + 3x^3 + x^4$$

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \ldots$$

Put $h = 2, then$

$$f(x+2) = f(2) + xf'(2) + \frac{x^2}{2!}f''(2) + \frac{x^3}{3!}f'''(2) + \frac{x^4}{4!}f^{iv}(2) + \ldots \qquad (1).$$

$f(x) = 7 + x + 3x^3 + x^4$	$f(2) = 49$
$f'(x) = 1 + 9x^2 + 4x^3$	$f'(2)$=69
$f''(x)$ = $18x + 12x^2$	$f''(2)$ =84
$f'''(x)$ = 18+24x	$f'''(2)$ =66
$f^{iv}(x)$ = 24	$f^{iv}(x)$ = 24
$f^{v}(x)$=0	

Substituting in (1) we have

$$f(x+2) = 49 + x.69 + \frac{x^2}{2!}.84 + \frac{x^3}{3!}.66 + \frac{x^4}{4!}.24 + 0\ldots$$

$$f(x+2) = 49 + 69x + 42x^2 + 11x^3 + x^4$$

Ex. 9) Using Taylor's series express 5 + $4(x-1)^2 - 3(x-1)^3 + (x-1)^4$ in ascending powers at x.

Sol. : let $f(x+h)$ = 5 + $4(x-1)^2 - 3(x-1)^3 + (x-1)^4$

Put h = -1

By Taylor's theorem ,

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!}f''(h) + \frac{x^3}{3!}f'''(h) + \frac{x^4}{4!}f^{iv}(h) + \ldots \ldots$$

$f(x) = 5 + 4x^2 - 3x^3 + x^4$ $\qquad f(-1) = 13$

$f'(x) = 8x - 9x^2 + 4x^3$ $\qquad f'(-1)$=-21

$f''(x)$ = 8-18$x + 12x^2$ $\qquad f''(-1)$ =38

$f'''(x)$ = -18+24x $\qquad f'''(-1)$ =-42

$f^{iv}(x)$ = 24 $\qquad f^{iv}(-1)$ = 24

$$\therefore \text{we get } f(x+h) = f(-1) + xf'(-1) + \frac{x^2}{2!}f''(-1) + \frac{x^3}{3!}f'''(-1) + \frac{x^4}{4!}f^{iv}(-1) + \ldots$$

= $13 - 21x$+19$x^2 - 7x^3 + x^4$

Ex.10) $\boldsymbol{Expand\ x^4 - 3x^3 + 2x^2 - x + 1\ in\ power\ of\ (x-3)}$.

Sol. By Taylor's theorem ,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots$$

$$f(x) = f(3) + (x-3)f'(3) + \frac{(x-3)^2}{2!}f''(3) + \ldots\ldots \qquad (1)$$

$f(x) = x^4 - 3x^3 + 2x^2 - x + 1$ $\qquad$ $f(3) = 16$

$f'(x) = 4x^3 - 9x^2 + 4x - 1$ $\qquad$ $f'(3) = 38$

$f''(x) = 12x^2 - 18x + 4$ $\qquad$ $f''(3) = 58$

$f^{iv}(x) = 24$ $\qquad$ $f^{iv}(3) = 24$

$f^{v}(x) = 0$ $\qquad$ $f^{v}(3) = 0$

Substituting in (1) ,we have

$$f(x) = 16 + 38(x-3) + \frac{58(x-3)^2}{2!} + \frac{54(x-3)^3}{3!} + 24\frac{(x-3)^4}{4!} + 0$$

$$f(x) = 16 + 38(x-3) + 29(x-3)^2 + 9(x-3)^3 + (x-3)^4$$

Ex.11) $\boldsymbol{Expand\ 3x^3 - 2x^2 + x - 4\ in\ powers\ of\ (x+2)\ using}$ **Taylors theorem.**

Sol. . By Taylor's theorem ,

$$f(x) = f(a) + (x\text{-}a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots\ldots$$

