A Math Filled Summer!

Part 3

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Unit References and Conversions

10^{12} = trillion
10^9 = billion
10^6 = million
10^3 = thousand(k)
10^{-2} = centi (c)
10^{-3} = milli (m)
10^{-6} = micro (μ)
10^{-9} = nano (n)
1000L of water = 1 m^3
1L of water = 1 kg

Problem-Solving Philosophy

Immerse in the known
Ponder the unknown
Expand into the bounds
A symbol is better than a number
An exact number is better than an approximate number
An approximate number is better than nothing
Sanity checks are for the sane

1 Calculus

1. Food for Thought
   (a) Define speed with a formula (consider the units)
   (b) How would a car measure it's "speed at time t". What does it mean to have a speed at a specific time?
   (c) Consider a boat traveling round trip from Florida to Puerto Rico and back. Suppose it reaches Puerto Rico and immediately turns around and heads to Florida. It travels with the same speed s during both parts of the trip. Did the boat at any point in time have a speed of 0? When?
2. Physical Intuition
   (a) Suppose we have a function x(t), where t is time and x(t) is the distance traveled at time t. Let x(t) = at + b. Draw a graph for any values of a,b. What is the velocity symbolically at any time t? What is the velocity graphically?
   (b) What are the units of a,b,t, and x(t)?
   (c) Suppose we have a function v(t), where t is time and v(t) is the velocity at time t. Graph v(t) as a straight line, v(t) = a. What is the total distance traveled at any time t? Find a symbolic and graphical explanation.
   (d) Suppose x(t) = t^2. What is the velocity at any given time?
3. Numerical Approximation
   (a) Suppose you are given the following table of distance traveled at various times. How would you approximate the speed at time 1.4 seconds?

Time(sec)	0.45	1	1.4	2.1	2.4
Distance(meters)	0.5	0.9	1.2	1.99	2.3

   (b) How would you approximate the time at 1.7 seconds? Justify your method
   (c) Suppose we have a femtosecond laser(a laser that shoots really fast) that measures how far we have traveled. For all purposes, we have a table of infinite precision of how far we have traveled at each second. How would you approximate our speed at any given time interval now?
   (d) Consider if we plotted a graph of the above table with time on the x axis and distance traveled on the y-axis. Translate what we found physically into mathematical terms.
   (e) Given the above numerical procedure, try to derive a formula for the velocity, given a continuously defined function v(t)

4. Cross Country Road Trip
   Ben, Nathan, Will, Corban are racing on a cross country road trip to see who can cross the continental U.S. in the shortest time. Their speed as a function of time is given below. Assume each takes the same path.

$$\begin{aligned}
&\text{Ben} && v(t) = 100 \\
&\text{Nathan} && v(t) = 10t \\
&\text{Will} && v(t) = \begin{cases} 0 & t = 0 \\ 200 & 0 < t \leq 1 \\ 7t & t > 1 \end{cases} \\
&\text{Nathan} && v(t) = 2t^2 + 3t - 10
\end{aligned}$$

   (a) Plot all of their respective velocities as a function of time
   (b) How far will each person have traveled by t=2? t=5?
   (c) Plot all of their respective distances traveled as a function of time
   (d) What geometrical intuition can you give about plotting velocity to plotting distance?

2 Applied

1. Visualization

   (a) Consider the following 3D function

$$z = f(x, y) = x^2 + y^2$$

   You can visualize this function in 3D as x,y parallel to the ground and the output of the function, z, as the height.

   i. What is the domain of this function?
   ii. What is the range of this function?
   iii. Is this function symmetric?
   iv. What is the minimum of this function i.e. smallest output? For which (x,y) pairs does this occur?
   v. Describe how the given function relates to the function g geometrically

$$g(x) = x^2$$

2. Gas Guzzlers

   (a) Describe a formula for the *total price* of a fill up for a car with some *volume* gas tank and some *price* of fuel in dollars per unit volume

(b) Describe a formula for total miles a car can travel on a full tank of gas, using the above variables and the *fuel efficiency* in units of unit distance per unit volume

(c) Suppose you have 4 cars with the following mile-per-gallon(mpg) performance: 5mpg, 15mpg, 25mpg, and 50mpg. How many gallons will each consume after traveling 100 miles? Plot the relationship between mpg and gallons consumed after 100 miles. What type of function is this?

