On the existence of a matchable double cover

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Abstract

We prove that every regular graph has a matchable double cover. As a corollary, we obtain that every connected $2d+1$-regular graph is isomorphic to a Schreier graph.

The aim of this note is to prove the following result on perfect matchings.

Theorem 1. *Let G be a d-regular graph. Then either G has a perfect matching, or there exists a double cover H of G that has a perfect matching.*

Moreover, in the second case, if G is connected then so is H.

As a corollary, we will also show

Theorem 2. *Let G be a $(2d+1)$-regular connected graph. Then either G is isomorphic to a Schreier graph of $F_d * (\mathbf{Z}/2\mathbf{Z})$ or G has a double-cover H which is isomorphic to a Schreier graph of $F_d * (\mathbf{Z}/2\mathbf{Z})$.*

Both Theorem 1 and Theorem 2 are constructive when restricted to finite graphs.

Observe that it isn't possible to ensure that G has a double cover isomorphic to a Schreier graph. Indeed, if T is a regular tree, then its only double cover is the disjoint union of two copies of T, which is not connected and hence not isomorphic to a Schreier graph.

While we were not able to find a reference to the above results in the literature, we do not claim any priority on it. For example, the following remark can be found at the end of Section 7 of [4]: "In fact up to the cover of degree 2 any regular graph can be realized as a Schreier graph [10]". However, [10] seems to treat only the even case.

This paper is organized as follow. The next section is dedicated to general graphs and perfect matchings, while Section 2 deals with Schreier graphs.

1 Graphs and matchings

Since one of our motivation is the study of Schreier graphs, we will look at graphs in a broad sense. We allow multi-edges and loops as well as semi-edges (or degenerated loops).

A *digraph* is a 4-tuple $(V, \vec{E}, \iota, \tau)$ where V and $\vec{E}$ are sets (of *vertices* and of *arcs*) and $\iota, \tau \colon \vec{E} \to V$ are two applications (*initial vertex* and *terminal vertex*) with no restriction on them.

A *graph* is a digraph together with an involution $\mathrm{inv}\colon \vec{E} \to \vec{E}$ such that $\iota \cdot \mathrm{inv} = \tau$ and $\tau \cdot \mathrm{inv} = \iota$. There is a natural equivalence relation on arcs defined by $\vec{E} \sim \vec{b}$ if and only if $\vec{e} = \vec{f}$ or $\vec{e} = \mathrm{inv}(\vec{f})$. The quotient set $\vec{E}/\sim$ is denoted by E and is the set of (semi)-edges, where $e = \{\vec{e}, \mathrm{inv}(\vec{e})\}$ is an *edge* if $\vec{e} \neq \mathrm{inv}(\vec{e})$ and a *semi-edge* (also called *degenerated loop*) otherwise.

A vertex v and an edge $e = \{\vec{e}, \mathrm{inv}(\vec{e})\}$ are *adjacent* if $v = \iota(\vec{e})$ or $v = \tau(\vec{e})$, in which case we also say the v is an *end* of e. An edge that is adjacent to only one vertex is called a *loop*. The *degree* of a vertex is the number of arcs $\vec{e}$ such that $v = \iota(\vec{e})$. In particular, a loop add two to the degree of its adjacent vertex, while a semi-edge add only 1. All graphs under consideration will be *locally finite*: the degree of any vertex is finite. A *multi-edge* is a set of at least two edges with the same ends. A graph is *simple* if it has no multi-edge, loop or semi-edge. Classical notions like *paths* and *connected components* are defined in a natural way. In particular, a *bipartite graph* has no loops nor semi-edge. A *cycle* is a non self-intersecting closed path, which might consists of a single loop. Finally, for a vertex v we define $\mathrm{Star}(v) \coloneqq \{\vec{e} \in \vec{E} \mid \iota(\vec{e}) = v\}$.

We will often write $G = (V, E)$ for a graph when it is not necessary to specify the underlying structure. A graph with no semi-edge corresponds to the definition of a graph by Serre and many results about such graphs from [11] can be extended easily to the general case, see [8, 7] for more details.

On pictures, semi-edges will be represented by dotted line and edges (including multi-edges and loops) with plain lines. Informally, a semi-edge is the index 2 quotient of the path of length 1, see Figure 1. It can either been seen as an edge with only one end, or as a loop that add only one to the degree. Allowing semi-edges is necessary to treat the case of Schreier graphs of groups with generators of order 2.

