Unit II : Fourier Series

Periodic Functions:

For every real $f(x)$ and there exists some positive number T such that $F(x+nT) = F(x)$ is callled Periodic Function.

T is called permitive period or fundamental period of $f(x)$

Example: The fundamental period of $sin\, x, cos\, x, sec\, x, cosec\, x$ is 2π and $tan\, x, cot\, x$ is π

Even Function: Function $f(x)$ is defined in $-l < x < l$ is said to be even if

$f(x) = f(-x)$ Example: $cos\, x$, x^2

Odd Function: Function $f(x)$ is defined in $-l < x < l$ is said to be odd if

$f(x) = -f(-x)$ Example: $sin\, x$, x^3, $tan\, x$

Note:

1) If $f(x)$ is even, the values of y for $-x$ and x are same, therefore graph of $y = f(x)$ is symmetric about $x-$ axis. 2) If $f(x)$ is odd, the values of y for $-x$ and x differ by sign only therefore graph of $y = f(x)$ is symmetric about origin(opposite quadrants).

3) If $f(x)$ is Even function of x, $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

4) If $f(x)$ is Odd function of x, $\int_{-a}^{a} f(x)dx = 0$

5) Any function $f(x)$ can be expressed as sum of even and odd functions

$$f(x) = \left[\frac{f(x)+f(-x)}{2}\right] + \left[\frac{f(x)-f(-x)}{2}\right]$$

6)

Sr. No	$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x) \begin{smallmatrix}\times \\ \div\end{smallmatrix} g(x)$
1)	Even	Even	Even	Even
2)	Odd	Odd	Odd	Even
3)	Even	Odd	Neither Odd nor Even	Odd
4)	Odd	Even	Neither Odd nor Even	Odd

"The Only things that will stop you from fulfilling your dreams is you"

Formula:

1) $\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx$

2) $\int \sin x \sin nx\ dx \ = \frac{1}{2}\int[\cos(1-n)x - \cos(1+n)\,x]\,dx$

3) $\int \cos x \cos nx\,dx \ = \frac{1}{2}\int[\cos(1+n)x + \cos(1-n)\,x]\,dx$

4) $\int \sin x \cos nx\,dx \ = \frac{1}{2}\int[\sin(1+n)x + \sin(1-n)\,x]\,dx$

5) $\int \cos x \sin nx\,dx \ = \frac{1}{2}\int[\sin(1+n)x + \sin(1-n)\,x]\,dx$

6) $\cos n\pi = (-1)^n$ $\qquad$ $\cos 2n\pi = 1$

7) $\sin n\pi = 0$ $\qquad$ $\sin 2n\pi = 0$

8) $\int e^{ax}\sin bx\ dx \ = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

9) $\int e^{ax}\cos bx\ dx \ = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$

10) $\int uv dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + u''''v_5 - u'''''v_6 + \ \dots\dots\dots$

where dashes($'\ ''\ '''\ ''''$ ...)indicate derivitives and suffixes (1,2,3..)indicates integrals

Dirchlet's Condition:

Let $f(x)$ be function defined in $C < x < C + 2L$ such that

i) $f(x)$is defined and single valued in the given interval also $\int_C^{C+2L} f(x)dx$ exits

ii) $f(x)$ may have finite number of finite discontinuities in the interval.

iii) $f(x)$ may have finite number of maxima or minima in the given interval.

Fourier Series :

Let $f(x)$ be periodic function of period $2L$ defined in the interval $C < x < C + 2L$ and satisfies Dirchlet's Conditions then $f(x)$ can be expressed as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$$

where a_0 , a_n , b_n are called Fourier constants or Fourier coeficients and given by

$$a_0 = \frac{1}{L}\int_C^{C+2L} f(x)dx$$

$$a_n = \frac{1}{L}\int_C^{C+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_C^{C+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Series For different interval

Fourier Series in the interval $(0, 2L)$	Fourier Series in the interval $(0, 2\pi)$
$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$
$a_0 = \frac{1}{L}\int_0^{2L} f(x)dx$	$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$
$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$	$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$
$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$

Example: *Find the Fourier series of the function* $f(x) = e^{-x};\ \ 0 \le x \le 2\pi$ *and* $f(x + 2\pi) = f(x)$

Solution: The Fourier series of $f(x)$ in $0 \le x \le 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$\mathbf{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} e^{-x}dx$$

$$= \frac{1}{\pi}(-e^{-x})_0^{2\pi} = \frac{1}{\pi}[-e^{-2\pi} - (-e^0)] = \frac{1}{\pi}[-e^{-2\pi} + 1] = \frac{1}{\pi}[1 - e^{-2\pi}]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\, dx \qquad \because \int e^{ax}\cos bx\ dx = \frac{e^{ax}}{a^2 + b^2}[a\cos bx + b\sin bx]$$

$a = -1$ **and** $b = n$

$$a_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1 + n^2}(-\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(-\cos 2n\pi + n\sin 2n\pi)\right] - \left[\frac{e^0}{1+n^2}(-\cos 0 + n\sin 0)\right]\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(-1+0)\right] - \left[\frac{1}{1+n^2}(-1+0)\right]\right\} \qquad \because \cos 2n\pi = 1 \quad \sin 2n\pi = 0$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-e^{-2\pi}}{1+n^2}\right] - \left[\frac{-1}{1+n^2}\right]\right\} = \frac{-e^{-2\pi}+1}{\pi(1+n^2)} = \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx \qquad \because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$$

