Partial Differentiation Introduction & Definition

Subject- Engg.Mathematcs-I

Faculty- Prof.Ms.S.A.Gurav

Sinhgad College Of Engineering

CONTENT OF THE TOPIC

1.Introduction to Partial Derivative, Definition, Physical Meaning and basic examples.

2. Properties of Partial Derivative and Higher order derivatives.

3.Derivative of Composite function

4.Variable to be treated as constant.

5.Homogeneous Function and Euler's Theorem.

6.Partial Derivative of Implicit function& Total Derivative.

PARTIAL DERIVATIVES

Introduction – Partial Derivative is used for functions with more than one independent variable.

If the function $z = f(x, y)$ be a function of two variables x, y then the rate of change of z with respect to x, keeping y const. is called partial derivative of z with respect to x & it is denoted by $\frac{\partial z}{\partial x}$.

Similarly the rate of change of z with respect to y, keeping x const. is called partial derivative of z with respect to x & it is denoted by $\frac{\partial z}{\partial y}$

Thus for $z = f(x, y)$ $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are called First Order Partial derivatives with respect to x & y respectively.

Sinhgad Institutes

Thus $$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

and $$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

In general $\frac{\partial z}{\partial x}\ \frac{\partial z}{\partial y}$ are function of both the variables x and y so we may obtain higher derivatives

GEOMETRICAL INTERPRETATION

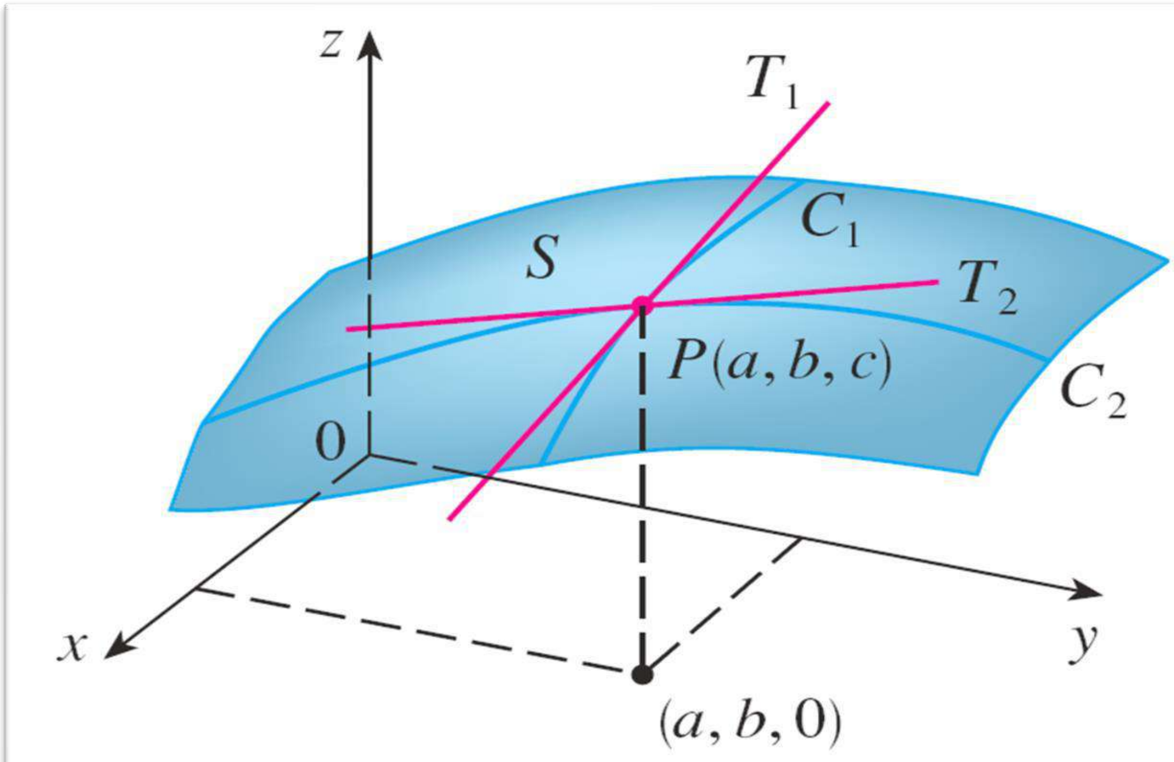
Geometrically z=f(x, y) represents a surface S ,so if y=k (constant) i.e. a plane parallel to X Z plane thus

z=f(x ,y) and y=k (constant) together represents a curve C which is the section of S by y=k plane.

Thus $\dfrac{\partial z}{\partial x}$ represents the slope of tangent to C at(x,k, z)

Similarly $\dfrac{\partial z}{\partial y}$ represents the slope of tangent drawn to the curve of intersection of z =f(x, y) & x=k.

Sinhgad Institutes



If f(a,b)=c then pt P(a,b,c) lies on S

y=b intersects surface S in C1

T1 is the slope of tangent to curve C1 i.e. $\dfrac{\partial z}{\partial x}$

Similarly T2 is the slope of tangent to curve C2 i.e. $\dfrac{\partial z}{\partial y}$

Sinhgad Institutes

Examples on Direct Partial Derivatives

Q1. If $x = r\cos\theta\ ,\ y = r\sin\theta$ show that

$$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$$

Sol $x = r\cos\theta\ ,\ y = r\sin\theta$ then

$(x)^2 + (y)^2 = r^2$ therefore

$$2r\frac{\partial r}{\partial x} = 2x \quad \& \quad 2r\frac{\partial r}{\partial y} = 2y$$

$$so\ \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta \quad \& \quad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

$$thus \left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \cos^2\theta + \sin^2\theta = 1$$

Sinhgad Institutes

Particular Example

1) If $z^3 - zx - y = 4 \; then \; find \; \frac{\partial z}{\partial x} \; \& \; \frac{\partial z}{\partial y}$

Solution : Given $z^3 - zx - y = 4$----(1)

Differentiate eq(1) partially w.r.t. x keeping y as constant

$$3z^2 \frac{\partial z}{\partial x} - z - x\frac{\partial z}{\partial x} = 0$$

$$(3z^2 - x)\frac{\partial z}{\partial x} = z$$

$$\frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

Also, differentiate eq (1) partially w.r.t. y, keeping x as constant

$$\left(3z^2\frac{\partial z}{\partial y} - x\frac{\partial z}{\partial y} - 1\right) = 0$$

$$(3z^2 - x)\frac{\partial z}{\partial y} = 1$$

$$\frac{\partial z}{\partial y} = \frac{1}{(3z^2 - x)}$$

3) 1)If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$ then find the value of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

