Controlled K-theory, part II

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Recall : Controlled K-theory

Let $A = (A_r)_{r>0}$ be a unital filtered C^*-algebra, $r > 0$ (propagation) and $0 < \varepsilon < 1/4$ (control):

- $p \in A$ is an ε-r-projection if $p \in A_r$, $p = p^*$ and $\|p^2 - p\| < \varepsilon$.
- $u \in A$ is an ε-r-unitary if $u \in A_r$, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$.

Replacing in the definition of K-theory projections by ε-r-projections and unitaries by ε-r-unitaries we obtain controlled K-theory

$$K_*^{\varepsilon,r}(A) = K_0^{\varepsilon,r}(A) \oplus K_*^{\varepsilon,r}(A)$$

with structure maps

$$\iota_*^{\varepsilon,r,r'} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon,r'}(A)$$

and

$$\iota_*^{\varepsilon,r,} : K_*^{\varepsilon,r}(A) \longrightarrow K_*(A).$$

Recall : controlled index

Let X be a compact metric space.

- A non degenerated representation $(\mathscr{H}_X, \rho_X)$ of $C(X)$ is an X-standard module if $\rho_X(f)$ compact implies $f = 0$
- If A is a C^*-algebra, we set $K_*(X, A) = KK_*(C(X), A)$.
- Let us fix a non degenerated standard X-module $(\rho_X, \mathscr{H}_X)$. For any $0 < \varepsilon < 1/100$ and any $r > 0$, there exists a controlled index map $\mathrm{Ind}^{\varepsilon,r}_{X,A} : K_*(X, A) \to K^{\varepsilon,r}_*(\mathscr{K}(\mathscr{H}_X) \otimes A)$ such that
    1. $\iota^{\varepsilon,r,r'}_* \circ \mathrm{Ind}^{\varepsilon,r}_{X,A} = \mathrm{Ind}^{\varepsilon,r'}_{X,A}$;
    2. the composition

$$K_*(X, A) \longrightarrow K^{\varepsilon,r}_*(\mathscr{K}(\mathscr{H}_X) \otimes A) \xrightarrow{\iota^{\varepsilon,r}_*} K_*(\mathscr{K}(\mathscr{H}_X) \otimes A) \cong K_*(A)$$

is the index map $\mathrm{Ind}_{X,A} : K_*(X, A) \to K_*(A)$ (induced by $X \mapsto \{pt\}$).

Next steps

- For Σ a finite metric space and $\mathscr{H}_d$ a non degenerated $P_d(\Sigma)$-standard module, complete

$$K_*(P_d(\Sigma), A) \xrightarrow{\mathrm{Ind}^{\varepsilon,r}_{P_d(\Sigma),A}} K_*^{\varepsilon,r_0}(\mathscr{K}(\mathscr{H}_d)\otimes A) \longrightarrow K_*^{\varepsilon,r}(\mathscr{K}(\ell^2(\Sigma))\otimes A)$$

in a controlled assembly map for r_0 small and $r > 2d$. The right arrow is obtained by conjugation by a covering isometry

$$V_{\Sigma,d} : \mathscr{H}_d \longrightarrow \ell^2(\Sigma, \mathscr{H}_d) = \ell^2(\Sigma)\otimes\mathscr{H}_d$$

- State a uniformly controlled version of the Coarse Baum-Connes assembly map for families of finite metric spaces;
- show that it is stable under ”cut-and pasting”.