$a = -1$ and $b = n$

$$b_n = \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(-\sin 2n\pi - n\cos 2n\pi)\right] - \left[\frac{e^0}{1+n^2}(-\sin 0 - n\cos 0)\right]\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{e^{-2\pi}}{1+n^2}(0-n)\right] - \left[\frac{1}{1+n^2}(0-n)\right]\right\} \qquad \because \cos 2n\pi = 1 \quad \sin 2n\pi = 0$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{-ne^{-2\pi}}{1+n^2}\right] - \left[\frac{-n}{1+n^2}\right]\right\} = \frac{-ne^{-2\pi}+n}{\pi(1+n^2)} = \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$$

$$e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{1-e^{-2\pi}}{\pi(1+n^2)}\cos nx + \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx\right]$$

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Example: Find the Fourier series of the functions $f(x) = x^2; \; 0 \le x \le 2\pi$ and $f(x + 2\pi) = f(x)$

Solution: The Fourier series of $f(x)$ in $0 \le x \le 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx$$

$$= \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi} = \frac{1}{\pi}\left[\frac{(2\pi)^3}{3} - 0\right) = \frac{1}{\pi}\left[\frac{8\pi^3}{3}\right] = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx$$

$$a_n = \frac{1}{\pi}\left\{x^2 \frac{\sin nx}{n} - (2x)\left(\frac{-\cos nx}{n*n}\right) + (2)\left(\frac{-\sin nx}{n*n*n}\right)\right\}_0^{2\pi}$$

$$a_n = \frac{1}{\pi}\left\{(2\pi)^2 \frac{\sin 2n\pi}{n} + (2*2\pi)\left(\frac{\cos 2n\pi}{n^2}\right) + (2)\left(\frac{-\sin 2n\pi}{n^3}\right)\right\}$$

$$-\left\{(0)^2 \frac{\sin 0}{n} + (2*0)\left(\frac{\cos 0}{n^2}\right) + (2)\left(\frac{-\sin 0}{n^3}\right)\right\}$$

$$a_n = \frac{1}{\pi}\left\{0 + \frac{4\pi}{n^2} + 0\right\} - \{0 - 0 + 0\} \quad \because \cos 2n\pi = 1 \quad \sin 2n\pi = 0$$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx$$

$$b_n = \frac{1}{\pi}\left\{x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n*n}\right) + (2)\left(\frac{\cos nx}{n*n*n}\right) - 0\right\}_0^{2\pi}$$

$$b_n = \frac{1}{\pi}\left\{(2\pi)^2\left(\frac{-\cos 2\pi n}{n}\right) + (2*2\pi)\left(\frac{\sin 2n\pi}{n^2}\right) + (2)\left(\frac{\cos 2n\pi}{n^3}\right)\right\}$$

$$-\left\{(0)^2\frac{\cos 0}{n}+(2*0)\left(\frac{\sin 0}{n^2}\right)+(2)\left(\frac{\cos 0}{n^3}\right)\right\}$$

$$b_n=\frac{1}{\pi}\left\{-\frac{4\pi^2}{n}\quad+0+\frac{2}{n^3}\right\}-\left\{0+0+\frac{2}{n^3}\right\}\qquad \because \boldsymbol{\cos 2n\pi=1}\quad \boldsymbol{\sin 2n\pi=0}$$

$$b_n=\frac{1}{\pi}\left\{-\frac{4\pi^2}{n}\quad+0+\frac{2}{n^3}-\frac{2}{n^3}\right\}$$

$$b_n=-\frac{4\pi}{n}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]$$

$$x^2=\frac{8\pi^2}{3}\quad+\sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right]$$

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Example: Find the Fourier expansion of the periodic function $f(x)=\cos ax\quad(0,2\pi)$

a is not an integer

Solution: $f(x)=\cos ax\quad(0,2\pi)\quad period=2\pi$

The Fourier series of $f(x)$ is given by $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]\ \ldots\ldots(1)$

$$\boldsymbol{a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx}\qquad \boldsymbol{a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx}\qquad \boldsymbol{b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx}$$

$$\boldsymbol{a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx}$$

$$=\frac{1}{\pi}\int_0^{2\pi}\cos ax\,dx$$

$$=\frac{1}{\pi}\left[\frac{\sin ax}{a}\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{\sin 2\pi a}{a}-\frac{\sin 0}{a}\right]$$

$$=\frac{\sin 2\pi a}{a\pi}-\frac{0}{a}=\frac{\sin 2\pi a}{a\pi}$$

$$\boldsymbol{a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\cos ax\cos nx\,dx\qquad \because \boldsymbol{\cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)]}$$

$A = ax$ and $B = nx$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}[\cos(ax+nx)+\cos(ax-nx)]dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}[\cos(a+n)x+\cos(a-n)x]dx$$

$$a_n = \frac{1}{2\pi}\left[\frac{\sin(a+n)x}{(a+n)}+\frac{\sin(a-n)x}{(a-n)}\right]_0^{2\pi}$$

$$a_n = \frac{1}{2\pi}\left\{\frac{\sin(a+n)2\pi}{(a+n)}+\frac{\sin(a-n)2\pi}{(a-n)}-\frac{\sin(a+n)0}{(a+n)}-\frac{\sin(a-n)0}{(a-n)}\right\}$$

$$a_n = \frac{1}{2\pi}\left\{\frac{\sin 2a\pi}{(a+n)}+\frac{\sin 2a\pi}{(a-n)}\right\}$$ $\because \sin(a+n)2\pi = \sin 2a\pi, \sin(a-n)2\pi = \sin 2a\pi$

$$a_n = \frac{\sin 2a\pi}{2\pi}\left\{\frac{1}{(a+n)}+\frac{1}{(a-n)}\right\}$$

$$a_n = \frac{\sin 2a\pi}{2\pi}\left\{\frac{(a-n)+(a+n)}{(a+n)(a-n)}\right\}$$ $\because (a^2-b^2) = (a+b)(a-b)$

$$a_n = \frac{\sin 2a\pi}{2\pi}\left\{\frac{2a}{a^2-n^2}\right\} = \frac{a\sin 2a\pi}{\pi(a^2-n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\sin nx\cos ax\,dx$$ $\because \sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}[\sin(nx+ax)+\sin(nx-ax)]dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi}[\sin(n+a)x+\sin(n-a)x]dx$$

$$b_n = \frac{1}{2\pi}\left[\frac{-\cos(n+a)x}{(n+a)}+\frac{-\cos(n-a)x}{(n-a)}\right]_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left\{\left[\frac{-\cos(n+a)2\pi}{(n+a)}+\frac{-\cos(n-a)2\pi}{(n-a)}\right]-\left[\frac{-\cos(n+a)0}{(n+a)}-\frac{\cos(n-a)0}{(n-a)}\right]\right\}$$

