Engineering Mathematics - II

Unit I

First Order Ordinary Differential Equations

Differential Equations (D.E.)

It is an equation involving dependent variables and their derivatives with respect to the independent variables.

Ordinary Differential Equations (O.D.E.)

It is a differential equation involving only one independent variable .

Partial Differential Equations (P.D.E.)

It is a differential equation involving two or more independent variables.

Order of a D.E.

It is the highest order derivative appearing in the equation.

Degree of a D. E.

It is the degree of the highest ordered derivative when the derivatives are free from radicals.

State order and degree of following D.E.

$$\left(\frac{dy}{dx}\right)^1 + x^3 = \sin y$$

Order = 1 Degree= 1

State order and degree of following D.E.

$$\left(\frac{dy}{dx}\right)^4 + x^3 = \left(\frac{d^3y}{dx^3}\right)^2 - 4xy$$

Order = 3 Degree= 2

State order and degree of following D.E.

$$\sqrt[3]{\left(\frac{d^2y}{dx^2}\right)^2 - x^2y} = \left(\frac{dy}{dx}\right)^{1/2}$$

Order = 2 Degree= 2

State order and degree of following D.E.

-
- $$\left[1+\left(\frac{dy}{dx}\right)^3\right]^{\frac{3}{2}}=\left(\frac{d^2y}{dx^2}\right)$$

- Order =

  2

- degree =

  2

Solution of a D.E.

It is a relation between the variables which satisfies the given D. E.

General Solution

It is a solution of a D.E. in which the number of arbitrary constants equals to the order of D.E.

Particular Solution

It is a solution of a D.E. obtained by assigning particular values to the arbitrary constants in general solution of D.E.

Ordinary D.E. of 1^{st} order and 1^{st} degree

It is the D.E. of the form

$$Mdx + Ndy = 0$$

where M and N are functions of x, y or constants

These types are:

1. Variable separable form. (V.S Form)
2. Reducible to V.S Form.

3. Homogeneous D.E.
4. Non- Hom D.E. Reducible to Hom. form

5. Exact D.E.
6. Reducible to exact form by using integrating factor.

7. Linear D.E. of the first order.
8. Equations reducible to linear form.

Methods of solving O.D.E. of 1st order and 1st degree

Variable separable form (V.S Form)

The given D.E. can be written as

$$f(x)dx = g(y)dy$$

G. S. is obtained by taking integration on both sides

$$\int f(x)dx = \int g(y)dy + C$$

Evaluate $\frac{dy}{dx}+\frac{1+y^2}{1+x^2}=0$

$$Given \quad \frac{dy}{dx}+\frac{1+y^2}{1+x^2}=0 \quad \Rightarrow \int\frac{dy}{1+y^2}+\int\frac{dx}{1+x^2}=0$$

$$\tan^{-1} y+\tan^{-1} x=\tan^{-1} C$$

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right)=\tan^{-1} C \quad \Rightarrow \quad \left(\frac{x+y}{1-xy}\right)=C \quad is \ the \ G.S.$$

Evaluate $\frac{dy}{dx}=e^{x-y}+3x^2e^{-y}$

$$\frac{dy}{dx}=e^{x-y}+3x^2e^{-y} \ \ Multiplying \ by \ e^y, \ we \ get$$

$$e^y\,dy=\left(e^x+3x^2\right)dx$$

$$\text{int}\,egrating \quad e^y=e^x+x^3+c \ is \ the \ G.S.$$

Exact D.E.

A D.E. $M\,dx + N\,dy = 0$ is said to be exact if there exist a function $f(x,y)$ such that

$$M\,dx + N\,dy = df.$$

The necessary and sufficient condition that the equation be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- The G.S. of exact D.E. is given by

Rule 1.

$$\int_{y\ constant} M\,dx + \int [terms\ of\ N\ not\ containing\ x]\ dy = C$$

Rule 2.

