Unit –IV
Curve Tracing

Curve Tracing

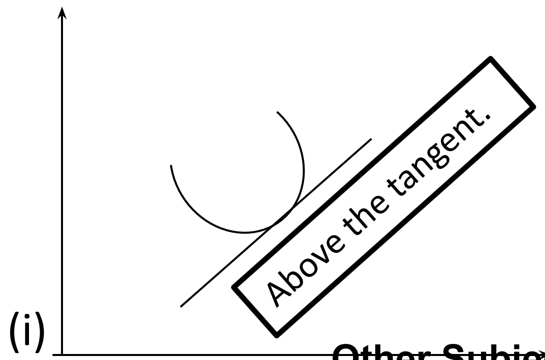
Tracing of curves means finding approximate shape of the curves using different properties like symmetry, Intercepts, tangents, asymptotes, region of existence etc.

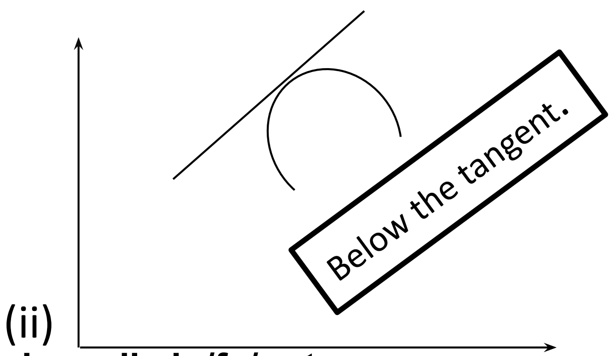
The knowledge of tracing of curves is useful in applications of integration in finding area, mass, center of gravity, volume etc.

1.Concavity.

(i) Concave upwards (Convex downwards.)

(ii) Convex upwards (Concave downwards.)



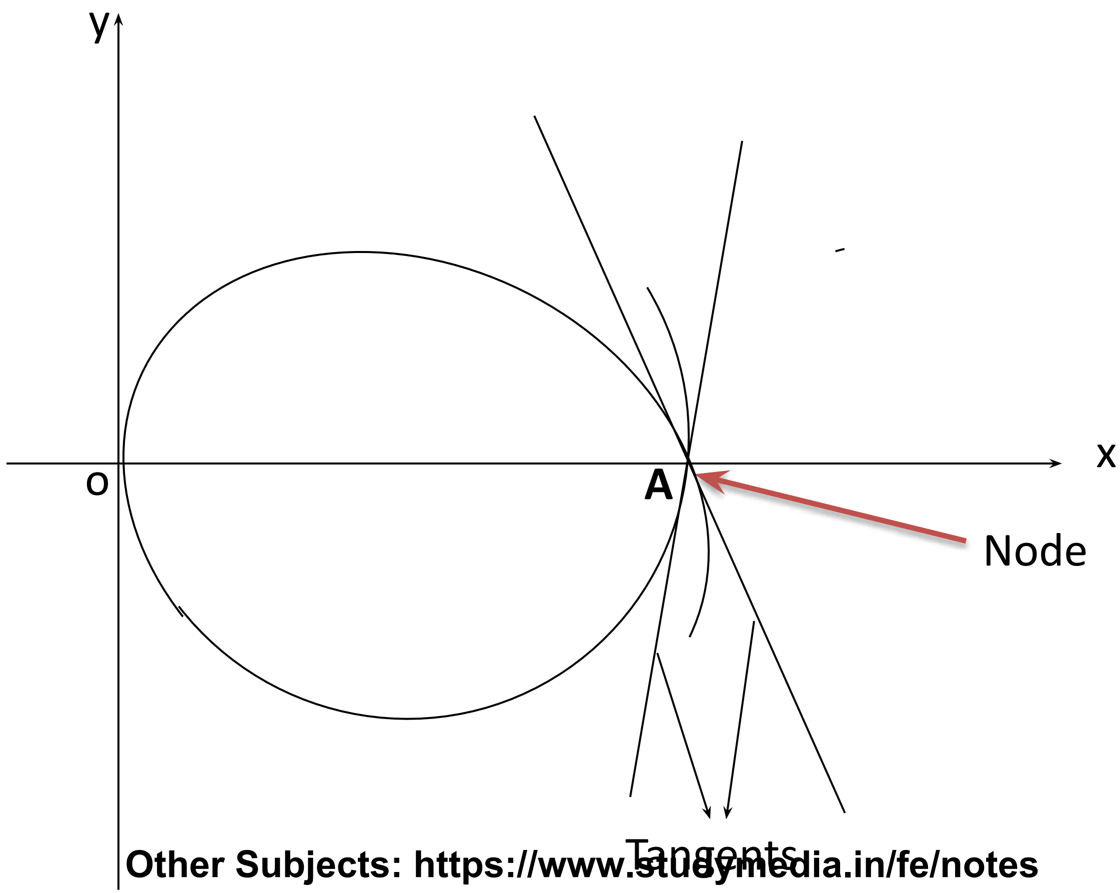


2. Singular Points. Following points are called as singular points.

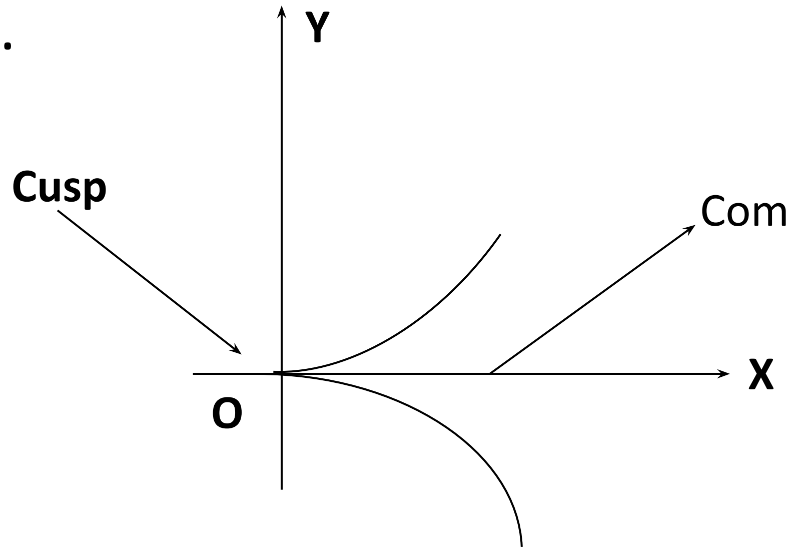
(i) **Double Point –** through which two branches of curve pass.

(ii) **Multiple Point -** through which more than one branches of curve pass.

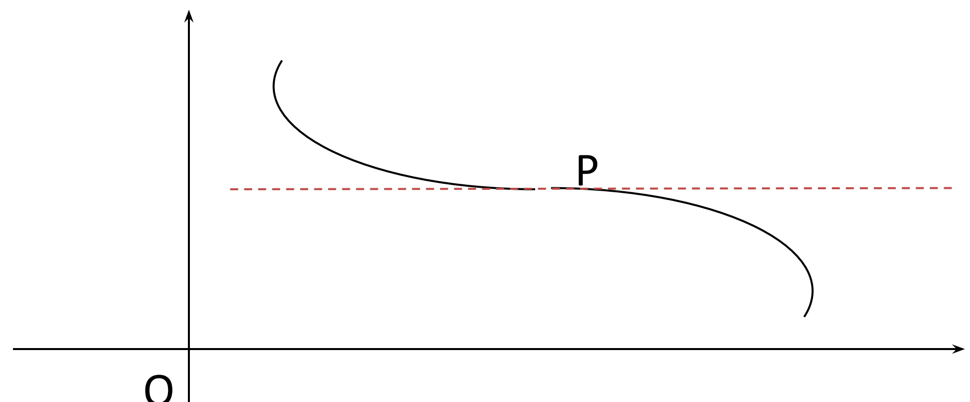
(iii) **Node –** A double point is called a node if distinct branches have distinct tangents.



IV. Cusp: A double point is called cusp if two branches have a common tangent.



V. Point Of inflexion : A curve has inflexion at P if it changes from concavity upwards to concavity downwards or vice versa.



VI. Isolated Point : A point P is called a isolated point or conjugate point if the co-ordinates of P satisfies the equation of the curve , but no branches passes through P.

The curve y = f(x)

a	Is increasing in [a , b] If $f'(x) > 0$, ∀ ▯∈[a , b]	Concave upwards in [a , b] If $f''(x) > 0$, ∀ ▯∈[a , b]
b	Is decreasing in [a , b] If $f'(x) < 0$, ∀ ▯∈[a , b]	Concave downwards in [a , b] If $f''(x) < 0$, ∀ ▯∈[a , b]
c	Has extreme point If $f'(x) = 0$, for some x ∈[a , b]	Has a point of inflexion If $f''(x) = 0$, for some x ∈[a , b]

Types of Curves

- Cartesian curves
- Parametric curves
- Polar curves
- Rose curves
- Rectification of curves

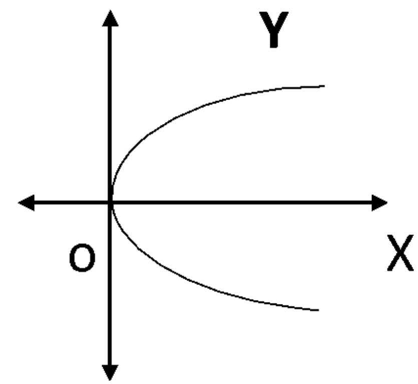
Cartesian curve
(Explicit relations)

The equation of curve given in x and y can be express as $y = f(x)$ or $x = g(y)$.

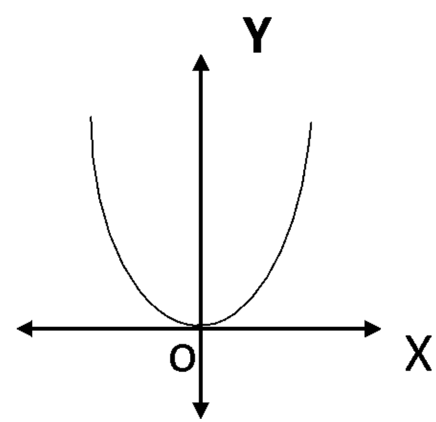
Rules For Tracing Of Cartesian Curves.

Rule 1 : Symmetry :

(a) **Symmetry about X- axis:** If equation of the curve remains unchanged by changing y to – y or all the powers of y in the equation are even. e.g. $y^2 = 4ax$.

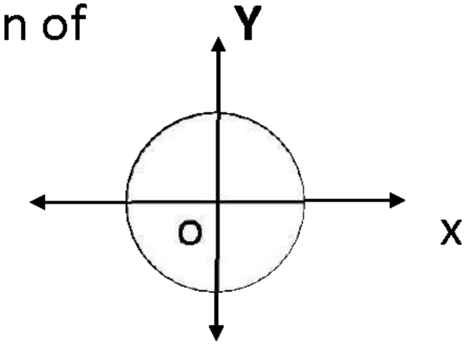


(b) **Symmetry about Y- axis:** If equation of the curve remains unchanged by changing x to – x **or** all the powers of x in the equation are even. e.g. $x^2 = 4ay$.



(c) Symmetry about both X and Y axes: If equation of the curve contains all even powers of x and y .

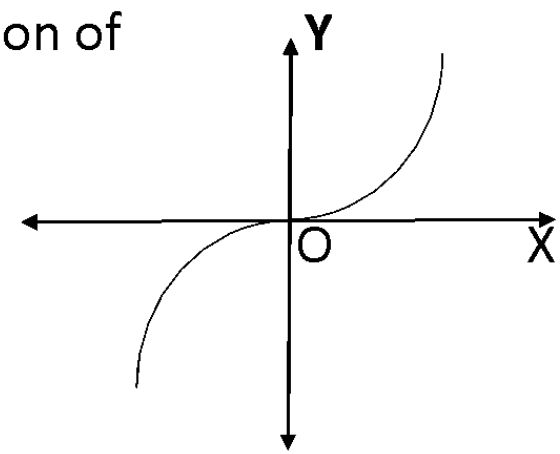
e.g. $x^2 + y^2 = a^2$.