$\because \cos(n+a)2\pi = \cos 2a\pi, \quad \cos(n-a)2\pi = \cos 2a\pi$

$$b_n = \frac{1}{\pi}\left\{\frac{-\cos 2a\pi}{(n+a)}+\frac{-\cos 2a\pi}{(n-a)}+\frac{1}{(n+a)}+\frac{1}{(n-a)}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{1-\cos 2a\pi}{(n+a)} + \frac{1-\cos 2a\pi}{(n-a)}\right\}$$

$$b_n = \frac{1-\cos 2a\pi}{2\pi}\left\{\frac{1}{(n+a)} + \frac{1}{(n-a)}\right\}$$

$$b_n = \frac{1-\cos 2a\pi}{2\pi}\left\{\frac{(n-a)+(n+a)}{(n+a)(n-a)}\right\}$$

$$b_n = \frac{1-\cos 2a\pi}{2\pi}\left\{\frac{2n}{n^2-a^2}\right\} = \frac{n(1-\cos 2a\pi)}{\pi(n^2-a^2)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$f(x) = \frac{1}{2}\frac{\sin 2\pi a}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{a \sin 2a\pi}{\pi(a^2-n^2)} \cos nx + \frac{n(1-\cos 2a\pi)}{\pi(n^2-a^2)} \sin nx\right]$$

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Example: Find the Fourier expansion of the periodic function

$$f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x-\pi & \pi < x < 2\pi \end{cases}$$

State the value of the series at $x = \pi$ $\quad i.e. f(\pi)$

Solution: $f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x-\pi & \pi < x < 2\pi \end{cases}$ here $(0, 2\pi)$

The Fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx}$$

$$\boldsymbol{a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx}$$

$$a_0 = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)dx + \int_{\pi}^{2\pi} f(x)dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi}(-\pi)dx + \int_{\pi}^{2\pi}(x-\pi)dx\right\}$$

$$= \frac{1}{\pi}\left\{(-\pi)\int_0^{\pi}dx + \int_{\pi}^{2\pi}(x-\pi)dx\right\}$$

$$= \frac{1}{\pi}\left\{[-\pi x]_0^{\pi} + \left[\frac{(x-\pi)^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{[(-\pi * \pi) - (0)] + \left[\frac{(2\pi-\pi)^2}{2} - \frac{(\pi-\pi)^2}{2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\pi^2 + \frac{\pi^2}{2} - 0\right\} = \frac{\pi^2}{\pi}\left\{-1 + \frac{1}{2}\right\}$$

$$= \frac{\pi^2}{\pi}\left\{-\frac{1}{2}\right\}$$

$$= -\frac{\pi}{2}$$

$$\boldsymbol{a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx}$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi}(-\pi)\cos nx\,dx + \int_{\pi}^{2\pi}(x-\pi)\cos nx\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\left[\frac{-\pi\sin nx}{n}\right]_0^{\pi} + \left[\left((x-\pi)\frac{\sin nx}{n}\right) - \left((1-0)\frac{-\cos nx}{n^2}\right) + (0)\frac{-\sin nx}{n^3}\right]_{\pi}^{2\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\begin{matrix}\frac{-\pi\sin n\pi}{n} - \frac{-\pi\sin 0}{n} + \left[\left((2\pi-\pi)\frac{\sin 2\pi n}{n}\right) - \left(\frac{-\cos 2\pi n}{n^2}\right)\right] \\ -\left[\left((\pi-\pi)\frac{\sin 0}{n}\right) - \left(\frac{-(-1)^n}{n^2}\right)\right]\end{matrix}\right\}$$

$$a_n = \frac{1}{\pi}\left\{0 - 0 + \left[\left((\pi)0\right) + \frac{1}{n^2}\right] - \left[(0) + \left(\frac{(-1)^n}{n^2}\right)\right]\right\} \qquad \because \cos n\pi = (-1)^n \quad \sin n\pi = 0$$

$$a_n = \frac{1}{\pi}\left\{\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right\}$$

$$a_n = \frac{1}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\}$$

$$\boldsymbol{b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx}$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi}(-\pi)\sin nx\,dx + \int_{\pi}^{2\pi}(x-\pi)\sin nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{(-\pi)(-\cos nx)}{n}\right]_0^{\pi} + \left[\left((x-\pi)\left(\frac{-\cos nx}{n}\right)\right) - \left((1-0)\left(\frac{-\sin nx}{n^2}\right)\right)\right]_{\pi}^{2\pi}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\begin{matrix}\frac{\pi\cos n\pi}{n} - \frac{\pi\cos 0}{n} + (2\pi-\pi)\left(\frac{-\cos 2n\pi}{n}\right) + \left(\frac{-\sin 2n\pi}{n^2}\right) \\ -(\pi-\pi)\left(\frac{-\cos n\pi}{n}\right) - \left(\frac{-\sin n\pi}{n^2}\right)\end{matrix}\right\}$$

$$b_n = \frac{1}{\pi}\left\{\frac{\pi(-1)^n}{n} - \frac{\pi}{n} + (\pi)\left(\frac{-1}{n}\right) + 0 - 0 + 0\right\} \qquad b_n = \frac{(-1)^n - 2}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1}{\pi}\left\{\frac{1-(-1)^n}{n^2}\right\} \cos nx + \frac{(-1)^n - 2}{n} \sin nx\right] \ldots\ldots\ldots\ldots\ldots (1)$$

