UNIT III: Partial Differentiation and Applications

Partial Differentiation:

Given a function of two variables, $f(x, y)$ the derivative with respect to x only (treating y as a constant) is called the partial derivative of f with respect to x

It is denoted by either $\frac{\partial f}{\partial x}$ $or\ f_x$

Similarly, the derivative of f with respect to y only (treating x as a constant) is called the partial derivative of f with respect to y. It is denoted by either $\frac{\partial f}{\partial y}$ or f_y

Differentiate f with respect to x twice. (i.e differentiate f with respect to x; then differentiate the result with respect to x again.) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$ $or\ f_{xx}$

Differentiate f with respect to y twice. (i.e. differentiate f with respect to y; then differentiate the result with respect to y again.) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$ $or\ f_{yy}$

Mixed partials derivative:

- First differentiate f with respect to x; then differentiate the result with respect to y.

  $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$ $or\ f_{xy}$

- First differentiate f with respect to y; then differentiate the result with respect to x.

  $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y\, \partial x}$ $or\ f_{yx}$

****

Example 1: If $f(x,y) = 3x^2y + 5x - 2y^2 + 1$ find f_x f_y f_{xx} f_{yy} f_{xy} f_{yx}

Solution: Let $f(x,y) = 3x^2y + 5x - 2y^2 + 1$

f_x means differentiate f with respect to x (treating y as a constant)

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(3x^2y + 5x - 2y^2 + 1)$$

$$f_x = \frac{\partial f}{\partial x} = 6xy + 5 - 0 + 0 = 6xy + 5$$

f_y *means* differentiate f with respect to y (treating x as a constant)

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(3x^2y + 5x - 2y^2 + 1)$$

$$f_y = \frac{\partial f}{\partial y} = 3x^2 + 0 - 4y + 0 = 3x^2 - 4y$$

$\frac{\partial^2 f}{\partial x^2}$ or f_{xx} Differentiate f with respect to x twice.

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(6xy + 5) = 6y$$

$\frac{\partial^2 f}{\partial y^2}$ or f_{yy} Differentiate f with respect to y twice.

$$\frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2 - 4y) = -4$$

First differentiate f with respect to x; then differentiate the result with respect to y.

$$\frac{\partial^2 f}{\partial y\,\partial x} \text{ or } f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(6xy + 5) = 6x$$

First differentiate f with respect to y; then differentiate the result with respect to x.

$$\frac{\partial^2 f}{\partial x\,\partial y} \text{ or } f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2 - 4y) = 6x$$

****

Example 2: If $z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$ then find the value of $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2}$

Solution: Let $z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$

To find $\frac{\partial^2 z}{\partial x^2}$ Differentiate z with respect to x twice

$$z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$$

Differentiating z with respect to x partially treating y as constant

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[\tan(y + ax) + (y - ax)^{\frac{3}{2}}\right] \quad \because \frac{d}{dx}\tan x = \sec^2 x \quad \frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{\partial z}{\partial x} = \sec^2(y + ax)\frac{\partial}{\partial x}(y + ax) + \frac{3}{2}(y - ax)^{\frac{3}{2}-1}\frac{\partial}{\partial x}(y - ax)$$

$$\frac{\partial z}{\partial x} = \sec^2(y + ax)\,(0 + a) + \frac{3}{2}(y - ax)^{\frac{1}{2}}(0 - a)$$

$$\frac{\partial z}{\partial x} = a\sec^2(y + ax) - \frac{3a}{2}(y - ax)^{\frac{1}{2}} \ldots\ldots\ldots\ldots (1)$$

Differentiating $\frac{\partial z}{\partial x}$ with respect to x partially treating y as constant

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[a\sec^2(y + ax) - \frac{3a}{2}(y - ax)^{\frac{1}{2}}\right]$$

$$\frac{\partial^2 z}{\partial x^2} = 2a\sec(y + ax)\frac{\partial}{\partial x}[\sec(y + ax)] - \frac{3a}{2}\frac{1}{2}(y - ax)^{\frac{1}{2}-1}\frac{\partial}{\partial x}(y - ax)$$

$$\frac{\partial^2 z}{\partial x^2} = 2a\sec(y + ax)\sec(y + ax)\tan(y + ax)\frac{\partial}{\partial x}(y + ax)$$
$$- \frac{3a}{4}(y - ax)^{-\frac{1}{2}}(-a)$$

$$\frac{\partial^2 z}{\partial x^2} = 2a\sec^2(y + ax)\tan(y + ax)\,(a) - \frac{3a}{4}(y - ax)^{-\frac{1}{2}}(-a)$$

$$\frac{\partial^2 z}{\partial x^2} = 2a^2\sec^2(y + ax)\tan(y + ax) + \frac{3a^2}{4}(y - ax)^{-\frac{1}{2}} \ldots\ldots\ldots\ldots (A)$$

To find $\frac{\partial^2 z}{\partial y^2}$ Differentiate z with respect to y twice

$$z = \tan(y + ax) + (y - ax)^{\frac{3}{2}}$$

Differentiating z with respect to y partially treating x as constant

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\tan(y + ax) + (y - ax)^{\frac{3}{2}}\right]$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax)\frac{\partial}{\partial y}(y + ax) + \frac{3}{2}(y - ax)^{\frac{3}{2}-1}\frac{\partial}{\partial y}(y - ax)$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax)\,(1 + 0) + \frac{3}{2}(y - ax)^{\frac{1}{2}}(1 - 0)$$

$$\frac{\partial z}{\partial y} = \sec^2(y + ax) + \frac{3}{2}(y - ax)^{\frac{1}{2}} \quad \dots\dots\dots\dots (2)$$

