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UNIT 3 - Integral Calculus

Reduction Formulae

Reduction Formula means a formula which reduces a given integral to a Known integration form by repeated applications of integration by parts.

1]
$$\int_{0}^{\pi/2} \cos^{n} x \, dx = \int_{0}^{\pi/2} \sin^{n} x \, dx$$

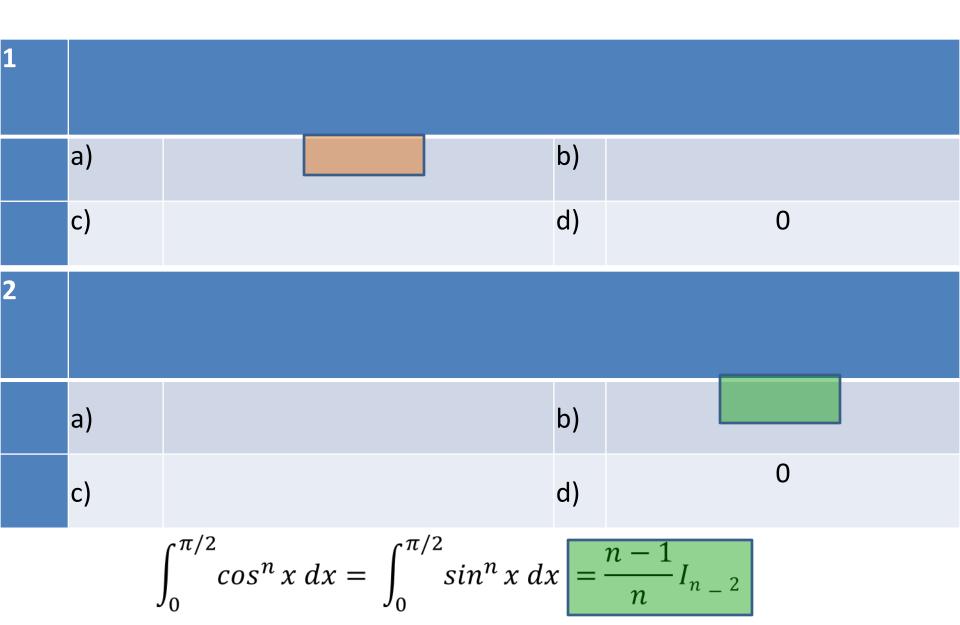
$$= \frac{[(n-1) \text{subtract } 2...... 2 \text{ or } 1]}{[(n) \text{ subtract } 2...... 2 \text{ or } 1]} \times (\frac{\pi}{2}) \text{ if n is even.}$$

$$= \frac{[(n-1) \text{subtract } 2...... 2 \text{ or } 1]}{[(n) \text{ subtract } 2...... 2 \text{ or } 1]} \times 1 \text{ if n is odd.}$$

$$\int_0^{\frac{\pi}{2}} \sin^9 x = ----- \int_0^{\frac{\pi}{2}} \cos^6 x = -----$$

$$= \frac{128}{315}$$

$$= \frac{5\pi}{32}$$



$$= \frac{1}{1} \frac{$$

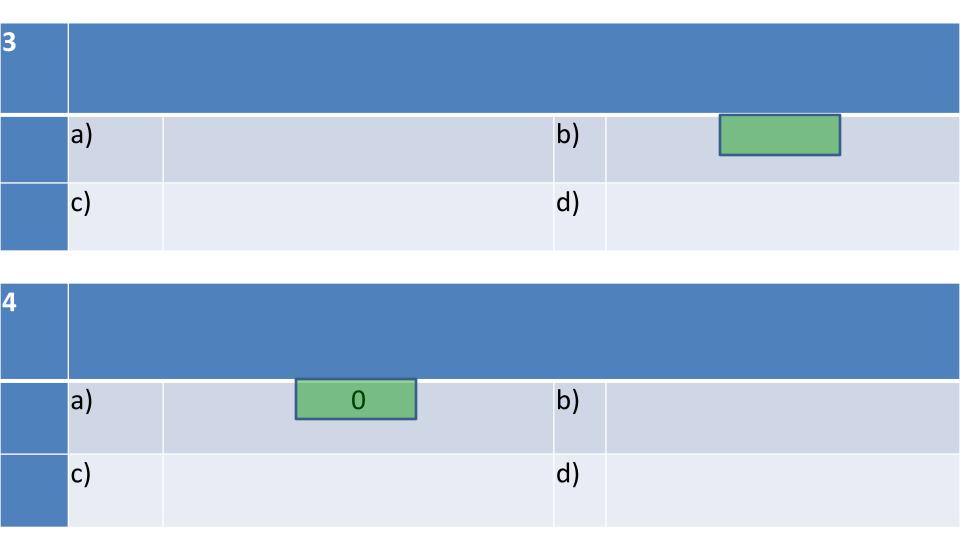
If m and n both are even.

$$= \frac{1}{3} \frac{1}{3} \frac{1}{3} \frac{1}{3} \times \frac{1}{3} \frac{1}{3} = \frac{1}{3} \frac{1}{3$$

Otherwise.

$$\int_0^{\frac{\pi}{2}} \sin^6 x \cos^4 x \, dx = --- - \int_0^{\frac{\pi}{2}} \sin^3 x \cos^5 x \, dx = ---- -$$

$$=\frac{3\pi}{512}$$
 $=\frac{1}{24}$



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$$\int_0^{\frac{\pi}{2}} \cos^5 x \sin x \, dx = -- - \int_0^{\frac{\pi}{2}} \cos x \sin^{10} x \, dx = -- -$$

$$= \frac{1}{6}$$

$$= \frac{1}{11}$$

Conversion Formulae:

1]
$$\Box_0^{2\Box}$$
 $\Box_0^{2\Box}$ \Box_0

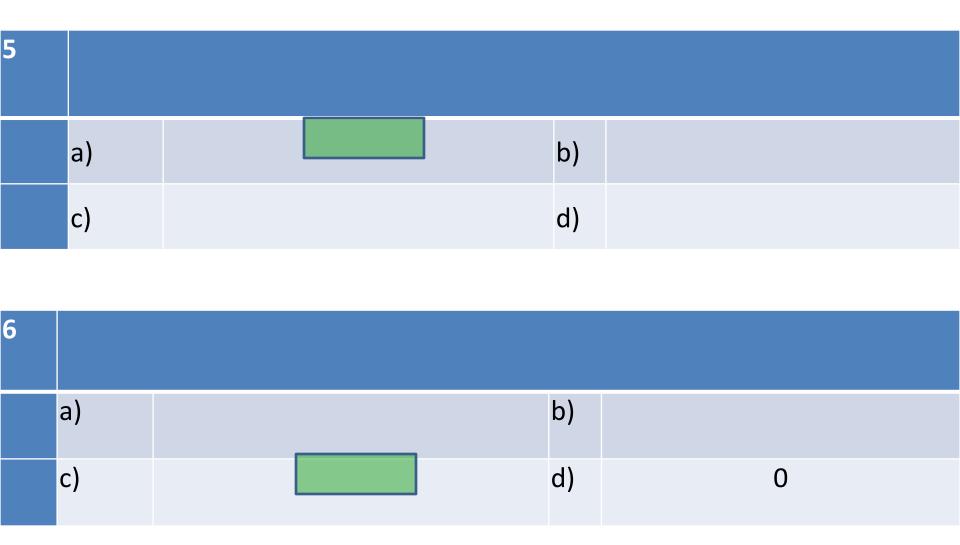
$$= \bigcap_{i=0}^{m/2} \bigcap_{i=0}^{m/2$$

$$\int_{0}^{2\pi} \sin^6 x \cos^4 x \, dx = \frac{3\pi}{128}$$

3]
$$\Box_0^{2\square}$$
 $\Box_0^{2\square}$ $\Box_0^{2\square}$

5]
$$\Box_0^{\square}$$
 \Box_0^{\square} \Box_0^{\square}

6]
$$\Box_0^{\square} \Box_0^{\square} \Box_0^{\square} = \Box_0^{\square} \Box_0^$$