(d) Symmetry in opposite quadrants: If equation of the curve remains unchanged by changing

$(x, y) \to (-x, -y)$ simultaneously.

e.g. $y = x^3$

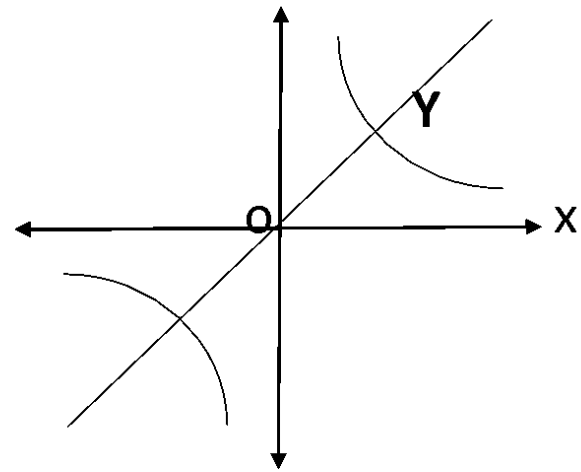


(e) Symmetry about the line y = x:

If equation of the curve remains unchanged

by changing $(x, y) \rightarrow (y, x)$

e.g. $xy = c^2$

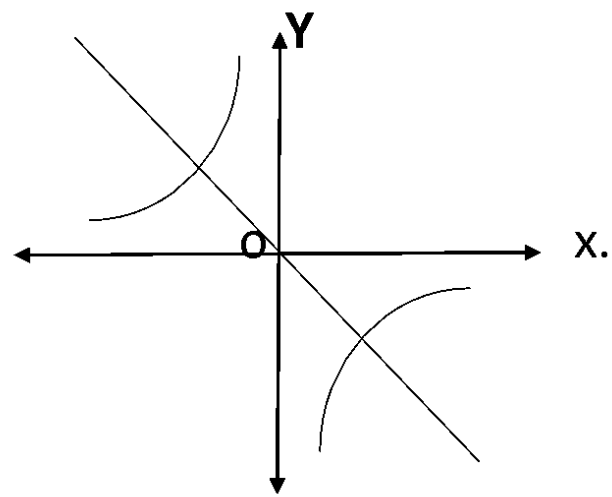


(f) Symmetry about the line y = - x :

If equation of the curve remains unchanged

by changing $(x, y) \rightarrow (-y, -x)$

e.g. $xy = -c^2$



Rule 2. Special Points on the Curve

- Intersection with co-ordinate axes

For x −intercept put $y = 0$ and

for y −intercept put $x = 0$ in the equation of curve

Origin : Curve passes through origin if $(0,0)$ satisfies equation of curve.

Rule 3 :Tangents:

1. Tangents at the origin : If a curve is passing through origin then : The equation of the tangent or tangents at origin can be obtained by equating to zero, the lowest degree terms taken together in the equation of the curve.

2. Tangents at any other points : To find nature of tangent at any point P. find $\frac{dy}{dx}$ at that point.

1. If $\left(\frac{dy}{dx}\right)_P = 0 \Longrightarrow$ Tangent at P is parallel to X- axis.
2. If $\left(\frac{dy}{dx}\right)_P = \infty \Longrightarrow$ Tangent at P is parallel to Y- axis.
3. If $\left(\frac{dy}{dx}\right)_P > 0 \Longrightarrow$ Tangent at P makes acute angle with X- axis.
4. If $\left(\frac{dy}{dx}\right)_P < 0 \Longrightarrow$ Tangent at P makes obtuse angle with X- axis.

4. Asymptotes

- Asymptotes are the tangents to the curve at infinity.

If $y \rightarrow \infty$ as $x \rightarrow a$ then $x = a$ is an asymptotes.

If $x \rightarrow \infty$ as $y \rightarrow b$ then $y = b$ is an asymptotes.

5. Region of Absence of the Curve

- For $y = f(x)$, find x for which y becomes imaginary.
- For $x = f(y)$, find y for which x becomes imaginary.

1)	**A double point is Node if**			
	a)	Distinct branches have a common tangent	b)	Distinct branches have distinct tangent
	c)	Tangent at double point is above the curve	d)	Tangent at double point is below the curve
2)	A double point is Cusp if			
	a)	Two branches have distinct tangents	b)	Tangent line cuts the curve unusually
	c)	Two branches have a common tangent	d)	None of the above
3)				
	a)		b)	
	c)		d)	

4)				
	a)		b)	
	c)		d)	
5)				
	a)		b)	
	c)		d)	
6)	**If the curve passes through origin then the tangent to the curve at origin is obtained by**			
	a)	Equating highest degree terms to zero	b)	Equating odd degree terms to zero
	c)	Equating even degree terms to zero	d)	Equating lowest degree terms to zero

Trace the curve $y^2 = (x-1)(x-2)(x-3)$

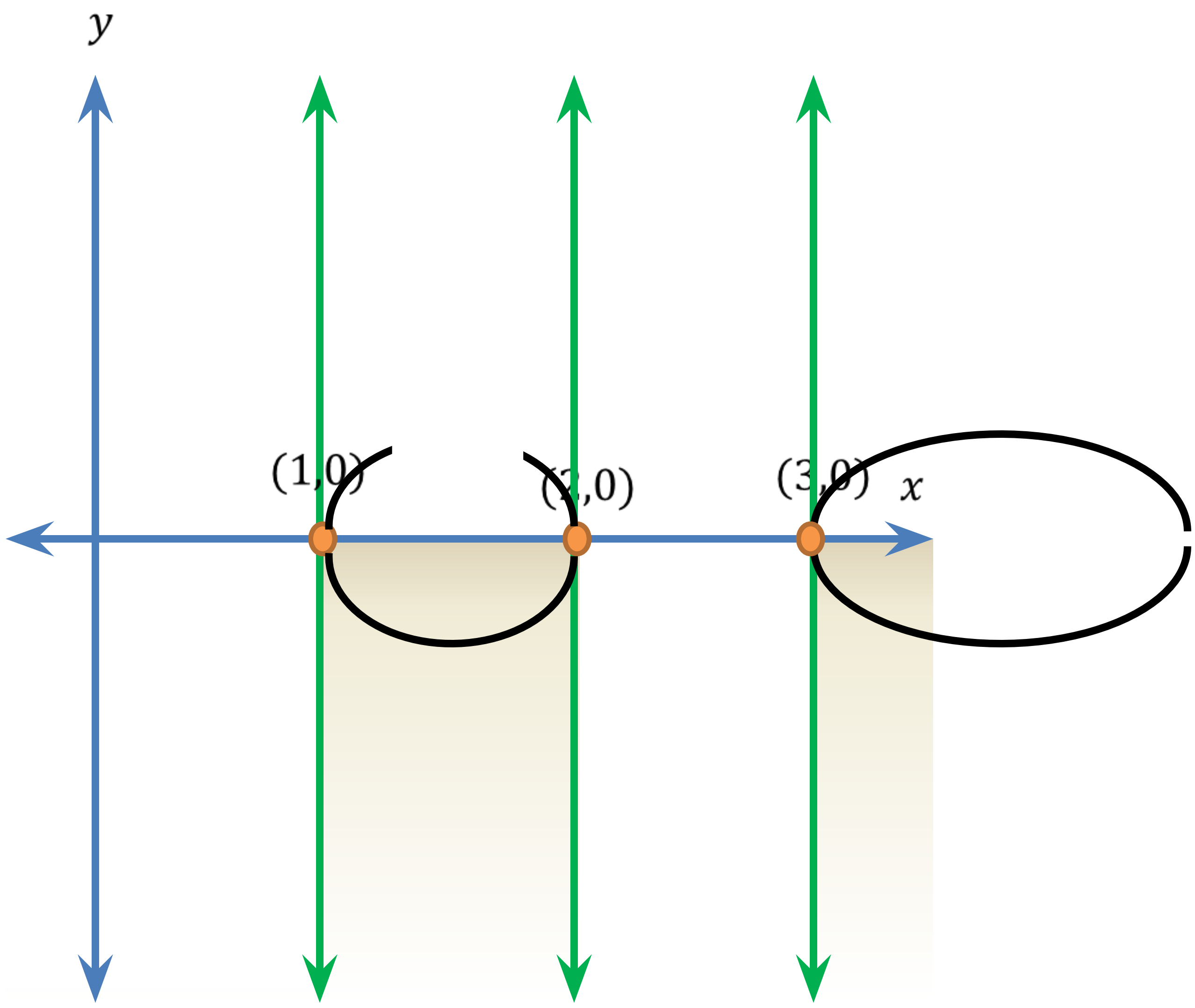
- Symmetry: about $x -$ axis
- Points of intersection : Not passing through origin
- Special points on curve:

  $x -$ intersects at points $(1,0), (2,0), (3,0)$

  $\frac{dy}{dx} = \infty$ for $(1,0), (2,0), (3,0)$

  Tangent parallel to y—axis passing through these points
- Asymptotes: no asymptote
- Region of absence:

$$x < 1, \qquad 2 < x < 3$$



y
$(1,0)$
$(2,0)$
$(3,0)$
x

- **Symmetry:**

Even powers of x – symmetric about y –axis

Even powers of y – symmetric about x –axis

- **Points of intersection:**

Origin: $(0,0)$ satisfies the equation

Tangent at origin: Lower degree terms $= 0$

- **Special points on the curve:**

For x –intercept $y = 0$, for y –intercept $x = 0$ (say (x_1, y_1))

If $\left(\frac{dy}{dx}\right)_{P(x_1,y_1)} = 0$ tangent at (x_1, y_1) parallel to x –axis.

If $\left(\frac{dy}{dx}\right)_{P(x_1,y_1)} = \infty$ tangent at (x_1, y_1) parallel to y –axis.

- **Asymptotes:**

$x \to \infty$ for $y = a$, $y \to \infty$ for $x = a$

- **Region of absence:**

Values of x for which y becomes imaginary

Values of y for which x becomes imaginary

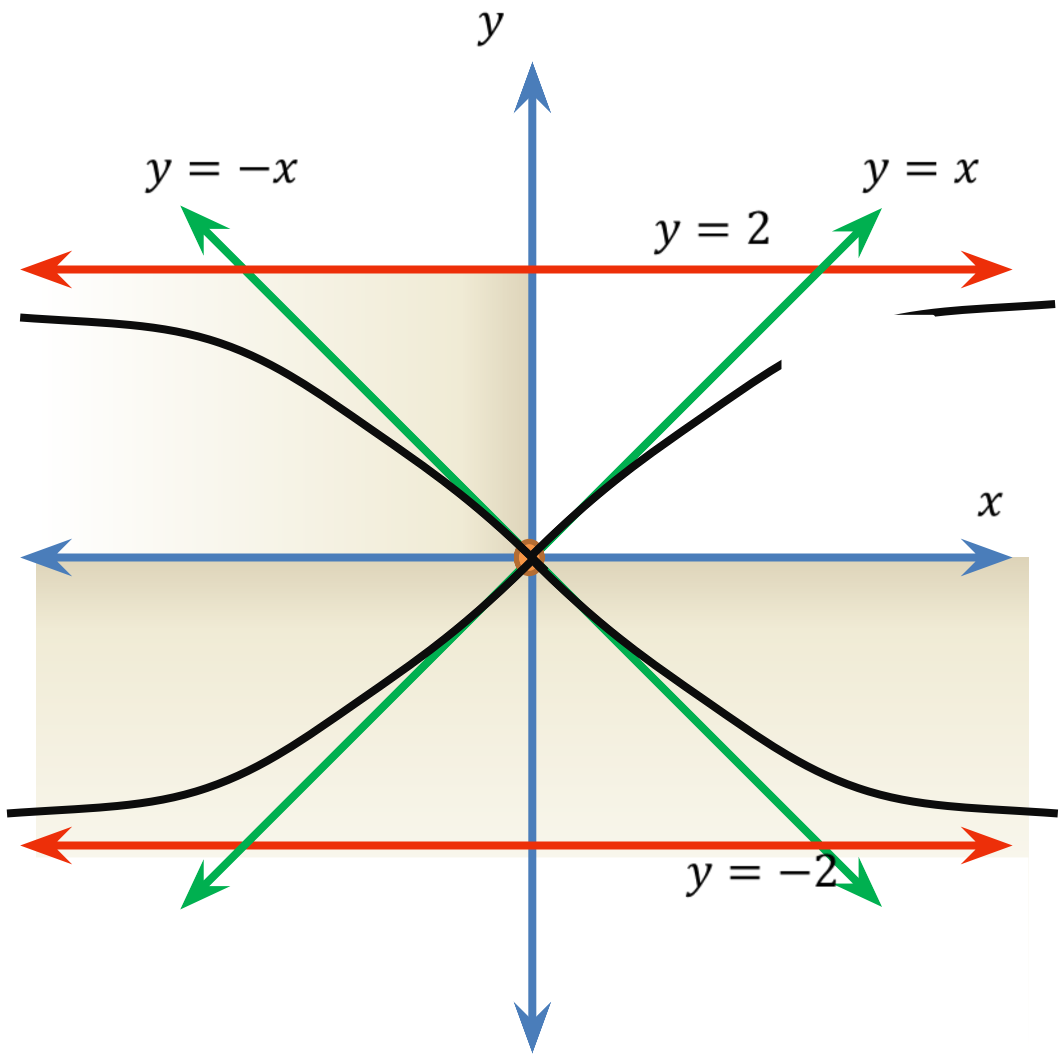
Trace the curve $y^2(4 + x^2) = 4x^2$

- Symmetry: about both axes
- Points of intersection

  Origin: Passes through origin

  Tangent at origin: $y = \pm x$
- Special points on curve: No x or y intercepts.
- Asymptotes: $y = \pm 2$
- Region of absence:

$$y < -2, \qquad y > 2$$



y
$y = -x$
$y = x$
$y = 2$
x
$y = -2$

Trace the curve $y(x^2 - 1) = x^2 + 1$

- Symmetry: about y −axis
- Points of intersection

Origin: not passing through origin

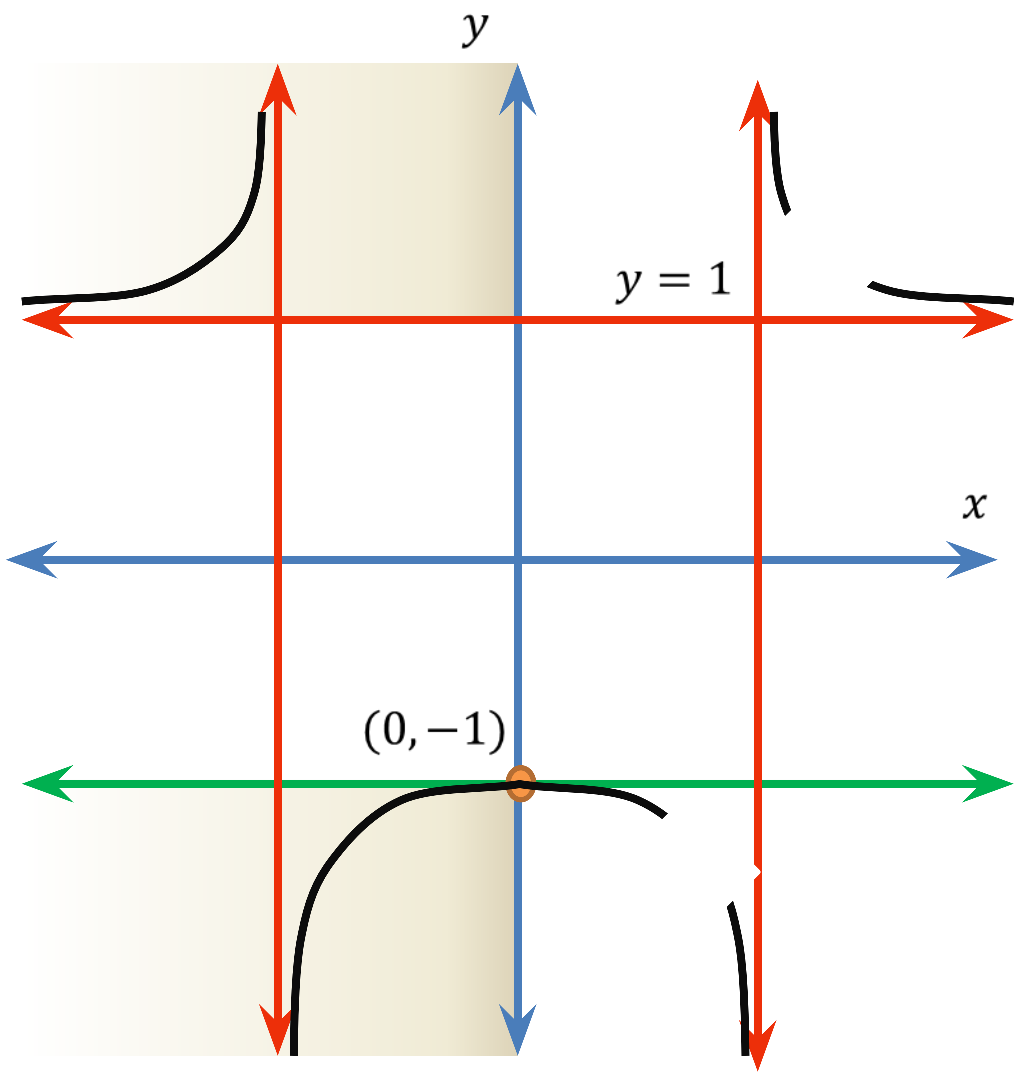
- Special points on curve:

y intercepts is $(0, -1)$

$$\left(\frac{dy}{dx}\right)_{(0,-1)} = 0$$

Tangent at $(0, -1)$ parallel to x −axis

- Asymptotes: $\mathrm{x} = \pm 1, y = 1$
- Region of absence: $-1 < y < 1$


y
y = 1
x
(0, −1)
x = −1
x = 1

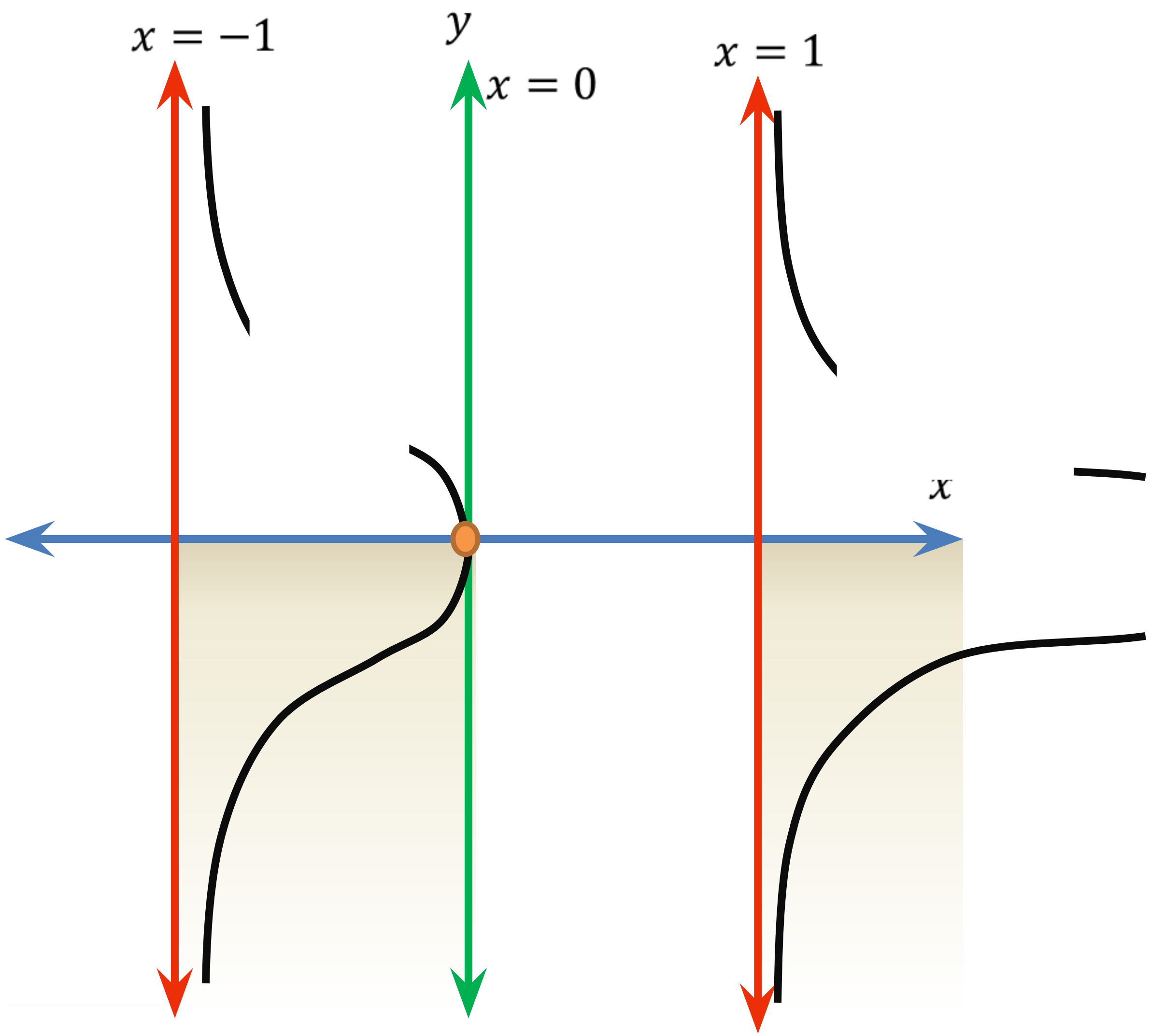
Trace the curve $y^2(x^2 - 1) = x$

- Symmetry: about $x -$ axis
- Points of intersection

Origin: Passes through origin

Tangent at origin: $x = 0$

- Special points on curve: No x or y intercepts.
- Asymptotes: $x = \pm 1$
- Region of absence: $x < -1,\ 0 < x < 1$



$x = -1$
y
$x = 0$
$x = 1$
x

Assignments

Trace the curve $x^2y^2 = a^2(y^2 - x^2)$ 2012

Trace the curve $y^2 = x^2(1 - x)$ 2013

Trace the curve $y^2 = x^5(2a - x)$ May 2015

Trace the curve $ay^2 = x^2(a - x)$ Nov 2015

	a)		b)	
	c)		d)	
	a)		b)	does not pass through origin
	c)	passes through the origin	d)	

	a)		b)	Lowest degree terms to zero
	c)	Highest degree terms to zero	d)	

a)		b)	
c)		d)	

	a)		b)	
	c)		d)	

a)		b)	
c)		d)	

Parametric Curve Equations, x = f (t) , y = g(t) Where t is a parameter.

Rules For Tracing parametric Curves.

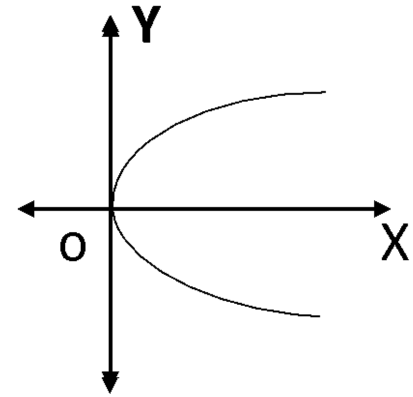
Rule 1 : Symmetry :

(a) Symmetry about X- axis: If equation of

x remains unchanged by changing ‘t’ to ‘ – t’

And y changes the sign then curve will be symmetric

About X – axis. e.g. $x = at^2$, $y = at$ i.e. $y^2 = 4ax$ Parabola.

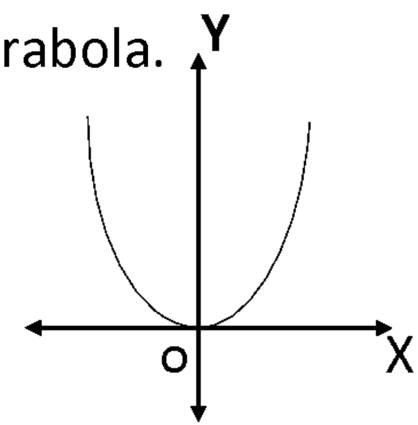


(b) Symmetry about Y- axis: If equation of

y remains unchanged by changing ‘t’ to ‘ – t’

And x changes the sign then curve will be symmetric

About Y – axis. e.g. $x = at$, $y = at^2$ i.e. $x^2 = 4ay$ Parabola.

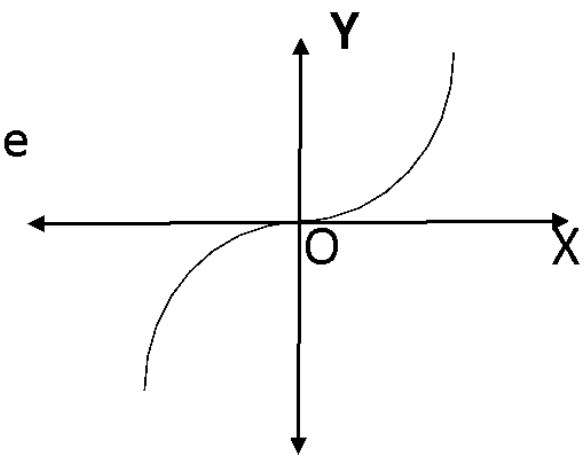


Note : For trigonometric equations if on replacing t to ▯▯− ▯▯, y remains unchanged and x changes the sign then also curve will be symmetric about Y – axis.

(c) Symmetry about origin: If on **replacing t by - t** if both x and y change the sign then curve is symmetric about otigin.

i.e. both x(t) and y(t)are odd functions of t.

e.g. $x = t$, $y = t^3$.



Rule 2 : Points Of Intersection :

1. If for some value of t both x and y become zero, then the curve passes through origin.
2. Find x and y intercepts if any.

Rule 3: Nature of tangents

1. [illegible] = [illegible] **2. Form the table of values of t , x, y ,** [illegible]

Rule 4: Asymptotes and Region

1. Find asymptotes if any. **2.** Find region of absence.

Ex 1 : Trace the curve $[\text{illegible}]^{2/3} + [\text{illegible}]^{2/3} = 1$

Sol : 1 . The parametric equations of the curve are

$$x = a\cos^3\theta \quad , \quad y = b\sin^3\theta \ ; \ \theta \text{ is parameter.}$$

2 . Symmetry : (i) By replacing θ by $-\theta$, x remains unchanged and y changes its sign $\Longrightarrow$ curve is symmetric about X – axis.

(ii) By replacing θ by $\pi - \theta$, y remains unchanged and x changes its sign $\Longrightarrow$ curve is symmetric about Y – axis.