Some times we may write the general solution by using the following rule

$$\int_{x\ constant} N\,dy + \int [terms\ of\ M\ not\ containing\ y]\ dx = C$$

Remark: Some times an equation of the form

$$\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$$

becomes Exact if $b_1 = -a_2$

Solve $\frac{dy}{dx} = \frac{2x-3y+1}{3x+4y-5}$

Here $b1 = -a2$

Given D.E is in Exact form

G.S is

$$\int_{y\ constant} M\, dx + \int [terms\ of\ N\ not\ containing\ x]\ dy = C$$

$$\int_{y\ constant} (2x - 3y + 1)\, dx + \int (-4y + 5)dy = C$$

Solve

$$(y^2e^{xy^2}+4x^3)dx+(2xy\,e^{xy^2}-3y^2)dy=0$$

Here $M=y^2e^{xy^2}+4x^3,\quad N=2xye^{xy^2}-3y^2$

$$\frac{\partial M}{\partial y}=2ye^{xy^2}+2xy^3e^{xy^2}=\frac{\partial N}{\partial x}$$

$\therefore$ Given D.E. is exact

$\therefore$ G.S. is $\int\limits_{y=const} M\,dx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$$\int\limits_{y=const}\left(y^2e^{xy^2}+4x^3\right)dx+\int\left(-3y^2\right)dy=c$$

$$\therefore e^{xy^2}+x^4-y^3=c$$

Solve $\left(\frac{y^2}{1+x^2} - 2y\right)dx = (2x - 2y\tan^{-1}x - \sinh y)dy$

$$\left(\frac{y^2}{1+x^2} - 2y\right)dx - \left(2x - 2y\tan^{-1}x - \sinh y\right)dy = 0$$

Here $M = \frac{y^2}{1+x^2} - 2y, \qquad N = -2x + 2y\tan^{-1}x + \sinh y$

$$\frac{\partial M}{\partial y} = \frac{2y}{1+x^2} - 2 = \frac{\partial N}{\partial x}$$

$\therefore$ Given D.E. is exact

$\therefore$ G.S. is $\int\limits_{y=const} M\,dx + \int (\text{terms of } N \text{ not containing } x)dy = c$

$$\int\limits_{y=const} \left(\frac{y^2}{1+x^2} - 2y\right)dx + \int (\sinh y)dy = c$$

$$\therefore\ y^2\tan^{-1}x - 2xy + \cosh y = c$$

Solve $(1+\log xy)dx+\left(1+\frac{x}{y}\right)dy=0$

Solve $\left(\log\left(x^2+y^2\right)+\frac{2x^2}{x^2+y^2}\right)dx+\frac{2xy}{x^2+y^2}dy=0$

Home work

D.E. Reducible to Exact Form By Using Integrating Factor.

If $M\ dx + N\ dy = 0$ is not exact then by multiplying the equation by function $k(x,y)$ called as Integrating Factor (I.F.)

The equation can be made exact, i.e. there exists a function $u(x,y)$ such that $k\ (M\ dx + N\ dy) = du$

Rules of finding I.F.

- Rule 1

If $xM + yN \neq 0$ and given D.E. is homo.

then $I.F. = \dfrac{1}{xM + yN}$

Solve $(x^2y - 2xy^2)dx + (-x^3 + 3x^2y)dy = 0$

Here $M = x^2y - 2xy^2, \quad N = -x^3 + 3x^2y$

$$\frac{\partial M}{\partial y} = x^2 - 4xy \neq \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

As $xM + yN = x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = x^2y^2 \neq 0$

and the given D.E. homogeneous I.F. $= \frac{1}{xM+yN} = \frac{1}{x^2y^2}$

$$\frac{1}{x^2y^2}\left[\left(x^2y - 2xy^2\right)dx - \left(x^3 - 3x^2y\right)dy\right] = 0$$

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$ which is exact D.E.

