UNIT 3 - Integral Calculus

Reduction Formulae

Reduction Formula means a formula which reduces a given integral to a Known integration form by repeated applications of integration by parts.

1] $\int_0^{\pi/2} cos^n x\, dx = \int_0^{\pi/2} sin^n x\, dx$

$$= \frac{[\,(n-1\,)\text{subtract } 2 \ldots\ldots\ldots\ldots\ldots\ldots 2 \text{ or } 1\,]}{[\,(\,n\,)\ \text{subtract } 2 \ldots\ldots\ldots\ldots\ldots\ldots 2 \text{ or } 1]} \times \left(\frac{\pi}{2}\right) \text{ if n is even.}$$

$$= \frac{[\,(n-1\,)\text{subtract } 2 \ldots\ldots\ldots\ldots\ldots\ldots 2 \text{ or } 1\,]}{[\,(\,n\,)\ \text{subtract } 2 \ldots\ldots\ldots\ldots\ldots\ldots 2 \text{ or } 1]} \times 1 \text{ if n is odd.}$$

$\int_0^{\frac{\pi}{2}} \sin^9 x = -------\,-$ $\qquad$ $\int_0^{\frac{\pi}{2}} \cos^6 x = --------\,-$

$= \frac{128}{315}$ $\qquad$ $= \frac{5\pi}{32}$

1				
	a)		b)	
	c)		d)	0

2				
	a)		b)	
	c)		d)	0

$$\int_0^{\pi/2} cos^n x\, dx = \int_0^{\pi/2} sin^n x\, dx \boxed{= \frac{n-1}{n} I_{n\,-\,2}}$$

2.(a) $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \left\{ \frac{[(m-1)(\text{subtract } 2 \ldots\ldots 2 \text{ or } 1)][(n-1)(\text{subtract } 2 \ldots\ldots 2 \text{ or } 1)]}{[(m+n)(\text{subtract } 2 \ldots\ldots 2 \text{ or } 1)]} \right\} \times \left(\frac{\pi}{2}\right)$$

If m and n both are even.

2.(b) $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

$$= \left\{ \frac{[(m-1)(\text{subtract } 2 \ldots\ldots 2 \text{ or } 1)][(n-1)(\text{subtract } 2 \ldots\ldots 2 \text{ or } 1)]}{[(m+n)(\text{subtract } 2 \ldots\ldots 2 \text{ or } 1)]} \right\} \times (1)$$

Otherwise .

$\int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x \, dx = ----\ -$ $\qquad$ $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^5 x \, dx = -----\ -$

$= \frac{3\pi}{512}$ $\qquad$ $= \frac{1}{24}$

3				
	a)		b)	
	c)		d)	

4				
	a)	0	b)	
	c)		d)	

3] $\int_0^{\pi/2} \sin^m x \cos x \, dx = \int_0^{[illegible]/2}$ [illegible] $\sin$ [illegible] $= \frac{1}{[illegible]+1}$

$\int_0^{\frac{\pi}{2}} \cos^5 x \; sinx \; dx = ---$ $\int_0^{\frac{\pi}{2}} cosx \sin^{10} x \; dx = ---$

$= \frac{1}{6}$ $= \frac{1}{11}$

Conversion Formulae :

1] $\int_0^{2[illegible]}$ [illegible] $= \Big\{ \begin{array}{l} = 4\int_0^{[illegible]/2} \text{[illegible]}, \text{[illegible]}, \text{[illegible]}. \\ = 0 \;, \quad \text{[illegible]} h \text{[illegible]}. \end{array}$

2] $\int_0^{[illegible]}$ [illegible]

$= \Big\{ \begin{array}{l} = 2\int_0^{[illegible]/2} \text{[illegible]}, \text{[illegible]}, \text{[illegible]}. \\ = 0 \;, \quad , \text{[illegible]}, \text{[illegible]}. \end{array}$

$$\int_0^{2\pi} \sin^6 x \cos^4 x \, dx = \frac{3\pi}{128}$$

3] [illegible] $= 4 \int_0^{[illegible]/2}$ [illegible], [illegible]
$= 0$, [illegible]

4] [illegible] $= 4 \int_0^{[illegible]/2}$ [illegible], [illegible]
$= 0$, [illegible]

5] [illegible] $= 2 \int_0^{[illegible]/2}$ [illegible], [illegible].

