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Abstract

These notes are based on Ron Donagi's lectures for the course "Complex Algebraic Geometry" at UPenn along with Daniel Huybrechts's *Complex Geometry*. Any mistake in what follows is my own.

Contents

1 A cursory overview of algebraic geometry

1.1 Lectures 1-4

These lectures consisted of informal surveys of certain fundamental concepts of algebraic geometry. They were meant as previews of various topics that we will cover rigorously.

2 Complex analysis

2.1 Lecture 5

First, let's review some basic concepts about functions of a single complex variable.

Definition 2.1.1. Let $z_0 \in \mathbb{C}$. A function $f = u + iv : U \subset \mathbb{C} \to \mathbb{C}$ is *holomorphic* or *analytic* if at least one of the following equivalent conditions holds.

- Both u and v are C^1, and f satisfies the Cauchy-Riemann equations, i.e.,

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

- $\frac{\partial f}{\partial \bar{z}} = 0$, where $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.
- The Cauchy integral formula holds, i.e.,

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{\eta - w} d\eta$$

  for any closed circular path γ centered at w in U.
- f has a power series representation on U.

Definition 2.1.2. A bijective function $f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is *biholomorphic* if it is holomorphic and its inverse is holomorphic. In this case, we say that U is *biholomorphic to* V, written as $U \approx V$.

Fact 2.1.3.

(a) (The maximum modulus principle) If $U \subset \mathbb{C}$ is a domain, $f : U \to \mathbb{C}$ is holomorphic, and $|f|$ has a local maximum, then f is constant.

(b) (Liouville's theorem) Any bounded entire function is constant.

(c) (The Riemann extension theorem) If $\epsilon > 0$ and $f : B_\epsilon(z) \setminus \{z\} \subset \mathbb{C} \to \mathbb{C}$ is bounded and holomorphic, then f can be extended to a holomorphic function on $B_\epsilon(z)$.

(d) (The Riemann mapping theorem) If $U \subsetneq \mathbb{C}$ is simply connected and open, then $U \approx B_1(0)$.

(e) (The residue theorem) If $f : B_\epsilon(0) \setminus \{0\}$ is holomorphic, then f can be expanded in a Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ such that $a_{-1} = \frac{1}{2\pi i}\oint f(z)dz$.

Next, let's look at some basic concepts about functions of several complex variables.

Definition 2.1.4. A function $f = u + iv : U \subset \mathbb{C}^n \to \mathbb{C}$ is *holomorphic* if at least one of the following equivalent conditions holds.

- f is holomorphic in each variable individually.

- Both u and v are C^1, and f satisfies the Cauchy-Riemann equations,

$$\begin{aligned} u_{x_i} &= v_{y_i} \\ u_{y_i} &= -v_{x_i} \end{aligned}$$

  for each $i = 1, \dots, n$.

- $\sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} = 0$.

- f has a power series representation on U,

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}.$$

Note 2.1.5. Statements (a), (b), and (c) from Fact 2.1.3 generalize to functions of several variables, as does the Cauchy integral formula:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\eta_i - z_i| = \epsilon_i} \frac{f(\eta_1, \dots, \eta_n)}{(\eta_1 - z_1) \cdots (\eta_n - z_n)} d\eta_1 \cdots d\eta_n$$

where $\eta_i > 0$ for each $i = 1, \dots, n$.

Theorem 2.1.6 (Hartog). *If $n > 1$, then any holomorphic function $f : B_\epsilon(0) \setminus \{0\} \subset \mathbb{C}^n \to \mathbb{C}$ extends to a holomorphic function on $B_\epsilon(0)$.*

Definition 2.1.7. Let X be a (topological) space. A sheaf F on X is a presheaf on X such that for any open $U \subset X$ and any open cover $\{U_i\}_{i \in J}$ of U, there is an exact sequence

$$0 \longrightarrow F(U) \longrightarrow F(U_i) \longrightarrow F(U_{ij})$$

where $U_{ij} \coloneqq U_i \cap U_j$.

Definition 2.1.8. A *ringed space* is a pair $(X, \mathcal{J})$ where X is a space and $\mathcal{J}$ is a sheaf of rings on X.

Remark 2.1.9. Given any standard object $(X, \mathcal{J})$, we can define a *geometric object* as a ringed space locally isomorphic to $(X, \mathcal{J})$.

Definition 2.1.10 (Vector bundle). Let X and V be complex manifolds. Let $\pi : V \to X$ be holomorphic. We say that π is a *(holomorphic) vector bundle of rank n* if for any $x \in X$, there exist an open set $U \ni x$ in X and an isomorphism $\pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}^n$ such that the *transition maps* $U_{ij} \times \mathbb{C}^n \to U_{ij} \times \mathbb{C}^n$ are holomorphic and fiber linear.

Any vector bundle $\pi : V \to X$ induces a sheaf on X given by

$$F(U) = \Gamma\left(U, \pi^{-1}(U)\right).$$

Example 2.1.11.

1. The sheaf induced by the trivial bundle $\mathbf{1} \coloneqq X \times \mathbb{C}$ is denoted by $\mathcal{O}_X$.

2. The tangent bundle TX of a smooth manifold X induces the sheaf of vector fields on X.

3. The cotangent bundle T^*X induces the sheaf $\Omega^1(X)$ of one-forms on X.

4. The alternating bundle $\bigwedge^p X$ of rank p induces the sheaf $\Omega^p(X)$ of p-forms on X.

3 Line bundles

3.1 Lecture 6

Definition 3.1.1. A *line bundle* is a vector bundle of rank 1.

