* Part 1 : Pen & Paper

(1) We need to show that, $p(\rho) = \dfrac{k_{nr}\rho^{5/3}}{\sqrt{1+(\rho/\rho_0)^{2/3}}}$

Since we already know that, $p = \dfrac{p_{nr}\,p_r}{\sqrt{p_{nr}^2+p_r^2}}$ where,

$$p_{nr} = k_{nr}\rho^{5/3}, \quad p_r = k_r \rho^{4/3}$$

We are also given that, $\rho_0 = \left(\dfrac{k_r}{k_{nr}}\right)^3 = 3.789 \times 10^6\ g/cm^3$

Now simply plugging this in the equation of state.

$$p = \frac{p_{nr}\,p_r}{\sqrt{p_{nr}^2+p_r^2}}$$

$$= \frac{(k_{nr}\rho^{5/3})(k_r\rho^{4/3})}{\sqrt{(k_{nr}\rho^{5/3})^2+(k_r\rho^{4/3})^2}}$$

$$= \frac{k_{nr}k_r\rho^{9/3}}{\sqrt{k_{nr}^2\rho^{10/3}+k_r^2\rho^{8/3}}}$$

$$= \frac{k_{nr}k_r\rho^{9/3}}{\sqrt{\rho^{8/3}(k_{nr}^2\rho^{2/3}+k_r^2)}}$$

$$= \frac{k_{nr}k_r\rho^{9/3}}{\rho^{4/3}\sqrt{k_{nr}^2\rho^{2/3}+k_r^2}}$$

$$= \frac{k_{nr}k_r\rho^{9/3}\rho^{-4/3}}{k_r\sqrt{1+\left(\frac{k_{nr}}{k_r}\right)^2\rho^{2/3}}}$$

$$= \frac{K_{nr}\, \rho^{5/3}}{\sqrt{1 + \frac{1}{\rho_0^{2/3}}\rho^{2/3}}} \qquad \left\{ \because \rho_0 = \left(\frac{k_r}{k_{nr}}\right)^3 \therefore \frac{k_{nr}}{k_r} = \frac{1}{\rho_0^{1/3}} \right\}$$

$$\therefore \boxed{P(\rho) = \frac{K_{nr}\, \rho^{5/3}}{\sqrt{1+(\rho/\rho_0)^{2/3}}}}$$

(2) The Hydrostatic equilibrium : $\frac{dP}{dr} = -\frac{GM\rho}{r^2}$, Mass: $\frac{dM}{dr} = 4\pi r^2 \rho$

$$\frac{dM}{dr} = 4\pi r^2 \rho$$

$$\int dM = \int 4\pi r^2 \rho \, dr$$

$$\therefore \frac{dP}{dr} = -\frac{G M \rho}{r^2}$$

$$\frac{dP}{dr} = -\frac{G\left(\int 4\pi r^2 \rho \, dr\right)\rho}{r^2}$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -G\int 4\pi r^2 \rho \, dr$$

If we differentiate both the sides the we get,

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G 4\pi r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G 4\pi \rho \qquad \text{or}$$

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho}$$

(3) $\because \rho(r) = \rho_0 \theta_n(r)$ and $r = \lambda\eta$ $\therefore \frac{d}{dr} = \frac{1}{\lambda}\frac{d}{d\eta}$

The equation of state $P(\rho)$ in terms of $\rho(r)$ becoms,

$$P = \frac{k_{rr}\,\rho^{5/3}}{\sqrt{1+(\rho/\rho_0)^{2/3}}} = \frac{k_{rr}\,\rho_0^{5/3}\,\theta_n^{5/3}}{\sqrt{1+\theta_n^{2/3}}}$$

$$\therefore \frac{dP}{dr} = \frac{d}{dr}\,\frac{k_{nr}\,\rho^{5/3}\,\theta_n^{5/3}}{\sqrt{1+\theta_n^{2/3}}}$$

$$= \frac{1}{\lambda}\frac{d}{d\eta}\,\frac{k_{nr}\,\rho^{5/3}\,\theta_n^{5/3}}{\sqrt{1+\theta_n^{2/3}}} \quad //$$

Now from the Hydrostatic equilibrium equation, Substitute everything

$$\frac{1}{\lambda^2\eta^2}\,\frac{1}{\lambda}\frac{d}{d\eta}\left(\frac{\lambda^2\eta^2}{\rho_0\,\theta_n}\,\frac{1}{\lambda}\frac{d}{d\eta}\,\frac{k_{nr}\,\rho^{5/3}\,\theta_n^{5/3}}{\sqrt{1+\theta_n^{2/3}}}\right) = -4\pi G\rho_0\theta_n$$

This becomes,

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left[\frac{\eta^2}{\theta_n}\,\frac{d\,\theta_n^{5/3}\left(1+\theta_n^{2/3}\right)^{-1/2}}{d\eta}\right] = -\theta_n$$