$\therefore$ G.S. is $\int\limits_{y=const}\left(\frac{1}{y} - \frac{2}{x}\right)dx + \int\left(\frac{3}{y}\right)dy = c \quad \therefore \frac{x}{y} - 2\log x + 3\log y = c$

- Rule 2

If $xM - yN \neq 0$ and

the given D.E. can be written as

$$y\, f_1(xy)\, dx + x\, f_2(xy)\, dy = 0$$

then $I.F. = \dfrac{1}{x\,M - y\,N}$

Solve $(x^2y^2 + xy)y\,dx + (x^2y^2 - 1)x\,dy = 0$

Here $M = (x^2y^2 + xy)y, \quad N = (x^2y^2 - 1)x$

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy \neq \frac{\partial N}{\partial x} = 3x^2y^2 - 1$$

As $Mx - Ny = (x^2y^2 + xy)xy - (x^2y^2 - 1)xy \;= (xy+1)xy \neq 0$

$y\,f_1(xy)\,dx + x\,f_2(xy)\,dy = 0 \quad I.F. = \dfrac{1}{xM - yN} = \dfrac{1}{(xy+1)xy}$

$$\frac{1}{(xy+1)xy}\left[(x^2y^2 + xy)ydx + (x^2y^2 - 1)xdy\right] = 0 \qquad ydx + \left(x - \frac{1}{y}\right)dy = 0$$

Which is exact diff. eq.

$\therefore$ G.S. is $\displaystyle\int_{y=const} Mdx + \int (\text{terms of } N \text{ not containing } x)dy = c$

$$\int_{y=const} ydx + \int\left(-\frac{1}{y}\right)dy = c \qquad \therefore xy - \log y = c$$

- Solve

$$(x^2y^3 + 5xy^2 + 2y)dx + (x^3y^2 + 4x^2y + 2x)dy = 0$$

Here $M = (x^2y^2 + 5xy + 2)y$, $N = (x^2y^2 + 4xy + 2)x$

$$y\, f_1(xy)\, dx + x\, f_2(xy)\, dy = 0$$

- Rule 3

$$\text{If } \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x) \text{ then } I.F. = e^{\int f(x)\,dx}$$

Solve $(x^4 e^x - 2xy^2)dx + 2x^2 y\, dy = 0$

Here $M = x^4 e^x - 2xy^2,\ N = 2x^2 y$ $\qquad \dfrac{\partial M}{\partial y} = -4xy \neq \dfrac{\partial N}{\partial x} = 4xy$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{-4xy - 4xy}{2x^2 y} = -\frac{4}{x} = f(x) \quad \therefore I.F. = e^{\int f(x)dx} = e^{\int -\frac{4}{x}dx} = \frac{1}{x^4}$$

$$\frac{1}{x^4}\left[\left(x^4 e^x - 2xy^2\right)dx + 2x^2 y\,dy\right] = 0$$

$$\left(e^x - \frac{2y^2}{x^3}\right)dx + \frac{2y}{x^2}dy = 0 \quad \text{which is exact D.E.}$$

$\therefore$ G.S. is $\displaystyle\int_{y=const} M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$

$$\int_{y=const} \left(e^x - \frac{2y^2}{x^3}\right)dx = c \qquad \therefore e^x + \frac{y^2}{x^2} = c$$

Solve

$$(x\sec^2 y - x^2\cos y)dy = (\tan y - 3x^4)dx$$

$$(tany\ - 3x^4)dx + (x^2cosy - xsec^2y)dy = 0$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = -\frac{2}{x} = f(x) \qquad \text{I.F.} = \frac{1}{x^2}$$

$$\left(\frac{1}{x^2}\right)\left((\tan y - 3x^4)dx + (-x\sec^2 y + x^2\cos y)dy\right) = 0$$

which is exact D.E.