Definition 3.1.2. Let X be a complex manifold. A *sheaf F of $\mathcal{O}_X$-modules* is a sheaf on X such that for any open set U in X,

- $F(U)$ is a module over $\mathcal{O}_X(U)$ and
- if $U \subset V \subset X$, then $(f \cdot a) \restriction_U = f \restriction_U \cdot a \restriction_U$.

Example 3.1.3 (Sheaf of sections). Let X be a complex manifold and J be a vector bundle over X. For any open $U \subset X$, let

$$\mathcal{L}_J(U) = \Gamma(U, L).$$

This inherits a vector space structure from the family of fibers of V. Also, any relation of the form $U_1 \subset U_2 \subset U$ induces a linear map $\Gamma(U_2, L) \to \Gamma(U_1, L)$ given by $\sigma \mapsto \sigma \restriction_{U_1}$. Thus, $\mathcal{L}_J(-)$ is a sheaf of vector spaces. Moreover, it is easily seen to be a sheaf of $\mathcal{O}_X$-modules.

Since any vector bundle is locally trivial, we see that $\mathcal{L}_J$ is *locally free*, i.e., for any $x \in X$, there exist an (open) neighborhood U of x in X and an isomorphism $\varphi : \mathcal{L}_J(U) \to \bigoplus_{i=1}^{\mathrm{rank}(J)} \mathcal{O}_X(U)$ such that for any open set $V \subset U$, the square

$$\begin{tikzcd}
\mathcal{L}_J(U) \arrow[r, "\cong"] \arrow[d] & \bigoplus_{i=1}^{\mathrm{rank}(J)} \mathcal{O}_X(U) \arrow[d] \\
\mathcal{L}_J(V) \arrow[r, "\cong"'] & \bigoplus_{i=1}^{\mathrm{rank}(J)} \mathcal{O}_X(V)
\end{tikzcd}$$

commutes. In other words, $\mathcal{L}_J$ is locally isomorphic to $(\mathcal{O}_X)^{\oplus \mathrm{rank}(J)}$.

Definition 3.1.4. A sheaf F on a complex manifold X is *invertible* if there exist an open cover $\{U_i\}$ of X and a family of holomorphic isomorphisms $\varphi_i : \mathcal{O}_{U_i} \to \mathcal{L}_J \restriction_{U_i}$.

Example 3.1.5. If J is a line bundle, then $\mathcal{L}_J$ is invertible.

Consider the composition

$$\mathcal{O}_{U_i \cap U_j} \xrightarrow{\varphi_i} \mathcal{L}_J \restriction_{U_i \cap U_j} \xrightarrow{\varphi_j^{-1}} \mathcal{O}_{U_i \cap U_j}\ , \qquad 1 \longmapsto g_{ij}.$$

From this, we can construct a line bundle L over X by defining the total space as

$$\coprod_i (U_i \times \mathbb{C}) \big/ \sim$$

where $(x, \lambda)_i \sim (y, \mu)$ if $x = y$ and $\mu = g_{ij}\lambda$.

Definition 3.1.6 (Divisor). A *divisor on a complex manifold* X is a locally finite $\mathbb{Z}$-combination of irreducible holomorphic hypersurfaces of X. Equivalently, it is a subset of X locally defined by the vanishing of a holomorphic function.

Example 3.1.7. If $X = \mathcal{A}^1$, then any divisor D on X is of the form

$$D = \sum m_i p_i, \quad p_i \in \mathcal{A}^1,\ m_i \in \mathbb{Z}.$$

Terminology. Each m_i is known as the *multiplicity of* p_i.

Any divisor D defines a line bundle $\mathcal{O}_X(D)$ on X and a holomorphic map $X \dashrightarrow \mathbb{P}(V^\vee)$ where $V \equiv \Gamma(X, \mathcal{O}_X(D))$. It is also true that any line bundle defines a divisor. It follows that

$$\text{(line bundles)} \xleftrightarrow{\sim} \text{(invertible sheaves)} \xleftrightarrow{\sim} \text{(divisors module linear equiv.)}\ . \tag{$\dagger$}$$

Consider the case where $D = \mathsf{pt}$. Let $f \in \Gamma(U, \mathcal{O}_U)$ and let $U_i = X \setminus D$, which is a tubular neighborhood of D. Note that $U_i = f^{-1}(\mathbb{C} \setminus \text{hyperplane})$. Define $\mathcal{O}_X(D)$ as the line bundle with transition functions of the form $f \restriction_{U_i \cap U_j}$.

Alternatively, let

$$(\mathcal{O}_X(D))(U) = \{g : U \to \mathbb{C} \mid g \text{ is meromorphic, } \overbrace{fg}^{\text{product}} \text{ is holomorphic}\}.$$

For example, let $X = \mathbb{P}^1$ and D be a point p. Let (x_0, x_1) denote local coordinates on X near p. Let g be meromorphic in these coordinates and let $f(x_0, x_1) = \frac{x_1}{x_0}$. Then fg is holomorphic, i.e., g has a pole of order at most one at p.

Question.

1. What is $\Gamma\left(\mathbb{P}^1, \mathcal{O}_X\right)$?
2. What is $\Gamma\left(\mathbb{P}^1, \mathcal{O}_X(D)\right)$?