The covering isometry

- Let Σ be a finite metric space. Recall that the Rips complex of degree d is the set $P_d(\Sigma)$ of probability measures on Σ with support of diameter less than d.
- Let $(\lambda_\sigma)_{\sigma\in\Sigma}$ be the family of coordinate functions for $P_d(\Sigma)$, i.e $\nu = \Sigma_{\sigma\in\Sigma}\lambda_\sigma(\nu)\delta_\sigma$ for any ν in $P_d(\Sigma)$.
- Fix a non degenerated $P_d(\Sigma)$-standard module $\mathscr{H}_d$ and define $V_{\Sigma,d} : \mathscr{H}_d \longrightarrow \ell^2(\Sigma,\mathscr{H}_d)$ by $V_{\Sigma,d}\cdot\xi(\sigma) = \lambda_\sigma^{1/2}\cdot\xi$ ($\xi\in\mathscr{H}_d$ and $\sigma\in\Sigma$);
- Since $\sum\lambda_\sigma = 1$ then $V_{\Sigma,d}$ is an isometry;
- If $T\in A\otimes\mathscr{K}(\mathscr{H}_d)$ has propagation r then $Ad_{V_{\Sigma,d}}\cdot T = (1\otimes V_{\Sigma,d})\cdot T\cdot(1\otimes V^*_{\Sigma,d})$ has propagation $r' = 2d(r+1)$ and hence induces a morphism

$$K_*^{\varepsilon,r}(A\otimes\mathscr{K}(\mathscr{H}_d)) \stackrel{Ad_{V_{\Sigma,d},*}}{\longrightarrow} K_*^{\varepsilon,r'}(A\otimes\mathscr{K}(\ell^2(\Sigma,\mathscr{H}_d))) \cong K_*^{\varepsilon,r'}(A\otimes\mathscr{K}(\ell^2(\Sigma))).$$

Quantitative coarse assembly maps

- The composition
$$\mathrm{Ind}^{\varepsilon,r}_{P_d(\Sigma),A}: K_*(P_d(\Sigma),A) \to K^{\varepsilon,r}_*(A \otimes \mathscr{K}(\mathscr{H}_d))$$
and
$$Ad_{V_{\Sigma,d},*}: K^{\varepsilon,r}_*(A \otimes \mathscr{K}(\mathscr{H}_d)) \longrightarrow K^{\varepsilon,2d(r+1)}_*(A \otimes \mathscr{K}(\ell^2(\Sigma)))$$
give rise to a family of **quantitative coarse assembly maps**
$$\mu^{\varepsilon,r,d}_{\Sigma,A,*}: K_*(P_d(\Sigma),A) \longrightarrow K^{\varepsilon,r}_*(A \otimes \mathscr{K}(\ell^2(\Sigma)))$$
for $r > 2d$.
- These maps are compatible with $q^*_{d,d'}: K_*(P_d(\Sigma),A) \longrightarrow K_*(P_{d'}(\Sigma),A)$ induced by inclusions $P_d(\Sigma) \hookrightarrow P_{d'}(\Sigma)$ and with structure maps $\iota^{\varepsilon,r,r'}_*: K^{\varepsilon,r}_*(A \otimes \mathcal{K}(\ell^2(\Sigma))) \to K^{\varepsilon,r'}_*(A \otimes \mathscr{K}(\ell^2(\Sigma)))$;
- $\mu^{\varepsilon,r,d}_{\Sigma,A,*}$ induces the index map in K-theory.

Quantitative statement

We consider for Σ a finite metric space and A a C^*-algebra the statements :

$QI_{\Sigma,A}(d,d',r,\varepsilon)$:

$$\begin{tikzcd} K_*(P_d(\Sigma)) \ni x \arrow[r, mapsto] \arrow[d, dotted] & 0 \in K_*^{\varepsilon,r}(A\otimes\mathscr{K}(\ell^2(\Sigma))) \\ K_*(P_{d'}(\Sigma)) \ni 0 & \end{tikzcd}$$

$\mu^{\varepsilon,r,d}_{\Sigma,A,*}(x)=0$ in $K_*^{\varepsilon,r}(A\otimes\mathscr{K}(\ell^2(\Sigma)))$ implies $q^*_{d,d'}(x)=0$ in $K_*(P_{d'}(\Sigma),A)$, $\forall x\in K_*(P_d(\Sigma),A)$.

