Example: If $u = \sec^{-1}\left[\frac{x+y}{x^{1/2}+y^{1/2}}\right]$ then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\cot u}{4}[3+\cot^2 u]$$

****

Example: If $u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$ then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u\,[1-4\sin^2 u]$$

****

Example: If $u = \sin^{-1}\left[\frac{x+y}{x^{1/2}+y^{1/2}}\right]$ then Prove that

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = \frac{-\sin u\cos 2u}{4\cos^3 u}$$

****

Example: If $u = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]^{\frac{1}{2}}$ then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{4}[\tan^2 u - 1]$$

****

UNIT-IV: JACOBIAN AND THEIR APPLICATIONS

Definition and properties of Jacobian & Illustrations.
Jacobian:

If u and v are differentiable functions of independent variables x and y , then the determinant

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u and v with respect to x and y.

Denoted by $J = \frac{\partial(u.v)}{\partial(x,y)}$

Thus, $\frac{\partial(u.v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Similarly, If u, v and w are differentiable functions of independent variables x ,y and z then

$\frac{\partial(u.v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$ and so on.

Properties of Jacobian:

1. If $J = \frac{\partial(u.v)}{\partial(x,y)}$ then $J' = \frac{\partial(x,y)}{\partial(u.v)}$ is called reciprocal of J.
2. $JJ' = 1$
3. .Chain rule **:**
   If u and v are functions of x and y , and x and y are functions of r and θ ,

   then $\frac{\partial(u.v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(r,\theta)}$ is chain rule for Jacobian.

Examples :

1.If $= rsin\theta cos\phi\ , y = rsin\theta sin\phi\ , z = rcos\theta$ then, find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

Soln **:** We have, $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$= \begin{vmatrix} sin\theta cos\phi & rcos\theta cos\phi & -rsin\theta sin\phi \\ sin\theta sin\phi & rcos\theta sin\phi & rsin\theta cos\phi \\ cos\theta & -rsin\theta & 0 \end{vmatrix}$$

$$= r^2 sin\theta$$

2. If $x = a\ coshu\ cosv,\ y = a\ sinhu\ sinv$, show that

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{a^2}{2}(cosh2u - cos2v)$$

Soln **:** We have , , $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$= \begin{vmatrix} a\ sinhu\ cosv & -a\ coshu\ sinv \\ a\ coshu\ sinv & a\ sinhu\ cosv \end{vmatrix}$$

$$= a^2(sinh\,^2u\ cos^2v\ +\ cosh\,^2u\ sin^2v)$$

Using ,$sinh\,^2u = \frac{cosh2u-1}{2}$, $cos^2v = \frac{1+cos2v}{2}$

$cosh\,^2u = \frac{cosh2u+1}{2}$, $sin^2v = \frac{1-cos2v}{2}$

We get ,

$$\frac{\partial(x,y)}{\partial(u,v)} = a^2\left[\left(\frac{cosh2u-1}{2}\right)\left(\frac{1+cos2v}{2}\right) + \left(\frac{cosh2u+1}{2}\right)\left(\frac{1-cos2v}{2}\right)\right]$$

Examples on $JJ^{'} = 1$

1) Verify $JJ^{'} = 1$ for the following

$x = v^2 + w^2 \quad y = u^2 + w^2 \quad z = u^2 + v^2$

Solution:- Let $x = v^2 + w^2$ ………………..(1)

$y = u^2 + w^2$ ……………….. (2)

$z = u^2 + v^2$ ………………….(3)

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix} = 16uvw$$

$$J = 16uvw$$

using equations (1), (2), & (3) we can find $J^{'}$

$$\therefore J^{'} = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-1}{4u} & \frac{1}{4u} & \frac{1}{4u} \\ \frac{1}{4v} & \frac{-1}{4v} & \frac{1}{4v} \\ \frac{1}{4w} & \frac{1}{4w} & \frac{-1}{4w} \end{vmatrix}$$

$$= \left(\frac{1}{4u}\right)\left(\frac{1}{4v}\right)\left(\frac{1}{4w}\right)\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = \frac{1}{16uvw}$$

$$\therefore \quad JJ^{'} = 1$$

3) $If \quad x=u,\ y=uv. \quad show \quad that \quad JJ' = 1$

Soln:-Given $x = u(1-v) \quad y = uv$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u(1-v) + uv = u$$

$x = u - uv \quad and \quad y = uv$

$\therefore x = u - y \quad \therefore u = x + y,$

$$\therefore y = (x+y)v, \quad \therefore v = \frac{y}{x+y}$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{1}{u}$$

$$\therefore JJ' = \frac{\partial(x,y)}{\partial(u,v)} * \frac{\partial(u,v)}{\partial(x,y)} = u * \frac{1}{u} = 1$$

4) Verify $JJ' = 1$ for the transformation $x = uv, \quad y = \frac{u}{v}$

Solution:-Given $x = uv, \quad y = \frac{u}{v}$

Let $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix} = \frac{-uv}{v^2} - \frac{u}{v} = \frac{-2u}{v}$

$\therefore x = uv \quad and \quad y = \frac{u}{v} \quad \therefore u = vy$

$$\therefore x = vy \Rightarrow \frac{x}{y} = v^2$$

$$\therefore v = \sqrt{\frac{x}{y}} \ \therefore u = y\sqrt{\frac{x}{y}} = \sqrt{xy}$$

$$\therefore J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{xy}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix} = \frac{-1}{2y}$$

But $y = \frac{u}{v} \quad \therefore J' = \frac{-1}{\frac{2u}{v}} = \frac{-v}{2u}$

$\therefore JJ' = 1$

5) $x = v^2 + w^2,\ y = w^2 + u^2,\ z = u^2 + v^2,\ \ prove\ \ that\ JJ' = 1$

Solution:- we have

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix} = 16uvw \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Since from given information we can write

$$\therefore u^2 + v^2 + w^2 = \frac{1}{2}(x+y+z) = \frac{x}{2} + \frac{y}{2} + \frac{z}{2}$$

$$\therefore u^2 = -\frac{x}{2} + \frac{y}{2} + \frac{z}{2},\ \ v^2 = \frac{x}{2} - \frac{y}{2} + \frac{z}{2},\ \ w^2 = \frac{x}{2} + \frac{y}{2} - \frac{z}{2}$$

$$\therefore J' = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{16uvw} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) & (2) we can write

$$\therefore \quad JJ' = 1$$

Illustrations on Jacobian of composite function

Jacobian of composite function

Chain rule :

If u and v are functions of x and y , and x and y are functions of r and θ , then

$\frac{\partial(u.v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(r,\theta)}$ is chain rule for Jacobian.