Differentiating $\frac{\partial z}{\partial y}$ with respect to y partially treating x as constant

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[\sec^2(y + ax) + \frac{3}{2}(y - ax)^{\frac{1}{2}}\right]$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec(y + ax)\frac{\partial}{\partial x}[\sec(y + ax)] + \frac{3}{2}\frac{1}{2}(y - ax)^{\frac{1}{2}-1}\frac{\partial}{\partial x}(y - ax)$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec(y + ax)\sec(y + ax)\tan(y + ax)\frac{\partial}{\partial x}(y + ax) + \frac{3}{4}(y - ax)^{-\frac{1}{2}}(1)$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec^2(y + ax)\ \tan(y + ax)\,(a) + \frac{3a}{4}(y - ax)^{-\frac{1}{2}} \dots\dots\dots\dots (3)$$

Multiplying by $-a^2$

$$-a^2\frac{\partial^2 z}{\partial y^2} = -2\,a^2\sec^2(y + ax)\ \tan(y + ax) - \frac{3a^2}{4}(y - ax)^{-\frac{1}{2}} \dots\dots\dots\dots (B)$$

Equation (A) + (B)

$$\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 2\,a^2\sec^2(y + ax)\ \tan(y + ax) + \frac{3a^2}{4}(y - ax)^{-\frac{1}{2}}$$

$$= -2\,a^2\sec^2(y + ax)\ \tan(y + ax) - \frac{3a^2}{4}(y - ax)^{-\frac{1}{2}}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0$$

Example 3: If $u = \log(x^3 + y^3 - x^2y - xy^2)$ then prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x+y)^2}$$

Solution: Let $u = \log(x^3 + y^3 - x^2y - xy^2)$

$$u = \log(x^2(x-y) - y^2(x-y))$$

$$u = \log[(x-y)\,(x^2 - y^2)]$$

$$u = \log[(x-y)\,(x-y)(x+y)] \qquad \because (a^2 - b^2) = (a+b)(a-b)$$

$$u = \log[(x-y)^2(x+y)]$$

$$u = \log(x-y)^2 + \log(x+y) \qquad \because \log ab = \log a + \log b$$

$$u = 2\log(x-y) + \log(x+y) \qquad \because \log a^n = n \log a$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial^2}{\partial x^2} + 2\,\frac{\partial^2}{\partial x\,\partial y} + \frac{\partial^2}{\partial y^2}\right) u = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} \qquad \ldots\ldots (A)$$

To find $\frac{\partial^2 u}{\partial x^2}$ means Differentiate u with respect to x twice

$u = 2\log(x-y) + \log(x+y)$

Differentiating u with respect to x partially y is constant

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\,[2\log(x-y) + \log(x+y)\,] \qquad \because \frac{d}{dx}\log x = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)}\,\frac{\partial}{\partial x}(x-y) + \frac{1}{(x+y)}\,\frac{\partial}{\partial x}(x+y)$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)}\,(1-0) + \frac{1}{(x+y)}\,(1+0)$$

$$\frac{\partial u}{\partial x} = \frac{2}{(x-y)} + \frac{1}{(x+y)} \ldots\ldots\ldots\ldots(1)$$

Differentiating $\frac{\partial u}{\partial x}$ with respect to x partially

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{2}{(x-y)} + \frac{1}{(x+y)}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\frac{-2}{(x-y)^2}\frac{\partial}{\partial x}(x-y) + \frac{-1}{(x+y)^2}\frac{\partial}{\partial x}(x+y)\right] \qquad \because \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\frac{-2}{(x-y)^2}(1-0) + \frac{-1}{(x+y)^2}(1+0)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

To find $\frac{\partial^2 u}{\partial y^2}$ Differentiate u with respect to y twice

$$u = 2\log(x-y) + \log(x+y)$$

Differentiating u with respect to y partially

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\,[2\log(x-y) + \log(x+y)\,]$$

$$\frac{\partial u}{\partial y} = \frac{2}{(x-y)}\frac{\partial}{\partial y}(x-y) + \frac{1}{(x+y)}\frac{\partial}{\partial y}(x+y)$$

$$\frac{\partial u}{\partial y} = \frac{2}{(x-y)}(0-1) + \frac{1}{(x+y)}(0+1)$$

$$\frac{\partial u}{\partial y} = \frac{-2}{(x-y)} + \frac{1}{(x+y)} \ldots\ldots\ldots\ldots.(3)$$

Differentiating $\frac{\partial u}{\partial y}$ with respect to y partially

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial}\left[\frac{-2}{(x-y)} + \frac{1}{(x+y)}\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \left[\frac{-(-2)}{(x-y)^2}\frac{\partial}{\partial y}(x-y) + \frac{-1}{(x+y)^2}\frac{\partial}{\partial y}(x+y)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \left[\frac{2}{(x-y)^2}(0-1) + \frac{-1}{(x+y)^2}(0+1)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

$\frac{\partial^2 u}{\partial x\,\partial y}$ Differentiating u with respect to y partially then x

