Notes on subgroups of the Grigorchuk group

Preliminary version

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1. Introduction

In 1980, Grigorchuk constructed in [10] a group $\mathcal{G}$, which is known as *the (first) Grigorchuk group*, and proved in [7] that $\mathcal{G}$ has intermediate growth between polynomial and exponential, thus providing an answer to an open problem posed by John Milnor in 1968.

Since then, this group continues to be extensively studied and gave rise to the definition of branch groups (which naturally appear in the classification of just-infinite groups). All these groups appear as automorphism groups of infinite rooted trees. Branch groups admit an intrinsic definition in terms of stabilizers of points and of stabilizers of rays (also called *parabolic subgroups*) for this action and a major open problem about them is to describe their lattice of subgroups. In the case of the Grigorchuk group, a deep result in this direction, see [13], is the rigidity of maximal subgroups.

On the other hand, Grigorchuk [8] and Bartholdi and Grigorchuk [1, 2] showed that in a branch group, all the parabolic subgroups are infinite, pairwise distinct and weakly maximal; where a subgroup is said to be *weakly maximal* if it is of infinite index and maximal for this property. It was recently proven in [4] that there exist uncountably many weakly maximal subgroups that are not parabolics, and not only in $\mathcal{G}$ but in an arbitrary regularly weakly branch group.

This discovery can be used to study the subgroup structure of branch groups from the statistical viewpoint. Namely, a very active area of research has lately developed around the notion of IRS – invariant random subgroup in a locally compact group. An IRS is a probability distribution on the space of all subgroups of the group which is required to be invariant under the natural action of the group on its subgroups by conjugation. This is a natural example of an invariant measure, but moreover it also has a nice algebraic meaning: it generalizes the notion of a normal subgroup that correspond to the Dirac delta measures. It is very interesting and important to understand how the algebraic subgroup structure of the group is related to the variety of IRSs on it. For example, recent new examples of Juschenko-Monod [11] provide first examples of simple groups (i.e. groups that have no non-trivial normal subgroups) with non-trivial

IRS. Benli-Grigorchuk-Nagnibeda [3] constructed first examples of groups of intermediate growth with uncountably many distinct IRS. It is still an open question whether a just-infinite group, and in particular the Grigorchuk group $\mathcal{G}$ can have uncountably many IRS (observe that being just-infinite it can only have countably many normal subgroups).

This problem is connected the result in [4] about variety of weakly maximal subgroups. There is indeed a strategy that potentially allows to construct an IRS out of a weakly maximal subgroup, as follows. Given a weakly maximal subgroups H in $\mathcal{G}$, it is possible to construct a binary rooted tree T_H on which $\mathcal{G}$ acts and such that H is the stabilizer of the leftmost ray. If H is not parabolic for the original action, then the action of $\mathcal{G}$ on T_H is not branched, but we can ask if, given the action of $\mathcal{G}$ on T_H, is it true that all parabolic subgroups for this action are weakly maximal? pairwise distinct? Positive answer to these questions will imply the construction of new non-trivial invariant random subgroups of $\mathcal{G}$.

In this note we give partial answer to this question and propose some directions of research.

Throughout this note and unless specified otherwise, G will denote a finitely generated group, T a locally finite rooted tree and $\mathcal{G}$ the first Grigorchuk group.

In Section 3 we introduce the necessary definitions. Section 4 is based on [4] and shows that there is at least 4 distinct classes of weakly maximal subgroups in the Grigorchuk group. In Section 5 we turns our attention on parabolic subgroups of groups acting on infinite rooted tree. The main result of this section is Proposition 8, where for each wmc H (see Definition 2) in a just infinite residually finite p-group G we construct a tree T_H such that all $\hat{G}$ stabilizers of rays are wmc and pairwise distinct. This allows us to find four pairwise distinct IRS on $\hat{\mathcal{G}}$ with uncountable support consisting of wmc subgroups. In Section 7 we investigate a particular example of a finitely generated weakly maximal subgroup of the Grigorchuk group and give some ideas on how to construct other such examples. Section 8 proposes some ideas (mainly due to Grigorchuk) for a classification of weakly maximal subgroups of the Grigorchuk group. The Appendix A proposes a description of the top of the lattice of subgroups in the Grigorchuk group, while Appendix B investigates the geometry of Schreier graphs of weakly maximal subgroups of the Grigorchuk group.

2. Overview of the results and and of the questions

In this section, we present a brief overview of the main results and of questions that are still open. A subgroup H of a topological group G is said to be *wmc* (stands for *weakly maximal closed*) if it is maximal among closed subgroups of infinite index. For the other definitions and proofs, see the next sections.

If a group is weakly branch, then all the parabolic subgroups are pairwise distinct and infinite [8]. If the action is branched, all parabolic subgroups are wmc [1, 2]. Therefore, it is natural to ask the following.

Question 1. Suppose that $G \leq \mathrm{Aut}(T)$ is such that all parabolic subgroups are infinite, pairwise distinct and wmc. Does it implies that the action of G is (weakly) branched?

This question is of a particular interest in the case of the Grigorchuk group $\mathcal{G}$ (and other groups considered in [9]) since there exists only one weakly branched action. This implies that if H is not a parabolic subgroup for the original action, then the action of $\mathcal{G}$ on T_H is not branched.

2.1. Residually finite groups

If G is a finitely generated residually finite group and H a wmc subgroup, it is possible to construct a coset tree T_H. This tree is a rooted infinite spherically homogeneous tree and can be chosen to be p-regular if G is a p-group. There is a natural action by multiplication of G on T_H, and $\mathrm{Stab}_G(\bar{0}) = H$. If G is just infinite, then the action is faithful. We want to answer the following question.

Question 2. Is it true that for the action of G on T_H,

1. All parabolic subgroups are pairwise distinct?

2. All parabolic subgroups are wmc?

It is possible to weaken the question in two direction. Firstly, by restricting the class of group G that we consider; for example, it would be nice to have the answer for $G = \mathcal{G}$ the first Grigorchuk group. Secondly, we could replace the statement "all parabolic subgroups" by a subset of ∂T with positive measure.

The answer to both this question is yes for $\hat{G}$, the profinite completion of G, if G is a residually finite p-group. But the questions remains open for G itself.

The following result solves the question for rays in the G-orbit of $\bar{0}$. But since we are mainly interested in finitely generated group, the G-orbit of $\bar{0}$ is countable and hence of zero measure.

Proposition 1. *Let G be a torsion group acting on an infinite rooted tree T. Then in each orbit of ∂T containing a wmc parabolic subgroup, all parabolic subgroups are pairwise distinct.*

Proof. See Corollary 4. □

Proposition 2. *Let T be a spherically homogeneous rooted tree and $G \leq Aut(T)$ be a finitely generated subgroup which is torsion and acts spherically transitively on T. Let $H = \mathrm{Stab}_G(\xi)$ be the stabilizer of a ray $\xi \in \partial T$. Suppose that H is a wmc subgroup of G and $\bar{H}$ is a wmc subgroup of $\hat{G}$.*

1. *Then all $\hat{G}$ parabolic subgroups are pairwise distinct and wmc and all G parabolic subgroups are distinct from H.*

2. *Suppose moreover that for all $g \in \hat{G}$ we have $\overline{\bar{H} \cap G^g} = \bar{H}$. Then all G parabolic subgroups are pairwise distinct.*

Proof. See Proposition 6. □

Observe that the condition $\overline{\bar{H}\cap G^g}=\bar{H}$ for all g is implied by the condition that the applications $\bar{\cdot}$ and $\cdot\cap G$ send wmc subgroups on wmc subgroups.

Another statement that implies positive answers to Question 2 is the following.

Lemma 1. *Let G be a residually finite just infinite p-group and H a wmc subgroup. Suppose that $\bar{H}$ is a wmc subgroup of $\hat{G}$ and that for all n and all vertex v of level n, the orbits of ∂T_v under $\mathrm{Stab}_G(n)$ and under G coincide. Then all G parabolic subgroups are pairwise distinct and wmc.*

The following theorem resumes what we are able to prove on parabolic subgroups of the action on T_H, without making any assumptions on the closure of H.

Theorem 1. *Let G be a finitely generated residually finite p-group and H be a wmc subgroup. Then, there exists an infinite rooted p-regular tree T such that*

1. *G acts faithfully and spherically transitively on T;*
2. *The G-stabilizer of $\bar{0}$, the leftmost ray in ∂T, is H;*
3. *Any two rays in the G-orbit of $\bar{0}$ have distinct G-stabilizers;*
4. *If g is in $N_{\hat{G}}(G)$, then $\mathrm{Stab}_G(g\bar{0})$ is distinct from H;*
5. *All $\hat{G}$-stabilizers are pairwise distinct and wmc.*

Proof. See Propositions 7 and 8. □

Observe that in the Grigorchuk group $\mathcal{G}$, $N_{\hat{G}}(G)$ is countable. Therefore, this theorem only provides countably many distincts parabolic subgroups and does not answer Question 2.

The following proposition, gives some sufficient criterion to have uncountably many distinct parabolic subgroups. Before that, we fix some notation. For every vertex w, $C_w\subseteq\partial T$ is the cylinder under w and $\mathrm{Stab}_G(C_v)$ denotes the pointwise stabilizer of C_v. That is, $\mathrm{Stab}_G(C_v)=\bigcap_{\xi\text{ s.t. }v\in\xi}\mathrm{Stab}_G(\xi)$.

Proposition 3. *Let G, H and T be as in Theorem 1. For $A\leq G$, let $S_A:=\{\xi\in\partial T\mid\mathrm{Stab}_G(\xi)=A\}$. The implications between the following assumptions are resumed in the diagram below.*

1. *All parabolic subgroups are wmc;*
2. *For all ξ and η in ∂T, if $\mathrm{Stab}_G(\xi)\leq\mathrm{Stab}_G(\eta)$, then they are equal;*
3. *For all $A\leq G$, the subset S_A is closed;*
4. *The group G is hereditarily just infinite and there is no ray with trivial stabilizer;*
5. *For all $\xi\in\partial T$ and all $v\in\xi$, we have $\mathrm{Stab}_G(C_v)\lneqq\mathrm{Stab}_G(\xi)$;*
6. *For all A and all v in T, there exists $w\leq v$, such that $C_w\subset\partial T\setminus S_A$;*

7. *There is an uncountable number of pairwise distinct parabolic subgroups.*

$$\begin{array}{ccccccc} 1 & \Longrightarrow & 2 & \Longrightarrow & 3 & \Longrightarrow & 6 & \Longrightarrow & 7 \\ & & & & & & \Uparrow & & \\ & & & & 4 & \Longrightarrow & 5 & & \end{array}$$

Proof. See Lemma 12. □

An important remark at this point is that the last proposition does not guaranty the existence of a subset X of ∂T of positive measure such that any two rays in X have distinct stabilizers. On the other hand, we do not know yet any examples of $H \leq G$ that is not a parabolic subgroup of a branch action but that still satisfy any assumption of the proposition.

3. Definitions

Let G be an infinite group. A *weakly maximal subgroup* H is an infinite index subgroup which is maximal for this property.

Let T be an infinite rooted tree — we will always assume that T is locally finite. We can divide T in level: $\mathcal{L}(n)$ consist of all vertices at distance n from the root. We also have a natural order on the vertices: $w \leq v$ if and only if there is "descending path" from v to w — i.e. a path $v = v_0, v_1, \dots, v_n = w$ with the level of v_i being 1 plus the level of v_{i-1}. For any vertex v, let T_v denotes the subtree (rooted at v) consisting of all vertices below v. If the degree of a vertex depends only on its level (that is if $\mathrm{Aut}(T)$ acts transitively on level), we say that T is *spherically homogeneous*.

Now, let G be a group acting on a infinite rooted tree T. If the action of G is transitive on each level, we say that G acts *level transitively*. Recall that for any vertex v of T, the *stabilizer* $\mathrm{Stab}_G(v)$ is the subgroup of elements fixing v. The stabilizer of a level, $\mathrm{Stab}_G(n)$ is equal to the intersection $\bigcap_{v\in\mathcal{L}(n)} \mathrm{Stab}_G(v)$, this is a normal subgroup. The *rigid stabilizer* $\mathrm{Rist}_G(v)$ of a vertex v consist of elements g fixing all vertices outside T_v. This is a normal subgroup of $\mathrm{Stab}_G(v)$. The *rigid stabilizer of level* n is $\mathrm{Rist}_G(n) = \langle \mathrm{Rist}_G(v) \mid v \in \mathcal{L}(n)\rangle = \prod_{v\in\mathcal{L}(n)} \mathrm{Rist}_G(v)$. An action of G on T is *weakly branched*, if G acts level transitively and $\mathrm{Rist}_G(v)$ is infinite for all v (equivalently, $\mathrm{Rist}_G(v)$ is never trivial). The action is said to be *branched* if G acts level transitively and $\mathrm{Rist}_G(n)$ is a finite index subgroup of G for all n.

Mettre def de regular branch !?!

For a group G acting on some infinite rooted tree, stabilizers of rays are called parabolic subgroups.

4. Counting weakly maximal subgroups in $\mathcal{G}$

Let G be a finitely generated group and H a subgroup of infinite index, then H is contained in a weakly maximal subgroup. If G acts on a spherically regular rooted tree in a weakly branched way, then all parabolic subgroups are pairwise distinct and infinite [8]. If the action is branched, then parabolic subgroups are weakly maximal[1, 2]. Moreover, if G is a regular branch group, then any finite subgroup is contained in uncountably many weakly maximal subgroups [4]. This remains true, even if we look at equivalence classes of weakly maximal subgroups under the action of $\mathrm{Aut}(G)$. Indeed, each equivalence class is countable.

Let G be a group and H a subgroup of infinite index. Suppose that $H = \bigcap_{i\geq 0} H_i$ where $G = H_0 > H_1 > H_2 > \dots$ are finite index subgroups. Then, it is possible to construct the associated *coset tree* $T_H = T_{(H_i)_i}$ in the following way. Let G be the root of the tree. The vertices of first level corresponds to the left classes $\{g_{1,1}H_1, \dots, g_{1,r_1}H_1\}$. The vertices of the second level under $g_{1,i}H_1$ are $\{g_{1,i}g_{2,1}H_2, \dots, g_{1,i}g_{2,r_2}H_2\}$, where $\{g_{2,j}\}_j$ is a transversal for $H_2 < H_1$. The tree T_H is infinite, rooted and spherically transitive.

As seen in Subsection 5.4, for every weakly maximal subgroup H of the Grigorchuk group (such a group is a wmc by Lemma 5) it is possible to construct T_H. Moreover, it is always possible to choose T_H to be a 2-regular rooted tree. and other group ?

Definition 1. Let G be a group. Two weakly maximal subgroups A and B are tree equivalent if there exists a coset tree T_A with B a parabolic subgroup for the G action on T_A.

Observe, that this is an equivalence relation. Indeed, if $B = \mathrm{Stab}_G(x_1x_2\dots)$ is a parabolic subgroup, then $B = \bigcap_{i\geq 1} \mathrm{Stab}_G(x_1x_2\dots x_i)$ and thus T_A is a coset tree for B. This implies that $\sim$ is symmetric. On the other hand, if $A \sim B$ and $B \sim C$, then A and C are both parabolic subgroups of $T_B = T_C$. For example, if $G \leq \mathrm{Aut}(T)$ is branch, then all parabolic subgroups (of the original action) are weakly maximal and tree equivalent.

Question 3. How many tree equivalence classes of weakly maximal subgroups has $\mathcal{G}$?

It is possible that such a class is uncountable (for example, for the original action, the class as the cardinality of the continuum), and therefore the number of such classes is a priori smaller than the number of classes under automorphism equivalence.