Note :

1. If $J = \frac{\partial(u.v)}{\partial(x,y)}$ then $J' = \frac{\partial(x,y)}{\partial(u.v)}$ is called reciprocal of J.
2. $JJ' = 1$

Examples :

1. If $u = e^x(x\,cosy - y\,siny), v = e^x(x\,siny + y\,cosy)$,
Where $= l\xi + m\eta$, $y = l\eta - m\xi$, find $\frac{\partial(u.v)}{\partial(\xi,\eta)}$.
Soln: Here $u, v \to x, y \to \xi, \eta$

$$\frac{\partial(u.v)}{\partial(\xi,\eta)} = \frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(\xi,\eta)}$$

$$= \begin{vmatrix} e^x(x\,cosy - y\,siny + cosy) & e^x(-x\,siny - y\,cosy - siny) \\ e^x(x\,siny + y\,cosy + siny) & e^x(x\,cosy + cosy - ysiny) \end{vmatrix} \begin{vmatrix} l & m \\ -m & l \end{vmatrix}$$

$$= e^{2x}(l^2 + m^2)[(x+1)^2 + y^2]$$

2. If $x = r\,cos\theta,\ y = r\,sin\theta$ then evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$.

Soln : Given that $x = r\,cos\theta, y = r\,sin\theta$ then $r^2 = x^2 + y^2$ and $\theta = tan^{-1}\frac{y}{x}$.

$$\therefore \frac{\partial x}{\partial r} = cos\theta, \quad \frac{\partial x}{\partial \theta} = -r\,sin\theta, \qquad \frac{\partial y}{\partial r} = sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\,cos\theta$$

Hence, $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -r\,sin\theta \\ sin\theta & r\,cos\theta \end{vmatrix} = r$

And $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial \theta}{\partial x} = \frac{-y}{r^2}$, $\frac{\partial \theta}{\partial y} = \frac{x}{r^2}$

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix} = \frac{1}{r}$$

3. For transformations $x = e^u\,cosv$, $y = e^u\,sinv$,

$$prove\ that\ \frac{\partial(x,y)}{\partial(u,v)} . \frac{\partial(u.v)}{\partial(x,y)} = 1$$

Soln: Given, $x = e^u\,cosv$, $y = e^u\,sinv$.

We first find

$$J = \frac{\partial(x,y)}{\partial(u.v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} e^u\,cosv & -e^u\,sinv \\ e^u\,sinv & e^u\,cosv \end{vmatrix}$$

$= e^{2u}$------ (1)

Since, $x = r\,cos\theta, y = r\,sin\theta$ $then\ v = tan^{-1}\frac{y}{x}$ $and\ u = \frac{1}{2}\log(x^2 + y^2)$

Next, we find

$J' = \frac{\partial(u.v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{x^2+y^2} = \frac{1}{e^{2u}}$------- (2)

From (1) and (2), we have

$$JJ' = \frac{\partial(x,y)}{\partial(u,v)} . \frac{\partial(u.v)}{\partial(x,y)} = e^{2u}\frac{1}{e^{2u}} = 1$$

Partial derivatives of implicit functions using Jacobian:

Definition : Implicit function or relation is a relationin which one variable can expressed as function of others, provided partial derivative with respect to that variable is non-zero.

Ex.1. for $f(x,y) = 0$ containing two variables ,y can be function of x , if $\frac{\partial f}{\partial y} \neq 0$.

2.for$f(x,y,z) = 0$ containing three variables , z can be function of x and y , if $\frac{\partial f}{\partial z} \neq 0$.

above two are implicit functions.

Consider an implicit relation$f(u,v,w,x) = 0$,this defines x as a function of u, v, w

If $\frac{\partial f}{\partial x} \neq 0$. (i.e. u, v, w are independent and x is dependent variable)

Then ,$\frac{\partial x}{\partial u} = -\frac{\partial f/_{\partial u}}{\partial f/_{\partial x}}$, $\frac{\partial x}{\partial v} = -\frac{\partial f/_{\partial v}}{\partial f/_{\partial x}}$, $\frac{\partial x}{\partial w} = -\frac{\partial f/_{\partial w}}{\partial f/_{\partial x}}$.

If$f_1(u, v, x, y) = 0$,$f_2(u, v, x, y) = 0$ are two implicit relations these relations define define $u\ and\ v$ as functions of independent variables $x\ and\ y$ provided

$\frac{\partial f_1}{\partial u} \neq 0$; $\frac{\partial f_1}{\partial v} \neq 0$; $\frac{\partial f_2}{\partial u} \neq 0$;$\frac{\partial f_2}{\partial v} \neq 0$

Therefore we get four partial derivative ,$\frac{\partial u}{\partial x}$,$\frac{\partial u}{\partial y}$,$\frac{\partial v}{\partial x}$,$\frac{\partial v}{\partial y}$ as below ,

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1,\ f_2)}{\partial(x,\ v)}}{\frac{\partial(f_1,\ f_2)}{\partial(u,\ v)}}\ ,\ \frac{\partial u}{\partial y} = -\frac{\frac{\partial(f_1,\ f_2)}{\partial(y,\ v)}}{\frac{\partial(f_1,\ f_2)}{\partial(u,\ v)}}\ ,$$

$$\frac{\partial v}{\partial x} = -\frac{\frac{\partial(f_1,\ f_2)}{\partial(u,\ x)}}{\frac{\partial(f_1,\ f_2)}{\partial(u,\ v)}}\ ,\quad \frac{\partial v}{\partial y} = -\frac{\frac{\partial(f_1,\ f_2)}{\partial(u,\ y)}}{\frac{\partial(f_1,\ f_2)}{\partial(u,\ v)}}$$

Note :

1. Each of the above derivatives has same denominator and is a Jacobian of dependent variables. Therefore these derivatives exist only if $\frac{\partial(f_1,\ f_2)}{\partial(u,\ v)} \neq 0$

2.If $u = f_1(x, y, z)$; $v = f_2(x, y, z)$; $w = f_3(x, y, z)$

Take ,$F_1 \equiv u - f_1(x, y, z) = 0$,$F_2 \equiv v - f_2(x, y, z) = 0$,$F_3 \equiv w - f_3(x, y, z) = 0$

Then, $\frac{\partial x}{\partial u} = (-1)\frac{\frac{\partial(F_1,\ F_2,\ F_3)}{\partial(u,\ y,\ z)}}{\frac{\partial(F_1,\ F_2,\ F_3)}{\partial(x,\ y,\ z)}}$(i.e. replace x by u)

$\frac{\partial y}{\partial w} = (-1)\frac{\frac{\partial(F_1,\ F_2,\ F_3)}{\partial(x,\ w,\ z)}}{\frac{\partial(F_1,\ F_2,\ F_3)}{\partial(x,\ y,\ z)}}$(i.e. replace y by w)

Examples :

1.) If $u^2 + xv^2 - uxy = 0,\quad v^2 - xy^2 + 2uv + u^2 = 0$, find $\frac{\partial u}{\partial x}$ by proper choice of dependent and independent variables.

