Derivations of IMU Calibration Equations

Setting up the Jacobians

In order to run the LM minimization, you need to define the jacobian of the error you want to minimize with respect to the parameters. For the first LM run, our error tern is $e = \|g\|^2 - \|T^a K^a (a^S + b^a)\|^2$. The parameters here are $[\alpha_{yz}, \alpha_{zy}, \alpha_{zx}, s^a_x, s^a_y, s^a_z, b^a_x, b^a_y, b^a_z]$, which means the jacobian will look like this:

$$\begin{bmatrix} \frac{\partial e_1}{\partial \alpha_{yz}} & \frac{\partial e_1}{\partial \alpha_{zy}} & \frac{\partial e_1}{\partial \alpha_{zx}} & \frac{\partial e_1}{\partial s^a_x} & \frac{\partial e_1}{\partial s^a_y} & \frac{\partial e_1}{\partial s^a_z} & \frac{\partial e_1}{\partial b^a_x} & \frac{\partial e_1}{\partial b^a_y} & \frac{\partial e_1}{\partial b^a_z} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial e_M}{\partial \alpha_{yz}} & \frac{\partial e_M}{\partial \alpha_{zy}} & \frac{\partial e_M}{\partial \alpha_{zx}} & \frac{\partial e_M}{\partial s^a_x} & \frac{\partial e_M}{\partial s^a_y} & \frac{\partial e_M}{\partial s^a_z} & \frac{\partial e_M}{\partial b^a_x} & \frac{\partial e_M}{\partial b^a_y} & \frac{\partial e_M}{\partial b^a_z} \end{bmatrix}$$

We can solve for these analytically.