$QS_{\Sigma,A}(d,r,r',\varepsilon)$:

$$\begin{tikzcd} K_*(P_d(\Sigma)) \ni x \arrow[r, mapsto] & \iota_*^{\varepsilon,r,r'}(y) \in K_*^{\varepsilon,r'}(A\otimes\mathscr{K}(\ell^2(\Sigma))) \\ & y \in K_*^{\varepsilon,r}(A\otimes\mathscr{K}(\ell^2(\Sigma))) \arrow[u, mapsto] \end{tikzcd}$$

For every y in $K_*^{\varepsilon,r}(A\otimes\mathscr{K}(\ell^2(\Sigma)))$, there exists $x\in K_*(P_d(\Sigma),A)$ such that $\mu^{\varepsilon,r',d}_{\Sigma,,A,*}(x)=\iota_*^{\varepsilon,r,r'}(y)$.

Uniform Quantitative Assembly Map (QAM) estimates

Definition

Let $(\Sigma_i)_{i\in I}$ be a family of finite metric spaces. We say that $(\Sigma_i)_{i\in\mathbb{N}}$ satisfies uniformly the QAM-estimates if

1. *for any $d > 0$, $0 < \varepsilon < 1/200$ and $r > 2d$, there exists d' with $d' \geqslant d$ such that $QI_{\Sigma_i,A}(d, d', r, \varepsilon)$ is satisfied for any i in I and any C^*-algebra A.*
2. *For any $0 < \varepsilon < 1/200$ and $r > 0$, there exist positive numbers d and $r' > \sup\{2d, r\}$, such that $QS_{\Sigma_i,A}(d, r, r', \varepsilon)$ is satisfied for any i in I and any C^*-algebra A.*

$(\Sigma_i)_{i\in I}$ has uniformly bounded geometry if $\forall r > 0\ \exists N_r > 0$ such that in all Σ_i, balls of radius r have cardinal less than N_r.

Example

A family $(\Sigma_i)_{i\in I}$ of uniformly bounded finite metric spaces with uniformly bounded geometry satisfies uniformly the QAM-estimates.

Examples

$(\Sigma_i)_{i\in I}$ uniformly coarsely embeds into a Hilbert space $\mathscr{H}$ if $\exists$ proper maps $\rho_\pm : \mathbb{R}^+ \to \mathbb{R}^+$ and $\forall\, i \in I$ maps $f_i : \Sigma_i \to \mathscr{H}$ s.t $\rho_-(\|f_i(x) - f_i(y)\|) \leqslant d(x,y) \leqslant \rho_+(\|f_i(x) - f_i(y)\|)$ for all x, y in Σ_i.

Theorem (O-Yu)

Let $(\Sigma_i)_{i\in I}$ be a family of finite metric spaces. Assume $(\Sigma_i)_{i\in I}$ has uniformly bounded geometry and uniformly coarsely embeds into a Hilbert space. Then $(\Sigma_i)_{i\in I}$ satisfies uniformly the QAM-estimates.

Theorem (O-Yu)

Let Γ be a finitely generated group provided with any word metric. Assume Γ satisfies the Baum-Connes conjecture with coeff. Then the family of all finite subsets of Γ satisfies uniformly the QAM-estimates.

Application to Novikov conjecture

Theorem (O-Yu)

Let Σ be a discrete space with bounded geometry. Assume that the family of all finite subsets of Σ satisfies uniformly the QAM-estimates. Then Σ satisfies the coarse Baum-Connes conjecture.

Corollary (Descent principle of Higson/Roe)

Let Γ be a finitely generated group equipped with any word metric whose classifying space B_Γ has homotopy type of a finite CW-complex. Assume that the family of all finite subsets of Γ satisfies uniformly the QAM-estimates. Then Γ satisfies the Novikov conjecture.