Lemma 2. *If $A \sim B$ are equivalent by a coset tree T_A corresponding to $A_0 > A_1 > \dots$, then they are equivalent by T'_A corresponding to $G > A'_1 > A'_2 > \dots$ where each A'_{i+1} is maximal in A_i.*

Proof. Refining the sequence $G = A_0 > A_1 > \dots$ does not change the set of rays and thus does not change the set of parabolic subgroups. □

Corollary 1. *The Grigorchuk group $\mathcal{G}$ has at least 4 tree equivalence classes of weakly maximal subgroups.*

Proof. In this special case, all maximal subgroups are of index two and hence normal. Now, take $A_1 = \langle b, c\rangle$, $A_2 = \langle a, b\rangle$, $A_3 = \langle a, c\rangle$ and $A_4 = \langle a, d\rangle$. These subgroups are finite. In fact, they are respectively isomorphic to the dihedral group of order 4, 16, 8 and 32 [6]. Therefore, each A_i is contained in a weakly maximal subgroups A'_i. Since $\mathcal{G} = \langle a, x, y\rangle$ for any choice of $x \neq y \in \{b, c, d\}$, all the A'_i are pairwise distinct.

It is easy to see that for each i, there exists only one index 2 subgroup $\tilde{A}_i$ of $\mathcal{G}$ containing A_i and that all the $\tilde{A}_i$ are pairwise distinct. Therefore, if $i \neq j$, the subgroups A'_i and mettre ref

A'_j are not tree equivalent. Indeed, if it was the case, we would have A'_j a parabolic subgroup of a coset tree $T_{A'_i}$. By Lemma 2, we could suppose that the stabilizer of any first level vertex in $T_{A'_i}$ is $\tilde{A}_i$. This implies that $A'_j \leq \tilde{A}_i$ which is absurd. □

Observe that with the same kind of proof we may hope to find up to 7 tree equivalence classes of weakly maximal subgroups, one for each maximal subgroup. Indeed, in order to have n different classes, it is sufficient to find n weakly maximal subgroups such that each of them is contained in a unique maximal subgroup (and that all these maximal subgroups are pairwise distinct).

5. Groups acting on rooted trees

In this section we will work with groups acting faithfully on locally finite rooted trees. Such a group is residually finite and its profinite completion also acts on the tree. First we begin with some results on topological groups and on residually finite group.

5.1. Topological groups

Throughout this subsection, G will be a topological group not necessarily (topologically) finitely generated. For us a subgroup of G will always be an abstract subgroup not necessarily closed.

Lemma 3. *Let X be a topological space, Y a subspace with the subspace topology, $\mathcal{F}(Y)$ the collection of closed subsets of Y and $\bar{\cdot}$ the closure in X. Then we have two applications:*

$$\begin{aligned} \Theta\colon \mathcal{F}(Y) &\to \mathcal{F}(X) \\ A &\mapsto \bar{A} \end{aligned} \qquad\qquad \begin{aligned} \Psi\colon \mathcal{F}(X) &\to \mathcal{F}(Y) \\ B &\mapsto B \cap Y \end{aligned}$$

which satisfy

1. $\Psi \cdot \Theta = \mathrm{Id}_{\mathcal{F}(Y)}$*;*
2. Ψ *is increasing, while* Θ *is strictly increasing;*
3. $\Theta \cdot \Psi(B) \subseteq B$*;*
4. *If we restrict ourself to* $\mathrm{Im}(\Theta)$*, then* Θ *and* Ψ *are isomorphism of lattices and* $\Theta = \Psi^{-1}$*;*
5. *If Y is dense and B open,* $\Theta \cdot \Psi(B) = B$*;*
6. *If Y is dense, B open and ϕ an homeomorphism of X, we have* $\phi(B) = \overline{\phi(B) \cap Y}$*.*

Proof. For $A \in \mathcal{F}(Y)$, we have $A \subseteq \bar{A}$ and $A \subseteq Y$, thus $A \subseteq \bar{A} \cap Y$. For the other direction, remember that $\overline{U \cap V} \subseteq \bar{U} \cap \bar{V}$. Now, by definition of the induced topology, $A = B \cap Y$ with B a closed subset of X. Therefore, $\bar{A} \cap Y = \overline{B \cap Y} \cap Y \subseteq \overline{B} \cap \overline{Y} \cap Y = B \cap Y = A$;

and this prooves the first statement. It is evident that both Ψ and Θ are increasing. The equality $\Psi \cdot \Theta = \mathrm{Id}_{\mathcal{F}(Y)}$ implies that Θ is injective and thus strictly increasing.

The third and fourth statement are trivial.

For the fifth statement, we have to prove that $B \cap Y$ is dense in B. Take any open set U of B (with the induced topology). Since B is open, U is also an open set of X. Therefore, $U \cap (B \cap Y) = (U \cap B) \cap Y = U \cap Y \neq \emptyset$ by density of Y.

For the last statement, since ϕ is an homeomorphism, $\phi^{-1}(Y)$ is dense in X and $\phi(B)$ is clopen. Therefore, $\overline{\phi(B) \cap Y} = \phi(\overline{B \cap \phi^{-1}(Y)}) = \phi(B)$. $\square$

This lemma means that (when restricted to closed subset of X and Y), $\bar{\cdot}$ is section $\cdot \cap Y$. Therefore, $\bar{\cdot}$ is injective and $\cdot \cap Y$ is surjective. Observe that even in the particular case of X a topological group, Y a dense subgroup and B a closed subgroup, it is not necessarily true that $\Theta \cdot \Phi(B) = B$. For example, take $X = \mathbf{R}$, $Y = \mathbf{Q}$ and $B = \sqrt{2}\mathbf{Z}$.

Proposition 4. *Let K be a topological group and $G \leq K$ an abstract subgroup. For every pair of subgroups $H \leq L$ of G, if $[L : H]$ is finite then $[\bar{L} : \bar{H}] \leq [L : H]$. If H is a closed subgroup of G, then $[L : H]$ is finite if and only if $[\bar{L} : \bar{H}]$ is finite, in which case they are equal. Moreover, if H is a closed subgroup of finite index of G and $\{g_i\}_{i=1}^n$ is a transversal for $H \leq L$, it is also a transversal for $\bar{H} \leq \bar{L}$*

Proof. First suppose that $[L : H]$ is finite. We have $L = \sqcup_{i=1}^n g_i H$. Thus $\bar{L} = \overline{\sqcup g_i H} = \cup_{i=1}^n \overline{g_i H} = \cup g_i \bar{H}$, where the second equality holds because a finite union of closed subset is closed and the last equality holds since left multiplication by g_i is an homeomorphism.

Moreover, if H is closed in G and $g_i \bar{H} \cap g_j \bar{H}$ is non-empty, then $g_i g_j^{-1} \in \bar{H} \cap G = H$ which implies $i = j$. This prove that in this case $\bar{L} = \sqcup g_i \bar{H}$.

Now suppose that H is closed in G and $\bar{H}$ is of finite index in $\bar{L}$ we have that $\bar{H} \cap L$ is of finite index in $\bar{L} \cap L = L$. We have $L \leq \bar{L}$ and H closed in G implies that H is closed in L for the subspace topology coming from L. Thus, we can apply Lemma 3 to $L \leq \bar{L}$ in order to have $\bar{H} \cap L = \bar{H}^L \cap L = H$ of finite index in L. $\square$

Let K be a topological group. We denotes by $\mathrm{Sub}_{\mathrm{cl}}(K)$ the collection of closed subgroups of K. This is a lattice with $A \wedge B = A \cap B$ and $A \vee B = \overline{\langle A \cup B \rangle}$.

Corollary 2. *If K is topological group and G a dense subgroup we have two applications:*

$$\begin{aligned} \Theta\colon \mathrm{Sub}_{\mathrm{cl}}(G) &\to \mathrm{Sub}_{\mathrm{cl}}(K) \\ H &\mapsto \bar{H} \end{aligned} \qquad\qquad \begin{aligned} \Psi\colon \mathrm{Sub}_{\mathrm{cl}}(K) &\to \mathrm{Sub}_{\mathrm{cl}}(G) \\ M &\mapsto M \cap G \end{aligned}$$

which satisfy

1. $\Psi \cdot \Theta = \mathrm{Id}_{\mathrm{Sub}_{\mathrm{cl}}(G)}$;
2. Ψ *is increasing, while* Θ *is strictly increasing;*
3. $\Theta \cdot \Psi(M) \subseteq M$;
4. *If we restrict ourself to* $\mathrm{Im}(\Theta)$, *then* Θ *and* Ψ *are isomorphism of lattices and* $\Theta = \Psi^{-1}$;

5. *If M is clopen, $\Theta \cdot \Psi(M) = M$;*

6. *For all H, L in $\mathrm{Sub}_{\mathrm{cl}}(G)$, the index $[L : H]$ is finite if and only if $[\bar{L} : \bar{H}]$ is finite, in which case they are equal;*

7. *For all M in $\mathrm{Sub}_{\mathrm{cl}}(K)$, the index $[K : M]$ is finite if and only if $[G : M \cap G]$ is finite. In which case they are equal;*

8. *For all H, L in $\mathrm{Sub}_{\mathrm{cl}}(G)$ and M, N in $\mathrm{Sub}_{\mathrm{cl}}(K)$ we have*

$$\begin{aligned} \Theta(H \vee L) &= \Theta(H) \vee \Theta(L) & \qquad \Psi(M \vee N) &\geq \Psi(N) \vee \Psi(N) \\ \Theta(H \wedge L) &\leq \Theta(H) \wedge \Theta(L) & \qquad \Psi(M \wedge N) &= \Psi(N) \wedge \Psi(N) \end{aligned}$$

Proof. Assertions 1 to 6 are obvious.

For 7, we already know that $[K : M] \geq [G : M \cap G]$. We also have that $\overline{M \cap G} \leq M$. Therefore, we have $[G : M \cap G] \leq [\bar{G} : M] \leq [\bar{G} : \overline{M \cap G}]$. If $[K : M]$ is finite, so is $[G : M \cap G]$ and we can use assertion 6. If $[K : M]$ is infinite, $[G : M \cap G]$ can not be finite, otherwise we would have $[K : M] \leq [\bar{G} : \overline{M \cap G}] = [G : M \cap G]$.

For Assertion 8, the only non-obvious inequality is $\Theta(A) \vee \Theta(B) \leq \Theta(A \vee B)$ If g is in $\langle \bar{A} \cup \bar{B} \rangle$, then $g = a_1 b_1 \dots a_n b_n$ with the a_i in $\bar{A}$ and the b_i in B. Therefore, $g = \lim_{j \to \infty} a_{1,j} b_{1,j} \dots a_{n,j} b_{n,j}$, where for all i, $a_i = \lim_{j \to \infty} a_{i,j}$ with the $a_{i,j}$ in A. This show that $\langle \bar{A} \cup \bar{B} \rangle \leq \overline{\langle A \cup B \rangle} = \overline{\langle A \cup B \rangle}^G = \Theta(A \vee B)$. Since this last subset is closed, we conclude that $\Theta(A) \vee \Theta(B) = \overline{\langle \bar{A} \cup \bar{B} \rangle} \leq \Theta(A \vee B)$.

□

Lemma 4. *If G is a topological group group, then for all closed subgroups H, the normalizer $N_G(H)$ is a closed subgroup.*

Proof. For any $h \in G$, let ϕ_h be the continuous fonction $\phi_h(g) = ghg^{-1}$. Then

$$\begin{aligned} N_G(H) &= \{g \mid gHg^{-1} \leq H \text{ and } g^{-1}Hg \leq H\} \\ &= \bigcap_{g \in H} \phi_g^{-1}(H) \cap (\bigcap_{g \in H} \phi_g^{-1}(H))^{-1} \end{aligned}$$

is closed. □

Definition 2. Let G be a topological group. A subgroup W is called *wmc* or *weakly maximal closed* if it is a closed subgroup of infinite index and maximal for this property.

Observe that if a closed subgroup is weakly maximal, then it is wmc. If G is discrete, then wmc subgroups are exactly weakly maximal subgroups. This is also true for groups where all weakly maximal subgroups are closed. An example of such group is the Grigorchuk group. More generally, we have.

Lemma 5. *Let G be a group. Assume that for each finite index subgroup $H \leq G$, maximal subgroups of H are of finite index. Then each weakly maximal subgroup of G is closed in the profinite topology.*

Proof. If W is a weakly maximal subgroup, it is contained in a maximal subgroup M_1 of G. By assumption this subgroup is of finite index in G, therefore $W \neq M_1$. This implies that W is contains in a maximal subgroup M_2 of M_1, which is by assumption of finite index. Thus we have that $W \leq \bigcap M_i$ with $G > M_1 > M_2 > \dots$ with all indices finite. But then $\bigcap M_i$ is an infinite index subgroup of G containing W. By maximality we have $W = \bigcap M_i$ is closed in the profinite topology. $\square$

The following lemma will be useful for torsion group.

Lemma 6. *Let G be a topological group, H a wmc subgroup, $g \notin H$ with $g^k \in H$ for some k (for example, g with finite order). Then $gHg^{-1} \neq H$.*

Proof. If it was the case, then $\langle H, g\rangle = H \sqcup gH \sqcup \cdots \sqcup g^{k-1}H$ is a closed subgroup of G. We have $H < \langle H, g\rangle$ of index $k > 1$, therefore $\langle H, g\rangle$ is of finite index in G and so is H. This is the desired contradiction. $\square$

Corollary 3. *Let G be a topological group and H a wmc subgroup such that for all $g \in G$, there exists k with $g^k \in H$. Then H is self-normalizing.*

A natural application of this result is to group acting on infinite rooted tree.

Corollary 4. *Let G be a topological group acting on a infinite rooted tree and $\xi \in \partial T$. If $H = \mathrm{Stab}(\xi)$ is wmc and $g \in G \setminus H$ is such that $g^k \in H$ for some k, then $\mathrm{Stab}(g\xi)$ is distinct from $\mathrm{Stab}(\xi)$. In particular, if G is torsion, in each orbit of ∂T with a wmc stabilizer, all stabilizers are pairwise distinct.*

Proof. The proof is by contradiction. If there were not distinct, then we have

$$\mathrm{Stab}(\xi) = \mathrm{Stab}(g\xi) = g\,\mathrm{Stab}(\xi)g^{-1}$$

and therefore $g \in N_G(H) \setminus H$. But this is not possible by last corollary. $\square$

5.2. Profinite groups

Definition 3. Let G be an abstract group. Its *profinite completion* $\hat{G}$ is the inverse limit of the system $(G/N)_{N\in\mathcal{N}}$ where $\mathcal{N}$ is the collection of finite index normal subgroup of G.

There is always a natural map $\eta\colon G \to \hat{G}$, which is injective if and only if G is residually finite. If G is residually finite, the profinite topology on G is the same as the subspace topology induced by $\hat{G}$ (this follows from a universal property of $\hat{G}$) and $\bar{G} = \hat{G}$, where $\bar{\cdot}$ denotes the closure in $\hat{G}$.

Put references + definitions

Fact 1 (On profinite groups)**.** *1. In a compact group $\mathcal{G}$, open subgroups are exactly finite index closed subgroups.*

2. *A group $\mathcal{G}$ is profinite if and only if it is Hausdorff, compact and totally disconnected.*

3. *In a profinite group, the intersection of all open normal subgroups is trivial.*

4. *In a profinite group $\mathcal{G}$, a subgroup H is closed if and only if it is an intersection of open subgroups. The subgroup H is normal and closed if and only if it is an intersection of open normal subgroups.*

5. *If $\mathcal{G}$ is a (topologically) finitely generated profinite group, then the element 1 of $\mathcal{G}$ has a countable fundamental system of neighborhoods. This implies that $\mathcal{G}$ is metrizable.*

6. *If $\mathcal{G}$ is infinite profinite group, then it has countably many clopen subsets (and a countable base of open subsets).*

Lemma 7. *In a group G with the profinite topology, finite index subgroups are exactly clopen subgroups.*

Proof. First, suppose that H is of finite index. Then it contains a normal subgroup of finite index (its normal core) and is therefore open. Since H is of finite index, we have a finite system of transversal: $g_1 = 1, \ldots, g_n$. This implies that H is also closed ($H = G - \bigcup_{i=2}^{n} g_i H$, a finite union).

For the other direction, if H is open it contains by definition a normal subgroup of finite index and is therefore of finite index. □

An important and non trivial fact about profinite group is the following.

Theorem 2 (Nikolov and Segal)**.** *Let $\mathcal{G}$ be a (topologically) finitely generated profinite group. Then every finite index subgroup is open.*

A direct consequence of this theorem is that in a finitely generated profinite group $\mathcal{G}$, finite index subgroups are exactly clopen subgroups.