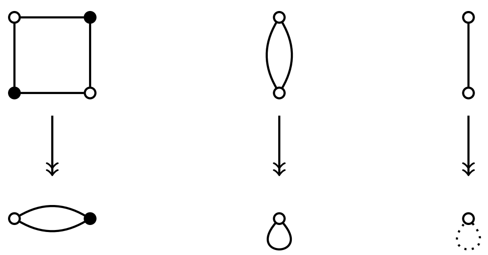

Figure 1: Double covers of some graphs. The leftmost one is isomorphic to the double cover from $\mathrm{Schrei}(\mathbf{Z}/4\mathbf{Z}, \{1\}; 1)$ to $\mathrm{Schrei}(\mathbf{Z}/4\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}; 1)$, the middle one to $\mathrm{Schrei}(\mathbf{Z}/4\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}; 1) \twoheadrightarrow \mathrm{Schrei}(\mathbf{Z}/4\mathbf{Z}, \mathbf{Z}/4\mathbf{Z}; 1)$ and the rightmost one to $\mathrm{Schrei}(\mathbf{Z}/2\mathbf{Z}, \{1\}; 1) \twoheadrightarrow \mathrm{Schrei}(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}/2\mathbf{Z}; 1)$.

Let $G = (V, E)$ be a graph. For $A \subseteq V$ we denote by $G[A]$ the *induced subgraph* of G, that is the subgraph with vertex set A and with (semi)-edges set $\{e \in E \mid \text{ends of } e \text{ are in } A\}$. For $B \subseteq V$, we denote by $G - B$ the subgraph $G[V \setminus B]$.

A *morphism* of graph from $(V, \vec{E})$ to $(W, \vec{F})$ is a map $\varphi\colon V \sqcup \vec{E} \to W \sqcup \vec{F}$ which is compatible with the graph structure. For any vertex $v \in V$ this induced a map $\varphi_v\colon \mathrm{Star}(v) \to \mathrm{Star}\big(\varphi(v)\big)$. A morphism $\varphi\colon G \to H$ between two graphs is a *covering* if all the induced map φ_v are bijections. If moreover $|\varphi^{-1}(v)| = d$ (the quantity $|\varphi^{-1}(v)|$ depends only on the connected component of v) for every vertex, then it is called a *cover of degree* d or a *double cover* if $d = 1$.. It is well-known that every $2d$-regular graph without semi-edge cover a graph with 1 vertex and d loops.

Definition 3. A *matching* (also called a 1-*factor*) of a graph G is a subgraph M of G such that every vertex of M has degree 1 in M. A vertex v is *covered* if it is in M and *missed* otherwise. A *perfect matching* is a matching covering every vertex of G. A graph is *matchable* if it admits a perfect matching. A graph G is *factor-critical* if it is not matchable, but for every vertex v the graph $G - \{v\}$ is matchable.

Observe that the empty graph is matchable and therefore that a graph consisting of a single vertex with no semi-edge but possibly with loops is factor-critical. On the other hand, even among regular graphs of odd degree, not all graphs are matchable. See Figure 2 for an example. However, every locally finite

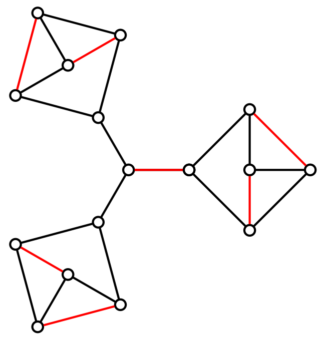

Figure 2: A simple 3-regular non matchable graph with a maximal matching.

vertex-transitive graph of odd degree is matchable [7]. We also have

Lemma 4. *Let G be a $(2d + 1)$-regular graph without semi-edge. Then G is matchable if and only if it covers a graph with one vertex, one semi-edge and d loops.*

Proof. If $\varphi\colon G \to R$ is a covering and R has a semi-edge $\vec{e}$, then $\varphi^{-1}(\vec{e})$ is a perfect matching of G.