Soln : Let $f_1 = u^2 + xv^2 - uxy = 0$, $f_2 = v^2 - xy^2 + 2uv + u^2 = 0$

$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1,\ f_2)}{\partial(x,\ v)}}{\frac{\partial(f_1,\ f_2)}{\partial(u,\ v)}}$ ....[1]

$\frac{\partial(f_1,\ f_2)}{\partial(x,\ v)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix} = \begin{vmatrix} v^2 - uy & 2xv \\ -y^2 & 2v + 2u \end{vmatrix}$

$= (v^2 - uy)(2v + 2u)(2xy^2v)$**...................................[2]**

$\frac{\partial(f_1,\ f_2)}{\partial(u,\ v)} = \begin{vmatrix} 2u - xy & 2xv \\ 2v + 2u & 2v + 2u \end{vmatrix} = 2(u + v)[2u - xy - 2xv]$**..................[3]**

Putting these values in (1) , we get

$$\frac{\partial u}{\partial x} = -\frac{(v^2 - uy)(2v + 2u)(2xy^2v)}{2(u + v)[2u - xy - 2xv]}$$

2) If $x = cos\,\theta - rsin\,\theta$, $y = \sin\theta + rcos\,\theta$, prove that

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Soln : Here$f_1 = x - cos\,\theta + rsin\,\theta$, $f_2 = y - \sin\theta + rcos\,\theta$,

$$\frac{\partial r}{\partial x} = -\frac{\frac{\partial(f_1,f_2)}{\partial(x,\theta)}}{\frac{\partial(f_1,f_2)}{\partial(r,\theta)}} = \frac{\begin{vmatrix} 1 & \sin\theta + rcos\,\theta \\ 0 & -cos\,\theta + rsin\,\theta \end{vmatrix}}{\begin{vmatrix} sin\,\theta & \sin\theta + rcos\,\theta \\ -cos\,\theta & -cos\,\theta + rsin\,\theta \end{vmatrix}}$$

$$\frac{\partial r}{\partial x} = \frac{cos\,\theta - rsin\,\theta}{r} = \frac{x}{r}$$

3) **.**If $x = u + v + w, y = u^2 + v^2 + w^2$,$z = u^3 + v^3 + w^3$ then show that

$$\frac{\partial u}{\partial x} = \frac{uv}{(u-v)(u-w)}$$

Soln : Here , $f_1 = x - u + v + w$

$$f_2 = y - u^2 + v^2 + w^2$$

$$f_3 = z - u^3 + v^3 + w^3$$

$$\frac{\partial u}{\partial x} = -\frac{\partial(f_1,f_2,f_3)\big/\partial(x,v,w)}{\partial(f_1,f_2,f_3)\big/\partial(u,v,w)} = \frac{\begin{vmatrix} 1 & -1 & -1 \\ 0 & -2v & -2w \\ 0 & -3v^2 & -3w^2 \end{vmatrix}}{\begin{vmatrix} -1 & -1 & -1 \\ -2u & -2v & -2w \\ -3u^2 & -3v^2 & -3w^2 \end{vmatrix}}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{uv}{(u-v)(u-w)}$$

Now $f_1 = u + v + w - x - y - z = 0$ $f_2 = uv + vw + uw - x^2 - y^2 - z^2 = 0$

$$f_3 = uvw - \left(\frac{x^3 + y^3 + z^3}{3}\right) = 0$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -1\frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

4) If $u = xyz, \; v = x^2 + y^2 + z^2, \; w = x + y + z$ then, find $\frac{\partial x}{\partial u}$

Soln. : $f_1 = xyz - u = 0,$

$f_2 = x^2 + y^2 + z^2 - v = 0,$

$f_3 = x + y + z - w = 0$

Group 2

$$\therefore \frac{\partial x}{\partial u} = \frac{\partial(f_1,f_2,f_3)/\partial(u,y,z)}{\partial(f_1,f_2,f_3)/\partial(x,y,z)} = \frac{-\begin{vmatrix} -1 & xz & xy \\ 0 & 2y & 2z \\ 0 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}}$$

$$= \frac{2(y-z)}{2(x-y)(y-z)(x-z)}$$

$$\frac{\partial x}{\partial u} = \frac{1}{(x-y)(x-z)}$$

Illustrations on Partial derivative of implicit function by using Jacobian

1. If $u^2 + xv^2 - uxy = 0, \; , \; v^2 - xy^2 + 2uv + u^2 = 0,$ find $\frac{\partial u}{\partial x}$ by proper choice of dependent and independent variables.

Soln: Let $f_1 = u^2 + xv^2 - uxy = 0, \quad f_2 = v^2 - xy^2 + 2uv + u^2 = 0$

Now $\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1, f_2)}{\partial(x,v)}}{\frac{\partial(f_1, f_2)}{\partial(u,v)}}$ ……………………………………..(1)

$$\frac{\partial(f_1,f_2)}{\partial(x,v)}=\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v}\end{vmatrix}=\begin{vmatrix}v^2-uy & 2xy\\ -y^2 & 2v+2u\end{vmatrix}$$

$$=(v^2-uy)(2v+2u)+2xy^2v \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\partial(f_1,f_2)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}=\begin{vmatrix}2u-xy & 2xv\\ 2v+2u & 2v+2u\end{vmatrix}=2(u+v)[2u-xy-2xv]\ldots\ldots(3)$$

Putting these values in (1), we get,

$$\frac{\partial u}{\partial x}=-\frac{(v^2-uy)(2v+2u)+2xvy^2}{2(u+v)(2u-xy-2xv)}$$

2) *If* $u^2+xv^2-uxy=a,$ and $x^2+y^2+z^2+u^2+v^2=b$ where a,b constants, use Jacobians to find $\left(\frac{\partial v}{\partial y}\right)_{x,u}$ and $\left(\frac{\partial y}{\partial v}\right)_{x,z}$.

Soln: Let $\begin{array}{l} f_1=x+y+z+u+v-a=0\\ f_2=x^2+y^2+z^2+u^2+v^2-b=0\end{array}$

$$\left(\frac{\partial v}{\partial y}\right)_{x,u}=-\frac{\frac{\partial(f_1,f_2)}{\partial(y,z)}}{\frac{\partial(f_1,f_2)}{\partial(v,z)}}=-\frac{\begin{vmatrix}1 & 1\\ 2y & 2z\end{vmatrix}}{\begin{vmatrix}1 & 1\\ 2v & 2u\end{vmatrix}}=\frac{y-z}{z-v}$$

$$\left(\frac{\partial v}{\partial v}\right)_{x,z}=-\frac{\frac{\partial(f_1,f_2)}{\partial(v,u)}}{\frac{\partial(f_1,f_2)}{\partial(y,u)}}=-\frac{\begin{vmatrix}1 & 1\\ 2v & 2u\end{vmatrix}}{\begin{vmatrix}1 & 1\\ 2y & 2u\end{vmatrix}}=\frac{u-v}{y-u}$$

