Computing the left-hand side of Baum-Connes

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Goal

The goal of this talk is to show how to compute the left-hand side of the Baum-Connes assembly map for discrete groups.

The main tool that is useful in the context is Bredon homology, via an equivariant version of the Atiyah-Hirzebruch spectral sequence.

We will illustrate the exposition with the computation of a number of examples.

The contents of this talk are mainly based on joint work with S. Pooya, A. Valette and A. Zumbrunnen.

Contents

- Classifying space for proper actions.
- Bredon homology and Atiyah-Hirzebruch spectral sequence.
- Ordinary homology and the 0-th group.
- Computations with the chain complex.
- Amalgams and Mayer-Vietoris sequence.
- Martínez spectral sequence.

Throughout this talk, G will always be a discrete group.

Baum-Connes conjecture

For $n = 0, 1$, there is an isomorphism

$$K_n^G(\underline{E}G) \simeq K_n^{top}(C_r^*(G)).$$

As in this talk we are interested in the left-hand side of the conjecture, we first briefly recall the definition of the classifying space $\underline{E}G$.

Classifying space for proper actions

Definition

Given a discrete group G, a classifying space for proper actions of G is a G-CW-complex $\underline{E}G$ such that the fixed-point set $\underline{E}G^H$ is contractible for every finite subgroup $H < G$, and empty otherwise.

The classifying space $\underline{E}G$ is unique up to G-homotopy equivalence.

The orbit category

In order to define Bredon homology, we first need the notion of orbit category. Given a discrete group G, denote by $\mathcal{F}$ the family of finite subgroups of G.

Definition

The orbit category $\mathfrak{D}_{\mathcal{F}}G$ is the category whose objects are the homogeneous G-spaces G/K, for K finite, and whose morphisms are the G-maps.

Definition

A $\mathfrak{D}_{\mathcal{F}}G$-module is a functor from $\mathfrak{D}_{\mathcal{F}}G$ to the category of abelian groups.

The category whose objects are the covariant $\mathfrak{D}_{\mathcal{F}}G$-modules and whose morphisms are the natural transformations is denoted by $G\text{-}\mathrm{Mod}_{\mathcal{F}}$.

We first define a chain complex associated to a G-CW-complex.

Let X be a G-CW-complex with finite isotropy groups, and $N \in G\text{-}\mathrm{Mod}_{\mathcal{F}}$. Given $d \geq 0$, consider a set e_α of representatives of the orbits of the d-cells under the action of G.

If we denote by S_α the isotropy group of e_α, then we define $C_d = \bigoplus_\alpha N(G/S_\alpha)$, which is an abelian group.

Bredon homology of G-spaces: the boundary

To define the boundary operator ∂_d, we first order the vertices of X. Then we consider a $(d-1)$-face ge_i of a d-cell e, and call S_e the isotropy group of e and S_i the isotropy group of e_i (which is conjugated to the isotropy group of ge_i). Then it is induced a G-map $f_i : G/S_e \to G/S_i$.

Now the boundary is defined over $N(G/S_e)$ as the alternate sum of the homomorphisms $N(f_i)$, and then over $C_d = \bigoplus_e N(G/S_e)$ by linear extension.

It can be verified that for every d, $\partial_{d-1} \circ \partial_d$ is trivial.

Bredon homology of G-spaces

Definition

The Bredon homology groups $H_d^{\mathcal{F}}(X, N)$ of a G-CW-complex X are the homology groups of the chain complex (C_d, ∂_d).

When G is torsion-free, Bredon homology with coefficients N coincides with ordinary homology with coefficients in $N(G/1)$.

Abusing language, we will sometimes refer to the groups $H_d^{\mathcal{F}}(\underline{E}G, N)$ as "the Bredon homology groups of G", as they do not depend on the concrete model of $\underline{E}G$.

The complex representation ring functor

The useful coefficient module for Bredon homology in the context of Baum-Connes is the complex representation ring functor $R_{\mathbb{C}}$.