Corollary 5. *If G is a residually finite group and $\hat{G}$ its profinite completion, the two applications*

$$\begin{aligned} \Theta\colon \operatorname{Sub}_{\mathrm{cl}}(G) &\to \operatorname{Sub}_{\mathrm{cl}}(\hat{G}) \\ H &\mapsto \bar{H} \end{aligned} \qquad\qquad \begin{aligned} \Psi\colon \operatorname{Sub}_{\mathrm{cl}}(\hat{G}) &\to \operatorname{Sub}_{\mathrm{cl}}(G) \\ M &\mapsto M \cap G \end{aligned}$$

of Corollary 2 satisfies that if M is of finite index, $\Theta \cdot \Psi(M) = M$.

Proof. This is Corollary 5 and the fact (Theorem 2) that finite index subgroups of $\hat{G}$ are clopen subgroups. □

Corollary 6. *If G is a residually finite group and $\hat{G}$ its completion, their lattice of clopen subgroups are isomorphic, by $\bar{\cdot}$ and $\cdot \cap G$.*

If moreover G is a finitely generated, their lattice of finite index subgroups are isomorphic, by $\bar{\cdot}$ and $\cdot \cap G$.

Lemma 8. *Let G be a residually finite group. Then*

1. *for any subgroup $M \leq \hat{G}$, we have $N_G(M \cap G) \geq N_{\hat{G}}(M) \cap G$;*
2. *for any subgroup $H \leq G$, we have $N_G(H) = N_{\hat{G}}(\bar{H}) \cap G$.*

Proof. Take $g \in N_{\hat{G}}(M) \cap G$ then, for all $h \in M \cap G \leq M$, ghg^{-1} is in M. But since g is in G and h too, we have $ghg^{-1} \in M \cap G$. This prove the first statement an half of the second statement.

On the other hand, if g is in $N_G(H)$ and $h \in \bar{H}$, we have $h = \lim_i h_i$ with all the h_i in H. Therefore, $ghg^{-1} = g(\lim h_i)g^{-1} = \lim gh_ig^{-1}$ belongs to $\bar{H}$ since all the gh_ig^{-1} are in H. $\square$

Corollary 7. *Let G be a finitely generated residually finite group. Then if $H \leq G$ is of finite index, we have $\overline{N_G(H)} = N_{\hat{G}}(\bar{H})$.*

Proof. Since H is of finite index, so is $\bar{H}$. Hence, both $N_{\hat{G}}(\bar{H})$ and $\overline{N_G(H)}$ are of finite index. Therefore, $N_{\hat{G}}(\bar{H}) = \overline{N_G(H)}$ if and only if $N_{\hat{G}}(\bar{H}) \cap G = N_G(H)$ and the conclusion follows. $\square$

The following lemma give us some link between wmc in G and wmc in $\hat{G}$.

Lemma 9. *Let G be a residually finite group and $M \leq \hat{G}$ a closed subgroup of the profinite completion.*

1. *If $\overline{M \cap G}$ is wmc in $\hat{G}$, then $M \cap G$ is wmc in G.*
2. *If $M \cap G$ is wmc in G, then $\overline{M \cap G}$ is weakly maximal among closed subgroups of the form $\overline{A \cap G}$.*

Proof. If $\overline{M \cap G}$ is wmc, then it has infinite index and $M \cap G$ has infinite index too. If $M \cap G$ is not wmc, then we have $M \cap G < H <_\infty G$, which gives us $\overline{M \cap G} < \bar{H} <_\infty G$.

On the other hand, suppose that $M \cap G$ is wmc. This implies that, $\overline{M \cap G}$ is of infinite index. If there is $\overline{M \cap G} \leq \overline{A \cap G} <_\infty \hat{G}$, then $M \cap G \leq A \cap G <_\infty G$. By maximality, $M \cap G = A \cap G$ and thus $\overline{M \cap G} = \overline{A \cap G}$. $\square$

Question 4. Is it possible to have more? Namely: $M \leq \hat{G}$ is wmc if and only if $M \cap H$ is wmc? Probably not...

5.3. Groups acting on infinite rooted tree

Recall that if G is a subgroup of $\mathrm{Aut}(T)$, then G is residually finite. In this case, we also have that $\bar{G}^T \coloneqq \bar{G}^{\mathrm{Aut}(T)}$, the closure of G in $\mathrm{Aut}(T)$, is a quotient of $\hat{G}$. This implies that $\hat{G}$ acts too on T, but this action is not necessarily faithful. If the action of G is transitive on each level, then $\hat{G}$ acts transitively on the boundary ∂T. The following lemma follows immediately from the definition of just infinite.

Lemma 10. *Let $G \leq \mathrm{Aut}(T)$ be an infinite residually finite group such that $\hat{G}$ is just infinite. Then $\hat{G} = \bar{G}^T$.*

Remark 1. Let H be any subgroup of G. Then we can consider different closure for H. Firstly the closure in $\hat{G}$, which we will denote by $\bar{H}$. There is also the closure in $\mathrm{Aut}(T)$ which we denote by $\bar{H}^T$. Finally, there is the closure of H in G, written $\bar{H}^G$, where the topology on G is the profinite one.

Lemma 11. *Let T be a spherically homogeneous rooted tree and let $G \leq \mathrm{Aut}(T)$ be a finitely generated subgroup. Then for all vertex v and all ray ξ we have*

1. $\mathrm{Stab}_G(v) = \mathrm{Stab}_{\bar{G}^T}(v) \cap G$ *and* $\mathrm{Stab}_G(\xi) = \mathrm{Stab}_{\bar{G}^T}(\xi) \cap G$*;*
2. $\overline{\mathrm{Stab}_G(v)}^T = \mathrm{Stab}_{\bar{G}^T}(v)$*.*

Proof. The first statement is obvious.

The statement on $\mathrm{Stab}(v)$ follows since it is a finite index subgroup and G is dense in $\bar{G}^T$ which is a finitely generated profinite group. □

Proposition 5. *Let T be a spherically homogeneous rooted tree and let $G \leq Aut(T)$ be a finitely generated subgroup. Suppose that this action if spherically transitive. Let ξ be any ray in T and $M = \mathrm{Stab}_{\hat{G}}(\xi)$ be the corresponding parabolic subgroup in $\hat{G}$. Then there exists a partition of ∂T in $[N_{\hat{G}}(M) \colon M]$ subsets (A_λ) of size $[\hat{G} : N_{\hat{G}}(M)]$ such that two rays in the same A_λ have distinct $\hat{G}$-stabilizers and $\{\mathrm{Stab}_{\hat{G}}(\xi) \mid \xi \in A_\lambda\}$ is the same for all β.*

Proof. Since G acts spherically transitive on T, $\hat{G}$ acts transitively on ∂T — this action is not necessarily faithfull. Let $N = N_{\hat{G}}(M)$. Infinite rays are in one-to-one correspondance with $\hat{G}/M$. Let $\{g_\beta \mid \beta \in B\}$ be a system of representatives of $\hat{G}/N$ and $\{n_\lambda \mid \lambda \in \Lambda\}$ a set of representatives of N/M. Therefore, for each rays ξ' there exists a unique g_β and a unique n_λ such that $\xi' = g_\beta n_\lambda \xi$. For all $\lambda \in \Lambda$, define $A_\lambda \coloneqq \{g_\beta n_\lambda \xi \mid \beta \in B\}$. The A_λ clearly form a partition of ∂T. Now, if ξ_1 and ξ_2 are two different rays in the same A_λ, we have $\xi_i = g_{\beta_i} n_\lambda \xi$, for $i = 1, 2$, with $\beta_1 \neq \beta_2$. Thus

$$\begin{aligned}\mathrm{Stab}_{\hat{G}}(\xi_i) &= (g_{\beta_i} n_\lambda)\,\mathrm{Stab}_{\hat{G}}(\xi)(g_{\beta_i} n_\lambda)^{-1}\\ &= g_{\beta_i} n_\lambda M n_\lambda^{-1} g_{\beta_i}^{-1}\\ &= g_{\beta_i} M g_{\beta_i}^{-1}\end{aligned}$$

Since B is a system of representative of $\hat{G}/N$, $\beta_1\beta_2^{-1}$ is not in $N = N_{\hat{G}}(M)$, which implies that $\mathrm{Stab}_{\hat{G}}(\xi_1)$ and $\mathrm{Stab}_{\hat{G}}(\xi_2)$ are distinct. Now, the set $\{\mathrm{Stab}_{\hat{G}}(\xi) \mid \xi \in A_\lambda\}$ is exactly

$$\{g_\beta n_\lambda M n_\lambda^{-1} g_\beta^{-1} \mid \beta \in B\} = \{g_\beta M g_\beta^{-1} \mid \beta \in B\}$$

which does not depend on λ. □

Remark 2. With the hypothesis of last proposition, the action of $\hat{G}$ on ∂T is totally nonfree if and only if $N_{\hat{G}}(M) = M$, that is if and only if all stabilizers are pairwise distinct. The action is extremely nonfree if and only if the set of rays corresponding to $\hat{G}/N_{\hat{G}}(M)$ is of positive measure. This is the case if and only if $[N_{\hat{G}}(M) : M]$ is finite. In this case, the maximal subset of ∂T with pairwise distinct stabilizers has measure $1/[N_{\hat{G}}(M) : M]$.

Def. of tot. non-free and extr. non-free

Proposition 6. *Let T be a spherically homogeneous rooted tree and $G \leq Aut(T)$ be a finitely generated subgroup which is torsion. Suppose that G acts transitively on levels and there exists a ray ξ such that $H = \mathrm{Stab}_G(\xi)$ is wmc and $\bar{H} \leq \hat{G}$ is wmc. Then*

1. *All $\hat{G}$-parabolic subgroups are pairwise distinct and wmc;*

2. *All G-parabolic subgroups are distinct from H;*

3. *Suppose moreover that for all $g \in \hat{G}$ we have $\overline{\bar{H} \cap G^g} = \bar{H}$. Then all G parabolic subgroups are pairwise distinct.*

Proof. Let $M = \mathrm{Stab}_{\hat{G}}(\xi)$. This is an infinite index closed subgroup of $\hat{G}$ and we have $M \cap G = H$. By maximality of $\bar{H}$, we have $M = \bar{H}$. Therefore, $H = N_G(H) = N_{\hat{G}}(M) \cap G$. Since $M \leq N_{\hat{G}}(M)$ is wmc, either $M = N_{\hat{G}}(M)$ or $N_{\hat{G}}(M)$ is of finite index in $\hat{G}$. But this would implies that $H = N_{\hat{G}}(M) \cap G$ is of finite index in G which is impossible. Thus $M = N_{\hat{G}}(M)$ and all $\hat{G}$-parabolic subgroups are wmc and pairwise distinct.

Now, let η be a ray. Then, there exists $g \in \hat{G}$ such that $M_\eta := \mathrm{Stab}_{\hat{G}}(\eta) = g\bar{M}g^{-1}$. Therefore, $H_\eta := \mathrm{Stab}_G(\eta) = M_\eta \cap G = gMg^{-1} \cap G$.

Suppose that $H_\eta = H$. This implies that $\overline{gMg^{-1} \cap G} = \bar{H} = M$ and thus, $M \leq M^g$. By maximality, we have $M = M^g$ which implies that g fixes η.

Finally, observe that $\overline{\bar{H} \cap G^g} = \bar{H}$ for all g if and only if $\overline{\bar{H}^g \cap G} = \bar{H}^g$ for all g. Hence, we can apply the same argument as above to prove that G stabilizers are pairwise distincts. ∎

Question 5. Let $\mathcal{G}$ be the Grigorchuk group. Is it true that for all wmc H, $\bar{H}$ is wmc?

5.4. P-Groups

Shalev showed (Lemma 1.2) that if $\hat{G}$ is virtually pro-p, then every non-open subgroup is contained in a wmc subgroup. Put ref

In the following, we will consider finitely generated residually finite p-group[1]. In this case, the pro-finite completion and the pro-p completion of G coincide. Let G be a finitely generated residually finite p-group and $M < \hat{G}$ be a wmc subgroup. We have $M = \bigcap M_\alpha$ where the intersection is over all finite index $M < M_\alpha < \hat{G}$. Since M is of infinite index, there exists a descending chain of finite index subgroups $\hat{G} = M_0 > M_1 > \dots$ such that $\bigcap M_i \geq M$. But $\bigcap M_i$ is a closed subgroup of infinite index and therefore we have $\bigcap M_i = M$. We can choose M_{i+1} to be maximal in M_i and thus of index p. We can therefore construct a p-regular coset tree $T_M = T_{(M_i)}$. The group $\hat{G}$ acts on T_M by left multiplication and this action is transitive on ∂T.

On the other hand, we have $H := M \cap G = \bigcap H_i$ where $H_i = M_i \cap G$. This gives us a coset tree T_H for H. Moreover, if $\{g_{i,j}\}$ is a transversal system for H_i, it is also a transversal system for M_i. Thus T_H and T_M are isomorphic by $\phi(gH_i) = gM_i$. If G is just infinite, its action on T_H is faithfull. Indeed, the image of G in $\mathrm{Aut}(T_H)$ is infinite

[1] In fact, it is sufficient to take G finitely generated residually finite torsion group such such that $\hat{G}$ is virtually pro-p.

since the action is spherically transitive. Therefore, $\bar{G}^{\mathrm{Aut}(T_H)}$ and $\hat{G}$ act transitively on ∂T_H. On the other hand, $\hat{G}$ also acts on $T_M \cong T_H$. These two actions are in fact the same.

Proposition 7. *Let G be a finitely generated residually finite p-group and $M < \hat{G}$ be a wmc subgroup such that $M \cap G$ is wmc. For the action of $\hat{G}$ on T_M, all parabolic subgroups are wmc and they are pairwise distinct.*

Proof. Let $H = M \cap G$. Since H is wmc and G is torsion, we have $H = N_G(H)$. On the other hand, $M \leq N_{\hat{G}}(M)$. By Lemma 8, $N_{\hat{G}}(M) \cap G \leq N_G(H) = H$ is of infinite index. Therefore, $M = N_{\hat{G}}(M)$ and we conclude by Proposition 5 and Remark 2. $\square$

Proposition 8. *Le G be a residually finite just infinite p-group and H be a wmc subgroup. Then there exists a coset tree T_H such that $\mathrm{Stab}_G(\bar{0}) = H$ and*

1. *All $\hat{G}$ parabolic subgroups are wmc and pairwise distinct.*
2. *Any two rays in the G-orbit of $\bar{0}$ have distinct stabilizers.*
3. *If g is in $N_{\hat{G}}(G)$, then $\mathrm{Stab}_G(g\bar{0})$ is distinct from H.*

Proof. We have $H = N_G(H)$ by Corollary 3. Since $N_{\hat{G}}(\bar{H}) \cap G = N_G(H)$, the subgroup $N_{\hat{G}}(\bar{H})$ is of infinite index in $\hat{G}$ and contained in some wmc M. We have $M \cap G = H$ (it is an infinite index closed subgroup containing H) and the descending intersection of finite index subgroups $\bigcap M_i = M$ gives us $H = \bigcap H_i$ with $H_i = M_i \cap G$, and thus $\bar{H}_i = M_i$. We can therefore construct two coset trees: T_H and T_M corresponding to the sequences $(H_i)_i$ and $(\bar{H}_i)_i = (M_i)_i$. The discussion before Proposition 7 shows that these trees coincide.

Now, $\mathrm{Stab}_G(\bar{0}) = H$ while $\mathrm{Stab}_{\hat{G}}(\bar{0}) = M$. By Proposition 7, all $\hat{G}$-parabolic subgroups are pairwise distinct and wmc. For all $g \in \hat{G}$, we have $\mathrm{Stab}_G(g\bar{0}) = \mathrm{Stab}_{\hat{G}}(g\bar{0}) \cap G = M^g \cap G$.