On the other hand, let M be a perfect matching of G and G' the graph obtained by removing all edges of M from G. Then G' is $2d$-regular and hence covers R' a graph with one vertex and d loops. Let R be the graph obtained by adding a semi-edge to R' Then the covering $G' \twoheadrightarrow R'$ naturally extends to $G \twoheadrightarrow R$ by sending all edges of M to the semi-edge of R. $\square$

Observe that this result can easily be generalized to characterize in $(2d + n)$-regular graphs the existence of n *orthogonal* matchings, that is matchings that share no edges, via the existence of a covering to the graph with one vertex, n semi-edges and d loops.

There is a natural partial order on the set of all matchings of a graph G defined by $M \leq N$ if every vertex covered by M is also covered by N. A maximal element for this order is called a *maximal matching.* If A and B are two disjoint subsets of V, we say that there exists a *matching from A into B* if there exists a matching M of G covering all vertices of A and such that each edge of M has one end in A end the other in B. In particular, the subgraph M is bipartite.

It is a direct consequence of the definition that a factor-critical graph has no semi-edge. On the other hand, a locally finite factor-critical graph is finite and has an odd number of vertices [12] .

We introduce one last notion that will be useful for the proof of our main result.

Definition 5. Let G be a graph and M a matching. An *alternating path* for M is a path in G that alternates between edges in M and edges not in M

A *near-alternating cycle* for $v \notin M$ is a cycle in G which is alternating except at v where the two incident edges are not in M.

It follows from the definitions that every near-alternating cycle is of odd length and that every loop is a near-alternating cycle. In important fact for us is the next lemma which follows from the existence of a so called *M-alternating odd-ear-decomposition* for factor-critical graphs.

Lemma 6 ([9])**.** *Let G be a finite factor-critical graph that is not reduce to a single vertex. Let v be any vertex of G and M any perfect matching of $G - \{v\}$. Then G contains a near-alternating cycle (with respect to M) for v.*

The proof of the following refinement of the Cantor-Berstein Theorem is similar to the standard proof of the Cantor-Berstein Theorem and is let as an exercice.

Lemma 7. *Let G be a bipartite graph with bipartition $V \sqcup W$. And let V_0 be a subset of V and let W_0 be a subset of W. If there is a matching covering V_0 and a matching covering W_0, then there is a matching covering both V_0 and W_0.*

Let $G = (V, E)$ be graph and $A, T \subseteq V$ two disjoint subsets of vertices. Let $\mathfrak{A}$ be the set of all factor-critical connected components of $G[A]$. We denote by $\Pi(G, A, T)$ the bipartite graph with vertex sets $(\mathfrak{A}, T)$ and with an edge from $F \in \mathfrak{A}$ to $t \in T$ if and only if there exists $a \in A$ that is adjacent to t in G. Finally, let $\mathfrak{A}_0$ be the set of connected components of $G[A]$ that are reduced to a single vertex (no loop nor semi-edge). There is a natural identification between $\mathfrak{A}_0$ and the subset $A_0 \subseteq A$ of vertices supporting it.

We now prove a technical lemma.

Lemma 8. *Let $G = (V, E)$ be a locally finite graph together with a decomposition of V as a disjoint union $V = A \sqcup B \sqcup T$. Let $\mathfrak{A}_0$ be the set of connected components of $G[A]$ that are reduced to a single vertex. Suppose that*

1. *The graph $G[B]$ is matchable;*

2. *Every connected component of $G[A]$ is factor-critical;*

3. *There exists a matching from some subset $\mathfrak{A}_0'$ of $\mathfrak{A}_0$ to T in $\Pi(G, A, T)$. Equivalently, there exists a matching from $A_0' \subset A_0$ to T in G;*

4. *There exists a matching from T to $\mathfrak{A}$ in $\Pi(G, A, T)$. Equivalently, for every $t \in T$ there exists $v_t \in A$ such that all the v_t belong to pairwise distinct connected components of $G[A]$.*

Then there exists a double cover $\pi\colon H \to G$ without loop nor semi-edge and a matching L of H such that the vertices missed by L are contained in $\pi^{-1}(A_0 \setminus A'_0)$ and such that H is simple if G is simple.

Moreover, if G is connected, then H is connected if G has at least one loop or semi-edge or if G is not matchable and $A'_0 = A_0$.

