Helicity formalism
for the
Amplitude Analysis
of the 3-body B-decays

Part – I

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CONTENTS' OVERVIEW

ARBITRARY ROTATION - I

Consider the following two reference frames Oxyz & Ox'y'z' :

x,y,z are fixed cartesian coordinates
x',y',z' is the coordinate system attached to the physical system which rotates w.r.t. xyz frame

The Euler angles (α, β, γ) **measure the rotation of x'y'z' w.r.t. xyz frame.**

It can be proved that an arbitrary complete rotation $R(\alpha, \beta, \gamma)$ **can be built by means of 3 subsequent rotations on a single axis:**

1) **Rotation about z-axis by an angle** α **: it takes Oy into Ou**

2) **Rotation about u-axis by an angle** β **: it takes Oz into Oz'**

3) **Rotation about z'-axis by an angle** γ **: it takes Ou into Oy'**

The complete rotation is therefore : $R(\alpha, \beta, \gamma) = R_{z'}(\gamma) R_u(\beta) R_z(\alpha)$

ARBITRARY ROTATION - II

Using the fact that a rotation about a given axis $\hat{n}$ is generated by the angular momentum operator $\vec{J}\cdot\hat{n}$ we can write the previous expression for $R(\alpha,\beta,\gamma)$ in the following way:

$$R(\alpha,\beta,\gamma) = R_{z'}(\gamma)R_u(\beta)R_z(\alpha) = e^{-i\gamma J_{z'}}e^{-i\beta J_u}e^{-i\alpha J_z}$$

However this expression is not very useful since it is not expressed in terms of rotations about the original coordinate axes x,y,z. The useful expression having these features is:

$$R(\alpha,\beta,\gamma) = e^{-i\alpha J_z}e^{-i\beta J y}e^{-i\gamma J_z}$$

Rotation operator in z-y-z convention, right-handed frame, right-hand screw rule

where the arbitrary rotation is expressed in terms of the rotations about the axes of the xyz frame.

This can be demonstrated - like in the following 2 slides - by exploiting the unitarity of the rotations operators.

The last expression is very important because it expresses an arbitrary rotation specified by the 3 Euler angles (α,β,γ) in terms of rotations about the fixed axes of the xyz coordinate frame.

It can be noticed that this expression appears in almost every paper on the helicity formalism and, as we will appreciate later, it is the origin of the Wigner functions.

ARBITRARY ROTATION - III

DEMONSTRATION

Let us introduce the generic… $|a\rangle$ vector state representing some physical system

Q observable

and recall that under a rotation R they transform in this way:

$$\boxed{|a\rangle \xrightarrow{R} |a'\rangle = R|a\rangle}$$

$$\langle a|Q|a\rangle = \langle a'|Q'|a'\rangle = \langle a|R^{+}Q'R|a\rangle$$

From the last we can write $Q = R^{+}Q'R$ and also $Q' = RQR^{+}$ (*)

We can apply the last rule to the sequence of the 3 rotations that build up $R(\alpha,\beta,\gamma)$!
From the (*) we can write:

$$J_u = R_z(\alpha)J_y R_z^{+}(\alpha) \qquad \textbf{(1)}$$

$$J_{z'} = \left[R_u(\beta)R_z(\alpha)\right]J_z\left[R_u(\beta)R_z(\alpha)\right]^{+}$$

Taking into account that the rotation around z-axis preserves J_z that is to say that $R_z(\alpha)J_z R_z^{+}(\alpha) = J_z R_z(\alpha)R_z^{+}(\alpha) = J_z$, we can rewrite the last expression as:

$$J_{z'} = R_u(\beta)J_z R_u^{+}(\beta) \qquad \textbf{(2)}$$

ARBITRARY ROTATION - IV

By using (2) we can write: $$R_{z'}(\gamma) = e^{-i\gamma J_{z'}} = e^{-i\gamma\left[R_u(\beta) J_z R_u^+(\beta)\right]}$$

and exploiting the unitarity of rotation operators we re-write: $$R_{z'}(\gamma) = e^{-i\gamma J_{z'}} = R_u(\beta) e^{-i\gamma J_z} R_u^+(\beta) \qquad \textbf{(3)}$$