$R_{\mathbb{C}}$ is a covariant module over the orbit category of G, that sends every homogeneous space G/H to the complex representation ring $R_{\mathbb{C}}H$, considered as an abelian group.

In turn, every G-map $G/H \to G/K$ which induces inclusion of subgroups is sent to the homomorphism $R_{\mathbb{C}}H \to R_{\mathbb{C}}K$ given by induction of representations, while maps which induce conjugations are sent to the corresponding isomorphisms.

The equivariant Atiyah-Hirzebruch spectral sequence

Given a discrete group G and a proper G-space X, there exists a first and fourth quadrant spectral sequence, such that:

- $E^2_{p,q} = H^{\mathcal{F}}_p(X, K^G_q(-))$.
- $E^{\infty}_{p,q} = K^G_{p+q}(X)$.
- The differential d_n has bidegree $(-n, n-1)$.

In general it is hard to compute differentials in the spectral sequence, but things get approachable when X is low-dimensional, as we will see next.

Mislin result

The following sequence is a collapse of the equivariant Atiyah-Hirzebruch spectral sequence, and is the key to obtain the equivariant K-homology of G in many cases.

Theorem

Let Γ be a group such that there exists a model for $\underline{E}\Gamma$ of dimension 3. Then there is a natural exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & H_1^{\mathcal{F}}(\Gamma; R_{\mathbb{C}}) \arrow[r] & K_1^{\Gamma}(\underline{E}\Gamma) \arrow[r] & H_3^{\mathcal{F}}(\Gamma; R_{\mathbb{C}}) \arrow[d] \\
0 & H_2^{\mathcal{F}}(\Gamma; R_{\mathbb{C}}) \arrow[l] & K_0^{\Gamma}(\underline{E}\Gamma) \arrow[l] & H_0^{\mathcal{F}}(\Gamma; R_{\mathbb{C}}) \arrow[l]
\end{tikzcd}$$

Proper geometric dimension

The minimal dimension of a model for $\underline{E}G$ is called the proper geometric dimension of G, and denoted $\underline{\mathrm{gd}}\ G$.

When computing Bredon homology, the first step is to find a model of $\underline{E}G$ as small and manageable as possible, ideally of the proper geometric dimension of G. Observe that the definition of the chain complex implies:

Proposition

Let m be the proper geometric dimension of a group G. Then, if $n > m$, $H_n^{\mathcal{F}}(\underline{E}G, N) = 0$ for every coefficient module N.

In general, a good knowledge of the action is also necessary in order to make the computations.

Example: wallpaper groups

The seventeen wallpaper groups act on the plane via isometries with bounded fundamental domain.

With this action, the plane turns out to be a model for the classifying space for proper actions of these groups.

In particular, if G is a wallpaper group, $H_n^{\mathcal{F}}(\underline{E}G, N) = 0$, for $n > 2$ and every coefficient module N.

The singular part

Frequently, the knowledge of the ordinary homology of the group gives good information about Bredon homology.

Given a group G, we denote by $(\underline{E}G)^{\mathrm{sing}}$ the singular part of $\underline{E}G$, i.e. the subcomplex of points with non-trivial isotropy.

Proposition

For every group G there is a natural homomorphism

$$H_i^{\mathcal{F}}(\underline{E}G, R_{\mathbb{C}}) \to H_i(\underline{B}G, \mathbb{Z})$$

which is an isomorphism when $i > \dim\,(\underline{E}G)^{\mathrm{sing}} + 1$ and a monomorphism in dimension $i = \dim\,(\underline{E}G)^{\mathrm{sing}} + 1$.

Here $\underline{B}G$ is the orbit space of $\underline{E}G$ under the action of G.

Non-torsion part of the homology

Proposition

Let G be a discrete group. Then

$$H_i^{\mathcal{F}}(\underline{E}G, R_{\mathbb{C}} \otimes \mathbb{Q}) = \bigoplus H_i(BC_G(x), \mathbb{Q})$$

.