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-2}{(x-y)} + \frac{1}{(x+y)}\right) \qquad from\ (3)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{-(-2)}{(x-y)^2}\frac{\partial}{\partial x}(x-y) + \frac{-1}{(x+y)^2}\frac{\partial}{\partial x}(x+y)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{2}{(x-y)^2}(1-0) + \frac{-1}{(x+y)^2}(1+0)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} \quad \dots\dots\dots\dots..(5)$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2}$$

From (2) (4) and (5)

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} + 2\left[\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}\right] - \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}$$

$$= -\frac{2}{(x-y)^2} - \frac{1}{(x+y)^2} + \frac{4}{(x-y)^2} - \frac{2}{(x+y)^2} - \frac{2}{(x-y)^2} - \frac{1}{(x+y)^2}$$

$$= -\frac{4}{(x+y)^2}$$

****

Type II $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

If *f(x, y)* is Homogeneous Function then $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$

Example 1: If $u = \log(x^2 + y^2)$ verify $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$

Solution: Let $u = \log(x^2 + y^2)$

$\frac{\partial u}{\partial x}$ Differentiating u with respect to x partially y is constant

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}\,\frac{\partial}{\partial x}(x^2 + y^2) \qquad \because \frac{d}{dx}\log x = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2}\,(2x + 0) = \frac{2x}{(x^2 + y^2)} \ldots\ldots\ldots\ldots(1)$$

$\frac{\partial^2 u}{\partial y\,\partial x}$ Differentiating $\frac{\partial u}{\partial x}$ with respect to y partially x is constant

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 + y^2}\right)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = 2x\,\frac{\partial}{\partial y}\left(\frac{1}{x^2 + y^2}\right) \qquad \because \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{2x\,(-1)}{(x^2 + y^2)^2}\,\frac{\partial}{\partial y}(x^2 + y^2)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{-\,2x}{(x^2 + y^2)^2}\,(0 + 2y)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{-4xy}{(x^2 + y^2)^2} \ldots\ldots\ldots(A)$$

$$u = \log(x^2 + y^2)$$

$\frac{\partial u}{\partial y}$ Differentiating u with respect to y partially

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\log(x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}\frac{\partial}{\partial y}(x^2 + y^2)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2}(0 + 2y) = \frac{2y}{(x^2 + y^2)} \ldots\ldots\ldots\ldots..(2)$$

$\frac{\partial^2 u}{\partial x\,\partial y}$ Differentiating $\frac{\partial u}{\partial y}$ with respect to x partially constant y

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2}\right) = 2y\frac{\partial}{\partial x}\left(\frac{1}{x^2 + y^2}\right)$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{2y\,(-1)}{(x^2 + y^2)^2}\frac{\partial}{\partial x}(x^2 + y^2)$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{-\,2y}{(x^2 + y^2)^2}(2x + 0)$$

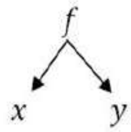
$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{-4xy}{(x^2 + y^2)^2} \ldots\ldots\ldots..(B)$$

From (A) and (B)

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$$

****

Variable to be treated as Constant:



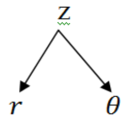
Let f be function of x and y $i.e.\ f(x,y)$

To find partial derivative with respect to ***x*** we have to treat ***y*** as constant

$$\left(\frac{\partial f}{\partial x}\right)_y$$

To find partial derivative with respect to ***y*** we have to treat ***x*** as constant $\left(\frac{\partial f}{\partial y}\right)_x$

For Example:



$\left(\frac{\partial z}{\partial r}\right)_\theta$ Means z is function of $r,\ \theta$ z is differentiable w. r .t $r,\ \theta$

here z is partially differentiated with respect to ***r*** and θ as constant.

$\left(\frac{\partial y}{\partial \theta}\right)_r$ Means y is function of $r,\ \theta$ y is differentiable w. r. t $r,\ \theta$

here y is partially differentiated with respect to θ and r as constant.

****

Example 1: If $\mathrm{u = 2x + 3y}$, $\mathrm{v = 3x - 2y}$ Find $\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u$

Solution:

$$\mathrm{u = 2x + 3y} \quad \ldots\ldots\ldots(1)$$

$$\mathrm{v = 3x - 2y} \quad \ldots\ldots\ldots(2)$$

u, v are functions of x, y variable $i.e.\ \ u, v = f(x,y)$

$\left(\frac{\partial u}{\partial x}\right)_y$ u is function of $x,\ y$ variable $i.e\ \ u = 2x + 3y$

Diff u partially w. r. t. x keep y as constant

$$\left(\frac{\partial u}{\partial x}\right)_y = 2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots.(A)$$

$\left(\frac{\partial x}{\partial u}\right)_v$ x is function of $u,\ v$ variable

$\mathrm{u = 2x + 3y}$ $\qquad$ $\mathrm{v = 3x - 2y}$ $\qquad$ Eliminate y

$\mathrm{u = 2x + 3y}$ $\qquad$ $\mathrm{2y = 3x - v}$

$\mathrm{u = 2x + 3y}$ $\qquad$ $\mathrm{y = \frac{3x - v}{2}}$

$$\mathrm{u = 2x + 3\left(\frac{3x - v}{2}\right)}$$

$$2u = 4x + 3(3x - v)$$

$$2u = 4x + 9x - 3v$$

$$2u + 3v = 13x$$

$$x = \frac{2u + 3v}{13}$$

$\left(\frac{\partial x}{\partial u}\right)_v$ Diff partially w. r. t. u keep v as constant

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{2}{13} \qquad \ldots\ldots\ldots\ldots (B)$$