Similarly we get: $$R_u(\beta) = e^{-i\beta J_u} \overset{\boxed{(1)}}{=} e^{-i\beta\left[R_z(\alpha) J_y R_z^+(\alpha)\right]} = R_z(\alpha)\, e^{-i\beta J_y} R_z^+(\alpha) \qquad \textbf{(4)}$$

Thus - finally - we can write :

$$R(\alpha,\beta,\gamma) = R_{z'}(\gamma) R_u(\beta) R_z(\alpha) \overset{\boxed{(3)}}{=} R_u(\beta) e^{-i\gamma J_z} \cancel{\boxed{R_u^+(\beta) R_u(\beta)}} R_z(\alpha) \overset{\boxed{(4)}}{=}$$

$$= R_z(\alpha)\, e^{-i\beta J_y} R_z^+(\alpha)\, e^{-i\gamma J_z} R_z(\alpha) = R_z(\alpha)\, e^{-i\beta J_y}\; e^{-i\gamma J_z} \cancel{\boxed{R_z^+(\alpha) R_z(\alpha)}} =$$

$$= R_z(\alpha)\, e^{-i\beta J_y}\; e^{-i\gamma J_z} = e^{-i\alpha J_z}\; e^{-i\beta J_y}\; e^{-i\gamma J_z}$$

ROTATION OF VECTORS - I

In order to derive the explicit matrix (3x3) representation $\Re$ of the unitary rotation operator R we consider the rotation of vectors in a multi-dimensional (3D) space.

Considered the reference frames Oxyz and O'x'y'z', the second rotating w.r.t the first, the effect of a generic unitary rotation is to take the basis unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ that point along Ox,Oy,Oz into three unit vectors $\hat{e}'_1, \hat{e}'_2, \hat{e}'_3$ that point along O'x', O'y', O'z':

$$\hat{e}' = R\hat{e} \quad \Leftrightarrow \quad \boxed{\hat{e}'_j = \sum_{i=1}^{3} \Re_{ij} \hat{e}_i \quad , \quad j = 1,2,3}$$

To evaluate the matrix elements $\Re_{ij}$ we use the orthogonality of the basis unit vectors:

$$\hat{e}_k \cdot \hat{e}'_j = \sum_{i=1}^{3} \Re_{ij} \hat{e}_k \cdot \hat{e}_i = \sum_{i=1}^{3} \delta_{ki} \Re_{ij} = \Re_{kj} \quad , \quad \forall k, j = 1,2,3 \quad \Rightarrow \quad \Re_{ij} = \hat{e}_i \cdot \hat{e}'_j$$

It is then possible to evaluate these matrix elements in terms of the Euler angles.

ROTATION OF VECTORS - II

Consider now the generic vector that can be expressed in the basis unit vectors: $\vec{V} = \sum_{j=1}^{3} V_j \hat{e}_j$
Under a rotation *R* it trasforms as:

$$\vec{V} \xrightarrow{R} \vec{V}' = R\left[\vec{V}\right] = R\left[\sum_{j-1}^{3} V_j \hat{e}_j\right] = \sum_{j=1}^{3} V_j R\left[\hat{e}_j\right] = \sum_{j=1}^{3} V_j \hat{e}'_j = \sum_{j=1}^{3} V_j \sum_{i=1}^{3} \Re_{ij} \hat{e}_i = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} \Re_{ij} V_j\right) \hat{e}_i$$

The components of the rotated vector in the xyz coordinate system are obtained by comparison …

… the previous result $\vec{V}' = \sum_{i=1}^{3}\left(\sum_{j=1}^{3} \Re_{ij} V_j\right) \hat{e}_i$ **with the general** $\vec{V}' = \sum_{i=1}^{3} V'_i \hat{e}_i$ **:**

$$\boxed{V'_i = \sum_{j=1}^{3} \Re_{ij} V_j \quad , \quad i = 1,2,3}$$

Note the difference between the rule for transforming a basis vector and this rule for obtaining the new components of a vector: the indices of the matrix are interchanged w.r.t the summing index.