In fact, it can be shown that

$$\Gamma\left(\mathbb{P}^1, \mathcal{O}_X(m, p)\right) = \begin{cases} \mathbb{C}\langle 1, x, \ldots, x^m\rangle & m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In general, D is defined locally, and thus so is $\mathcal{O}_U(D)$. Specifically, $\Gamma(U, \mathcal{O}_U(D))$ consists of all holomorphic functions $f : U \setminus \operatorname{supp}(D) \to \mathbb{C}$ such that if $D = \sum m_i Y_i$ and $Y_i \cap U = \{f_i = 0\}$, then $g \prod_i f_i^{m_i}$ is holomorphic in U.

Example 3.1.8 (Veronese embedding). Let $X = \mathbb{P}^1$ and p be as before.

1. Let $D = \mathcal{O}(2p)$. Consider the space $V \coloneqq \Gamma\left(\mathbb{P}^1, \mathcal{O}(2p)\right) = \mathbb{C}\langle 1, x, x^2\rangle$. Define the map $\varphi_{\mathcal{O}(2p)} : \mathbb{P}^1 \to \mathbb{P}^2$ by
$$\varphi_{\mathcal{O}(2p)}(x) = \underbrace{\left(1, x, x^2\right)}_{(x,y,z)}.$$
This is an embedding. Its image is precisely the smooth curve given by $y^2 = xz$.
2. Let $D = \mathcal{O}(3p)$. Then the image of the map $\varphi_{\mathcal{O}(3p)} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $x \mapsto \left(1, x, x^2, x^3\right)$ is a so-called twisted cubic.

The line bundle L on X determines the map $X \dashrightarrow \mathbb{P}\left(\Gamma(X, L)^\vee\right)$ directly, as follows.

$$x \mapsto \ker\left(\Gamma(X, L) \xrightarrow{\mathrm{eval}_x} L_p\right)$$

Definition 3.1.9. The *base locus of* L is $\mathcal{BL}(L) \equiv \{x \in X \mid s(x) = 0 \text{ for each } s \in \Gamma(X, L)\}$.

Note that we get a map $X \setminus \mathcal{BL}(L) \to \mathbb{P}\left(\Gamma(X, L)^\vee\right)$.

Now, let's consider a slight generalization of our preceding discussion. Let $V \subset \Gamma(X, L)$. This induces a map

$$\begin{tikzcd} X \arrow[r, dashed] & \mathbb{P}(V^\vee) \\ X \setminus \mathcal{BL}(V) \arrow[u, hook] \arrow[ur] & \end{tikzcd}.$$

Let $X = \mathbb{P}^1$ and $p = \{x = 0\}$. Then $V \subset \Gamma\left(\mathbb{P}^1, \mathcal{O}(2)\right) = \mathbb{C}\langle 1, x, x^2\rangle$, and

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\varphi_{\mathcal{O}(2)}"] \arrow[dr, dashed, "\varphi_V"'] & \mathbb{P}^2 \arrow[d, dashed, "\rho"] \\ & \mathbb{P}^1 \end{tikzcd}$$

commutes where ρ denotes the linear projection. Note that φ_V is a morphism so long as the center of ρ is not in the image of $\varphi_{\mathcal{O}(2)}$. In this case, we have that

$$\begin{aligned} \varphi_{\mathcal{O}(2)}(x) &= \frac{a + by + cx^2}{d + ex + fx^2} \\ \rho(x) &= \frac{a + bx}{c + dx}. \end{aligned}$$

3.2 Lecture 7

Let L_1 and L_2 be line bundles over X with transition functions $\{g_1^{kl} : U_{kl} \to \mathbb{C}^*\}$ and $\{g_2^{ij} : U_{ij} \to \mathbb{C}^*\}$, respectively. We can take a refinement $\{U_i \cap U_k\}$ where both L_1 and L_2 are trivial. Define $L^1 \otimes L^2$ as the line bundle with transition functions $\{g_1^{kl} g_2^{ij} : U_{ij} \cap U_{kl} \to \mathbb{C}^*\}$. Further, define $\left(L^1\right)^{-1}$ as the line bundle with transition functions $\{\left(g_1^{kl}\right)^{-1} : U_{kl} \to \mathbb{C}^*\}$. Note that, locally, $L^1 \otimes \left(L^1\right)^{-1} \cong \mathcal{O}_X$.

Definition 3.2.1. We say that a divisor $D = \sum_i m_i Y_i$ is effective if $m_i \geq 0$ for each i.

Let $V = \Gamma\left(X, \mathcal{O}_X(D)\right)$ and let D be effective. Note that $\mathbb{C}\langle D\rangle \subset V$. We have that $\operatorname{supp}(D) = \varphi^{-1}$ (hyperplane) where $\left(\mathbb{C}\langle 0\rangle\right)^{\perp}$ is precisely the hyperplane in $\mathbb{P}\left(V^{\vee}\right)$.

Example 3.2.2. Let $X = \mathbb{P}^1$.

