Numerical Integration Methods
Applied Stochastic Processes (FIN 514)

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Numerical Integration

- In stochastic finance, numerical integral is often required to calculate the expectation (over the probability density):

$$E\left(f(X)\right) = \int_{-\infty}^{\infty} f(x)w(x)dx,$$

where $w(x)$ is the PDF of X.

Trapezoid Rule

One period (2 points):

$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+f(b)}{2}$$

Chained rule:

$$\int_a^b f(x)dx = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right) + \varepsilon,$$

where $x_0 = a$, $x_n = b$, and $\Delta x = (b-a)/n$.

The error is given by

$$\varepsilon = -\frac{(b-a)(\Delta x)^2}{12} f''(\xi) \quad \text{for} \quad \xi \in (a,b)$$

Simpson's Rule

One period (3 points):

$$\int_a^b f(x)dx \approx (b-a)\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}$$

For derivation, see 2023-24 exam.

Chained rule (for even N):

$$\int_a^b f(x)dx = \frac{\Delta x}{3}(f(x_0)+4f(x_1)+2f(x_2)+\cdots$$
$$+2f(x_{n-2})+4f(x_{n-1})+f(x_n))+\varepsilon,$$

where $x_0=a$, $x_n=b$, and $\Delta x=(b-a)/n$.

The error is given by

$$\varepsilon = -\frac{(b-a)(\Delta x)^4}{180}f^{(4)}(\xi) \quad \text{for} \quad \xi\in(a,b)$$

Lagrange Interpolation

- (WIKIPEDIA)
- We want to interpolate $f(x)$ with order $n-1$ polynomial :

$$\{(x_k, y_k := f(x_k)) : k = 1, \ldots, n \text{ and } a \le x_k \le b\}.$$

- Define

$$L(x) = (x - x_1)(x - x_2)\cdots(x - x_N)$$
$$\text{and} \quad l_k(x) = \frac{L(x)}{L'(x_k)(x - x_k)}$$

- Then, $l_k(x)$ has nice properties:

$$l_k(x_j) = 0 \text{ for } j \ne k \quad \text{and} \quad l_k(x_k) = 1 \;\; \text{(L'hopital rule).}$$

  So $\{l_k(x)\}$ serve as a basis of the interpolation.
- Interpolating polynomial:

$$p(x) = y_1 l_1(x) + \cdots + y_n l_n(x) \quad \text{satisfies} \quad p(x_k) = y_k = f(x_k).$$

Integral using Lagrange Interpolation

- How to approximate the integral

$$\int_a^b f(x)dx = ?$$

- We use the Lagrange polynomial $p(x)$:

$$f(x) \approx p(x) = y_1 l_1(x) + \cdots + y_n l_n(x).$$

- Pre-calculate the integral of the $l_k(x)$:

$$w_k = \int_a^b l_k(x)dx.$$

- Integral of $f(x)$ is approximated as

$$\int_a^b f(x)dx \approx \int_a^b p(x)dx = \sum_{k=1}^{n} y_k\, w_k.$$

- Exact when $f(x)$ is a polynomial of order up to $n-1$.

Gaussian Quadrature

- A set of point and weight $\{(x_k, w_k) : k = 1, \ldots n\}$ associated with a weight function $w(x)$.
- When $w(x)$ is a PDF of X,

$$E(f(X)) = \int_a^b f(x)w(x)dx \approx \sum_{k=1}^{n} f(x_k)w_k$$

- $\{(x_k, w_k)\}$ are the most optimal in that they exactly evaluate the integral when $f(x)$ is a polynomial of order up to $2n-1$.
- When $w(x)$ is a PDF, the moments up to order $2n-1$ is exact!

$$E(X^n) = \sum_{k=1}^{n} x_k^n w_k \quad \text{for} \quad n = 0,\ 1,\ \ldots,\ 2n-1$$

Quadratures and Special Polynomials

- How to find the points and weights with respect to $w(x)$?
- Orthogonal polynomials $p_n(x)$ (of degree n) are well-known for various $w(x)$:

$$\int_a^b p_i(x)p_j(x)w(x)dx = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j) \end{cases}$$

- The points $\{x_k\}$ of the Gaussian quadrature is chosen as the roots of $p_n(x)$:

$$p_n(x) = (x-x_1)\cdots(x-x_n).$$

- The weights $\{w_k\}$ are pre-calculated as

$$w_k = \frac{1}{p_n'(x_k)} \int_a^b \frac{p_n(x)}{(x-x_k)} w(x)dx.$$

- Integral of $f(x)$ is approximated as (exact up to order $n-1$):

$$\int_a^b f(x)dx \approx \int_a^b p_n(x)dx = \sum_{k=1}^n y_k\, w_k.$$

Why $2n-1$?

- $\{p_0(x), p_1(x), \cdots, p_n(x)\}$ form basis for polynomial of order up to n.
- If $f(x)$ is a polynomial of order $2n-1$, it can be divided by $p_n(x)$:

$$f(x) = q(x)p_n(x) + r(x),$$

where $q(x)$ and $r(x)$ are of order $n-1$.

- For any polynomial $f(x)$ of order $2n-1$, integral is exact!

$$\int_a^b f(x)w(x)dx = \underbrace{\int_a^b q(x)p_n(x)w(x)dx}_{=0} + \int_a^b r(x)w(x)dx$$

$$= \sum_{k=1}^{n} r(x_k)w_k.$$

- The first integral is zero because $p_n(x)$ is orthogonal to $q(x)$.
- The second integral is exact because $r(x)$ is of order $n-1$.

Various Gaussian Quadratures

Various Gaussian quadratures for $w(x)$:

Polynomial	Quadrature	(a, b)	$w(x)$	Distribution
Hermite	Gauss–Hermite	$(-\infty, \infty)$	$n(x)$	Normal
Legendre	Gauss–Legendre	$[-1, 1]$	1	Uniform
(Gen) Laguerre	Gauss–Laguerre	$[0, \infty)$	$x^{a-1}e^{-x}$	Gamma
Jacobi	Gauss–Jacobi	$(-1, 1)$	$(x-1)^\alpha$ $\cdot(x+1)^\beta$	Beta

- See Gaussian quadrature (WIKIPEDIA)