ROTATION OF ANGULAR MOMENTUM EIGENSTATES - I

The angular momentum eigenstates $|j,m\rangle_{m=-j,\ldots,+j}$ transform irreducibly under rotations because $\left[R,J^2\right]=0$! Thus the matrix representation of the rotation is labeled by the total angular momentum *j* : D^j .

The effect of the rotation *R* on the basis states $|j,m\rangle$ can be written by exploiting the parallelism:

$$\hat{e}' = R\hat{e} \quad \Leftrightarrow \quad |j,m'\rangle = R|j,m\rangle \quad \forall j$$

$$\hat{e}'_j = \sum_{i=1}^{3} \Re_{ij}\hat{e}_i \quad \forall j \quad \Leftrightarrow \quad |j,m'\rangle = \sum_{m''=-j}^{j} D^j_{m''m'}|j,m''\rangle \quad \forall j$$

$$\hat{e}_k \cdot \hat{e}'_j = \sum_{i=1}^{3} \Re_{ij}\hat{e}_k \cdot \hat{e}_i = \quad \Leftrightarrow \quad \langle j,m''\|j,m'\rangle = \langle j,m''|\sum_{m''=-j}^{j} D^j_{m''m'}|j,m''\rangle$$

$$= \sum_{i=1}^{3} \delta_{ki}\Re_{ij} = \Re_{kj} \quad \forall k,j \qquad = \sum_{m''=-j}^{j} D^j_{m''m'}\langle j,m''\|j,m''\rangle = D^j_{m''m'} \; \forall j,m''$$

ROTATION OF ANGULAR MOMENTUM EIGENSTATES - II

Let us re-write again the previous result:

$$\left|j,m'\right\rangle = R(\alpha,\beta,\gamma)\left|j,m\right\rangle = \sum_{m'=-j}^{j} D^{j}_{m'm}(\alpha,\beta,\gamma)\left|j,m'\right\rangle \quad \forall j$$

$$\left\langle j,m''\middle\|j,m'\right\rangle = \left\langle j,m''\right|R(\alpha,\beta,\gamma)\left|j,m\right\rangle = \sum_{m''=-j}^{j} D^{j}_{m''m'}(\alpha,\beta,\gamma)\left\langle j,m''\middle\|j,m''\right\rangle = D^{j}_{m''m'}(\alpha,\beta,\gamma)$$

$$\forall j,m''$$

These equations express the rotated angular momentum state in terms of the original basis angular momentum eigenstates; specifically this is provided by the 1st equation whereas the 2nd allows to calculate the matrix representation of the rotation, as we will effectively see in next slide.

ROTATION OF ANGULAR MOMENTUM EIGENSTATES - III

Let's conveniently re-write the last equation of previous slide:

$$\langle j,m'|R(\alpha,\beta,\gamma)|j,m\rangle = D^{j}_{m'm}(\alpha,\beta,\gamma)$$

In order to calculate $D^{j}_{m'm}(\alpha,\beta,\gamma)$ **we need to express explicitely the rotation operator borrowing one of the expressions derived previously for the rotation operator** $R(\alpha,\beta,\gamma)$**, namely:**

$$R(\alpha,\beta,\gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$$

Therefore: $D^{j}_{m'm}(\alpha,\beta,\gamma) = \langle j,m'|e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}|j,m\rangle$

Since a rotation around z-axis doesn't change J_z **and thus** ***m, m'*** **:**

$$D^{j}_{m'm}(\alpha,\beta,\gamma) = e^{-i\alpha m'}\langle j,m'|e^{-i\beta J_y}|j,m\rangle e^{-i\gamma m} \equiv e^{-i\alpha m'} d^{j}_{m'm}(\beta) e^{-i\gamma m}$$

The matrix element $d^{j}_{m'm}(\beta) = \langle j,m'|e^{-i\beta J_y}|j,m\rangle$ **is given by Wigner Formula [A]:**

$$d^{j}_{m'm}(\beta) = \sum_n \left\{ \frac{(-1)^n\left[(j+m)!(j-m)!(j+m')!(j-m')!\right]^{1/2}}{(j-m'-n)!(J+m-n)!(n+m'-m)!n!} \cdot \left(\cos\frac{\beta}{2}\right)^{2j+m-m'-2n} \cdot \left(-\sin\frac{\beta}{2}\right)^{m'-m+2n} \right\}$$

The sum includes all integers ***n*** **for which all of the arguments of the factorials are positive.**

WIGNER MATRICES

Let us summarize the last slides.