$$\begin{aligned}
\frac{\partial e_i}{\partial \alpha_{yz}} &= \frac{\partial}{\partial \alpha_{yz}} \left(\|g\|^2 - \|T^a K^a (a^S + b^a)\|^2 \right) \\
&= \frac{\partial}{\partial \alpha_{yz}} \|g\|^2 - \frac{\partial}{\partial \alpha_{yz}} \|T^a K^a (a^S + b^a)\|^2 \\
&= -\frac{\partial}{\partial \alpha_{yz}} \|T^a K^a (a^S + b^a)\|^2 \\
&= -\frac{\partial}{\partial \alpha_{yz}} \left(\sqrt{\sum \left(T^a K^a (a^S + b^a) \right)^2} \right)^2 \\
&= -\frac{\partial}{\partial \alpha_{yz}} \sum \left(T^a K^a (a^S + b^a) \right)^2 \\
&= -\frac{\partial}{\partial \alpha_{yz}} \sum \left(\begin{bmatrix} 1 & -\alpha_{yz} & \alpha_{zy} \\ 0 & 1 & -\alpha_{zx} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^a_x & 0 & 0 \\ 0 & s^a_y & 0 \\ 0 & 0 & s^a_z \end{bmatrix} \left(\begin{bmatrix} a^s_{x,i} \\ a^s_{y,i} \\ a^s_{z,i} \end{bmatrix} + \begin{bmatrix} b^a_x \\ b^a_y \\ b^a_z \end{bmatrix} \right) \right)^2 \\
&= -\frac{\partial}{\partial \alpha_{yz}} \sum \left(\begin{bmatrix} 1 & -\alpha_{yz} & \alpha_{zy} \\ 0 & 1 & -\alpha_{zx} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^a_x & 0 & 0 \\ 0 & s^a_y & 0 \\ 0 & 0 & s^a_z \end{bmatrix} \begin{bmatrix} a^s_{x,i} + b^a_x \\ a^s_{y,i} + b^a_y \\ a^s_{z,i} + b^a_z \end{bmatrix} \right)^2 \\
&= -\frac{\partial}{\partial \alpha_{yz}} \sum \left(\begin{bmatrix} 1 & -\alpha_{yz} & \alpha_{zy} \\ 0 & 1 & -\alpha_{zx} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s^a_x(a^s_{x,i} + b^a_x) \\ s^a_y(a^s_{y,i} + b^a_y) \\ s^a_z(a^s_{z,i} + b^a_z) \end{bmatrix} \right)^2 \\
&= -\frac{\partial}{\partial \alpha_{yz}} \sum \left(\begin{bmatrix} s^a_x(a^s_{x,i} + b^a_x) - \alpha_{yz} s^a_y(a^s_{y,i} + b^a_y) + \alpha_{zy} s^a_z(a^s_{z,i} + b^a_z) \\ s^a_y(a^s_{y,i} + b^a_y) - \alpha_{zx} s^a_z(a^s_{z,i} + b^a_z) \\ s^a_z(a^s_{z,i} + b^a_z) \end{bmatrix} \right)^2 \\
&= -\frac{\partial}{\partial \alpha_{yz}} \left(\left(s^a_x(a^s_{x,i} + b^a_x) - \alpha_{yz} s^a_y(a^s_{y,i} + b^a_y) + \alpha_{zy} s^a_z(a^s_{z,i} + b^a_z) \right)^2 + \left(s^a_y(a^s_{y,i} + b^a_y) - \alpha_{zx} s^a_z(a^s_{z,i} + b^a_z) \right)^2 + \left(s^a_z(a^s_{z,i} + b^a_z) \right)^2 \right) \\
&= -\frac{\partial}{\partial \alpha_{yz}} \left(s^a_x(a^s_{x,i} + b^a_x) - \alpha_{yz} s^a_y(a^s_{y,i} + b^a_y) + \alpha_{zy} s^a_z(a^s_{z,i} + b^a_z) \right)^2 \\
&= -2\left(s^a_x(a^s_{x,i} + b^a_x) - \alpha_{yz} s^a_y(a^s_{y,i} + b^a_y) + \alpha_{zy} s^a_z(a^s_{z,i} + b^a_z) \right) \frac{\partial}{\partial \alpha_{yz}} \left(s^a_x(a^s_{x,i} + b^a_x) - \alpha_{yz} s^a_y(a^s_{y,i} + b^a_y) + \alpha_{zy} s^a_z(a^s_{z,i} + b^a_z) \right) \\
&= -2\left(s^a_x(a^s_{x,i} + b^a_x) - \alpha_{yz} s^a_y(a^s_{y,i} + b^a_y) + \alpha_{zy} s^a_z(a^s_{z,i} + b^a_z) \right) \frac{\partial}{\partial \alpha_{yz}} \left(\alpha_{yz} s^a_y(a^s_{y,i} + b^a_y) \right)
\end{aligned}$$

$$= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\big(s_y^a(a_{y,i}^s + b_y^a)\big)$$

And then we do this for all the other variables

$$\begin{aligned}
\frac{\partial e_i}{\partial \alpha_{zy}} &= -\frac{\partial}{\partial \alpha_{zy}}\Bigg(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_z^a(a_{z,i}^s + b_z^a)\big)^2\Bigg) \\
&= -\frac{\partial}{\partial \alpha_{zy}}\Bigg(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2\Bigg) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial \alpha_{zy}}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial \alpha_{zy}}\big(\alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\big(s_z^a(a_{z,i}^s + b_z^a)\big)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e_i}{\partial \alpha_{zx}} &= -\frac{\partial}{\partial \alpha_{zx}}\Bigg(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_z^a(a_{z,i}^s + b_z^a)\big)^2\Bigg) \\
&= -\frac{\partial}{\partial \alpha_{zx}}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2 \\
&= -2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial \alpha_{zx}}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big) \\
&= -2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial \alpha_{zx}}\big(-\alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big) \\
&= -2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\big(-s_z^a(a_{z,i}^s + b_z^a)\big)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e_i}{\partial s_x^a} &= -\frac{\partial}{\partial s_x^a}\Bigg(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2\Bigg) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_x^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_x^a}\big(s_x^a(a_{x,i}^s + b_x^a)\big) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_x^a}(a_{x,i}^s + b_x^a)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e_i}{\partial s_y^a} &= -\frac{\partial}{\partial s_y^a}\Bigg(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2\Bigg) \\
&= -\frac{\partial}{\partial s_y^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 - \frac{\partial}{\partial s_y^a}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2 \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_y^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big) \\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_y^a}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_y^a}\big(-\alpha_{yz}s_y^a(a_{y,i}^s + b_y^a)\big) \\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_y^a}\big(s_y^a(a_{y,i}^s + b_y^a)\big) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\big(-\alpha_{yz}(a_{y,i}^s + b_y^a)\big) - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)(a_{y,i}^s + b_y^a)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e_i}{\partial s_z^a} &= -\frac{\partial}{\partial s_z^a}\Bigg(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_z^a(a_{z,i}^s + b_z^a)\big)^2\Bigg) \\
&= -\frac{\partial}{\partial s_z^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 - \frac{\partial}{\partial s_z^a}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2 \\
&\quad - \frac{\partial}{\partial s_z^a}\big(s_z^a(a_{z,i}^s + b_z^a)\big)^2
\end{aligned}$$