If ξ and ζ are in the same G-orbit,

Suppose that we have $\overline{\mathrm{Stab}_G(g\bar{0})} = \bar{H}^g$. In this case, the parabolic subgroups $\mathrm{Stab}_G(\bar{0})$ and $\mathrm{Stab}_G(g\bar{0})$ are distinct. Indeed, if it was not the case, we would have $\bar{H} = \bar{H}^g$, but this implies that g belongs to $N_{\hat{G}}(\bar{H}) \leq M$ and thus g fixes $\bar{0}$. Note that $\overline{\mathrm{Stab}_G(g\bar{0})} = \bar{H}^g$ if and only if $\overline{M \cap G^{g^{-1}}} = \bar{H}$. In particular, for all $g \in N_{\hat{G}}(G)$ we have that $\mathrm{Stab}_G(g\bar{0})$ is distinct from $\mathrm{Stab}_G(\bar{0})$. $\square$

Corollary 8. *There exists at least four IRS with uncountable disjoint supports in $\hat{\mathcal{G}}$, the profinite completion of the first Grigorchuk group.*

Proof. If M and K are tree-equivalent, so are $M \cap \mathcal{G}$ and $K \cap \mathcal{G}$. Therefore, by Corollary 1 and 8, there is at least four non-tree equivalent wmc M_i in $\hat{\mathcal{G}}$. Since, by Proposition 8, for a given i, all the $\hat{\mathcal{G}}$ parabolic subgroups of T_{M_i} are pairwise distinct, the measure on ∂T gives raise to an IRS μ_i of $\mathcal{G}$. The support of μ_i consist of all conjugates of M_i. Finally, since the M_i are not tree-equivalent, they are not conjugate. $\square$

Remark 3. For $\mathcal{G}$ the grigorchuk group, $N_{\hat{\mathcal{G}}}(\mathcal{G})$ is countable. Indeed, Grigorchuk and Sidki showed that for the original action on T we have $N_{\mathrm{Aut}(T)}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})$ is countable and we obviously have $N_{\hat{\mathcal{G}}}(\mathcal{G}) \leq N_{\mathrm{Aut}(T)}(\mathcal{G})$.

5.5. Some computations and ideas

Proposition 8 is mainly a result on actions of $\hat{G}$ while we would like a result on action of G. This raises the following question.

Question 6. Let $G \leq Aut(T)$ be a finitely generated group acting transitively on levels and such that all $\hat{G}$ parabolic subgroups are wmc and at least one G parabolic subgroup is wmc. What can be say about G parabolic subgroups ? Are they pairwise distinct, wmc? If not, is this true for a subset of positive measure? Or an uncountable subset?

One possible direction to answer this question is to look at the restriction of the function $\Psi(M) = M \cap G$ to the set of conjugate of M. Is this function injective? And if not, how many preimage can have a subgroup $M^g \cap G$?

Another possibility is to look directly at $\mathrm{Stab}_G(\xi)$ and at the partion of ∂T by the subset $S_A \coloneqq \{\xi \mid \mathrm{Stab}_G(\xi) = A\}$. We have the following lemma.

Lemma 12. *Let G be a residually finite just infinite p-group, H a wmc subgroup and T_H the coset tree constructed in Proposition 8. Suppose that one of the following conditions is true.*

1. *All parabolic subgroups are wmc;*
2. *For all ξ and η in ∂T, if $\mathrm{Stab}_G(\xi) \leq \mathrm{Stab}_G(\eta)$, then they are equal;*
3. *For all $A \leq G$, the subset S_A is closed;*
4. *The group G is hereditarily just infinite and there is no ray with trivial stabilizer;*
5. *For all $\xi \in \partial T$ and all $v \in \xi$, we have $\mathrm{Stab}_G(C_v) \lneqq \mathrm{Stab}_G(\xi)$;*
6. *For all A and all v in T, there exists $w \leq v$, such that $C_w \subset \partial T \setminus S_A$.*

Then, there is an uncountable number of pairwise distinct parabolic subgroups.

Proof. First of all, for all A, S_A contains at most one ray in the G-orbit of $\bar{0}$. In particular, for all A and all $v \in T$, $C_v \not\subset S_A$. Indeed, by Proposition 8, all rays in the G-orbit of $\bar{0}$ have pairwise distinct stabilizers.

It is clear that Hypothesis 1 implies Hypothesis 2. If ξ_i are all in S_A and ξ_i converge to ξ, we have $A \leq \mathrm{Stab}_G(\xi)$. Hence, Hypothesis 2 implies Hypothesis 3.

Now, take A such that S_A is closed. Since S_A is closed, $S_A = \partial T \setminus \bigcup_{\lambda\in\Lambda} C_{w_\lambda}$. We already know that for any v we have $C_v \not\subset S_A$ and thus, there exists ξ in $C_v \setminus S_A$. Therefore, there exists $\lambda \in \Lambda$ such that ξ belongs to C_{w_λ}. Since ξ belongs to C_v and to C_{w_λ}, we have either $v \leq w_\lambda$ or $w_\lambda \leq v$. Let z be the smallest of the two vertices. Then $z \leq v$ and $C_z \subseteq \partial T \setminus S_A$. This shows that Hypothesis 3 implies Hypothesis 6.

Observe that $\mathrm{Stab}_G(C_v)$ is a normal subgroup of $\mathrm{Stab}_G(v)$, a finite index subgroup of G. Therefore, if G is herediteraly just infinite, we have $\mathrm{Stab}_G(C_v) = \{1\}$ for all v. Hence, Hypothesis 4 implies Hypothesis 5.

For all ξ and all $v \in \xi$ we have $\mathrm{Stab}_G(C_v) \leq \mathrm{Stab}_G(\xi)$. Suppose that Hypothesis 6 does not holds. That implies that there exists A and v such that for all $w \leq v$ there exists ξ_w with $w \in \xi_w$ and $\xi_w \in S_A$. This implies that $\mathrm{Stab}_G(\xi_w) = A$ and that $\mathrm{Stab}_G(\xi_v) = A \leq \mathrm{Stab}_G(C_v)$. We just proved that Hypothesis 5 implies Hypothesis 6.

Finally, suppose that Hypothesis 6 holds and that there is only a countable number of pairwise distinct parabolic subgroups $(A_i)_{i\geq 0}$. We thus have that $\partial T = \bigcup_{i\geq 0} S_{A_i}$. Let v_0 be the root of the tree. By hypothesis, there exists $v_1 \leq v_0$ with $C_{v_1} \subset \partial T \setminus S_{A_1}$. There also exists $v_2 \leq v_1$ with $C_{v_2} \subset \partial T \setminus S_{A_2}$, and so on. Let $\xi = (v_1 v_2 v_2 \dots)$. By definition, ξ belongs to ∂T but does not belongs to any S_{A_i}, which is absurd. $\square$

In order to prove that all G parabolic subgroups are distinct from H, we want $\overline{\mathrm{Stab}_G(g\bar{0})} = \bar{H}^g$ (Proposition 8). But in general, $\overline{\mathrm{Stab}_G(g\bar{0})} = \overline{M^g \cap G} \geq \overline{\bar{H}^g \cap G}$ and $\overline{\bar{H}^g \cap G} \leq \bar{H}^g$.

Remark 4. We have $M_i = \bar{H}_i$, $M^g = \bigcap \bar{M}_i^g$ and $\bar{H}^g = \overline{\bigcap H^g} \leq M^g$.

We also have that for $g = \lim g_j$, for all i, the sequence $g_j H_i g_j^{-1}$ stabilizes. But this does not implies that $g_j H_i g_j^{-1} = gHg^{-1}$ for j big enough. For example, $H_0 = G$ and as said before, $N_{\hat{G}}(G)$ is countable and therefore not equal to $\hat{G}$.

Remark 5. Suppose that if $M \not\subset M^g$ then $M \cap G \not\subset M^g \cap G$. This implies that all G parabolic subgroups are pairwise distinct.

6. Questions, conjectures and more

Question 7. Let X be an infinite topological space such that in every infinite quotient of X, there is no dense finite subset. Suppose that we have a just infinite action of G on X. Is it true that all stabilizers are weakly maximal subgroups of G ?

Question 8. Are all weakly maximal subgroups of $\mathcal{G}$ conjugated in $\hat{\mathcal{G}}$. If not, what can we say.

Conjecture 1. *Let H be a wmc of G a residually finite p-group. We know that $H = \bigcap_{n\geq 1} H_n$ with $[H_n : H_{n+1}] = p$. Then $\bar{H} = \bigcap_{n\geq 1} \bar{H}_n$.*

Proof. We already know (true for all top space) that $\bar{H} \subseteq \bigcap_{n\geq 1} \bar{H}_n$. Moreover, here the intersection is made over a descending family. If it was a finite family, then the claim is true. General case ?

We also have: "If $X \subseteq \hat{G}$ a profinite group, then $\bar{X} = \bigcap_{N \lhd_0 \hat{G}} X \cdot N$". Now, $H < G < \hat{G}$ and thus

In general, $(A \cap B)X \neq AX \cap BX$.

$$
\begin{aligned}
\bar{H} &= \bigcap_{N \lhd_0 \hat{G}} H \cdot N = \bigcap_{N \lhd_0 \hat{G}} \left(\bigcap_n H_n \right) \cdot N \\
&\subseteq \bigcap_{N \lhd_0 \hat{G}} \left(\bigcap_n H_n \cdot N \right) = \bigcap_n \left(\bigcap_{N \lhd_0 \hat{G}} H_n \cdot N \right) \\
&= \bigcap_n \bar{H}_n.
\end{aligned}
$$

That is, we have $\bar{H} \subseteq \bigcap_n \bar{H}_n$... $\square$

In general, we have $(\bigcap_n H_n)N \subset \bigcap (H_n N)$ but not the other inclusion. Maybe we can use the fact that N is of finite index, or clopen. Observe, that this would imply that for all M with $M \cap G = H$ wmc, then $\overline{M \cap G} = M$.

7. Finitely generated weakly maximal subgroups

In this section we investigate a particular example, due to Pervova, of a weakly maximal subgroup of the Grigorchuk group. We then turn our attention to finitely generated weakly maximal subgroups.

Recall that $\mathcal{G} = \langle a, b, c, d \rangle$ is the first Grigorchuk group. This group is branch over $K := \langle (ab)^2 \rangle^{\mathcal{G}}$ and we have[2] $K <_2 B := \langle b \rangle^G <_2 \tilde{B} := \langle B, (ad)^2 \rangle <_2 4\mathcal{G}$, where all this subgroups are normal. Another important subgroup of $\mathcal{G}$ is $H := \mathrm{Stab}_{\mathcal{G}}(1) <_2 \mathcal{G}$, the stabilizer of the first level.

The left and right projections are denoted by $\pi_0, \pi_1 \colon G \to G$.

7.1. Pervova's example

Lemma 13. *Let $1 \neq x \in \mathrm{Stab}_{\mathcal{G}}(1)$. Then $\langle x \rangle^{\tilde{B}}$ has infinite index in $\mathcal{G}$ if and only if $\pi_i(x) = 1$ for some $i \in \{0, 1\}$.*

Proof. If $\pi_i(x) = 1$, then $\langle x \rangle^{\tilde{B}} \leq \{1\} \times \mathcal{G}$ (or $\langle x \rangle^{\tilde{B}} \leq \mathcal{G} \times \{1\}$) and thus cannot contain $\mathrm{Stab}_{\mathcal{G}}(n)$ for any n. Therefore, if $\pi_i(x) = 1$, then $\langle x \rangle^{\tilde{B}}$ has infinite index in $\mathcal{G}$ by the congruence subgroup property.

On the other hand, suppose that $x = (x_0, x_1)$ with $x_i \neq 1$. In this case, by [14], the centralizer $C_{\mathcal{G}}(x_i)$ has infinite index in $\mathcal{G}$. This implies that there exists $y_i \in K$, $i \in \{0, 1\}$ such that y_i does not belongs to the centralizer of x_i. We then have $[x_i, y_i] \neq 1$ and thus $[x, (1, y_1)] = (1, [x_1, y_1])$ and $[x, (y_0, 1)] = ([x_0, y_0], 1)$ with both $[x_i, y_i]$ belonging to K. Since $\pi_i(\tilde{B}) = \mathcal{G}$, we have $\langle x \rangle^{\tilde{B}} \geq \langle [x_0, y_0] \rangle^{\mathcal{G}} \times \langle [x_1, y_1] \rangle^{\mathcal{G}}$. Both $\langle [x_i, y_i] \rangle^{\mathcal{G}}$ are non-trivial normal subgroups of $\mathcal{G}$ and therefore of finite index since $\mathcal{G}$ is just-infinite. This shows that $\langle x \rangle^{\tilde{B}}$ itself is of finite index in $\mathcal{G}$. $\square$

Conjecture 2. *Let $1 \neq X$ be any element of $\bar{\mathcal{G}}$. Then $[\mathcal{G} : C_{\mathcal{G}}(x)] = \infty$.*

[2] The fact that $K <_2 B <_8 \mathcal{G}$ is well-known. The other indices are easily computed on Schreier graphs.

Observe that the above conjecture is true if x belongs to $\mathcal{G}$. In general, this implies that for every $1 \neq x \in \bar{\mathcal{G}}$, there exists $y \in \bar{K}$ such that $[x,y] \neq 1$. Indeed, $[\bar{\mathcal{G}} : C_{\bar{\mathcal{G}}}(x)] = \infty$ if and only if $[\mathcal{G} : C_{\bar{\mathcal{G}}}(x) \cap \mathcal{G}] = [\mathcal{G} : C_{\mathcal{G}}(x)] = \infty$. In this case, the finite index subgroup $\bar{K}$ cannot be contained in $C_{\bar{\mathcal{G}}}(x)$.

Lemma 14 (Assuming Conjecture 2)**.** *Let $1 \neq x \in \mathrm{Stab}_{\bar{\mathcal{G}}}(1)$. Then $\overline{\langle x\rangle^{\tilde{B}}}$ has infinite index in $\bar{\mathcal{G}}$ if and only if $\pi_i(x) = 1$ for some $i \in \{0,1\}$.*

Proof. Assuming the conjecture true, the proof is the same as for Lemma 13. □

We now turn our attention on the following example of weakly maximal subgroup due to Pervova:

$$W := \langle a, \mathrm{diag}(\tilde{B} \times \tilde{B}), \{1\} \times K \times \{1\} \times K\rangle$$

where $\mathrm{diag}(\tilde{B} \times \tilde{B}) = \{(x,x) \mid x \in \tilde{B}\} \leq \mathcal{G} \times \mathcal{G}$ is the diagonal subgroup.

Lemma 15. *The group W is a subgroup of $\mathcal{G}$.*

Proof. By definition a belongs to $\mathcal{G}$ and it is well-known that $\{1\} \times K \times \{1\} \times K$ is a subgroup of $\mathcal{G}$. Therefore, it remains to check that $\mathrm{diag}\,\tilde{B} \times \tilde{B}$ is also a subgroup of $\mathcal{G}$. We have $B = \langle K, b, (ad)^2\rangle$ and we know that for every $k \in K$ the element (k,k) belongs to $\mathcal{G}$. On the other hand, $(b,b) = d \cdot d^a$ and $((ad)^2, (ad)^2) = ((ad)^2, (da)^2) = (c \cdot c^a)^2$ also belong to $\mathcal{G}$. □

Theorem 3. *W is a finitely generated weakly maximal subgroup of $\mathcal{G}$.*

Proof. The subgroup K and $\tilde{B}$ being of finite index in $\mathcal{G}$, a finitely generated group, are finitely generated and so is W. Now, if g is an element of $W \cap (K \times \{1\} \times \{1\} \times \{1\})$, we have $g = (k,1,1,1)$ and also $g = (g_0, g_1, g_0, g_3)$ which implies $g = 1$.We have shown that $W \cap (K \times \{1\} \times \{1\} \times \{1\}) = \{1\}$ and therefore that W is of infinite index.

We now want to prove that W is weakly maximal. That is, for all $x \in \mathcal{G} \setminus W$, the subgroup $\widetilde{W} := \langle W, x\rangle$ is of finite index in $\mathcal{G}$. Since a belongs to W, we can assume that x belongs to H and $x = (x_0, x_1)$. We have $\mathcal{G}/B = \{1, a, d, ad, ada, \dots, (ad)^3a\} \cong D_{2\cdot 4}$, the dihedral group of order 8, and hence $\mathcal{G}/\tilde{B} = \{1, a, d, ad\}$. By factorizing the first coordinate by $\pi_0(W) \geq \tilde{B}$ we can assume that x_0 belongs to $\{1, a, d, ad\}$ which leave us with four cases to check. If x_0 is not in H, then $\widetilde{W}$ contains $(\{1\} \times K \times \{1\} \times \{1\})^{x_0} = K \times \{1\} \times \{1\} \times \{1\}$ and a. In this case, $\widetilde{W}$ contains $K \times K \times K \times K$ and is therefore of finite index. We can hence suppose that x_0 is in H, and by symmetry, that x_1 is also in H. It thus remains to check to cases: $(1, x_1)$ and (d, x_1) with x_1 in H.