3).If $x=u+v+w,\ \ y=u^2+v^2+w^2,\ \ z=u^3+v^3+w^3$ then show that

$$\frac{\partial u}{\partial x}=\frac{vw}{(u-v)(u-w)}$$

Soln: Here $\begin{array}{l} f_1=x-u-v-w\\ f_2=y-u^2-v^2-w^2\\ f_3=x-u^3-v^3-w^3\end{array}$

$$\frac{\partial u}{\partial x}=-\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}=-\frac{\begin{vmatrix}1 & -1 & -1\\ 0 & -2v & -2w\\ 0 & -3v^2 & -3w^2\end{vmatrix}}{\begin{vmatrix}-1 & -1 & -1\\ -2u & -2v & -2w\\ -3u^2 & -3v^2 & -3w^2\end{vmatrix}}$$

$$= \frac{vw(w-v)}{(u-v)(v-w)(w-u)}$$

$$\frac{\partial u}{\partial x} = \frac{vw}{(u-v)(u-w)}$$

4).If $ux + vy = a,\ \frac{u}{x} + \frac{v}{y} = 1,$ then, prove that

$$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = 0$$

Soln: Here $f_1 = ux + vy - a = 0$

$$f_2 = \frac{u}{x} + \frac{v}{y} - 1 = 0$$

$$\frac{\partial(f_1, f_2)}{\partial(x, y)} = -\begin{vmatrix} u & v \\ -\frac{u}{x^2} & -\frac{v}{y^2} \end{vmatrix}$$

Now, $= uv\left(-\frac{1}{y^2} + \frac{1}{x^2}\right) = \frac{uv(y^2 - x^2)}{x^2 y^2}$ ....(1)

$$\left(\frac{\partial x}{\partial u}\right)_v = -\frac{\frac{\partial(f_1, f_2)}{\partial(u, y)}}{\frac{\partial(f_1, f_2)}{\partial(x, y)}}$$

$= -\frac{\frac{-v(x^2 + y^2)}{y^2 x}}{\frac{uv(y^2 - x^2)}{x^2 y^2}} = \frac{x}{u}\frac{x^2 + y^2}{y^2 - x^2}$ ....(2)

$$\left(\frac{\partial y}{\partial v}\right)_u = -\frac{\frac{\partial(f_1, f_2)}{\partial(x, v)}}{\frac{\partial(f_1, f_2)}{\partial(x, y)}}$$

And

$= -\frac{\frac{u(x^2 + y^2)}{x^2 y}}{\frac{uv(y^2 - x^2)}{x^2 y^2}} = -\frac{y}{v}\frac{x^2 + y^2}{y^2 - x^2}$ ....(3)

From (2) and (3), we get,

$$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = \frac{x^2 + y^2}{y^2 - x^2} - \frac{x^2 + y^2}{y^2 - x^2} = 0$$

Functional dependence, independence by using Jacobian:

Functional Dependence:

Let $u = f_1(x, y)$ and $v = f_2(x, y)$ be any two functions of $x$and y, $u = f_1(x, y)$ and $v = f_2(x, y)$ are functionally dependent if their Jacobian vanishes identically.

i.e.$J = \frac{\partial(f_1, f_2)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)} = 0$

Notes :

1. This fact can be generalized in case of n functions $f_r(x_1, x_2, \ldots \ldots \ldots, x_n)$ in $(r = 1,2,3, \ldots \ldots n)$

$n$independent variables $x_1, x_2, \ldots \ldots \ldots, x_n$ i.e. these n functions are functionally dependent

if and only if their Jacobian $J = \frac{\partial(f_1, f_2, \ldots\ldots\ldots, f_n)}{\partial(x_1, x_2, \ldots\ldots\ldots, x_n)} = 0$ . ...(1)

2. However, if number of functions are less than the number of variables then several relations of

type (1) are required to be satisfied for the functional dependence.

viz. for functions $f_1(x, y, z)$ and $f_2(x, y, z)$ of three variables it can be seen that for their

functional dependence,

$$\frac{\partial(f_1,\ f_2)}{\partial(x,y)} = 0,\quad \frac{\partial(f_1,\ f_2)}{\partial(y,z)} = 0,\quad \frac{\partial(f_1,\ f_2)}{\partial(z,x)} = 0$$

Examples :

1. Ascertain whether the following functions are functionally dependence, if so find the

relation between them. $u = \frac{x+y}{1-xy}$, $v = tan^{-1}x + tan^{-1}y$.

Soln. : We have,

$$\frac{\partial u}{\partial x} = \frac{1 + y^2}{(1 - xy)^2} \quad ; \quad \frac{\partial u}{\partial y} = \frac{1 + x^2}{(1 - xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1 + x^2} \quad ; \quad \frac{\partial v}{\partial y} = \frac{1}{1 + y^2}$$

$$\therefore \quad J = \frac{\partial(u,\ v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1 + y^2}{(1 - xy)^2} & \frac{1 + x^2}{(1 - xy)^2} \\ \frac{1}{1 + x^2} & \frac{1}{1 + y^2} \end{vmatrix}$$

$$= \frac{1}{(1 - xy)^2} - \frac{1}{(1 - xy)^2}$$

$$= 0$$

Thus, $J = \frac{\partial(u,v)}{\partial(x,y)} = 0$.

Hence, u and v are functionally dependent.

Relation between u and v : We have,

$$v = tan^{-1}x + tan^{-1}y = tan^{-1}\left(\frac{x+y}{1-xy}\right) = tan^{-1}u$$

$$\therefore \;\; v = tan^{-1}u$$

2. If u =x+y+z, $v = x^2 + y^2 + z^2$, $w = xy + yz + zx$ examine whether u, v, w are functionally dependent. If so find the relation between them.

Soln. : For functional dependence, we must have $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

$$\frac{\partial u}{\partial x} = 1 \;\; ; \;\; \frac{\partial u}{\partial y} = 1 \;\; ; \;\; \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = 2x \;\; ; \;\; \frac{\partial v}{\partial y} = 2y \;\; ; \;\; \frac{\partial v}{\partial z} = 2z$$

$$\frac{\partial w}{\partial x} = y + z \;\; ; \;\; \frac{\partial w}{\partial y} = x + z \;\; ; \;\; \frac{\partial w}{\partial z} = x + y$$

$$\frac{\partial(u,\ v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & x+z & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0$$

Hence, u, v, w are functionally dependent.