G.S

$$\frac{tany}{x} + \mathrm{x}^3 - \mathrm{siny} = \mathrm{C}$$

- Rule 4

$$\text{If } \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = g(y) \text{ then } I.F. = e^{\int g(y)dy}$$

Solve $(2xy + y\,e^x \log y)dx + e^x\,dy = 0$

Here $M = \left(2x + e^x \log y\right)y, \quad N = e^x$

$$\frac{\partial M}{\partial y} = 2x + e^x\left(1 + \log y\right) \neq \frac{\partial N}{\partial x} = e^x$$

As $$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{e^x - \left[2x + e^x\left(1 + \log y\right)\right]}{\left(2x + e^x \log y\right)y} = -\frac{1}{y} = g(y)$$

$$\therefore I.F. = e^{\int g(y)dy} = e^{\int -\frac{1}{y}dy} = e^{-\log y} = e^{\log\left(\frac{1}{y}\right)} = \frac{1}{y}$$

$$\frac{1}{y}\left[\left(2x + e^x \log y\right)y\,dx + e^x dy\right] = 0 \quad \left(2x + e^x \log y\right)dx + \frac{e^x}{y}dy = 0$$

which is exact D.E.

$\therefore$ G.S. is $$\int_{y=const} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$$

$$\int_{y=const} \left(2x + e^x \log y\right)dx = c \qquad \therefore x^2 + e^x \log y = c$$

• $Solve\ \ y(2x^2y + e^x)dx = (e^x + y^3)dy$

Here $M = 2x^2y^2 + e^x y\ , N = -e^x - y^3$

$$\frac{\partial M}{\partial y} = 4\,x^2 y + e^x \neq \frac{\partial N}{\partial x} = -e^x$$

As $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \dfrac{-2e^x - 4x^2y}{(2x^2y+e^x)y} = -\dfrac{2}{y} = g(y)$

$$I.F = e^{\int g((y))dy} = e^{\int -\frac{2}{y}dy} = \frac{1}{y^2}$$

$$\frac{1}{y^2}(y(2x^2y + e^x)dx - (e^x + y^3)dy) = 0$$

$$\left(2x^2 + \frac{e^x}{y}\right)dx - \left(\frac{e^x}{y^2} + y\right)dy = 0$$ Which is exact diff. eq.

G.S. is $\int\left(2x^2 + \frac{e^x}{y}\right)dx\ - \int y\,dy = C\quad i.e\,\dfrac{2x^3}{3} + \dfrac{e^x}{y} - \dfrac{y^2}{2} = C$

- Rule 5

If the given D.E. can be written as

$$x^a y^b (m(y\,dx) + n(x\,dy)) + x^{a_1} y^{b_1} (m_1(y\,dx) + n_1(x\,dy)) = 0$$

Where $a, b, m, n, a_1, b_1, m_1, n_1$ are all constant.

$m, n, m_1, n_1 \neq 0$ & $mn_1 - nm_1 \neq 0$

then $I.F. = x^h y^k$ choose h and k such that condition of exactness is satisfied.

Solve $(x^7 y^2 + 3y)dx + (3x^8 y - x)dy = 0$

Here $M = x^7 y^2 + 3y, \quad N = 3x^8 y - x$

$$\frac{\partial M}{\partial y} = 2x^7 y + 3 \neq \frac{\partial N}{\partial x} = 24x^7 y - 1$$

$$x^7 y^2 dx + 3ydx + 3x^8 ydy - xdy = 0$$

$$x^7 y^1 (ydx + 3xdy) + x^0 y^0 (3ydx - xdy) = 0$$

Hence the given differential equation of the form

$$x^a y^b (m(y\,dx) + n(x\,dy)) + x^{a_1} y^{b_1} (m_1(y\,dx) + n_1(x\,dy)) = 0$$

$$\therefore I.F. = x^h y^k$$

$$x^h y^k \left[(x^7 y^2 + 3y) dx + (3x^8 y - x) dy \right] = 0$$

$$\left(x^{h+7}y^{k+2}+3x^{h}y^{k+1}\right)dx+\left(3x^{h+8}y^{k+1}-x^{h+1}y^{k}\right)dy=0$$

