Sinhgad Institutes

STES's

SKN Sinhgad Institute of Technology and Science, Lonavala.

F.E., Semester - I (2020-21)

Engineering Mathematics I

Notes Unit VI

(Eigen values and eigen vectors)

For a given non-zero square matrix A if there exists a scalar λ and a non-zero vector $X = [x_1, x_2, \dots, x_n]$ such that $\boldsymbol{AX} = \boldsymbol{\lambda X}$ then λ is called *eigen value* or characteristic root and X is called *eigen vector* or characteristic vector.
The set of all eigen values of A is called *spectrum* A.

For square matrix A of order 3

Characteristic equation of A is

$$|A - \lambda I| = \lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$$

where

$$S_1 = a_{11} + a_{22} + a_{33}$$

$$S_2 = M_{11} + M_{22} + M_{33} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

Roots of Ch. Eqn gives eigen values, for finding eigen vector consider matrix equation

$$(A - \lambda I)X = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

After putting eigen values if we get any two distinct rows use Cramer's rule to get eigen vetor and if all the rows are similar rows then put arbitrary values for two variables say $x_2 = s,\ x_3 = t$ then separate the matrix in to two matrices one containing only s terms and other only t terms which gives two eigen vectors.

Verification eigen values and eigen vectors:

$$S_1 = \lambda_1 + \lambda_2 + \cdots + \lambda_n \text{ and } |A| = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

For **symmetric matrix,** $X_1^T \cdot X_2 = 0 = X_2^T \cdot X_3 = X_3^T \cdot X_1$ where X_1, X_2, X_3 are eigen vectors corresponding to *distinct eigen values*.

Cayley Hamilton Theorem:

Every square matrix satisfies its own characteristic equation.

If $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$ is characteristic equation of A then by Cayley Hamilton theorem we have $A^3 - S_1A^2 + S_2A - |A|I = 0$.

1) Find the eigen values and eigen vectors for the matrix

i) $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & -4 \\ 1 & -1 & -2 \end{bmatrix}$

iv)$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

v) $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

vi) $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -5 & -2 \end{bmatrix}$

vii) $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

viii) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

ix) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

x) $A = \begin{bmatrix} 7 & -2 & -2 \\ -2 & 1 & 4 \\ -2 & 4 & 1 \end{bmatrix}$

2) Verify the Cayley-Hamilton theorem for the matrix

i) $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

iii) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$, Hence find A^{-1}.

iv) $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, Hence find A^{-1}.

v) $A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$, Hence find A^{-1}.

Diagonalization of a Matrix

For a given square matrix A of order n having n linearly independent eigen vectors can be written as

$$D = P^{-1}AP$$

where D is a diagonal matrix called as spectral matrix with eigen values of A as entries on diagonal and P is non-singular matrix called as modal matrix having eigen vectors of A as columns.

Let A be a 3×3 matrix with X_1, X_2, X_3 be the eigen vectors corresponding to eigen values $\lambda_1, \lambda_2, \lambda_3$ then

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad P = [\,X_1 \quad X_2 \quad X_3].$$

1. Find a matrix P such that $P^{-1}AP$ is a diagonal matrix where $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

2. Find the modal matrix P which transform the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ to the diagonal form.

3. Find the modal matrix P which diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Quadratic Form

A homogenous polynomial of the second degree in any number of variables is called a quadratic form

A quadratic form

$$Q(x) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3$$

can be written in matrix form as

$$Q(x) = X'AX = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

where $a_{ij} = a_{ji} \ (i \neq j)$.

The quadratic form $Q(x) = X'AX$ can be reduce to another quadratic form $Q'(x) = Y'BY = c_1y_1^2 + c_2y_2^2 + \cdots + c_ry_r^2$ called as sum of the square terms form by non-singular transformation $X = PY$, then the reduced quadratic form $Q'(x) = Y'BY$ is called Canonical form or sum of the square form.

In this case, matrix B will be a diagonal matrix.

The number of positive terms in canonical form is called index and is denoted by , the rank r of B is called rank of the quadratic form.

The number of negative terms in the canonical form $= r - p$

The difference between positive terms and negative terms is called as signature of quadratic form and is denoted by s

To reduce the given quadratic form $Q(x) = X'AX$ to canonical or sum of the square form $Q'(x) = Y'BY$ and to find matrix P of the linear transformation $X = PY$, consider $A = IA$.

By performing identical row and column transformation on matrix A on L.H.S. to obtain diagonal matrix B, while perform only corresponding row transformation on prefactor matrix I on R.H.S. Then we get $B = P'A$

1. Reduce the following quadratic form to the "sum of the squares form". Find the corresponding linear transformation. Also find the index and signature

$$Q(x) = 2x_1^2 + 9x_2^2 + 6x_3^2 + 8x_1x_2 + 8x_2x_3 + 6x_3x_1$$

2. Find the transformation which will reduce the following quadratic form to "sum of the squares form"

$$Q(x) = 10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_3x_1 - 4x_1x_2$$