$\left(\frac{\partial y}{\partial v}\right)_x$ y is function of v, x variable

From (2) $v = 3x - 2y \Rightarrow 2y = 3x - v \Rightarrow y = \frac{3x - v}{2}$

Diff partially w. r. t. v keep x as constant

$$\left(\frac{\partial y}{\partial v}\right)_x = \frac{-1}{2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (C)$$

$\left(\frac{\partial v}{\partial y}\right)_u$ v is function of y, u variable

$u = 2x + 3y \qquad v = 3x - 2y \qquad$ Eliminate x

$u = 2x + 3y \qquad v + 2y = 3x$

$u = 2x + 3y \qquad x = \frac{v + 2y}{3}$

$$u = 2\left(\frac{v + 2y}{3}\right) + 3y$$

$$3u = 2v + 4y + 9y$$

$$3u = 2v + 13y$$

$2v = 3u - 13y \qquad v = \frac{3u - 13y}{2}$

Diff partially w. r. t. y keep u as constant

$$\left(\frac{\partial v}{\partial y}\right)_u = -\frac{13}{2} \qquad \ldots\ldots\ldots\ldots (D)$$

From (A), (B), (C) and (D)

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v \left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = 2\left(\frac{2}{13}\right)\left(-\frac{1}{2}\right)\left(-\frac{13}{2}\right) = 1$$

****

Example 2: If $x = \frac{\cos\theta}{r}$, $y = \frac{\sin\theta}{r}$ Find $\left(\frac{\partial x}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial y}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial y}\right)_x$

Solution:

$$x = \frac{\cos\theta}{r} \quad \dots\dots\dots (1)$$

$$y = \frac{\sin\theta}{r} \quad \dots\dots\dots (2)$$

x, y are function of r, θ variable $i.e.$ $x, y = f(r, \theta)$

$\left(\frac{\partial x}{\partial r}\right)_\theta$ x is function of r, θ variable $i.e$ $x = \frac{\cos\theta}{r}$

Diff x partially w. r. t. r keep as θ constant

$$\left(\frac{\partial x}{\partial r}\right)_\theta = -\frac{\cos\theta}{r^2} \quad \dots\dots\dots\dots\dots\dots (A)$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$\left(\frac{\partial r}{\partial x}\right)_y$ r is function of x, y variable

$x = \frac{\cos\theta}{r}$ $y = \frac{\sin\theta}{r}$ Eliminate θ

$x^2 = \frac{\cos^2\theta}{r^2}$ $y^2 = \frac{\sin^2\theta}{r^2}$ squaring

adding

$$x^2 + y^2 = \frac{\cos^2\theta}{r^2} + \frac{\sin^2\theta}{r^2}$$

$$x^2 + y^2 = \frac{\cos^2\theta + \sin^2\theta}{r^2}$$

$$x^2 + y^2 = \frac{1}{r^2}$$

$$r^2 = \frac{1}{x^2 + y^2}$$

$r^2 = (x^2 + y^2)^{-1}$ taking square root on both side

$$r = (x^2 + y^2)^{-1/2}$$

$$\left(\frac{\partial r}{\partial x}\right)_y = -\frac{1}{2}(x^2 + y^2)^{\frac{-3}{2}} \frac{\partial}{\partial x}(x^2 + y^2)$$

$$\frac{d}{dx}x^n = nx^{n-1}\frac{dx}{dx}$$

$$\left(\frac{\partial r}{\partial x}\right)_y = -\frac{1}{2}(x^2 + y^2)^{\frac{-3}{2}} (2x + 0)$$

$$= (-1)\left(\frac{1}{r^2}\right)^{\frac{-3}{2}}\left(\frac{\cos\theta}{r}\right) \qquad \mathbf{x^2 + y^2 = \frac{1}{r^2}} \ \boldsymbol{and\ x} = \frac{\mathbf{cos\,\theta}}{\mathbf{r}}$$

$$= (-1)\frac{1}{r^{-3}}\left(\frac{\cos\theta}{r}\right)$$

$$= -\frac{\cos\theta}{r^{-2}}$$

$$= -r^2\cos\theta \qquad \dots\dots\dots(B)$$

$\left(\frac{\partial y}{\partial r}\right)_\theta$ $\quad y$ is function of r,θ variable $\quad i.e\ \ y = \frac{\sin\theta}{r}$

Diff partially w. r. t. r keep as θ constant

$$\left(\frac{\partial x}{\partial r}\right)_\theta = -\frac{\sin\theta}{r^2} \qquad \dots\dots\dots\dots\dots\dots.(C) \qquad \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$\left(\frac{\partial r}{\partial y}\right)_x$ $\quad$ r is function of x,y variable $\quad r = (x^2 + y^2)^{-1/2}$

$$\left(\frac{\partial r}{\partial y}\right)_x = -\frac{1}{2}(x^2 + y^2)^{\frac{-3}{2}}\frac{\partial}{\partial y}(x^2 + y^2) \qquad \frac{d}{dx}x^n = nx^{n-1}\frac{dx}{dx}$$

$$\left(\frac{\partial r}{\partial y}\right)_x = -\frac{1}{2}(x^2 + y^2)^{\frac{-3}{2}}(0 + 2y)$$

$$= \left(-\frac{1}{2}\right)\left(\frac{1}{r^2}\right)^{\frac{-3}{2}}\left(\frac{2\sin\theta}{r}\right) \qquad \mathbf{x^2 + y^2 = \frac{1}{r^2}}\ \boldsymbol{and} \quad \boldsymbol{y} = \frac{\mathbf{sin\ \theta}}{\mathbf{r}}$$

$$= -\frac{\sin\theta}{r^{-2}}$$

$$\left(\frac{\partial r}{\partial y}\right)_x = -r^2\sin\theta \dots\dots\dots(D)$$