Proof. Let $\mathfrak{A}$ be the set of connected components of $G[A]$. By hypothesis it contains only factor-critical graphs. By Lemma 7, there exists a matching in $\Pi(G, A, T)$ that covers T and a subset $\mathfrak{A}_1$ of $\mathfrak{A}$ such that $\mathfrak{A}'_0 \subseteq \mathfrak{A}_1$. Let $\mathfrak{A}_2 \coloneqq \mathfrak{A} \setminus \mathfrak{A}_1 = \{F_i \mid i \in I\}$ for some index set I.

For each $t \in T$ there is an element F_t of $\mathfrak{A}_1$ and a vertex v_t of F_t such that $\{t, v_t\}$ is an edge of G, the F_t are pairwise distinct and $\mathfrak{A}_1 = \{F_t \mid t \in T\}$. For every $t \in T$, let N_t be a perfect matching of $F_t - \{v_t\}$ and let N be a perfect matching of $G[B]$. Finally, for each $i \in I$, choose an arbitrary vertex v_i of F_i and let N_i be a perfect matching of $F_i - \{v_i\}$. Then

$$M \coloneqq N \cup \big(\bigcup_{i\in I} N_i\big) \cup \big(\bigcup_{t\in T} N_t\big) \cup \big(\bigcup_{t\in T} \{t, v_t\}\big)$$

is a matching of G which misses exactly the vertices of $\{v_i \mid i \in I\}$.

Let $I_0 \subseteq I$ be the subset of indices such that F_i is reduced to a single vertex. For each $i \in I \setminus I_0$, let c_i be a near-N_i-alternating cycle in F_i for v_i. The existence of such a cycle is guarantee by Lemma 6. We now construct the desired double cover H of G. The vertex set of H is simply the Cartesian product $V \times \{0, 1\}$. It remains to explain how (semi)-edges of G are lifted to (semi)-edges of H. First of all, every loop adjacent to a vertex v is lifted to a pair of edges between $(v, 0)$ and $(v, 1)$, while every semi-edge adjacent to a vertex v is lifted to an edge between $(v, 0)$ and $(v, 1)$. Let G' be the graph obtained from G by removing all edges belonging to $\bigcup_{i\in I} c_i$. If e is in G' and is not a loop nor a semi-edge, then it is lifted to $e \times \{0, 1\}$. On the other hand, a cycle $c_i = (v_1, \ldots, v_d)$ (d odd) is lifted to the cycle $\big((v_1, 0), \ldots, (v_d, 0), (v_1, 1), \ldots, (v_d, 1)\big)$ of length $2d$. By construction, the canonical projection $\pi\colon H \to G$ is a double cover and it remains to define the desired matching L of H.

Let $L_0 \coloneqq \pi^{-1}(M \cap G')$ and for each $i \in I \setminus I_0$ choose a perfect matching L_i of $\pi^{-1}(c_i)$. We claim that $L \coloneqq L_0 \cup \bigcup_{i\in I\setminus I_0} L_i$ is a matching of H that misses only the vertices in $\pi^{-1}(\{F_i \mid i \in I_0\})$. Let $\tilde{v}$ be a vertex of $H \setminus \bigcup_{i\in I} \pi^{-1}(c_i)$ and $v = \pi(\tilde{v})$ its projection. Then v is not in any of the c_i and has degree 0 or 1 in M depending if it belongs to $\{F_i \mid i \in I_0\}$ or not. Therefore, $\tilde{v}$ has degree 0 or 1 in L depending if it belongs to $\pi^{-1}(\{F_i \mid i \in I_0\})$. On the other hand, suppose that $\tilde{v}$ belongs to c_i for some i (such a i is unique). Then it has degree 1 in L_i and degree 0 in L_j for $j \neq i$. On the other hand, $\pi(\tilde{v})$ is not incident to any (semi)-edge of $M \cap G'$ which implies that $\tilde{v}$ has degree 0 in L_0 and hence degree 1 in L. We have just proved that L a matching of H that misses exactly the vertices of $\pi^{-1}(\{F_i \mid i \in I_0\}) \subseteq \pi^{-1}(A_0 \setminus A'_0)$.

By construction, simple edges of G gives simple edges of H, multi-edges give multi-edges, loops give multi-edges and semi-edges give simple edges.