Sol:
$$U_n = \int_0^{\pi/4} tan^n x \, dx : U_{n+1} = \int_0^{\pi/4} tan^{n+1} x \, dx$$

$$\therefore U_{n+1} = \int_0^{\pi/4} \tan^{n-1} x \tan^2 x \, dx$$

$$= \int_0^{\pi/4} \tan^{n-1} x \, (\sec^2 x - 1) \, dx$$

$$\therefore \ U_{n+1} = \int_0^{\pi/4} tan^{n-1}x \ sec^2x \ dx - \int_0^{\pi/4} tan^{n-1}x \ dx$$
$$= \left[\frac{tan^n x}{n}\right]_0^{\pi/4} - U_{n-1}$$

$$\therefore$$
 n ($U_{n+1} + U_{n-1}$) = 1

If
$$U_n = \int_0^{\pi/4} tan^n x \, dx$$
 and if n ($U_{n+1} + U_{n-1}$) = 1 then $\int_0^{\pi/4} tan^6 x \, dx$ is $\frac{13}{15} - \frac{\pi}{4}$

$$U_{n+1} = \frac{1}{n} - U_{n-1}$$

Put n = 5,3,1

$$U_{6} = \left(\frac{1}{5} - U_{4}\right), U_{4} = \left(\frac{1}{3} - U_{2}\right), U_{2} = (1 - U_{0})$$

$$U_{0} = \int_{0}^{\frac{\pi}{4}} dx \qquad U_{6} = \frac{13}{15} - \frac{\pi}{4}$$

$$\mathbf{U}_{n} = \frac{\mathbf{U}_{n-2} + \mathbf{U}_{n-2}}{(\mathbf{U}_{n-1})} \mathbf{I}_{n-2}$$

Then
$$\Box_0^{11/4}$$
 $\boxed{}$ $= \frac{28}{15}$

Now let
$$U_n = \Box_0^{\square / 4}$$

And
$$: U_n = \left[\frac{sec^{n-2}x tanx}{(n-1)}\right]_0^{\pi/4} + \frac{(n-2)}{(n-1)} U_{n-2}$$

$$= \frac{\left(\sqrt{2}\right)^{n-2}}{(n-1)} + \frac{(n-2)}{(n-1)} U_{n-2} - \dots (1)$$

To find $\int_0^{\pi/4} sec^6 x \ dx$, put n = 6, 4, 2 in (2)

$$U_6 = \frac{\mathbb{E}^{\frac{7}{2}} \mathbb{I}^{6-2}}{(6-1)} + \frac{\mathbb{I}^{6-2}}{(6-1)} U_{6-2} = \frac{\mathbb{E}^{\frac{7}{2}} \mathbb{I}^4}{5} + \frac{4}{5} U_4$$
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$$U_4 = \frac{(\sqrt{2})^2}{3} + \frac{2}{3} U_2$$
, $U_2 = \int_0^{\pi/4} sec^2 x \ dx = [tanx]_0^{\pi/4} = 1$

$$\therefore \ \ U_6 = \int_0^{\pi/4} sec^6 x \ dx = \left[\frac{\left(\sqrt{2}\right)^4}{5} + \frac{4}{5} \left(\frac{\left(\sqrt{2}\right)^2}{3} + \frac{2}{3} \right) \right]$$

$$= \left[\frac{4}{5} + \frac{4}{5} \left(\frac{2}{3} + \frac{2}{3} (1) \right) \right]$$

$$\therefore \int_0^{\pi/4} sec^6 x \ dx = \frac{28}{15}$$

If I_n =
$$\prod_0^{\pi/2} \cos^n x \cos x \, dx$$
 , Prove that I_n = $\frac{1}{2}$ I_{n-1} = $\frac{\pi}{2^{n+1}}$

Sol:
$$I_n = \prod_0^{\pi/2} \cos^n x \cos nx \, dx$$

$$= \left[\cos^{n} x \, \frac{\sin nx}{n} \, \right]_{0}^{\pi/2} - \int_{0}^{\pi/2} n \, \cos^{n-1} x \, (-\sin x) \, \frac{\sin nx}{n} \, dx$$
$$= 0 + \int_{0}^{\pi/2} \cos^{n-1} x \, \sin nx \, dx$$

 \because cos (n - 1) x = cosnx cosx + sinnx sinx

$$I_n = \int_0^{\pi/2} \cos^{n-1}x \left(\cos\left(n-1\right)x - \cos x \cos x\right) dx$$

$$= \int_0^{\pi/2} \cos^{n-1} x \cos((n-1)x) dx - \int_0^{\pi/2} \cos^n x \cos x dx$$

$$= I_{n-1} - I_n$$
, $: I_n = I_{n-1} - I_n$ or $: I_n = \frac{1}{2} I_{n-1}$

$$\ \, : \quad I_n = \frac{1}{2} \ I_{n-1} \ \, , \quad I_{n-1} = \frac{1}{2} \ I_{n-2} \quad , \quad \quad I_{n-2} = \frac{1}{2} \ I_{n-3} \quad , ----- \, , \, I_1 = \frac{1}{2} \ I_0$$

$$I_n = \frac{1}{2^n} I_0$$
 and $I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2}$

Hence
$$I_n = \frac{\pi}{2^{n+1}}$$

If
$$I_n = \prod_{0}^{\pi/4} \frac{\sin \left(\frac{\pi}{2} n - 1 \right) x}{\sin x} dx$$
, Prove that $\left(\prod_{n+1} - I_n \right) = \sin \left(\frac{\pi}{2} \right)$

Hence find I₃.

$$\begin{split} \text{Sol}: \ I_n &= \int_0^{\pi/4} \frac{\sin{(2\,n-1)x}}{\sin{x}} \ dx \\ & \therefore \ (\ I_{n+1} - \ I_n \,) = \int_0^{\pi/4} \frac{\sin{(2\,n+1)x}}{\sin{x}} \ dx - \int_0^{\pi/4} \frac{\sin{(2\,n-1)x}}{\sin{x}} \ dx \\ &= \int_0^{\pi/4} \frac{\sin{(2\,n+1)x} - \sin{(2\,n-1)x}}{\sin{x}} \ dx \\ &= \int_0^{\pi/4} \frac{\sin{(2\,n+1)x} - \sin{(2\,n-1)x}}{\sin{x}} \ dx \\ \end{split}$$

$$\begin{aligned} \text{We know} \quad & \sin{c} - \sin{d} = 2\cos{\frac{c+d}{2}}\sin{\frac{c-d}{2}} \\ &= \int_0^{\pi/4} \frac{2\cos{2nx}\sin{x}}{\sin{x}} \ dx = 2\left[\frac{\sin{2nx}}{2n}\right]_0^{\pi/4} \end{aligned}$$

$$\therefore n(I_{n+1} - I_n) = \sin(\frac{n\pi}{2})$$
 -----(1)