$Find \quad f(\pi)$ **:** $\quad f(x) = \begin{cases} -\pi & 0 < x < \pi \\ x - \pi & \pi < x < 2\pi \end{cases}$ here $(0, 2\pi)$

$$f(\pi^-) = \log_{x\to\pi^-} -\pi \;= -\pi \;\; and \quad f(\pi^+) = \log_{x\to\pi^+} x - \pi \;= \pi - \pi = 0$$

As $f(x)$ is discontinuous at $x = \pi$,

$$f(\pi) = \frac{f(\pi^-) + f(\pi^+)}{2} = \frac{-\pi + 0}{2} = \frac{-\pi}{2}$$

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Example: Find the Fourier expansion of the function $: f(x) = 2x - x^2; \;\; 0 \le x \le 3$

Solution: $f(\mathrm{x}) = 2\mathrm{x} - \mathrm{x}^2; \qquad 0 \le \mathrm{x} \le 3$

Here $0 \le \mathrm{x} \le 3$ i.e. $0 \le \mathrm{x} \le 2\mathrm{L}$

$2\mathrm{L} = 3 \;\; \Rightarrow \;\; \mathrm{L} = \frac{3}{2}$

The Fourier series of $f(x)$ in $0 \le \mathrm{x} \le 2\mathrm{L}$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$\boldsymbol{a_0 = \frac{1}{L}\int_0^{2L} f(x)dx \qquad a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$\boldsymbol{a_0 = \frac{1}{L}\int_0^{2L} f(x)dx}$$

$$= \frac{1}{3/2}\int_0^3 (2x - x^2)dx$$

$$= \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3 \;\; = \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left\{\left[3^2 - \frac{3^3}{3}\right] - \left[0^2 - \frac{0^3}{3}\right]\right\} \;\; = \frac{2}{3}[9 - 9]$$

$$a_0 = 0$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$a_n = \frac{1}{3/2}\int_0^{3} (2x - x^2)\cos\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3}\int_0^{3} (2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left\{\frac{(2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} - \frac{(2-2x)\left[-\cos\left(\frac{2n\pi x}{3}\right)\right]}{\left(\frac{2n\pi}{3}\right)^2} + \frac{(0-2)\left[-\sin\left(\frac{2n\pi x}{3}\right)\right]}{\left(\frac{2n\pi}{3}\right)^3} - (0)\right\}_0^3$$

$$= \frac{2}{3}\left\{\frac{(2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} + \frac{(2-2x)\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$= \frac{2}{3}\left\{\frac{(2(3)-3^2)\sin\left(\frac{2n\pi(3)}{3}\right)}{\frac{2n\pi}{3}} + \frac{(2-2(3))\cos\left(\frac{2n\pi(3)}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\sin\left(\frac{2n\pi\,(3)}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3} - \frac{(2(0)-0^2)\sin(0)}{\frac{2n\pi}{3}} - \frac{(2-2(0))\cos(0)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\sin(0)}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{\frac{(-3)\sin(2n\pi)}{\frac{2n\pi}{3}} + \frac{(-4)\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\sin(2n\pi)}{\left(\frac{2n\pi}{3}\right)^3} - \frac{(0)\sin(0)}{\frac{2n\pi}{3}} - \frac{(2)\cos(0)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\sin(0)}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{0 + \frac{(-4)}{\left(\frac{2n\pi}{3}\right)^2} + 0 - 0 - \frac{2}{\left(\frac{2n\pi}{3}\right)^2} - 0\right\}$$

$\sin(2n\pi) = 0, \quad \cos 2n\pi = 1$

$$= \frac{2}{3}\left\{\frac{-4}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2}{\left(\frac{2n\pi}{3}\right)^2}\right\} = \frac{2}{3}\left\{\frac{-6}{\left(\frac{2n\pi}{3}\right)^2}\right\}$$

$$= \frac{-12}{3\left(\frac{4n^2\pi^2}{9}\right)} = \frac{-9}{n^2\pi^2}$$

$$\mathbf{b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$b_n = \frac{1}{3/2}\int_0^3 (2x - x^2)\sin\left(\frac{n\pi x}{3/2}\right)dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left\{\frac{(2x - x^2)\left[-\cos\left(\frac{2n\pi x}{3}\right)\right]}{\frac{2n\pi}{3}} - \frac{(2 - 2x)\left[-\sin\left(\frac{2n\pi x}{3}\right)\right]}{\left(\frac{2n\pi}{3}\right)^2} + \frac{(0 - 2)\left[\cos\left(\frac{2n\pi x}{3}\right)\right]}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$= \frac{2}{3}\left\{\frac{-(2x - x^2)\cos\left(\frac{2n\pi x}{3}\right)}{\frac{2n\pi}{3}} + \frac{(2 - 2x)\sin\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\cos\left(\frac{2n\pi x}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right\}_0^3$$

$$= \frac{2}{3}\left\{\frac{-(2(3) - 3^2)\cos\left(\frac{2n\pi(3)}{3}\right)}{\frac{2n\pi}{3}} + \frac{(2 - 2(3))\sin\left(\frac{2n\pi(3)}{3}\right)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\cos\left(\frac{2n\pi\,(3)}{3}\right)}{\left(\frac{2n\pi}{3}\right)^3}\right.$$