1. Let $x = \frac{x_1}{x_0}$ and $D = p \coloneqq \{x = 0\}$. Then $V = \mathbb{C}\langle 1, x\rangle$, and the map $\varphi_V : \mathbb{P}^1 \to \mathbb{P}\left(V^{\vee}\right)$ is given by $c \mapsto y \coloneqq \frac{x}{1}$.
2. Let $D = m\left(\infty\right)$ with $m > 0$. Then $V = \mathbb{C}\langle x_1, \ldots, x_m^m\rangle$, and the map $\varphi_{m\infty} : \mathbb{P}^1 \to \mathbb{P}^m$ is given by
$$\begin{aligned} (x_0, x_1) &\mapsto \left(x_0^m, x_0^{m-1}x_1, \ldots, x_0 x_1^{m-1}, x_1^m\right) \\ x &\mapsto \left(1, x, \ldots, x^m\right). \end{aligned}$$
3. Let $D = p_1 + \cdots + p_m$ where $p_i = [1 : t_i]$. Let $x = \frac{x_1}{x_0}$, so that ∞ is given by $x_0 = 0$. Then $V = \mathbb{C}\langle \underbrace{1, \frac{1}{x - t_1}, \ldots, \frac{1}{x - t_m}}_{a_0, \frac{a_1}{x - t_1}, \ldots, \frac{a_m}{x - t_m}}\rangle$. This can be viewed as the space of all regular meromorphic functions on open subsets of $\mathbb{P}^1$ having poles of order at most m. The image of $\varphi : \mathbb{P}^1 \to \mathbb{P}^m$ is precisely the hyperplane $\{a_0 = 0\}$.

Example 3.2.3. Let X be an elliptic curve, i.e., a space of the form $\mathbb{C}/\Lambda$. Let p be the image of 0 and let $D = mp$.

1. Let $m = 1$. Then $V = \Gamma\left(X, \mathcal{O}_X(D)\right)$, which consists of all maps $f : X \to \mathbb{P}^1$ such that $f^{-1}\left(\infty\right) = \{0\}$. These are precisely the constant maps, so that $V \cong \mathbb{C}\langle s\rangle$ where s is a holomorphic section of $\mathcal{O}_X(D)$ vanishing at p and is meromorphic on $\mathcal{O}_X$.

$$\begin{tikzcd} X \arrow[r, dashed] & \mathbb{P}^0 \\ X \setminus p \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

It follows that $\mathcal{BL}\left(\mathcal{O}_X(D)\right) = p$.

2. Let $m = 2$. Then $V = \mathbb{C}\langle 1, p\rangle$, and $\varphi_{2p} : X \to \mathbb{P}^1$ is precisely the D-th Weierstrass function. Moreover, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(p) \longrightarrow \mathcal{O}_X(2p) \longrightarrow \mathcal{O}_p \longrightarrow \cdots .$$

3. Let $m = 3$. Then $V = \langle 1, p, p'\rangle$, and the image of $\varphi_{3p} : X \to \mathbb{P}^2$ is given by $y^2 = x^3 + ax + b$.

Example 3.2.4. Let $X = \mathbb{P}^2$. Let $D = m\underbrace{(\text{line at } \infty)}_{\{z=0\}}$.

1. Let $m = 0$. Then $V = \mathbb{C}\langle 1\rangle$, and $\mathcal{BL} = \emptyset$.
2. Let $m = 1$. Then $C = \mathbb{C}\langle \frac{x}{z}, \frac{y}{z}, \frac{z}{z}\rangle \cong \mathbb{C}\langle 1, X, Y\rangle$, and $\mathcal{BL} = \emptyset$. The map $\varphi_D : \mathbb{P}^2 \to \mathbb{P}^2$ is precisely the identity.
3. Let $m = 2$. Then $V = \langle \frac{x^2}{z^2}, \frac{x^4}{z^2}, \frac{y^2}{z^2}. \frac{x}{z}, \frac{y}{z}, \frac{z}{z}\rangle$, and the map $\varphi_D : \mathbb{P}^2 \to \mathbb{P}^5$ is an embedding given by $(x, y, z) \mapsto \langle x^2, xy, y^2, xz, yz, z^2\rangle$.

In general, if $H \subset \mathbb{P}^n$ is a hyperplane, then $\varphi_{\mathcal{O}(dH)} : \mathbb{P}^n \to \mathbb{P}^{\binom{d+n}{n}-1}$ is given by

$$(x_0, \ldots, x_n) \mapsto (d\text{-th order homogenous polynomials}),$$

known as the d-th order Veronese embedding on $\mathbb{P}^n$.

Example 3.2.5. Let $X = \mathbb{P}^2$ with coordinates (x, y, z). Let H denote the hyperplane given by $z = 0$ and let $D = 2H$. Then $V = \{s \in \Gamma(\mathcal{O}(2H)) \mid s(0, 0, 1) = 0\}$, and

$$\begin{tikzcd}
V \arrow[r, hook] & \Gamma(\mathcal{O}(2H)) \\
\mathbb{C}\langle x^2, xy, y^2, xz, yz\rangle \arrow[u, "\cong"] \arrow[r, hook] & \mathbb{C}\langle x^2, xy, y^2, xz, yz, z^2\rangle \arrow[u, "\cong"']
\end{tikzcd}$$

commutes. Further, $\mathcal{BL}(V) = \{0\} = [0, 0, 1]$, and φ_V is a map $\mathbb{P}^2 \setminus \{0\} \to \mathbb{P}^4$ but does not extend to $\mathbb{P}^2$. Indeed, we have that

$$\begin{aligned}
\lim_{\substack{(0,y,1)\\ y\to 0}} \varphi_V &= \lim_{y\to 0} \left(0, 0, y^2, 0, y\right) = (0, 0, 0, 0, 1) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \not\parallel \\
\lim_{\substack{(x,0,1)\\ x\to 0}} \varphi_V &= \lim_{x\to 0} \left(x^2, 0, 0, x, 0\right) = (0, 0, 0, 1, 0).
\end{aligned}$$

Note that for any $p \in X$, there exist $\widetilde{X}$ and $\pi : \widetilde{X} \to X$ such that π restricted to $\pi^{-1}(X \setminus p)$ is an isomorphism and $\pi^{-1}(p)$ is a divisor on $\widetilde{X}$ that is isomorphic to $\mathbb{P}^1$.