(d) Carefully consider what information mpg is designed to convey to the customer. What metric do we really care about? Is mpg a well-designed metric? Tread carefully - you've just discovered how simple, yet subtle, mathematics can greatly affect public perception and public policy. If only we had informed leaders who understood these nuances!

(e) What would be a better metric to use instead of mpg? Why?

(f) Notice in the previous question we had 4 different cars with varying mpgs. Use your new metric to rank their performance. What do you notice? What implications does this have for carbon emissions in the automobile industry? And be careful, you've discovered a secret that many automobile companies try to cover up!
If you're still interested, take a peek at this link or this one.

3. Go!
Note: The board game descriptions here are simplifications
The board game is believed to be one of the oldest continuously played board games in history. One of the reasons Go has endured for so long is its simplicity. At a very basic level, the board is a square grid of vertices. There are two players and each player takes turns placing a stone at a vertex. Once a stone is placed a vertex, it cannot be moved nor can another stone be placed at that vertex.
Obviously there is much more nuance, but for our purposes, this is the only understanding we will need. For the purposes of this question, assume the game ends after a win-condition is reached, rather than a player quitting, not showing up, etc. We use the term turn to mean a player moves and the term move to define player 1 moving once then player 2 moving once
Recently, researchers at Google presented a machine learning algorithm, AlphaGo Zero, which easily defeated some of the best human Go players in the world last year. Let's see why Go has been such a difficult game for AI and why this result is so impressive.

(a) The smaller boards in Go have 13x13 size. Calculate the number of possible moves at the very first turn i.e. Player 1's turn.

(b) After the first move, a stone is placed at any vertex. Calculate the number of possible moves at the second turn i.e. Player 2's turn.

(c) Assuming the game ends when there are no more vertices left, **write out but do not calculate** how many possible turns there are for the entire game.

(d) How big do you think the above number is? A million? A billion? Try to use approximations to get a rough order of magnitude understanding

(e) A normal size board in Go is 19x19 size. **Write out but do not calculate** the total number of possible turns now. How much bigger do you think this is than for the 13x13 size board?

(f) Try to calculate the total number of turns for a 13x13 and 19x19 board. It should not be possible on any regular calculator.

(g) Given that I have not told you the actual strategy or win-condition of this game, do you think that we have given an underestimate or overestimate of the number of moves that would **normally occur** in a game of Go?

(h) Let's compare the number of possible moves in Go to chess, a game that computers have solved decades earlier. A chess board is 8x8 and has 32 pieces on the board. During each players turn, they move the one piece to a different square. Assume each piece can move to any open square and that no pieces are ever removed. How many open squares should there be at any point in the game?

(i) Calculate how many moves we would need to make on such a board game as specified as above until there were as many possible turns as in Go for both the 13x13 and 19x19 board.

(j) Now in chess, pieces on the board are actually removed and in many cases, games end with 5 or fewer pieces. Given this new information, do you think any estimate using 32 pieces on the board that assumes pieces are never removed will be an overestimate or underestimate of the actual number of possible turns?

(k) The longest chess game was 264 moves i.e. player 1 moved 264 times and player 2 moved 264 times. It is theoretically possible to take a piece off the board after move 2 i.e. after player 1 moves once and player 2 moves once. Given this information, write out a procedure that will give the tightest or best possible estimate of the total number of possible turns in a game of chess.

(l) How does this new answer compare to the number of possible moves in Go?