$$\left. + \frac{(2(0) - 0^2)\cos(0)}{\frac{2n\pi}{3}} - \frac{(2 - 2(0))\sin(0)}{\left(\frac{2n\pi}{3}\right)^2} + \frac{2\cos(0)}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{\frac{-(-3)\cos(2n\pi)}{\frac{2n\pi}{3}} + \frac{(-4)\sin(2n\pi)}{\left(\frac{2n\pi}{3}\right)^2} - \frac{2\cos(2n\pi)}{\left(\frac{2n\pi}{3}\right)^3} + \frac{(0)\cos(0)}{\frac{2n\pi}{3}} - \frac{(2)\sin(0)}{\left(\frac{2n\pi}{3}\right)^2}\right.$$

$$\left. + \frac{2\cos(0)}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{\frac{3}{\frac{2n\pi}{3}} + 0 - \frac{2}{\left(\frac{2n\pi}{3}\right)^3} + 0 - 0 + \frac{2}{\left(\frac{2n\pi}{3}\right)^3}\right\}$$

$$= \frac{2}{3}\left\{\frac{3}{\frac{2n\pi}{3}} - \frac{2}{\left(\frac{2n\pi}{3}\right)^3} + \frac{2}{\left(\frac{2n\pi}{3}\right)^3}\right\} = \frac{2}{3}\frac{3}{\frac{2n\pi}{3}} = \frac{3}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$2x - x^2 = \frac{0}{2} + \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{n\pi x}{3/2}\right) + \frac{3}{n\pi}\sin\left(\frac{n\pi x}{3/2}\right)\right]$$

$$2x - x^2 = \sum_{n=1}^{\infty}\left[\frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)\right]$$

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Example: Find the Fourier expansion of the function :

$$f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ \pi(2 - x) & 1 \le x \le 2 \end{cases}$$

Solution: $f(x) = \begin{cases} \pi x & 0 \le x \le 1 \\ 2\pi - x\pi & 1 \le x \le 2 \end{cases}$

Here $0 \le x \le 2$ i.e. $0 \le x \le 2L$

$2L = 2 \Rightarrow L = 1$

The Fourier series of $f(x)$ in $(0, 2L)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)dx$$

$$= \frac{1}{1}\left\{\int_0^1 \pi x dx + \int_1^2 (2\pi - x\pi)\, dx\right\}$$

$$= \left[\frac{\pi x^2}{2}\right]_0^1 + \left[2\pi x - \frac{\pi x^2}{2}\right]_1^2$$

$$= \left\{\left[\frac{\pi}{2} - 0\right] + \left[2\pi(2) - \frac{\pi(2)^2}{2} - 2\pi(1) + \frac{\pi(1)^2}{2}\right]\right\}$$

$$= \left\{\frac{\pi}{2} + 4\pi - 2\pi - 2\pi + \frac{\pi}{2}\right\}$$

$$= \left\{\frac{\pi}{2} + \frac{\pi}{2}\right\}$$

$$a_0 = \pi$$

$$\mathbf{a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx}$$

$$a_n = \int_0^1 \pi x\cos(n\pi x)\,dx + \int_1^2 (2\pi - \pi x)\cos(n\pi x)\,dx$$

$$= \left\{\frac{\pi x\sin(n\pi x)}{n\pi} - \frac{-\pi\cos(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{(2\pi - \pi x)\sin(n\pi x)}{n\pi} - \frac{(0-\pi)(-\cos n\pi x)}{(n\pi)^2}\right\}_1^2$$

$$= \left\{\frac{\pi x\sin(n\pi x)}{n\pi} + \frac{\pi\cos(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{(2\pi - \pi x)\sin(n\pi x)}{n\pi} - \frac{\pi\cos n\pi x}{(n\pi)^2}\right\}_1^2$$

$$= \left\{\frac{\pi\sin(n\pi)}{n\pi} + \frac{\pi\cos(n\pi)}{(n\pi)^2} - \frac{\pi(0)\sin(0)}{n\pi} - \frac{\pi\cos(0)}{(n\pi)^2}\right\}$$
$$+ \left\{\frac{(2\pi - 2\pi)\sin(2n\pi)}{n\pi} - \frac{\pi\cos 2n\pi}{(n\pi)^2} - \frac{(2\pi-\pi)\sin(n\pi)}{n\pi} + \frac{\pi\cos n\pi}{(n\pi)^2}\right\}$$

$$= \left\{0 + \frac{\pi\cos(n\pi)}{(n\pi)^2} - 0 - \frac{\pi\cos(0)}{(n\pi)^2}\right\} + \left\{0 - \frac{\pi\cos 2n\pi}{(n\pi)^2} - 0 + \frac{\pi\cos n\pi}{(n\pi)^2}\right\}$$

$$= \frac{\pi(-1)^n}{(n\pi)^2} - \frac{\pi}{(n\pi)^2} - \frac{\pi}{(n\pi)^2} + \frac{\pi(-1)^n}{(n\pi)^2} \qquad \cos(n\pi) = (-1)^n \quad \cos 2n\pi = 1$$

$$= \frac{2\pi(-1)^n}{(n\pi)^2} - \frac{2\pi}{(n\pi)^2} \qquad = \frac{2\pi(-1)^n - 2\pi}{(n\pi)^2}$$

$$a_n = \frac{2(-1)^n - 2}{\pi n^2}$$

$$\mathbf{b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$b_n = \int_0^1 \pi x\sin(n\pi x)\,dx + \int_1^2 (2\pi - \pi x)\sin(n\pi x)\,dx$$

$$= \left\{\frac{\pi x(-\cos(n\pi x))}{n\pi} - \frac{-\pi\sin(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{(2\pi - \pi x)(-\cos(n\pi x))}{n\pi} - \frac{(-\pi)(-\sin n\pi x)}{(n\pi)^2}\right\}_1^2$$

$$= \left\{\frac{-\pi x\cos(n\pi x)}{n\pi} + \frac{\pi\sin(n\pi x)}{(n\pi)^2}\right\}_0^1 + \left\{\frac{-(2\pi - \pi x)\cos(n\pi x)}{n\pi} - \frac{\pi\sin n\pi x}{(n\pi)^2}\right\}_1^2$$