(i) If $x_0 = 1$, then $x_1 \neq 1$. In this case, $\widetilde{W}$ contains $\{1\} \times \langle x_1\rangle^{\tilde{B}}$ and $\mathrm{diag}(\langle x_1\rangle^{\tilde{B}} \times \langle x_1\rangle^{\tilde{B}}) \leq \mathrm{diag}(\tilde{B} \times \tilde{B})$. This implies $\widetilde{W} \geq \langle x_1\rangle^{\tilde{B}} \times \langle x_1\rangle^{\tilde{B}}$. If $\langle x_1\rangle^{\tilde{B}}$ has finite index in $\mathcal{G}$, then $\widetilde{W}$ has also finite index in $\mathcal{G}$. We can therefore assume that $\langle x_1\rangle^{\tilde{B}}$ has infinite index in $\mathcal{G}$, which implies by Lemma 13 that $x_1 = (1,z)$ or $x_1 = (z,1)$, $z \neq 1$. In both cases, we have $x = (1, x_1)$ is an element of $\mathrm{Rist}_{\mathcal{G}}(2) = K \times K \times K \times K$ which implies that z is in K. This rules out the case $x_1 = (1,z)$, since in this case we would have $x = (1, 1, 1, z) \in \{1\} \times K \times \{1\} \times K \leq W$. On the other hand, $\langle z \rangle^{\mathcal{G}}$ is a non-trivial normal subgroup of $\mathcal{G}$ and thus of finite index. Therefore, $\widetilde{W}$ contains $A := \langle z \rangle^{\mathcal{G}} \times K$, a finite index subgroup of $\tilde{B}$, and hence it also contains $\operatorname{diag}(A \times A)$. Altogether, we have $\widetilde{W} \geq A \times A$, where $A = \langle z \rangle^{\mathcal{G}} \times K$ is a finite index subgroup of $\mathcal{G}$. This implies that $\widetilde{W}$ is of finite index.

(ii) We will now show that the case $x_0 = d$ cannot happen if x_i is in H. Indeed, (d, x_1) belongs to $\mathcal{G}$ if and only if $(1, ax_1) = c^a \cdot (d, x_1)$ is in $\mathcal{G}$. But in this case, $(1, ax_1)$ belongs to $\operatorname{Rist}_{\mathcal{G}}(1) = B \times B$ and so ax_1 is in $B \leq H$, which is impossible if $x_1 \in H$.

□

Lemma 16. *The subgroup $W \cap H$ is a weakly maximal subgroup of H with both left and right projections equal to $\tilde{B} <_4 \mathcal{G}$.*

Proof. We have $W \cap H = \langle \operatorname{diag}(\tilde{B} \times \tilde{B}), \{1\} \times K \times \{1\} \times K \rangle$ and the result on projections follows directly.

For the weak maximality, let x be in $H \setminus W$ and look at $\widetilde{W} := \langle x, W \rangle$. Then $x = (x_0, x_1)$ and factorizing the first factor by $\tilde{B}$, we can assume that x_0 is either 1 or d. The rest of the proof is the same as the proof of the weak maximality of W in $\mathcal{G}$. □

Conjecture 3. *1. $\overline{W}$ is a (topologically) finitely generated weakly maximal closed subgroup of $\bar{\mathcal{G}} = \hat{\mathcal{G}}$;*

2. There is a continuum of two-by-two distinct conjugates of $\overline{W}$ in $\bar{\mathcal{G}}$ (and all of them are weakly maximal).

3. ?? There is a continuum of different subgroups of type $\mathcal{G} \cap \overline{W}^{\bar{g}}$, $\bar{g} \in \bar{\mathcal{G}}$ and all of them are weakly maximal ??

Proof. We will prove assertions 1 and 2 under the assumption that Conjecture 2 is true.

The application $\bar{\ }$: $\operatorname{Sub}_{\mathrm{cl}}(G) \to \operatorname{Sub}_{\mathrm{cl}}(\bar{\mathcal{G}})$ that send a close (in the profinite topology) subgroup of $\mathcal{G}$ to its closure in $\bar{\mathcal{G}}$ send infinite index subgroups to infinite index subgroups. Since W is weakly maximal in $\mathcal{G}$, it is close and we have that $\overline{W}$ is an infinite index subgroup of $\bar{\mathcal{G}}$. The element a normalizes both $\operatorname{diag}(\tilde{B} \times \tilde{B})$ and $\{1\} \times K \times \{1\} \times K$ and since K is normal and $\tilde{B} \leq H$, the subgroup $\{1\} \times K \times \{1\} \times K$ is normalized by $\operatorname{diag}(\tilde{B} \times \tilde{B})$ Therefore, $W = \{1, a\} \cdot \operatorname{diag}(\tilde{B} \times \tilde{B}) \cdot (\{1\} \times K \times \{1\} \times K)$ and $\overline{W} = \overline{\{1, a\}} \cdot \overline{\operatorname{diag}(\tilde{B} \times \tilde{B})} \cdot \overline{\{1\} \times K \times \{1\} \times K}$. The subgroup $\{1, a\}$ is closed and we have $\overline{\{1\} \times K \times \{1\} \times K} = \{1\} \times \bar{K} \times \{1\} \times \bar{K}$ and $\overline{\operatorname{diag}(\tilde{B} \times \tilde{B})} = \operatorname{diag}(\bar{\tilde{B}} \times \bar{\tilde{B}})$. Altogether, we have

$$\overline{W} = \langle a, \operatorname{diag}(\bar{\tilde{B}} \times \bar{\tilde{B}}), \{1\} \times \bar{K} \times \{1\} \times \bar{K} \rangle$$

is a (topologically) finitely generated subgroup of $\bar{\mathcal{G}}$.

For all n we have $\operatorname{Stab}_{\mathcal{G}}(n) = \operatorname{Stab}_{\bar{\mathcal{G}}}(n) \cap \mathcal{G}$ and $\operatorname{Rist}_{\mathcal{G}}(n) = \operatorname{Rist}_{\bar{\mathcal{G}}}(n) \cap \mathcal{G}$. Since these subgroups are of finite index, for all n we have

$$\overline{\operatorname{Stab}_{\mathcal{G}}(n)} = \operatorname{Stab}_{\bar{\mathcal{G}}}(n) \qquad \overline{\operatorname{Rist}_{\mathcal{G}}(n)} = \operatorname{Rist}_{\bar{\mathcal{G}}}(n)$$

In particular, we have

$$\mathrm{Rist}_{\bar{\mathcal{G}}}(1) = \bar{B} \times \bar{B} \qquad \mathrm{Rist}_{\bar{\mathcal{G}}}(2) = \bar{K} \times \bar{K} \times \bar{K} \times \bar{K}$$

On the other hand, the closure preserves transversals for finite index subgroups. In particular, $\bar{\mathcal{G}}/\bar{H} = \{1, a\}$ and $\bar{\mathcal{G}}/\bar{\bar{B}} = \{1, a, ad, d\}$. Let x be an element from $\mathcal{G} \setminus \overline{W}$ and look at $\widetilde{\overline{W}} := \langle \overline{W}, x\rangle$. The proof that $\widetilde{\overline{W}}$ is of finite index is the same as the one for $\widetilde{W}$, where Lemma 13 is replaced by Lemma 14.

Since W is weakly maximal we can construct a 2-regular coset tree T_W. Since $\overline{W}$ is weakly maximal closed, then for the action of $\bar{\mathcal{G}}$ on T_W, parabolic subgroups are 2-by-2 distincts; and there is $2^{\aleph_0}$ many such subgroups. □

Remark 6. We have te following alternative. Either some parabolic subgroups for the action $\mathcal{G}$ on T_W are finitely generated and some are infinitely generated, or there is only a countable number of distincts parabolic subgroups. Indeed, since Γ is finitely generated, there are only countably many finitely generated subgroups.

In both cases we have a comportment that is far away from the classical action, where all parabolic subgroups are of infinite rank and two-by-two distincts.

7.2. Block subgroups

Since $\mathcal{G}$ is finitely generated, it has at most countably many finitely generated subgroups. In particular, it has at most countably many finitely generated weakly maximal subgroups.

Conjecture 4. *The group $\mathcal{G}$ has infinitely many distincts finitely generated weakly maximal subgroups, up to* $\mathrm{Aut}(\mathcal{G})$.

An important tool for the study of finitely generated subgroups of $\mathcal{G}$ is the notion of block subgroups due to Grigorchuk and Nagnibeda (to appear). Write K_u for the subgroup of $\mathcal{G}$ acting as K on T_u and trivially outside. Observe that if u is the root, then $K_u = K$ and if u is of level at least 2, then $K_u = \mathrm{Rist}_{\mathcal{G}}(u)$.

Definition 4. Let S be a section of T and $S = E \sqcup \{E_i\}_{i\in I} \sqcup S'$ a covering of S with all E_i of cardinality at least 2. Observe that S is necessarily finite. If $E_i = u_1, \dots, u_j$ we can define a *diagonal subgroup* $D_i := \mathrm{diag}(K_{u_1} \times \phi_2(K_{u_2}) \times \cdots \times \phi_j(K_{u_j}))$ where the ϕ_t are automorphism of $\mathcal{G}$. These datas (the covering of S and the choices of ϕ_t) determine a *block subgroup* U of $\mathcal{G}$ by

$$U := \prod_{u\in E} K_u \times \prod_{i\in I} D_i \times \prod_{u\in S'} \{1\}$$

It is obvious that block subgroups are finitely generated. On the other hand, we have the following result.

Proposition 9 (Grigorchuk-Nagnibeda)**.** *A subgroup of $\mathcal{G}$ is finitely generated if and only if it contains a block subgroup as a finite index subgroup.*

In order to find weakly maximal subgroups that are finitely generated, it may be interesting to start with infinite index block subgroups that are maximal among infinite index block subgroups. The following lemma directly follows from the congruence subgroup property.

Lemma 17. *A block subgroup*

$$U = \prod_{u\in E} K_u \times \prod_{i\in I} D_i \times \prod_{u\in S'} \{1\}$$

is of finite index if and only if $E = S$, that is if $S' = I = \emptyset$

Lemma 18. *Let*

$$U = \prod_{u\in E} K_u \times \prod_{i\in I} D_i \times \prod_{u\in S'} \{1\}$$

be a block subgroup that is maximal among block subgroup of infinite index. Then, $S = E \sqcup E_1$ with $E_1 = \{v, w\}$ and S is the "uppermost" section that contains v and w.

Proof. By assumption, U is of infinite index and either S' or I is non-empty. If S' is non-empty, then U is not maximal. Indeed, let $u \in S'$ and v and w be the two children of u. Then U is properly contained in U', the block subgroup obtained from U by replacing the factor $\{1\}_u$ by $\{1\}_v \times \{1\}_w$. By the lemma, U' is still of infinite index. Therefore, S' is empty and I is non-empty.

If I has more than one element, than U is contained in U', the block subgroup obtained by replacing D_2 by $K_{u_1} \times K_{u_j}$, which is still of infinite index. This implies that I has exactly one element.

Now, if E_1 has more than two elements, we can construct $U < U'$ by replacing D_1 by $\operatorname{diag}(K_{u_1} \times \phi_2(K_{u_2})) \times K_{u_3} \times \cdots \times K_{u_t}$.

Finally, we want to prove that S is the "uppermost" section containing v and w. To be precise, we can put a partial order on section by saying that $S \leq \tilde{S}$ if every element of S is below some elements of $\tilde{S}$. Among sections containing v and w, there is a greatest element $\tilde{S}$ which consist of v, w and all roots of maximal trees in $T \setminus$ path from v to the root, path from w to the root. It is evident that if our block subgroup correspond to a section $S < \tilde{S}$, then it is a subgroup of the block subgroup corresponding to $\tilde{S}$. □

We know turn our attention to a special case of block subgroups. Define U_n by $E = \{0, 10, \ldots, 1^{n-1}0\}$, $E_1 = \{1^n0, 1^{n+1}\}$ and $D_1 = \operatorname{diag}(K_{1^n0} \times K_{1^{n+1}})$. The subgroups U_n are maximal among block subgroups of infinite index and they are in the stabilizer of 1^n. Moreover, U_0 is the block subgroup corresponding to W, the weakly maximal subgroup of the last subsection. In order to construct more weakly maximal subgroup, we are going to extend U_n to a bigger subgroup W_n which will be weakly maximal. Since U_n is of infinite index, such a W_n can always be founded, but we want to ensure that it will be finitely generated.

In order to do that, it is useful to have a description of $\operatorname{Stab}_{\mathcal{G}}(1^n)$. This was done by Kravchenko (see also [2, Theorem 4.4]) who describes $\operatorname{Stab}_{\mathcal{G}}(1^n)$ has an iterated semi-direct product, as depicted in Figure 1.

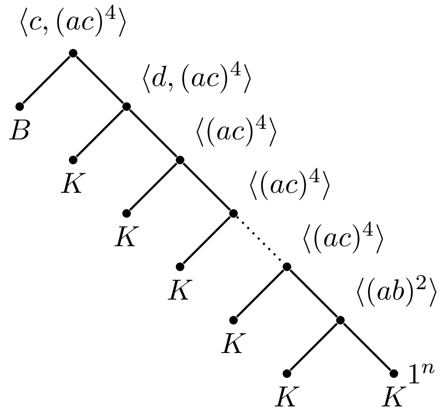

Figure 1: The stabilizer in $\mathcal{G}$ of the vertex 1^n.

Observe that the projection of $\mathrm{Stab}_{\mathcal{G}}(1^n)$ on 1^n is equal to $\mathcal{G}$. Therefore, for all weakly maximal subgroup M of Γ, we can define M_n as the subgroup of $\mathrm{Stab}_{\mathcal{G}}(1^n)$ consisting of elements g such that the projection of g on 1^n is in M.

Lemma 19. *For all weakly maximal subgroup M and all n, the subgroup M_n is weakly maximal.*

Proof. Let g be in $\mathcal{G}\setminus M_n$. If g does not stabilizes 1^n, then M_n contains $K_{g(1^n)}$. Therefore, $\langle g, M_n\rangle$ contains K_{1^n} and also K_v for every $v \neq 1^n$ of level n. This implies that $\langle g, M_n\rangle$ has finite index in $\mathcal{G}$. On the other hand, if g is in $\mathrm{Stab}_{\mathcal{G}}(1^n)$, then by definition, g_{1^n} does not belong to M. This implies that the projection of M_n on q^n, which is equal to $\langle g_{1^n}, M\rangle$, has finite index in $\mathcal{G}$. Therefore, $\langle g, M\rangle$ is a finite index subgroup of $\mathcal{G}$. This conclude the proof of the weak maximality of M_n. $\square$

Lemma 20. *Let $(W_n)_{n\in\mathbf{N}}$ be the weakly maximal subgroups constructed from W, the Pervova's example. Then, if $n \neq m$, the subgroups W_n and W_m are distincts, non-conjugate and even not in the same class of equivalence under* $\mathrm{Aut}(\mathcal{G})$.

Proof. Grigorchuk and Sidki showed that $N_{\mathrm{Aut}(T)}(\mathcal{G}) = \mathrm{Aut}(\mathcal{G})$. We are going to prove that if $m > n$, then for every $1 \neq \phi \in \mathrm{Aut}(\mathcal{G})$, the subgroup $\langle W_n, \phi(W_m)\rangle$ has finite index in $\mathcal{G}$. If ϕ does not stabilizes 1^n, then $\langle W_n, \phi(W_m)\rangle$ contains K_v for every vertex of level n and is thus of finite index.