From (A), (B),(C) and (D)

$$\left(\frac{\partial x}{\partial r}\right)_\theta\left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial y}{\partial r}\right)_\theta\left(\frac{\partial r}{\partial y}\right)_x = \left(-\frac{\cos\theta}{r^2}\right)(-r^2\cos\theta) + \left(-\frac{\sin\theta}{r^2}\right)(-r^2\sin\theta)$$

$$\left(\frac{\partial x}{\partial r}\right)_\theta\left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial y}{\partial r}\right)_\theta\left(\frac{\partial r}{\partial y}\right)_x = \cos^2\theta + \sin^2\theta = 1$$

****

Composite Function:

A composite function is a function that depends on another function.

The chain rule exists for differentiating a function of another function.

The chain rule is a formula to compute the derivative of a composite function. If $f(x, y)$ and $x = h(r,\ \theta),\ y = \text{g}(r,\ \theta)$ are differentiable functions.

$f(x, y)$ is differentiable functions,

Diff partially w. r. t. to x $\frac{\partial f}{\partial x}$

Diff partially w. r. t. to $x,\ y$ $\frac{\partial f}{\partial y}$

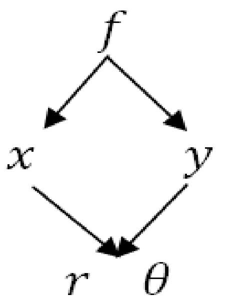
x, y are differentiable function,

Diff partially w. r. t. to $r,\ \theta$ $\frac{\partial x}{\partial r}, \frac{\partial x}{\partial \theta}$ and $\frac{\partial y}{\partial r}, \frac{\partial y}{\partial \theta}$

By Chain

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

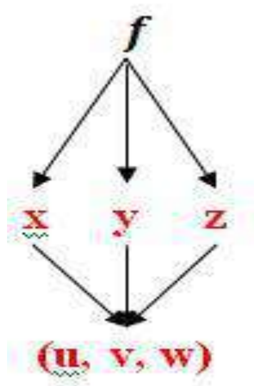


If $f = f_1(x, y, z)$ *and* $x, y, z = f_2(u, v, w)$ then

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial f}{\partial w} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial w} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial w}$$



****

Examples 1: If $z = f(u,v)$ and $u = x\cos t - y\sin t$, $v = x\sin t + y\cos t$,
where t is a constant, prove that : $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$

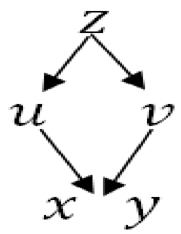
Solution: $z = f(u,v)$

$u = x\cos t - y\sin t \quad \ldots\ldots\ldots (1)$

$v = x\sin t + y\cos t \quad \ldots\ldots\ldots (2)$

u and v is differentiable functions of x and y

Diff u and v partially w. r. t. to x and y respectively



$$\frac{\partial u}{\partial x} = \cos t \quad \text{and} \quad \frac{\partial u}{\partial y} = -\sin t$$

$$\frac{\partial v}{\partial x} = \sin t \quad \text{and} \quad \frac{\partial v}{\partial y} = \cos t$$

By chain rule $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cos t + \frac{\partial z}{\partial v}\sin t$$

Multiplying by x $\quad x\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial u}\cos t + x\frac{\partial z}{\partial v}\sin t$

$$x\frac{\partial z}{\partial x} = x\cos t\frac{\partial z}{\partial u} + x\sin t\frac{\partial z}{\partial v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}(-\sin t) + \frac{\partial z}{\partial v}\cos t$$

Multiplying by y $\quad y\frac{\partial z}{\partial y} = y\frac{\partial z}{\partial u}(-\sin t) + y\frac{\partial z}{\partial v}\cos t$

$$y\frac{\partial z}{\partial y} = -y\sin t\frac{\partial z}{\partial u} + y\cos t\frac{\partial z}{\partial v} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B)$$

Adding equations (A) and (B)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\cos t\frac{\partial z}{\partial u} + x\sin t\frac{\partial z}{\partial v} - y\sin t\frac{\partial z}{\partial u} + y\cos t\frac{\partial z}{\partial v}$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = (x\cos t - y\sin t)\frac{\partial z}{\partial u} + (x\sin t + y\cos t)\frac{\partial z}{\partial v}$$

From (1) and (2)

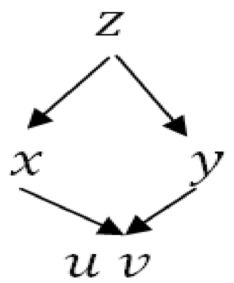
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}$$

****

Examples 2: $If\ \ z = f(x,y)\ \ where\ \ x = u + v\ \ \ and\ \ \ y = uv$

$$then\ prove\ that:\quad u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$$

Solution: $z = f(x,y)\quad i.e.\ z\ is\ function\ of\ x,\ y$



$$x = u + v \qquad \ldots\ldots\ldots(1)$$

$$y = uv \qquad \ldots\ldots\ldots.(2)$$

x *and* y is differentiable functions of u and v

Diff partially w. r. t. to u and v respectively

$$\frac{\partial x}{\partial u} = 1 \qquad and \qquad \frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = v \qquad and \qquad \frac{\partial y}{\partial v} = u$$