6] [illegible] $= 2 \int_0^{[illegible]/2}$ [illegible], [illegible]
$= 0$, [illegible]

5				
	a)		b)	
	c)		d)	

6				
	a)		b)	
	c)		d)	0

If $U_n = \int_0^{\pi/4} \tan^n x \, dx$, Show that $n (U_{n+1} + U_{n-1}) = 1$

Sol : $U_n = \int_0^{\pi/4} tan^n x \, dx \;\; \therefore \;\; U_{n+1} = \int_0^{\pi/4} tan^{n+1} x \, dx$

$\therefore \;\; U_{n+1} = \int_0^{\pi/4} tan^{n-1} x \; tan^2 x \, dx$

$= \int_0^{\pi/4} tan^{n-1} x \, (sec^2 x - 1) \, dx$

$\therefore \;\; U_{n+1} = \int_0^{\pi/4} tan^{n-1} x \;\; sec^2 x \, dx \; - \; \int_0^{\pi/4} tan^{n-1} x \;\; dx$

$= \left[\frac{tan^n x}{n} \right]_0^{\pi/4} - \; U_{n-1}$

$\therefore \;\; n (U_{n+1} + U_{n-1}) = 1$

If $U_n = \int_0^{\pi/4} tan^n x\, dx$ and if n (U_{n+1} + U_{n-1}) = 1 then $\int_0^{\pi/4} tan^6 x\, dx$ is $\frac{13}{15} - \frac{\pi}{4}$

$$U_{n+1} = \frac{1}{n} - U_{n-1}$$

Put $n = 5,3,1$

$$U_6 = \left(\frac{1}{5} - U_4\right), U_4 = \left(\frac{1}{3} - U_2\right), U_2 = (1 - U_0)$$

$$U_0 = \int_0^{\frac{\pi}{4}} dx \qquad U_6 = \frac{13}{15} - \frac{\pi}{4}$$

IF $I_n = \int_0^{\pi/4} \sec^n x \, dx$ and if

$$U_n = \frac{\sec^{n-2} x \tan x}{(n-1)} + \frac{(n-2)}{(n-1)} I_{n-2}$$

Then $\int_0^{\pi/4} \sec^6 x \, dx = \frac{28}{15}$

Now let $U_n = \int_0^{\pi/4} \sec^n x \, dx$

And $\therefore$ $U_n = \left[\frac{sec^{n-2}x \, tanx}{(n-1)} \right]_0^{\pi/4} + \frac{(n-2)}{(n-1)} U_{n-2}$

$$= \frac{(\sqrt{2})^{n-2}}{(n-1)} + \frac{(n-2)}{(n-1)} U_{n-2} \text{ -------(1)}$$

To find $\int_0^{\pi/4} sec^6 x \, dx$, put n = 6 , 4 , 2 in (2)

$$U_6 = \frac{(\sqrt{2})^{6-2}}{(6-1)} + \frac{(6-2)}{(6-1)} U_{6-2} = \frac{(\sqrt{2})^4}{5} + \frac{4}{5} U_4$$

$$U_4 = \frac{\left(\sqrt{2}\,\right)^2}{3} + \frac{2}{3}\,U_2 \quad , \; U_2 = \int_0^{\pi/4} sec^2x\;\,dx \;=\; [tanx]_0^{\pi/4} \;= 1$$

$$\therefore \;\; U_6 = \int_0^{\pi/4} sec^6x\;\,dx = \left[\frac{\left(\sqrt{2}\,\right)^4}{5} + \;\frac{4}{5}\;(\;\frac{\left(\sqrt{2}\,\right)^2}{3} + \;\frac{2}{3}\;\;(1)\;)\right]$$

$$= \left[\frac{4}{5} + \;\frac{4}{5}\;(\;\frac{2}{3} + \;\frac{2}{3}\;\;(1)\;)\right]$$

$$\therefore \;\; \int_0^{\pi/4} sec^6x\;\,dx \;\; = \frac{28}{15}$$

If $I_n = \int_0^{\pi/2} \cos^n x \cos nx \, dx$, Prove that $I_n = \frac{1}{2} I_{n-1} = \frac{\pi}{2^{n+1}}$

Sol : $I_n = \int_0^{\pi/2} \cos^n x \cos nx \, dx$

$$= \left[\cos^n x \, \frac{\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} n \cos^{n-1} x \, (-\sin x) \, \frac{\sin nx}{n} \, dx$$

$$= 0 + \int_0^{\pi/2} \cos^{n-1} x \, \sin x \sin nx \, dx$$

$\because \cos(n-1)x = \cos nx \cos x + \sin nx \sin x$

$$\therefore I_n = \int_0^{\pi/2} \cos^{n-1} x \, (\cos(n-1)x - \cos nx \cos x) \, dx$$

$$= \int_0^{\pi/2} \cos^{n-1} x \cos(n-1)x \, dx - \int_0^{\pi/2} \cos^n x \cos nx \, dx$$

$$= I_{n-1} - I_n \quad , \therefore\ I_n = I_{n-1} - I_n \quad \text{or} \quad \therefore\ \mathbf{I_n = \frac{1}{2}\, I_{n-1}}$$

$$\therefore\quad I_n = \frac{1}{2}\, I_{n-1}\ ,\quad I_{n-1} = \frac{1}{2}\, I_{n-2}\quad ,\quad I_{n-2} = \frac{1}{2}\, I_{n-3}\quad ,\text{-------}, I_1 = \frac{1}{2}\, I_0$$

$$\therefore\ I_n = \frac{1}{2^n} I_0 \ \text{ and }\ I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$$