QAM-ESTIMATES AND COARSE DECOMPOSITIONS (”cut-and-pasting”)

Coarse decomposability

Definition

Let $\mathcal{X}$ and $\mathcal{Y}$ be families of discrete proper metric spaces and $R > 0$. We say that $\mathcal{X}$ is R-decomposable relatively to $\mathcal{Y}$ if for every X in $\mathcal{X}$ there exist two subsets $X^{(1)}$ and $X^{(2)}$ of X such that

- $X = X^{(1)} \cup X^{(2)}$;
- $X^{(i)}$ *is a R-disjoint union of spaces in $\mathcal{Y}$, i.e $\exists (X_k^{(i)})_{k\in\mathbb{N}}$ family of spaces in $\mathcal{Y}$ s.t $X^{(i)} = \bigsqcup_{k\in\mathbb{N}} X_k^{(i)}$ and $d(X_k^{(i)}, X_l^{(i)}) > R$ if $k \neq l$;*

Remark

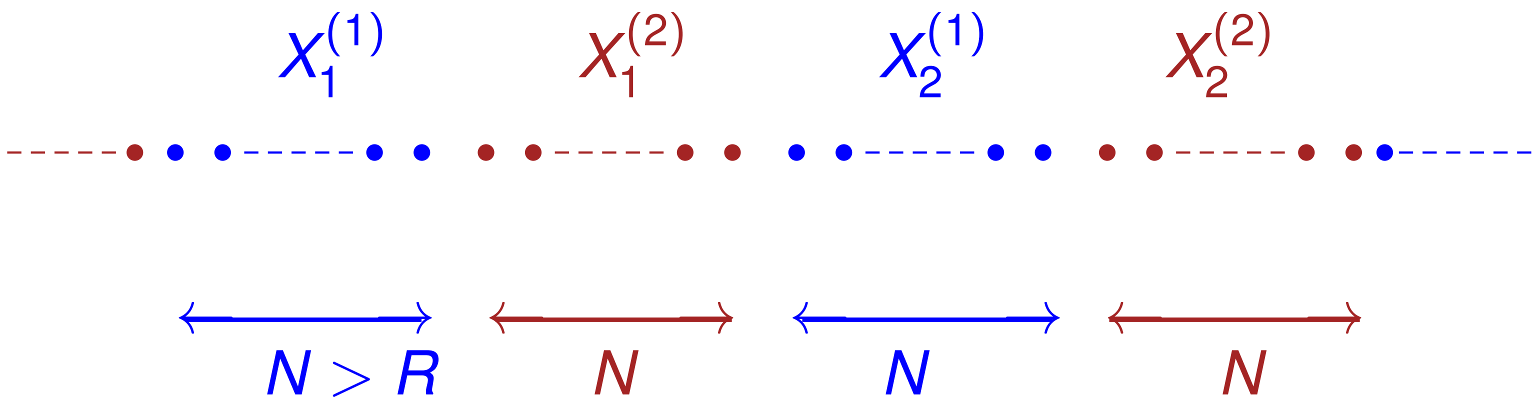
The single element family $\{X\}$ is R-decomposable for every $R > 0$ relatively to a family of uniformly bounded metric spaces $\overset{def}{\iff}$ X has **asymptotic dimension** 1.

Asymptotic dimension 1

Definition

A metric space X has asymptotic dimension 1 if for every $R > 0$, there exist two subsets $X^{(1)}$ and $X^{(2)}$ of X such that

- $X = X^{(1)} \cup X^{(2)}$;
- $X^{(i)}$ *is a R-disjoint union of a family of uniformly bounded subsets*



$\mathbb{Z}$ has asymptotic dimension 1.In the same way, free groups have asymptotic dimension 1

QAM-estimates and coarse decomposability

For $\mathcal{Y}$ a family of finite sets, let $\widetilde{\mathcal{Y}}$ be the family of all subsets of all sets in $\mathcal{Y}$.