By chain rule $\quad \dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}1 + \frac{\partial z}{\partial y}v$$

Multiplying by $u \quad u\dfrac{\partial z}{\partial u} = u\dfrac{\partial z}{\partial x}1 + u\dfrac{\partial z}{\partial y}v$

$$u\frac{\partial z}{\partial u} = u\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}1 + \frac{\partial z}{\partial y}u$$

Multiplying by v $\quad v\dfrac{\partial z}{\partial v} = v\dfrac{\partial z}{\partial x}1 + v\dfrac{\partial z}{\partial y}u$

$$v\frac{\partial z}{\partial v} = v\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(B)$$

Adding equations (A) and (B)

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = u\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y} + v\frac{\partial z}{\partial x} + uv\frac{\partial z}{\partial y}$$

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = (u+v)\frac{\partial z}{\partial x} + (uv + uv)\frac{\partial z}{\partial y}$$

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = (u+v)\frac{\partial z}{\partial x} + 2uv\frac{\partial z}{\partial y}$$

From (1) and (2)

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} + 2y\frac{\partial z}{\partial y}$$

****

Homogeneous Function:

A function $F(x, y)$ is said to be homogeneous function in which the power of each term is same.

$$F(x, y, z) = x^3 + y^3 + z^3 - 9x^1y^2 + 19x^2y^1 - \sqrt{3}y^2z^1$$

Note:

- To verify homogeneous function put $x = xt, \quad y = yt \quad and \quad z = zt$

  $F(x, y, z) = (xt)^3 + (yt)^3 + (zt)^3 - 9xt^1yt^2 + 19xt^2yt^1 - \sqrt{3}yt^2zt^1$

  $F(x, y, z) = x^3t^3 + y^3t^3 + t^3z^3 - 9xy^2t^{\,3} + 19x^2t^3y - \sqrt{3}y^2t^3z$

  $F(x, y, z) = t^3F(x, y, z)$

  F is homogeneous function of degree 3

- $f(x, y, z) = t^nf(x, y, z)$ f is homogeneous function of degree n

****

Euler's Theorem of Homogeneous Function:

Let $f(x, y)$ be a homogenous function of degree n in x and y

For 2 variables and 1st order $\quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\, f(x, y)$

Let $f(x, y, z)$ be a homogenous function of degree n so that $f(x, y, z) = t^nf(x, y, z)$

For 3 variables and 1st order $\quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf(x, y, z)$

Let f be a homogenous function of degree n in $x_1, x_2, x_3, \dots \dots . x_n$

For n variables and 1st order

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} + \dots\dots\dots\dots + x_n\frac{\partial f}{\partial x_n} = n\, f$$

Euler's Theorem for second order

Let $f(x,y)$ be a homogenous function of two variables x, y of degree n

For 2 variables and 2nd order $$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f$$

****

Example: If $f(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$ then prove that

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f=0 \qquad OR \qquad Find \quad x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}$$

Solution: Consider $f(x,y)=\frac{1}{x^2}+\frac{1}{xy}+\frac{\log x-\log y}{x^2+y^2}$ $\quad \because \log a-\log b=\log\left(\frac{a}{b}\right)$

$$f(x,y)=\frac{xy+x^2}{x^3y}+\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2}$$

$$f(x,y)=\frac{y+x}{x^2y}+\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2}$$

Let $u=\frac{y+x}{x^2y}$ $\quad v=\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2}$

$$f(x,y)=u+v\ \ldots\ldots\ldots(A)$$

Put $x=xt,\quad y=yt$

$$u=\frac{yt+xt}{(xt)^2yt} \qquad v=\frac{\log\left(\frac{xt}{yt}\right)}{(xt)^2+(yt)^2}$$

$$u=\frac{t(y+x)}{t^3x^2y} \qquad v=\frac{\log\left(\frac{x}{y}\right)}{t^2(x^2+y^2)}$$

$$u=\frac{(y+x)}{t^2x^2y} \qquad v=\frac{\log\left(\frac{x}{y}\right)}{t^2(x^2+y^2)}$$

$$u=t^{-2}\frac{(y+x)}{x^2y} \qquad v=t^{-2}\frac{\log\left(\frac{x}{y}\right)}{(x^2+y^2)}$$

$$u=t^{-2}u(x,y) \qquad v=t^{-2}v(x,y)$$

$\therefore$ u and v are homogeneous function of degree $n=-2$

$\therefore$ by Eulers theorem

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \quad \Rightarrow \quad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-2u \qquad \dots\dots\dots(1)$$

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=nv \quad \Rightarrow \quad x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=-2v \qquad \dots\dots\dots(2)$$

$$f=u+v \qquad diff\ w.r.t\ \ x\ ,\ \ y$$

$$\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x} \qquad \& \qquad \frac{\partial f}{\partial y}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$x\frac{\partial f}{\partial x}=x\frac{\partial u}{\partial x}+x\frac{\partial v}{\partial x} \qquad\qquad y\,\frac{\partial f}{\partial y}=y\frac{\partial u}{\partial y}+y\frac{\partial v}{\partial y}$$