The Wigner D-matrix is a square matrix of dimension 2j+1 with general element

$$D^j_{m'm}(\alpha,\beta,\gamma)=\langle j,m'|R(\alpha,\beta,\gamma)|j,m\rangle \equiv e^{-i\alpha m'} d^j_{m'm}(\beta) e^{-i\gamma m}$$

and the matrix with general element $d^j_{m'm}(\beta)=\langle j,m'|e^{-i\beta J_y}|j,m\rangle$ **given by Wigner formula is known as Wigner's (small) d-matrix .**

The d-matrix elements are real and this realness is one of the reasons for which the z-y-z convention is typically preferred in quantum-mechanical applications.

WIGNER MATRICES and POLYNOMIALS - I

Relation to Jacobi polynomials:

The d-matrix elements are related to Jacobi polynomials $P_k^{(a,b)}(\cos\beta)$ with nonnegative a and b.[2] Let

$$k = \min(j+m,\, j-m,\, j+m',\, j-m').$$

$$\text{If} \quad k = \begin{cases} j+m: & a = m'-m; \quad \lambda = m'-m \\ j-m: & a = m-m'; \quad \lambda = 0 \\ j+m': & a = m-m'; \quad \lambda = 0 \\ j-m': & a = m'-m; \quad \lambda = m'-m \end{cases}$$

Then, with $b = 2j-2k-a$, the relation is

$$d^j_{m'm}(\beta) = (-1)^\lambda \binom{2j-k}{k+a}^{1/2} \binom{k+b}{b}^{-1/2} \left(\sin\frac{\beta}{2}\right)^a \left(\cos\frac{\beta}{2}\right)^b P_k^{(a,b)}(\cos\beta),$$

where $a, b \geq 0$.

WIGNER MATRICES and POLYNOMIALS - II

Relation to Legendre polynomials:

For integer values of l, the D-matrix elements with second index equal to zero are proportional to spherical harmonics and associated Legendre polynomials, normalized to unity and with Condon and Shortley phase convention:

$$D^\ell_{m0}(\alpha,\beta,0) = \sqrt{\frac{4\pi}{2\ell+1}} Y^{m*}_\ell(\beta,\alpha) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P^m_\ell(\cos\beta)\, e^{-im\alpha}$$

This implies the following relationship for the d-matrix:

$$d^\ell_{m0}(\beta) = \sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P^m_\ell(\cos\beta)$$

When both indices are set to zero, the Wigner D-matrix elements are given by ordinary Legendre polynomials:

$$D^\ell_{0,0}(\alpha,\beta,\gamma) = d^\ell_{0,0}(\beta) = P_\ell(\cos\beta).$$

In the present convention of Euler angles, α is a longitudinal angle and β is a colatitudinal angle (spherical polar angles in the physical definition of such angles). This is one of the reasons that the *z-y-z* convention is used frequently in molecular physics. From the time-reversal property of the Wigner D-matrix follows immediately

$$(Y^m_\ell)^* = (-1)^m Y^{-m}_\ell.$$

There exists a more general relationship to the spin-weighted spherical harmonics:

$$D^\ell_{-ms}(\alpha,\beta,-\gamma) = (-1)^m \sqrt{\frac{4\pi}{2\ell+1}}\, {}_sY_{\ell m}(\beta,\alpha) e^{is\gamma}.$$

Unitarity of rotation oerator

Before to explore the properties of the Wigner matrices it is useful to have in mind that the rotation operator is a unitary operator; in other words: $R^{-1} = R^{+}$ **.**
This can be proven as follows:

We know that : $R(\alpha,\beta,\gamma) = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z}$