Sinhgad Institutes

Solution:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left[(x^2 + y^2 + z^2)^{-1/2}\right]$$

$$\frac{\partial u}{\partial x} = \frac{-1}{2}\,\frac{2x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial u}{\partial x} = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = (-x)\left(\frac{-3}{2}\right)(x^2+y^2+z^2)^{-5/2} +$$

$$(-1)(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{3x^2}{(x^2+y^2+z^2)^{5/2}} - \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{-x^2+2y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} =$$

$$\frac{2x^2-y^2-z^2-x^2+2y^2-z^2-x^2-y^2+2z^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

3) If $u = \log(tanx + tany + tanz)$ then prove that $sin2x\frac{\partial u}{\partial x} + sin2y\frac{\partial u}{\partial y} + sin2z\ \frac{\partial u}{\partial z} = 2$

Solution : Given $u = \log(tanx + tany + tanz)$

$$\frac{\partial u}{\partial x} = \frac{sec^2 x}{tanx+tany+tanz}$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{sec^2 y}{tanx+tany+tanz}$$

$$\frac{\partial u}{\partial z} = \frac{sec^2 z}{tanx+tany+tanz}$$

Now

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \ \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x \sec^2 x + \sin 2y \sec^2 y + \sin 2z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{\frac{2\sin x \cos x}{\cos^2 x} + \frac{2\sin y \cos y}{\cos^2 y} + \frac{2\sin z \cos z}{\cos^2 z}}{\tan x + \tan y + \tan z}$$

$$= \frac{2\tan x + 2\tan y + 2\tan z}{\tan x + \tan y + \tan z} = 2$$

4) If $u = \emptyset(x+ay)+\varphi(x-ay) then\ show\ that\ \frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2}$

Solution: $\frac{\partial u}{\partial y} = \emptyset'(x+ay).a+\varphi'(x-ay).(-a)$

Again differentiating w.r.t.y

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial y}(\emptyset'(x+ay).a+\varphi'(x+ay).(-a))$$

Sinhgad Institutes

$$= a.\emptyset''(x+ay).a + (-a)\varphi''(x-ay)(-a)$$

$$= a^2\left(\emptyset''(x+ay) + \varphi''(x-ay)\right)$$

Now $\dfrac{\partial u}{\partial x} = \emptyset'(x+ay) + \varphi'(x-ay)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \emptyset''(x+ay). + \varphi''(x-ay)$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial x^2}$$

Partial derivative of higher order:-

Let z = f(x ,y), then $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = z_{xx}$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = z_{xy} \qquad \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{yx}$$

In general $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$ Order of differentiation is commutative where z is a continuous function

Therefore further differentiation of any of these w .r. t either x or y or both possible.

Sinhgad Institutes

To find the value of Parameter ‘n

Ex.1)Find the value of ‘n’ for which $u = A\, e^{-gx}\sin(nt - gx)$ satisfies the partial differential equation $\frac{\partial u}{\partial t} = m\frac{\partial^2 u}{\partial x^2}$ where A, g, m are constant.

Solution :Given $u = A\, e^{-gx}\sin(nt - gx)$

Differentiating w.r.t. ‘t ‘ keeping x constant

$$\frac{\partial u}{\partial t} = A\, e^{-gx}\cos(nt - gx)(n)$$

Now differentiating.w.r.t. ‘x’ keeping t constant

Now differentiating.w.r.t. ‘x’ keeping t constant

$$\frac{\partial u}{\partial x} = Ae^{-gx}(-g)\sin(nt - gx) + Ae^{-gx}.\cos(nt - gx)(-g)$$

Again differentiating w.r.t. x

$$\frac{\partial^2 u}{\partial x^2} = A(-g)e^{-gx}(-g).\sin(nt - gx) +$$
$$Ae^{-gx}(-g).\cos(nt - gx)\,(-g) +$$
$$Ae^{-gx}(-g)\cos(nt - gx)\,(-g) + Ae^{-gx}(-\sin(nt -$$
$$gx))(-g)(-g)$$

Sinhgad Institutes

$=A\, g^2 e^{-gx}[\sin(nt-gx)+\cos(nt-gx)+\cos(nt-gx)-\sin(nt-gx)]$

$$\frac{\partial^2 u}{\partial x^2}=Ag^2e^{-gx}2.\cos(nt-gx)$$

Now as $\frac{\partial u}{\partial t}= \frac{\partial^2 u}{\partial x^2}$

$A\, e^{-gx}\cos(nt-gx)(n)= mAg^2e^{-gx}2\cos(nt-gx)$

Thus, $n=2mg^2$

Sinhgad Institutes

Q2 $Find\ \ 'n'$such that v= $r^n(3\cos^2\theta - 1)$

satisfies the equation $\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) +$

$\frac{1}{sin\theta}\ \frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial v}{\partial\theta}\right) = 0$

$solution$:

v= $r^n(3\cos^2\theta - 1)$ $\therefore \frac{\partial v}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$.

$$\therefore \frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) = n(n+1)v$$

Also $\frac{\partial v}{\partial\theta} = r^n[3.2\cos^1\theta(-sin\theta)]$.

$\therefore \frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v}{\partial\theta}\right) = -6r^n[-sin\theta\sin^2\theta + cos\theta.\,2sin\theta cos\theta]$

Sinhgad Institutes

$$\therefore \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v}{\partial\theta}\right) = -6r^n[2\,cos^2\theta - \sin^2\theta]$$

$$= -6r^n[3\cos^2\theta\text{-}1] \;\;=\;\; \text{-}6\ v$$

$$\therefore \frac{\partial}{\partial r}\left(r^2\ \frac{\partial v}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial v}{\partial\theta}\right) = 0$$

⇨ $n(n+1)v - 6v = 0$

⇨ $n^2 + n - 6 = 0\ , v \neq 0$

⇨ $n = -3\ , 2.$

Example on verification of $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Ex.1) If $u = x^y + y^x$ then prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solution : $u = x^y + y^x$-----(1)

Differentiating equation (1) partially w.r.t. y keeping ‘x’ as constant

$$\frac{\partial u}{\partial y} = x^y logx + xy^{x-1}$$

Differentiating above partially w.r.t. x keeping y as constant

Sinhgad Institutes

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[x^y logx + xy^{x-1}\right]$$

$$= x^y \frac{1}{x} + logx.\, yx^{y-1} + xy^{x-1}logy + y^{x-1}$$

$$\frac{\partial^2 u}{\partial x \partial y} = x^{y-1} + yx^{y-1}logx + xy^{x-1}logy + y^{x-1}$$

Now diff.eq(1)partially w.r.t. x keeping 'y' as constant

$$\frac{\partial u}{\partial x} = y\, x^{y-1} + y^x logy$$

Again diff. above w.r.t.y keeping x as constant

Sinhgad Institutes

$$\frac{\partial^2 u}{\partial y.\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}[y\, x^{y-1} + y^x logy]$$

$= yx^{y-1}logx + x^{y-1} + y^x\frac{1}{y} + logy.x.y^{x-1}$

$=yx^{y-1}logx + x^{y-1} + y^{x-1} + xy^{x-1}logy$ -----(B)

From (A) and (B)

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x}$$

Application Of P D

Partial Differentiation is used in the problems of Jacobian, vibration of string, Theory of approximation, Maxima &Minima .