Hence $\mathbf{I_n = \frac{\pi}{2^{n+1}}}$

If $I_n = \square_0^{\pi/4} \frac{\sin \square 2n-1 \square x}{\sin x} dx$, Prove that $\square\square\square(\ I_{n+1} - I_n\ ) = \sin\left(\frac{\square\square\square}{2}\right)$

Hence find I_3.

$$\text{Sol : } \quad I_n = \int_0^{\pi/4} \frac{\sin\ (2n-1)x}{\sin x}\, dx$$

$$\therefore\ (\ I_{n+1} - I_n\ ) = \int_0^{\pi/4} \frac{\sin\ (2n+1)x}{\sin x}\, dx - \int_0^{\pi/4} \frac{\sin\ (2n-1)x}{\sin x}\, dx$$

$$= \int_0^{\pi/4} \frac{\sin(2n+1)x - \sin\ (2n-1)x}{\sin x}\, dx$$

We know $\quad sinc - sind = 2\cos\frac{c+d}{2}\sin\frac{c-d}{2}$

$$= \int_0^{\pi/4} \frac{2\cos 2nx \sin x}{\sin x}\, dx = 2\left[\frac{\sin 2nx}{2n}\right]_0^{\pi/4}$$

$$\therefore\ n\,(\, I_{n+1} - I_n\,) = \sin\left(\frac{n\pi}{2}\right) \text{ ------------------------------- } (1)$$

To find I_3, from (1),

Put $n = 2$, $\quad 2\,(\, I_3 - I_2\,) = \sin(\pi) \implies I_3 - I_2 = 0$ ----- (2)

Similarly putting $n = 1$, we get $I_2 - I_1 = 1 \implies I_2 = I_1 + 1$ --- (3)

$$I_1 = \frac{\pi}{4} \qquad \therefore I_3 = \frac{\pi}{4} + 1$$

If $I_n = \int_0^{\pi/2} x\,.\cos^n x\,.dx$, and if $I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$, then I_3 is ………

$$\Longrightarrow \quad I_n = \frac{-1}{n^2} + \frac{(n-1)}{n} I_{n-2} \quad \text{---------------- (1)}$$

From (1), $I_3 = \frac{-1}{9} + \frac{2}{3} I_1$

and $I_1 = \int_0^{\pi/2} x.\cos x\,dx = [x\sin x + \cos x]_0^{\pi/2} = \frac{\pi}{2} - 1$

$$I_3 = \frac{-1}{9} + \frac{2}{3} I_1 = I_3 = \left\{ \frac{-1}{9} + \frac{2}{3} \left(\frac{\pi}{2} - 1\right) \right\}$$

$$\therefore\ I_3 = \frac{\pi}{3} - \frac{7}{9}$$

Evaluate $\int_4^6 sin^4 \pi x . cos^2 2\pi x . dx$

$$\textbf{Sol}: \text{ Let } \mathrm{I} = \int_4^6 sin^4 \pi x . cos^2 2\pi x . dx$$

$$\text{Put } \pi x = 4\pi + t \Longrightarrow \pi\, dx = dt$$

$$\sin(\pi x) = \sin(4\pi + t) = \sin t \quad \text{and } \cos(2\pi x) = \cos(8\pi + 2t) = \cos(2t)$$

X	4	6
t	0	

$$\therefore \quad \mathrm{I} = \frac{1}{\pi}\int_0^{2\pi} \sin^4 t . \cos^2 2t . dt = \frac{1}{\pi}\int_0^{2\pi} \sin^4 t . (1 - 2\sin^2 t)^2 . dt$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin^4 t\,.(1 - 4\sin^2 t + 4\sin^4 t)\,.dt$$

$$\because \int_0^{2\pi} \sin^n x\,dx = \begin{cases} = 4\int_0^{\pi/2} \sin^n x\,dx\,, & \textbf{if n is even}. \\ = 0 & \textbf{, if n is odd}. \end{cases}$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \sin^4 t\ -4\sin^6 t + 4\sin^8 t)\,.dt$$