Therefore curve is symmetric about both the axes.

3 . x and y are not zero for any value of θ , therefore the curve does not pass through Origin.

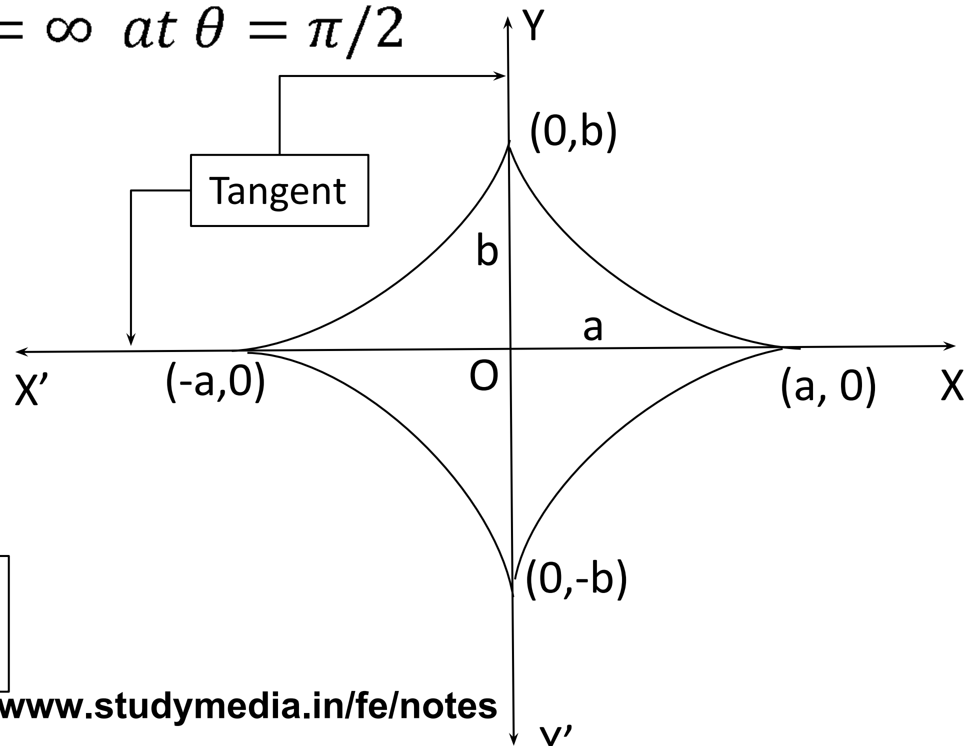
4 . since $|cos\theta| \leq 1 \ and \ |sin\theta| \leq 1$

$\therefore$ Range of x is $-a \leq \theta \leq a$ and Range of y is $-b \leq \theta \leq b$

5 . $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{3bsin^2\theta.cos\theta}{-3acos^2\theta.sin\theta} = \frac{-b.sin\theta}{a.cos\theta}$

$$\therefore \frac{dy}{dx} = 0 \ at \ \theta = 0 \quad and \quad \frac{dy}{dx} = \infty \ at \ \theta = \pi/2$$

X and Y axes are tangents.

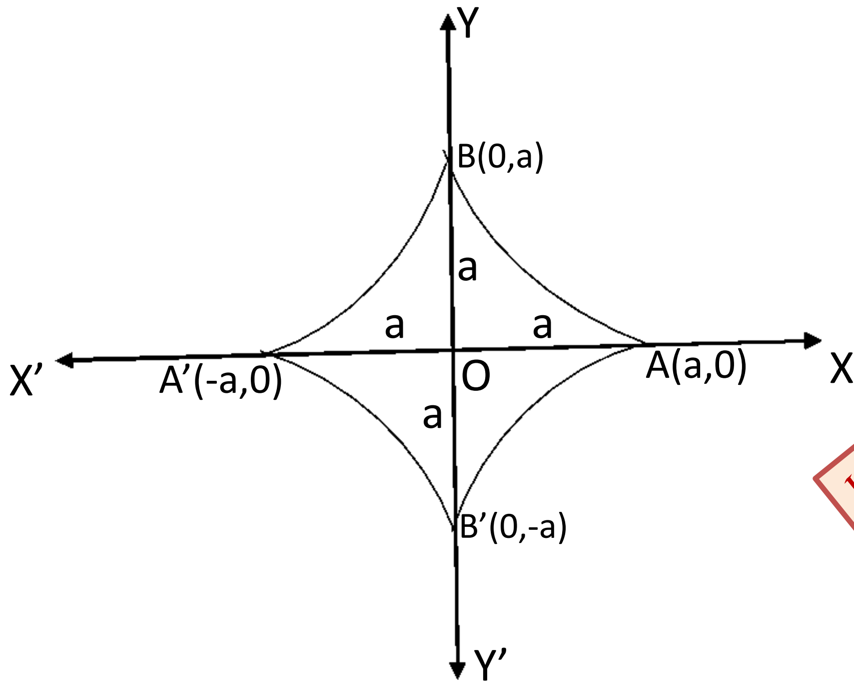


6. Table:

θ	0	π/2
x	a	0
y	0	b

We considered this table because the curve is symmetric about both axes.

Ex 2 : Trace the curve $x^{2/3} + y^{2/3} = a^{2/3}$



Home Work

Cycloid : A locus of a fixed point on the circumference of a circle when rolled on a plane along a given line is called as cycloid

Ex 3: Trace the curve $x = a(\theta + \sin\theta)$, $y = a(1 + \cos\theta)$

1 . Symmetry : By replacing θ by $-\theta$, y remains unchanged and x changes its sign $\Longrightarrow$ curve is symmetric about Y – axis.

2 . x and y are not zero for any value of θ , therefore the curve does not pass through Origin.

3 . y is periodic function of period 2π.

4 . $\frac{dy}{dx} = \frac{(dy/d\theta)}{(dx/d\theta)} = \frac{-a.\sin\theta}{a\,(1+\cos\theta)} = -\tan\frac{\theta}{2}$

5. Table:

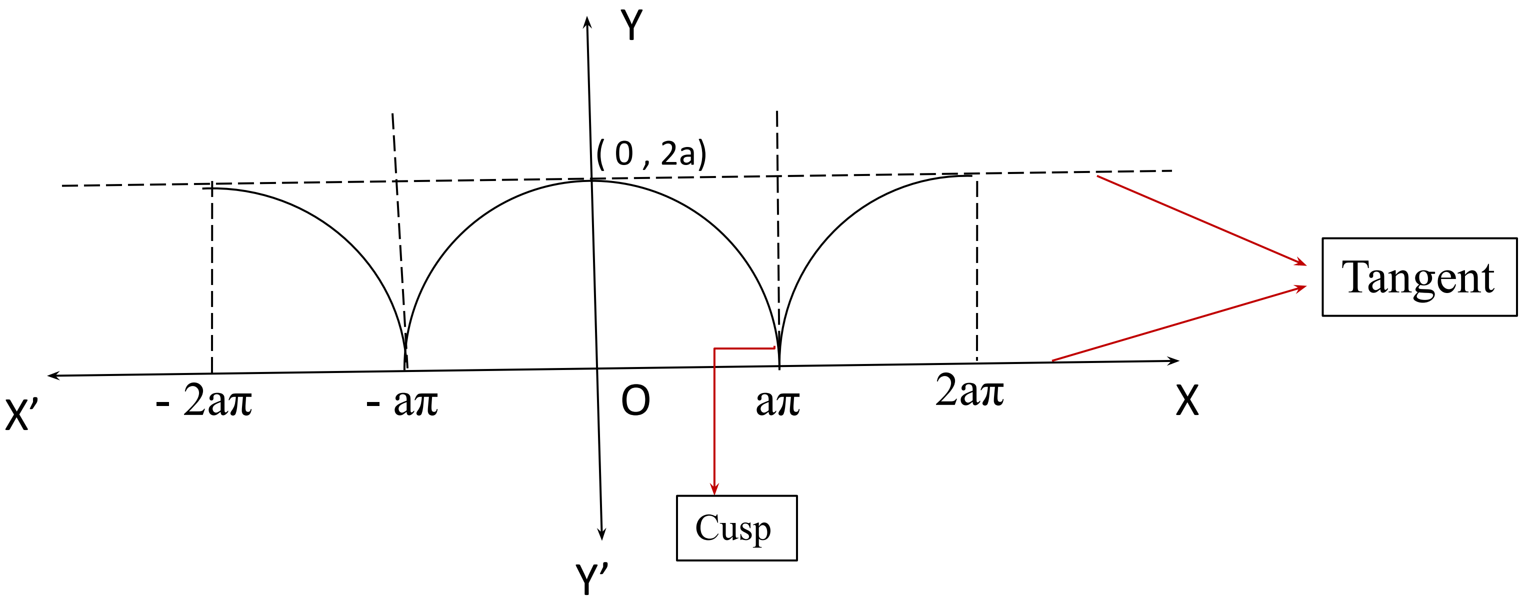
θ	$-\pi$	0	π	2π
x	$-a\pi$	0	$a\pi$	$2a\pi$
y	0	2a	0	2a
dy/dx	∞	0	$-\infty$	0

6. At $(0, 2a)$ and $(2a\pi, 2a)$; $dy/dx = 0 \Longrightarrow$ Tangent is parallel to X- axis at these points.

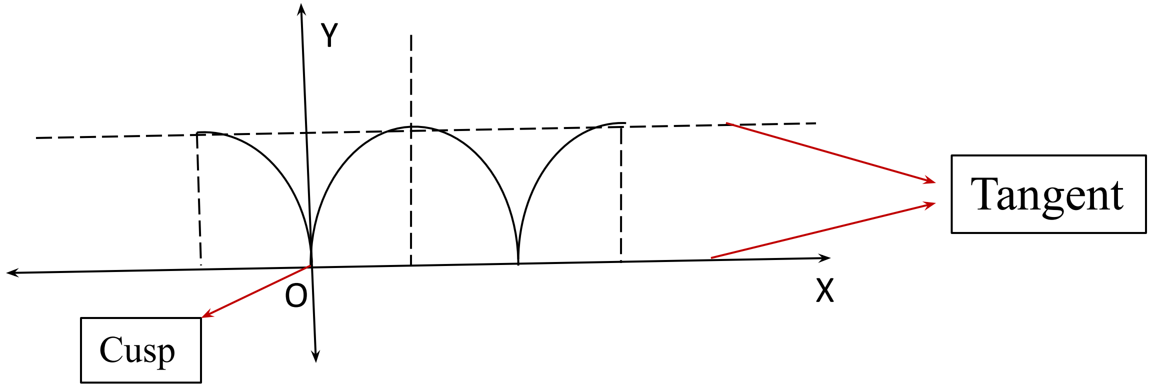
At $(-a\pi, 0)$ and $(a\pi, 0)$; $dy/dx = \infty \Longrightarrow$ Tangent is parallel to Y- axis at these points.

7. Ragion : Since $-1 \leq \cos\theta \leq 1 \Longrightarrow 0 \leq y \leq 2a$

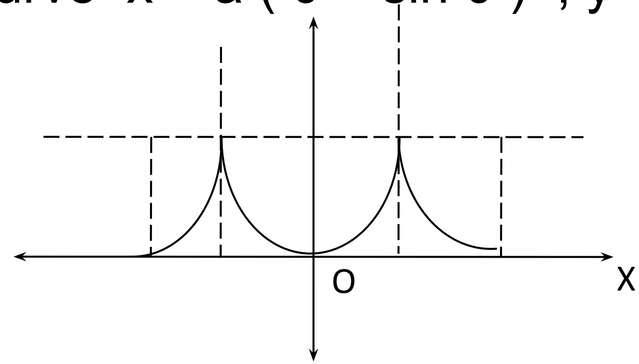
$\therefore$ The curve does not exist below $X -$ axis and above $y = 2a$



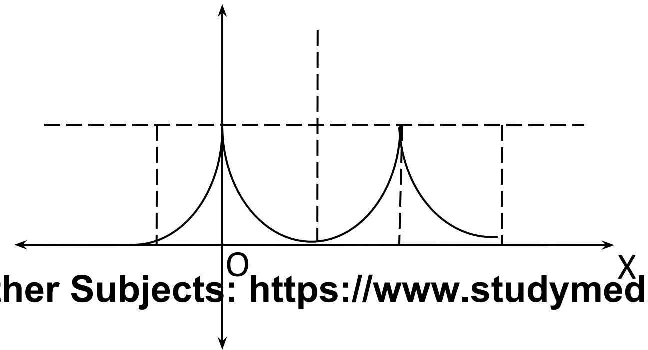
Ex 4 : Trace the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ Nov 2014