For exactness $\frac{\partial}{\partial y}\left(x^{h+7}y^{k+2}+3x^{h}y^{k+1}\right)=\frac{\partial}{\partial x}\left(3x^{h+8}y^{k+1}-x^{h+1}y^{k}\right)$

$$(k+2)x^{h+7}y^{k+1}+3(k+1)x^{h}y^{k}=3(h+8)x^{h+7}y^{k+1}-(h+1)x^{h}y^{k}$$

$$(k+2)=3(h+8),\quad 3(k+1)=-(h+1)\quad \therefore h=-7,k=1$$

$$\therefore \left(x^{-7+7}y^{1+2}+3x^{-7}y^{1+1}\right)dx+\left(3x^{-7+8}y^{1+1}-x^{-7+1}y^{1}\right)dy=0$$

$$\therefore \left(y^{3}+3x^{-7}y^{2}\right)dx+\left(3xy^{2}-x^{-6}y\right)dy=0$$

which is exact D.E.

$\therefore$ G.S. is $\int\limits_{y=const} M dx+\int(\text{terms of } N \text{ not containing } x)dy=c$

$$\int\limits_{y=const}\left(y^{3}+3x^{-7}y^{2}\right)dx=c \qquad \therefore xy^{3}-\frac{x^{-6}y^{2}}{2}=c$$

Linear D.E.

A D.E. of the form

$$\frac{dy}{dx} + Py = Q$$

where P, Q are functions of x or constants, is called a linear D.E. in y

$$I.F. = e^{\int P\,dx}$$

G.S. of linear D.E. is

$$ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$$

A D.E. of the form

$$\frac{dx}{dy} + Px = Q$$

where P, Q are functions of y or constants, is called a linear D.E. in x

$$I.F. = e^{\int P\,dy}$$

G.S. of linear D.E. is

$$xe^{\int Pdy} = \int Qe^{\int Pdy}\,dy + c$$

- Note
- If a D.E. contain single term y then go for linear in y

- If a D.E. contain single term x then go for linear in x

Solve $1+y^2+\left(x-e^{-tan^{-1}y}\right)\frac{dy}{dx}=0$

$$\left(1+y^2\right)\frac{dx}{dy}+\left(x-e^{-\tan^{-1}y}\right)=0,\quad \frac{dx}{dy}+\frac{x}{\left(1+y^2\right)}=\frac{e^{-\tan^{-1}y}}{\left(1+y^2\right)}$$

which is linear D.E.

$$\text{where } P=\frac{1}{1+y^2},\quad Q=\frac{e^{-\tan^{-1}y}}{1+y^2}$$

$$\text{G.S. is}\quad xe^{\int\frac{1}{1+y^2}dy}=\int\frac{e^{-\tan^{-1}y}}{1+y^2}e^{\int\frac{1}{1+y^2}dy}dy+c$$

$$xe^{\tan^{-1}y}=\int\frac{e^{-\tan^{-1}y}}{1+y^2}e^{\tan^{-1}y}dy+c\quad\therefore xe^{\tan^{-1}y}=\tan^{-1}y+c$$

Solve $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$

$$\frac{dy}{dx}+\frac{(2-x)}{x(x-1)}y=\frac{x^3(2x-1)}{x(x-1)}$$

Which is linear D.E

$$P=\frac{2-x}{x(x-1)},\ Q=\frac{x^3(2x-1)}{x(x-1)}$$

G.S is $y\,e^{\int P\,dx}=\int Q.e^{\int P\,dx}dx+C$

$$\int P\,dx=\int\frac{2-x}{x(x-1)}dx=\int -\frac{2}{x}dx+\int\frac{1}{x-1}dx$$

$$=-2logx+\log(x-1)=\log(\frac{x-1}{x^2})$$

I.F$=e^{\int P\,dx}=e^{\log(\frac{x-1}{x^2})}=\frac{x-1}{x^2}$

G.S is $\frac{y(x-1)}{x^2}=\int\frac{x^2(2x-1)}{x-1}\left(\frac{x-1}{x^2}\right)dx+C=x^2-x+C$

Bernoulli's D.E.