$$\because \int_0^{\pi/2} \sin^n x\,dx = \frac{[\,(n-1\,)\text{subtract } 2 \ldots\ldots\ldots\ldots\ldots\ldots\ 2 \text{ or } 1\,]}{[\,(n\,)\ \text{subtract } 2 \ldots\ldots\ldots\ldots\ldots\ldots\ 2 \text{ or } 1]} \times \left(\frac{\pi}{2}\right) \text{if n is even.}$$

$$\therefore\ \ I = \frac{4}{\pi}\left[\ \frac{3.1}{4.2}\,\frac{\pi}{2} - 4\,\frac{5.3.1}{6.4.2}\,\frac{\pi}{2} + 4\,\frac{7.5.3.1}{8.6.4.2}\,\frac{\pi}{2}\ \right]$$

$$= \frac{4}{\pi}\,\frac{3.1}{4.2}\,\frac{\pi}{2}\left[\ 1 - \frac{4.5}{6} + \frac{4.7.5}{8.6}\ \right] \qquad \therefore\ \ I = \frac{\mathbf{7}}{\mathbf{16}}$$

Show that $\int_0^{\pi} x \,.\, \sin^7 x \,.\, \cos^4 x \,.\, dx = \frac{16\pi}{1155}$

Sol : $I = \int_0^{\pi} x \,.\, sin^7 x \,.\, cos^4 x \,.\, dx$ ------------------- (1)

$= \int_0^{\pi} (\pi - x) \,.\, sin^7(\pi - x) \,.\, cos^4(\pi - x) \,.\, dx$

$\because \quad \int_0^{a} f(x) \,.\, dx = \int_0^{a} f(a - x) \,.\, dx$

And using , $\sin(\pi - x) = \sin x$; $\cos(\pi - x) = -\cos x$

$= \int_0^{\pi} (\pi - x) \,.\, sin^7 x \,.\, cos^4 x \,.\, dx$ ------------------------- (2)

Adding (1) and (2)

$2I = \int_0^{\pi} \pi \,.\, sin^7 x \,.\, cos^4 x \,.\, dx$

$$\because \int_0^{\pi} sin^m x\, cos^n x\, dx = \begin{cases} = 2\int_0^{\pi/2} sin^m x\, cos^n x\, dx\ , \boldsymbol{if\ n\ even\ , for\ any\ m.} \\ = 0\ . \qquad \boldsymbol{if\ n\ is\ odd}. \end{cases}$$

$$= 2\pi \int_0^{\pi/2} sin^7 x\ .\ cos^4 x\ .\ dx$$

Using the formula,

$$\int_0^{\pi/2} sin^m x\, cos^n x\, dx$$

$$= \left\{\frac{[(m-1)\text{subtract } 2 \ldots\ldots\ 2 \text{ or } 1].[(n-1)\text{subtract } 2 \ldots\ldots\ 2 \text{ or } 1]}{[(m+n)\text{subtract } 2 \ldots\ldots\ldots\ 2 \text{ or } 1]}\right\} \times 1$$

If m & n both are not even

$$\mathrm{I} = \pi\, \frac{(6.4.2)(3.1)}{11.9.7.5.3.1} = \boldsymbol{\frac{16\,\pi}{1155}}$$

a)	3	b)	2
c)		d)	

a)		b)	
c)		d)	

	a)		b)	
	c)		d)	

Beta and Gamma Function

1. Gamma Function

Definition: The integral $\int_0^\infty e^{-x} x^{n-1} dx$ is called as Gamma function

and denoted by $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$ (n > 0)

Properties :

$$\boxed{1.\ \Gamma 1 = 1}$$

Proof: $\Gamma 1 = \int_0^\infty e^{-x} x^{1-1} dx$

$$= \int_0^\infty e^{-x} x^0 dx = \int_0^\infty e^{-x} dx$$

$$= [-e^{-x}]_0^\infty = -e^{-\infty} + e^0 = 0 + 1 = 1. = R.H.S.$$

2. Reduction formula: $\Gamma(n+1) = n\,\Gamma(n)$

$= n!\,,\text{if } n \text{ is} + ve \text{ integer}$

Proof : $\Gamma(n+1) = \int_0^\infty e^{-x} x^n dx$

$$= \left[x^n \left(-e^{-x}\right) \right]_0^\infty - \int_0^\infty \left(-e^{-x}\right) n\, x^{n-1} dx$$