Relation between u, v and w : We have,

$$u^2 = (x+y+z)^2$$

$$= x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$= v + 2w$$

$$\therefore\ u^2 = v + 2w$$

3) Examine the functions are functionally dependent or not, if so find the relation between them

$u = \dfrac{x-y}{x+y}\ ;\ v = \dfrac{x+y}{x}$

Soln: Given $u = \dfrac{x-y}{x+y}\ ;\ v = \dfrac{x+y}{x}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{(x+y)-(x-y)}{(x+y)^2} & \dfrac{(x+y)(-1)-(x-y)(-1)}{(x+y)^2} \\ \dfrac{x-(x+y)}{(x)^2} & \dfrac{1}{x} \end{vmatrix}$$

$$= \frac{1}{(x+y)^2}\begin{vmatrix} 2y & -2x \\ \dfrac{-y}{x^2} & \dfrac{1}{x} \end{vmatrix} = \frac{1}{(x+y)^2}\left[\frac{2y}{x} - \frac{2yx}{x^2}\right] = \frac{1}{(x+y)^2} \times 0$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

This shows that the given functions are functionally dependent.
Relation:

$$\therefore u = \frac{x-y}{x+y}\ ;\ v = \frac{x+y}{x}$$

$$\Rightarrow v = 1 - \frac{y}{x}$$

$$\Rightarrow \frac{y}{x} = 1 - v$$

$$u \cdot v = \left(\frac{x-y}{x+y}\right) \cdot \left(\frac{x+y}{x}\right) = \frac{x-y}{x} = 1 - \frac{y}{x}$$

$$= 1 - (v - 1)$$

$$u \cdot v = 2 - v$$

Errors and Approximations:

For $u = f(x, y, z)$ the total differential is,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Where, If dx, dy, dz represent increments in x, y, z and du the corresponding increment in u.

If dx, dy, dz represents small errors in independent variables x, y, z .

Then du given by above relation, represent approximate error in u .

Relative errors in x, y, z are denoted by $\frac{dx}{x}, \frac{dy}{y}, \frac{dz}{z}$ and corresponding relative error in u is $\frac{du}{u}$.

Multiplying relative errors by 100, we get percentage errors in variables x, y, z

Examples:-

1) The diameter and height of a right circular cylinder are measured to be 5 and 8 inches respectively. If each of these dimensions were in error by 0.1 inch, find the percentage error in the volume of the cylinder.

Solution:-let the diameter of cylinder be ‘ d ’.

d =5 with df =0.1 height= h =8 with dh =0.1

$\% \frac{df}{f}$ =2% $\% \frac{dh}{h}$ =1.25%

Volume of cylinder = $v = \pi r^2 h$

But $d = 2r$ $\therefore r = \frac{d}{2}$

$$\therefore v = \frac{\pi}{4} d^2 h$$

$$\therefore \log v = \log \pi + 2\log d + \log h - \log 4$$

$$\therefore \frac{dv}{v} = 0 + 2\frac{dd}{d} + \frac{dh}{h} - 0$$

$$\therefore 100\frac{dv}{v} = 2\frac{100dd}{2} + \frac{100dh}{h} = 5.25\%$$

2) The focal length of a mirror is found from $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find the percentage error in $'f'$.if u and v are both of error 2 % each.

Solution:- consider,

$$\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$$

$$\therefore 2\frac{df}{-f^2} = \frac{-dv}{v^2} + \frac{du}{u^2}$$

$$\therefore \frac{-2}{f}\frac{df}{f} = \frac{-1}{v}\frac{dv}{v} + \frac{1}{u}\frac{du}{u}$$

$\therefore\ \frac{-2}{f}\frac{100df}{f} = \frac{-1}{v}\frac{100dv}{v} + \frac{1}{u}\frac{100du}{u}$, after simplifying we get

$\frac{-2}{f}\frac{100df}{f} = -2\left(\frac{2}{f}\right) \therefore \frac{100df}{f} = 2$

$\therefore$ % error in f =2

3) Find the percentage error in the area of an ellipse, when the error of 2% and 3% are made in measuring its major and minor axes respectively.

Solution:- Area of an ellipse = πab, $a =$ major axis , $b =$ minor axis

$\frac{\partial a}{a} * 100 = 2, \qquad \frac{\partial b}{b} * 100 = 3......given$

$A = \pi ab$

$\therefore \log A = \log \pi + \log a + \log b$

$\therefore \frac{\partial A}{A} = 0 + \frac{\partial a}{a} + \frac{\partial b}{b}$

$\therefore 100\frac{\partial A}{A} = 100\frac{\partial a}{a} + 100\frac{\partial b}{b} = 2 + 3 = 5$

%error in area =5

4) In calculating volume of right circular cylinder, errors of 2% and 1% are found in measuring height and base radius respectively. .Find the percentage error in calculated volume of the cylinder.

Solution:-Let $r =$ radius of cylinder, $h =$ height of cylinder

$\therefore$ volume of cylinder = $v = \pi r^2 h$

Taking log on both sides we get,

$\log v = \log \pi + 2\log r + \log h$, differentiating we get

$\therefore \quad \frac{\partial v}{v} = 0 + 2\frac{\partial r}{r} + \frac{\partial h}{h}$

$\frac{100\partial v}{v} = 2\frac{100\partial r}{r} + \frac{100\partial h}{h} = 2(1) + 2 = 4$

$\therefore$ % error in volume = 4

5)Find the percentage error in the area of an sllipse when an error of 1% is made in measuring its major and minor axes.

Soln: If A is area and $2a$ and $2b$ are the major and minor axes of the ellipse having equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then,

$A = \pi ab$

Taking log on both sides,

$\log A = \log \pi + \log a + \log b$

Differentiating we get,

$$\frac{dA}{A} = 0 + \frac{da}{a} + \frac{db}{b}$$

$$\therefore \frac{100dA}{A} = \frac{100da}{a} + \frac{100db}{b}$$

$\because$ % in each area is equal to 1, we grt

$$\therefore \frac{100dA}{A} = 1 + 1 = 2$$

%error in the area $A = 2\%$

Maxima and Minima of function of two variables:

Let $u = f(x, y)$ be a continuous and differentiable function of two independent variables x and y .The function $u = f(x, y)$ is said to have maximum value at $x = a, y = b$ if $f(a, b) > f(a + h, b + k)$ and it have minimum value if $f(a, b) < f(a + h, b + k)$, here h and k are very small positive or negative values.

Stationary Point: The point at which function $u = f(x, y)$ is either maximum or minimum is known as stationary point.

Extreme value: The maximum or minimum value of a function is called its extreme value.

Let f is function of x and y of two variables which is continuous and differentiable in $x$and y.

To find maximum or minimum value, use following procedure.

1. Given function, $f(x, y) = 0$. 2. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ 3. Solve :$\frac{\partial f}{\partial x} = 0$; $\frac{\partial f}{\partial y} = 0$ 4. Solve these equations simultaneously and find values of x and $y(x, y) \equiv (a_1, b_1), (a_2, b_2), (a_3, b_3)$ … … … .. are called stationary values.
5. Find $r = \frac{\partial^2 f}{\partial x^2}$; $s = \frac{\partial^2 f}{\partial x \partial y}$; $t = \frac{\partial^2 f}{\partial y^2}$ 6. Find r, s, t for each pair. For (a_1, b_1) Find $r_{(a_1,b_1)}, s_{(a_1,b_1)}, t_{(a_1,b_1)}$ 7. (i) If $(rt - s^2) > 0$and $r > 0$, then function is minimum at (a_1, b_1). $\therefore f_{min} = f(a_1, b_1)$is called extreme value. (ii) If $(rt - s^2) > 0$ and $r < 0$, then function is maximum. $f_{max} = f(a_1, b_1) =$ Extreme value(iii) If $(rt - s^2) < 0$, then function is neither maximum nor minimum, then the point(a_1, b_1) is called saddle point.
8. If $(rt - s^2) = 0$;we need more investigation/require more study.

