Controlled K-theory, part I

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Les Diablerets

Conjecture (Novikov)

Let M be a compact oriented manifold. The higher signature $\langle \mathbf{f}_*(\mathbb{L}(\mathbf{M}) \cap [\mathbf{M}]), \mathbf{x}\rangle$ is homotopy invariant, where

- $[M]$ *is the* **fundamental class** *of M;*
- $\mathbb{L}(\mathbf{M}) \in H^*(M, \mathbb{Q})$ *is the* **Pontrjagin-Hirzebruch** *class of M;*
- $\Gamma = \pi_1(M)$ *and $f : M \longrightarrow B_\Gamma$ is the* **classifying map***;*
- *x is an element in $H^*(B_\Gamma, \mathbb{Q}) \cong H^*(\Gamma, \mathbb{Q})$.*

G. Yu proved the Novikov conjecture for a very large class of finitely generated groups Γ only using the geometry of Γ:

- to any finite, symmetric generating set S is associated a length $\ell(\gamma) = \inf\{n \text{ such that } \gamma = \gamma_1 \cdots \gamma_n \text{ with } \gamma_1, \ldots, \gamma_n \text{ in } S\}$;
- then Γ is provided with the metric $d(\gamma, \gamma') = \ell(\gamma^{-1}\gamma')$ and the proof is carried out using "**cut-and-pasting**" on the metric space (Γ, d).

The Coarse Baum-Connes assembly map

- Let Σ be a proper discrete metric space (balls are finite). Fix $\mathscr{H}$ a separable Hilbert space of infinite dimension. The Roe algebra $C^*(\Sigma)$ is the closure in $\mathcal{L}(\ell^2(\Sigma)\otimes\mathscr{H})$ of the algebra of locally compact operators T with finite propagation, i.e
  - $T=(T_{\sigma,\sigma'})_{(\sigma,\sigma')\in\Sigma^2}$ with $T_{\sigma,\sigma'}\in\mathscr{K}(\mathscr{H})$;
  - there exists $r>0$ such that $T_{x,y}$ if $d(x,y)>r$.
- The Rips complex of degree d is the set $P_d(\Sigma)$ of probability measures on Σ with support of diameter less than d.
- There is a family of assembly maps

$$\mu^d_{\Sigma,*}:K_*(P_d(\Sigma))\longrightarrow K_*(C^*(\Sigma))$$

  compatible with $P_d(\Sigma)\subseteq P_{d'}(\Sigma)$
- We obtain taking inductive limit the Coarse BC assembly map

$$\mu_{\Sigma,*}:\lim_{d>0}K_*(P_d(\Sigma))\longrightarrow K_*(C^*(\Sigma).$$

- Σ satisfies the Coarse BC conjecture if $\mu_{\Sigma,*}$ is an isomorphism.

The descent principle of Higson/Roe

Theorem

- *Let Γ be a finitely generated group equipped with any word metric whose classifying space B_Γ has homotopy type of a finite CW-complex.*
- *Assume that Γ as a metric space satisfies the Coarse BC Conjecture.*

Then Γ satisfies the Novikov conjecture.

The case of a finite metric space!

- Assume that X is a finite metric space.
  - $C^*(X) = \mathscr{K}(\ell^2(X)\otimes\mathscr{H})$ and hence $K_*(C^*(X)) \cong \mathbb{Z}$.
  - The assembly map

    $$\mu^d_{X,*} : K_*(P_d(X)) \longrightarrow K_*(C^*(X)) \cong \mathbb{Z}$$

    is the index map (induced by $P_d(X) \mapsto \{pt\}$).
  - For $d > diam\, X$, then $P_d(X)$ is contractible and hence $K_*(P_d(X)) \cong \mathbb{Z}$;
- Indices have finite propagation!
- Can we take into account this propagation?