$\therefore$ Adding (1) and(2)

$$\Rightarrow\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}\ =-2u\ -2v$$

$$\Rightarrow\ x\frac{\partial u}{\partial x}+x\frac{\partial v}{\partial x}+y\frac{\partial u}{\partial y}+y\frac{\partial v}{\partial y}\ =-2(u+v)$$

$$\Rightarrow x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\ =-2f$$

$$\Rightarrow\ x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+2f=0$$

****

Example: If $u=\frac{x^3+y^3}{x+y}+\frac{1}{x^5}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$ then find the value of

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+\mathrm{x}^2\frac{\partial^2\mathrm{f}}{\partial\mathrm{x}^2}+2\mathrm{xy}\frac{\partial^2\mathrm{f}}{\partial\mathrm{x}\,\partial\mathrm{y}}+\mathrm{y}^2\frac{\partial^2\mathrm{f}}{\partial\mathrm{y}^2} \qquad \text{at the point } (1,2)$$

Solution: Consider $f(x,y)=\frac{x^3+y^3}{x+y}+\frac{1}{x^5}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$

$$\text{Let}\quad u=\frac{x^3+y^3}{x+y} \qquad v=\frac{1}{x^5}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$$

$$f(x,y)=u+v\ \ \dots\dots\dots(A)$$

$$\text{Put}\quad x=xt,\quad y=yt$$

$$u=\frac{(xt)^3+(yt)^3}{xt+yt} \qquad v=\frac{1}{(xt)^5}\sin^{-1}\left(\frac{(xt)^2+(yt)^2}{(xt)^2+2xtyt}\right)$$

$$u = \frac{t^3(x^3+y^3)}{t(x+y)} \qquad v = \frac{1}{x^5t^5}\sin^{-1}\left(\frac{t^2(x^2+y^2)}{t^2(x^2+2xy)}\right)$$

$$u = t^2\,\frac{(x^3+y^3)}{(x+y)} \qquad v = t^{-5}\,\frac{1}{x^5}\sin^{-1}\left(\frac{x^2+y^2}{x^2+2xy}\right)$$

$$u = t^2\,u(x,y) \qquad v = t^{-5}v(x,y)$$

$\therefore$ u is homogeneous function of degree $n = 2$

$\therefore$ v is homogeneous function of degree $n = -5$

$\therefore$ by Eulers theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Rightarrow\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u \qquad \dots\dots\dots(1)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2(2-1)u \quad = 2u \qquad \dots\dots\dots.(2)$$

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

$$\Rightarrow \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = -5v \qquad \dots\dots\dots(3)$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = n(n-1)v$$

$$x^2\frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x\,\partial y} + y^2\frac{\partial^2 v}{\partial y^2} = -5(-5-1)v \ = 30v \qquad \dots\dots\dots(4)$$

$f = u + v \qquad diff\ w.r.t\ \ x, y$

$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$ multiply by x $\quad x\frac{\partial f}{\partial x} = x\frac{\partial u}{\partial x} + x\frac{\partial v}{\partial x}$

$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$ multiply by y $\quad y\,\frac{\partial f}{\partial y} = y\frac{\partial u}{\partial y} + y\frac{\partial v}{\partial y}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x^2} \quad \text{multiply by } x^2 \quad x^2 \frac{\partial^2 f}{\partial x^2} = x^2 \frac{\partial^2 u}{\partial x^2} + x^2 \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y^2} \quad \text{multiply by } y^2 \quad y^2 \frac{\partial^2 f}{\partial y^2} = y^2 \frac{\partial^2 u}{\partial y^2} + y^2 \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial y} \quad \text{multiply by } 2xy \quad 2xy \frac{\partial^2 f}{\partial x \partial y} = 2xy \frac{\partial^2 u}{\partial x \partial y} + 2xy \frac{\partial^2 v}{\partial x \partial y}$$

$\therefore$ Adding (1), (2), (3)and (4)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + +x^2 \frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + x^2 \frac{\partial^2 v}{\partial x^2} + 2xy\frac{\partial^2 v}{\partial x \partial y} + y^2 \frac{\partial^2 v}{\partial y^2}$$

$$= 2u - 5v + 2u + 30v$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4u + 25v$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4\,\frac{x^3 + y^3}{x+y} + 25\frac{1}{x^5}\,\sin^{-1}\left(\frac{x^2 + y^2}{x^2 + 2xy}\right)$$

At point $(1,2)$ put $x = 1$ and $y = 2$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4\,\frac{1^3 + 2^3}{1+2} + 25\frac{1}{1^5}\,\sin^{-1}\left(\frac{1^2 + 2^2}{1^2 + 2(1)(2)}\right)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 4\left(\frac{9}{3}\right) + 25\sin^{-1}\left(\frac{5}{5}\right)$$

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = 12 + 25\frac{\pi}{2}$$

****

Example: If $u = x^n f\left(\frac{y}{x}\right) + y^{-n} g\left(\frac{x}{y}\right)$ then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + x^2 \frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2 \frac{\partial^2 f}{\partial y^2} = n^2 u$$

****

Euler's Theorem of Non-Homogeneous Function:

Let $u = f^{-1}(x,\ y)$ be a non-homogenous function but $f(u)$ is homogenous function of order n so that $f(u) = t^n f(u)$

For 2 variables and 1st order $\quad x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\frac{f(u)}{f'(u)} = g(u)$

For 2 variables and 2nd order $\quad x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = g(u)[g'(u) - 1]$