To find I_3 , from (1),

Put
$$n = 2$$
, $2(I_3 - I_2) = \sin(\pi) \implies I_3 - I_2 = 0 -----(2)$

Similarly putting
$$n = 1$$
, we get $I_2 - I_1 = 1 \implies I_2 = I_1 + 1 - (3)$

$$I_1 = \frac{\pi}{4} \qquad \therefore I_3 = \frac{\pi}{4} + 1$$

If $I_n = \prod_0^{\pi/2} x \cdot \cos^n x \cdot dx$, and if $I_n = -\frac{1}{n^2} + \frac{n-1}{n} I_{n-2}$, then I_3 is

$$\Rightarrow I_{n} = \frac{-1}{n^{2}} + \frac{(n-1)}{n} I_{n-2} - \cdots (1)$$
From (1), $I_{3} = \frac{-1}{9} + \frac{2}{3} I_{1}$
and $I_{1} = \int_{0}^{\pi/2} x \cdot \cos x \, dx = \left[x \sin x + \cos x\right]_{0}^{\pi/2} = \frac{\pi}{2} - 1$

$$I_{3} = \frac{-1}{9} + \frac{2}{3} I_{1} = I_{3} = \left\{\frac{-1}{9} + \frac{2}{3} \left(\frac{\pi}{2} - 1\right)\right\}$$

$$\therefore I_{3} = \frac{\pi}{3} - \frac{7}{9}$$

Evaluate $\int_4^6 \sin^4 \pi x \cdot \cos^2 2\pi x \cdot dx$

Sol: Let
$$I = \int_4^6 \sin^4 \pi x \cdot \cos^2 2\pi x \cdot dx$$

Put
$$\pi x = 4\pi + t \Rightarrow \pi dx = dt$$

$$\sin(\pi x) = \sin(4\pi + t) = \sin t \text{ and } \cos(2\pi x) = \cos(8\pi + 2t) = \cos(2t)$$

X	4	6
t	0	

$$I = \frac{1}{\pi} \int_0^{2\pi} \sin^4 t \cdot \cos^2 2t \cdot dt = \frac{1}{\pi} \int_0^{2\pi} \sin^4 t \cdot (1 - 2\sin^2 t)^2 \cdot dt$$

$$= \frac{1}{\pi} \int_0^{2\pi} \sin^4 t \cdot (1 - 4\sin^2 t + 4\sin^4 t) \cdot dt$$

$$\therefore \int_0^{2\pi} \sin^n x \, dx = \begin{cases} = 4 \int_0^{\pi/2} \sin^n x \, dx \text{, if n is even.} \\ = 0 & \text{, if n is odd.} \end{cases}$$

$$= \frac{4}{\pi} \int_0^{\pi/2} \sin^4 t - 4\sin^6 t + 4\sin^8 t \cdot dt$$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \frac{[(n-1) \text{subtract } 2 \text{.....} 2 \text{ or } 1]}{[(n) \text{ subtract } 2 \text{.....} 2 \text{ or } 1]} \times \left(\frac{\pi}{2}\right) \text{ if n is even.}$$

$$\therefore I = \frac{4}{\pi} \left[\frac{3.1}{4.2} \frac{\pi}{2} - 4 \frac{5.3.1}{6.4.2} \frac{\pi}{2} + 4 \frac{7.5.3.1}{8.6.4.2} \frac{\pi}{2} \right]$$

$$= \frac{4}{\pi} \frac{3.1}{4.2} \frac{\pi}{2} \left[1 - \frac{4.5}{6} + \frac{4.7.5}{8.6} \right] \qquad \therefore I = \frac{7}{16}$$

Sol:
$$I = \int_0^{\pi} x \cdot \sin^7 x \cdot \cos^4 x \cdot dx$$
 -------(1)
 $= \int_0^{\pi} (\pi - x) \cdot \sin^7 (\pi - x) \cdot \cos^4 (\pi - x) \cdot dx$
 $\therefore \int_0^a f(x) \cdot dx = \int_0^a f(a - x) \cdot dx$
And using, $\sin(\pi - x) = \sin x$; $\cos(\pi - x) = -\cos x$
 $= \int_0^{\pi} (\pi - x) \cdot \sin^7 x \cdot \cos^4 x \cdot dx$ ------(2)
Adding (1) and (2)
 $2I = \int_0^{\pi} \pi \cdot \sin^7 x \cdot \cos^4 x \cdot dx$

$$\because \int_0^{\pi} \sin^m x \cos^n x \, dx = \begin{cases} = 2 \int_0^{\pi/2} \sin^m x \cos^n x \, dx, & \text{if } n \text{ even }, \text{for any } m. \\ = 0. & \text{if } n \text{ is odd.} \end{cases}$$

$$=2\pi\int_0^{\pi/2}\sin^7x\cdot\cos^4x\cdot dx$$

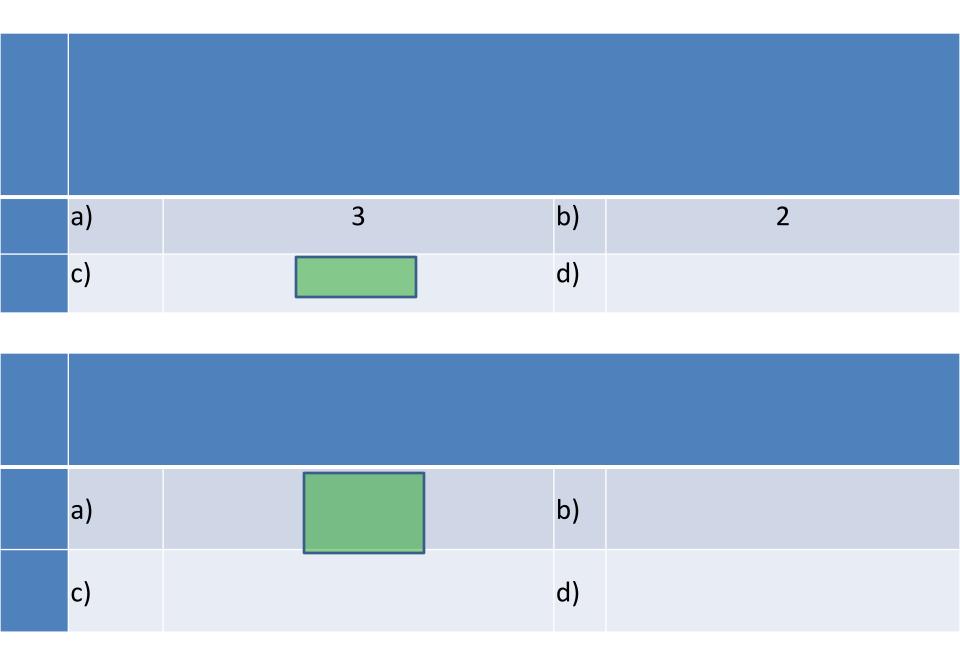
Using the formula,

$$\int_{0}^{\pi/2} \sin^{m} x \cos^{n} x \, dx$$

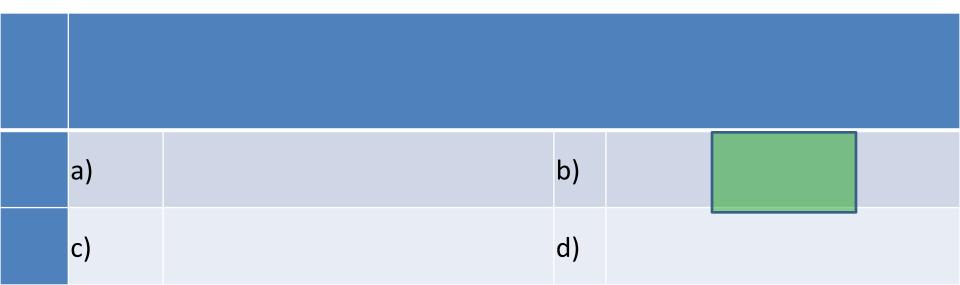
$$= \left\{ \frac{[(m-1) \text{subtract } 2 \dots 2 \text{ or } 1].[(n-1) \text{subtract } 2 \dots 2 \text{ or } 1]}{[(m+n) \text{subtract } 2 \dots 2 \text{ or } 1]} \right\} \times 1$$

If m & n both are not even

$$I = \pi \frac{(6.4.2)(3.1)}{11.9.7.5.3.1} = \frac{16 \,\pi}{1155}$$



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Beta and Gamma Function

1. Gamma Function

Definition: The integral $\int_{0}^{\infty} e^{-x} x^{n-1} dx$ is called as Gamma function and denoted by $m = \int_{0}^{\infty} e^{-x} x^{n-1} dx$ (n > 0)