Propagation and indices I

- Let D be an elliptic differential operator on a compact manifold M.
- Let Q be a parametrix for D.
- Then $S_0 := Id - QD$ and $S_1 := Id - DQ$ are smooth kernel operators on $M \times M$, i.e of the form $K \cdot f(x) = \int k(x,y)f(y)dy$;
- $$P_D = \begin{pmatrix} {S_0}^2 & S_0(Id + S_0)Q \\ S_1 D & Id - {S_1}^2 \end{pmatrix}$$

  is an idempotent with coefficients in smooth kernel operators on $M \times M$ and we can choose Q such that P_D has arbitrary small propagation i.e $P_D - I_2$ is given by a smooth kernel $k : M \times M \to M_2(\mathbb{C})$ with support arbitrary close to the diagonal.
- D is a Fredholm operator and

$$\text{Ind}\, D = [P_D] - \left[\begin{pmatrix} 0 & 0 \\ 0 & Id \end{pmatrix}\right] \in K_0(\mathscr{K}(L^2(M)) \cong \mathbb{Z}.$$

Operator propagation

Definition

Let (X, d) be a compact metric space, let $(\mathscr{H}_X, \rho_X)$ be a non degenerated representation of $C(X)$ and $T \in \mathcal{L}(\mathscr{H}_X)$

- *$\operatorname{supp} T$ is the complementary of the open subset of $X \times X$*

$$\{(x, y) \in X \times X \text{ s.t } \exists f \text{ and } g \in C(X) \text{ s.t } \\ f(x) \neq 0,\ g(y) \neq 0 \text{ and } \rho_X(f) \cdot T \cdot \rho_X(g) = 0\}$$

- *T has propagation less than r if $d(x, y) < r$ for all (x, y) in $\operatorname{supp} T$.*

Example

if μ is a borelian measure on X and if $k : X \times X \to \mathbb{C}$ is continuous and supported in $\{(x, y) \in X \times X \text{ s.t } d(x, y) < r\}$ then the kernel operator $L^2(X, \mu) \to L^2(X, \mu);\ f \mapsto \int k(\cdot, y) f(y) d\mu(y)$ has propagation $< r$.

Let X be a compact metric space and let $(\mathscr{H}_X, \rho_X, T)$ be an even K-cycle for $K_*(X) = KK_*(C(X), \mathbb{C})$ with $\mathscr{H}_X$ non degenerated and $T = \left(\begin{smallmatrix} 0 & D \\ D^* & 0 \end{smallmatrix}\right)$. Recall that T commutes with functions modulo compact. Replacing T by the compact deformation

$$\rho_X(f_1^{1/2}) \cdot T \cdot \rho_X(f_1^{1/2}) + \ldots + \rho_X(f_n^{1/2}) \cdot T \cdot \rho_X(f_n^{1/2})$$

where $(f_i)_{i=1,\ldots,n}$ is a partition of the unit for X, we can assume that T has propagation arbitrary small. Set

$$P_D = \begin{pmatrix} DD^* & -(1 - D^*D)^{1/2}D^* \\ -D(1 - D^*D)^{1/2} & Id - D^*D \end{pmatrix}$$

Then the index of $[\mathscr{H}_X, \rho_X, T]$ is $[P_D] - \left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]$.

Controlled Index

Replacing in

$$P_D = \begin{pmatrix} DD^* & -(1-D^*D)^{1/2}D \\ -D(1-D^*D)^{1/2} & Id - D^*D \end{pmatrix}$$

the coefficient $(1-D^*D)^{1/2}$ by a power serie, we obtain for any $0<\varepsilon<1/4$ and $r>0$ an approximation $Q_D^{\varepsilon,r}$ such that

- $Q_D^{\varepsilon,r}$ is an ε-r-projection i.e.
  - $Q_D^{\varepsilon,r}$ is self-adjoint and $\|(Q_D^{\varepsilon,r})^2 - Q_D^{\varepsilon,r}\| < \varepsilon$;
  - $Q_D^{\varepsilon,r}$ has propagation less than r.
- The spectral projection $\kappa(Q_D^{\varepsilon,r})$ of $Q_D^{\varepsilon,r}$ is close to P_D and hence the index of $[\mathscr{H}_X, \rho_X, T]$ is

$$[\kappa(Q_D^{\varepsilon,r})] - \left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\right].$$

Propagation in coarse assembly maps

First step : define for finite metric space controlled assembly maps

$$K_*(P_d(X)) \longrightarrow \text{"classes in } K_*(\mathscr{K}(\ell^2(X))) \text{ of propagation } < r_d\text{"}$$