Ex 5: Trace the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ 2012



Ex 6: Trace the curve $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$



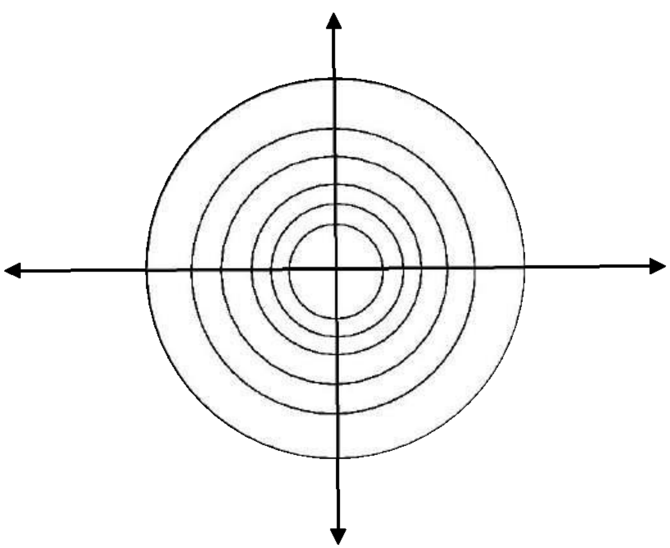
Polar Curves

If it is not possible to trace a curve in Cartesian co-ordinates change equation into polar co-ordinates by using the transformations,

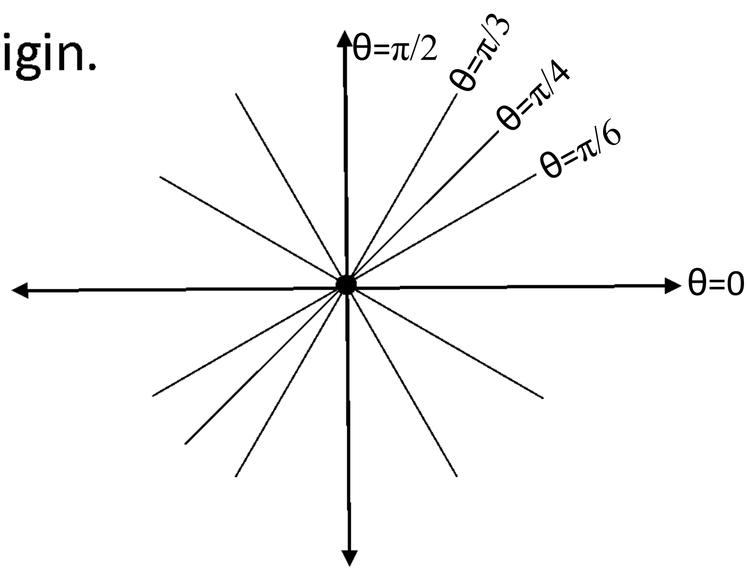
$$x = r\cos\theta\,, \quad y = r\sin\theta$$

$$r^2 = x^2 + y^2\,, \quad \theta = \tan^{-1}\frac{y}{x}$$

r = parameter, represents concentric circles with centre at origin.



θ = parameter,represents a family of straight lines passing through origin.



Rules For Tracing Polar Curves.

Terminology : In polar curves

a) $\theta = 0$, positive $X -$ axis is called as

b) Equation of polar curves is often given by $r = f(\theta)$.

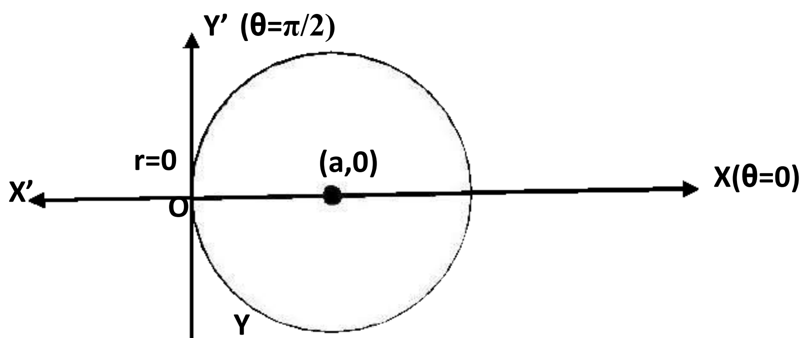
c) Origin O is called as ***Pole***.

d) R is called as radius vector.

Symmetry :

a) If on changing θ to $-\theta$, equation of the curve remains unchanged then curve is symmetric about initial line (X - axis).

e.g. $x^2 + y^2 = 2ax$, Polar equation $r = 2a\cos\theta$



b) If on changing r to - r ,equation of the curve remains unchanged then curve is symmetric about the pole.

e.g. $(x^2 + y^2) = a^2(x^2 - y^2)$, Polar equation $r^2 = a^2 \cos 2\theta$

c) If on changing r to - r and θ to - θ,equation of the curve remains unchanged then curve is symmetric about Y - axis.

OR

If on changing θ to $\pi - \theta$, equation of the curve remains unchanged then curve is symmetric about Y - axis.

e.g. (i) $(x^2 + y^2) = 2ay$, Polar equation $r = 2a\sin\theta$

(ii) $r = (1 + \sin\theta)$, First rule fails but second rule gives symmetry about Y – axis.

Pole : If for some values of θ, r becomes zero then the pole lies on the curve.

e.g. (i) $(x^2 + y^2) = a^2(x^2 - y^2)$, Polar equation $r^2 = a^2 \cos 2\theta$

$$\theta = \frac{\pi}{4}, \quad r = 0 \quad \Longrightarrow \text{curve passes through the pole.}$$

Tangents at Pole : To find tangents at pole, put r = 0 in the equation, the value of θ gives tangent at the pole.

e.g. r = a sin3 θ , r = 0 ⟹ [illegible] $3\theta = 0$

$3\theta = 0, \pi, 2\pi, 3\pi, \ldots\ldots\ \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \ldots\ldots$ are all tangents at pole.

- **Prepare a table showing the values of r and** [illegible]
- **Find the angle between radius vector and the tangent (∅)**
- $\tan\varnothing =$ [illegible], find the value of θ, for which $\varnothing = 0$ [illegible] $\frac{[illegible]}{2}$
- The values of θ for which

$\varnothing = 0,$ [illegible]

$\varnothing = \frac{[illegible]}{2},$ [illegible]

Trace the curve $r^2 = a^2 \cos 2\theta$

- Symmetry: about Initial line , Pole , Y- Axis
- Pole: For $\theta = \pi/4,\ r = 0$

$\therefore$curve passes through pole

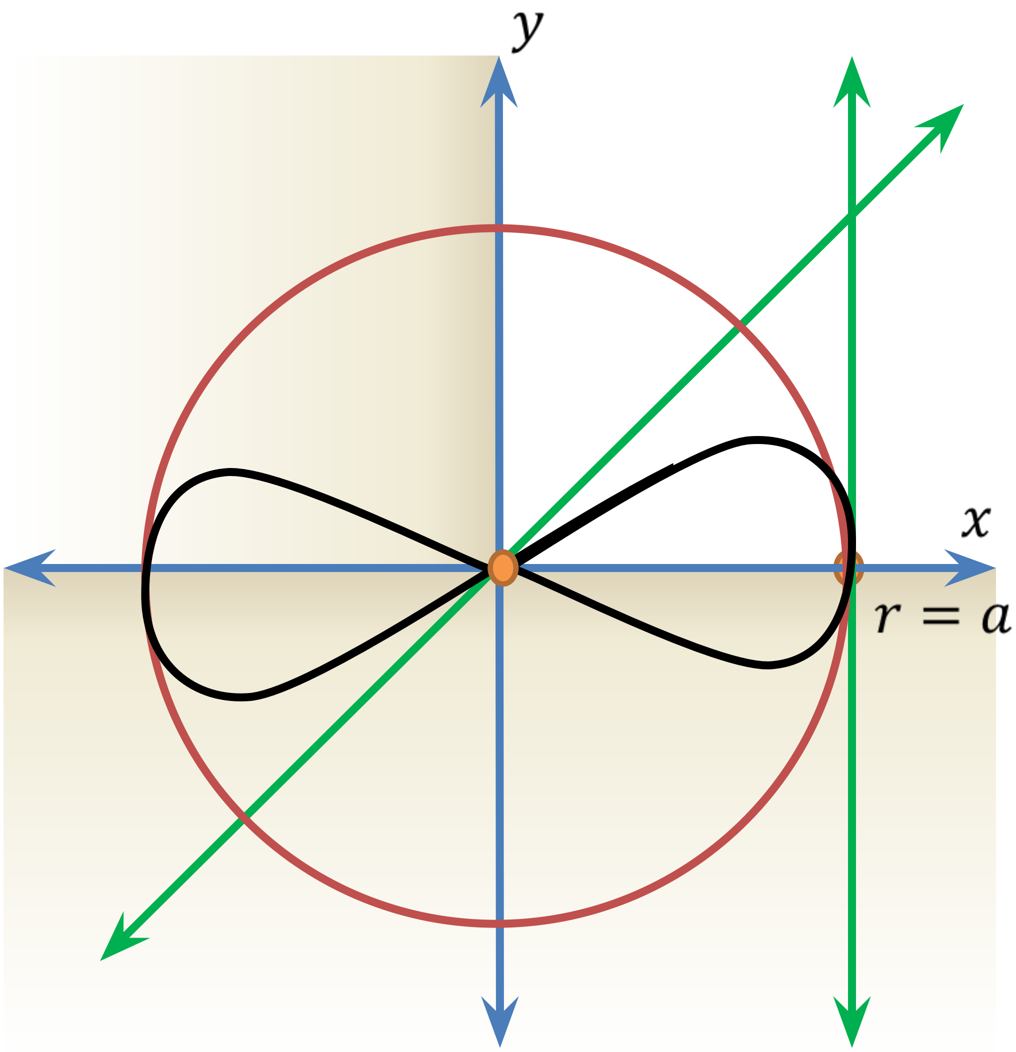
Tangent at pole: For $r = 0, \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \dots\dots..$

Angle ϕ :

As $\tan\phi = r\frac{d\theta}{dr} = \tan\left(\frac{\pi}{2} + 2\theta\right)$ $\quad\quad \therefore \phi = \frac{\pi}{2} + 2\theta,$

For $\theta = 0, \phi = \frac{\pi}{2}, r = \pm a$

Limitations of curve: $r \leq a,$



y
x
r = a

Trace the curve $r = a + b\cos\theta$ for $a > b,\ a < b,\ a = b$

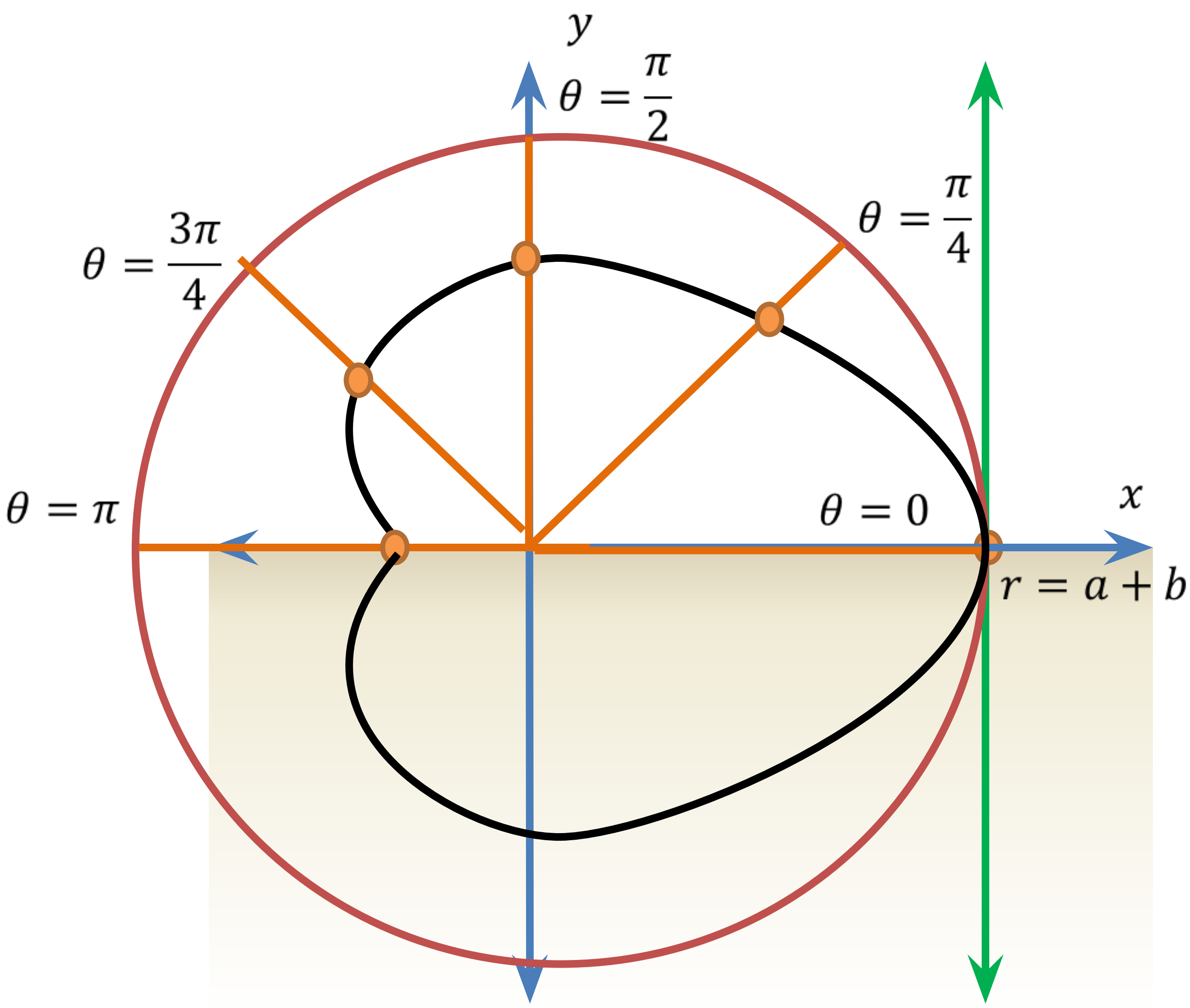
Let $a > b$

- Symmetry: about x –axis
- Pole: not passing through pole

As $\tan\phi = \frac{a+b\cos\theta}{-b\sin\theta}$

For $\theta = 0, \phi = \frac{\pi}{2}, r = a + b$

Limitations of curve: $r \leq a + b$,


y
$\theta = \frac{\pi}{2}$
$\theta = \frac{3\pi}{4}$
$\theta = \frac{\pi}{4}$
$\theta = \pi$
$\theta = 0$
x
$r = a + b$