$$= [0-0] + n\int_0^\infty e^{-x} x^{n-1} dx$$

$$= n\,\Gamma(n)$$

$$3.\ \boxed{\Gamma 0 = \infty}$$

$$\text{Proof} : \square \quad \Gamma n = \frac{\Gamma(n+1)}{n}$$

$$\therefore \quad \Gamma 0 = \frac{\Gamma(0+1)}{0} = \infty$$

$$4.\ \boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

$$5.\ \boxed{\Gamma P \; \Gamma(1-P) = \frac{\square\square}{\square\square\square\square\square\square\square\square\square\square}}$$

$$6.\ \int_0^\infty e^{-kx} \, x^{n-1} \, dx = \frac{\Gamma n}{k^n}$$

Problems

$$\text{Prove the following}$$

$$1. \quad \int_0^\infty e^{-\sqrt[3]{x}}\sqrt{x}\,dx = \frac{315}{16}\sqrt{\pi}$$

$$Sol: Put\ \ x = t^3 \;\Rightarrow\; dx = 3t^2\,dt$$

$$L.H.S. = \int_0^\infty e^{-t}\, t^{3/2}\, \left(3t^2\right) dt$$

$$= 3\int_0^\infty e^{-t}\, t^{7/2}\, dt$$

$$= 3\,\Gamma\left(\tfrac{7}{2}+1\right)$$

$$= 3\left(\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\right)\times\sqrt{\pi}$$

$$= \frac{315}{16}\sqrt{\pi}$$

2. $$\int_0^\infty e^{-2x^2} x^7 dx = \frac{3}{16}$$

$Proof:$ $Put\ 2x^2 = t \Rightarrow 4x\,dx = dt$

$$L.H.S. = \int_0^\infty e^{-2x^2}\ x^6\ (xdx)$$

$$= \int_0^\infty e^{-t}\left(\frac{t}{2}\right)^3 \left(\frac{dt}{4}\right)$$

$$= \frac{1}{32}\int_0^\infty e^{-t}\ t^3\ \ dt$$

$$= \frac{1}{32}\,\overline{|4} = \frac{1}{32} \times 3! = \frac{6}{32} = \frac{3}{16}$$

$$3. \int_0^{\infty} \frac{x^4}{4^x}\, dx = \frac{24}{(\log 4)^5}$$

$$\text{Proof: } \quad Let \quad 4 = e^m$$

$$L.H.S. = \int_0^{\infty} \frac{x^4}{e^{mx}}\, dx = \int_0^{\infty} e^{-mx}\, x^4\, dx$$

$$Put \quad mx = t \Rightarrow m\,dx = dt$$

$$L.H.S. = \int_0^{\infty} e^{-t} \left(\frac{t}{m}\right)^4 \frac{dt}{m}$$

$$= \frac{1}{m^5} \int_0^{\infty} e^{-t}\, t^4\, dt$$

$$= \frac{1}{m^5} \Gamma 5 = \frac{1}{m^5}\, 4! = \frac{24}{(\log 4)^5}$$

4. $\int_0^1 \frac{1}{\sqrt{-logx}} dx = \sqrt{\pi}$

$Put \quad -logx = t$

1				
	a)		b)	
	c)		d)	

2				
	a)		b)	
	c)		d)	

3				
	a)		b)	
	c)		d)	

4				
	a)		b)	
	c)		d)	

5				
	a)		b)	
	c)		d)	

6				
	a)		b)	
	c)		d)	

7				
	a)		b)	
	c)		d)	

Beta Function.

$$Definition : \boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx} \; ; \; where \;\; m, n \; are \; +ve \; \text{integers}$$

Properties Of Beta Function.

1. $$\beta(m,n) = \beta(n,m)$$

2. $$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy$$

3. $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \; d\theta$$

4. $$\int_0^{\pi/2} \sin^p \theta \cos^q \theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

5. Relation Between Beta and Gamma Function.

$$\boxed{\beta(m, n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}}$$

6. Legendre's duplication formula :

$$\boxed{\Gamma m \, \Gamma(m + 1/2) = \frac{\sqrt{\pi}}{2^{2m-1}} \Gamma(2m)}$$

PROBLEMS.

Ex 1 : Prove that $\int_0^1 (1 - x^{1/n})^m \, dx = \dfrac{m!\, n!}{(m+n)!}$

Sol : Let $I = \int_0^1 (1 - x^{1/n})^m \, dx$

Put $x^{1/n} = t \,, x = t^n \,; \; dx = n\, t^{n-1}\, dt$

x	0	1
t	0	1

$$I = \int_0^1 (1-t)^m \; n\, t^{n-1}\, dt = n \int_0^1 t^{n-1} . (1-t)^m \, dt$$

$$I = n\beta(n, m+1) = n \frac{\Gamma(n) \; \Gamma(m+1)}{\Gamma(n+m+1)} = \frac{m!\, n!}{(m+n)!}$$