Here the direct sum is extended to all the conjugacy classes of elements of finite order in G, and $C_G(x)$ denotes the centralizer of x in G.

In particular, if the Bredon homology groups are finitely generated, this result identifies them up to torsion.

The 0-th group

In general, the 0-th Bredon homology group can be computed out of the orbit category of the group G, i.e. from the knowledge of the inclusion and conjugation relations between its subgroups, and the induced maps between the images of the coefficient module.

Proposition

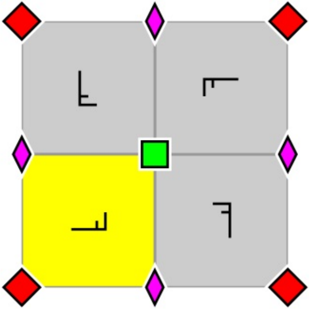
For every group G, $H_0^{\mathcal{F}}(\underline{E}G, N) = \mathrm{colim}_{\mathfrak{D}_{\mathcal{F}}G} N$.

In the next example, the previous results permit the computation of the Bredon homology without constructing the chain complex.

Example: the group **p4**

This wallpaper group is given by the presentation:

$$\mathbf{p4} = \{x, y, z \mid xyx^{-1}y^{-1}, z^4, zyz^{-1}x, zxz^{-1}y^{-1}\}.$$



Example: the group **p4**

It is not hard to see that the singular part of **p4** is zero dimensional, and that the quotient of the plane under the action of **p4** is homeomorphic to the sphere S^2.

In turn, the 0-th Bredon homology group of **p4** is given by the colimit of the diagram:

$$\begin{tikzcd}
R_{\mathbb{C}}(\mathbb{Z}/4) & R_{\mathbb{C}}(\mathbb{Z}/2) & R_{\mathbb{C}}(\mathbb{Z}/4) \\
R_{\mathbb{C}}(\mathbb{Z}/2) \arrow[u] & R_{\mathbb{C}}\{1\} \arrow[l] \arrow[r] \arrow[u] & R_{\mathbb{C}}(\mathbb{Z}/2). \arrow[u]
\end{tikzcd}$$

Joining these two facts and taking into account the previous results, we obtain the following.

Example: the group **p4**

Proposition

We have $H_0^{\mathcal{F}}(\mathbf{p4}, R_{\mathbb{C}}) = \mathbb{Z}^8$, $H_2^{\mathcal{F}}(\mathbf{p4}, R_{\mathbb{C}}) = \mathbb{Z}$ and $H_i^{\mathcal{F}}(\mathbf{p4}, R_{\mathbb{C}}) = 0$ otherwise.

In particular, by the collapsed version of Atiyah-Hirzebruch sequence, we obtain:

Proposition

- $K_1^{\mathbf{p4}}(\underline{E}\mathbf{p4}) = 0$, $K_0^{\mathbf{p4}}(\mathbf{p4}) = \mathbb{Z}^9$.

This result was obtained previously by other methods by Lück-Stamm and Yang.

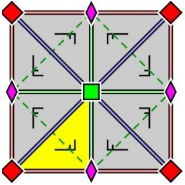
Steps to compute Bredon homology

- To build a manageable low-dimensional model of $\underline{E}G$ where the action of G is well-understood.
- To compute the isotropy groups of the action and the corresponding representation rings, in order to get the abelian groups C_n of the chain complex.
- To describe the boundary operator, using the face relations in the G-cellular structure de $\underline{E}G$, and the induced homomorphisms between the representation rings. Characters are very useful at this point.
- To identify the kernels and images and compute the corresponding homology groups.