Since ***(AB)⁺ =B⁺A⁺*** **we can write:**

Pauli representation

$$R^{+}(\alpha,\beta,\gamma) = (e^{-i\gamma J_z})^{+}(e^{-i\beta J_y})^{+}(e^{-i\alpha J_z})^{+} = e^{+i\gamma J_z^{+}} e^{+i\beta J_y^{+}} e^{+i\alpha J_z^{+}} = e^{+i\gamma J_z} e^{+i\beta J_y} e^{+i\alpha J_z}$$

On the other hand :

$$R^{-1}(\alpha,\beta,\gamma) = (e^{-i\gamma J_z})^{-1}(e^{-i\beta J_y})^{-1}(e^{-i\alpha J_z})^{-1} = e^{+i\gamma J_z} e^{+i\beta J_y} e^{+i\alpha J_z}$$

WIGNER MATRICES : Properties - I

P1) From the Wigner formula one gets: $\boxed{d^j_{m'm}(-\beta)} = (-1)^{m'-m} d^j_{m'm}(\beta)$

P2) We can also demonstrate that: $d^j_{m'm}(-\beta) = d^j_{m\,m'}(\beta)$
Indeed…

- for the properties of the inner product : $\langle a|O|a'\rangle = \langle a'|O^+|a\rangle^*$

- being the rotation an unitary operator it holds : $R^{-1} = R^+$

$$d^j_{m\,m'}(\beta) = \langle j,m|e^{-i\beta J_y}|j,m'\rangle = \langle j,m|R(0,\beta,0)|j,m'\rangle =$$

$$\langle j,m'|R^+(0,\beta,0)|j,m\rangle^* =$$

$$\langle j,m'|R^{-1}(0,\beta,0)|j,m\rangle^* \equiv \langle j,m'|(e^{-i\beta J_y})^{-1}|j,m\rangle^* =$$

$$= \langle j,m'|e^{-i(-\beta)J_y}|j,m\rangle^* = (d^j_{m'm}(-\beta))^* = \boxed{d^j_{m'm}(-\beta)}$$

d-matrix is real

P3) Putting together (P1) and (P2) one gets: $d^j_{m\,m'}(\beta) = (-1)^{m'-m} d^j_{m'm}(\beta)$

WIGNER MATRICES : Properties -II

P4) **Rewriting from few slides before :** $D^{j}_{m'm}\left(\alpha,\beta,\gamma\right)=e^{-i\alpha m'}d^{j}_{m'm}\left(\beta\right)e^{-i\gamma m}$

On the other hand from (P2) we have: $d^{j}_{m\,m'}(-\beta)=d^{j}_{m'm}(\beta)$

Thus: $D^{j}_{m'm}\left(\alpha,\beta,\gamma\right)=e^{-i\alpha m'}d^{j}_{m\,m'}\left(-\beta\right)e^{-i\gamma m}=e^{-i\gamma m}d^{j}_{m\,m'}\left(-\beta\right)e^{-i\alpha m'}=D^{j}_{m\,m'}(\gamma,-\beta,\alpha)$

P5) **From the Wigner formula one can also calculate** $d^{j}_{m'm}(\beta)$ **for** $\beta=\pi,2\pi$ **:**

$$d^{j}_{m'm}(\pi)=(-1)^{j-m}\delta_{m',-m}$$

$$d^{j}_{m'm}(2\pi)=(-1)^{2j}d^{j}_{m'm}(0)=(-1)^{2j}\delta_{m'm}$$

P6) **There is an useful orthogonality relation (that will be used later) :**

$$\int_0^{2\pi}d\alpha\int_0^{2\pi}d\gamma\int_0^{\pi}\sin\beta\,d\beta\left[D^{j*}_{m\,n}\left(\alpha,\beta,\gamma\right)D^{j}_{m'n'}\left(\alpha,\beta,\gamma\right)\right]=\frac{8\pi^2}{2j+1}\delta_{m\,m'}\delta_{n\,n'}\delta_{j\,j'}$$