Properties:

Proof:
$$\boxed{1} = \int_{0}^{\infty} e^{-x} x^{1-1} dx$$

$$= \int_{0}^{\infty} e^{-x} x^{0} dx = \int_{0}^{\infty} e^{-x} dx$$

$$= [-e^{-x}]_{0}^{\infty} = -e^{-\infty} + e^{0} = 0 + 1 = 1. = R.H.S.$$

2. Re duction formula: n+1 = n n= n!, if n is + ve integer

Proof:
$$n+1 = \int_{0}^{\infty} e^{-x} x^{n} dx$$

$$= \left[x^{n} \left(-e^{-x} \right) \right]_{0}^{\infty} - \int_{0}^{\infty} \left(-e^{-x} \right) n \ x^{n-1} dx$$

$$= \left[0 - 0 \right] + n \int_{0}^{\infty} e^{-x} x^{n-1} dx$$

$$= n \overline{n}$$

$$3.\boxed{0}=\infty$$

$$\operatorname{Pr} oof: \square \overline{n} = \frac{|n+1|}{n}$$

$$\therefore \quad \boxed{0} = \frac{|0+1|}{0} = \infty$$

$$4. \left| \frac{1}{2} \right| = \sqrt{\pi}$$

$$6. \int_{0}^{\infty} e^{-kx} x^{n-1} dx = \frac{n}{k^n}$$

Problems

Prove the following

1.
$$\int_{0}^{\infty} e^{-\sqrt[3]{x}} \sqrt{x} dx = \frac{315}{16} \sqrt{\pi}$$

$$Sol: Put \ x = t^3 \implies dx = 3t^2 dt$$

$$L.H.S. = \int_{0}^{\infty} e^{-t} t^{\frac{3}{2}} (3t^{2}) dt$$

$$=3\int_{0}^{\infty}e^{-t}t^{\frac{7}{2}}dt$$

$$= 3 \frac{7}{2} + 1$$

$$= 3\left(\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}\right) \times \sqrt{\pi}$$

$$= \underbrace{\frac{315}{\text{Other Subjects: https://www.studymedia.in/fe/notes}}^{315}}_{\text{Other Subjects: https://www.studymedia.in/fe/notes}}$$

$$\int_{0}^{\infty} e^{-2x^{2}} x^{7} dx = \frac{3}{16}$$

Proof: Put
$$2x^2 = t \implies 4x \, dx = dt$$

$$L.H.S. = \int_{0}^{\infty} e^{-2x^2} x^6 \left(x dx\right)$$

$$= \int_{0}^{\infty} e^{-t} \left(\frac{t}{2}\right)^{3} \left(\frac{dt}{4}\right)$$

$$=\frac{1}{32}\int_{0}^{\infty}e^{-t} t^{3} dt$$

$$=\frac{1}{32} \boxed{4} = \frac{1}{32} \times 3! = \frac{6}{32} = \frac{3}{16}$$

$$3. \int_{0}^{\infty} \frac{x^4}{4^x} dx = \frac{24}{(\log 4)^5}$$

Proof: Let $4 = e^m$

$$L.H.S. = \int_{0}^{\infty} \frac{x^4}{e^{mx}} dx = \int_{0}^{\infty} e^{-mx} x^4 dx$$

 $Put \quad mx = t \implies m \, dx = dt$

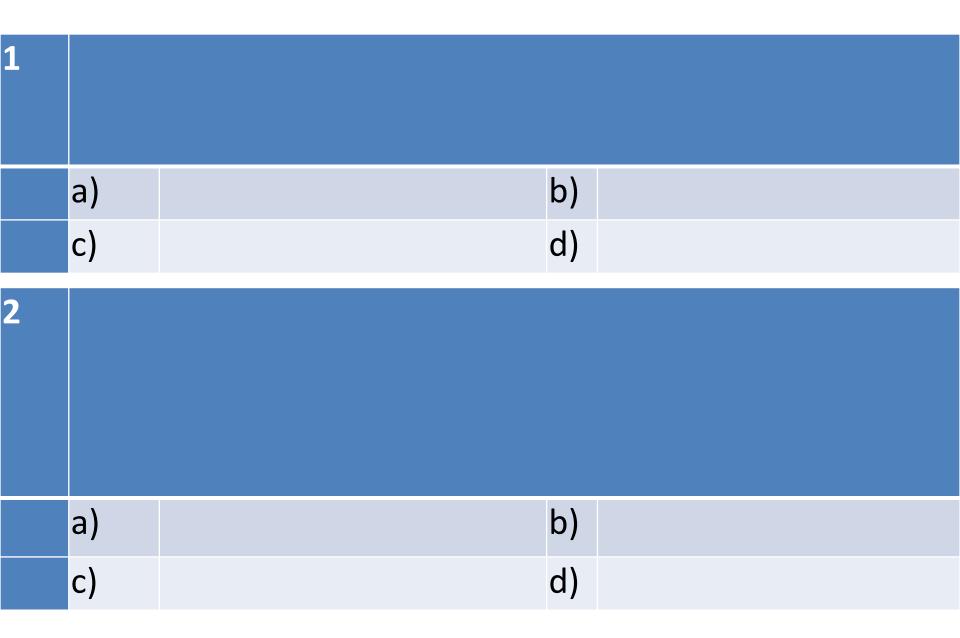
$$L.H.S. = \int_{0}^{\infty} e^{-t} \left(\frac{t}{m}\right)^{4} \frac{dt}{m}$$

$$=\frac{1}{m^5}\int_{0}^{\infty}e^{-t}t^4dt$$

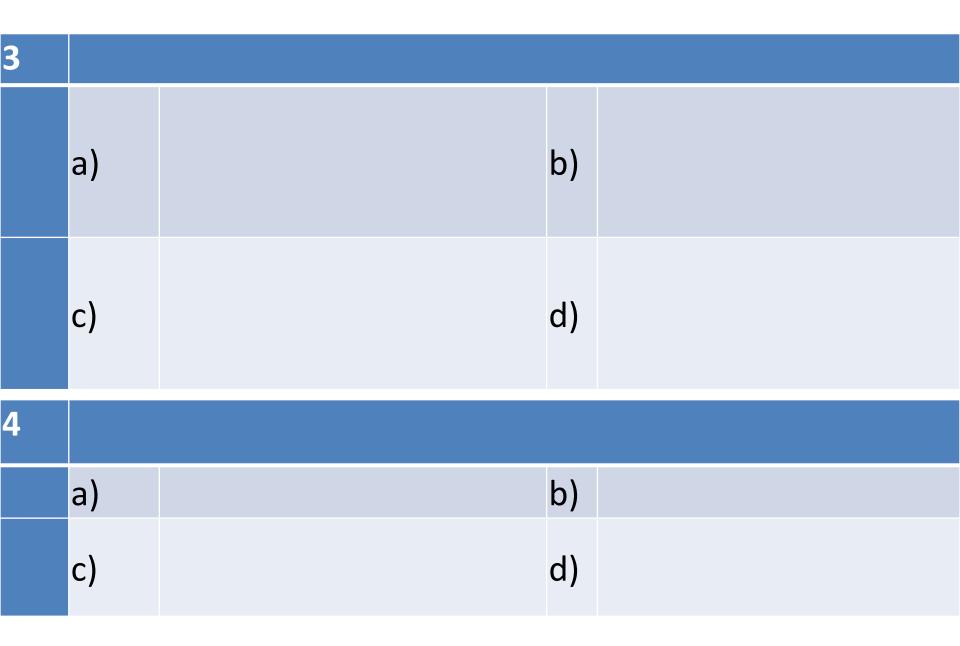
$$= \frac{1}{m^5} \sqrt{5} = \frac{1}{m^5} 4! = \frac{24}{(\log 4)^5}$$