Proposition 3.2.6. *Let $Y \subset X$ be a submanifold of codimension $k \geq 2$. Let $\varphi : X \setminus Y \to Z$. Then there exist $\widetilde{X}$ and $\pi : \widetilde{X} \to X$ such that π restricted to $\pi^{-1}(X \setminus Y)$ is an isomorphism and restricted to $\underbrace{\pi^{-1}(Y)}_{\text{divisor on } X}$ is a bundle with each fiber isomorphic to $\mathbb{P}^{k-1}$.*

Notation. In this case, the space $\widetilde{X}$ is denoted by $\mathrm{Bl}_Y(X)$.

3.3 Lecture 8

Recall our correspondence (†). We can add to it the class of all maps

$$X \setminus \mathcal{BL}(L) \to \mathbb{P}\left(\Gamma(X, L)^\vee\right).$$

Let's turn now to some higher-dimensional examples.

Example 3.3.1. Let $X = \mathbb{P}^2$, $L = \mathcal{O}(2)$, and $V = \{s \in \Gamma(X, \mathcal{O}(2)) \mid \text{linearity condition}\}$. Then $\varphi_V : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. Consider any homogenous polynomial $\sum a_{ijk} x^i y^j z^k$. Then our linearity condition may take any of the following forms.

- $\sum a_{ijk} x^i y^j z^k = 0$ where a_{ijk} ranges over

$$\{a_{000}, a_{120}, a_{020}, a_{101}, a_{011}, a_{002}\}.$$

- $a_{002} = 0$
- $a_{002} + a_{001} = 0$.

In the case of either of these last two, we get a map

$$\mathbb{P}^2 \xrightarrow{\varphi_V} \mathbb{P}^5 \overset{\psi}{\dashrightarrow} \mathbb{P}^4$$

for any $p \in \mathbb{P}^5$. There are two scenarios to consider.

(a) Suppose that $p \notin \operatorname{im} \varphi_V$. Then $\psi \circ \varphi_V$ is a morphism.

(b) Suppose that $p = \varphi_V(001)$. Then ψ blows up at p. Consider the map $\varphi_V : \mathbb{P}^2 \setminus p \hookrightarrow \mathbb{P}^4$ given by $(x, y, z) \mapsto \underbrace{(x^2, xy, y^2, xz, y^2)}_{(x,y,z,u,v)}$. The image of this map is precisely $\operatorname{im} \varphi_V \coprod \underbrace{\mathbb{P}^1}_{\{x=y=z=0\}} \subset \mathbb{P}^4$.

Terminology. In this setting, $\mathbb{P}^1$ is called an *exceptional divisor.*

Note that the equations

$$\begin{aligned} xz &= y^2 \\ zu &= yv \\ xv &= yu \end{aligned}$$

together generate the relevant ideal.

Remark 3.3.2. If we took L to be $\mathcal{O}(n)$ with $n \neq 2$, then our generators would still be quadratic.

Now, fix a and b and let $x = \epsilon a$, $y = \epsilon b$, and $z = 1$ where $\epsilon \to 0$. Then

$$\begin{aligned} \varphi_V(x, y, z) &= \left(\epsilon^2 a^2, \epsilon^2 ab, \epsilon^2 b^2, \epsilon a, \epsilon b\right) \\ &\sim \left(\epsilon a^2, \epsilon ab, \epsilon b^2, a, b\right) \\ &\to (0, 0, 0, a, b). \end{aligned}$$

Question. Is $\operatorname{im} \varphi_V$ a manifold at $00010 = \varphi_V(1, b, a)$?

We have that

$$\begin{aligned} zu - yv &\to \frac{z}{u} = \frac{y}{u}\frac{v}{u} \\ xv = yu &\to \frac{x}{u}\frac{v}{u} = \frac{y}{u}. \end{aligned}$$

More generally, let X be a complex n-manifold and let $p \in X$. Then $\operatorname{Bl}_p X = (X \setminus p) \coprod \underbrace{\mathbb{P}^{n-1}}_{\mathbb{P}(T_p X)}$. There at least two ways of extending the map

$$X \setminus p \xrightarrow{\varphi_V} \mathbb{P}^n \overset{\psi}{\dashrightarrow} \mathbb{P}^{n-1}$$

so that its image is a manifold at every point.

(a) Provided that $\psi \circ \varphi_V$ is an embedding, then we can take $\mathrm{Bl}_p(X)$ to be the closure of $X \setminus p$ in $\mathbb{P}^{n-1}$.

(b) Let U is any polydisk containing the origin. We can replace $(X \setminus p) \cup U$ with $(X \setminus p) \cup \widetilde{U}$ where $\widetilde{U}$ denotes the blow-up of U at 0.