If ϕ does stabilizes 1^n, then $\phi(W_m)$ contains either $K_{1^{n+1}}$ or $K_{1^n 0}$. On the other hand, W_n contains $\mathrm{diag}(K_{1^{n+1}} \times K_{1^n 0})$. Hence, $\langle W_n, \phi(W_m)\rangle$ contains K_v for every vertex of level $n+1$ and is thus of finite index. $\square$

Conjecture 5. *For each n, the subgroup W_n is finitely generated.*

Remark 7. For $n = 0$ we have $W_0 = W$ which is obviously finitely generated. In general, $\mathrm{Stab}_{\mathcal{G}}(1^n)$ is generated by the normal subgroup $N \coloneqq B_0 \times K_{10} \times \cdots \times K_{1^{n-1}0}$ and by $L \coloneqq \langle K_{1^n}, (c)_1, (b)_2, ((ab)^2)_{n-1}((ac)^4)_i, 0 \leq i \leq n-2\rangle$, where $(g)_n$ denotes the element

$(1, \dots, 1, g)$ in $\mathrm{Stab}_{\mathcal{G}}(n)$ — for example $(b)_2 = (1, b) = d$. Therefore, $W_n = \langle N, L' \rangle$ where $L' := \{g \in L \mid g_{1^n} \in W\}$ and W_n is finitely generated if L' is finitely generated.

8. Classifying weakly maximal subgroups of $\mathcal{G}$

In this section, we try to classify weakly maximals subgroups of $\mathcal{G}$ using their left and right projections, following an idea of Grigorchuk.

Here $\mathcal{G} = \langle a, b, c, d \rangle$ is the first Grigorchuk group, $B := \langle b \rangle^{\mathcal{G}}$ the normal closure of b and $H := \mathrm{Stab}_{\mathcal{G}}(1) = \langle b, c, d, aba, aca, ada \rangle$ the stabilizer of the first level. Let $A < \mathcal{G}$ be a weakly maximal subgroup. There is two possibilities. Either A is a subgroup of H or not.

8.1. Weakly maximal subgroups of $\mathcal{G}$ contained in H

In this case, we can look at A_0 and A_1, the action of A on the left and right subtrees. Since $\mathcal{G}$ is self-replicating, A_0 and A_1 are subgroups of $\mathcal{G}$. Thus $A \leq_S A_0 \times A_1$ is a subgroup of the product.

Lemma 21. *Suppose that $[\mathcal{G} : A_0]$ is of infinite index. Then A_0 is a weakly maximal subgroup of $\mathcal{G}$ and A_1 contains B.*

Proof. Suppose that $A_0 < A_0'$ is of infinite index. Then for $\alpha \in A_0' \setminus A_0$ there exists $\beta \in \mathcal{G}$ such that $(\alpha, \beta) \in \mathcal{G}$ and we have

$$A < \langle A, (\alpha, \beta) \rangle \leq_S A_0' \times \langle A_1, \beta \rangle \qquad [\mathcal{G} : \langle A, (\alpha, \beta) \rangle] = \infty$$

since A_0' is of infinite index. This prove the weak maximality of A_0.

On the other hand, if there exists $\beta \in B \setminus A_1$, then $(1, \beta) \in \mathcal{G}$ and

$$A < \langle A, (1, \beta) \rangle \leq_S A_0 \times \langle A_1, \beta \rangle \qquad [\mathcal{G} : \langle A, (1, \beta) \rangle] = \infty$$

since A_0' is of infinite index.

We even have:

$$B \leq D := \{\beta \in \mathcal{G} \mid \exists \alpha \in A_0 : (\alpha, \beta) \in \mathcal{G}\} \leq A_1.$$

□

Since in this case A_1 contains B, it is interesting to know all subgroups of $\mathcal{G}$ containing B. The classification of such subgroups is done in [5] for normal subgroups and in the appendix for the general case. There is exactly 10 such subgroups: $\mathcal{G}$, 3 subgroups of index 2 — $J_{0,2}$, H, $J_{0,5}$ — 5 of index 4 — $J_{1,5}$ (normal), $S_{2,3,0,0}$, $S_{2,3,0,1}$, $S_{2,4,0,0}$ and $S_{2,4,0,1}$ and B itself.

Question 9. What about the case where both A_i are of finite index? Is it possible? Does it implies that A is finitely generated? Which of the 10 subgroups containing B could appear as A_1? Is it possible that $A \leq H$ if A is not parabolic?

8.2. Weakly maximal subgroups of $\mathcal{G}$ not contained in H

In this case, we can look at $A' = A \cap H$, A_0 and A_1, the action of A' on the left and right subtrees. Since $\mathcal{G}$ is self-replicating, A_0 and A_1 are subgroups of $\mathcal{G}$. Thus $A' \leq_S A_0 \times A_1$ is a subgroup of the product.

Lemma 22. *Suppose that $\tilde{A}$ is weakly maximal in H and that $[\mathcal{G} : A_0]$ is of infinite index. Then A_0 is a weakly maximal subgroup of $\mathcal{G}$ and A_1 contains B.*

The proof is the same as the for Lemma 21.

Question 10. What about the general case: i.e. A' not weakly maximal in H? Can it happens?

Example 1. Let $\tilde{B} \coloneqq \langle B, (ad)^2 \rangle$, $W \coloneqq \langle a, \operatorname{diag}(\tilde{B} \times \tilde{B}), \{1\} \times K \times \{1\} \times K \rangle$ be the Pervova's example. Then, $W' = W \cap H = \langle \operatorname{diag}(\tilde{B} \times \tilde{B}), \{1\} \times K \times \{1\} \times K \rangle$ has both projections equal to $\tilde{B}$ (of index 4) and is weakly maximal in H.

The only non-obvious part of these assertion is the weak maximality of W'. Let x be in $H \setminus W'$ and look at $\widetilde{W} \coloneqq \langle x, W' \rangle$. Then $x = (x_0, x_1)$ and, factorizing the first factor by $\tilde{B}$, we can assume that x_0 is either 1 or d. The rest of the proof is the same as the proof of the weak maximality of W in $\mathcal{G}$.

8.3. Parabolic subgroups

Let A be a weakly maximal subgroup of Γ. We can look as before at A_0 and A_1 the left and right projections (of $A \cap \operatorname{Stab}_{\mathcal{G}}(1)$). But, for any vertex $v \in T$ of level n we can also look at A_v the projection on v (of $A \cap \operatorname{Stab}_{\mathcal{G}}(n)$). It is obvious that if $v \leq w$, then $A_v = (A_w)_v$.

Lemma 23. *For all subgroup $A \leq \mathcal{G}$ and all $v \in T$, we have $(\bar{A})_v = \overline{A_v}$.*

Proof. Observe that since H is maximal of index 2 in $\mathcal{G}$ we have $A \leq H$ — in which case $\bar{A} \leq \bar{H}$ — or $A \cap H <_2 A$ — in which case $\bar{A} \cap \bar{H} <_2 \bar{A}$. Since the closure preserves indices, if $A \not\leq H$, we have $\overline{A \cap H} <_2 A$ and $\overline{A \cap H} \leq \bar{A} \leq \bar{H} <_2 A$. Therefore, we always have $\overline{A \cap H} \leq \bar{A} \leq \bar{H}$.

Suppose now, that v is a vertex of the first level; say $v = 0$. Now, if g belongs to $(\bar{A})_0$, there exists $h \in \bar{\mathcal{G}}$ such that (g, h) is an element of $\bar{A} \cap \bar{H} = \overline{A \cap H}$. This implies that $(g, h) = \lim_i (g_i, h_i)$ with $(g_i, h_i) \in A \cap H$. In particular, $g = \lim g_i$ with $g_i \in A_0$ and therefore $(\bar{A})_0 \leq \overline{A_0}$.

On the other hand, if g is in $\overline{A_v}$, then $g = \lim_i g_i$ for some g_i in A_v. For all i, there exists h_i such that (g_i, h_i) is in A. By compacity, there exists a converging subsequence of the h_i such that $\lim_i (g_i, h_i)$ belongs to $\bar{A}$. This proves that $(\bar{A})_v \geq \overline{A_v}$.

For the general case where v is of level n, we do an induction, using the fact that for $v \leq w$ we have $A_v = (A_w)_v$. $\square$

Lemma 24. *Let $A \leq \mathcal{G}$ be a weakly maximal subgroup. Then the following are equivalent.*

1. *A is a parabolic subgroup of the original action;*

2. *There exists a infinite ray $\xi = x_1x_2\dots$ such that for all $j \geq 0$, the subgroup $A_{x_1\dots x_j}$ is contained in H (where $A_\emptyset = A$).*

Proof. The condition on $A_{x_1\dots x_j} \leq H$ implies that A stabilizes the ray ξ for the original action. Therefore, $A \leq \mathrm{Stab}_{\mathcal{G}}(\xi)$ and we conclude by weak maximality. The other direction is obvious. $\square$

Lemma 25. *If $A = \mathrm{Stab}_{\mathcal{G}}(\xi)$, then for all j we have*

$$A_{x_1\dots x_j} = \mathrm{Stab}_{\mathcal{G}}(\sigma^j(\xi)) \qquad A_{x_1\dots x_{j-1}\bar{x}_j} = J_{0,2} <_2 \mathcal{G}$$

Proof. If $A = \mathrm{Stab}_{\mathcal{G}}(\xi)$ with $\xi = x_1x_2\dots$, then A_{x_1} stabilizes $\sigma(\xi)$. Therefore, A_{x_1} is an infinite index subgroup of H. This implies that $A_{\bar{x}_1}$ contains B and that A_{x_1} is weakly maximal and hence equal to $\mathrm{Stab}_{\mathcal{G}}(\sigma(\xi))$. An easy induction shows that $A_{x_1\dots x_j} = \mathrm{Stab}_{\mathcal{G}}(\sigma^j(\xi))$ and $B \leq A_{x_1\dots x_{j-1}\bar{x}_j} \leq \mathcal{G}$.

Let $P \coloneqq \mathrm{Stab}_{\mathcal{G}}(1^\infty)$. We have the following description of P from [2, Theorem 4.4]

$$P = \Big(B \times ((K \times ((K \times \dots) \rtimes \langle (ac)^4 \rangle)) \rtimes \langle b, (ac)^4 \rangle)\Big) \rtimes \langle c, (ac)^4 \rangle$$

This immediately implies that for all j we have

$$P_{1^j} = P \qquad P_{1^{j-1}0} = \langle B, (ad)^2, a \rangle = J_{0,2} <_2 \mathcal{G}$$

For the general case, let ξ be any ray in T and $A = \mathrm{Stab}_{\mathcal{G}}(\xi)$. There exists $g \in \bar{\mathcal{G}}$ such that $\bar{A} = \mathrm{Stab}_{\bar{\mathcal{G}}}(\xi) = \mathrm{Stab}_{\bar{\mathcal{G}}}(1^\infty) = \bar{P}$. Therefore, for all j, we have $\bar{A}_{x_1\dots x_{j-1}\bar{x}_j} = \overline{J_{0,2}}^g = \overline{J_{0,2}}$. $\square$

A. Subgroups of $\mathcal{G}$ of finite index

The aim of this appendix is to describe subgroups of $\mathcal{G}$ via their Schreier graph. This allows to describe the top of the lattice of subgroups of the Grigorchuk group.

A.1. Introduction

The description of the top of the lattice of normal subgroups was done in [5]. In the following, we will use the notation of [5] for normal subgroups. There is seven subgroups of index 2, all of them normal and seven normal subgroups of index 4.

In the following, we will make extensive use of the following presentation of the Grigorchuk group due to Lysenok [12]. Let A be the set of words on $\{a, b, c, d\}$ such that one letter out of two is an a and the other belongs to $\{b, c, d\}$. Define $\sigma\colon A \to A$ by $\sigma(ww') = \sigma(w)\sigma(w')$ and

$$\begin{aligned} \sigma\colon a &\mapsto aca & b &\mapsto d \\ c &\mapsto b & d &\mapsto c. \end{aligned}$$

Let $w_0 = ad$ and for all n, define $w_{n+1} = \sigma(w_n) = \sigma^n(ad)$. The the first Grigorchuk group admits the following presentation

$$\mathcal{G} = \left\langle a,b,c,d \,\middle|\, \begin{array}{c} a^2 = b^2 = c^2 = d^2 = bcd = 1 \\ \forall n \geq 0, (w_n)^4 = (w_n w_{n+1})^4 = 1 \end{array} \right\rangle$$

Notice that the relators in this presentation come naturally in two distincts groups:

$$a^2 = b^2 = c^2 = d^2 = bcd = 1 \tag{1}$$

$$\forall n \geq 0, (w_n)^4 = (w_n w_{n+1})^4 = 1. \tag{2}$$

An *X-graph* (V,E) is a graph with a function $l\colon E \to X^{\pm}$ such that for all e we have $l(\bar{e}) = l(e)^{-1}$ and for all vertices and all $s \in X^{\pm}$ there is exactly one outgoing and one ingoing edge with label s. A morphism between two X-graphs is called a *X-morphism* if it respects the labeling. In our case $X = \{a,b,c,d\}$ contains only element of order 2. Therefore, a X-graph is a 4-regular graph such that at every vertex there is exactly one edge with label x for all $x \in X$.

In the following, label edges will be drawn in color, according to the following table.

a	red
b	black
c	blue
d	green

Recall that for any group $G = \langle X \mid R\rangle$ and subgroups H_1 and H_2, we have $H_1 = H_2$ if and only if the corresponding Schreier graphs are isomorphic as labeled rooted graphs. Moreover, H_1 and H_2 are conjugated if and only if their Scheier graphs are isomorphic as labeled graphs. We also have that $H_1 \leq H_2$ of and only if there is a X-covering sending root to root from the Schreier graph of H_1 to the one of H_2. Therefore, there is a X-covering from the Schreier graph of H_1 to the one of H_2 if and only if H_1 is a subgroup of a conjugate of H_2. Recall also that a (rooted) X-graph is a Schreier graph of $\langle X \mid R\rangle$ if and only if for all vertices v and all $r \in R$, the unique path with initial vertex v and label r has final vertex v.

For a X-graph (V,E) and $T \subseteq X$, the *T-components* of (V,E) are the connected components of the subgraph of (V,E) consisting of all edges labeled by elements of T.

Now, let us have a look at the relators of (1). They exactly state that $\langle b,c,d\rangle$ is the Vierergruppe. If (V,E) a a Schreier graph of $\mathcal{G}$, then the condition (1) implies that each $\{b,c,d\}$-component is isomorphic to a quotient of the Cayley graph of $\{b,c,d\}$. See Figure 2. In this figure, violet, gray and brown edges correspond to any choice of 3 distincts letters in $\{b,c,d\}$. This gives us a total of $1+3+1=5$ possibilities.

Before looking at subgroups of index 4, let us draw the graphs of normal subgroups of index at most 4. See Figures 3 and 5.

In the notation of [5], the first line of figure 3 correspond (from the left to the right) to subgroups $J_{0,1}$ to $J_{0,4}$, the second line to $J_{1,2} = H$ (the stabilizer of the first level of the 2 regular rooted tree) and subgroups $J_{0,5}$ to $J_{0,7}$. The third line correspond to subgroups $J_{0,8}$ to $J_{0,11}$ and the last line to subgroups $J_{1,3}$, $J_{1,9}$ and $J_{1,5}$.

The following lemma will be useful when we will investigate the lattice of subgroups.

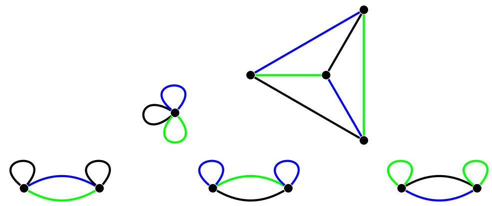

Figure 2: $\{b, c, d\}$-components arising in Schreier graphs of the Grigorchuk group: the rose, the tetrahedron and 3 "mickeys" Black edges are labeled by b, blue by c and green by d.

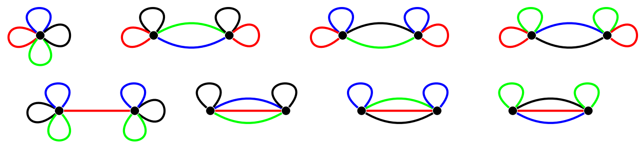

Figure 3: Schreir graphs of normal subgroups of index at most 2. Edges in red are labeled by a, in black by b, in blue by c and in green by d.