Ex 2 : Prove that $\int_0^{\infty} \frac{[illegible]}{1+[illegible]^4} = \frac{[illegible]}{2\,\xi\,\overline{2}}$

Sol : Let $I = \int_0^{\infty} \frac{[illegible]}{1+[illegible]^4}$, Put $x^2 =$ [illegible], [illegible] $= \frac{1}{2}$ [illegible] $^{-1/2}$. [illegible] ^{2} [illegible]

	0	
	0	$\pi/2$

$I = \int_0^{[illegible]/2} \frac{1}{1+[illegible]^2\,[illegible]2} \frac{1}{2}$ [illegible] $^{-1/2}$. [illegible] ^{2} [illegible]

$$= \frac{1}{2}\int_0^{\pi/2} (tan\ \theta)^{-1/2} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} cot^{\,1/2}\ \theta\ d\theta = \frac{1}{2}\int_0^{\pi/2} sin^{\,-1/2}\theta\ cos^{\,1/2}\ \theta\ d\theta$$

$$I = \frac{1}{2}\,\frac{1}{2}\,\beta\left(\frac{-\frac{1}{2}+1}{2}\,,\ \frac{\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{4}\,\beta\left(\frac{1}{4}\,,\ \frac{3}{4}\right)$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$$

$$= \frac{1}{4}\,\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)}{\Gamma(1)} \qquad \because \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$= \frac{1}{4}\,\frac{\pi}{\sin\left(\frac{\pi}{4}\right)} = \frac{\sqrt{2}\,\pi}{4} = \frac{\pi}{2\sqrt{2}}$$

Ex3: Prove that $\int_{-1}^{1}(1+x)^{m}(1-x)^{n}\,dx = 2^{m+n+1}\cdot\frac{m!\,n!}{(m+n+1)!}$

Where m and n are positive integers.

Sol: Let $I=\int_{-1}^{1}(1+x)^{m}(1-x)^{n}\,dx$

Put $1+x=2t$, $dx=2dt$

x	- 1	1
t	0	1

$$I=\int_{0}^{1}(2t)^{m}(2-2t)^{n}\,2dt$$

$$=2^{m+n+1}\int_{0}^{1}t^{m}(1-t)^{n}\,dt = 2^{m+n+1}\,\beta(m+1,\ n+1)$$

$$I=2^{m+n+1}\frac{\Gamma(m+1)\,\Gamma(n+1)}{\Gamma(m+n+2)}$$

$$=2^{m+n+1}\cdot\frac{m!\,n!}{(m+n+1)!}$$

Ex 4 : Evaluate $\int_3^7 (x-3)^{1/4}(7-x)^{1/4}\,dx$

Sol : Let $I = \int_3^7 (x-3)^{1/4}(7-x)^{1/4}\,dx$

Put $x - 3 = 4t$, $dx = 4dt$

x	3	7
t	0	1

$$I = \int_3^7 (4t)^{1/4}(4-4t)^{1/4}\,4dt = 8\int_3^7 (t)^{1/4}(1-t)^{1/4}\,dt$$

$$I = 4^2\,\beta\left(\frac{5}{4},\frac{5}{4}\right) = 8\,\frac{\Gamma(5/4)\,\Gamma(5/4)}{\Gamma(5/2)} = 16\,\frac{\left(\frac{1}{4}\Gamma(1/4)\right)^2}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{4}{3\sqrt{\pi}}\left(\Gamma(1/4)\right)^2$$

1				
	a)		b)	
	c)		d)	

2				
	a)		b)	
	c)		d)	

3				
	a)		b)	
	c)		d)	

4				
	a)		b)	
	c)		d)	

5				
	a)		b)	
	c)		d)	

6				
	a)	1	b)	2
	c)	3	d)	4
7				
	a)		b)	
	c)		d)	
8				
	a)		b)	
	c)		d)	

Differentiation Under Integral Sign (DUIS)

1. Introduction.

- In addition to variables, additional parameters

$$I(\alpha) = \int_a^b f(x,\alpha)\,dx$$ ⟹ α = Parameter , x = Variable.