A D.E. of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called Bernoulli's D.E. in y

$Divide\ by\ y^n$

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

$Put \quad y^{1-n} = u\ and\ solve$

Similarly, a D.E. of the form

$$\frac{dx}{dy} + P(y)x = Q(y)x^n$$

is called Bernoulli's D.E. in x

$Divide\ by\ x^n$

$$x^{-n}\frac{dx}{dy} + P(y)x^{1-n} = Q(y)$$

$Put \quad x^{1-n} = u\ and\ solve$

Solve $x\frac{dy}{dx} + 3y = x^4 e^{\frac{1}{x^2}} y^3$

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{3}{xy^2} = x^3 e^{1/x^2}$$

Putting $\frac{1}{y^2} = u, \therefore -\frac{2}{y^3}\frac{dy}{dx} = \frac{du}{dx}$

$$-\frac{1}{2}\frac{du}{dx} + \frac{3u}{x} = x^3 e^{1/x^2} \qquad \frac{du}{dx} - \frac{6u}{x} = -2x^3 e^{1/x^2}$$

which is linear D.E. where $P = -\frac{6}{x}, \quad Q = -2x^3 e^{1/x^2}$

G.S. is $ue^{\int \frac{-6}{x}dx} = \int -2x^3 e^{1/x^2} e^{\int \frac{-6}{x}dx} dx + c$

$$\frac{u}{x^6} = \int \frac{-2x^3 e^{1/x^2}}{x^6} dx + c \qquad \frac{u}{x^6} = \int \frac{-2e^{1/x^2}}{x^3} dx + c \qquad \frac{u}{x^6} = e^{1/x^2} + c$$

$$\frac{1}{x^6 y^2} = e^{1/x^2} + c$$

- Solve $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

Put $y^{-2} = u$

$$\frac{du}{dx} + (2x)u = 2e^{-x^2}$$

$$P = 2x \quad , Q = 2e^{-x^2}$$

$$\text{I.F.} = e^{\int 2x\, dx} = e^{x^2}$$

$$\text{G.S} \quad \frac{e^{x^2}}{y^2} = 2x + C$$

Equation reducible to linear form

The D.E. of the form

$$f'(y)\frac{dy}{dx}+P(x)f(y)=Q(x)$$

can be reduce to linear D.E. by substituting

$$f(y)=u, \quad \therefore f'(y)\frac{dy}{dx}=\frac{du}{dx}$$

$$\frac{du}{dx}+P(x)u=Q(x)$$

which is linear D.E. in u

$$\therefore \text{G.S. is } ue^{\int Pdx}=\int Qe^{\int Pdx}dx+c$$

Similarly, the D.E. of the form

$$f'(x)\frac{dx}{dy}+P(y)f(x)=Q(y)$$

can be reduce to linear D.E. by substituting

$$f(x)=u, \quad \therefore f'(x)\frac{dx}{dy}=\frac{du}{dy}$$

$$\frac{du}{dy}+P(y)u=Q(y)$$

which is linear D.E. in u

Solve $\cos x \dfrac{dy}{dx} = y(\sin x - y)$

$$\cos x \frac{dy}{dx} = y \sin x - y^2 \qquad \therefore \frac{dy}{dx} - y\frac{\sin x}{\cos x} = -\frac{y^2}{\cos x}$$

$$\therefore \frac{1}{y^2}\frac{dy}{dx} - \frac{\tan x}{y} = -\sec x$$ Putting $-\dfrac{1}{y} = u, \therefore \dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{du}{dx}$

$$\therefore \frac{du}{dx} + u\tan x = -\sec x$$ which is linear D.E.