$$= \left\{\frac{-\pi\cos(n\pi)}{n\pi} + \frac{\pi\sin(n\pi)}{(n\pi)^2} + \frac{\pi(0)\cos(0\ )}{n\pi} - \frac{\pi\sin(0)}{(n\pi)^2}\right\}$$
$$+ \left\{\frac{-(2\pi - 2\pi)\cos(2n\pi\ )}{n\pi} - \frac{\pi\sin 2n\pi}{(n\pi)^2} + \frac{(2\pi - \pi)\cos(n\pi\ )}{n\pi} + \frac{\pi\sin\ n\pi}{(n\pi)^2}\right\}$$

$$= \left\{\frac{-\pi(-1)^n}{n\pi} + \ 0 + 0 - \ 0\right\} + \left\{0 + \frac{\pi(-1)^n}{n\pi} + 0 + 0\right\}$$

$$= \frac{-\pi(-1)^n}{n\pi} + \frac{\pi(-1)^n}{(n\pi)^2} \qquad = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

$$f(x) = \begin{cases}\pi x & 0 \le x \le 1\\ \pi(2-x) & 1 \le x \le 2\end{cases} = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n - 2}{\pi n^2}\cos(\ n\pi x) + (0)\sin(\ n\pi x)\right]$$

$$f(x) = \begin{cases}\pi x & 0 \le x \le 1\\ \pi(2-x) & 1 \le x \le 2\end{cases} = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2(-1)^n - 2}{\pi n^2}\cos(\ n\pi x)$$

****

Fourier Series in the interval $(-\pi, \pi)$ *or* $(-L, L)$

Fourier Series in the interval $(-L, L)$

function neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Series in the interval $(-\pi, \pi)$

function neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Fourier Series in the interval $(-L, L)$

f(x) is EVEN Function

OR Half Range Cosine Series $(0, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

Where

$$a_0 = \frac{2}{L}\int_{0}^{L} f(x)dx$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = 0$$

Fourier Series in the interval $(-\pi, \pi)$

f(x) is EVEN Function

OR Half Range Cosine Series $(0, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

$$b_n = 0$$

Fourier Series in the interval $(-L, L)$

f(x) is ODD Function

OR Half Range Sine Series $(0, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Fourier Series in the interval $(-\pi, \pi)$

f(x) is ODD Function

OR Half Range Sine Series $(0, \pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = 0 \qquad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

"The Only things that will stop you from fulfilling your dreams is you"

Remark :

Whenever the function is defined in the interval $(-\pi, \pi)$ or $(-L, L)$ we have to check if the function is **even** or **odd** or **neither even nor odd**

****

Example: Find the Fourier series for $f(x) = \pi^2 - x^2$ in $(-\pi, \pi)$ and hence deduce that

i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \dots \dots = \frac{\pi^2}{12}$ $\quad$ $ii)$ $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots \dots = \frac{\pi^2}{6}$ $\quad$ $iii)$ $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{5^2} \dots \dots = \frac{\pi^2}{8}$

Solution: $f(x) = \pi^2 - x^2 \quad in \ (-\pi, \pi)$

$f(x) = \pi^2 - x^2 \dots \dots (1) \qquad f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 \ \dots \dots (2)$

$f(x) = f(-x) \quad \therefore f(x)$ is even function

The Fourier series of $f(x)$ is given by $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx \qquad b_n = 0$

$\boldsymbol{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)dx$$

$$= \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} \qquad = \frac{2}{\pi}\left[\pi^2\pi - \frac{\pi^3}{3} - 0 + \frac{0^3}{3}\right]$$

$$= \frac{2}{\pi}\left[\frac{3\pi^3 - \pi^3}{3}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$= \frac{4\pi^2}{3}$$

$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi^2 - x^2)\cos nx \, dx$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi^2 - x^2)\sin nx}{n} - \frac{(0-2x)(-\cos nx)}{n^2} + \frac{(-2)(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[\frac{(\pi^2 - \pi^2)\sin n\pi}{n} - \frac{2\pi(\cos n\pi)}{n^2} + \frac{2(\sin n\pi)}{n^3}\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 - \frac{2\pi(-1)^n}{n^2} + 0\right] - [0 - 0 + 0]\right\}$$

$$a_n = -\frac{4(-1)^n}{n^2}$$

$$\mathbf{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2}\cos(nx) \quad \dots\dots\dots(1)$$

Put $x = 0$ in (1)

$$\pi^2 = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(0)$$

$$\pi^2 = \frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = -\left\{\frac{4(-1)^1}{1^2} + \frac{4(-1)^2}{2^2} + \frac{4(-1)^3}{3^2} + \frac{4(-1)^4}{4^2} + \frac{4(-1)^5}{5^2} + \dots\dots\dots\dots\right\}$$

$$\frac{\pi^2}{3} = -\left\{-\frac{4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \frac{4}{4^2} - \frac{4}{5^2} + \dots\dots\dots\dots\right\}$$

$$\frac{\pi^2}{3} = \frac{4}{1^2} - \frac{4}{2^2} + \frac{4}{3^2} - \frac{4}{4^2} + \frac{4}{5^2} \dots\dots\dots\dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots(A)$$

Put x = π in (1)

$$0 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2}\cos(n\pi)$$

$$-\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} -\frac{4(-1)^n}{n^2}\,(-1)^n$$

$$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3} = \frac{4(-1)^2}{1^2} + \frac{4(-1)^4}{2^2} + \frac{4(-1)^{12}}{3^2} + \frac{4(-1)^{16}}{4^2} + \frac{4(-1)^{20}}{5^2} + \ldots\ldots\ldots\ldots$$

$$\frac{2\pi^2}{(3)(4)} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ \ldots\ldots\ldots\ldots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ \ldots\ldots\ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots(B)$$