Finally, suppose that G is connected. By construction H has at most two connected components. Moreover, it has exactly one connected component if and only if there exists a vertex v of G such that $(v, 0)$ and $(v, 1)$ are connected in H. This last condition holds by construction as soon as G has a loop or a semi-edge. On the other hand if G is not matchable and $A'_0 = A_0$, then $\mathfrak{A}_2$ is

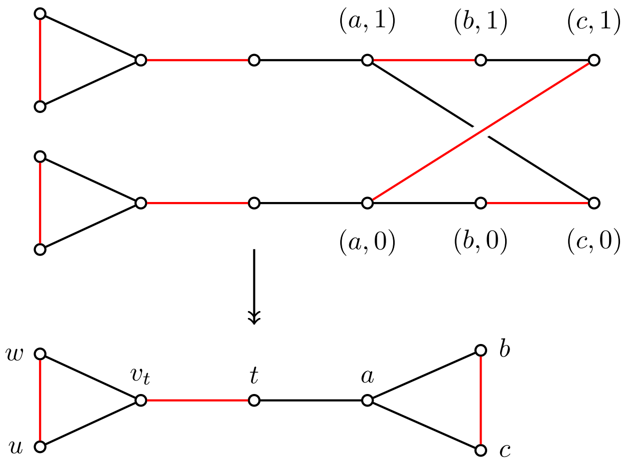

Figure 3: On bottom: a graph G with a maximal matching in red. On top: a matchable double cover of G with a perfect matching in red.

not empty and disjoint from $\mathfrak{A}_0$. In particular, $I \setminus I_0$ is not empty and there exists a cycle c_i based at v_i which is lifted to a cycle containing both $(v_i, 0)$ and $(v_i, 1)$. $\square$

Example 9. Let G be the graph depicted at the bottom of Figure 3. It is connected and unmatchable. We have $T = \{t\}$, B is empty as well as $\mathfrak{A}_0$, $\mathfrak{A}_1$ contains only $G[\{v_t, u, w\}]$ and $\mathfrak{A}_2$ has only one element: $G[\{a, b, c\}]$, which is a cycle of length 3. The matching $M = \{\{u, w\}, \{v_t, t\}, \{b, c\}\}$ is depicted in red in Figure 3 and misses only the vertex a.

The double cover H of G given by Lemma 8 and its perfect matching L are depicted at the top of Figure 3.

Remark 10. The graph H produce by the proof of Lemma 8 is not necessarily the only double cover of G that admits a matching L satisfying the desired condition. For example, an alternative algorithm would be to lift loops that are not one of the c_i to a pair of loops rather than to a pair of parallel edges. Similarly, semi-edges can be lifted either to a pair of semi-edges or to a single edge.

Figure 4 shows a 3-regular non-matchable graphs together with two matchable double covers. The graph on the top left is obtained by a direct application of the construction of Lemma 8 while the one on the top right is obtained by the alternative version of the algorithm.

In order to apply Lemma 8 we will rely on the following Edmonds-Gallai decomposition for locally finite graphs.

Proposition 11 ([1])**.** *Let $G = (V, E)$ be a locally finite graph with no semi-edges. Let A be the set of vertices of G that are missed by at least one maximal matching, T be the set of vertices in $V \setminus A$ adjacent to A and $B \coloneqq V \setminus (A \cup T)$. Then*

1. *Every connected component of $G[A]$ is factor-critical and $G[B]$ is matchable;*

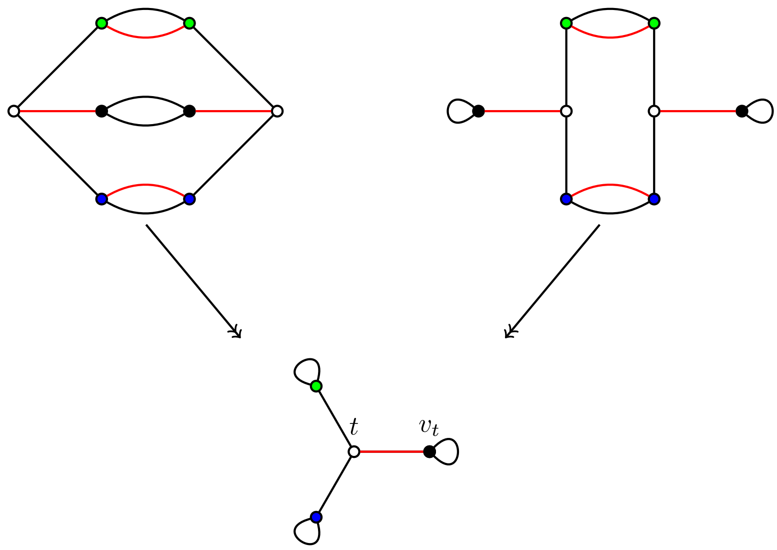

Figure 4: Two matchable double covers of a non-matchable 3-regular graph.