Lemma 26. *Let G be a p-group. And $S \geq T$ two subgroups of index p^i and p^j. Then for all $i \leq s \leq j$, there exists a subgroup L of index p^s and with $S \geq L \geq T$.*

Proof. Observe that since G is a p-group, every finite index subgroup has index a power of p. Let $N = \mathrm{Core}_S(T)$ the normal core of T in S. The normal subgroup N has finite index in S, and therefore the group S/N is a finite p-group, and hence nilpotent. This implies that $\bar{T}$, the image of T in S/N is subnormal. Therefore, there exists a composition series $S_0 = S \geq S_1 \geq \cdots \geq S_{j-i} = \bar{T} \geq \cdots \geq \{1\}$. The image of this series in S gives the wanted subgroups. $\square$

A.2. Subgroups of small index

In this subsection, we give a general method that helps to find subgroups of index 2^i for all i. It consist of an algorithm that take all triples of involutions of S_{2^i} as entry and answer yes, no or maybe. If the answer is yes, the algorithm gives the corresponding subgroup and if the answer is maybe, the algorithm gives a "candidat subgroup" but is not able to decide if it is indeed a subgroup of $\mathcal{G}$. The interest of this method is that it works for any index and can be passed to a computer. But there is three flaws in it. Firstly, the algorithm sometimes answer "maybe" and give a candidat subgroup S. In this case, we can test some relators to see if they hold in S, but with no garantie that this will terminate. Secondly, this algorithm is not efficient. Finally, there is multiple triples of involutions corresponding to the same subgroup. Therefore, after we have run the algorithm, we still have to test the subgroup for equality. The good news is that since the index is finite, this can be done in finite time.

Recall that $A = \{a, b, c, d\}$ and let G be a A-graph. Then every word x on A induce a permutation p_x on the vertices of G. Moreover, we have $p_{xy} = p_x p_y$. As seen before, G is a Schreier graph of $\mathcal{G}$ if and only if for each relator r, $p_r = \mathrm{Id}$. Let P be the subgroup of S_8 generated by $\{p_a, p_b, p_d\}$. Then, for all word $x \in A^*$, we have $p_x \in P$. Now, if G is a A-graph with $\{b, c, d\}$-components as in Figure 3, we have $p_{bcd} = \mathrm{Id}$. In this case, it remains to test if for all n we have $p_{w_n}^4 = p_{w_n w_{n+1}}^4 = \mathrm{Id}$. Instead of testing all this relation, it may be simpler to look at the order of elements of P. Indeed, if for all p in P, the order of p is a divisor of 4, then we have $p_{w_n}^4 = p_{w_n w_{n+1}}^4 = \mathrm{Id}$. If it is not the case, we may look at $\tilde{P} := \langle p_{aca}, p_b, p_c, p_d \rangle \leq P$. Indeed, for all $n \geq 1$, both p_{w_n} and $p_{w_n w_{n+1}}$ are elements of $\tilde{P}$. We have thus proved the following lemma.

Lemma 27. *Let G be a A-graph with $\{b, c, d\}$-components as in Figure 3. Suppose that at least one of the following is true:*

- *All elements of P are of order a divisor of 4;*
- *All elements of $\tilde{P}$ are of order a divisor of 4 and $p_{w_0}^4 = p_{w_0 w_1}^4 = \mathrm{Id}$.*

Then, G is a Schreier graph of $\mathcal{G}$.

If we have a A-graph G, then $\{a, b, c\}$ define 3 involutions of the vertex set. On the other hand, given 3 involutions $\{p_a, p_b, p_c\}$ of $\{1, \dots, n\}$, we can construct a A-graph G by putting an edge labeled by a between i and j if and only if $p_a(i) = j$. Let q_a be the involution defined by a on this graph G. Then $p_a = q_a$. Therefore, one way to classify the subgroup of order 8 is to look at all triples of involution $\{p_a, p_b, p_c\} \leq S_8$, to construct the corresponding graph. We then need to verify that the graph is indeed connected and a Schreier graph of $\mathcal{G}$ and we also need to identify the obtained graphs up to label preserving automorphisms. The annex contains the code for subgroups of index 8. A computation on a portable computer takes approximately 1 hour to treat the case of index 8 and return 61 conjugacy classes.

The main flaw of this method is that the number of triples of involutions of S_{2^n} grows very fast. For example, for $n = 8$, there is nearly 500 millions of such triples.

A more efficient way to find subgroups of index 2^n using a computer is to comput double coverings of Schreier graphs of subgroups of index 2^{n-1} and then to verify if the graphs obtained by this method are Schreier graphs of $\mathcal{G}$.

A.3. Subgroups of index 4

In this case, the corresponding Schreier graph has 4 vertices. The following technical proposition will be useful in order to classify $\{a, b, c, d\}$-graphs over 4 vertices that are Schreier graphs of $\mathcal{G}$.

Proposition 10. *Let $X = \{a_1, \dots a_n\}$ be a finite alphabet (we allow $a_i^2 = 1$ for some i's). Let (V, E) be a X-graph on 4 vertices such that under its group of X-automorphisms, there is either only one orbit, or two orbits consisting each of them of two vertices. Then for all $w \in X^*$, p_{w^4}, the permutation induced on V by w^4 is trivial.*

Proof. First, observe that for any X-automorphism α of the graph, if $p_w\colon i \mapsto j$, then $p_w\colon \alpha(i) \mapsto \alpha(j)$.

If the graph is not connected, then it consist of two X-transitive graphs on 2 vertices each. Since Sym_2 has only elements of order 1 or 2, the proposition is trivially true. Therefore, we may assume (V,E) to be connected.

Now, if there is 2 orbits of 2 vertices each, there is an X-automorphism of order 2. If there is only 1 orbits, the groups of X-automorphism is a transitive subgroup of S_4 and thus also have an element of order 2. Let $\bar{\ }$ be this X-automorphism of order 2. This allow us to label the vertices of (V,E) by $0,\bar{0},1,\bar{1}$. Indeed, if the vertices are $\{0,1,2,3\}$ we may assume, up to relabeling that $\bar{0}=3$. This implies that either $\bar{1}=2$ or $\bar{1}=1$. Assume fo the sake of contradiction that $\bar{1}=1$. This implies that $\bar{2}=2$. Now, if there is an edge from 1 to 0, say labeled by l, then there is an edge labeled by l from $\bar{1}=1$ to $\bar{0}\neq 0$. This is not possible. Repeating the same argument shows that 1 and 2 do not belong to the connected component of $\{0,\bar{0}\}$, which is absurd.

Now, take $w \in X^*$ and $x \in \{0,\bar{0},1,\bar{1}\}$ If $p_w(x)=x$ or $p_w(x)=\bar{x}$, then ${p_w}^4(x) = {p_w}^2(x) = x$. Otherwise, $p_w(x)=y$ with $y \neq x,\bar{x}$. This implies that $p_w(\bar{x})=\bar{y}$ and therefore that $p_w(y) \in \{x,\bar{x}\}$. In both case, we have $p_{w^4}(x)=x$. □

We now look at all 4 vertices X-graphs such that every path labeled by a relation from (1) is closed. Such graphs are exactly the one such that every $\{b,c,d\}$ connected component is one of the 5 graphs in figure 2. A quick investigation gives us 16 possible graphs. Among these 16 graphs, 7 are X-transitive. They correspond to normal subgroup of index 4, see Figure 3. The 9 remaining graphs split into 3 family consisting of 3 graphs each, see Figure 4. Such a family consist of graphs that can be obtained one from another by changing the labeling according to any permutation of $\{b,c,d\}$. The graphs of the first two families have 2 X-orbits, while the graph in the third family have 4 X-orbits.

Theorem 4. *All the* 12 *graphs defined by Figure 4 are Schreier graphs of the Grigorchuk group.*

Proof. For all these graphs, the paths labeled by relators in (1) are closed. The 9 graphs defined by the first three diagrams satisfy the hypothesis of the Proposition 10. Therefore, for these graphs, the paths labeled by relators in (2) are also closed.

Finally, we look at the last 3 graphs, corresponding to the last diagram in Figure 4. These graphs have 4 X-orbits. But we know that any letter in $\{b,c,d\}$ is a product of the two others. Therefore, for any $x \in \{b,c,d\}$, a $\{a,b,c,d\}$-graph is a Schreier graph of $\mathcal{G}$ if and only if the graph obtained by deleting all edges with label x is a Schreier graph of $\mathcal{G}$. In our case, if we delete gray edges of (V,E), we obtain a graph (V,E') satisfying the hypothesis of Proposition 10. This implies that (V,E') is a Schreier graph of $\mathcal{G}$ (paths labeled by relators in (1) and (2) are closed) and hence that (V,E) is itself a Schreier graph of $\mathcal{G}$. □

Theorem 5. *There is* 37 *subgroups of index* 4 *of the Grigorchuk group, which belong to* 19 *conjugacy classes. More precisely, there is* 7 *normal subgroups,* 9 *conjugacy classes of size* 2 *and* 3 *conjugacy classes of size* 4.

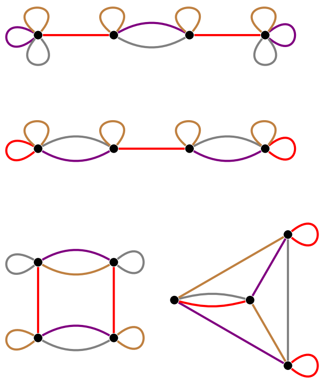

Figure 4: Schreir graphs of non-normal subgroups of index 4. Edges in red are labeled by a. Violet, gray and brown edges correspond to any choice of 3 distincts letters in $\{b, c, d\}$.

Proof. Using $\{b, c, d\}$-components described in Figure 2, it is easy to see that there is only 19 $\{a, b, c, d\}$-graphs such that every path labeled by relators in (1) is closed. By Theorem 4, all these graphs are Schreier graphs of $\mathcal{G}$. Recall that there is a one-to-one correspondance between Schreier graphs of $G = \langle X \rangle$ and conjugacy classes of subgroups. Under this correspondance, the index of the subgroup is the number of vertices in the graph, and the number of conjugate of H is the number of X-orbits. Therefore, there is exactly 19 conjugacy classes of subgroups of $\mathcal{G}$ of index 4, for a total of $7 \cdot 1 + 3 \cdot 2 + 3 \cdot 2 + 3 \cdot 2 + 3 \cdot 4 = 31$ subgroups. □

A.3.1. Graphical representations

Recall that edges in red are labeled by a, in black by b, in blue by c and in green by d.

First, let us recall the Schreier graphs of normal subgroups of index 4. From left to right we have (in the notation frome [5]), $J_{0,8}$ to $J_{0,11}$ on the first line and $J_{1,3}$, $J_{1,9}$ and $J_{1,5}$.

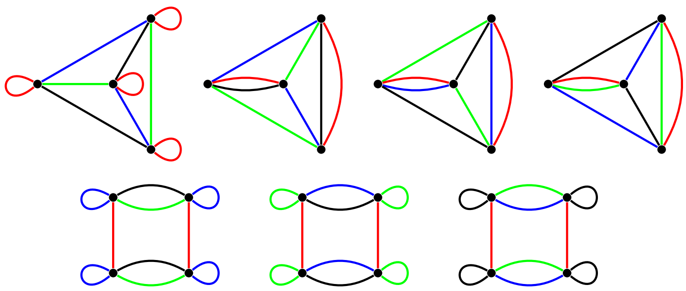

Figure 5: Schreir graphs of normal subgroups of index 4.

In order to classify all subgroups of $\mathcal{G}$, it is useful to have a unified notation. We will adopt the following convention. A subgroup of finite index of $\mathcal{G}$ will be denoted by $S_{i,j,k,l}$ with the following conventions. The index i will denote the "level" of this subgroup, that is $S_{i,j,k,l}$ is of index 2^i. The index j will classify class of $\{a,b,c,d\}$-length-isomorphic subgroups (i.e. subgroups with isomorphic unlabeled Schreier graphs on generating set $\{a,b,c,d\}$), the index k class of conjugated subgroups (i.e. subgroups with isomorphic labeled Schreier graphs) and the index l distincts subgroups in the same conjugation class. We will omit the last index for normal subgroups and the third index if there is only one subgroup in the length-isomorphic class. By convention, for each level we will firstly list the normal subgroups and then the non-normal ones.

With this notation, subgroups of Figure 3 correspond from left to right to $\mathcal{G}$, and $S_{1,0,0}$ to $S_{1,0,3}$ for the first line and $S_{1,1} = H$ and $S_{1,2,0}$ to $S_{1,2,3}$ for the second line. Subgroups of Figure 5 correspond from left to right to $S_{2,0}$, $S_{2,1,1}$, $S_{2,1,2}$ and $S_{2,1,3}$ for the first line, and $S_{2,2,1}$ to $S_{2,2,3}$ for the second line.

Now, let us take a look at the 30 non normal subgroups of index 4. Their Schreier graphs are depicted in Figures 6 to 9.

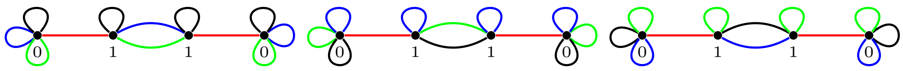

Figure 6: From left to right, the subgroups $S_{2,3,0,l}$ to $S_{2,3,2,l}$ with $0 \leq l \leq 1$ depending on the choice of the root.

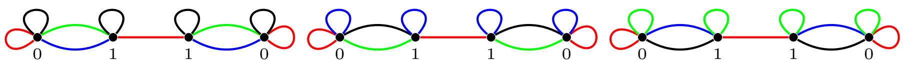

Figure 7: From left to right, the subgroups $S_{2,4,0,l}$ to $S_{2,4,2,l}$ with $0 \leq l \leq 1$ depending on the choice of the root.

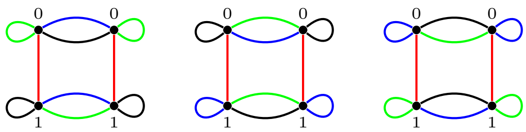

Figure 8: From left to right, the subgroups $S_{2,5,0,l}$ to $S_{2,5,2,l}$ with $0 \leq l \leq 1$ depending on the choice of the root.

A.3.2. Generators

There is a well-know and easy link between Schreier graphs of a subgroup S of a group G and generators of S as a subgroup of G. The algorithm is the following. Note v_0 the root of the Schreier graph of S and choose a spanning tree T. For each pair of edges $\{e, \bar{e}\}$

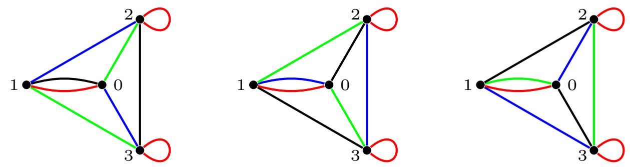

Figure 9: From left to right, the subgroups $S_{2,6,0,l}$ to $S_{2,6,2,l}$ with $0 \leq l \leq 3$ depending on the choice of the root.

not in T there is a unique path in T from v_0 to the initial vertex of e and a unique path in T from the final vertex of e to v_0. The set of labels of path $[v_0, \iota(e)]_T e[\tau(e), v_0]_T$ form a generating system for S. Note that this generating system need not to be minimal. In order to reduce the number of generators, and their length, it is possible to use relations true in $\mathcal{G}$. For example, $(ac)^3 = ca$.

In our case, since $\mathcal{G} = \langle a, b, c\rangle$, we can look at the $\{a, b, c\}$ (or $\{a, b, d\}$ or $\{a, c, d\}$)-components of our graphs in order to find generating system. In order to minimise the number of generators, and their length, it is useful to use the relation in $\mathcal{G}$. For example, the generators for $S_{2,3,0,0}$ could be computed as $\{b, b^a, b^{ac}, \dots\}$, but $b^{ac} = acbca = b^a$ and thus, this generator is not necessary. This give us the table in Table 1 for non-normal subgroups, where $g^h = hgh^{-1}$. The generators of normal subgroups can be found in [5]. In order to keep the table short and readable, we give the generators only for one subgroup in each conjugacy class.