Second step : investigate for family of finite metric space the existence of an "inverse up to rescaling"

$$K_*(P_{d_r}(X)) \longleftarrow \text{"classes in } K_*(\mathscr{K}(\ell^2(X))) \text{ of propagation } < r\text{"}$$

i.e

$$\begin{tikzcd}
K_*(P_d(X)) \arrow[r] \arrow[d, dotted] & \text{propagation } < r_d \arrow[dl] \\
K_*(P_{d_{r_d}}(X)) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_*(P_{d_r}(X)) \arrow[dr] & \text{propagation } < r \arrow[l] \arrow[d, dotted] \\
& \text{propagation } < r_{d_r}
\end{tikzcd}$$

Main goal

Our aim is to provide (coarse) Baum-Connes type statements for family of finite discrete metric spaces such that:

- it takes into account (uniformly) finite propagation;
- for the family of finite subsets of Γ, it implies the Novikov conjecture.
- the statements are stable under ”cut-and-pasting” (**coarse decomposition**).

QUANTITATIVE K-THEORY

(support for ”finite propagation indices”)

The framework : Filtered algebras

Definition

A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces:

- $A_r \subseteq A_{r'}$ if $r \leqslant r'$;
- A_r is closed under involution;
- $A_r \cdot A_{r'} \subseteq A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A.

- If A is unital, we also require that the identity 1 is an element of A_r for every positive number r.
- The elements of A_r are said to have propagation less than r.

Examples

- Let (X, d) be a finite metric space. Then $\mathscr{K}(\ell^2(X))$ is a filtered C^*-algebra : an element $T = (T_{x,y})_{(x,y)\in X^2}$ in $\mathscr{K}(\ell^2(X))$ has propagation less than r if $T_{x,y} = 0$ whenever $d(x, y) > r$. .
- In the same way, for any C^*-algebras A then $A \otimes \mathscr{K}(\ell^2(X))$ is a filtered C^*-algebra;
- More generally, if X is a compact metric space, $(\mathscr{H}_X, \rho_X)$ a non degenerated representation, then $\mathscr{K}(\mathscr{H}_X)$ is filtered using operator propagation.
- Roe algebras;
- C^*-algebras of finitely generated groups and cross-products algebra by such a group action;
- C^*-algebra of étale groupoids (C. Dell'Aiera);
- Compact quantum groups (C. Dell'Aiera).

Almost projections and almost unitaries

Let $A = (A_r)_{r>0}$ be a unital filtered C^*-algebra, $r > 0$ (propagation) and $0 < \varepsilon < 1/4$ (control):

- $p \in A$ is an ε-r-projection if $p \in A_r$, $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- an ε-r projection p has a spectral gap around $1/2$ and hence gives rise by functional calculus to a projection $\kappa(p)$ s.t $\|p - \kappa(p)\| < 2\varepsilon$.
- $u \in A$ is an ε-r-unitary if $u \in A_r$, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. (in particular, ε-r-unitaries are invertible).

Remark

- if q and q' are ε-r-projections of A, then $\operatorname{diag}(q, q')$ and $\operatorname{diag}(q', q)$ are homotopic ε-r-projections in $M_2(A)$;
- if u and v are ε-r-unitaries in A, then $\operatorname{diag}(u, v)$, $\operatorname{diag}(v, u)$ and $\operatorname{diag}(uv, 1)$ are homotopic as **3ε-2r-unitaries** in $M_2(A)$;
- If u is an ε-r-unitary in A, then $\operatorname{diag}(u, u^*)$ and I_2 are homotopic as 3ε-$2r$-unitaries in $M_2(A)$.

Notations

- $P^{\varepsilon,r}(A)$ is the set of ε-r-projections of A.
- $U^{\varepsilon,r}(A)$ is the set of ε-r-unitaries of A.
- $P_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} P^{\varepsilon,r}(M_n(A))$ for $P^{\varepsilon,r}(M_n(A)) \hookrightarrow P^{\varepsilon,r}(M_{n+1}(A));\ x \mapsto \mathrm{diag}(x,0)$.
- $U_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} U^{\varepsilon,r}(M_n(A))$ for $U^{\varepsilon,r}(M_n(A)) \hookrightarrow U^{\varepsilon,r}(M_{n+1}(A));\ x \mapsto \mathrm{diag}(x,1)$.