$f(x) = 3x^3 - 2x^2 + x - 4$ $\qquad f(-2) = -38$

$f'(x) = 9x^2 - 4x + 1$ $\qquad f'(-2) = 45$

$f''(x) = 18x - 4$ $\qquad f''(-2) = 40$

$f''(x) = 18$ $\qquad f''(-2) = 18$

$$f(x) = -38 + 45(x+2) - 20\,(x+2)^2 + 3(x+2)^3$$

Ex.12) Use Taylor's theorem to $\sqrt{25.15}$

Sol.Let $f(x+h) = \sqrt{x+h} = \sqrt{25.15} = \sqrt{25+0.15}$

$\therefore$ $f(x) = \sqrt{x}$ and h = 0.15 , x =25

$f'(x) = \frac{1}{2\sqrt{x}}$, $f''(\mathrm{x}) = -\frac{1}{4}x^{-3/2}$, $f'''(\mathrm{x}) = \frac{3}{8}x^{-5/2}$

$\therefore$ By Taylor's theorem

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots.$

$= \sqrt{x} + \frac{h}{2}\frac{1}{\sqrt{x}} - \frac{h^2}{8(\sqrt{x}^3)} + \frac{h^3}{16(\sqrt{x}^5)}$

Put x= 25 and h= 0.15

Let $f(x+h) = \sqrt{25.15} = 5 + \frac{0.15}{2} \times \frac{1}{5} - \frac{0.15^2}{8(5)^3} + \frac{0.15^3}{16(5)^5} = \sqrt{25.15}$ =5.01478

INDETERMINATE FORMS

Let f(x) and g(x) be any two functions of x such that f(a) = 0 and g(a) = 0,then the ratio $\frac{f(x)}{g(x)}$ is said to assume The indeterminate form $\frac{0}{0}$ at x= a.

There are seven indeterminate forms, $\frac{0}{0}$, $\frac{\infty}{\infty}, 0 \times \infty,\ \infty - \infty,\ \infty^0, 1^\infty,\ 0^0$

*L'Hospital's Rule :

Let f(x) and g(x) be functions of x such that f(a) = 0 and g(a) = 0,

i.e. $\lim_{x \to a} f(x) = 0 \ \ and \ \ \lim_{x \to a} g(x) = 0$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided derivative of f(x) and g(x) exists.

$= \frac{f'(a)}{g'(a)}$ (g'(a) $\neq 0$).

Important Formulae of limits :

1) $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$

2) $\lim\limits_{x\to 0}\dfrac{\tan x}{x} = 1$

3) $\lim\limits_{x\to 0}\dfrac{\sin^{-1} x}{x} = 1$

4) $\lim\limits_{x\to 0}(1+x)^{1/x} = e$

5) $\lim\limits_{x\to 0}\dfrac{a^{x}-1}{x} = \log a$

6) $\lim\limits_{x\to 0}\dfrac{e^{x}-1}{x} = 1$

1) Evaluate $\displaystyle\lim_{x\to 0} \frac{e^{ax}-e^{-ax}}{\log(1+bx)}$

Let ,

$$L = \lim_{x\to 0} \frac{e^{ax}-e^{-ax}}{\log(1+bx)} \qquad \ldots\ldots\left(\frac{0}{0} form\right)$$

$\therefore$ *Applying L'hospital Rule , we get*

$$= \lim_{x\to 0} \frac{ae^{ax}-e^{-ax}(-a)}{\frac{1}{(1+bx)}(b)}$$

$$= \lim_{x\to 0} \frac{ae^{ax}+ae^{-ax}}{\left(\frac{b}{1+bx}\right)}$$

$$= \frac{a+a}{b}$$

$$= \frac{2a}{b}$$

2) Evaluate $\lim\limits_{x \to 0} \frac{(1+x)^n - 1}{x}$

Sol. Let L = $\lim\limits_{x \to 0} \frac{(1+x)^n - 1}{x}$ .... $\left(\frac{0}{0}\ form\right)$