Adding (A)and (B)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \ \ldots\ldots \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ \ldots\ldots\ldots\ldots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \ \ldots\ldots\ldots\ldots$$

$$\frac{\pi^2}{4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \ \ldots\ldots\ldots\ldots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ \ldots\ldots\ldots\ldots(C)$$

****

Example: Find the Fourier series for $f(x) = x^2 \quad in\ (-\pi, \pi)$ and hence deduce that

i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \dots\dots = \frac{\pi^2}{12}$ ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} \dots\dots = \frac{\pi^2}{6}$ iii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{5^2} \dots\dots = \frac{\pi^2}{8}$

Solution: $f(x) = x^2 \quad in\ (-\pi, \pi)$

$f(x) = x^2 \dots\dots (1)$ $\qquad f(-x) = (-x)^2 = x^2 \dots\dots (2)$

$f(x) = f(-x) \quad \therefore f(x)$ is even function

The Fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \qquad b_n = 0$

$\boldsymbol{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx \qquad = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \qquad = \frac{2}{\pi}\left[\frac{\pi^3}{3} - \frac{0^3}{3}\right] \quad = \frac{2\pi^2}{3}$$

$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$a_n = \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} - \frac{2x(-\cos nx)}{n^2} + \frac{2(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[\frac{\pi^2 \sin n\pi}{n} - \frac{2\pi(-\cos n\pi)}{n^2} + \frac{2(-\sin n\pi)}{n^3}\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 + \frac{2\pi(-1)^n}{n^2} + 0\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$\boldsymbol{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx) \quad \dots\dots\dots (1)$$

Put $x = 0$ in (1)

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(0)$$

$$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3} = \frac{4(-1)^1}{1^2} + \frac{4(-1)^2}{2^2} + \frac{4(-1)^3}{3^2} + \frac{4(-1)^4}{4^2} + \frac{4(-1)^5}{5^2} + \dots\dots\dots\dots$$

$$-\frac{\pi^2}{3} = \frac{-4}{1^2} + \frac{4}{2^2} - \frac{4}{3^2} + \frac{4}{4^2} - \frac{4}{5^2} + \dots\dots\dots\dots$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots (A)$$

Put $x = \pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi)$$

$$\pi^2 - \frac{\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} (-1)^n$$

$$\frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4(-1)^{2n}}{n^2}$$

$$\frac{2\pi^2}{3} = \frac{4(-1)^2}{1^2} + \frac{4(-1)^4}{2^2} + \frac{4(-1)^{12}}{3^2} + \frac{4(-1)^{16}}{4^2} + \frac{4(-1)^{20}}{5^2} + \dots\dots\dots\dots$$

$$\frac{2\pi^2}{(3)(4)} = \frac{4}{1^2} + \frac{4}{2^2} + \frac{4}{3^2} + \frac{4}{4^2} + \frac{4}{5^2} + \dots\dots\dots\dots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots \qquad \dots\dots\dots\dots\dots (B)$$

Adding (A)and (B)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots\dots\dots\dots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \quad \dots \dots \dots \dots$$

$$\frac{\pi^2}{4} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \quad \dots \dots \dots \dots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \quad \dots \dots \dots \dots (C)$$

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Example: Find the Fourier series for $f(x) = x \quad in\ (-\pi, \pi)$and hence deduce that

i) $1 - \frac{1}{3} + \frac{1}{5} \ \dots \dots = \frac{\pi}{4}$

Solution: $f(x) = x \quad in\ (-\pi, \pi)$

$f(x) = x \dots \dots (1)$ $\qquad$ $f(-x) = -x \ \dots \dots (2)$

$f(x) = -f(-x)$ $\qquad$ $\therefore\ f(x)$ is odd function

The Fourier series of $f(x)$ is given by $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$

$$\boldsymbol{b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx}$$

$$\mathrm{b_n} = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$\mathrm{b_n} = \frac{2}{\pi}\left[\frac{x(-\cos nx)}{n} - \frac{1(-\sin nx)}{n^2}\right]_0^{\pi}$$

$$\mathrm{b_n} = \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$\mathrm{b_n} = \frac{2}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right] - \left[\frac{-0\cos 0}{n} + \frac{\sin 0}{n^2}\right]\right\}$$

$$\mathrm{b_n} = \frac{2}{\pi}\left\{\left[\frac{-\pi(-1)^n}{n} + 0\right] - [0 - 0\,]\right\}$$

$$\mathrm{b_n} = \frac{-2(-1)^n}{n}$$

"The Only things that will stop you from fulfilling your dreams is you"

The Fourier series of $f(x)$ is given by $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx) \quad \ldots\ldots\ldots (1)$$

$Put\ x = \frac{\pi}{2}\ in\ (1)$

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{2} = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{-4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin\left(\frac{n\pi}{2}\right)$$

$$\frac{\pi}{-4} = \frac{(-1)^1}{1} \sin\left(\frac{\pi}{2}\right) + \frac{(-1)^2}{2} \sin(\pi) + \frac{(-1)^3}{3} \sin\left(\frac{3\pi}{2}\right) + \frac{(-1)^4}{4} \sin(4\pi) + \frac{(-1)^5}{5} \sin\left(\frac{5\pi}{2}\right) + \ldots\ldots\ldots$$

$$\frac{\pi}{-4} = (-1)\,(1) + \frac{1}{2}(0) - \frac{1}{3}(-1) + \frac{1}{4}(0) - \frac{1}{5}(1) \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\pi}{-4} = -1 + \frac{1}{3} - \frac{1}{5} \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \ldots\ldots\ldots\ldots\ldots\ldots$$

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Example: Find the Fourier series for $f(x) = \begin{cases} \pi + x\ ;\ -\pi \le x \le 0 \\ \pi - x\ ;\ 0 \le x \le \pi \end{cases}$