$$4. \qquad \int_0^1 \frac{1}{\sqrt{-\log x}} \, dx = \sqrt{\pi}$$

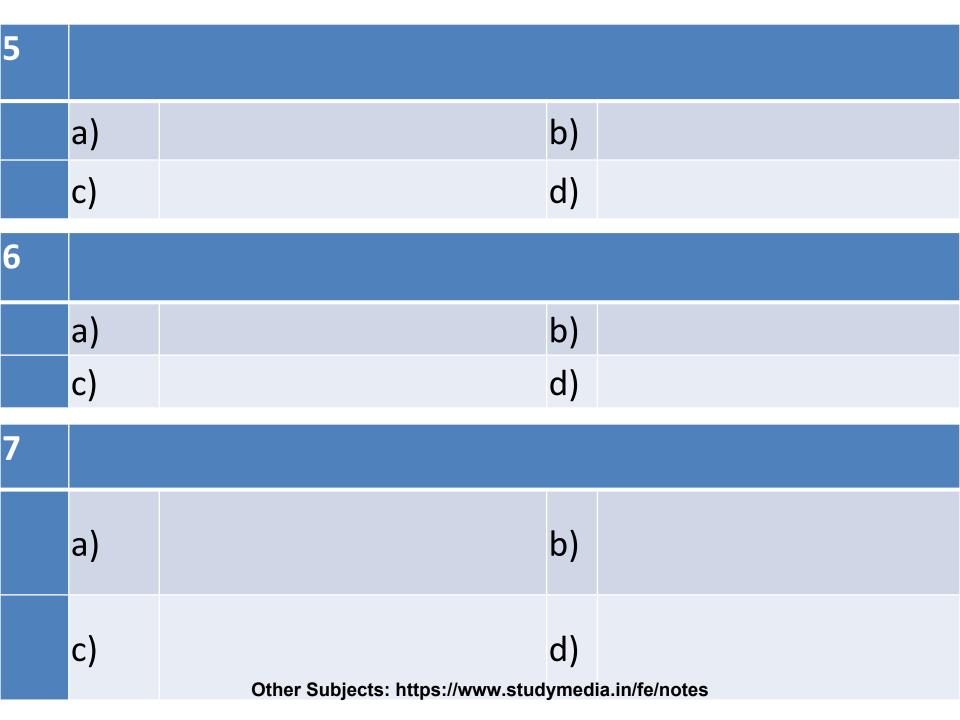
$$Put - log x = t$$



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Beta Function.

Definition:
$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$
; where m,n are +ve integers

Properties Of Beta Function.

1.
$$\beta(m,n) = \beta(n,m)$$

2.
$$\beta(m,n) = \int_{0}^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

3.
$$\beta(m,n) = 2 \int_{0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \ d\theta$$

4.
$$\int_{0}^{\frac{\pi}{2}} \sin^{p}\theta \cos^{q}\theta \, d\theta = \frac{1}{2}\beta \left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

5. Relation Between Beta and Gamma

Function.

6. Legendre's duplication formula:

$$\overline{m} \quad \overline{m+1/2} = \frac{\xi \overline{\square}}{2^2 \square -1} \quad \overline{2} \square$$

PROBLEMS.

Ex1:Prove that
$$\Box_0^1 \Box_1 - \Box_1^{-1} \Box_{-1}^{-1} = \frac{\Box_1 \Box_1}{\Box_{-1} + \Box_{-1}}$$

Sol: Let
$$I = \Box_0^1 \Box 1 - \Box_0^1 / \Box_0^{\square} \Box$$

Put
$$x^{1/n} = t$$
, $x = t^n$; $dx = n t^{n-1} dt$

X	0	1
t	0	1

$$I = \int_0^1 (1-t)^m \ n \ t^{n-1} \ dt = n \int_0^1 \ t^{n-1} \cdot (1-t)^m \ dt$$

$$I = n\beta(n, m+1) = n \frac{\sqrt{n (m+1)}}{\sqrt{n+m+1}} = \frac{m! n!}{(m+n)!}$$

Ex2:Prove that
$$\Box_0^{\infty} \frac{\square}{1+\square^4} = \frac{\square}{2 \, \xi \, \overline{2}}$$

Sol: Let
$$I = \bigcup_{0}^{\infty} \frac{1}{1+ ||||^{4}}$$
, Put $x^{2} = |||||||||$, $|||| = \frac{1}{2} |||||||||^{-1/2} \cdot ||||||||||||$

0	
0	$\pi/2$

$$I = \begin{bmatrix} \frac{1}{2} & \frac{1}{1+||\mathbf{m}||^2 + |\mathbf{m}|^2} \end{bmatrix} \underbrace{1}_{1+|\mathbf{m}||\mathbf{m}|^2 + |\mathbf{m}|^2} \underbrace{1}_{1+|\mathbf{m}|^2 + |\mathbf{m}|^2} \underbrace{1}_{1+|\mathbf{m}||\mathbf{m}|^2 + |\mathbf{m}|^2} \underbrace{1}_{1$$

$$= \frac{1}{2} \int_0^{\pi/2} (\tan \theta)^{-1/2} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \cot^{1/2} \theta \ d\theta = \frac{1}{2} \int_0^{\pi/2} \sin^{-1/2} \theta \cos^{1/2} \theta \ d\theta$$

$$I = \frac{1}{2} \frac{1}{2} \left[\prod \frac{-\frac{1}{2}+1}{2} , \frac{\frac{1}{2}+1}{2} \right]$$

$$=\frac{1}{4} \boxed{1} \frac{1}{4}, \frac{3}{4} \boxed{1}$$

$$=\frac{1}{4} \frac{\boxed{1/_4} \boxed{3/_4}}{\boxed{1}}$$

$$=\frac{1}{4} \quad \boxed{\frac{1}{4} \quad (1-\frac{1}{4})} \quad \because \boxed{1-\boxed{}} = \frac{\boxed{}}{\boxed{}}$$

$$=\frac{1}{4}\frac{\Box}{\sin\Box\Box_{4}}=\frac{\xi^{\overline{2}}\Box}{4}=\frac{\Box}{2\xi^{\overline{2}}}$$

Where m and n are positive integers.

Sol: Let
$$I = \prod_{1}^{1} \prod_{1} + \prod_{2}^{2} \prod_{1} - \prod_{3}^{2} \prod_{4}^{2}$$

Put 1 + x = 2t, dx = 2dt

х	- 1	1
t	0	1

$$I = \prod_{i=1}^{1} 2 \boxed{12} \boxed{12} - 2 \boxed{12} \boxed{12} 2 dt$$

$$=2^{\square+\square+1}\,\square^1\square\square\square\square-\square\square\square dt = 2^{\square+\square+1}\,\square\square\square+1\,,\,\square\square+1\,\square$$

$$I = 2^{\square + \square + 1} \frac{\square + 1 \square \square (\square + 1)}{\square \square + \square \square + 2}$$
$$= 2^{\square + \square + 1} \cdot \frac{\square ! \square !}{\square \square + \square \square !}$$

Ex4: Evaluate
$$\Box_3^7 \Box \Box - 3\Box^{1/4} \Box 7 - \Box \Box^{1/4} \Box \Box$$