$\therefore\ Applying\ \ L'hospital\ Rule\ , we\ get$

L = $\lim\limits_{x \to 0} \frac{n(1+x)^{n-1}}{1}$

= $n(1+0)^{n-1}$

= n

3) Evaluate $\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$

Sol. Let L = $\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$ .... $\left(\frac{0}{0}\ form\right)$

By $L'hospital\ Rule$,

L = $\lim\limits_{x\to 0} \dfrac{(xe^x + e^x) - \frac{1}{1+x}}{2x}$ .... $\left(\frac{0}{0}\ form\right)$

L = $\lim\limits_{x\to 0} \dfrac{(xe^x + e^x + e^x) + 1/(1+\mathrm{x})^2}{2}$

L = $\dfrac{1+1+1}{2}$

= $\dfrac{3}{2}$

Ex.4) If $\lim\limits_{x\to 0}\frac{\sin 2x+p\sin x}{x^3}$ is finite then find the value of p and hence the value of limit.

Sol. $\lim\limits_{x\to 0}\frac{\sin 2x+p\sin x}{x^3}$

$$= \lim_{x\to 0}\frac{\sin 2x + p\sin x}{x^3}$$

$$= \lim_{x\to 0}\frac{sinx}{x}\times\frac{2cosx+p}{x^2}$$

$$= \lim_{x\to 0}\frac{2cosx+p}{x^2} \qquad \left(\because \lim_{x\to 0}\frac{sinx}{x} = 1\right)$$

Here the denominater being zero for x = 0 and numerator becomes 2+p. Therefore,if the limit is to be finite, The numerator must be zero for x = 0.this requires

$$2+p=0 \qquad \Longrightarrow \qquad p = -2$$

With this value of p ,required limit

$$= \lim_{x\to 0}\frac{2cosx-2}{x^2} \qquad \left(form\ \frac{0}{0}\right)$$

$$= \lim_{x\to 0}\left(-\frac{sinx}{x}\right) \qquad \text{by L'hospital rule}$$

=-1

$$\therefore\ p = -2\ and\ limit = -1$$

Ex.5) Evaluate $\lim\limits_{x\to 0} \dfrac{\log tanx}{logx}$

Solutin: $\lim\limits_{x\to 0} \dfrac{\log tanx}{logx}$ $\left(form\ \frac{\infty}{\infty}\right)$

$\therefore\ Applying\ \ L'hospital\ Rule\ ,we\ get$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan x}\ \ sec^2 x}{\frac{1}{x}} \qquad \left(form\ \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{x}{\sin x \cos x} \qquad \left(form\ \frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{2x}{\sin 2x}$$

= 1

Ex.6) Evaluate $\lim_{x \to 0} sinxlogx$.

Sol. $\lim_{x \to 0} sinxlogx \qquad (form\ 0 \times \infty)$

$$= \lim_{x \to 0} \frac{logx}{cosecx} \qquad \left(form \frac{\infty}{\infty}\right)$$

$$= \lim_{x \to 0} \frac{1/x}{-cosecxcotx} = -\lim_{x \to 0} \frac{sin^2x}{xcosx} \qquad \left(form \frac{0}{0}\right)$$

$$= -\lim_{x \to 0} \frac{2sinxcosx}{cosx - xsinx} \qquad (by\ Lhospital\ rule)$$

$$= -\left(\frac{0}{1-0}\right) = 0$$

Ex.7) Evaluate $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{logx}\right]$