where $P = \tan x, \quad Q = -\sec x$

G.S. is $ue^{\int \tan x dx} = \int -\sec x e^{\int \tan x dx} dx + c$

$$ue^{\log \sec x} = \int -\sec x e^{\log \sec x} dx + c$$

$$u \sec x = \int -\sec^2 x dx + c \qquad -\frac{\sec x}{y} = -\tan x + c$$

Solve $\frac{dy}{dx}+x\sin 2y = x^3\cos^2 y$

$$\frac{1}{\cos^2 y}\frac{dy}{dx}+x\frac{\sin(2y)}{\cos^2 y}=x^3 \qquad \sec^2 y\frac{dy}{dx}+2x\tan y=x^3$$

Putting $\tan y=u, \therefore \sec^2 y\frac{dy}{dx}=\frac{du}{dx}$

$\frac{du}{dx}+2xu=x^3$ which is linear D.E.

where $P=2x, \quad Q=x^3$

G.S. is $ue^{\int 2x dx}=\int x^3 e^{\int 2x dx}dx+c$

$$ue^{x^2}=\int x^3 e^{x^2}dx+c \qquad ue^{x^2}=(1/2)\left(x^2-1\right)e^{x^2}+c$$

$$(\tan y)e^{x^2}=(1/2)\left(x^2-1\right)e^{x^2}+c$$

Solve $x\cos x\frac{dy}{dx}+y\,(\,x\sin x+\cos x)=1$

$$\frac{dy}{dx}+y\frac{(x\sin x+\cos x)}{x\cos x}=\frac{1}{x\cos x}$$

which is linear D.E. $P=\frac{(x\sin x+\cos x)}{x\cos x},Q=\frac{1}{x\cos x}$

$\therefore$ G.S. is $ye^{\int\frac{(x\sin x+\cos x)}{x\cos x}dx}=\int\frac{1}{x\cos x}e^{\int\frac{(x\sin x+\cos x)}{x\cos x}dx}dx+c$

$$ye^{-\log\cos x+\log x}=\int\frac{1}{x\cos x}e^{-\log\cos x+\log x}dx+c$$

$$\frac{xy}{\cos x}=\int\frac{1}{x\cos x}\frac{x}{\cos x}dx+c \qquad \frac{xy}{\cos x}=\tan x+c$$

Solve $y(xy+2x^2y^2)dx+x(xy-x^2y^2)dy=0$

$$M=y\left(xy+2x^2y^2\right),\ N=x\left(xy-x^2y^2\right)$$

$$\frac{\partial M}{\partial y}=2xy+6x^2y^2\neq\frac{\partial N}{\partial x}=2xy-3x^2y^2$$

$$xM-yN=x^2y^2+2x^3y^3-x^2y^2+x^3y^3=3x^3y^3\neq 0$$

and as given D.E. is of the form $yf(xy)dx+xg(xy)dy=0$

$$I.F.=\frac{1}{xM-yN}=\frac{1}{3x^3y^3}$$

$$\frac{1}{3x^3y^3}\left[y\left(xy+2x^2y^2\right)dx+x\left(xy-x^2y^2\right)dy\right]=0$$

$$\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\left(\frac{1}{3xy^2}-\frac{1}{3y}\right)dy=0$$ which is exact

$$\int\limits_{y=const}\left(\frac{1}{3x^2y}+\frac{2}{3x}\right)dx+\int\left(-\frac{1}{3y}\right)dy=c \qquad -\frac{1}{3xy}+\frac{2}{3}\log x-\frac{1}{3}\log y=c$$

Solve $\left[\frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}\right] dx - \frac{x}{(x-y)^2} dy = 0$

$$M = \frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}, N = -\frac{x}{(x-y)^2}$$

$$\frac{\partial M}{\partial y} = \frac{x+y}{(x-y)^3} = \frac{\partial N}{\partial x}$$