Examples :1.Discuss maxima and minima of the function $x^2 + y^2 + 6x + 12$ **.**
Soln. : Let, $f = x^2 + y^2 + 6x + 12$ - - - - - - - - - - - - - - - (1)

$$\frac{\partial f}{\partial x} = 2x + 6 \quad and \quad \frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial x} = 0 \qquad and \qquad \frac{\partial f}{\partial y} = 0$$

$$\therefore \quad 2x + 6 = 0 \qquad and \qquad 2y = 0$$

$$\Longrightarrow \quad x = -3 \qquad and \qquad y = 0$$

$$(x, y) \equiv (-3\,, 0)$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2 \quad ; \quad s = \frac{\partial^2 f}{\partial x \partial y} = 0 \quad ; \quad t = \frac{\partial^2 f}{\partial y^2} = 2$$

$$(rt - s^2)_{(-3,0)} = (2 \times 2 - 0) = 4 > 0 \; and \quad r = 2 > 0$$

$\Longrightarrow$ Function is minimum ;

From equation (1)

$$f_{min} = (-3)^2 + 0 + 6(-3) + 12 = 9 - 18 + 12 = 3$$

$f_{min} = 3$be the extreme value.

2. Discuss maxima and minima for $x^3 + y^3 - 3axy$; $a > 0$. Soln. :Let,

$f = x^3 + y^3 - 3axy$ - - - - - - - - - - - - - -(1)

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay \quad and \quad \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial f}{\partial x} = 0 \qquad and \qquad \frac{\partial f}{\partial y} = 0$$

$$\therefore \quad 3x^2 - 3ay = 0 \qquad and \qquad 3y^2 - 3ax = 0$$

$$\Longrightarrow \quad x^2 = ay \qquad and \qquad y^2 = ax$$

$$\Longrightarrow \quad x^2 = a\sqrt{ax} \qquad and \qquad y = \sqrt{ax}$$

$$x^4 = a^2(ax)$$

$$\Longrightarrow \; x^4 - a^3 x = 0$$

$$\Longrightarrow \; x = 0 \quad or \quad x = a$$

$$x = 0 \Longrightarrow y = 0$$

$$x = a \Longrightarrow y = a$$

Hence, $(x, y) \equiv (0, 0), (a, a)$ be the stationary points.

Now, $r = \frac{\partial^2 f}{\partial x^2} = 6x$; $s = \frac{\partial^2 f}{\partial x \partial y} = -3a$; $t = \frac{\partial^2 f}{\partial y^2} = 6y$

(i) For (0,0)$r = 0$; $s = -3a$; $t = 0$

$$(rt - s^2) = 0 - (-3a)^2 = -9a^2 < 0.$$

The function is neither maximum nor minimum. The point $(0,0)$be saddle point. (ii) For $(a,a)r = 6a$; $s = -3a$; $t = 6a$

$$(rt - s^2) = 27a^2 > 0 \quad and \quad r = 6a > 0$$

Function is minimum.

$f(a,b) = f_{min} = a^3 + a^3 - 3a^3 = -a^3$be the extreme value.

3) Find the maximum and minimum value of $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Soln. : For $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y \quad and \quad \frac{\partial u}{\partial y} = 4y^3 - 4y + 4x$$

$$r = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4 \; ; \quad s = \frac{\partial^2 u}{\partial x \partial y} = 4 \quad ; \quad t = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4$$

Now for stationary values u,

$$\frac{\partial u}{\partial x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow 4y^3 - 4y + 4x = 0 \Rightarrow y^3 - y + x = 0$$

Eliminating y using $y = x - x^3$

We get, $(x - x^3)^3 - (x - x^3) + x = 0$

$\therefore \quad x = 0, \quad x = \pm\sqrt{2}$

$\therefore$ corresponding $y = 0, \quad y = \mp\sqrt{2}$

$\therefore$ possible points are (0,0), $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$

Now at $x = 0, y = 0, r = -4, s = 4, t = -4$

$\therefore (rt - s^2) = 0$

$\therefore$ Test fails .

At $x = \pm\sqrt{2},\ y = \mp\sqrt{2},\ r = 20,\ s = 4,\ t = 20$

$\therefore (rt - s^2) = 400 - 16 = 384 > 0$, and $r > 0$

$\therefore u$ is minimum at $x = \pm\sqrt{2},\ y = \mp\sqrt{2}$

$\therefore\ u_{min} = -8$

Illustration on Maxima and Minima

Examples :

1.Discuss maximum and minimum values of $f(x,y) = xy(a - x - y),\ x > 0; y > 0; a > 0.$

Soln. : Let, $f(x,y) = xy(a - x - y) = axy - x^2y - xy^2$ - - - - - - - (1)

$$\frac{\partial f}{\partial x} = ay - 2xy - y^2 \quad and \quad \frac{\partial f}{\partial y} = ax - x^2 - 2xy$$

For stationary values, $\frac{\partial f}{\partial x} = 0 \quad and \quad \frac{\partial f}{\partial y} = 0$

$$\therefore \quad ay - 2xy - y^2 = 0 \quad and \quad ax - x^2 - 2xy = 0$$

$$\Longrightarrow \quad y = 0; \; 2x + y = a \; and \; x = 0; \; x + 2y = a$$

Consider,

$$y = 0 \quad ; \quad x = 0$$

$$y = 0 \quad ; \quad x + 2y = a$$

$$2x + y = a \quad ; \quad x = 0$$

$$2x + y = a \quad ; \quad x + 2y = a$$

Therefore, stationary values are $(0,0), (a,0), (0,a), \left(\frac{a}{3}, \frac{a}{3}\right)$,

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = -2y \quad ; \quad s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y \quad ; \quad t = \frac{\partial^2 f}{\partial y^2} = -2x$$

(i) For $(x\,,y) \equiv (0,0)$;

$$(rt - s^2) = -a^2 < 0 \quad (x\,,y) \equiv (a,0); \; (rt - s^2) < 0$$

$$(x\,,y) \equiv (0,a); \; (rt - s^2) < 0$$

At all these points, function is neither maximum nor minimum. These points are called saddle points.