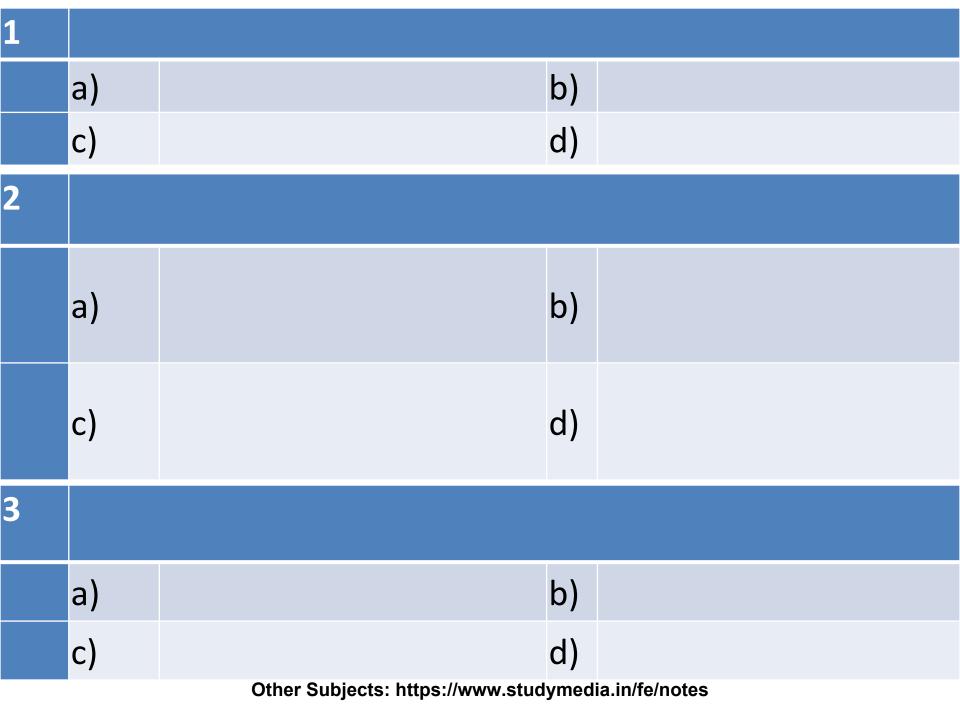
Sol: Let
$$I = \frac{1}{3} \frac{1}{1} - 3\frac{1}{4} \frac{1}{7} - \frac{1}{1} \frac{1}{4} \frac{1}{1}$$

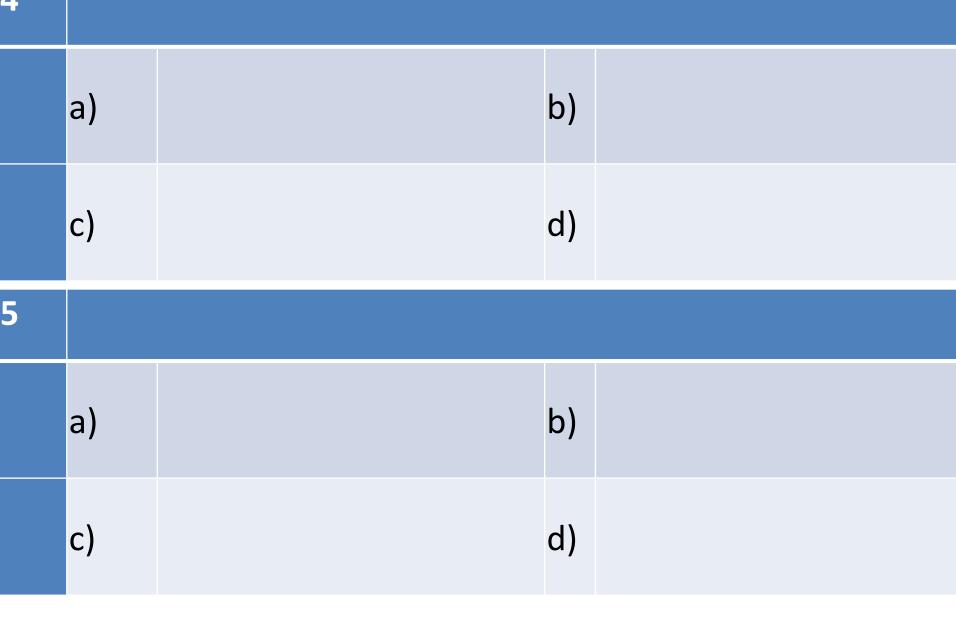
Put x-3=4t, dx=4dt

X	3	7
t	0	1

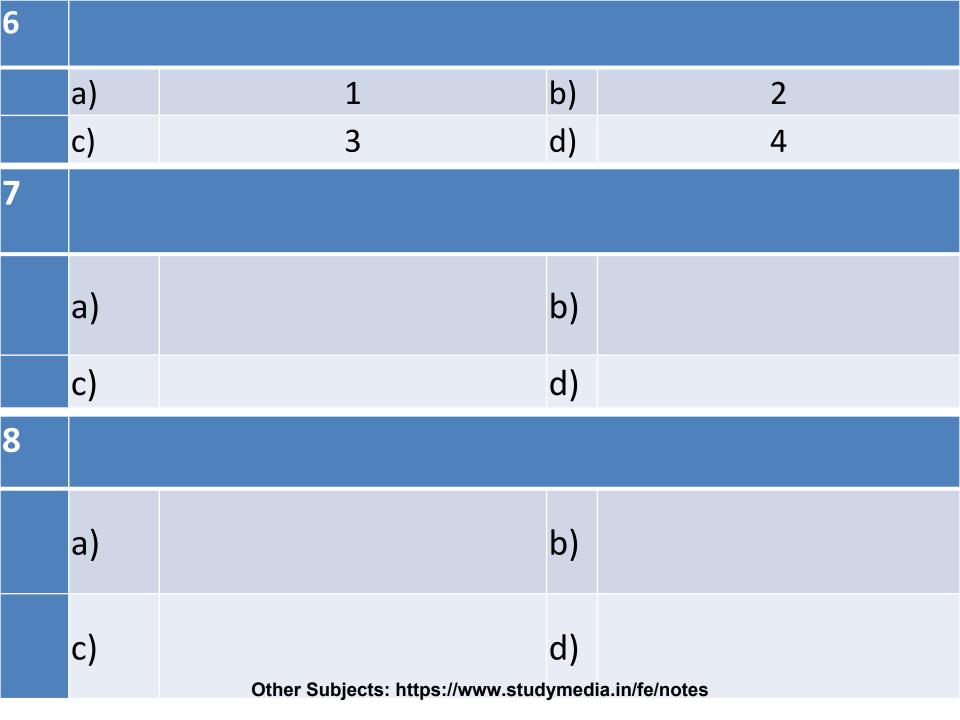
$$I = \Box_{3}^{7} \Box_{4} \Box_{1}^{1/4} \Box_{4} - 4 \Box_{1}^{1/4} \Box_{4} = 8 \Box_{3}^{7} \Box_{1}^{1/4} \Box_{1} - \Box_{1}^{1/4} \Box_{1}$$

$$I = 4^{2} \square \lceil \frac{5}{4}, \frac{5}{4} \rceil = 8 \quad \frac{\boxed{5/4} \quad \boxed{5/4}}{\boxed{5/2}} = 16 \quad \frac{\boxed{\frac{1}{4}} \quad \boxed{1/4} \quad \boxed{\frac{2}{1}}}{\boxed{\frac{3}{2}} \quad \frac{1}{2} \xi \boxed{\square}} = \frac{4}{3\xi \boxed{\square}} \boxed{\boxed{1/4}} \quad \boxed{\boxed{1/4}}$$





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Differentiation Under Integral Sign (DUIS)

1.

Introduction to variables, additional parameters

$$I(\alpha) = \int_{a}^{b} f(x, \alpha) dx$$
 $\longrightarrow \alpha = \text{Parameter}, x = \text{Variable}.$

Rule 1: Integrals with constant limits.