Sol. $\lim_{x\to 1}\left[\frac{x}{x-1}-\frac{1}{logx}\right]$ $\qquad (form\ \infty-\infty)$

$$=\lim_{x\to 1}\left[\frac{xlogx-x+1}{(x-1)logx}\right] \qquad \left(form\ \frac{0}{0}\right)$$

$$=\lim_{x\to 1}\left[\frac{(1+logx)-1}{\frac{(x-1)}{x}+logx}\right] \qquad (by\ Lhospital\ rule)$$

$$=\lim_{x\to 1}\left[\frac{logx}{1-\frac{1}{x}+logx}\right] \qquad \left(form\ \frac{0}{0}\right)$$

$$=\lim_{x\to 1}\left[\frac{\left(\frac{1}{x}\right)}{\frac{1}{x^2}+\frac{1}{x}}\right] \qquad (by\ Lhospital\ rule)$$

$$=\frac{1}{2}$$

Ex.8) Evaluate $\lim_{x \to \frac{\pi}{2}} secx^{cotx}$

Sol. $\lim_{x \to \frac{\pi}{2}} secx^{cotx}$

Ex.9) Evaluate $\lim_{x\to\frac{\pi}{2}} cosx^{cosx}$

Sol. Let L = $\lim_{x\to\frac{\pi}{2}} cosx^{cosx}$ $(form 0^0)$

Log L= $\lim_{x\to\frac{\pi}{2}}$ cosx log(cosx) $(form 0\times\infty)$

$$= \lim_{x\to\frac{\pi}{2}} \frac{\log(cosx)}{secx} \qquad \left(form\ \frac{\infty}{\infty}\right)$$

$$=\lim_{x\to\frac{\pi}{2}} \frac{(-sinx/cosx)}{(\sec x\, tanx)}$$

= $\lim_{x\to\frac{\pi}{2}}$ (-cosx) = 0

L = e^0 = 1

Ex.10) Solve $\lim_{x\to 0}\left(\frac{2^x+3^x}{2}\right)^{1/x}$

Sol. Let L= $\lim_{x\to 0}\left(\frac{2^x+3^x}{2}\right)^{1/x}$ $\quad (form 1^{\infty})$

Log L = $\lim_{x\to 0}\frac{1}{x}.\log\left(\frac{2^x+3^x}{2}\right)$ $\quad (form \infty\times 0)$

=

$\lim_{x\to 0}\frac{\log\left(\frac{2^x+3^x}{2}\right)}{x}$ $\quad form\left(\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{\left(\frac{2}{2^x+3^x}\right)\left(\frac{2^x log2+3^x log3}{2}\right)}{1}$ $\quad (by\ L'hospital\ rule)$

= $\lim_{x\to 0}\frac{2^x log2+3^x log3}{2^x+3^x}$

= $\frac{log2+log3}{2}$ = $\frac{1}{2}log6 = log\sqrt{6}$

L = $e^{log\sqrt{6}}$ = $\sqrt{6}$

Ex.11) Evaluate $\lim\limits_{x \to \infty}\left[\dfrac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right]^{x}$

Sol. Put $\frac{1}{x} = y, then\ y \to 0\ as\ x \to \infty\ and$

$$\lim_{x\to\infty}\left[\frac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right]^{x} = \lim_{y\to0}\left[\frac{a^{y}+b^{y}+c^{y}}{3}\right]^{1/y}$$

Let L = $\lim\limits_{y\to0}\left[\dfrac{a^{y}+b^{y}+c^{y}}{3}\right]^{1/y}$

$$\log L = \lim_{y\to0}\frac{1}{y}\ log\left[\frac{a^{y}+b^{y}+c^{y}}{3}\right] = \lim_{y\to0}\frac{log\left[\frac{a^{y}+b^{y}+c^{y}}{3}\right]}{y} \qquad ....\left(\frac{0}{0}form\right)$$

Apply $L'hospital\ rule$

$$=\lim_{y\to0}\frac{\left(\frac{3}{a^{y}+b^{y}+c^{y}}\right)\left(\frac{a^{y}loga+b^{y}logb+c^{y}logc}{3}\right)}{1}$$

$$=\lim_{y\to0}\frac{a^{y}loga+b^{y}logb+c^{y}logc}{a^{y}+b^{y}+c^{y}} = \lim_{y\to0}\frac{loga+logb+logc}{1+1+1}$$

$$=\frac{logabc}{3} = \log abc^{1/3}$$

$$\therefore L = abc^{1/3}$$