Rule 1 : Integrals with constant limits.

$$If\;\; I(\alpha) = \int_a^b f(x,\alpha)\,dx \quad then \quad \boxed{\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)\,dx}$$

a & b constants ⟹ LHS derivative ⟹ Partial derivative RHS

EXAMPLES

2013

$$Ex\ 1: Show\ \ that\ \int_0^1 \frac{x^a - 1}{\log x} dx = \log\ (a+1)\ \ ,(a \geq 0)$$

$$Solution: \quad Let \quad I(a) = \int_0^1 \frac{x^a - 1}{\log x} dx$$

$$I'(a) = \frac{d}{da}\int_0^1 \frac{x^a - 1}{\log x} dx = \int_0^1 \frac{\partial}{\partial a}\frac{x^a - 1}{\log x} dx \qquad applying\ \ DUIS$$

$$I'(a) = \int_0^1 \frac{x^a \log x}{\log x} dx \quad = \int_0^1 x^a dx$$

Integrating w.r.t. x $\quad = \left[\dfrac{x^{a+1}}{a+1}\right]_0^1 = \dfrac{1}{a+1} \qquad I'(a) = \dfrac{1}{a+1}$

Integrating w.r.t. a $I(a) = \log(a+1) + c$

$$Put\ a = 0, \quad I(0) = \log(0+1) + c \rightarrow c = 0$$

$$\therefore I(a) = \log(a+1), a \geq 0$$

==

$$Ex\ 2:\ Prove\ that\ \int_0^\infty e^{-x^2} \cos 2\lambda x dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}$$

$$Solution:\ Let\ I(\lambda) = \int_0^\infty e^{-x^2} \cos 2\lambda\ x\, dx\ ; \quad Where\ \lambda\ is\ a\ parameter$$

$$By\ DUIS\ I'(\lambda) = \int_0^\infty \frac{\partial}{\partial \lambda} e^{-x^2} \cos 2\lambda\, x dx$$

$$I'(\lambda) = \int_0^\infty e^{-x^2}(-2x).\sin 2\lambda\, x\, dx \quad = \int_0^\infty (\sin 2\lambda\, x).(e^{-x^2}(-2x))\, dx$$

 Integration by parts & using $\int \mathrm{e}^{\mathrm{f(x)}} f'(x)dx = e^{f(x)}$

$$= \left[\sin 2\lambda\, x\, e^{-x^2}\right]_0^\infty - \int_0^\infty 2\lambda e^{-x^2} \cos 2\lambda\, x dx$$

$$= (0 - 0) - 2\lambda\, \mathrm{I}(\lambda) \qquad \therefore\ \frac{I'(\lambda)}{I(\lambda)} = -2\lambda$$

Integrating w.r.t. λ , $\log \mathrm{I}(\lambda) = -\lambda^2 + c$ *i.e.* $I(\lambda) = e^{-\lambda^2}.e^c$

and for $\lambda = 0,\ I(0) = \int_0^\infty e^{-x^2}.dx = \frac{\sqrt{\pi}}{2} = e^c \qquad \therefore I(\lambda) = \frac{\sqrt{\pi}}{2} e^{-\lambda^2}$

==

PROBLEMS INVOLVING TWO PARAMETERS.

Procedure is exactly same as that of problems involving one parameter.

$$4: \textit{Show that} \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log \frac{b}{a} \;\; ; \;\; a \text{ and } b \text{ are two parameters.}$$

$$\textit{Solution: Let } I(a) = \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx \;\; \textit{Differentiating w.r.t.a and applying } \textbf{DUIS}$$

$$I'(a) = \int_0^\infty \frac{\partial}{\partial a} \frac{e^{-ax} - e^{-bx}}{x}\, dx = \int_0^\infty \frac{(-x)e^{-ax} - 0}{x}\, dx = -\int_0^\infty e^{-ax}\, dx = -\left[\frac{e^{-ax}}{-a}\right]_0^\infty = -\frac{1}{a}$$

$\therefore I'(a) = -\dfrac{1}{a}$ Integrating w.r.t. a ; I(a) = - log a + c

Put a = b ; I(b) = - log b + c , c = log b   since I(b) = 0

Thus   I(a) = - log a + logb ,

$$I(a) = \log \frac{b}{a}$$

==

Using DUIS prove that :

$$\int_0^{\infty} \frac{e^{-x} - e^{-ax}}{x \sec x}\, dx = \frac{1}{2} \log\left(\frac{a^2+1}{2}\right) \text{ for } a > 0$$

May 2015

DUIS Rule II: Integrals with limits as function of the parameter (Leibnitz's Rule)