Boundary value problems, Laplace Equation, Vectors, Heat ,Wave equations & solving some problems in Electrical engineering.

PDEs can be used to describe a wide variety of phenomena such as Sound ,Heat, Electrostatics, Electrodynamics , Fluid flow,

Elasticity or Quantum Mechanics

University Questions

1) If $u = \log(x^3 + y^3 - x^2y - y^2x)$ then Show that

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

2) If $u = \log(x^2 + y^2)$, $verify \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y}$

3) If $\theta = t^n . e^{\frac{-r^2}{4t}}$ Find n such that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

4) Prove that at a point of surface $x^x y^y z^z = c$ $where\ x = y = z$, $\frac{\partial^2 z}{\partial x \partial y} =$

$$-(xlogex)^{-1}$$

Sinhgad Institutes

Partial Derivative of Composite Function

- If u is a function of r i.e. u=f(r) and r again is a function of two independent variables x, y i.e. r=g(x ,y) .Thus u becomes a composite function of x and y .

- u$\rightarrow r \rightarrow (x, y) \therefore \frac{\partial u}{\partial x} = \frac{du}{dr}.\frac{\partial r}{\partial x}$, $\frac{\partial u}{\partial y} = \frac{du}{dr}.\frac{\partial r}{\partial y}$

- u$\rightarrow r \rightarrow (x, y, z) \therefore \frac{\partial u}{\partial x} = \frac{du}{dr}.\frac{\partial r}{\partial x}$ $\frac{\partial u}{\partial y} = \frac{du}{dr}.\frac{\partial r}{y}$

$$\frac{\partial u}{\partial z} = \frac{du}{dr}.\frac{\partial r}{\partial z}$$

- $z \rightarrow x, y \rightarrow u, v$

then z is a composite function of two variables u & v

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial u} \qquad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\,\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\,\frac{\partial y}{\partial v}$$

- $l \rightarrow x, y, z \rightarrow u, v$,w

$$\therefore \frac{\partial l}{\partial u} = \frac{\partial l}{\partial x}\,\frac{\partial x}{\partial u} + \frac{\partial l}{\partial y}\,\frac{\partial y}{\partial u} + \frac{\partial l}{\partial x}\,\frac{\partial x}{\partial u}$$ &so on..

- Note: For a function of single variable ,derivative is total whereas for a function of more than one variable derivative is Partial.

Example on Composite Function

1) If $z = f(u, v) and \quad u = xcost - ysint$;

$$v = xsint + \ +ycost$$

$where\ t\ is\ a\ constant$ Then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

Solution : $z = f(u, v) and$

$u = xcost - ysint$-------(1)

$v = xsint + ycost$--------(2)

Differentiating z w.r.t. x keeping y constant

$$\left(\frac{\partial z}{\partial x}\right) = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

&

$$\left(\frac{\partial z}{\partial y}\right) = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

Differentiating equation (1) w.r.t. ‘x’ keeping t as constant

$$\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(xcost - ysint) = cost$$

$$\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(xcost - ysint = -sint$$

Now $\left(\frac{\partial v}{\partial x}\right) = sint$; $\left(\frac{\partial v}{\partial y}\right) = cost$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(cost) + \frac{\partial z}{\partial v}(sint)$$---------(A)

$$\left(\frac{\partial z}{\partial y}\right) = \frac{\partial z}{\partial u}(-sint) + \frac{\partial z}{\partial v}(cost)$$------(B)

- Multiplying (A) by x & (B) by y

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xcost\frac{\partial z}{\partial u} + x\ sint\frac{\partial z}{\partial v} - ysint\ \frac{\partial z}{\partial u} + ycost\frac{\partial z}{\partial v}$$

$$= \frac{\partial z}{\partial u}(xcost - usint) + \frac{\partial z}{\partial v}(\ xsint + ycost)$$

Thus $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$

Sinhgad Institutes

2) If $u = f(x-y, y-z, z-x)$ then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$ = 0

- Solution : Let $x - y = l$; $y - z = m$; $z - x = n$

Differentiating by chain rule

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial x} \text{ ------ (A)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial y} \text{ --------(B)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\frac{\partial n}{\partial z} \text{ ---------(C)}$$

Since $l = x - y$; $m = y - z$; $n = z - x$

Sinhgad Institutes

$$\frac{\partial l}{\partial x} = 1 \;;\; \frac{\partial m}{\partial x} = 0 \;;\quad \frac{\partial n}{\partial x} = -1$$

$$\frac{\partial l}{\partial y} = -1 \;;\; \frac{\partial m}{\partial y} = 1 \;;\quad \frac{\partial n}{\partial y} = 0$$

$$\frac{\partial l}{\partial z} = 0 \;;\; \frac{\partial m}{\partial z} = -1 \;;\; \frac{\partial n}{\partial z} = 1$$

Equation (A), (B),(C) becomes

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(1) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}(-1) \text{ ------ } (1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(-1) + \frac{\partial u}{\partial m}(1) + \frac{\partial u}{\partial n}(0) \text{ --------- } (2)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-1) + \frac{\partial u}{\partial n}(1) \text{ --------- } (3)$$

- Adding (1),(2) ,(3) we get
- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Example on Composite Function

Q3) If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$,Prove that

$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$

Solution : Let $x^2 - y^2 = l\,, y^2 - z^2 = m\,, z^2 - x^2 = n,$

$\therefore u = f(l, m, n)$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}.\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}.\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}.\frac{\partial n}{\partial x}$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}.(2x) + \frac{\partial u}{\partial m}.(0) + \frac{\partial u}{\partial n}.(-2x)$

similarly $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}.(2x) + \frac{\partial u}{\partial m}.(0) + \frac{\partial u}{\partial n}.(-2x)$

- similarly
- $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}$.(-2y) $+ \frac{\partial u}{\partial m}.(2y) + \frac{\partial u}{\partial n}.(0)$

$\& \frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}.(0) + \frac{\partial u}{\partial m}.(-2z) + \frac{\partial u}{\partial n}.(2z)$

$$\therefore \frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$$

Hence Proved

Sinhgad Institutes

Variable to be Treated As Constant

Consider the

Notation $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial r}{\partial x}\right)_\theta, \left(\frac{\partial v}{\partial y}\right)_{x,u}$ all of them have definite meaning.

- These notations arise when there are more than one functional relations given.
- $\left(\frac{\partial u}{\partial x}\right)_y$ means express u in terms of x & y and differentiate

u partially w . r. t. x keeping y as constant using elimination method.