Quantitative K-groups

Define for a unital C^*-algebra A, $r > 0$ and $0 < \varepsilon < 1/100$ the (stably)-homotopy equivalence relations on $\mathrm{P}_\infty^{\varepsilon,r}(A) \times \mathbb{N}$ and $\mathrm{U}_\infty^{\varepsilon,r}(A)$ (with $\mathrm{P}_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,r}(M_n(A))$ and $\mathrm{U}_\infty^{\varepsilon,r}(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,r}(M_n(A))$):

- $(p,l) \sim (q,l')$ if there exists $k \in \mathbb{N}$ such that $\mathrm{diag}(p, I_{k+l'})$ and $\mathrm{diag}(q, I_{k+l})$ are homotopic as 25ε-r-projections.
- $u \sim v$ if u and v are homotopic as 25ε-$2r$-unitaries.

Definition

1. $K_0^{\varepsilon,r}(A) = \mathrm{P}^{\varepsilon,r}(A)/\sim$ *and* $[p,l]_{\varepsilon,r}$ *is the class of* (p,l) *mod.* $\sim$*;*
2. $K_1^{\varepsilon,r}(A) = \mathrm{U}^{\varepsilon,r}(A)/\sim$ *and* $[u]_{\varepsilon,r}$ *is the class of* u *mod.* $\sim$*.*

- $K_0^{\varepsilon,r}(A)$ is an abelian group for $[p,l]_{\varepsilon,r} + [p',l']_{\varepsilon,r} = [\mathrm{diag}(p,p'), l+l']_{\varepsilon,r}$;
- $K_1^{\varepsilon,r}(A)$ is an abelian group for $[u]_{\varepsilon,r} + [v]_{\varepsilon,r} = [\mathrm{diag}(u,v)]_{\varepsilon,r}$.
- if A is not unital, we use its unitarization to define $K_0^{\varepsilon,r}$ and $K_1^{\varepsilon,r}$.

Structure homomorphisms

For any filtered C^*-algebra A, $0 < \varepsilon < 1/100$ and $0 < r \leqslant r'$, we have natural (compatible) structure homomorphisms

- $\iota_0^{\varepsilon,r,r'} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon,r'}(A)$; $[p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon,r'}$;
- $\iota_1^{\varepsilon,r,r'} : K_1^{\varepsilon,r}(A) \longrightarrow K_1^{\varepsilon,r'}(A)$; $[u]_{\varepsilon,r} \mapsto [u]_{\varepsilon,r'}$.
- $\iota_*^{\varepsilon,r,r'} = \iota_0^{\varepsilon,r,r'} \oplus \iota_1^{\varepsilon,r,r'}$.

We set

$$K_*^{\varepsilon,r}(A) = K_0^{\varepsilon,r}(A) \oplus K_1^{\varepsilon,r}(A).$$

Approximation of K-theory

Remark

For $\varepsilon = 0$ and $r = +\infty$, then $K_*^{\varepsilon,r}(\bullet) = K_*(\bullet)$.

For any filtered C^*-algebra A, $0 < \varepsilon < 1/4$ and $0 < r$, we have natural homomorphisms (compatible with the structure homomorphisms)

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0(A)$; $[p,l]_{\varepsilon,r} \mapsto [\kappa(p)] - [I_l]$;
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1(A)$; $[u]_{\varepsilon,r} \mapsto [u]$; ($\varepsilon$-$r$-unitaries are invertible);
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r}$.

For any $\varepsilon \in (0, 1/4)$ and any projection p in A, there exists $r > 0$ and q an ε-r-projection of A such that $\kappa(q)$ and p are closed and hence homotopic projections. We have a similar result for unitaries

Consequence

For every $\varepsilon \in (0, 1/4)$ and $y \in K_*(A)$, there exists r and x in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = y$.

Standard module

Definition

Let X be a compact space. A non degenerated representation $(\mathscr{H}_X, \rho_X)$ of $C(X)$ is an X-standard module if $\rho_X(f)$ compact implies $f = 0$;

Example

- if X is a connected compact riemannian manifold of dim $\geqslant 1$, the representation on $L^2(X)$ by pointwise multiplication is standard.
- if $(\mathscr{H}_X, \rho_X)$ is a faithfull representation of $C(X)$ and if $\mathscr{H}_0$ is an infinite dimension Hilbert space. Then the diagonal representation $(\mathscr{H}_X \otimes \mathscr{H}_0, \rho_X \otimes Id_{\mathscr{H}_0})$ is standard.