List of WIGNER d-matrix elements

Using sign conventions of Wigner et al., the d-matrix elements for j=1/2, 1, 3/2, 2 are [B] :

for $j = 1/2$

- $d^{1/2}_{1/2,1/2} = \cos(\theta/2)$
- $d^{1/2}_{1/2,-1/2} = -\sin(\theta/2)$

for $j = 1$

- $d^{1}_{1,1} = \frac{1+\cos\theta}{2}$
- $d^{1}_{1,0} = \frac{-\sin\theta}{\sqrt{2}}$
- $d^{1}_{1,-1} = \frac{1-\cos\theta}{2}$
- $d^{1}_{0,0} = \cos\theta$

for $j = 3/2$

- $d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$
- $d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$
- $d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$
- $d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$
- $d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$
- $d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$

for $j = 2$ [3]

- $d^{2}_{2,2} = \frac{1}{4}(1+\cos\theta)^2$
- $d^{2}_{2,1} = -\frac{1}{2}\sin\theta\,(1+\cos\theta)$
- $d^{2}_{2,0} = \sqrt{\frac{3}{8}}\sin^2\theta$
- $d^{2}_{2,-1} = -\frac{1}{2}\sin\theta\,(1-\cos\theta)$
- $d^{2}_{2,-2} = \frac{1}{4}(1-\cos\theta)^2$
- $d^{2}_{1,1} = \frac{1}{2}\left(2\cos^2\theta+\cos\theta-1\right)$
- $d^{2}_{1,0} = -\sqrt{\frac{3}{8}}\sin 2\theta$
- $d^{2}_{1,-1} = \frac{1}{2}\left(-2\cos^2\theta+\cos\theta+1\right)$
- $d^{2}_{0,0} = \frac{1}{2}\left(3\cos^2\theta-1\right)$

Note : elements with swapped lower indices can be obtained with the property

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}.$$

Note : see also [C]

1-PARTICLE PLANE-WAVE HELICITY STATES - I

Let us consider the massive particle case and begin with the rest state where $\vec{p} = 0$.

Let us denote by
- S the SPIN OPERATOR
- Λ the HELICITY OPERATOR where $\Lambda = \vec{S} \cdot \frac{\vec{p}}{|\vec{p}|} \equiv \vec{S} \cdot \hat{p}$

What does $\hat{p}$ represent if we are in the rest frame ?
The direction in which we would apply a Lorentz boost to see the particle (together with its associated rest-frame) moving in an inertial frame.

In the rest frame the spin projection along an axis and the helicity λ are equivalent because it is always possible to consider $\hat{p}$ along the *z*-axis [in other words $\lambda = s_z$ since tipically the *z*-axis is chosen as the "spin-quantization axis"].

In the rest frame the rest state can be written as $|\vec{p} = 0, s, \lambda\rangle$ since the particle has spin s , spin projection $\lambda = s_z$, and the state can be expressed in the basis of the eigenstates of the operators $\vec{p}$, S^2, Λ .

The importance of the helicity (used to label the state) comes from the fact that the helicity operator is invariant under both generic rotations and boost along $\hat{p}$ of the state.

1-PARTICLE PLANE-WAVE HELICITY STATES - II

To obtain the generic state $|\vec{p},s,\lambda\rangle$ we need to :
1) rotate $|\vec{p}=0,s,\lambda\rangle$ so that its quantization axis points along the generic direction $\hat{p}(\vartheta,\varphi)$
2) apply a Lorentz boost along $\hat{p}(\vartheta,\varphi)$

We will demonstrate in next-to-next slide [Dim-B] that this procedure to obtain the generic state from the rest state is completely equivalent to the one in which we firstly boost along the *z*-axis and then rotate to the generic direction !

In the rotation the rest frame is rotated in order to point to the generic (ϑ,φ) direction; in other words the $\hat{p}\equiv\hat{z}$ rotates to $\hat{p}(\vartheta,\varphi)$. In this rotation the spin of the system $\vec{S}$ rotates as well and thus the helicity $\lambda=\vec{S}\cdot\hat{p}$ remains invariant!