Sinhgad Institutes

$\left(\frac{\partial r}{\partial x}\right)_{\theta}$ means express r in terms of x & θ and differentiate r partially w.r.t. x keeping θ as constant using elimination method.

$\left(\frac{\partial v}{\partial y}\right)_{x,u}$ means express v in terms of x, y & u and differentiate v partially w.r.t. y keeping x, u as constant using elimination method.

Note

Sinhgad Institutes

In case the elimination of certain variables is not possible or difficult then we interpret these notations as follows :

1) For a given notation variables in denominator are to be treated as independent variables & remaining variables as dependent .For e.g. $(\frac{\partial u}{\partial z})_{x,v}$ x,z,v are independent variables & u, z, y are dependent variables if there are two equations in 5 variables u,v,x,y,z .

2 In these cases we differentiate all the given equations w.r.t. one of the independent variables keeping other independent variables as constants. Therefore we get simultaneous equations in terms of number of derivatives.

3 We eliminate other derivatives to get the required derivative.

Sinhgad Institutes

Examples on variable to be treated as constant

Q1 If $x^2 = au + bv,\ \ y^2 = au - bv$ then show that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

solution: $x^2 = au + bv,\ \ y^2 = au - bv$ $\quad \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2x}$ $\quad \left(\frac{\partial y}{\partial v}\right)_u = -\frac{b}{2y}$

Since $x^2 + y^2 = 2au \qquad x^2 - y^2 = 2bv$

$$u = \frac{x^2 + y^2}{2a} \left(\frac{\partial u}{\partial x}\right)_y = \frac{x}{a}$$

$$v = \frac{x^2 - y^2}{2b} \ \left(\frac{\partial v}{\partial y}\right)_x = -\frac{y}{b}$$

- $\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a} \quad \frac{a}{2x} = \frac{1}{2}$

- $\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \left(-\frac{y}{b}\right)\left(-\frac{b}{2y}\right) = \frac{1}{2}$

Sinhgad Institutes

2) $If\ x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right); y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$

then prove that $\left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}$

Solution : Given $x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right); y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$

$\therefore x = rcosh\theta\ ; y = rsinh\theta$

Differentiating w.r.t. r keeping θ as constant

$\therefore \left(\frac{\partial x}{\partial r}\right)_{\theta} = cosh\theta$ ----(1)

Now we express r as a function of x and y

$x = rcosh\theta\ ; y = rsinh\theta$

$\therefore\ x^2 - y^2 = r^2cosh^2\theta - r^2sinh^2\theta = r^2\ (1) = r^2$

Sinhgad Institutes

$$\therefore\ r^2 = x^2 - y^2$$

Differentiating r with respect to x

keeping y constant

$$2r\left(\frac{\partial r}{\partial x}\right)_y = 2x \therefore \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \frac{r\cosh\theta}{r} = \cosh\theta$$

Thus from (1) and (2)

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y$$

Sinhgad Institutes

3) $x = utanv$; $y = usecv, then\ prove\ that$:

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$$

- Solution : Given $x = utanv; y = usecv$

We have to find u,v are functions of x,y

i.e.we have to find $u\ = f(x, y); v = f(x, y)$

squaring x ,y and subtracting

- $y^2 - x^2 = \ u^2sec^2v - u^2tan^2v$
- $\qquad = \ u^2[sec^2v - tan^2v] = u^2$
- $y^2 - x^2 = u^2$

Sinhgad Institutes

$differnetiate\ u^2$ w.r.t.x keeping y as constant

$$u^2 = y^2 - x^2$$

$$2u\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\ (y^2 - x^2)$$

$$2u\frac{\partial u}{\partial x} = -2x$$

$$\left(\frac{\partial u}{\partial x}\right)_y = \frac{-2x}{2u} = \frac{x}{u} = -\left[\frac{utanv}{u}\right]$$

$\left(\frac{\partial u}{\partial x}\right)_y = -tanv$ -----(1)

Sinhgad Institutes

differentiate u^2 w.r.t. y , keeping x as constant

$$2u\left(\frac{\partial u}{\partial y}\right)_x = \frac{\partial}{\partial y}(y^2 - x^2) = 2y$$

$$\left(\frac{\partial u}{\partial y}\right)_x = \frac{y}{u} = \frac{usecv}{u} = secv \text{-----(2)}$$

Now to find v as function of x,y

$$\frac{x}{y} = \frac{utanv}{usecv} = \frac{sinv}{cosv\left(\frac{1}{cosv}\right)} = sinv$$

$$\frac{x}{y} = sinv \text{-------(I)}$$

Differentiate eq(1) w.r.t.x, keeping y as constant

$$\frac{1}{y}\frac{\partial}{\partial x}(x) = \frac{\partial}{\partial x}(sinv)$$

$$\frac{1}{y} = cosv \ \left(\frac{\partial v}{\partial x}\right)_y$$

$$\left(\frac{\partial v}{\partial x}\right)_y = \frac{1}{ycosv} \text{ -----(3)}$$

Differentiate (I) w.r.t.y keeping x as constant

$$x.\frac{\partial}{\partial y}\left(\frac{1}{y}\right) = \frac{\partial}{\partial y}(sinv)$$

$$\frac{-x}{y^2} = cosv.\left(\frac{\partial v}{\partial y}\right)_x$$

$$\left(\frac{\partial v}{\partial y}\right)_x = \frac{-x}{y^2 cosv} \text{-----}(4)$$

From eq(1) and (3) we get,

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = -tanv \frac{1}{ycosv} = \frac{-sinv}{ycos^2 v}$$

And from eq(2) and(4) ,we get

$$\left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x = secv.\frac{-x}{y^2 cosv} = \frac{-x}{y^2 cos^2 v}$$

This shows that $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial v}{\partial x}\right)_y = \left(\frac{\partial u}{\partial y}\right)_x \left(\frac{\partial v}{\partial y}\right)_x$

4) If $u = 2x + 3y; v = 3x - 2y$ then find the value of

$i)$ $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u$

$ii)$ $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_x = \frac{13}{4}$

$iii)$ $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{4}{13}$

Sinhgad Institutes

Solution : given $u = 2x + 3y$----(1)

$v = 3x - 2y$---(2)

We have to find

$y \ \rightarrow f(x, v); v \rightarrow f(y, u)$

$u \ \rightarrow f(x, y); x \rightarrow f(u, v)$

Differentiate eq(1) w.r.t.x keeping y as constant

$\left(\frac{\partial u}{\partial x}\right)_y = 2$----(I)

To find $y \ \rightarrow f(x, v)$ From eq(2)

$y = \frac{nx - v}{m}$

Sinhgad Institutes

$\therefore \left(\frac{\partial y}{\partial v}\right)_x = \left(\frac{-1}{2}\right)$ ----(II)

To find $v \to f(y,u)$ from eq(1)