(ii) $(x,y) = \left(\frac{a}{3}, \frac{a}{3}\right)$,

$$r = \frac{-2a}{3} \quad ; \quad s = \frac{-a}{3} \quad ; \quad t = \frac{-2a}{3}$$

$$\therefore rt - s^2 = \left(\frac{-2a}{3}\right)\left(\frac{-2a}{3}\right) - \left(\frac{-a}{3}\right)^2 = \frac{a^2}{3} > 0$$

and $r < 0 \quad if \quad a > 0 \quad ; \quad r > 0 \quad if \quad a < 0$

$$F_{extreme\ value} = f\left(\frac{a}{3}, \frac{a}{3}\right) = \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^2}{27}$$

2. Find the minimum value of $xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$

Soln. : Let $f = xy + \left(\frac{a^3}{x} + \frac{a^3}{y}\right)$ ---------- (1)

For stationary values, $\frac{\partial f}{\partial x} = 0 \quad and \quad \frac{\partial f}{\partial y} = 0$

$$y - \frac{a^3}{x^2} = 0 \; \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$x - \frac{a^3}{y^2} = 0 \; \ldots\ldots\ldots\ldots\ldots (3)$$

From equations (2) and (3), $x = a \quad and \quad y = a$

Therefore, (a, a) be the stationary point.

Now,

$$r = \frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3} \quad ; \quad s = \frac{\partial^2 f}{\partial x \partial y} = 1 \quad ; \quad t = \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$$

At (a, a)

$$r = 2 \; ; \quad s = 1 \quad ; \quad t = 2$$

$$\Rightarrow rt - s^2 = 4 - 1 = 3 \quad and \quad r = 2 > 0$$

Hence, function is minimum.

$\therefore$ $f_{min}$from equation (1)

$$f_{min} = f(a, a) = (a)(a) + a^3\left(\frac{1}{a} + \frac{1}{a}\right) = a^2 + 2a^2 = 3a^2$$

3. Find the stationary value of $f(x, y) = sin\, x + sin\, y + \sin(x + y)$.

Soln. : Given, $f(x, y) = sin\, x + sin\, y + \sin(x + y)$ …………(1)

For stationary values,

$\frac{\partial f}{\partial x} = 0 \;\Longrightarrow \cos x + \cos(x + y) = 0$ ……………(2)

And $\frac{\partial f}{\partial y} = 0 \Longrightarrow \cos y + \cos(x + y) = 0$ ……………..(3)

From equations (2) and (3),

$$\cos x + \cos(x + y) = \cos y + \cos(x + y)$$

$$\Longrightarrow \quad \cos x = \cos y$$

$$\Longrightarrow \quad x = y$$

Put $x = y$ in equation (1)

$$\cos x + \cos 2x = 0 \quad \Longrightarrow \quad \cos x = -\cos 2x$$

$\Longrightarrow \quad \cos x = \cos(\pi - 2x) \text{or} \cos(\pi + 2x)$

$\Longrightarrow \quad x = \frac{\pi}{3} \text{and} x = -\pi$

Hence, $y = \frac{\pi}{3}$ and $y = -\pi$

Stationary points are $\left(\frac{\pi}{3}, \frac{\pi}{3}\right), (\pi, -\pi)$

Now, $r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x + y)$; $s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x + y)$;

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x + y)$$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$r = -\sqrt{3} \quad ; \quad s = -\frac{\sqrt{3}}{2} \quad ; \quad t = -\sqrt{3}$$

$\therefore \; rt - s^2 = \frac{9}{4} > 0 \text{and} r = -\sqrt{3} < 0$

Hence, the function is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$F_{max} = f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = 3\left(\frac{\sqrt{3}}{2}\right)$$

At $(-\pi, -\pi)$

$$r = 0, \quad s = 0, \quad t = 0$$

$rt - s^2 = 0.$ We need more information.

4. Discuss the maxima and minima of $x^3 y^2 (1 - x - y)$.

Soln: Here $f(x, y) = x^3 y^2 - x^4 y^2 - x^3 y^3$

The points of maxima and minima are given by

$\frac{\partial f}{\partial x} = 3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3 = 0$ ……………………………….(1)

$\frac{\partial f}{\partial y} = 2x^3 y - 2x^4 y - 3x^3 y^2 = 0$ …………………………………..(2)

Equations (1) and (2 are equivalent to

$x^2 y^2 (3 - 4x - 3xy) = 0$ and $x^3 y (2 - 2x - 3y) = 0$

$i.e. x = 0,\; y = 0,\; 3 - 4x - 3y = 0$ and $i.e. x = 0,\; y = 0,\; 2 - 2x - 3y = 0$

Solving these equations, we obtain the following stationary points:

(0,0) and $\left(\frac{1}{2}, \frac{1}{3}\right)$

Now, we calculate

$$r = \frac{\partial^2 f}{\partial x^2} = 6xy^2 - 12x^2 y^2 - 6xy^3$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 6x^2 y - 8x^3 y - 9x^2 y^2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 2x^3 - 2x^4 - 6x^3 y$$

At $\left(\frac{1}{2},\frac{1}{3}\right)$, $r=\frac{1}{9}$, $s=\frac{-1}{12}$, $t=\frac{-1}{8}$

At $(0,0)$, $r=0,\ s=0,\ t=0$

Now, At $\left(\frac{1}{2},\frac{1}{3}\right)$, $rt-s^2>0\ \ and\ \ r<0$

Hence $f(x,y)$ is maximum at $\left(\frac{1}{2},\frac{1}{3}\right)$ $f(x,y)_{\max}=\frac{1}{432}$

At (0,0) $rt-s^2=0$

Here condition of maxima and minima is not satisfied. Hence this case is undecidedand further investigation is required.

Lagrange's method of undetermined multipliers, Illustrations:

Let $f(x,y,z)$ be the function of three variables and these variables are also connected by relation $\phi(x,y,z)=0$.

By Lagrange's Method, we get only stationary points not the nature of the function.

There are two types :

1.Functions having only one constraints. 2. Functions having two or more constraints.

Type I : Functions having only one constraints :

Procedure :

(i) Let $u=f(x,y,z)$ ………………………..(1) be given function under the condition $\phi(x,y,z)=0$ ………………………..(2)

(ii) Construct a new function $F=u+\lambda\phi$; λ is undermined multiplier. For stationary values ;$\frac{\partial F}{\partial x}=0$; $\frac{\partial F}{\partial y}=0$; $\frac{\partial F}{\partial z}=0$

(iii) Solve these equations simultaneously and find values of x,y,z interms of k.

$x=f_1(k)$ …………………………(3)

$y=f_1(k)$ …………………………(4)

$z=f_1(k)$ …………………………(5)

(iv) Substitute this values in equation (2) in given condition and find value of k.

(v) Again put value of k in equations (3), (4) and(5), we get x,y,z.