Therefore: $$|\vec{p},s,\lambda\rangle = \underbrace{L(\vec{p})}_{\text{BOOST}}\,\underbrace{R(\alpha=\varphi,\ \beta=\vartheta,\ \gamma=-\varphi)}_{\text{ROTATION}}\,|\vec{p}=0,s,\lambda\rangle$$

To determine a generic rotation two angles must be defined out of the three and thus there is the freedom to choose the third angle. The choice $\gamma=-\varphi$ by Jacob-Wick is conventional and in the next [Dim-A] slide we will show that it is convenient because as $\vartheta\to 0$ one gets:

$$R(\varphi,\vartheta=0,-\varphi)\,|\vec{p}=0,s,\lambda\rangle = |\vec{p}=0,s,\lambda\rangle \quad \textbf{independently of } \varphi$$

1-PARTICLE PLANE-WAVE HELICITY STATES - III

Dim-A

As a consequence of the helicity operator invariance under rotations/boosts it is possible to build relativistic basis vectors that are eigenstates of linear momentum and helicity or eigenstates of total angular momentum and helicity.

In the rest frame $\ell = 0$ **so that** $j = \ell + s = s$ **. Let us also use the notation** $m \equiv \lambda$ **.**

Let us remember that in general:

$$\left|j,m'\right\rangle = R(\alpha,\beta,\gamma)\left|j,m\right\rangle = \sum_{m'=-j}^{j} D^{j}_{m'm}(\alpha,\beta,\gamma)\left|j,m'\right\rangle \quad \textbf{(1)}$$

$$D^{j}_{m'm}(\alpha,\beta,\gamma) = e^{-i\alpha m'} d^{j}_{m'm}(\beta) e^{-i\gamma m} \quad \textbf{(2)}$$

Therefore:

$$R(\varphi,0,-\varphi)\ \left|\vec{p}=0,s,\lambda\right\rangle \equiv R(\varphi,0,-\varphi)\ \left|\vec{p}=0,j,m\right\rangle \overset{[1]}{=} \sum_{m'=-j}^{+j} D^{j}_{m'm}(\varphi,0,-\varphi)\left|\vec{p}=0,j,m'\right\rangle =$$

$$\overset{[2]}{=} \sum_{m'=-j}^{+j} e^{-i\varphi m'} d^{j}_{m'm}(0)\ e^{-i(-\varphi)m}\left|\vec{p}=0,j,m'\right\rangle =$$

$$= \sum_{m'=-j}^{+j} e^{-i\varphi m'} \underbrace{\left\langle j,m'\right| \cancel{e^{-i0J_y}} \left|j,m\right\rangle}_{\boxed{\delta_{m'\,m}}} e^{+i\varphi m}\left|\vec{p}=0,j,m'\right\rangle =$$

$$= \sum_{m'=-j}^{+j} e^{-i\varphi(m'-m)} \underbrace{\delta_{m'\,m}}_{\boxed{=1 \text{ if } m'=m}} \left|\vec{p}=0,j,m'\right\rangle = \left|\vec{p}=0,j,m\right\rangle \equiv \left|\vec{p}=0,j,\lambda\right\rangle$$

1-PARTICLE PLANE-WAVE HELICITY STATES - IV

Dim-B

Taking into account that it holds:

$$L(\vec{p}) = \underbrace{R(\varphi,\vartheta,-\varphi)}_{\text{ROTATION to generic direction}} \underbrace{L(\vec{p}_z = p\hat{z})}_{\text{BOOST along z-axis}} \underbrace{R^{-1}(\varphi,\vartheta,-\varphi)}_{\text{ROTATION back to z-axis}}$$

We can write:

$$\left|\vec{p},s,\lambda\right\rangle = L(\vec{p})\ R(\varphi,\vartheta,-\varphi)\ \left|\vec{p}=0,s,\lambda\right\rangle =$$

$$= [R(\varphi,\vartheta,-\varphi)\ L(\vec{p}_z = p\hat{z})\ \cancel{R^{-1}(\varphi,\vartheta,-\varphi)]\ R(\varphi,\vartheta,-\varphi)}\ \left|\vec{p}=0,s,\lambda\right\rangle =$$

$$= R(\varphi,\vartheta,-\varphi)\ L(\vec{p}_z = p\hat{z})\ \left|\vec{p}=0,s,\lambda\right\rangle$$