$x = \frac{u-3y}{2}$ put in eq(2)

$$v = n\left[\frac{u-3y}{2}\right] - 2y$$

$$\left(\frac{\partial v}{\partial y}\right)_u = \frac{3}{2}(-3) - 2 = \frac{-3^2}{2} - 2$$

$$\left(\frac{\partial v}{\partial y}\right)_u = -\left[\frac{(2^2+3^2)}{2}\right] = \left(\frac{-13}{2}\right)\text{----(III)}$$

Sinhgad Institutes

To find $x \rightarrow f(u, v)$

We do, $m\ x\ eq(1) + n\ x\ eq(2)$

$$2u = 2^2x + 6y$$

$$3v = 3^2x - 6y$$

$$2u + 3v = 13x$$

$$x = \frac{2u+3v}{13}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{2}{13}$$------(IV)

Sinhgad Institutes

From (I),(II),(III),(IV)

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = 2\left(\frac{-1}{2}\right)\left(\frac{-13}{2}\right)\frac{2}{13} = 1$$

from(II)and(III)

ii) $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_x = \frac{13}{4}$

from(I) and(IV)

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{4}{13}$$

University Questions

1) If $z = f(u,v)\,; u = x^2 - 2xy - y^2 \quad and \quad v = y$

Show that $(x+y)\dfrac{\partial z}{\partial x} + (x-y)\dfrac{\partial z}{\partial y} = (x-y)\dfrac{\partial z}{\partial v}$

2) If $x = e^u tanv \quad and\ y = e^u secv$ find the value of

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$$

3) If $z = e^{ax+by} f(ax - by)\,, prove\ that \quad b\dfrac{\partial z}{\partial x} +$

$a\dfrac{\partial z}{\partial y} = 2abz$

Sinhgad Institutes

4) If $x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right)$ & $y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$

Prove that

$(\partial x/\partial r)_{\theta} = (\partial r/\partial x)_{y}$

5) If $ux + vy = 0$ & $\frac{u}{x} + \frac{v}{y} = 1$ prove that

$$\left(\frac{\partial u}{\partial x}\right)_y - \left(\frac{\partial v}{\partial y}\right)_x = \frac{x^2+y^2}{y^2-x^2}$$

Sinhgad Institutes

UNIT III: Partial Differentiation
Topic: Euler’s Theorem

Sinhgad Institutes

HOMOGENEOUS FUNCTIONS

A function z of two variables x and y is said to be *homogeneous function of degree 'n'* if it is possible to express z in the form

$$\mathbf{z = x^n . f\left(\frac{y}{x}\right) \quad OR \quad z = y^n . f\left(\frac{x}{y}\right)}$$

Euler's Theorem : If z is a *homogeneous* expression of two variables x and y *of degree n then ,*

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\,z$$

$$z = x^n f\left(\frac{y}{x}\right) \Rightarrow \frac{\partial z}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + n x^{n-1} f\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial z}{\partial x} = -yx^{n-1}f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) - - - (1)$$

$$\frac{\partial z}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y\frac{\partial z}{\partial y} = yx^{n-1}f'\left(\frac{y}{x}\right) - - - - (2)$$

Adding (1) and (2)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -yx^{n-1}f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) + yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$\text{Hence } x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n f\left(\frac{y}{x}\right) = nz$$

Sinhgad Institutes

Example On Euler's Theorem

1) $If\ T = sin\left[\frac{xy}{x^2+y^2}\right] + \sqrt{x^2 + y^2} + \frac{x^2y}{x+y}$ then find the value of $x\ \frac{\partial T}{\partial x} + y\ \frac{\partial T}{\partial y}\ at\ (3,4\ )$

Solution : $T = sin\left[\frac{xy}{x^2+y^2}\right] + \sqrt{x^2 + y^2} + \frac{x^2y}{x+y}$

Let T = U +V + W

Where $U = sin\left[\frac{xy}{x^2+y^2}\right]$; $V = \sqrt{x^2 + y^2}$; $W = \frac{x^2y}{x+y}$

Sinhgad Institutes

Put $x = xt; y = yt; in\ U, V, W\ we\ get$

$$U = sin\left[\frac{xt\ .yt}{(xt)^2 + (yt)^2}\right] = t^0\ sin\left[\frac{xy}{x^2+y^2}\right]$$

$$V = \sqrt{(xt)^2 + (yt)^2} = t^1\sqrt{x^2 + y^2} = t\ V$$

$$W = \frac{xt^{2\cdot}yt}{xt + yt} = t^2\ \frac{x^2y}{x+y} = t^2w$$

Thus U, V, W are homogeneous functions of degree 0,1,2 respectively

Thus by Euler's theorem

$$x\,\frac{\partial U}{\partial x}+y\frac{\partial U}{\partial y}=nU=0.U=0\text{------}(1)$$

$$x\,\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=nV=1.V=V\text{ ---------}(2)$$

$$x\,\frac{\partial W}{\partial x}+y\frac{\partial W}{\partial y}=nW=2.w=2W\text{ ------}(3)$$

Adding,We get

$$x\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial x}+\frac{\partial W}{\partial x}\right)+y\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial y}\right)=0+1V+2W\text{ ------}(4)$$

But as T = U +V + W

Sinhgad Institutes

- Differentiating T w.r.t. x & y

$$\therefore \frac{\partial T}{\partial x} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial x} + \frac{\partial W}{\partial x}$$

$$\frac{\partial T}{\partial y} = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial y}$$

Equation (4) becomes

$$x.\frac{\partial T}{\partial x} + y.\frac{\partial T}{\partial y} = V + 2W$$

$$\therefore x.\frac{\partial T}{\partial x} + y.\frac{\partial T}{\partial y} = \sqrt{x^2 + y^2} + 2\,\frac{x^2 y}{x+y}$$

At (x , y) = (3,4)

Sinhgad Institutes

$$x.\frac{\partial T}{\partial x} + y.\frac{\partial T}{\partial y} = (5) + 2\ \left(\frac{36}{7}\right) = \ \frac{107}{7}$$

Sinhgad Institutes

Deductions From Euler's Theorem :

(1) If z is a *homogeneous* expression of two variables x and y *of degree n then ,*

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)\,z$$

(2) If z is a *homogeneous* function of two variables x and y of degree n and z = f(u) then ,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\,\frac{f(u)}{f'(u)}$$

(3) $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=g(u)[g'(u)-1]$

Where $g(u)=n\,\frac{f(u)}{f'(u)}$

Sinhgad Institutes

HOMOGENEOUS FUNCTIONS OF THREE VARIABLES.