If
$$I(\alpha) = \int_{a}^{b} f(x,\alpha) dx$$
 then
$$\left[\frac{d}{d\alpha} \int_{a}^{b} f(x,\alpha) dx = \int_{a}^{b} \frac{\partial}{\partial \alpha} f(x,\alpha) dx \right]$$

a & b constants

► LHS derivative ► Partial derivative RHS

Ex 1: Show that
$$\int_{0}^{1} \frac{x^{a}-1}{\log x} dx = \log (a+1)$$
, $(a \ge 0)$

2013

Solution: Let
$$I(a) = \int_{0}^{1} \frac{x^{a} - 1}{\log x} dx$$

$$I'(a) = \frac{d}{da} \int_{0}^{1} \frac{x^{a} - 1}{\log x} dx = \int_{0}^{1} \frac{\partial}{\partial a} \frac{x^{a} - 1}{\log x} dx$$

applying DUIS

$$I'(a) = \int_{0}^{1} \frac{x^{a} \log x}{\log x} dx = \int_{0}^{1} x^{a} dx$$

Integrating w.r.t.
$$x = \left[\frac{x^{a+1}}{a+1}\right]_0^1 = \frac{1}{a+1}$$
 $I'(a) = \frac{1}{a+1}$

Integrating w.r.t. a $I(a) = \log(a+1) + c$

Put
$$a = 0$$
, $I(0) = \log(0+1) + c \rightarrow c = 0$

$$\therefore I(a) = \log(a+1), a \ge 0$$

Ex 2: Prove that
$$\int_{0}^{\infty} e^{-x^{2}} \cos 2\lambda x dx = \frac{\sqrt{\pi}}{2} e^{-\lambda^{2}}$$

Solution: Let
$$I(\lambda) = \int_{0}^{\infty} e^{-x^2} \cos 2\lambda \, x \, dx$$
; Where λ is a parameter

By DUIS
$$I'(\lambda) = \int_{0}^{\infty} \frac{\partial}{\partial \lambda} e^{-x^{2}} \cos 2\lambda x dx$$

$$I'(\lambda) = \int_{0}^{\infty} e^{-x^{2}} (-2x) \cdot \sin 2\lambda x \, dx = \int_{0}^{\infty} (\sin 2\lambda x) \cdot (e^{-x^{2}} (-2x)) \, dx$$

 \Box Integration by parts & using $\int e^{f(x)} f'(x) dx = e^{f(x)}$

$$= \left[\sin 2\lambda \, x \, e^{-x^2} \right]_0^\infty - \int_0^\infty 2\lambda e^{-x^2} \cos 2\lambda \, x \, dx$$

$$= (0-0) - 2\lambda I(\lambda) \quad \therefore \frac{I'(\lambda)}{I(\lambda)} = -2\lambda$$

Integrating w.r.t. λ , $\log I(\lambda) = -\lambda^2 + c$ i.e. $I(\lambda) = e^{-\lambda^2} e^{-c}$

and for
$$\lambda = 0$$
, $I(0) = \int_{0}^{\infty} e^{-x^{2}} . dx = \frac{\sqrt{\pi}}{2} = e^{c}$ $\therefore I(\lambda) = \frac{\sqrt{\pi}}{2} e^{-\lambda^{2}}$

PROBLEMS INVOLVING TWO PARAMETERS.

Procedure is exactly same as that of problems involving one parameter.

4: Show that
$$\int_{0}^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx = \log \frac{b}{a}$$
; a and b are two parameters.

Solution: Let
$$I(a) = \int_{0}^{\infty} \frac{e^{-ax} - e^{-bx}}{x} dx$$
 Differentiating w.r.t.a and applying **DUIS**

$$I'(a) = \int_{0}^{\infty} \frac{\partial}{\partial a} \frac{e^{-ax} - e^{-bx}}{x} dx = \int_{0}^{\infty} \frac{(-x)e^{-ax} - 0}{x} dx = -\int_{0}^{\infty} e^{-ax} dx = -\left[\frac{e^{-ax}}{-a}\right]_{0}^{\infty} = -\frac{1}{a}$$

$$\therefore I'(a) = -\frac{1}{a}$$
 Integrating w.r.t. a; I(a) = - log a + c

Put a = b; I(b) = - log b + c, c = log b since I(b) = 0

Thus I(a) = - log a + logb,

$$I(a) = \log \frac{b}{a}$$

Using DUIS prove that:

$$\int_0^\infty \frac{e^{-x} - e^{-ax}}{x \sec x} \ dx = \frac{1}{2} \log(\frac{a^2 + 1}{2}) \text{ for } a > 0$$

May 2015

DUIS Rule II: Integrals with limits as function of the parameter (Leibnitz's Rule)

l₹

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha) dx$$

Then

$$\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x, \alpha) \, dx + f(b, \alpha) \frac{db}{d\alpha} - f(a, \alpha) \frac{da}{d\alpha}$$

Solution:
$$f(x) = \int_0^x (x - t)^2 G(t) dt$$

$$\frac{df}{dx} = \frac{d}{dx} \int_0^x (x - t)^2 G(t) dt$$

By DUIS =
$$\int_0^x \frac{\partial}{\partial x} (x - t)^2 G(t) dt + \frac{dx}{dx} \cdot (0) - \frac{d0}{dx} (x^2 G(0))$$

$$\therefore \frac{df}{dx} = \int_0^x 2(x-t) \ G(t)dt$$

Again applying DUIS

$$\frac{d^2f}{dx^2} = \int_0^x \frac{\partial}{\partial x} 2(x-t) G(t)dt + 0 - 0$$

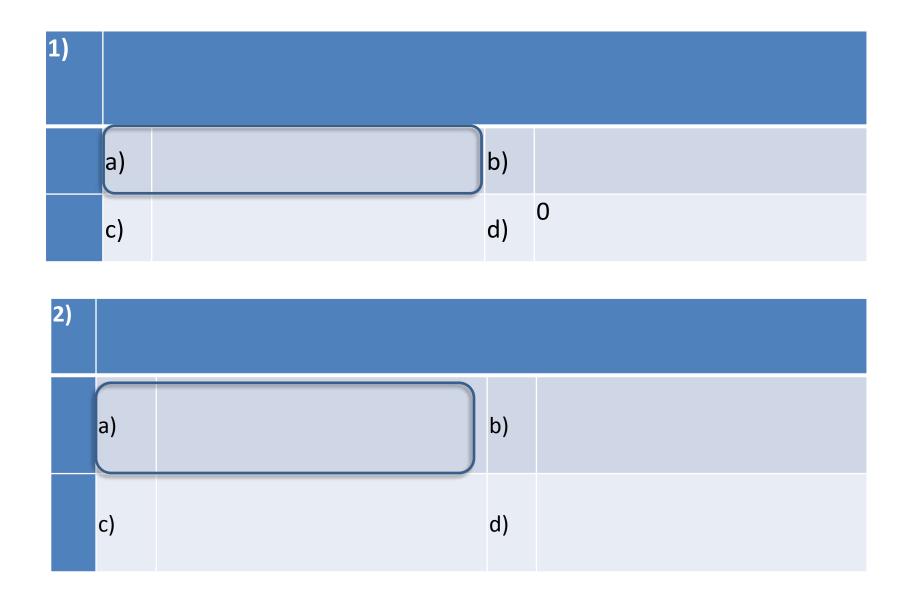
$$\frac{d^2f}{dx^2} = \int_0^x 2G(t)dt$$