Example: the group **p4mm**

This is another wallpaper group, given by the presentation:

$\mathbf{p4mm} = \{x, y, m, r \mid xyx^{-1}y^{-1}, r^4, m^2, ryr^{-1}x, rxr^{-1}y^{-1}, mxm^{-1}y^{-1}, mrm^{-1}r\}.$



Example: the group **p4mm**

In this model for $\underline{E}$**p4mm** the isotropy groups of the vertices of the fundamental domain are given by D_4, D_4 and $\mathbb{Z}/2 \times \mathbb{Z}/2$, the isotropy groups of the edges are given by three copies of $\mathbb{Z}/2$, and the isotropy groups of the 2-cell is trivial. This produces the chain complex:

$$0 \to R_{\mathbb{C}}\{1\} \xrightarrow{\partial_2} (R_{\mathbb{C}}\mathbb{Z}/2)^3 \xrightarrow{\partial_1} R_{\mathbb{C}}(\mathbb{Z}/2 \times \mathbb{Z}/2) \oplus (R_{\mathbb{C}} D_4)^2 \to 0.$$

Writing now the well-known values of these representation rings, the chain complex has the shape:

$$0 \to \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z}^6 \xrightarrow{\partial_1} \mathbb{Z}^{14} \to 0.$$

Example: the group **p4mm**

Now using the definition of the boundary, the character tables of the finite groups involved and the induction formula, it can be checked that:

Bredon homology of $\underline{E}\mathbf{p4mm}$

The only non-trivial Bredon homology group of $\underline{E}\mathbf{p4mm}$ is $H_0^{\mathcal{F}}(\underline{E}\mathbf{p4mm}, R_{\mathbb{C}}) = \mathbb{Z}^9$.

Mayer-Vietoris sequence

When the group of interest is given as an amalgam, there is a very useful Mayer-Vietoris sequence in Bredon homology.

Proposition

Let $G = H *_L K$ an amalgam of discrete groups, and N a coefficient module. Then there is a long exact sequence:

$$\ldots \to H_n^{\mathcal{F}}(L,N) \to H_n^{\mathcal{F}}(H,N) \oplus H_n^{\mathcal{F}}(K,N) \to$$
$$\to H_{n-1}^{\mathcal{F}}(G,N) \to H_{n-1}^{\mathcal{F}}(L,N) \to \ldots$$

Example: affine groups

The general affine group $GA(2,\mathbb{Z})$ of degree two over the integers is the semi-direct product

$$\mathbb{Z}\oplus\mathbb{Z}\rtimes GL(2,\mathbb{Z}),$$

where every element of the free abelian group is identified with a vertical vector $\binom{x}{y}$ and the (left) action is by product of matrices.

In turn, the special affine group $SA(2,\mathbb{Z})$ is defined by restricting the action to the subgroup $SL(2,\mathbb{Z}) < GL(2,\mathbb{Z})$.

In the remaining of the talk, these groups will be respectively called GA and SA.

More wallpaper groups

In order to get SA as an amalgam, we will need two additional wallpaper groups:

$\mathbf{p2} = \{x, y, z \mid z^2, xyx^{-1}y^{-1}, (zx)^2, (zy)^2\}$.

$\mathbf{p6} = \{x, y, z \mid xyx^{-1}y^{-1}, z^6, zxz^{-1}y^{-1}, zyz^{-1}yx^{-1}\}$.

The special affine group as an amalgam

SA can be described as an amalgam of wallpaper groups.

To see this, recall the famous decomposition

$$SL(2,\mathbb{Z}) = \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$$

obtained by Serre.

Now taking semi-direct products with $\mathbb{Z}^2$ in the factors of the previous push-out, we obtain:

$$SA = \mathbf{p4} *_{\mathbf{p2}} \mathbf{p6}.$$

A low-dimensional model for $\underline{E}SA$

Using that models for $\underline{E}SL(2,\mathbb{Z})$ and $\underline{E}GL(2,\mathbb{Z})$ are given by Bass-Serre trees, the following can be proved.