$$
\begin{aligned}
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_z^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_z^a}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_z^a}\big(s_z^a(a_{z,i}^s + b_z^a)\big)\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_z^a}\big(\alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_z^a}\big(-\alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial s_z^a}\big(s_z^a(a_{z,i}^s + b_z^a)\big)\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\big(\alpha_{zy}(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\big(-\alpha_{zx}(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_z^a(a_{z,i}^s + b_z^a)\big)(a_{z,i}^s + b_z^a)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial e_i}{\partial b_x^a} &= -\frac{\partial}{\partial b_x^a}\Big(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2\Big)\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_x^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_x^a}\big(s_x^a(a_{x,i}^s + b_x^a)\big)\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)s_x^a
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial e_i}{\partial b_y^a} &= -\frac{\partial}{\partial b_y^a}\Big(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2\Big)\\
&= -\frac{\partial}{\partial b_y^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 - \frac{\partial}{\partial b_y^a}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_y^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_y^a}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_y^a}\big(-\alpha_{yz}s_y^a(a_{y,i}^s + b_y^a)\big)\\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_y^a}\big(s_y^a(a_{y,i}^s + b_y^a)\big)\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)(-\alpha_{yz}s_y^a) - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)s_y^a
\end{aligned}
$$

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\begin{aligned}
\frac{\partial e_i}{\partial b_z^a} &= -\frac{\partial}{\partial b_z^a}\Big(\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2 + \big(s_z^a(a_{z,i}^s + b_z^a)\big)^2\Big)\\
&= -\frac{\partial}{\partial b_z^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)^2\\
&\quad - \frac{\partial}{\partial b_z^a}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)^2\\
&\quad - \frac{\partial}{\partial b_z^a}\big(s_z^a(a_{z,i}^s + b_z^a)\big)^2\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_z^a}\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_z^a}\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx}s_z^a(a_{z,i}^s + b_z^a)\big)\\
&\quad - 2\big(s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_z^a}\big(s_z^a(a_{z,i}^s + b_z^a)\big)\\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz}s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_z^a}\big(\alpha_{zy}s_z^a(a_{z,i}^s + b_z^a)\big)
\end{aligned}
$$

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\begin{aligned}
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx} s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_z^a}\big(-\alpha_{zx} s_z^a(a_{z,i}^s + b_z^a)\big) \\
&\quad - 2\big(s_z^a(a_{z,i}^s + b_z^a)\big)\frac{\partial}{\partial b_z^a}\big(s_z^a(a_{z,i}^s + b_z^a)\big) \\
&= -2\big(s_x^a(a_{x,i}^s + b_x^a) - \alpha_{yz} s_y^a(a_{y,i}^s + b_y^a) + \alpha_{zy} s_z^a(a_{z,i}^s + b_z^a)\big)(\alpha_{zy} s_z^a) \\
&\quad - 2\big(s_y^a(a_{y,i}^s + b_y^a) - \alpha_{zx} s_z^a(a_{z,i}^s + b_z^a)\big)(-\alpha_{zx} s_z^a) \\
&\quad - 2\big(s_z^a(a_{z,i}^s + b_z^a)\big)s_z^a
\end{aligned}
$$