2. *There exists a matching from T to $\mathfrak{A}$ in* $\Pi(G, A, T)$.

While Proposition 11 is stated in [1] for simple graphs, it applies *mutatis mutandis* to graphs where loops and multi-edges are allowed but with no semi-edge.

Lemma 8 together with Proposition 11 imply

Corollary 12. *Let $G = (V, E)$ be a locally finite graph with no semi-edge. Let A be the set of vertices of G that are missed by at least one maximal matching, T be the set of vertices in $V \setminus A$ adjacent to A and $A_0 := \{a \in A \mid \text{every neighbour of } a \text{ is in } T\}$. If there exists a matching from A_0 to T, then G has a matchable double cover H which is simple if G is simple.*

We finally prove our result for regular graphs. For regular graphs with no semi-edge it follows quickly from Corollary 12 and Hall's Theorem about matching in bipartite locally finite graphs. When we allow semi-edges the situation is a bit more complicated.

Theorem 13. *Let G be a d-regular graph, $d \in \mathbf{N}$. Then G has a matchable double cover H without loop nor semi-edge and which is simple if G is simple.*

Moreover, if G is connected, then H is connected if G has at least one loop or semi-edge or if G is not matchable.

Proof. Let K be the graph obtained from G by erasing all semi-edges. Let A be the set of vertices of K that are missed by at least one maximal matching of K, T be the set of vertices in $V \setminus A$ adjacent to A and $B := V \setminus (A \cup T)$. Finally, let $\mathfrak{A}$ be the set of connected components of $K[A]$. By Proposition 11, every connected component of $K[A]$ is factor-critical, $K[B]$ is matchable and there exists a matching from T to $\mathfrak{A}$ in $\Pi(K, A, T)$.

Let $\mathfrak{A}_0$ be the set of connected components of $K[A]$ and let A_0 be the corresponding subset of V. Let A_0' be the subset of vertices of A_0 that are not adjacent to a semi-edge in G. Therefore, a vertex v of A_0 is in A_0' if and only if it has degree d in K. In the bipartite graph $K[A_0', T]$ every vertex of A_0 has degree d, while vertices in T have degree at most d. The existence of a matching from A_0' to T is hence assured by the locally finite version of Hall's Theorem [6]. By Lemma 8 we obtain a double cover H' of K together with a matching L' that misses exactly the vertices in $A_0 \setminus A_0'$. For every semi-edge of G adjacent to some vertex v, we add an edge in H' between $(v, 0)$ and $(v, 1)$. The resulting graph H is a double cover of G. For every v in $A_0 \setminus A_0'$, choose a semi-edge e_v adjacent to v. Then

$$L \coloneqq L' \cup \bigcup_{v \in A_0 \setminus A_0'} \pi^{-1}(e_v)$$

is a perfect matching of H. □

Recall that two matchings are orthogonal if they do not share a (semi)-edge.

Corollary 14. *Let G be a d-regular connected graph, $d \in \mathbf{N}$. For every $1 \leq i \leq d$, there exists $j \leq i$ and a covering G_i of G of degree 2^j such that G_i is connected and has i pairwise orthogonal perfect matchings.*

Proof. The proof is by induction on i. For $i = 1$ this is Theorem 13. Now suppose that we have constructed the covering $G_i = (V_i, E_i)$ and we want to construct G_{i+1}. Choose i pairwise orthogonal perfect matchings M_r of G_i all let H_i be the graph obtained from G_i by erasing all the edges of the M_r. The graph H_i is $(d-i)$-regular. If it is matchable, then G_i has $i+1$ pairwise orthogonal perfect matchings and there is nothing to do. Otherwise, let H_{i+1} be the matchable double cover of H_i given by Theorem 13. Such a graph exists as soon as $d - i \geq 1$. The vertex set of H_{i+1} is $V_i \times \{0, 1\}$. Let G_{i+1} be the graph obtained from H_{i+1} by adding a pair of edge $e \times \{0, 1\}$ for every edge e in one of the M_t. Then G_{i+1} is a double cover of G_i that has $i+1$ pairwise orthogonal perfect matchings. □