More generally still, let $Y^m \subset X^n$ be a closed submanifold. Then $\widetilde{X} \coloneqq \mathrm{Bl}_Y(X) = (X \setminus Y) \coprod \mathbb{P}\,(\underbrace{N_Y X}_{\text{normal bundle}})$.

$$\begin{tikzcd}
\mathbb{P}^{n-m-1} \arrow[r] & \mathbb{P}\,(N_Y X) \arrow[d] \\
& Y
\end{tikzcd}$$

We wish to find a line bundle L over Y and a subspace $V \subset \Gamma\,(X, L)$ such that $\mathcal{BL}_V = Y$. In this case, the closure of the image of $\varphi_V : X \setminus Y \to \mathbb{P}\,(V^\vee)$ determines $(X \setminus Y) \cup \widetilde{U}$ on U where U denotes any tubular neighborhood of Y in X.

Alternatively, if we are given an embedding

$$\begin{tikzcd}
Y \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\
\mathbb{P}^{n-c} \arrow[r, hook] & \mathbb{P}^n
\end{tikzcd}$$

where c denotes the codimension of Y, then we can take $\mathrm{Bl}_Y(X)$ to be the closure of $\mathrm{Bl}_{\mathbb{P}^{n-c}}\,(\mathbb{P}^n \cap (X \setminus Y))$.

Example 3.3.3. Consider $\mathbb{P}^3$ with coordinates (x, y, z, w). We wish to resolve the cone $\{x^2 = y^2\} \subset \mathbb{P}^3$. Let $p = \{x = z = 0\}$. We have a commutative diagram

$$\begin{tikzcd}
& Y \arrow[r, hook] \arrow[d, hook] & \text{cone} \arrow[d, hook] & \\
\{x = z = 0\} \arrow[r] & \mathbb{P}^0 \arrow[r, hook] & \mathbb{P}^3 \arrow[r, hook] & \mathbb{P}^{\binom{5}{2}-1} \arrow[d] \\
& X \arrow[ur, hook] & \mathrm{Bl}_{\mathbb{P}^0}\left(\mathbb{P}^3\right) \arrow[u] \arrow[r, hook] \arrow[d, hook] & \mathbb{P}^8 \\
& & X \setminus p &
\end{tikzcd}.$$

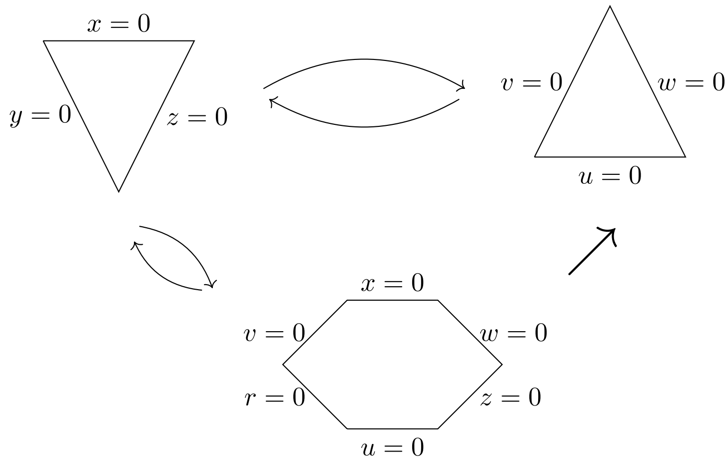
Then the exceptional divisor in $\mathrm{Bl}_p\left(\mathbb{P}^2\right)$ is isomorphic to $\mathbb{P}^2 \cong \mathbb{P}\left(T_p\mathbb{P}^2\right)$, and the exceptional divisor in $\mathrm{Bl}_p(X)$ is isomorphic to the cone.

Example 3.3.4. Consider the quadratic map $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$ given by $(x, y, z) \mapsto \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) = \underbrace{(yz, xz, xy)}_{(u,v,w)}$. Let

$$V = \{s \in \Gamma\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)\right) \mid s(001) = 0,\ s(010) = 0,\ s(100) = 0\},$$

which is isomorphic to $\Gamma\big(\underbrace{\mathcal{J}_{3\text{ points}}}_{\text{ideal sheaf on 3 points}} \otimes \mathcal{O}(2)\big)$. The fact that $\varphi^{-1} = \varphi$ yields the following properties.

- The line $z = 0$ collapses to the point $u = v = 0$.
- The line $y = 0$ collapses to the point $u = v = 0$.
- The point $y = z = 0$ blows up to the line $u = 0$.



This hexagon is called the *del Pazzo surface of degree three*, denoted by dP_3. Each of its lines is isomorphic to $\mathbb{P}^1$.

Note 3.3.5. Suppose that C is a smooth curve and that $\dim X < 2$. Then $\varphi : C \setminus \mathsf{pt} \to X$ automatically extends. But if C were singular or $\dim X \geq 2$, then this would be false.

3.4 Lecture 9

Definition 3.4.1 (Picard group). Let X be a complex manifold. The *Picard group* $\mathrm{Pic}(X)$ *of* X is the group of all isomorphism classes of line bundles over X under $\otimes$.

Let $n \in \mathbb{N}$ and consider the family of line bundles $\{\mathcal{O}(k) \mid k \in \mathbb{Z}\}$ over $\mathbb{P}^n$.

Proposition 3.4.2. $\mathrm{Pic}\left(\mathbb{P}^n\right) \cong \mathbb{Z}$ *with generator* $\mathcal{O}(1)$.