Subgroup	Generators	Transversal System
$S_{2,3,0,0}$	$b, c, b^a, b^{aca}, c^{aca}$	$\{1, a\}$
$S_{2,3,1,0}$	$c, d, c^a, c^{ada}, d^{ada}$	$\{1, a\}$
$S_{2,3,2,0}$	$d, b, d^a, d^{aba}, b^{aba}$	$\{1, a\}$
$S_{2,4,0,0}$	$a, b, b^d, b^{da}, a^{dad}$	$\{1, d\}$
$S_{2,4,1,0}$	$a, c, c^b, c^{ba}, a^{bab}$	$\{1, b\}$
$S_{2,4,2,0}$	$a, d, d^c, d^{ca}, a^{cac}$	$\{1, c\}$
$S_{2,5,0,0}$	$d, (ca)^2, b^{ca}$	$\{1, a\}$
$S_{2,5,1,0}$	$b, (da)^2, c^{da}$	$\{1, a\}$
$S_{2,5,2,0}$	$c, (ba)^2, d^{ba}$	$\{1, a\}$
$S_{2,6,0,0}$	ab, a^d, a^c	$\{1, a, b, c\}$
$S_{2,6,1,0}$	ac, a^b, a^d	$\{1, a, c, d\}$
$S_{2,6,2,0}$	ad, a^c, a^b	$\{1, a, b, d\}$

Table 1: The 30 non-normal subgroups of index 4 and their generators.

A.3.3. Lattice

For the lattice of normal subgroups up to index 2^8, see [5]. Inclusion between subgroups correspond to coverings that respect the label and the root and the index of one subgroup in the other is exactly the degree of the covering.

For each subgroup $S_{2,3,k,l}$, it's Schreier graph is a covering of degree 2 of only one graph. The resulting graph is obtained by identifying vertices of "type" 0 together and vertices of "type" 1 together. This is exactly the Schreier graph of H.

For each subgroup $S_{2,4,k,l}$, it's Schreier graph is a covering of degree 2 of only one graph. The resulting graph is obtained by identifying vertices of "type" 0 together and vertices of "type" 1 together. This is exactly the Schreier graph of $S_{1,0,k}$.

For each subgroup $S_{2,5,k,l}$, it's Schreier graph is a covering that respect label of degree 2 of only one graph. As before, the resulting graph is obtained by identifying vertices of "type" 0 together and vertices of "type" 1 together and correspond to the Schreier graph of H.

For each subgroup $S_{2,6,k,l}$, it's Schreier graph is a covering that respect label of degree 2 of only one graph. Indeed, the vertex 2 can only be identified with the vertex 3 (the only other vertex with a loop labeled by a). The resulting graph is a Schreier graph of $S_{1,0,k}$.

We hence have that H contains 12 non-normal subgroups of index 4 and for each $0 \leq k \leq 2$, the subgroup $S_{1,0,k}$ contains 6 non-normal subgroups. The 3 remaining subgroups of index 2 do not have non-normal subgroups of index 4.

A.4. Subgroups of index 8

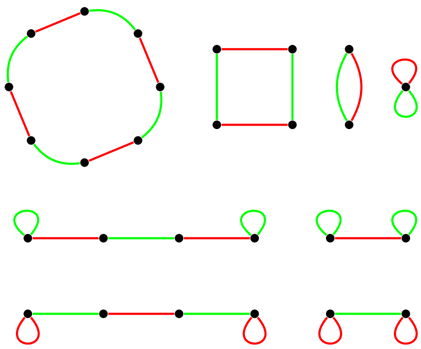

Figure 10: $\{a, d\}$-components arising in Schreier graphs of the Grigorchuk group.

A.4.1. The subgroup B

One of the subgroup of index 8 of $\mathcal{G}$ is of particular interest.

Since $B = \langle b \rangle^{\mathcal{G}}$, there is a loop labeled by b at each vertex of it's Schreier graph. Since the $\{a, b, d\}$-component of the Schreier graph of B is connected, its $\{a, d\}$ is also connected and is therefore an octogone. This give us the graph of Figure 11.

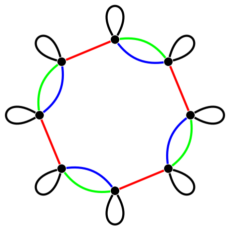

Figure 11: Schreier graph of the subgroup $B = \langle b \rangle^{\mathcal{G}}$.

Since there is a loop labeled by b at each vertex of it's Schreier graph, any subgroup S contained in B has also a loop labeled by b at each vertex of it's Schreier graph. This implies that S is either $S_{2,2,3} = J_{1,5}$, $S_{2,3,0,l}$ or $S_{2,4,0,l}$, $l \in \{0, 1\}$. An easy verification shows that the Schreier graph of B indeed covers the Schreier graphs of $S_{2,3,0}$ and $S_{2,4,0}$. Therefore, there is exactly 10 subgroups containing B, of which 5 are of index 4. The lattice of Schreier graphs of $\mathcal{G}$ covered by the Schreier graph corresponding to B is shown in Figure 12.

A.4.2. Conjugacy classes

A computation on GAP give us 61 conjugacy classes of subgroups of index 8.

A.4.3. Coverings

A.5. To do

1. Write an introduction for the GAP code.

2. Copy the GAP code and add comments.

3. Speak about index 16 : between 205 and 415 conjugacy classes. I.e. now the code is not able to answer fo 210 graphs. N.B. right now, the code test $w_0^4 = 1$, $(w_0 w_1)^4 = 1$, the $\{b, c, d\}$ relations and the fact that $\langle a, b, c \rangle$ is a 2-group.

4. Write the code for number of conjugate in a conjugacy class + system of transversal.

5. Write the code for coverings between explicit subgroups (and not only for conjugacy classes) (use strong isomorphisme).

6. Write the code for generating set (use covering tree).

7. Write a code that accept a subgroup S as entry, in the form $[a, b, c, index]$, and return 2 lists. One consisting of subgroups containing S and one consisting of subgroup of index at most 8 (16) contained in S.

8. For subgroup of H, try find the left and right projections.

9. If time : maybe write the code in Python (faster?).

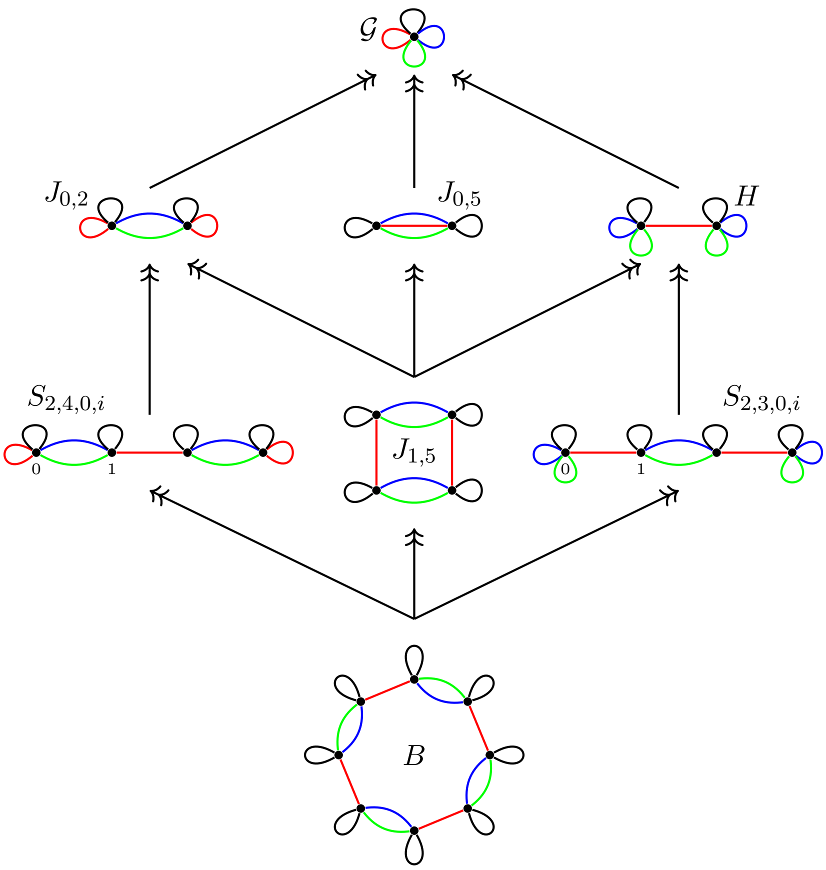

Figure 12: Schreier graphs of $\mathcal{G}$ covered by the Schreier graph corresponding to B. Two of these graphs have two orbits under the group of X-automorphisms. Therefore, each of them corresponds to two (conjugated) subgroups of $\mathcal{G}$.

B. “Non-linear” subgroups

In this section, we focus on subgroups with *non-linear* Schreier graphs, where a Schreier graph is *linear* if after erasing loops and identifying multiple edges it is isomorphic to a (possibly infinite or bi-infinite) path.

For the relevant definitions, see Appendix A.

Our conclusion will follows from the fact that in $\mathcal{G}$ the following relations holds

$$a^2 = b^2 = c^2 = d^2 = 1 \tag{3}$$

$$bcd = 1 \tag{4}$$

$$(ad)^4 = (ac)^8 = 1 \tag{5}$$

In the following, G will be any group that is a quotient of

$$\langle a, c, d \,|\, a^2, b^2, c^2, d^2, bcd, (ad)^4, (ac)^8 \rangle \tag{6}$$

with generating set $X = \{a, b, c, d\}$.

B.1. Restrictions on components

We will list all possible T-components for $T = \{b, c, d\}$, $\{a, d\}$ or $\{a, c\}$. Observe that since G is a quotient of the group given by Presentation (6), some listed T-components may actually never appear in Schreier graphs of G.

Let us have a look at the relators of (3) and (4). They exactly state that $\langle b, c, d\rangle$ is a quotient of the vier-gruppe $V_4 = (\mathbf{Z}/2\mathbf{Z})^2$. If (V, E) a a Schreier graph of G, then the Relations (3) and (4) implies that each $\{b, c, d\}$-component is isomorphic to a quotient of the Cayley graph of $\{b, c, d\}$. See Figure 2. This gives us a total of $1 + 3 + 1 = 5$ possibilities: a rose, a tetrahedron and three “mickeys”.

Now, for the $\{a, d\}$-components of Schreier graphs of G, we have that $(ad)^4 = 1$. Therefore, they are all quotients of an octagon, leaving us with eight possibilities, see Figure 10.

On the other hand, we have $(ac)^8 = 1$ which implies that in Schreier graphs of G, $\{a, c\}$-components are quotients of a 16-gone. See Figure 13 for the list of possible $\{a, c\}$-components with a loop labeled by a.

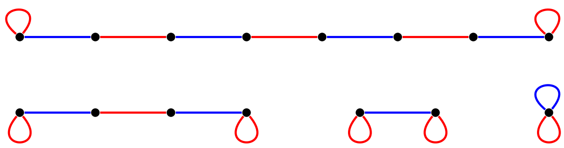

Figure 13: $\{a, c\}$-components with a a-loop that may arise in Schreier graphs of G.

B.2. The result

Proposition 11. *Let $H \leq G$ be a subgroup that contains a conjugate of a. Then either the corresponding Schreier graph as at least one $\{b, c, d\}$-component which is a tetrahedron, or H has index 2^i for some $0 \leq i \leq 3$.*

More precisly, if H is not of index 2^i for some $0 \leq i \leq 3$, then each a-loop is at distance 0, 2, 4 or 6 from a tetrahedron. On the other hand, if H is of index at most 8, then the corresponding Schreier graph is linear.

Proof. The subgroup H contains a conjugate of a if and only if the corresponding Schreier graphs as a loop labeled by a. There is only 3 choices for the corresponding $\{a, d\}$-component and 4 choices for the $\{a, c\}$-component. We now assume that the graph has no tetrahedron. Therefore, there is only 4 possibilities for the $\{b, c, d\}$-components.

If both the $\{a, d\}$-component and the $\{a, c\}$-component are rose with 2 petals, then $H \geq \langle a, d, c\rangle = G$.

If the $\{a, d\}$-component is not a rose with 2 petals, it is a “path” of length 1 or 3. In both cases, gluing $\{b, c, d\}$-components on it does not add more vertex and we obtain the full Schreier graph. In this case, H is of index 2 or 4.

The same argument shows that if the $\{a, c\}$-component is not a rose with 2 petals, then H is of index 2, 4 or 8.

We proved that each a-loop is in a connected component of at most 8 vertices, or at distance 0, 2, 4 or 6 from a tetrahedron.

Now, if H is of index at most 8, a quick check of possible components give us the graphs of Figure 14 as only possibilities. All these graphs are linear. □

Corollary 9. *There exists uncountably many weakly-maximal subgroups $H \leq \mathcal{G}$ with non strongly-isomorphic Schreier graphs such that the graphs are non-linear.*

Proof. By [4] there exists uncountably many weakly-maximal subgroups of $\mathcal{G}$ containing a. For all this subgroups, the corresponding Schreier graph contain a tetrahedron and is therefore non-linear. Since $\mathcal{G}$ is countable, there is uncountably many non strongly isomorphic such graphs. □

Remark 8. It is possible to list all H that contains a conjugate of a such that the Schreier graph does not have a tetrahedron. Indeed, the proof of Proposition 11 give us the list of all candidates, it then remains to show that they indeed correspond to subgroups of G. Table 2 give the list of all these subgroups that contains a for the overgroup group $\langle a, b, c, d \mid a^2, b^2, c^2, d^2, abc, (ad)^4, (ac)^8\rangle$ of Presentation 6. Observe that all subgroups of index at most 4 are subgroups of $\mathcal{G}$, the first Grigorchuk group.

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Subgroup	index	# of conjugates
G	1	1
$\langle a, b, a^c\rangle$	2	1
$\langle a, c, a^d\rangle$	2	1
$\langle a, d, a^b\rangle$	2	1
$\langle a, b, b^d, b^{da}, a^{dad}\rangle$	4	2
$\langle a, c, c^b, c^{ba}, a^{bab}\rangle$	4	2
$\langle a, d, d^c, d^{ca}, a^{cac}\rangle$	4	2
$\langle a, d, d^c, b^{ca}, b^{cac}, d^{(ca)^2}, d^{(ca)^2c}, d^{(ca)^3}, d^{(ca)^3c}, a^{(ca)^3c}\rangle$	8	8
$\langle a, d, d^c, d^{ca}, d^{cac}, d^{(ca)^2}, d^{(ca)^2c}, d^{(ca)^3}, d^{(ca)^3c}, a^{(ca)^3c}\rangle$	8	4

Table 2: Subgroups of $G = \langle a, b, c, d \,|\, a^2, b^2, c^2, d^2, abc, (ad)^4, (ac)^8\rangle$ that contains a a and do not have tetrahedron in the Schreier graph.

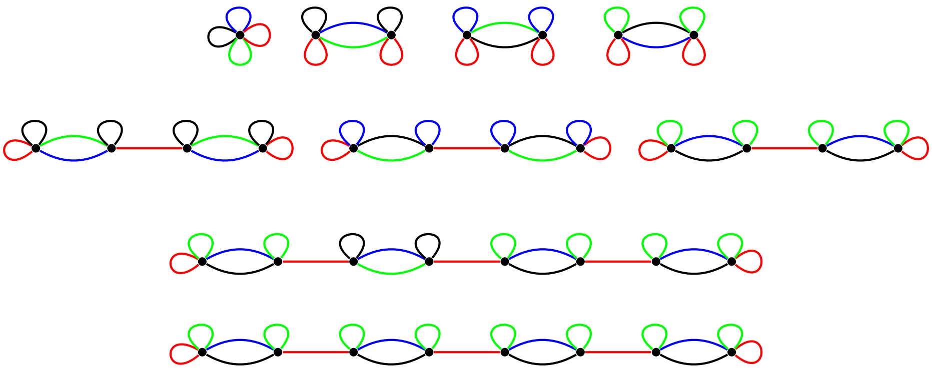

Figure 14: Schreier graphs of $G = \langle a, b, c, d \,|\, a^2, b^2, c^2, d^2, abc, (ad)^4, (ac)^8\rangle$ that contains a-loop and do not have tetrahedron.

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