2 An application to Schreir graphs

Let Γ be a group with a symmetric generating system S and let Λ be a subgroup of Γ. The corresponding (right) *Schreier graph* $\mathrm{Schrei}(\Gamma, \Lambda; S)$ is the graph with vertex set the right cosets $\{\Lambda g \mid g \in \Gamma\}$ and with an arc from Λg to Λh for every s in S such that $\Lambda g s = \Lambda h$. If X is a set with a right action $X \curvearrowleft \Gamma$, the corresponding (right) *orbital graph* $\mathrm{Schrei}_{\mathcal{O}}(\Gamma, X; S)$ is the graph with vertex set X with an arc from x to y for every s in S such that $x.s = y$.

These two definitions are the two faces of the same coin and we have $\mathrm{Schrei}_{\mathcal{O}}(\Gamma, X; S) = \mathrm{Schrei}(\Gamma, \mathrm{Stab}_G(x); S)$ for every $x \in X$ and $\mathrm{Schrei}(\Gamma, \Lambda; S) = \mathrm{Schrei}_{\mathcal{O}}(\Gamma, \Lambda\backslash\Gamma; S)$. See Figure 1 for some examples.

Schreier graphs appears naturally in the context of the lattice of subgroups as well as in the context of group actions and are interesting objects per se. On the other hand, realizing abstract graphs as Schreier graphs endow them with a more algebraic structure and can be fruitful both from a graph theoretic and from a group theoretic point of view.

By a theorem of Gross [5], every finite $2d$-regular connected graph without semi-edge is isomorphic to a Schreier graph of the free group F_d. This result extends by compacity to infinite $2d$-regular graphs, see [3] for a proof. On the other hand, Gross' Theorem does not carry over graphs of odd regular degree, see

Figure 2. However, every connected transitive graph of degree $2d+1$ and without semi-edge is isomorphic to a Schreier graph of $F_d * (\mathbf{Z}/2\mathbf{Z})$[2, 7]. For general connected regular graphs of odd degree, the situation is more complicated. In fact, it turns out that a connected regular graph of odd degree without semi-edge is isomorphic to a Schreier graph if and only if it is matchable. In this case it is isomorphic to a Schreier graph of $F_d * (\mathbf{Z}/2\mathbf{Z})$, where F_d is the free group of rank d. A proof of this fact can be found in [7], but the result itself was probably already general knowledge at that time and essentially (at least for finite graphs) follows from [5].

We know extend the above result to the most general case.

Theorem 15. *Let G be a connected d-regular graph, $d \in \mathbf{N}$. Then either G is isomorphic to a Schreier graph, or there exists a double cover of G that is isomorphic to a Schreier graph.*

Proof. if G has no semi-edge and is of even-degree or is matchable, then it is isomorphic to a Schreier graph.

On the other hand, if G has a semi-edge or is not matchable, then by Theorem 13 there exists a double covers of G which is connected, d-regular, with no semi-edge and matchable. Such a graph is always isomorphic to a Schreier graph. □

Observe that a Schreier graph is necessarily connected and of regular degree. On the other hand, there are examples of connected regular graphs of odd degree and without semi-edges that are not isomorphic to a Schreier graph (see Figure 2) There are also examples of connected regular graphs of even degree (but with at least one semi-edge) that are not isomorphic to a Schreier graph[7, Figure 3.2]. Hence, Theorem 15 is the "best possible" for locally finite graphs.

We conclude with

Proposition 16. *Let G be a d-regular connected graph. Then there exists $i \leq d$ and a covering H of G of degree 2^i such that H is isomorphic to a Schreier graph of $\Gamma = (\mathbf{Z}/2\mathbf{Z})^{*d}$, the free product of d copies of $\mathbf{Z}/2\mathbf{Z}$.*

Proof. The graph H is isomorphic to Schreier graph of $(\mathbf{Z}/2\mathbf{Z})^{*d}$ if and only if it has d pairwise orthogonal perfect matchings [7]. Corollary 14 finishes the proof. □

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