Let $\mathbb{P}^n = \mathbb{P}(V)$. We have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{\mathbb{P}^n} \otimes_{\mathbb{C}} V \longrightarrow \cdots$$

We have that

1. $\Gamma\left(\mathbb{P}^n, \mathcal{O}(1)\right) = \mathbb{C}\langle z_0, \ldots, z_n\rangle = V^\vee$,
2. $\Gamma\left(\mathbb{P}^n, \mathcal{O}(-1)\right) = 0$, and
3. $\Gamma\left(\mathbb{P}^n, \mathcal{O}(k)\right) = \begin{cases} \mathrm{Sym}^k\left(V^\vee\right) & k \geq 0 \\ 0 & k < 0 \end{cases}.$

Let $U_i = \{z \in \mathbb{P}^n \mid z_i \neq 0\}$ for each $i \in \{0, 1, \ldots, n\}$, so that $\mathbb{P}^n = \bigcup_{i=0}^n$. Let $Z_{ij} = \frac{z_j}{z_i}$, thereby endowing each U_i with local coordinates. Let s be a section of $\mathcal{O}$, so that

$$s = \left(s_i \in \Gamma\left(U_i, \mathcal{O}\right)\right)_{i=0}^n.$$

Note that Z_i defines a section on U_j with $s_j = \frac{z_i}{z_j} = Z_{ji}$ for each $j = 0, \ldots, n$.

$$\begin{tikzcd} s_j \arrow[r, equal] \arrow[d, equal] & Z_{jk} \cdot s_k \arrow[d, equal] \\ \frac{z_i}{z_j} \arrow[r, equal] & Z_{jk} \cdot \frac{z_i}{z_k} \end{tikzcd}$$

We can establish the following properties.

1. If $\mathcal{O} = \mathcal{O}(1)$, then $s_i = Z_{ij} s_j$.
2. If $\mathcal{O} = \mathcal{O}(-1)$, then $s_i = Z_{ji} s_j$.
3. If $\mathcal{O} = \mathcal{O}(k)$, then $s_i = (Z_{ij})^k \, s_j$.

In summary,

	$\mathcal{O}$ (trivial)	$\mathcal{O}(-1)$	$\mathcal{O}(1)$	$\mathcal{O}(k)$
LB	$\mathbb{P}^n \times \mathbb{C}$	tautological	dual	
Sheaf	1	Z_{ji}	Z_{ij}	$(Z_{ij})^k$
Divisor	0	$\underset{\text{h.p.}}{-H}$	$+H$	kH
Map	pt	undefined	id	$\begin{cases} \text{Veronese} & k > 0 \\ \text{undefined} & k < 0 \\ \mathsf{pt} & k = 0 \end{cases}$

Let X be a complex n-manifold. Then T_X consists of all local sections on an open set U with coordinates, say, $z_1, \ldots, z_n$. The set $\{\frac{\partial}{\partial z_i}\}$ is a basis for this, with each section of the form $\sum_{i=1}^n f_i \frac{\partial}{\partial z_i}$ where each f_i belongs to $\Gamma(U, \mathcal{O})$. For any other basis $\{\frac{\partial}{\partial w_i}\}$, we have that

$$\frac{\partial}{\partial w_i} = \sum \frac{\partial z_j}{\partial w_i} \frac{\partial}{\partial z_j}.$$

Note that $T_V \cong \mathcal{O}_V \otimes_{\mathbb{C}} V$. In general, $\Omega_V^i \cong \mathcal{O}_V \otimes \bigwedge^i V^\vee$.

Question. What is $T_{\mathbb{P}(V)}$?

Note 3.4.3 (Bundle associated to an n-manifold).

1. $T_X^\vee = \Omega \equiv \Omega^1$, whose transition functions are precisely the inverses of the transposes of those for T_X.
2. Let $\Omega^i = \bigwedge^i \Omega^1$. If $i = n$, then we call this space the *canonical sheaf* K_X or the *dualized sheaf* ω_X.
3. Recall the map $\bigwedge^i : \mathrm{GL}(n) \to \mathrm{GL}\binom{n}{i}$. If $i = n$, then this is precisely the determinant map.

Consider the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] & \mathcal{O}(1)^{\oplus n+1} \arrow[r] & T_{\mathbb{P}^n} \arrow[r] & 0 \\ & 1 \arrow[r, mapsto] & (z_i) & & \\ & & (a_i) \arrow[r, mapsto] & \sum a_i \frac{\partial}{\partial z_i} & \end{tikzcd}.$$

Terminology. The vector field given by $\sum z_i \frac{\partial}{\partial z_i}$ is known as the *Euler vector field.*

Moreover, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \underbrace{\mathcal{O}_{\mathbb{P}(V)}}_{\mathbb{C}} \arrow[r] & \underbrace{\mathcal{O}_{\mathbb{P}(V)}(1)}_{V^\vee} \otimes V \arrow[r] \arrow[d, hook] & T_{\mathbb{P}(V)} \arrow[r] \arrow[d, hook] & , \\ & & \mathcal{O}_V(1) \otimes V \arrow[r, "\cong"] & T_V \arrow[r] & 0 \end{tikzcd}.$$

Terminology. The top row of this diagram is known as the *Euler sequence*.

Therefore, the *weight* of V equals -1, whereas the weight of $V^\vee$ equals $+1$.

Informally, any holomorphic function f on V is the same as a direct sum of homogenous functions of degree k, i.e., has the form

$$\bigoplus_{k-0}^{\infty}\Gamma\left(\mathbb{P}(V),\mathcal{O}(k)\right),$$

called the *Taylor expansion of f*.