If

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx$$

Then

$$\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

5. If $f(x) = \int_0^x (x-t)^2 G(t)\,dt$ then show that $\frac{d^3f}{dx^3} = 2G(x)$

Solution: $f(x) = \int_0^x (x-t)^2\, G(t)\,dt$

$$\frac{df}{dx} = \frac{d}{dx}\int_0^x (x-t)^2\, G(t)\,dt$$

By DUIS $= \int_0^x \frac{\partial}{\partial x}(x-t)^2\, G(t)\,dt + \frac{dx}{dx}.(0) - \frac{d0}{dx}(x^2 G(0))$

$$\therefore \frac{df}{dx} = \int_0^x 2(x-t)\quad G(t)\,dt$$

Again applying DUIS

$$\frac{d^2 f}{dx^2} = \int_0^x \frac{\partial}{\partial x} 2(x-t) \quad G(t)dt \ + \ 0 \ - \ 0$$

$$\frac{d^2 f}{dx^2} = \int_0^x 2G(t)dt$$

Again applying DUIS,

$$\frac{d^3 f}{dx^3} = \int_0^x \frac{\partial}{\partial x} 2G(t)dt + \frac{dx}{dx}.2G(x) - 0$$

$$= 0 + 2G(x) - 0$$

$$\therefore \frac{d^3 f}{dx^3} = 2G(x)$$

=================================

1)				
	a)		b)	
	c)		d)	0

2)				
	a)		b)	
	c)		d)	

3)

a)		b)	
c)		d)	

4)

a)		b)	
c)		d)	

5)			
a)		b)	
c)		d)	

6)			
a)		b)	
c)		d)	

Error Function

- Error function of x is denoted by $\text{erf}(x)$ and is defined by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

Complementary Error Function

Complementary error function of x is denoted by $\text{erfc}(x)$ and defined by

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du$$

Properties of error function

1. $\text{erf}(\infty) = 1$

2. $\text{erf}(0) = 0$

3. $\text{erf}(x) + \text{erfc}(x) = 1$

4. Error function is an odd function

$i.e \quad \text{erf}(-x) = -\text{erf}(x)$

5. Series for error function is

$$erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \square\right]$$

Show that $\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\,[erf(b) - erf(a)]$

Solution: We know that $1 = erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\,dx$

$$= \frac{2}{\sqrt{\pi}}\left[\int_0^a e^{-x^2}\,dx + \int_a^b e^{-x^2}\,dx + \int_b^\infty e^{-x^2}\,dx\right]$$

By definition of erf (x) and erf_c (x)

$$1 = erf(a) + \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}\,dx + erf_c(b)$$

$$[1 - erf_c(b)] - erf(a) = \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}\,dx$$

$$erf(b) - erf(a) = \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}\,dx$$

i.e. $\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\,[erf(b) - erf(a)]$

==

$$\text{Prove that } erf_c(-x) + erf_c(x) = 2$$

: $erf(x) + erf_c(x) = 1$ ------------------ (1)

$$\therefore \; erf(-x) + erf_c(-x) = 1$$

But $erf(-x) = -erf(x)$

$\Longrightarrow$ $-erf(x) + erf_c(-x) = 1$ ------------------- (2)

Adding eqn (1) and (2) we get

$$erf_c(-x) + erf_c(x) = 2$$

===

4 : Show that $\frac{d}{dx}\, erf(\text{ax}) = \frac{2a}{\sqrt{\pi}}\, e^{-a^2x^2}$,Hence evaluate $\int_0^t \text{erf}(ax)dx$

$$As \quad \text{erf}(ax) = \frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du$$

$$\therefore \frac{d}{dx}\text{erf}(ax) = \frac{d}{dx}\left(\frac{2}{\sqrt{\pi}} \int_0^{ax} e^{-u^2} du \right)$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_0^{ax} \frac{\partial}{\partial x} e^{-u^2} du + e^{-a^2x^2} \frac{d}{dx}(ax) - 0 \right]$$

$$\therefore \frac{d}{dx}\text{erf}(ax) = \frac{2ae^{-a^2x^2}}{\sqrt{\pi}}$$

5: Prove that $\frac{d}{dx}\,\text{erf}(x) = \frac{2}{\sqrt{\pi}} \cdot e^{-x^2}$

Now find Integration of Error Function

$$\int_0^t \operatorname{erf}(ax)dx = \int_0^t 1 \cdot \operatorname{erf}(ax)dx$$

$$= [\operatorname{erf}(ax) \cdot x]_0^t - \int_0^t \left(\frac{d}{dx}\operatorname{erf}(ax)\right) x dx$$

$$= \operatorname{erf}(at) \cdot t - \int_0^t \frac{2ae^{-a^2x^2}}{\sqrt{\pi}} x dx$$

$$\int_0^t \operatorname{erf}(ax)dx = \operatorname{erf}(at) \cdot t + \frac{1}{a\sqrt{\pi}}[e^{-a^2t^2} - 1]$$

1)				
	a)		b)	
	c)		d)	
2)	Error function is			
	a)	Even function	b)	Neither ever nor odd function
	c)	Odd function	d)	Constant function
3)				
	a)		b)	
	c)		d)	

a)	3	b)	2
c)	1	d)	0