4. Approximating Life

(a) Charles eats roughly 2 granola bars everyday. Each box of 12 granola bars costs \$4.50. How much does he spend on granola bars over the summer?

(b) The perimeter of the United States is 8878 miles. The length of the United States (east to west) is 2680 miles while the width/height (north to south) is 1590 miles. The cost of gasoline is \$2.9 dollars per gallon. Suppose we take a road trip in a 25 mile-per-gallon car starting from virginia to california and back (assume the path taken is unknown i.e. do not asssume straight line path).

   i. Give 3 different methods of approximating the cost of this trip. For each method, find the cost and justify the method, as well as potential drawbacks.
   ii. Which method is the most accurate answer?

3 Pure

1. Squares in Circles in Triangles
   Note: I actually haven't done these problems, so I don't know what's going to happen, but I feel like there might be some cool stuff.

   (a) Consider a circle of radius r. Circumscribe a square a1 around the circle i.e. the circle should be completely within the square. Inscribe a square a1 inside the circle i.e. the circle should be completely within the square.

      i. Find the area of a1 and a2
      ii. Find the perimeter of a1 and a2
      iii. Compare the ratio of the perimeters and area

   (b) Consider a square of side length s. Circumscribe a circle 1around the square and inscribe a circle a2 inside the square.

      i. Find the circumference of a1,a2
      ii. Find the area of a1,a2
      iii. Compare the ratio of the circumference and area

   (c) Consider a sphere of radius r. Circumscribe a cube a1 around the sphere and inscribe a cube a2 inside the sphere.

      i. Find the surface area of a1 and a2
      ii. Find the volume of a1 and a2
      iii. Compare the ratio of the perimeters and area

   (d) Consider a cube of side length s. Circumscribe a sphere a1 around the cube and inscribe a sphere a2 inside the cube.

      i. Find the surface area of a1 and a2
      ii. Find the volume of a1 and a2
      iii. Compare the ratio of the perimeters and area

   (e) Consider a circle of radius r, centered at the origin. Construct a right triangle, with a base lying on the positive x-axis with a length of r and a height of r.

   i. Find the area of the triangle
   ii. Find the area of the quarter-circle in the positive x,y axis quadrant

2. Absolutely not Proof
   Consider the following equation

$$|a + b| = |a| + |b|$$

   (a) Find a counterexample where this is not true
   (b) Prove that this statement is not true (Hint: consider cases where a,b are positive and negative
   (c) Consider the following:

$$(|a + b|)^2 = (|a| + |b|)^2$$

   Give a simple argument why this is not true (Hint: just use logic)

3. Divisors and Dividing (Challenge Problem)
   A number is divisible by 3 if and only if the sum of the digits of the number is divisible by 3. Confirm this is true for yourself. Make sure to check both cases where a number is divisble by 3 and cases where it is not. Then prove that is true.
   Hint 1: 12345 = 1 * 10000 + 2 * 1000 + 3 *100 + 4 * 10 + 5
   Hint 2: 10 = 3 * 3 + 1

4 Reading

Pure: Prime Conspiracy
Klarreich, E., Mathematicians Discover Prime Conspiracy, Quanta Magazine, 2016.

Applied: Hyperuniformity in nature
Wolchover, N., A Bird's-Eye View of Nature's Hidden Order, Quanta Magazine, 2016.

5 Alphabet Soup Fun!

Ever wondered why we always assign variables to certain letters? Scared that people might laugh at you for using the wrong letter for your variable? Well your worries are no more, now that you have our handy dandy math alphabet soup table!

Math Alphabet Soup Table

a,b,c,d	constants
e	Euler's number
f,g,h	functions
i,j	imaginary numbers
i,j,k	counters, unit vectors
l	line,length
m,n	counters
o	not used, too similar to 0
p,q	lines or points
r	radius, sometimes distance
s	speed, free parameter
t	time, free parameter
u,v,w	vectors
x,y,z	variables