Examples :

1. Find points on surface $z^2 = xy + 1$ nearest to origin by using Lagrange's method.

Soln. : $O = (0,0,0)$ be the origin and $P = (x,y,z)$ be any point on the surface

$$d(OP) = \sqrt{x^2 + y^2 + z^2}$$

$$\left(d(OP)\right)^2 = x^2 + y^2 + z^2$$

$\therefore u = x^2 + y^2 + z^2$ ……………….(1)

$\phi = z^2 - xy - 1 = 0$ ………………(2)

Let, $F = u + \lambda\phi$

$$F = (x^2 + y^2 + z^2) + \lambda(z^2 - xy - 1) = 0$$

To find :$\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$; $\frac{\partial F}{\partial z} = 0$

$$\therefore \quad \frac{\partial F}{\partial x} = 2x - \lambda y = 0$$

$$\frac{\partial F}{\partial y} = 2y - \lambda x = 0$$

$$\frac{\partial F}{\partial z} = 2z + 2\lambda z = 0$$

$$\Longrightarrow 2x - \lambda y = 0$$

$$\frac{2x}{y} = \lambda \quad ; \quad \frac{2y}{x} = \lambda \ ; \quad \frac{2z}{2z} = -\lambda$$

$$\frac{2x}{y} = \lambda \quad ; \quad \frac{2y}{x} = \lambda \ ; \ -1 = \lambda$$

$$\therefore \quad \frac{2x}{y} = -1 \quad and \quad \frac{2y}{x} = -1$$

$$\Longrightarrow y = -2x \quad and \quad x = -2y$$

$\Longrightarrow y = 0 \quad and \quad x = 0$also

Equation (2) becomes,

$$z^2 - 1 = 0 \quad ; \quad z^2 = 1 \quad ; \quad z = \pm 1$$

Hence, $(0,0,\pm 1)$ is the nearest point from origin.

2. Find the minimum distance from the origin to the plane $3x + 2y + z = 12$.

Soln. : Let $P(x,y,z)$ be any point on the surface $3x + 2y + z - 12 = 0$

$\therefore$ The distance from origin to that point

$$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

$u = d^2 = x^2 + y^2 + z^2$ ………………..(1)

and$\phi = 3x + 2y + z - 12 = 0$ …………………(2)

Let, $F = u + \lambda\phi$

$$F = (x^2 + y^2 + z^2) + \lambda(3x + 2y + z - 12) = 0$$

To find :$\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$; $\frac{\partial F}{\partial z} = 0$

$\therefore \quad 2x + 3\lambda = 0 \Longrightarrow \frac{2x}{3} = -\lambda$ ………………(3)

$2y + 2\lambda = 0 \Longrightarrow y = -\lambda$ ……………….(4)

and$2z + \lambda = 0 \Longrightarrow 2z = -\lambda$ ………………..(5)

From equations (3), (4), (5) we get,

$$\frac{2x}{3} = y = 2z = -\lambda = k$$

$\Longrightarrow x = \frac{3k}{2}$, $y = k$, $z = \frac{k}{2}$ ………………….(6)

Put these values of x in equation (2) ,

$$3\left(\frac{3k}{2}\right) + 2(k) + \left(\frac{k}{2}\right) = 12 \Longrightarrow \quad k = \frac{12}{7}$$

Put in equation (6) ,

$$x = \frac{3k}{2} = \frac{18}{7}\ , \quad y = k = \frac{12}{7}\ , \quad z = \frac{k}{2} = \frac{6}{7}$$

$\therefore$ The minimum distance $(d) = \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{504}{7}}$

3. As the dimensions of a triangle ABC are varied, show that the maximum value of $\cos A \cos B \cos C$ is obtained when the triangle is equivalent.

Soln: Let $u = f(A,B,C) = \cos A \cos B \cos C$ ……………………..(1)

Under the condition $\phi = A + B + C - \pi = 0$ ……………………..(2)

Construct the function $F = ut + \lambda\phi$

$$F = \cos A \cos B \cos C + \lambda(A + B + C - \pi)$$

Now ,

$\frac{\partial F}{\partial A} = 0 \qquad \therefore -\sin A \cos B \cos C + \lambda = 0$............(3)

$\frac{\partial F}{\partial B} = 0 \qquad \therefore -\cos A \sin B \cos C + \lambda = 0$............(4)

$\frac{\partial F}{\partial C} = 0 \qquad \therefore -\cos A \cos B \cos C + \lambda = 0$............(5)

We eliminate A, B, C and λ using equations (1) and (5), From equations (3), (4) and (5)

$\sin A\cos B\cos C = \cos A\sin B\cos C = \cos A\cos b\sin C$

Dividing by $\cos A\cos B\cos C$,

$\tan A = \tan B = \tan C$

$\Rightarrow \quad A = B = C$

$\Rightarrow \quad \Delta ABC$ is equilateral.

Illustrations on Lagrange's method, Assignment, Discussions of previous University question paper.

1) Find the stationary values of $u = x^2 + y^2 + z^2$ if $xy + yz + zx = 3a^2$

Soln. : du = must be zero $\quad xdx + ydy + zdz = 0$ ----------------(1)

$xy + yz + zx = 3a^2$ ----------------(2)

$(y+z)dx + (z+x)dy + (x+z)dz = 0$ -----------------(3)

$x + \lambda(y+z) = 0$ ---------------(4)

$y + \lambda(z+x) = 0$ ---------------(5)

$z + \lambda(x+y) = 0$ ---------------(6)

Adding (4), (5) and (6) we get, Equation (7)

$\therefore \quad (x+y+z) + 2\lambda(x+y+z) = 0, \qquad \therefore 2\lambda = -1$ ---------------(4)

Also multiplying Equation (4),(5)(6) by x, y, z respectively and adding we get,

$u + 2\lambda(-a^2) = 0$

From (7) we get $u - 3a^2 = 0, \qquad u = 3a^2$ (stationary value of u)

2)If $ax^2 + by^2 = ab,$ show that the extreme values of $u = x^2 + y^2 + xy$ are the roots of $4(u-a)(u-b) = ab.$

Soln: Let $u = f(x,y) = x^2 + y^2 + xy$.....................................(1)

Under the condition $\phi = ax^2 + by^2 - ab = 0$................................(2)

Construct the function $\quad F = u + \lambda\phi$

$$= x^2 + y^2 + xy + \lambda\left(ax^2 + by^2 - ab\right)$$

Form the equations:

$\frac{\partial F}{\partial x} = 0 \quad \therefore 2x + y + \lambda 2ax = 0$....................................(3)

$$\frac{\partial F}{\partial y} = 0 \quad \therefore 2y + x + \lambda 2by = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

We eliminate x, y and λ using equations (1) to (4) we get

$\lambda = \frac{-u}{ab}$ $\therefore$ equation (3) becomes $2x\left(1 - \frac{u}{b}\right) + y = 0 \ldots\ldots\ldots\ldots\ldots(5)$

Also equation (4) becomes $2y\left(1 - \frac{u}{b}\right) + x = 0 \ldots\ldots\ldots\ldots\ldots(6)$

Eliminating x, y using (5) and (6)

We get, $\frac{4(b-u)}{b}\frac{(a-u)}{a} = 1$

$$\Rightarrow 4(u-a)(u-b) = ab$$

Which is quadratic in u, giving the values of u.