Theorem (O-Yu)

Let $\mathcal{X}$ be a family of finite metric spaces. Assume that for every $R > 0$, there exists $\mathcal{Y}$ with uniformly bounded geometry such that

- *$\mathcal{X}$ is R-decomposable with respect to $\mathcal{Y}$;*
- *$\widetilde{\mathcal{Y}}$ satisfies uniformly the QAM-estimates;*

Then $\mathcal{X}$ satisfies uniformly the QAM-estimates.

Corollary

The family of finite subsets of a tree satisfies uniformly the QAM-estimates.

Example :

Definition

Let X be a proper discrete metric space. Then X has asymptotic dimension m if for every $r > 0$ there exist $m+1$ subsets $X^{(1)}, \ldots X^{(m+1)}$ of X such that

- $X = \cup_{i=1}^{m+1} X^{(i)}$;
- $X^{(i)}$ is a r-disjoint union of a uniformly bounded family of subsets, i.e $X^{(i)} = \bigsqcup_{k\in\mathbb{N}} X_k^{(i)}$ with $(diam\, X_k^{(i)})_{k\in\mathbb{N}}$ bounded and $d(X_k^{(i)}, X_l^{(i)}) > r$ if $k \neq l$;

Example

1. $\mathbb{Z}^n$ has asymptotic dimension n;
2. Gromov hyperbolic spaces have finite asymptotic dimension;
3. Discrete subgroups in Lie groups have finite asymptotic dimension.

QAM and finite asymptotic dimension

Theorem (Dranishnikov-Zarichnyi)

Let Σ be a proper discrete metric space with bounded geometry. If X has finite asymptotic dimension, then Σ admit a coarse embedding into a product of trees $T_1 \times \cdots \times T_n$
(i.e there exists proper maps $\rho_\pm : \mathbb{R}^+ \to \mathbb{R}^+$ and a map $f : \Sigma \to T_1 \times \cdots \times T_n$ such that $\rho_-(d(f(x), f(y))) \leqslant d(x, y) \leqslant \rho_+(d(f(x), f(y)))$ for all x, y in Σ).

Corollary

Let Σ be a discrete metric space with bounded geometry and finite asymptotic dimension. Then the family of all finite subsets of Σ satisfies uniformly the QAM-estimates.

Finite decomposition complexity

Definition

A class $\mathcal{C}$ of families of proper discrete metric spaces is closed under coarse decomposability if the following is satisfied:

let $\mathcal{X}$ be a family of proper discrete metric spaces. Assume that for any $R > 0$, there exists $\mathcal{Y}$ in $\mathcal{C}$ such that $\mathcal{X}$ is R-decomposable with respect to $\mathcal{Y}$. Then $\mathcal{X}$ is in $\mathcal{C}$.

Consider $\mathcal{C}_{fdc}$ the smallest class of families of proper discrete metric spaces which is closed under coarse decomposability and which contains uniformly bounded families of metric spaces.

Definition

A proper discrete metric space X has **finite decomposition complexity** if the single element family $\{X\}$ is in $\mathcal{C}_{fdc}$.

Examples : countable subgroups in $GL_n(\mathbb{F})$ (Guentner-Tessera-Yu).

QAM-estimates and coarse decomposability

Theorem (O-Yu)

Let $\mathcal{X}$ be a family of finite metric spaces in $\mathcal{C}_{fdc}$ with uniformly bounded geometry. Then $\mathcal{X}$ satisfies uniformly the QAM-estimates.

Corollary

If X has finite decomposition complexity and bounded geometry, the family of all finite subsets of X satisfies the QAM-estimates.

Corollary (Guentner-Tessera-Yu)

If Γ is a finitely generated group whose classifying space B_Γ has homotopy type of a finite CW-complex and with finite decomposition complexity with respect to any word metric, then Γ satisfies the Novikov conjecture.

THANK YOU FOR YOUR ATTENTION !!!