Proposition

If T is a 1-dimensional model for $\underline{E}SL(2,\mathbb{Z})$ (respectively $\underline{E}GL(2,\mathbb{Z})$), $\mathbb{R} \times \mathbb{R} \times T$ is a model for $\underline{E}SA$ (resp. $\underline{E}GA$).

In fact, it is not hard to see that the proper geometric dimension of SA and GA is exactly 3.

A similar computation to the referred one for **p4** produces the Bredon homology of $\underline{E}$**p2** and $\underline{E}$**p6**, and also the value of the left-hand side of Baum-Connes.

Proposition

- For $n = 2, 4$ or 6, $H_1^{\mathcal{F}}(\underline{E}\mathbf{pn}, R_{\mathbb{C}}) = 0$ and $H_2^{\mathcal{F}}(\underline{E}\mathbf{pn}, R_{\mathbb{C}}) = \mathbb{Z}$.
- $H_0^{\mathcal{F}}(\underline{E}\mathbf{p2}, R_{\mathbb{C}}) = \mathbb{Z}^5$, $H_0^{\mathcal{F}}(\underline{E}\mathbf{p4}, R_{\mathbb{C}}) = \mathbb{Z}^8$ and $H_0^{\mathcal{F}}(\underline{E}\mathbf{p6}, R_{\mathbb{C}}) = \mathbb{Z}^9$.

Proposition

- For $n = 2, 4$ or 6, $K_1^{\mathbf{pn}}(\underline{E}\mathbf{pn}) = 0$,
- $K_0^{\mathbf{p2}}(\underline{E}\mathbf{p2}) = \mathbb{Z}^6$, $K_0^{\mathbf{p4}}(\underline{E}\mathbf{p4}) = \mathbb{Z}^9$ and $K_0^{\mathbf{p6}}(\underline{E}\mathbf{p6}) = \mathbb{Z}^{10}$.

Bredon homology of $\underline{E}SA$

Now the Mayer-Vietoris sequence of the push-out of wallpaper groups breaks in two shorter sequences:

$$0 \to \mathbb{Z} \to \mathbb{Z}^2 \to H_2^{\mathcal{F}}(\underline{E}SA, R_{\mathbb{C}}) \to 0.$$

$$0 \to H_1^{\mathcal{F}}(\underline{E}SA, R_{\mathbb{C}}) \to \mathbb{Z}^5 \to \mathbb{Z}^8 \oplus \mathbb{Z}^9 \to H_0^{\mathcal{F}}(\underline{E}SA, R_{\mathbb{C}}) \to 0.$$

And analyzing carefully the homomorphisms in this sequences, it is obtained:

Theorem

The only non-trivial Bredon homology groups of $\underline{E}SA$ (with coefficients in $R_{\mathbb{C}}$) are $H_0^{\mathcal{F}}(\underline{E}SA, R_{\mathbb{C}}) = \mathbb{Z}^{13}$, $H_1^{\mathcal{F}}(\underline{E}SA, R_{\mathbb{C}}) = \mathbb{Z}$ and $H_2^{\mathcal{F}}(\underline{E}SA, R_{\mathbb{C}}) = \mathbb{Z}$.

The equivariant K-homology of SA

Now we can describe the left-hand side of Baum-Connes for the group SA:

Theorem

The equivariant K-groups of SA are $K_0^{SA}(\underline{E}SA) = \mathbb{Z}^{14}$ and $K_1^{SA}(\underline{E}SA) = \mathbb{Z}$.

This is a consequence of the previous results on Bredon homology and the construction of a model of $\underline{E}SA$ of dimension three.

Bredon homology of $\underline{E}GA$

In a similar way, the amalgam $D_4 *_{D_2} D_6$, which is isomorphic to $GL(2,\mathbb{Z})$, produces another amalgam **p4mm** $*_{\mathbf{cmm}}$ **p6mm** of wallpaper groups that turns to be isomorphic to GA.