1-PARTICLE PLANE-WAVE HELICITY STATES - V

For the Lorentz invariant normalization we choose :

$$\left\langle \vec{p}',s',\lambda' \middle\| \vec{p},s,\lambda \right\rangle = (2\pi)^3 2E\ \delta^3(\vec{p}'-\vec{p})\ \delta_{s's}\delta_{\lambda'\lambda}$$

1-PARTICLE PLANE-WAVE HELICITY STATES - VI

Now, how the state $\left|-\vec{p},s,\lambda\right\rangle$ can be obtained?
A rotation by $\vartheta=\pi$ brings from *z*-axis to –*z*-axis !
However there is no unique choice for φ :
a rotation φ around *z*-axis is equivalent to a rotation $2\pi-\varphi$ around –*z*-axis .

This implies that the states $\left|-\vec{p}_z,s,\lambda\right\rangle$, $\left|\vec{p}_z,s,-\lambda\right\rangle$ differ by a phase (with no physical meaning).
In order to eliminate this degree of freedom the following condition is conventionally requested to be satisfied:

$$\lim_{-p_z\to 0}\left|-\vec{p}_z,s,\lambda\right\rangle=\lim_{p_z\to 0}\left|\vec{p}_z,s,-\lambda\right\rangle \quad (*)$$

Let us calculate the effect of a $R(0,\vartheta=\pi,0)=e^{-i\pi J_y}$ rotation on the rest state $\left|\vec{p}=0,s,\lambda\right\rangle$:

$$e^{-i\pi J_y}\left|\vec{p}=0,s,\lambda\right\rangle=$$

$$R(\alpha,\beta,\gamma)\left|j,m\right\rangle=\sum_{m'=-j}^{j}D^j_{m'm}(\alpha,\beta,\gamma)\left|j,m'\right\rangle$$

$$=\sum_{\lambda'}D^s_{\lambda'\lambda}(0,\pi,0)\left|\vec{p}=0,s,\lambda'\right\rangle=$$

$$D^j_{m'm}(\alpha,\beta,\gamma)=e^{-i\alpha m'}d^j_{m'm}(\beta)e^{-i\gamma m}$$

$$=\sum_{\lambda'}d^s_{\lambda'\lambda}(\pi)\left|\vec{p}=0,s,\lambda'\right\rangle=$$

$$d^j_{m'm}(\pi)=(-1)^{j-m}\delta_{m',-m}$$

$$=(-1)^{s-\lambda}\delta_{\lambda',-\lambda}\left|\vec{p}=0,s,\lambda'\right\rangle = (-1)^{s-\lambda}\left|\vec{p}=0,s,-\lambda\right\rangle$$

1-PARTICLE PLANE-WAVE HELICITY STATES - VII

Multiplying both members by $(-1)^{s-\lambda}$ **we can write equivalently :**

$$(-1)^{s-\lambda} e^{-i\pi J_y} \left|\vec{p}=0,s,\lambda\right\rangle = \underbrace{\cancel{(-1)^{2(s-\lambda)}}}_{\boxed{1}} \left|\vec{p}=0,s,-\lambda\right\rangle$$

Let us take now the limit of both members:

$$(-1)^{s-\lambda} e^{-i\pi J_y} \lim_{p_z \to 0} \left|\vec{p}_z,s,\lambda\right\rangle = \lim_{p_z \to 0} \left|\vec{p}_z,s,-\lambda\right\rangle$$

[(*)]

$$\lim_{-p_z \to 0} \left|-\vec{p}_z,s,\lambda\right\rangle$$

$$\Updownarrow$$

$$\lim_{p_z \to 0} \left|-\vec{p}_z,s,\lambda\right\rangle$$

Thus we obtain :

$$\left|-\vec{p}_z,s,\lambda\right\rangle = (-1)^{s-\lambda} e^{-i\pi J_y} \left|\vec{p}_z,s,\lambda\right\rangle$$

2-PARTICLE PLANE-WAVE HELICITY STATES - I

THIS ITEM & THEIR DEVELOPMENTS ARE FORWARDED TO NEXT PARTS

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[B] en.wikipedia.org/wiki/Wigner_D-matrix

[C] external link from [B] to the *PDG table of C.-G. coeff., spherical harmonics & d-functions*

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

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