A function u = f(x , y , z) is said to be homogeneous function of three variables x , y , z of degree ‘n’ if it is possible to express u in the form

$$\mathbf{u = x^n . f\left(\frac{y}{x} , \frac{z}{x}\right) \quad OR \quad u = y^n . f\left(\frac{x}{y} , \frac{z}{y}\right)}$$

Example On Deduction (1)

1) $If\ z = x^n f\left(\frac{y}{x}\right) + y^{-n}\emptyset\left(\frac{x}{y}\right)$ then prove that

$x^2 z_{xx} + 2xy\, z_{yy} + x.z_x + y.z_y = n^2 z$

Solution: Given $z = x^n f\left(\frac{y}{x}\right) + y^{-n}\emptyset\left(\frac{x}{y}\right)$

$Z = U + V$ --------(1)

Where $U = x^n f\left(\frac{y}{x}\right)$; $V = y^{-n}.\emptyset\left(\frac{x}{y}\right)$

Sinhgad Institutes

$\therefore$ U and V are homogeneous functions

in x & y of degree n and –n respectively

$\therefore$ By Euler's theorem

$x\, U_x + y\, U_y = nU$ ------(2)

$x\, V_x + y\, V_y = -nV$-------(3)

And

$x^2U_{xx} + 2xyU_{xy} + y^2U_{yy} = n\,(n-1)U$-------(4)

$x^2V_{xx} + 2xyV_{xy} + y^2V_{yy} = -n\,(-n-1)V$

$= n(n+1)$-------(5)

Sinhgad Institutes

Adding (2) ,(3) ,(4) and (5) we get

$$x^2(U_{xx}+V_{xx})+2xy\left(U_{xy}+V_{xy}\right)$$
$$+\quad y^2\left(U_{yy}+V_{yy}\right)+x(U_x+V_x)+y\left(U_y+V_y\right)$$
$$=n(n-1)U+n(n+1)V+nU-nV$$
$$=n^2U-nU+n^2V+nV+nU-nV$$
$$=n^2(U+V)\text{-------}(6)$$

Now from equation (1)

$$Z=U+V$$

Differentiating Z w.r.t. x & y

$$Z_x=U_x+V_x \text{ and } Z_y=U_y+V_y$$

Sinhgad Institutes

$$\therefore Z_{xy} = U_{xy} + V_{xy}; Z_{xx} = U_{xx} + V_{xx};$$
$$Z_{xy} = U_{yy} + V_{yy};$$

$$x^2 Z_{xx} + 2xy\, Z_{xy} + y^2 Z_{yy} + x\, Z_x + y Z_y = n^2 Z$$

Equation (6) becomes

EXAMPLES On Deduction (2)

EX 1) : If $x = \mathrm{e^u tanv}$ and $\mathrm{y} = \mathrm{e^u \sec v}$ find the value of

$$\left(\mathrm{x}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{u}}{\partial \mathrm{y}}\right)\left(\mathrm{x}\frac{\partial \mathrm{v}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{v}}{\partial \mathrm{y}}\right) = 0$$

Eliminating v between the given relation

$$y^2 - \mathrm{x}^2 = \mathrm{e}^{2\mathrm{u}} = \mathrm{z}\ (\mathrm{say})$$

Then $z = x^2\left(\frac{y^2}{x^2} - 1\right) = x^2 f\left(\frac{y}{x}\right)$ *and also* $z = f(u) = e^{2u}$

$\Rightarrow z\ is\ a\ Homogeneous\ function\ of\ x\ and\ y\ of\ degree\ 2.$

$\therefore$ from deduction of Euler$'$s theorem

$$x\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} = n\ \frac{f(u)}{f'(u)} = 2\frac{e^{2u}}{2e^{2u}} = 1\ ---(1)$$

Sinhgad Institutes

Now eliminating u between the given relation

$\frac{y}{x} = \frac{\sec v}{\tan v} = \operatorname{cosec} v = z \text{ (say)}, \text{Here } z = x^0 \left(\frac{y}{x}\right)$, also $z = g(v) = \operatorname{cosec} v$

$\Rightarrow z$ *is a Homogeneous function of x and y of degree* 0.

$\therefore$ from deduction of Euler′s theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = n\ \frac{g(v)}{g'(v)} = 0 \times \frac{cosec\ v}{-cot^2 v} = 0 \ ---(2)$$

$\therefore$ from (1)and (2)we get the required result.

$$\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 1 \times 0 = 0$$

Sinhgad Institutes

2) $If\ u = \ sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\ then\ prove\ that$

$$2x\ \frac{\partial u}{\partial x} + 2y\ \frac{\partial u}{\partial y} = \tan u$$

Solution : Given $u = \ sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ --------(1)

Put $x = xt : y = yt\ in\ equation\ (1) we\ get$

$$u = \ sin^{-1}\left(\frac{xt\ + yt}{\sqrt{xt} + \ \sqrt{yt}}\right) = \ sin^{-1}\left[t^{1/2}\left(\frac{x + y}{\sqrt{x} + \ \sqrt{y}}\right)\right]$$

Is not homogeneous function

But $f(u) = \sin u = t^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) = t^{1/2}u$ is homogeneous function in x & y of degree ½

Thus, by Euler's theorem

$$x\,\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,\frac{f(u)}{f'(u)} = \frac{1}{2}\,\frac{\sin u}{\cos u}$$

$$\therefore\ 2x\,\frac{\partial u}{\partial x} + 2y\,\frac{\partial u}{\partial y} = \tan u$$

EX : If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, Show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (\,1 - 4sec^2u\,)\sin 2u$$

Given $\tan u = \frac{x^3+y^3}{x-y} = x^2\left[\frac{1+y^3/x^3}{1-y/x}\right] = x^2 f\left(\frac{y}{x}\right) = z(say)$

Also z = f (u) = tan u

Homogeneous function of x and y of degree 2.

By $\text{Euler}'\text{s theorem}$

$$\mathrm{x}\frac{\partial \mathrm{u}}{\partial \mathrm{x}} + \mathrm{y}\frac{\partial \mathrm{u}}{\partial \mathrm{y}} = \mathrm{n}\;\frac{\mathrm{f(u)}}{\mathrm{f}'(\mathrm{u})} = 2\frac{\tan \mathrm{u}}{\sec^2 \mathrm{u}}$$

$$= 2\sin \mathrm{u}\cos \mathrm{u} = \mathrm{g(u)}\,\mathrm{say} - -(1)$$

Sinhgad Institutes

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = g(u)[g^{'}(u) - 1] \text{ , Where } g(u) = n \frac{f(u)}{f^{'}(u)}$$

$$= \sin 2u \left[2\cos 2u - 1\right]$$

$$= \sin 2u \left[2(1 - 2\sin^2 u) - 1\right]$$

$$= \sin 2u \left[1 - 4\sin^2 u\right]$$

EX : If $T = \sin\left(\frac{xy}{x^2+y^2}\right) + \sqrt{x^2+y^2} + \frac{x^2y}{x+y}$ then

find $x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y}$

Let $T = u + v + w$ Then

$$x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} = \left(x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y}\right) + \left(x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right) + \left(x\frac{\partial W}{\partial x} + y\frac{\partial W}{\partial y}\right) - -(1)$$

$$u = \sin\left(\frac{xy}{x^2+y^2}\right) \Rightarrow \sin^{-1} u = \frac{xy}{x^2+y^2}$$

$$= \frac{x^2(y/x)}{x^2\left(1+\frac{y^2}{x^2}\right)} = x^0 f\left(\frac{y}{x}\right) = z$$

and also $\sin^{-1} u = z$

u is homogeneous function of x, y order 0.