Hence the corresponding computation of the Bredon homology of the wallpaper groups involved and the subsequent application of Mayer-Vietoris gives the Bredon homology of GA:

Theorem

The only non-trivial Bredon homology group of GA (with coefficients in $R_{\mathbb{C}}$) is $H_0^{\mathcal{F}}(\underline{E}GA, R_{\mathbb{C}}) = \mathbb{Z}^{11}$.

Now we obtain the desired K-homology groups:

Theorem

We have $K_0^{GA}(\underline{E}GA) = \mathbb{Z}^{11}$ and $K_1^{GA}(\underline{E}GA) = 0$.

Martínez spectral sequence

One of the main tools that can be used to compute Bredon homology of an extension is the version of Lyndon-Hochschild-Serre spectral sequence developed by C. Martínez. We review it in a particular version.

Let $N \to G \to \bar{G}$ be a group extension, and we denote by $Fin(G)$ and $Fin(\bar{G})$ the corresponding families of finite subgroups.

In this way it is defined a first quadrant spectral sequence such that

$$E^2_{p,q} = H_p^{Fin(\bar{G})}(\bar{G}, \overline{H_q^{Fin(G)\cap -}(-, D)}),$$

that converges to $E^\infty_{p,q} = H_{p+q}^{Fin(G)}(G, D)$.

- In the page E_2, $\overline{H_q^{Fin(G)\cap -}(-,D)}$ is a module in $G\text{-}\mathrm{Mod}_{Fin(\bar{G})}$.

- The values of $\overline{H_q^{Fin(G)\cap -}(-,D)}$ are computed in the following way: first take an element $\bar{V} < \bar{G}$ in $Fin(\bar{G})$, and consider a preimage V in G. Then consider the family $\mathfrak{F}_V$ of the finite subgroups of V.

- Now the value of the functor $\overline{H_q^{Fin(G)\cap -}(-,D)}$ over $\bar{V}$ is $H_q^{\mathfrak{F}_V}(V,D)$, and it can be seen that it is independent of the choice of V.

Lamplighter groups

The classic lamplighter group can be expressed as a (restricted) wreath product

$$L = B \wr \mathbb{Z}$$

being $B = \bigoplus_{\mathbb{Z}} C_2$ and the integers acting over the kernel by translation.

Any group that can be obtained changing C_2 by a finite group F in the previous extension is called a lamplighter group of finite groups in the sequel.

Theorem

Let $G = B \rtimes \mathbb{Z}$ be a lamplighter group of finite groups, $\mathcal{F}$ the family of finite subgroups of G. Then:

- $H_0^{\mathcal{F}}(G; R_{\mathbb{C}})$ is a free abelian group over a countable base.
- $H_1^{\mathcal{F}}(G; R_{\mathbb{C}}) = \mathbb{Z}$.
- $H_i^{\mathcal{F}}(G; R_{\mathbb{C}}) = 0$ otherwise.

As there always exists a 2-dimensional model for the classifying space for proper actions of lamplighter groups, we obtain:

Corollary

Let $G = B \rtimes \mathbb{Z}$ be a lamplighter group of finite groups. Then:

- $K_0^G(\underline{E}G)$ is a free abelian group over a countable base.
- $K_1^G(\underline{E}G) = \mathbb{Z}$.

Final remarks

- In this talk we have recalled some recipes that frequently permit to compute the left-hand side of Baum-Connes for discrete groups.
- All the examples belong to joint work with Pooya-Valette-Zumbrunnen whose general goal is to describe the assembly map as precise as possible.
- In particular, Baum-Connes conjecture holds for all the groups that appear as examples in the talk.

A sample of computations of Bredon homology

- Coxeter groups (Sánchez-García)
- Groups with cyclic torsion (joint work with Antolín)
- Hyperbolic groups (Lafont-Ortiz-Rahm-Sánchez-García)
- One-relator groups (Mislin)
- Special linear groups (Hughes, Sánchez-García)
- Abstract calculations by computer (Bui-Ellis)

THANK YOU!!!