$$r = a + b\cos\theta$$

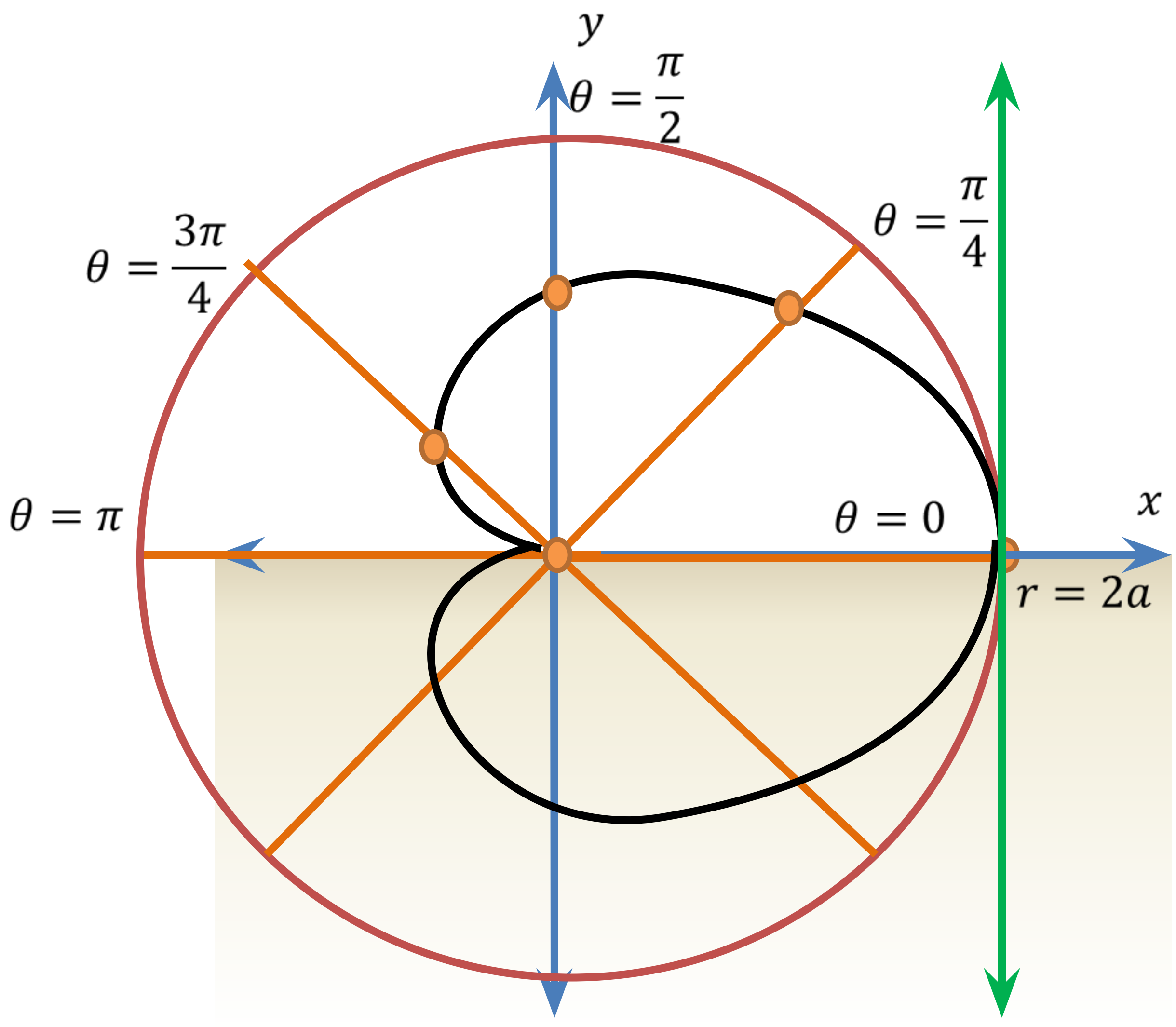
$$\text{let } a = b \text{ then } r = a + a\cos\theta$$

- Symmetry: about x –axis
- Pole: passes through pole

As $\tan\phi = \frac{1+\cos\theta}{-sin\ \theta}$

For $\theta = 0, \phi = \frac{\pi}{2}$, $\quad r = 2a$

Limitations of curve: $\quad r \leq 2a,$



y
$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$
$\theta = \frac{3\pi}{4}$
$\theta = \pi$
$\theta = 0$
x
$r = 2a$

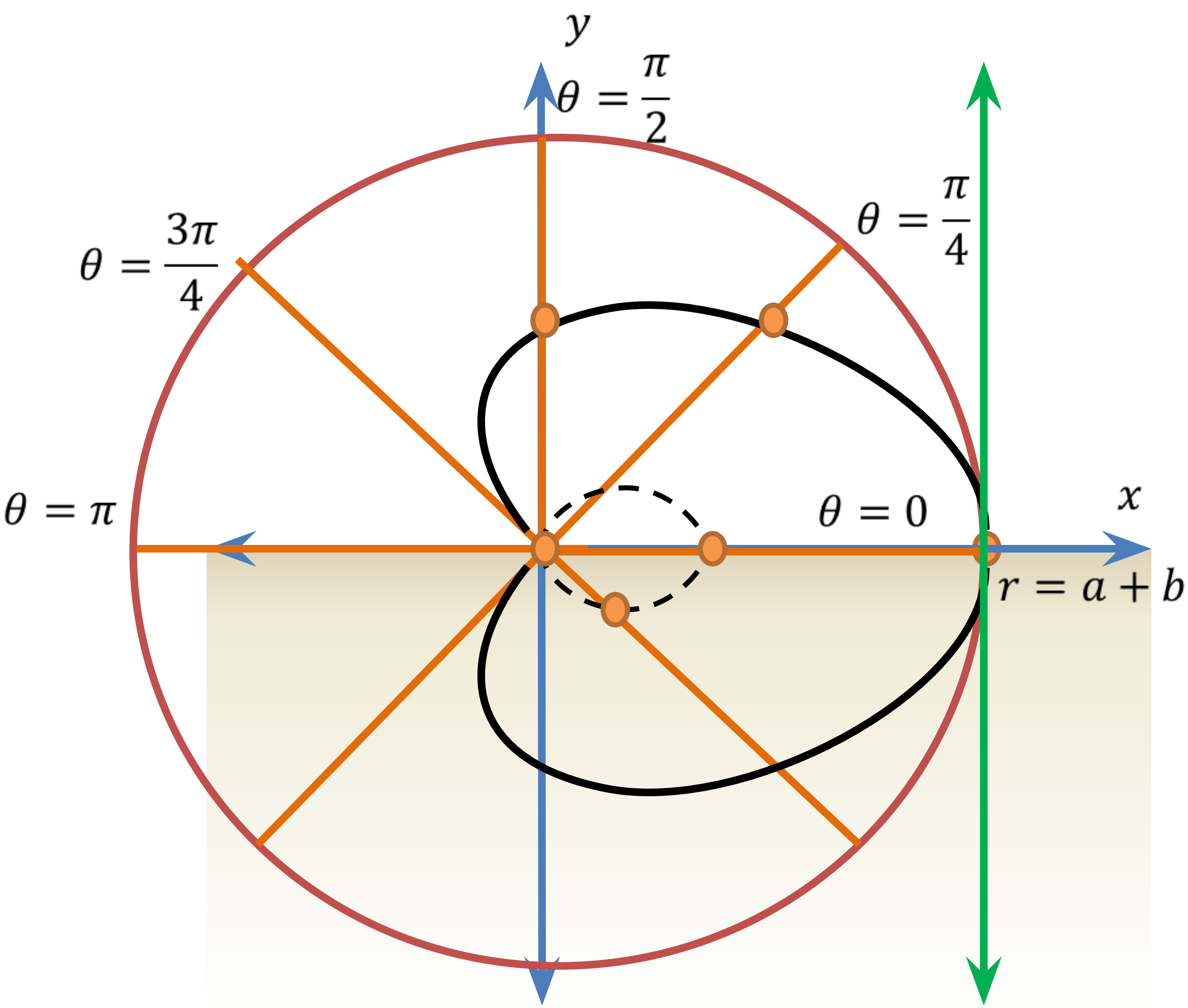
$r = a + b\cos\theta$ For a < b

- Symmetry: about x –axis
- Pole: not passing through pole

As $\tan\phi = \dfrac{a+b\cos\theta}{-b\sin\theta}$

For $\theta = 0, \phi = \dfrac{\pi}{2}, r = a + b$

Limitations of curve: $r \leq a + b,$



y
$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{4}$
$\theta = \frac{3\pi}{4}$
$\theta = \pi$
$\theta = 0$
x
$r = a + b$

a)		b)	line passing through pole and perpendicular to initial line
c)	pole	d)	

a)		b)	
c)		d)	

Rose Curves

(Curves given by Polar equations of the type $r = a\sin n\theta$ or $r = a\cos n\theta$)

- Symmetric:

About $x-$axis : Equation of curve remains unchanged by changing θ to $-\theta$

About $y-$axis : Equation of curve remains unchanged by changing θ to $-\theta$ and r to $-r$

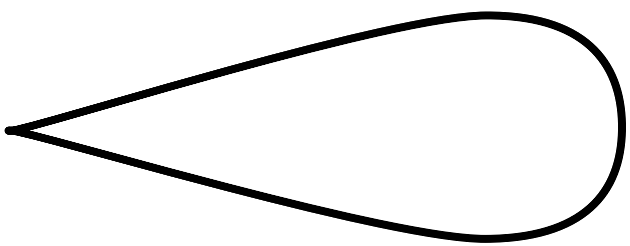
- Pole: Rose curves passes through pole.
- For tangent at pole find θ for $r = 0$.

- Limitations of the curve: Maximum value of r is a.

$\therefore$curve lies within circle of radius a.

The curve consists of

- n equal loops if n is odd.
- $2n$ equal loops if n is even.



- For drawing the loops, divide each quadrant in to n equal parts

- Draw first loop along $\theta = \frac{\pi/2}{n}$ for $r = a \sin n\theta$
- Draw first loop along $\theta = 0$ for $r = a \cos n\theta$
- If n is even draw loops in two sectors consecutively from $\theta = 0$ to $\theta = 2\pi$.
- If n is odd, draw loops in two sectors alternatively keeping two sectors between the loops vacant.