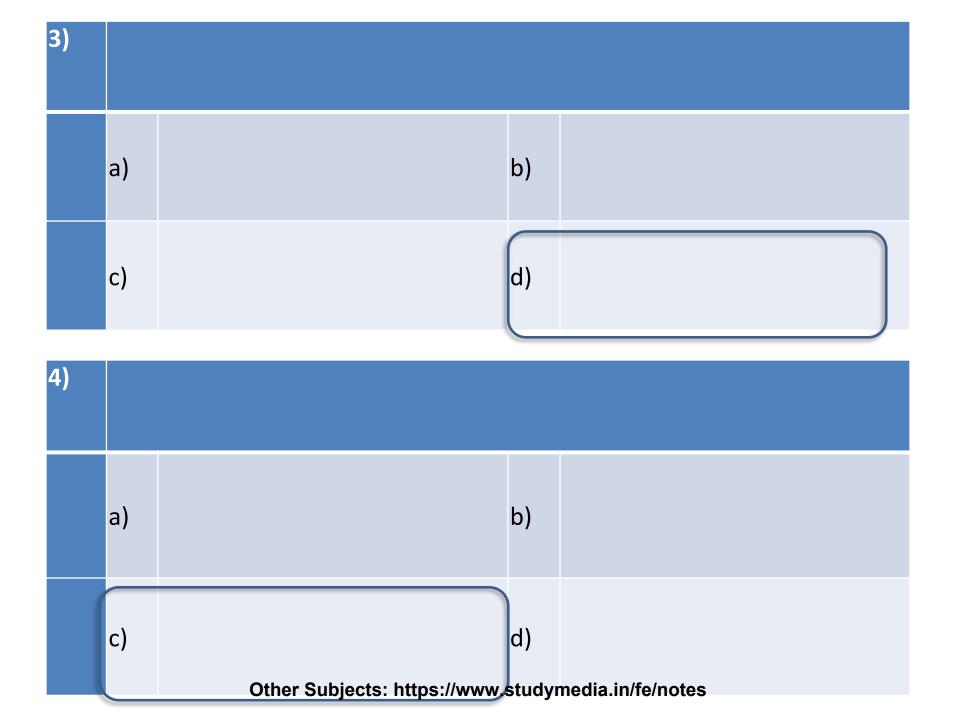
Again applying DUIS,

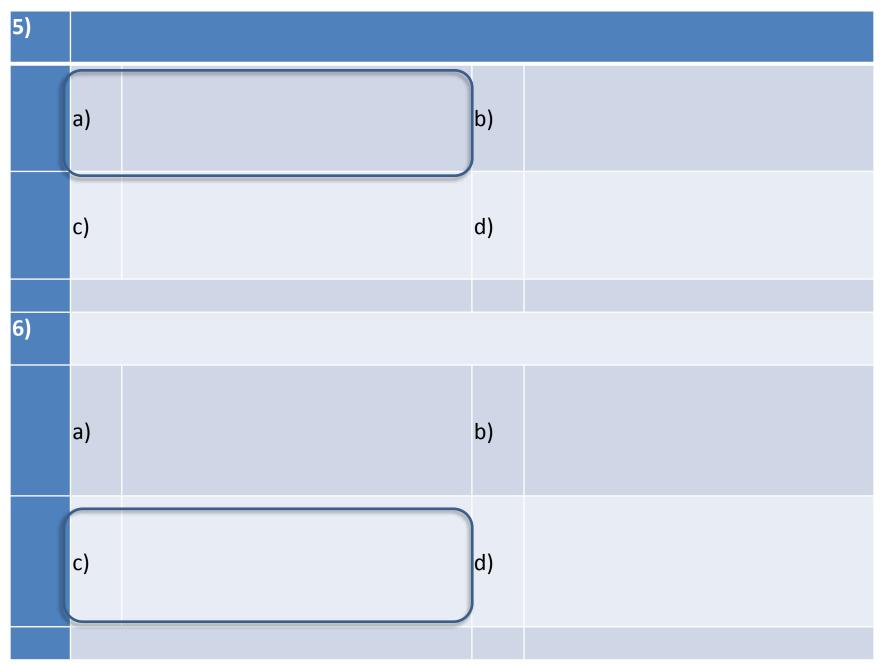
$$\frac{d^3f}{dx^3} = \int_0^x \frac{\partial}{\partial x} 2G(t)dt + \frac{dx}{dx} \cdot 2G(x) - 0$$

$$= 0 + 2G(x) - 0$$

$$\therefore \frac{d^3f}{dx^3} = 2G(x)$$







Other Subjects: https://www.studymedia.in/fe/notes

Error Function

Error function of x is denoted by erf(x) and is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-u^{2}} du$$

Complementary Error Function

Complementary error function of x is denoted by $\operatorname{erfc}(x)$ and defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^{2}} du$$

Properties of error function

$$1. \operatorname{erf}(\infty) = 1$$

$$2. \operatorname{erf}(0) = 0$$

$$3.\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$$

4.Error function is an odd function

$$i.e \operatorname{erf}(-x) = -\operatorname{erf}(x)$$

5. Series for error function is

$$erf(x) = \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \Box \right]$$

$$\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \left[erf(b) - erf(a) \right]$$

Solution: We know that
$$1 = \text{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2} dx$$

= $\frac{2}{\sqrt{\pi}} \left[\int_0^a e^{-x^2} dx + \int_a^b e^{-x^2} dx + \int_b^\infty e^{-x^2} dx \right]$

By definition of erf (x) and $erf_c(x)$

$$1 = \operatorname{erf}(a) + \frac{2}{\sqrt{\pi}} \int_{a}^{b} e^{-x^{2}} dx + \operatorname{er} f_{c}(b)$$

$$[1 - \operatorname{er} f_{c}(b)] - \operatorname{erf}(a) = \frac{2}{\sqrt{\pi}} \int_{a}^{b} e^{-x^{2}} dx$$

$$\operatorname{erf}(b) - \operatorname{erf}(a) = \frac{2}{\sqrt{\pi}} \int_{a}^{b} e^{-x^{2}} dx$$
i.e.
$$\int_{a}^{b} e^{-x^{2}} dx = \frac{\sqrt{\pi}}{2} \left[\operatorname{erf}(b) - \operatorname{erf}(a) \right]$$

Prove that $\operatorname{erf}_{c} \square + \operatorname{erf}_{c} \square \times \square = 2$

But
$$\square \square \square = -\square \square \square \square \square$$

Adding eqn (1) and (2) we get

$$\boxed{ } (-x) + \boxed{ } (x) = 2$$

4 : Show that
$$\frac{d}{dx} erf(ax) = \frac{2a}{\sqrt{\pi}} e^{-a^2x^2}$$
 ,Hence evaluate $\int_0^t erf(ax) dx$

$$As \operatorname{erf}(ax) = \frac{2}{\sqrt{\pi}} \int_{0}^{ax} e^{-u^{2}} du$$

$$\therefore \frac{d}{dx} \operatorname{erf}(ax) = \frac{d}{dx} \left(\frac{2}{\sqrt{\pi}} \int_{0}^{ax} e^{-u^{2}} du \right)$$

$$= \frac{2}{\sqrt{\pi}} \left[\int_{0}^{ax} \frac{\partial}{\partial x} e^{-u^{2}} du + e^{-a^{2}x^{2}} \frac{d}{dx} (ax) - 0 \right]$$

$$\therefore \frac{d}{dx} \operatorname{erf}(ax) = \frac{2ae^{-a^2x^2}}{\sqrt{\pi}}$$

Now find Integration of Error Function

$$\int_{0}^{t} \operatorname{erf}(ax)dx = \int_{0}^{t} 1 \cdot \operatorname{erf}(ax)dx$$

$$= [\operatorname{erf}(ax) \cdot x]_{0}^{t} - \int_{0}^{t} \left(\frac{d}{dx}\operatorname{erf}(ax)\right)xdx$$

$$= \operatorname{erf}(at) \cdot t - \int_{0}^{t} \frac{2ae^{-a^{2}x^{2}}}{\sqrt{\pi}}xdx$$

$$\int_{0}^{t} \operatorname{erf}(ax)dx = \operatorname{erf}(at) \cdot t + \frac{1}{a\sqrt{\pi}} \left[e^{-a^{2}t^{2}} - 1\right]$$

1)				
	a)		b)	
	c)		d)	
2)	Error f	unction is		
	a)	Even function	b)	Neither ever nor odd function
	c)	Odd function	d)	Constant function
3)				
	a)		b)	
	c)		d)	

a)	3	b)	2
c)	1	d)	0