$and\ f(x + 2\pi) = f(x)$

Solution: $Interval\ is\ (-\pi\ \pi)$ $\therefore$ Check even or odd

Given $f(x) = \begin{cases} \pi + x\ ;\ -\pi \le x \le 0 \\ \pi - x\ ;\quad 0 \le x \le \pi \end{cases}$

put $x = -x$

$$f(-x) = \begin{cases} \pi - x\ ; & -\pi \le -x \le 0 \\ \pi - (-x)\ ; & 0 \le -x \le \pi \end{cases}$$

$$f(-x) = \begin{cases} \pi - x \ ; & \pi \geq x \geq 0 \\ \pi + x \ ; & 0 \geq x \geq -\pi \end{cases}$$

$f(x) = f(-x) \quad \therefore$ function is even

The Fourier series of $f(x)$ is given by $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad b_n = 0$

$$\boldsymbol{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)dx$$

$$a_0 = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$a_0 = \frac{2}{\pi}\left\{\left[\pi\pi - \frac{\pi^2}{2}\right] - \left[0 - \frac{0}{2}\right]\right\}$$

$$a_0 = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi - x)\sin nx}{n} - \frac{(0-1)(-\cos nx)}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[\frac{(\pi - x)\sin nx}{n} - \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\frac{(\pi - \pi)\sin n\pi}{n} - \frac{\cos n\pi}{n^2} - \frac{(\pi - 0)\sin 0}{n} + \frac{\cos 0}{n^2}\right\}$$

$$a_n = \frac{2}{\pi}\left\{0 - \frac{(-1)^n}{n^2} - 0 + \frac{1}{n^2}\right\}$$

$$a_n = \frac{2}{\pi}\left[\frac{1 - (-1)^n}{n^2}\right]$$

"The Only things that will stop you from fulfilling your dreams is you"

The Fourier series of $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{1-(-1)^n}{n^2}\right] \cos nx$$

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Example: Find the half range cosine series for $f(x) = x^2 \ in \ 0 < x < \pi$

Solution: The Fourier half range cosine for $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \qquad b_n = 0$$

$$\boldsymbol{a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx \quad = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} \quad = \frac{2}{\pi}\left[\frac{\pi^3}{3} - \frac{0^3}{3}\right] \quad = \frac{2\pi^2}{3}$$

$$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$a_n = \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} - \frac{2x(-\cos nx)}{n^2} + \frac{2(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left\{\left[\frac{\pi^2 \sin n\pi}{n} - \frac{2\pi(-\cos n\pi)}{n^2} + \frac{2(-\sin n\pi)}{n^3}\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{2}{\pi}\left\{\left[0 + \frac{2\pi(-1)^n}{n^2} + 0\right] - [0 - 0 + 0]\right\}$$

$$a_n = \frac{4(-1)^n}{n^2}$$

$$\boldsymbol{b_n = 0}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

****

Example: Find the half range sine series for $f(x) = x \quad in \quad 0 < x < \pi$

Solution: The half range sine series for $f(x)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$a_0 = 0 \qquad a_n = 0 \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\, dx$$

$$b_n = \frac{2}{\pi}\left[\frac{x(-\cos nx)}{n} - \frac{1(-\sin nx)}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right] - \left[\frac{-0\cos 0}{n} + \frac{\sin 0}{n^2}\right]\right\}$$

$$b_n = \frac{2}{\pi}\left\{\left[\frac{-\pi(-1)^n}{n} + 0\right] - [0 - 0]\right\}$$

$$b_n = \frac{-2(-1)^n}{n}$$

The Fourier series of $f(x)$ is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n}\sin(nx)$$

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Example: Find the half range cosine series for $f(x) = \pi x - x^2 \ in \ 0 < x < \pi$

Solution: The Fourier half range cosine for $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \qquad b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \pi x - x^2 dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi\pi^2}{2} - \frac{\pi^3}{3} - 0 + 0\right] = \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{\pi^2}{3}$$

$$\boldsymbol{a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx}$$

$$\mathrm{a_n} = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\, dx$$

$$\mathrm{a_n} = \frac{2}{\pi}\left[\frac{(\pi x - x^2)\sin nx}{n} - \frac{(\pi - 2x)(-\cos nx)}{n^2} + \frac{(-2)(-\sin nx)}{n^3}\right]_0^{\pi}$$

$$\mathrm{a_n} = \frac{2}{\pi}\left\{\left[\frac{(\pi^2 - \pi^2)\sin n\pi}{n} - \frac{(\pi - 2\pi)(-\cos n\pi)}{n^2} + \frac{(-2)(-\sin n\pi)}{n^3}\right] - \left[\frac{(0)\sin 0}{n} - \frac{(\pi - 0)(-\cos 0)}{n^2} + \frac{(-2)(-\sin 0)}{n^3}\right]\right\}$$

$$\mathrm{a_n} = \frac{2}{\pi}\left\{\left[0 - \frac{\pi(-1)^n}{n^2} + 0\right] - \left[0 + \frac{\pi}{n^2} + 0\right]\right\}$$

$$\mathrm{a_n} = \frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n^2} - \frac{\pi}{n^2}\right]$$

$$\mathrm{a_n} = \frac{-2[(-1)^n + 1]}{n^2}$$

$$\mathbf{b_n = 0}$$

$$\mathrm{f(x)} = \frac{\mathrm{a_0}}{2} + \sum_{n=1}^{\infty} \mathrm{a_n}\cos \mathrm{nx}$$

$$\pi x - x^2 = \frac{1}{2}\frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{-2[(-1)^n + 1]}{n^2}\cos(\mathrm{nx}) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{-2[(-1)^n + 1]}{n^2}\cos(\mathrm{nx})$$

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