1. $r = a\sin 2\theta$ May 2015

- Symmetry: about $y -$ axis
- Pole: passes through pole

Tangent at pole:

$$2\theta = 0, \pi, 2\pi, \cdots$$

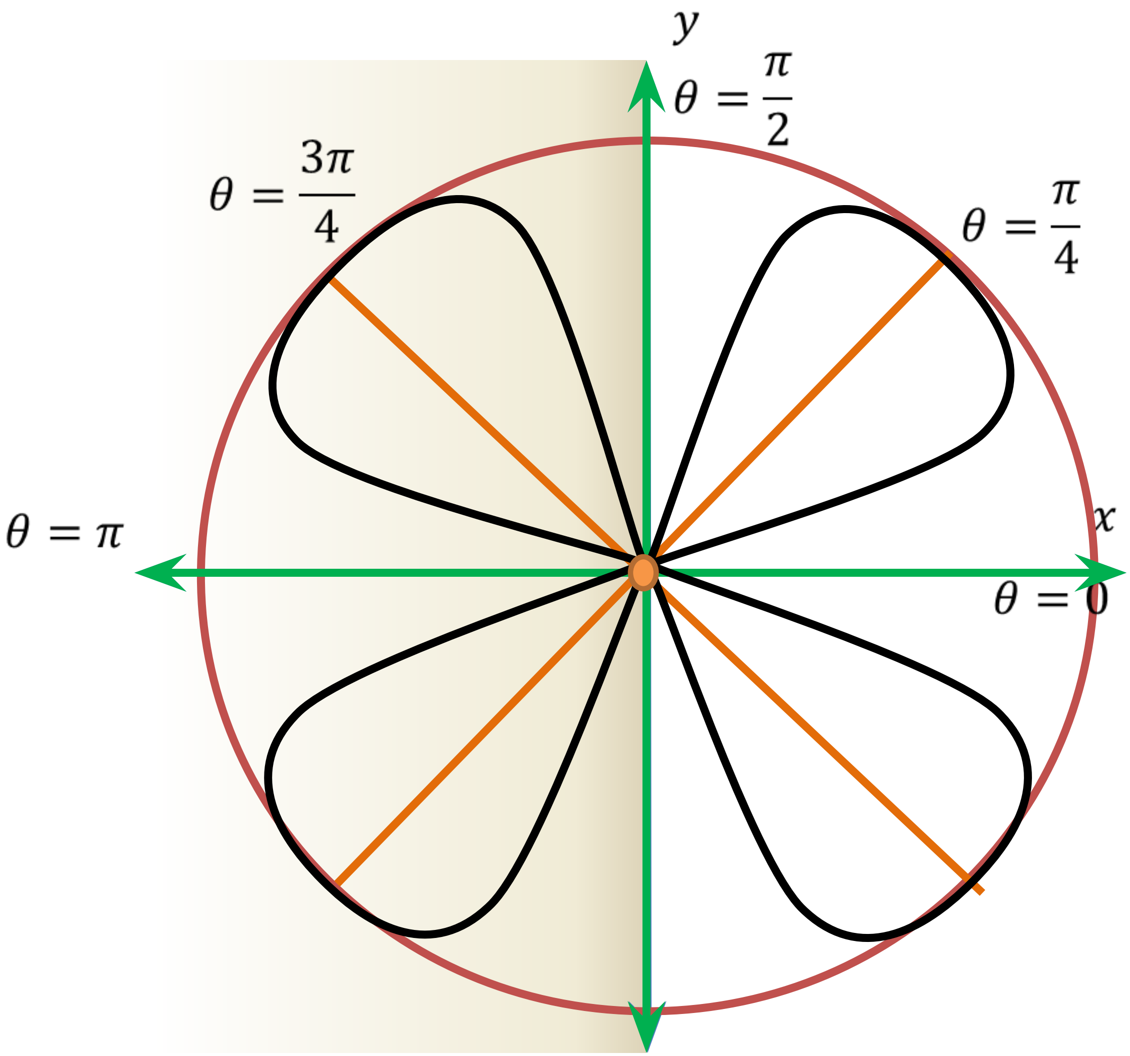
i.e. $\theta = 0, \frac{\pi}{2}, \pi, \cdots$

Limitations of curve: $r \leq a$

Dividing each quadrant into 2 equal parts

The curve consist of $2n = 4$ equal loops

First loop along $\theta = \frac{\pi/2}{n} = \frac{\pi}{4}$



y
$\theta = \frac{\pi}{2}$
$\theta = \frac{3\pi}{4}$
$\theta = \frac{\pi}{4}$
$\theta = \pi$
x
$\theta = 0$

$2. r = a \cos 3\theta$

- Symmetry: about $x -$ axis
- Pole: passes through pole

Tangent at pole:

$$3\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \cdots$$

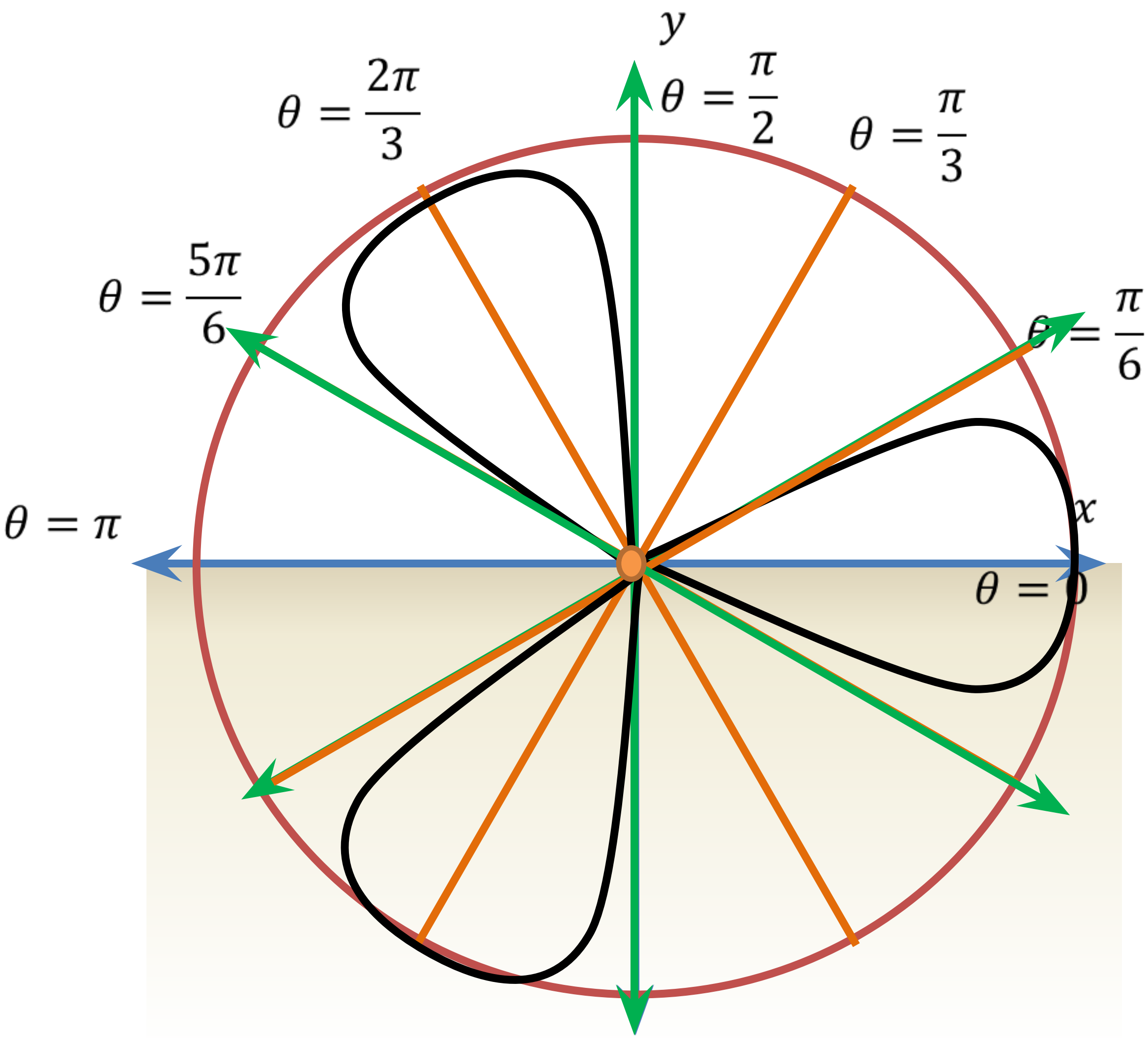
i.e. $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6} \cdots$

Limitations of curve: $r \leq a$

Dividing each quadrant into 3 equal parts

The curve consist of $n = 3$ equal loops

First loop along $\theta = 0$


y
$\theta = \frac{2\pi}{3}$
$\theta = \frac{\pi}{2}$
$\theta = \frac{\pi}{3}$
$\theta = \frac{5\pi}{6}$
$\theta = \frac{\pi}{6}$
$\theta = \pi$
x
$\theta = 0$

$3. r = a \sin 4\theta$

- Symmetry: about $y -$ axis
- Pole : passes through pole

Tangent at pole:

$$4\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \cdots$$

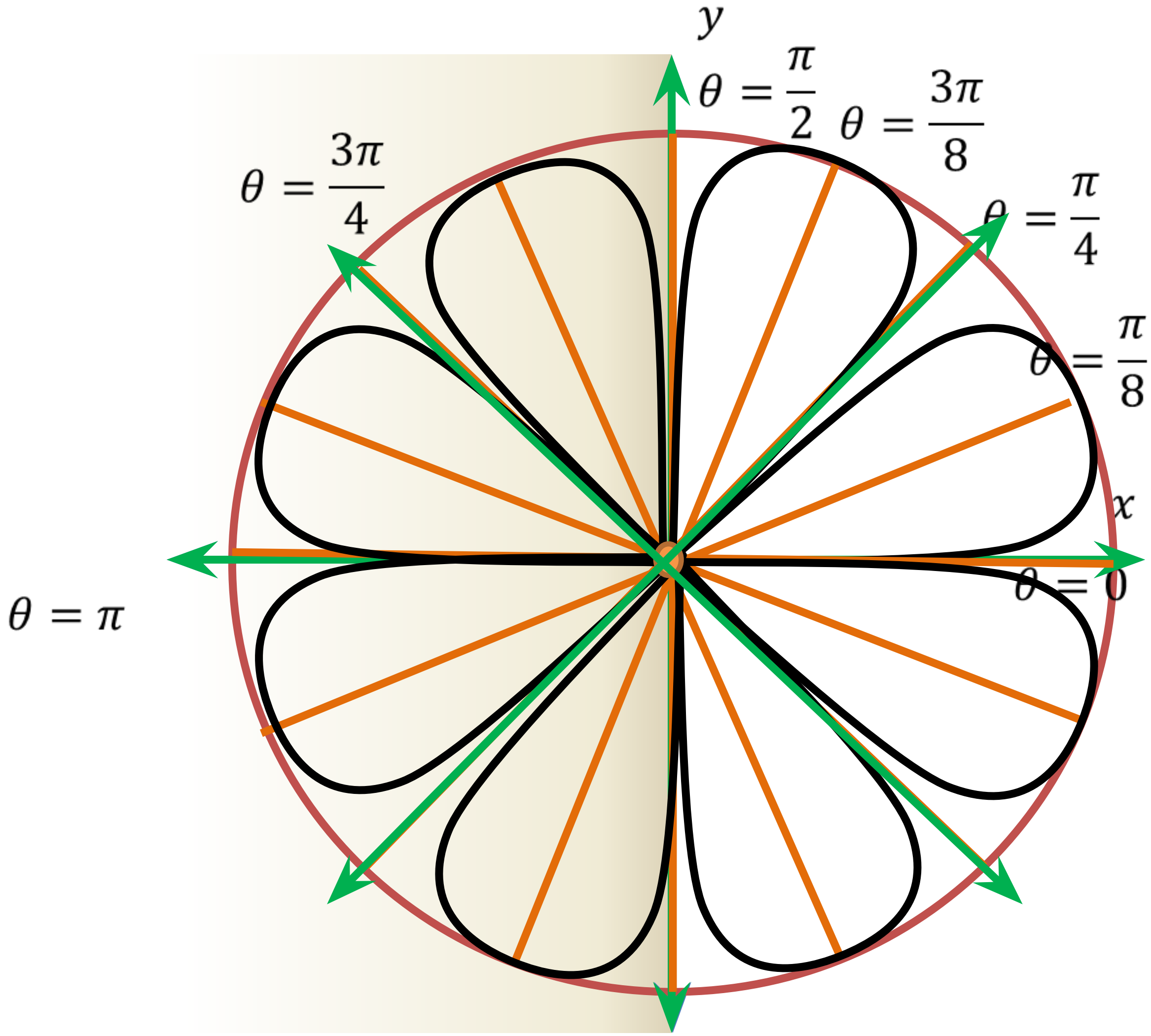
i.e. $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \cdots$

Limitations of curve: $r \leq a$

Dividing each quadrant into 4 equal parts

The curve consist of $2n = 8$ equal loops

First loop along $\theta = \frac{\pi/2}{n}$



Trace the curve $r = 2\sin 3\theta$ 2013

Trace the curve $r = a\sin 3\theta$ Nov 2014

Rectification Of Curves.