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\ \frac{f(u)}{f'(u)} = 0 \ \text{-}-(1)$$

$$v = \sqrt{x^2+y^2} = x\sqrt{1+\frac{y^2}{x^2}} = xf\left(\frac{y}{x}\right)$$

It is homogeneous function of x, y, of order 1

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,v = 1.\sqrt{x^2+y^2} = \sqrt{x^2+y^2} \quad --(2)$$

$$w = \frac{x^2y}{x+y} = \frac{x^3(y/x)}{x(1+y/x)} = x^2 f\left(\frac{y}{x}\right)$$

So w is homogeneous function of x ,y of order 2.

By Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\,v = 2.\frac{x^2y}{x+y} \quad --(3)$$

Putting the values from (1) , (2) and (3)

$$x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} = 0 + \sqrt{x^2+y^2} + \frac{2x^2y}{x+y} = \sqrt{x^2+y^2} + \frac{2x^2y}{x+y}$$

Example On Deduction (3)

- $If\ u = cosec^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

Put $x = xt$; $y = yt\ in\ u$,we get

$$u = cosec^{-1}\sqrt{\frac{(xt)^{1/2}+(yt)^{1/2}}{(xt)^{1/3}+(yt)^{1/3}}}$$

$$u = cosec^{-1}\sqrt{\frac{t^{1/2}\left(x^{1/2}+y^{1/2}\right)}{t^{1/3}\left(x^{1/3}+y^{1/3}\right)}}$$

$$u = cosec^{-1}\sqrt{\frac{t^{1/4}\sqrt{x^{1/2}+y^{1/2}}}{t^{1/6}\sqrt{x^{1/3}+y^{1/3}}}}$$

$$u = cosec^{-1}\left[t^{1/2}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right]$$

U is not homogeneous function

But $f(u) = cosecu = t^{1/2}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

Sinhgad Institutes

is homogeneous function in x & y of degree ½

Thus, by Euler's theorem on homogeneous function, we get

$$x^2u_{xx} + 2xy\,u_{xy} + \ y^2u_{yy} = G(u)[G'(u) - 1\,]$$ -----(1)

Where $G(u) = \ \frac{n\,f(u)}{f'(u)} = \ \frac{1}{12}.\frac{cosec\,u}{-cosecu.\cot u} = \ \frac{-1}{12}\ sec^2u$

Equation (1) becomes

$$x^2u_{xx} + 2xy\,u_{xy} + \ y^2u_{yy} = \ \frac{-1}{12}\ tanu\left[\frac{-1}{12}\ sec^2u - 1\right]$$

$$= \frac{1}{144} \tan u \, [sec^2 u + 12]$$

$$= \frac{1}{144} \tan u \; [tan^2 u + 1 + 12]$$

$$= \frac{1}{144} \tan u \; [tan^2 u + 13]$$

TOTAL DERIVATIVE

If $u = f(x\,,\; y)$, where $x = \emptyset(t)$ and $y = \psi(t)$ then

u can be expressed as function of single variable t.

and derivative of u w.r.t. t is ordinary differential coefficient $\frac{du}{dt}$

This $\frac{du}{dt}$ is called as Total Derivative.

If we put $t = x$ in (1) then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

Similarly if we put $t = y$ in (1) then

$$\frac{du}{dy} = \frac{\partial u}{\partial x}\frac{dx}{dy} + \frac{\partial u}{\partial y}$$

DIFFERENTIATION OF IMPLICIT FUNCTION

If $z = f(x, y) = 0$, y is function of x then

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \text{i.e.} \quad \frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} - - - (1)$$

Since $z = f(x, y) = 0 \Longrightarrow \frac{dz}{dx} = 0$

$$\therefore \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} = 0 \Longrightarrow \frac{dy}{dx} = -\frac{\frac{\partial z}{\partial x}}{\frac{\partial z}{\partial y}} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{p}{q}$$

$$\text{as} \ \ p = \frac{\partial z}{\partial x} \ , \ \ q = \frac{\partial z}{\partial y} \ \ , \ \ r = \frac{\partial^2 z}{\partial x \ \partial y} \ \ , \ \ s = \frac{\partial^2 z}{\partial x} \ \ , \ \ t = \frac{\partial^2 z}{\partial y}$$

$$or \ \ p = \frac{\partial f}{\partial x} \ , \ \ q = \frac{\partial f}{\partial y} \ \ , \ \ r = \frac{\partial^2 f}{\partial x \ \partial y} \ \ , \ \ s = \frac{\partial^2 f}{\partial x} \ \ , \ \ t = \frac{\partial^2 f}{\partial y}$$

Sinhgad Institutes

Differentiating once more we get

$$\therefore \frac{d^2y}{dx^2} = -\left[\frac{q^2r - 2pqs + p^2t}{q^3}\right]$$

This can also be remembered as

$$\therefore \frac{d^2y}{dx^2} = \frac{\det\begin{bmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{bmatrix}}{q^3}$$

EXAMPLES.

Sinhgad Institutes

EX : If $u = x \log xy$ and $x^3 + y^3 + 3xy = 0$ then find $\frac{du}{dx}$

Given $u = x \log xy$ and $x^3 + y^3 + 3xy = 0$

u is composite function of x

$$\therefore \quad \frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} - - - (1)$$

$$u = x \log xy \quad \text{and} \quad f(x, y) \equiv x^3 + y^3 + 3xy = 0$$

$$\frac{\partial u}{\partial x} = x\left(\frac{1}{xy}\right)y + \log xy = 1 + \log xy - - - (2)$$

$$\frac{\partial u}{\partial y} = x\left(\frac{1}{xy}\right)x = \frac{x}{y} - - - \quad (3)$$

$$\frac{dy}{dx} = -\frac{p}{q} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{3x^2+3y}{3y^2+3x} \quad - - - (4)$$