$\therefore$ given D.E. is exact

$\therefore$ G.S. is $\displaystyle\int_{y=const} \left[\frac{y}{(x-y)^2} - \frac{1}{2\sqrt{1-x^2}}\right] dx = c$

$$-\frac{y}{(x-y)} - \frac{1}{2}\sin^{-1} x = c$$

Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)=\frac{dy}{dx} \qquad \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

which is linear D.E. where $P=\frac{1}{\sqrt{x}}, Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$$\therefore \text{G.S. is} \quad ye^{\int \frac{1}{\sqrt{x}}dx}=\int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}e^{\int \frac{1}{\sqrt{x}}dx}dx+c$$

$$ye^{2\sqrt{x}}=\int \frac{e^{-2\sqrt{x}}}{\sqrt{x}}e^{2\sqrt{x}}dx+c$$

$$ye^{2\sqrt{x}}=\int \frac{dx}{\sqrt{x}}+c \qquad ye^{2\sqrt{x}}=2\sqrt{x}+c$$

Solve $\frac{dx}{dy} + xy = x^2 e^{\frac{y^2}{2}} \log y$

$$\frac{1}{x^2}\frac{dx}{dy} + \frac{y}{x} = e^{y^2/2}\log y \quad \text{Putting} \quad \frac{1}{x} = u, \therefore -\frac{1}{x^2}\frac{dx}{dy} = \frac{du}{dy}$$

$$\frac{du}{dy} - uy = -e^{y^2/2}\log y$$

which is linear D.E.

$$P = -y, Q = -e^{y^2/2}\log y$$

$$\therefore \text{G.S. is } ue^{\int -y dy} = \int -e^{y^2/2}\log y e^{\int -y dy} dy + c$$

$$ue^{-y^2/2} = \int -e^{y^2/2}\log y e^{-y^2/2} dy + c$$

$$\frac{e^{-y^2/2}}{x} = (1 - \log y)y + c$$

Solve $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^3 x$

$$\frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^3 x \qquad \sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^3 x$$

Putting $\sec y = u, \sec y \tan y \frac{dy}{dx} = \frac{du}{dx}$

$$\frac{du}{dx} + u \tan x = \cos^3 x$$

which is linear D.E. where $P = \tan x, Q = \cos^3 x$

$$ue^{\int \tan x dx} = \int \cos^3 x e^{\int \tan x dx} dx + c$$

$$u \sec x = \int \cos^3 x \sec x dx + c \qquad u \sec x = \int \cos^2 x dx + c$$

$$\sec y \sec x = \frac{1}{2}\left(\frac{\sin 2x}{2} + x\right) + c$$

Solve $\sin y \frac{dx}{dy} + 2x = \tan^3(\frac{y}{2})$

$$\frac{dx}{dy} + x\left(\frac{2}{\sin y}\right) = \frac{1}{\sin y}\tan^3\left(\frac{y}{2}\right)$$

which is linear D.E. where $P = \frac{2}{\sin y}$, $Q = \frac{1}{\sin y}\tan^3\left(\frac{y}{2}\right)$

G.S. is $xe^{\int \frac{2}{\sin y}dy} = \int \frac{1}{\sin y}\tan^3\left(\frac{y}{2}\right)e^{\int \frac{2}{\sin y}dy}dy + c$

$$xe^{2\log(\cos ec y - \cot y)} = \int \frac{1}{\sin y}\tan^3\left(\frac{y}{2}\right)e^{2\log(\cos ec y - \cot y)}dy + c$$

$$x(\cos ec y - \cot y)^2 = \int \frac{1}{\sin y}\tan^3\left(\frac{y}{2}\right)(\cos ec y - \cot y)^2 dy + c$$

$$x(\operatorname{cosec} y - \cot y)^2 = \int \frac{1}{\sin y} \tan^3\left(\frac{y}{2}\right)\left(\frac{1-\cos y}{\sin y}\right)^2 dy + c$$

$$= \int \frac{\tan^3\left(\frac{y}{2}\right)}{2\sin(y/2)\cos(y/2)}\left(\frac{2\sin^2(y/2)}{2\sin(y/2)\cos(y/2)}\right)^2 dy + c$$

$$= \int \frac{1}{2} \tan^4\left(\frac{y}{2}\right) \sec^2\left(\frac{y}{2}\right) dy + c$$

$$= \frac{1}{5} \tan^5\left(\frac{y}{2}\right) + c$$