Theorem (Voiculescu)

Let $(\mathscr{H}_X, \rho_X)$ be a X-standard module. Then any element of $K_*(X) = KK_*(C(X), \mathbb{C})$ can be represented by a K-cycle $(\mathscr{H}_X, \rho_X, T)$.

Controlled index

Let X be a compact metric space and let us fix a X-standard module $(\mathscr{H}_X, \rho_X)$. Let $(\mathscr{H}_X, \rho_X, T)$ be an even K-cycle for $K_*(X) = KK_*(C(X), \mathbb{C})$ with $\mathscr{H}_X$ non degenerated and $T = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$. Recall that we can assume that D has propagation arbitrary small. We define

$$Ind^{\varepsilon,r}(D) = [Q_D^{\varepsilon,r}, 1] \in K_0^{\varepsilon,r}(\mathscr{K}(\mathscr{H}_X)),$$

$Q_D^{\varepsilon,r}$ being an ε-r-projection close to $P_D = \begin{pmatrix} DD^* & (1-D^*D)^{1/2}D^* \\ -D(1-D^*D)^{1/2} & Id-D^*D \end{pmatrix}$

Lemma

We have a homomorphism

$$\begin{aligned} Ind_{X,*} : K_*(X) &\to K_*^{\varepsilon,r}(\mathscr{K}(\mathscr{H}_X)) \\ [\mathscr{H}_X, \rho_X, T] &\mapsto Ind^{\varepsilon,r}(D) \end{aligned}$$

Controlled index map with coefficients

If X is a metric compact space and A is a C^*-algebra, we set $K_*(X,A) = KK_*(C(X),A)$. The previous construction can be extended to K-cycle for $K_*(X,A)$:

Lemma

Let X be a metric compact space and fix a non degenerated standard X-module $(\rho_X, \mathscr{H}_X)$. For any $0 < \varepsilon < 1/100$ and any $r > 0$, there exists a controlled index map $\mathrm{Ind}_{X,A}^{\varepsilon,r} : K_*(X,A) \to K_*^{\varepsilon,r}(\mathscr{K}(\mathscr{H}_X) \otimes A)$ such that

1. $\iota_*^{\varepsilon,r,r'} \circ \mathrm{Ind}_{X,A}^{\varepsilon,r} = \mathrm{Ind}_{X,A}^{\varepsilon,r'}$;
2. *the composition*

$$K_*(X,A) \longrightarrow K_*^{\varepsilon,r}(\mathscr{K}(\mathscr{H}_X) \otimes A) \xrightarrow{\iota_*^{\varepsilon,r}} K_*(\mathscr{K}(\mathscr{H}_X) \otimes A) \cong K_*(A)$$

is the index map $\mathrm{Ind}_{X,A} : K_*(X,A) \to K_*(A)$ (induced by $X \mapsto \{pt\}$).

Behaviour for small propagation

Theorem

Let X be a finite simplicial complex equipped with a metric and let A be a C^*-algebra. For every $0 < \varepsilon < 1/200$, there exists $r_\varepsilon > 0$ such that for any $0 < r < r_\varepsilon$

$$\mathrm{Ind}_{X,A}^{\varepsilon,r} : K_*(X,A) \to K_*^{\varepsilon,r}(\mathscr{K}(\mathscr{H}_X)\otimes A)$$

is an isomorphism.

Remark

For the simplicial metric, r_ε only depends only on the dimension of X;

Next steps

- For X a finite metric space, complete

$$K_*(P_d(X), A) \xrightarrow{\mathrm{Ind}_{X,A}^{\varepsilon,r}} K_*^{\varepsilon,r}(\mathscr{K}(\mathscr{H}_{P_d(X)}) \otimes A) \longrightarrow K_*^{\varepsilon,r}(\mathscr{K}(\ell^2(X)) \otimes A)$$

  in a controlled assembly map;
- State a uniformly controlled version of the Coarse Baum-Connes asssembly map for families of finite metric spaces;
- show that it is stable under ”cut-and pasting”.