Where $g(u) = n\frac{f(u)}{f'(u)}$

****

Example: If $u = \sin^{-1}(x^2 + y^2)^{1/5}$ then Prove that

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x\,\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = \frac{2\tan u}{5}\left[\frac{2\tan^2 u}{5} - \frac{3}{5}\right]$$

Solution: Consider $u(x,y) = \sin^{-1}(x^2 + y^2)^{1/5}$

Put $\quad x = xt, \quad y = yt$

$$u = \sin^{-1}[(xt)^2 + (yt)^2]^{1/5}$$

$$u = \sin^{-1}[t^2(x^2 + y^2)]^{1/5} \qquad (a^n)^m = a^{nm}$$

$$u = \sin^{-1}\left[t^{2/5}(x^2 + y^2)^{1/5}\right] \qquad \sin^{-1}(ax) \neq a\sin^{-1}(x)$$

$u(x,y) \neq t^n u(x,y) \quad \therefore$ u is not homogeneous function

$$\sin u = t^{2/5}(x^2 + y^2)^{1/5}$$

$\therefore\ f(u) = \sin u$ is homogeneous function of degree $n = \frac{2}{5}$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = g(u)[g'(u) - 1] \quad \ldots\ldots\ldots(A)$$

where $g(u) = n\frac{f(u)}{f'(u)}$

here $f(u) = \sin u \quad \Rightarrow f'(u) = \cos u$

$$\Rightarrow g(u) = n\frac{f(u)}{f'(u)} = \frac{2\sin u}{5\cos u} = \frac{2}{5}\tan u$$

$$g'(u) = \frac{2}{5}\sec^2 u$$

Equation (A)

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=\frac{2}{5}\tan u\left[\frac{2sec^2u}{5}-1\right]$$

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=\frac{2}{5}\tan u\left[\frac{2(tan^2u+1)}{5}-1\right]$$

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=\frac{2}{5}\tan u\left[\frac{2\,tan^2u+2}{5}-1\right]$$

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=\frac{2}{5}\tan u\left[\frac{2\,tan^2u+2-5}{5}\right]$$

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=\frac{2}{5}\tan u\left[\frac{2\,tan^2u}{5}-\frac{3}{5}\right]$$

****

Example: If $u=\operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$ then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{12}\left[\frac{\tan^2 u}{12}+\frac{13}{12}\right]$$

Solution: Consider $u=\operatorname{cosec}^{-1}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

Put $x=xt,\quad y=yt$

$$u=\operatorname{cosec}^{-1}\sqrt{\frac{(xt)^{1/2}+(yt)^{1/2}}{(xt)^{1/3}+(yt)^{1/3}}}$$

$$u=\operatorname{cosec}^{-1}\sqrt{\frac{t^{1/2}(x^{1/2}+y^{1/2})}{t^{1/3}(x^{1/3}+y^{1/3})}}$$

$$u=\operatorname{cosec}^{-1}\left[\frac{t^{1/2}(x^{1/2}+y^{1/2})}{t^{1/3}(x^{1/3}+y^{1/3})}\right]^{1/2}$$

$$u=\operatorname{cosec}^{-1}\left[\frac{t^{1/2}t^{-1/3}(x^{1/2}+y^{1/2})}{(x^{1/3}+y^{1/3})}\right]^{1/2}$$

$$u = \operatorname{cosec}^{-1}\left[\frac{t^{1/6}(x^{1/2} + y^{1/2})}{(x^{1/3} + y^{1/3})}\right]^{1/2}$$

$$u = \operatorname{cosec}^{-1}\left[t^{1/12}\sqrt{\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}}\right]$$

$u(x,y) \neq t^n u(x,y) \quad \therefore$ u is not homogeneous function

$$\operatorname{cosec} u = t^{1/12}\sqrt{\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$$

$\therefore\ f(\mathrm{u}) = \operatorname{cosec}\mathrm{u}$ is homogeneous function of degree $\mathrm{n} = \frac{1}{12}$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$$

$$where\ \ g(u) = n\frac{f(u)}{f'(u)}$$

here $f(u) = cosec\,u \qquad \Rightarrow f'(u) = -cosec\,u\cot u$

$$\Rightarrow g(u) = n\frac{f(u)}{f'(u)} = -\frac{1}{12}\frac{cosec\,u}{cosec\ u\cot u}$$

$$g(u) = -\frac{1}{12}\frac{1}{\cot u} = -\frac{1}{12}\tan u$$

$$g'(u) = -\frac{1}{12}sec^2 u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{1}{12}\tan u\left[\frac{-sec^2 u}{12} - 1\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left[\frac{sec^2 u}{12} + 1\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left[\frac{tan^2 u + 1}{12} + 1\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left[\frac{tan^2 u + 1 + 12}{12}\right]$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{12}\tan u\left[\frac{tan^2 u + 13}{12}\right]$$

****

Example: If $u = \sec^{-1}\left[\frac{x+y}{x^{1/2}+y^{1/2}}\right]$ then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\frac{\cot u}{4}[3+\cot^2 u]$$

****

Example: If $u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$ then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u\,[1-4\sin^2 u]$$

****

Example: If $u = \sin^{-1}\left[\frac{x+y}{x^{1/2}+y^{1/2}}\right]$ then Prove that

$$x^2\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\,\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=\frac{-\sin u\cos 2u}{4\cos^3 u}$$

****

Example: If $u = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]^{\frac{1}{2}}$ then Prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{4}[\tan^2 u-1]$$

****