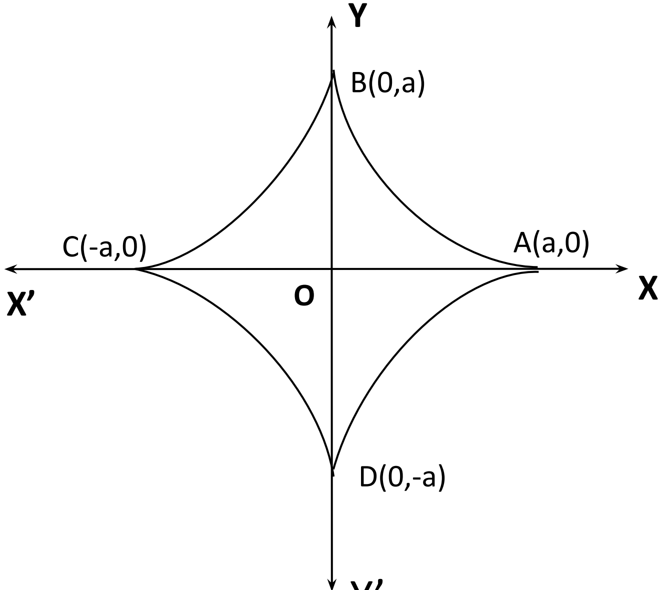
Definition : The process of determination of lengths of the plane curves whose equations are in Cartesian, Parametric and Polar forms is known as **Rectification of curves.**

If 's' is length of the curve from A to B then rectification formulae are

Equation.	ds	s
y = f (x)	$\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$	$\int_{x_1}^{x_2}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\ dx$
x = f (y)	$\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$	$\int_{y_1}^{y_2}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\ dy$
$x = f_1(t)$ $y = f_2(t)$	$\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$	$\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$

Equation.	ds	s
r = f (θ)	$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$	$\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$
θ = f (r)	$\sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\ dr$	$\int_{r_1}^{r_2} \sqrt{1 + r^2 \left(\frac{d\theta}{dr}\right)^2}\ dr$

Ex 1 : Find complete arc length of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ (Astorid)



The parametric equations of the arc are.

$$x = a\cos^3\theta \ \text{ and } \ y = a\sin^3\theta.$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \ ; \ \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 9a^2\sin^2\theta\cos^2\theta$$

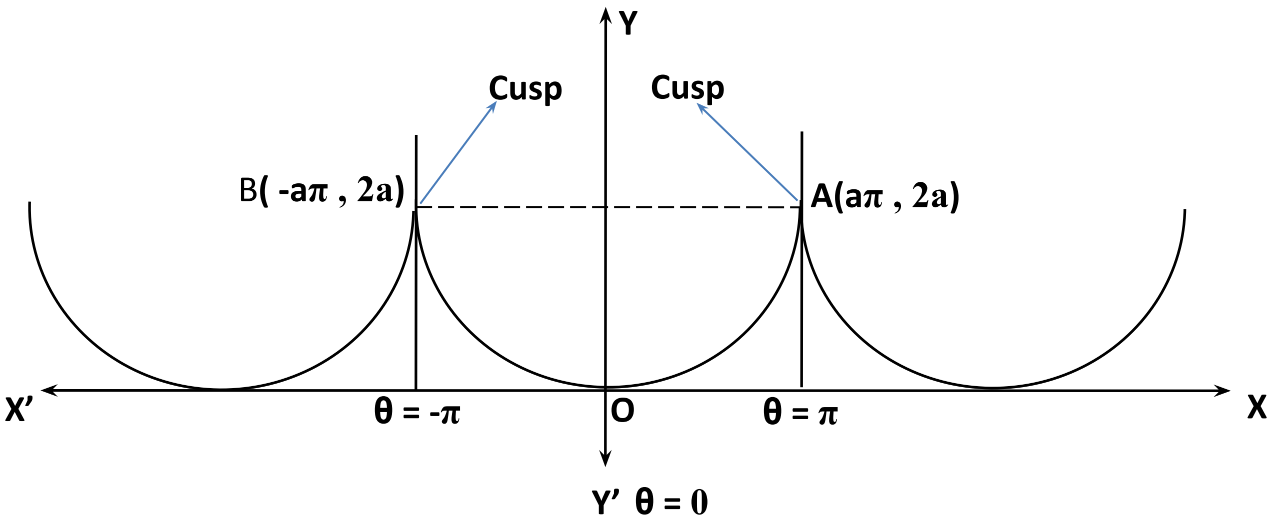
Since the curve is symmetric about both the axis

$$S = 4\int_{AB}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta = 4\int_0^{\pi/2}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$S = 4\int_0^{\pi/2}\sqrt{9a^2\sin^2\theta\cos^2\theta}\,d\theta = 6a\int_0^{\pi/2}\sin 2\theta\,d\theta = 6a\left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2} = 6a$$

---M---

Ex 2 : Find the arc length of the cycloid $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ from one cusp to another cusp.



Sol : given parametric equations of the cycloid $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$

$$\therefore \frac{dx}{d\theta} = a(1 + \cos\theta) \; ; \; \frac{dy}{d\theta} = a\sin\theta$$

And $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2(1 + \cos\theta)^2 + a^2\sin^2\theta = 2a^2(1 + \cos\theta) = 4a^2\cos^2\left(\frac{\theta}{2}\right)$

Length of arc AB = 2 length of arc OA, since the curve is symmetric about Y – axis.

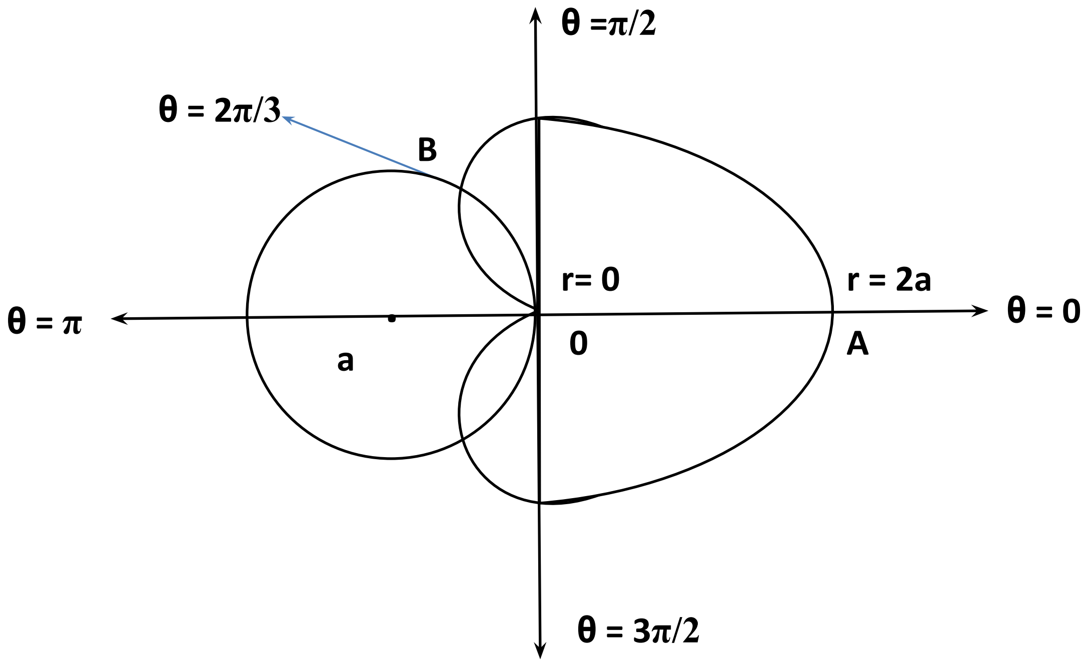
We have $S = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = 2\int_0^{\pi} \sqrt{4a^2 \cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 2\int_0^{\pi} 2a\cos\left(\frac{\theta}{2}\right) d\theta$

$\therefore\ S = 4a\left[2\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi} = 8a$

is required length of the arc from one cusp to another cusp.

---**M** ---

Ex 3 : Find the arc length of the cardiod $r = a(1 + \cos\theta)$ Which lies outside the circle $r + a\cos\theta = 0$



Sol : The point of intersection of the cardiod $r = a\,(1 + cos\theta)$ and circle r + a $cos\theta = 0$

is given by $a\,(1 + cos\theta)$ + a $cos\theta = 0$ $i.e.$ 1+2 $cos\theta = 0$; $cos\theta = -\frac{1}{2}$; $\theta = \frac{2\pi}{3}$

Required arc length outside the circle is twice the length of arc BA.

$$\therefore\ Required\ arc\ length = L = \int ds = \int \frac{ds}{d\theta}\, d\theta$$

Required arc length outside the circle is twice the length of arc BA.

$$\therefore Required\ arc\ length = L = 2 \int_{AB} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

$$\therefore\ L = 2 \int_0^{2\pi/3} \sqrt{a^2(1+cos\theta)^2 + (-a\ sin\theta)^2}\ d\theta$$

$$= 2\ a \int_0^{2\pi/3} \sqrt{1 + 2cos\theta + \cos^2\theta + sin^2\theta}\ d\theta = 2\ a \int_0^{2\pi/3} \sqrt{2 + 2\ cos\theta}\ d\theta$$

$$= 2\sqrt{2}a \int_0^{2\pi/3} \sqrt{1 +\ cos\theta}\ d\theta = 2\sqrt{2}a \int_0^{2\pi/3} \sqrt{1 +\ 2cos^2(\frac{\theta}{2}) - 1}\ d\theta$$

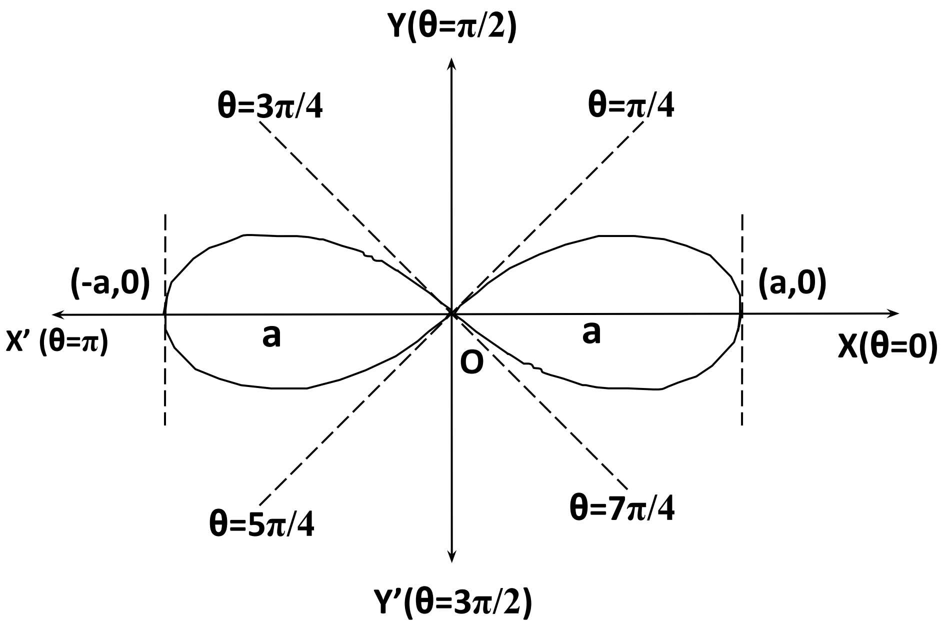
$$= 4a \int_0^{2\pi/3} \cos(\theta/2)\ d\theta = 4a[2\sin(\theta/2)]_0^{2\pi/3}$$

$$= 8a\sin\left(\frac{\pi}{3}\right)$$

$$= 8\ a\frac{\sqrt{3}}{2} = 4a\sqrt{3}$$

-- **M** --

Ex 4: Find the arc length of the upper arc of one loop of Lemiscate $r^2 = a^2 \cos 2\theta$



For upper arc of the curve θ *varies from* 0 *to* $\pi/4$

$$\therefore S = \int_0^{\pi/4} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

r = a $\sqrt{cos2\theta}$; $\frac{dr}{d\theta} = a\,\frac{(-2\ sin2\vartheta\ )}{2\sqrt{cos2\theta}}$

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 cos2\theta + a^2\,\frac{sin^2 2\theta}{cos2\theta}$$

$$= \frac{a^2}{cos2\theta}$$

$$\therefore S = \int_0^{\pi/4} \sqrt{\frac{a^2}{cos2\theta}}\, d\theta$$

$$\therefore S = a\int_0^{\pi/4} \frac{1}{\sqrt{cos2\theta}}\, d\theta \quad ; Put\ 2\theta = t, d\theta = \frac{dt}{2}\ ;\ \theta = 0, t = 0\ ; \theta = \frac{\pi}{4}, t = \frac{\pi}{2}$$

$$\therefore S = \frac{a}{2}\int_0^{\pi/2} \frac{1}{\sqrt{cost}}\, dt = \frac{a}{2}\int_0^{\pi/2} sin^0 t\, cos^{-1/2} t\, dt$$

$$= \frac{a}{4}\, \beta\left(\frac{0+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) = \frac{a}{4}\, \beta\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{a}{4}\frac{\overline{|1/2}\ \overline{|1/4}}{\overline{|3/4}} = \frac{a}{4}\frac{\overline{|1/4}\ \left(\overline{|1/4}\right)^2}{\overline{|1/4}\ \overline{|1-1/4}} = \frac{a}{4}\frac{\sqrt{\pi}\ \left(\overline{|1/4}\right)^2}{\pi}\sin\left(\frac{\pi}{4}\right)$$

$$= \frac{a}{4\sqrt{2}\sqrt{\pi}}\left(\overline{|1/4}\right)^2$$

--- **M** ---

THANK YOU