Sinhgad Institutes

Putting the values from (2) , (3) and (4) in (1)

$$\frac{du}{dx} = 1 + \log xy + \frac{x}{y}\left(-\frac{3x^2 + 3y}{3y^2 + 3x}\right)$$

$$= 1 + \log xy - \left(\frac{x^3 + xy}{y^3 + xy}\right)$$

$$= \log xy + \frac{y^3 + xy - x^3 - xy}{y^3 + xy}$$

$$\therefore \frac{du}{dx} = \log xy + \frac{y^3 - x^3}{y^3 + xy}$$

EX: If $(\cos x)^y = (\sin y)^x$ then find $\frac{dy}{dx}$

EXAMPLES ON TOTAL DERIVATIVES

Ex : If $x^2y - e^z + x\sin z = 0$ and $x^2 + y^2 + z^2 = a^2$

Then evaluate $\frac{dy}{dx}$ and $\frac{dx}{dz}$

$$\text{Let } f(x,y,z) = x^2y - e^z + x\sin z = 0$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$0 = (2xy + \sin z)dx + (x^2)dy + (-e^z + x\cos z)dz — (1)$$

$$\text{Let } \emptyset(x,y,z) = x^2 + y^2 + z^2 - a^2 = 0$$

$$d\emptyset = \frac{\partial \emptyset}{\partial x}dx + \frac{\partial \emptyset}{\partial y}dy + \frac{\partial \emptyset}{\partial z}dz$$

$$0 = x\,dx + y\,dy + z\,dz - - - -(2)$$

From (1) and (2) by Cramer's rule for dx , dy , dz

$$\frac{dx}{y(-e^z + x\cos z) - zx^2} = \frac{dy}{z(2xy + \sin z) - x(-e^z + x\cos z)}$$
$$= \frac{dz}{x^3 - y(2xy + \sin z)}$$

$$\frac{dy}{dx} = \frac{z(2xy + \sin z) - x(-e^z + x\cos z)}{y(-e^z + x\cos z) - zx^2}$$

$$\frac{dx}{dz} = \frac{y(-e^z + x\cos z) - zx^2}{x^3 - y(2xy + \sin z)}$$

Sinhgad Institutes

2) If $u = \mathrm{x}\log(xy)$ and $x^3 + y^3 + 3xy = 0$ then find $\frac{du}{dx}$ at (1,1)

Solution: Given $u = x\log(xy) = x[logx + logy]$

&

$x^3 + y^3 + 3xy = 0$

By differentiating u w.r.t.x by chain rule

$\frac{du}{dx} = \frac{\partial u}{\partial x}.\frac{dx}{dx} + \frac{\partial u}{\partial y}.\frac{dy}{dy}$ ------(1)

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}x[logx + logy] = 1 + logx + logy$

Sinhgad Institutes

And

$$\frac{\partial u}{\partial y} = \frac{x}{y}$$

Eq(1) becomes

$$\frac{du}{dx} = [1 + logx + logy] + \frac{x}{y}.\frac{dy}{dx} \quad \text{------(2)}$$

As $f = x^3 + y^3 + 3xy - 1$

Sinhgad Institutes

We have $\frac{dy}{dx} = -\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right] = -\left[\frac{3x^2+3y}{3y^2+3x}\right]=$

$$= -\left[\frac{x^2+y}{y^2+x}\right]$$

$eq(2)$ becomes

$$\frac{du}{dx} = [1 + logx + logy] + \frac{x}{y}$$

Sinhgad Institutes

$$=\left[1+logx+logy\right]-\frac{x}{y}.\left[\frac{x^2+y}{y^2+x}\right]$$

$$\left(\frac{du}{dx}\right)_{(1,1)}=(1+0+0)-\frac{1}{1}\left[\frac{1+1}{1+1}\right]$$

$$\left(\frac{du}{dx}\right)_{(1,1)}= 1-1$$

$$\left(\frac{du}{dx}\right)_{(1,1)}=0$$

Sinhgad Institutes

3) If $f(x, y) = 0; \emptyset(x, z) = 0$ then prove that

$$\frac{\partial \emptyset}{\partial x}\frac{\partial f}{\partial y}\frac{dy}{dz} = \frac{\partial f}{\partial x}\frac{\partial \emptyset}{\partial z}$$

Solution : Given

$f(x, y) = 0; \emptyset(x, z) = 0$

$f \rightarrow x, y \rightarrow x \;\&\; \emptyset \rightarrow x, z \rightarrow x$

Means $f \rightarrow x \;\&\; y \rightarrow x$

$$\emptyset \to x \;\&\; z \to x$$

Thus by chain rule

$$df = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$df = \frac{\partial \emptyset}{\partial x}\frac{dx}{dx} + \frac{\partial \emptyset}{\partial y}\frac{dy}{dx}$$

$$f = 0 \; and \; \emptyset = 0$$

Thus df = 0 and d$\emptyset = 0$

Sinhgad Institutes

$$\therefore\ \frac{dy}{dx} = -\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right] \text{ And } \frac{dz}{dx} = -\left[\frac{\frac{\partial \emptyset}{\partial x}}{\frac{\partial \emptyset}{\partial z}}\right]$$

$$\frac{\left(\frac{dy}{dx}\right)}{\left(\frac{dz}{dx}\right)} = \frac{-\left[\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}\right]}{-\left[\frac{\frac{\partial \emptyset}{\partial x}}{\frac{\partial \emptyset}{\partial z}}\right]}$$

Sinhgad Institutes

Therefore $\frac{dy}{dz} = \frac{\frac{\partial f}{\partial x} \cdot \frac{\partial \emptyset}{\partial z}}{\frac{\partial \emptyset}{\partial x} \cdot \frac{\partial f}{\partial y}}$

And $\frac{\partial \emptyset}{\partial x} \frac{\partial f}{\partial y} \frac{dy}{dz} = \frac{\partial f}{\partial x} \frac{\partial \emptyset}{\partial z}$

3) Find $\frac{dz}{dx}$ $if\ z = x^2 y\ \& x^2 + xy + y^2 = 1$

Solution : $z = x^2 y\ \& x^2 + xy + y^2 = 1$

Differentiate z w.r.t x ,keeping y as constant and then differentiate z w.r.t y,keeping x as constant

$$\frac{\partial z}{\partial x} = 2xy \;\&\; \frac{\partial z}{\partial y} = x^2$$

As $f(x, y) = x^2 + xy + y^2 - 1$

$$\frac{dy}{dx} = -\left[\frac{2x+y}{x+2y}\right]$$

By chain rule $z \to x, y \to x$

Sinhgad Institutes

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx} = (2xy)(1) + (x^2)\left(-\left[\frac{2x+y}{x+2y}\right]\right)$$

Thus $\frac{dz}{dx} = 2xy - x^2\left[\frac{2x+y}{x+2y}\right]$

Sinhgad Institutes

Ex : If $ax^2 + by^2 + cz^2 = 1$ and $lx + my + nz = 0$

Then prove that

$$\frac{dx}{bny - cmz} = \frac{dy}{clz - anx} = \frac{dz}{amx - bly}$$

Sinhgad Institutes

References:-

1) Higher Engineering Mathematics –

   B. V. Raman

2) Advanced Engineering Mathematics -

   H. K. Dass

3) Advanced Engineering Mathematics --