Note 3.4.4. In general, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}^{(n+1)} \longrightarrow T_{\mathbb{P}}(-1) \longrightarrow 0\ ,$$

which becomes the Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)\oplus\mathcal{O}(1) \longrightarrow T_{\mathbb{P}^1} \longrightarrow 0$$

in the case where $n=1$. It follows that

$$K_{\mathbb{P}^n}=\mathcal{O}_{\mathbb{P}^n}(-n-1)\,.$$

Lemma 3.4.5. *If $0\to A\to B\to C\to 0$ is an exact sequence of vector spaces, then*

$$\det(B)=\det(A)\otimes\det(C).$$

Corollary 3.4.6. $\mathcal{O}(2)\cong\det\left(\mathcal{O}(1)\oplus\mathcal{O}(1)\right)=\det(\mathcal{O})\otimes\det(T)=\det(T)$.

Remark 3.4.7. Similarly, we can show that $\det\left(T_{\mathbb{P}^n}\right)=\mathcal{O}_{\mathbb{P}^n}(n+1)$.

Suppose that $X\subset Y$ is a submanifold of codimension 1. Then we have a short exact sequence

$$0 \longrightarrow T_X \longrightarrow \left(T_Y\right)\big|_X \longrightarrow N_{X/Y} \longrightarrow 0\ .$$

Lemma 3.4.8. $N_{X/Y}\cong\mathcal{O}_Y\left(X\right)\big|_X$.

In other words, if $L\in\operatorname{Pic}(Y)$, $s\in\Gamma\left(Y,L\right)$, and $X=\{s=0\}$, then $N_{X/Y}\cong L\big|_X$.

Theorem 3.4.9 (Adjunction formula). $K_X\cong\left(K_Y\otimes_{\mathcal{O}_Y}\mathcal{O}_Y(X)\right)\big|_X$.

Proof. Note that $\left(K_Y^{-1}\right)\big|_X=K_X^{-1}\otimes N_{X/Y}$. Thus,

$$\begin{aligned}K_X&\cong K_Y\big|_X\otimes N_{X/Y}\\&\cong K_Y\big|_X\otimes\mathcal{O}_Y(X)\big|_X\\&\cong\left(K_Y\otimes\mathcal{O}_Y(X)\right)\big|_X.\end{aligned}$$

□

3.5 Lecture 10

Proof of Lemma 3.4.8. Let $s\in\Gamma\left(Y,L\right)$. We can write $s=fs_0$, so that $ds=s_0df+fds_0$. Consider the short exact sequence

$$0 \longrightarrow T_X \longrightarrow T_Y\big|_X \xrightarrow{ds} L \longrightarrow 0\ .$$

Thus, ds transforms just as s_0 does. □

Example 3.5.1.

1. Let $Y = \mathbb{P}^3$. Suppose that $\widetilde{X}$ is a smooth curve of degree d. Then $K_Y = \mathcal{O}(-3)$, and $K_X = \mathcal{O}(d-3)\big|_X$. Further, if g denotes the genus of a surface, then Bézout's theorem implies that

$$2g-2 = \deg(K_X) = d(d-3)$$
$$\Downarrow$$
$$g = 1 + \frac{d(d-3)}{2} = \frac{(d-1)(d-2)}{2}.$$

In particular,

d	g
1	0
2	0
3	1
4	3
5	6

.

2. Let $Y = \mathbb{P}^n$ and let $X \subset Y$ be of dimension d. Note that $K_X = \mathcal{O}_X$ precisely when $d = n+1$. In particular,

n	X
2	cubic / elliptic curve
3	quartic (*a K_3 surface*)
4	quintic

.

Let $p_1, \ldots, p_n \in \mathbb{P}^N$, let $m_1, \ldots, m_n \in \mathbb{Z}_{\geq 1}$, and let $d \in \mathbb{Z}$. We wish to describe

$$\Gamma\left(\mathcal{J}_{\Sigma_{m_i p_i}}(d)\right) := \left(\mathcal{J}_{\Sigma_{m_i p_i}} \otimes \mathcal{O}(d)\right).$$

For simplicity, let $N = 2$.

Definition 3.5.2. If $n = 1$, then *imposition* is $\mathsf{Imp}_m \equiv \operatorname{codim}(\Gamma(\mathcal{J}_{mp}(d), \Gamma(\mathcal{O}(d))))$.

Proposition 3.5.3. $\mathsf{Imp}_m = \binom{m+1}{2}$.

Definition 3.5.4. Consider the space Γ.

1. The *actual dimension of* Γ is the dimension of Γ as a vector space.
2. The *virtual dimension* $\operatorname{vd}(\Gamma)$ *of* Γ is the quantity $\binom{d+2}{2} - 1 - \sum_i \binom{m_i+1}{2}$.
3. The *expected dimension of* Γ is the quantity $\max(\operatorname{vd}(\Gamma), 0)$.

Conjecture 3.5.5. *The actual dimension always equals the expected dimension.*

Answer. This is **false**. For example, let $N = 2$, $d = 1$, $m_i = 1$, and $n = 3$. Then $\Gamma = 0$, so that $\mathbb{P}(\Gamma) = \emptyset$. Hence the expected dimension is zero, but the actual dimension is positive whenever the p_i are co-linear. $\square$

This leads us to the following modification of Conjecture 3.5.5.

Conjecture 3.5.6. *If the p_i are in general position, then the actual dimension equals the expected dimension.*

Answer. This is **false**. To see this, let $d = 2$ and $N = n = m_i = 2$. Consider a conic C through five points. Here, our conjecture holds. But if instead $N = 2$, $d = 4$, $n = 5$, and $m_i = 2$, then the virtual dimension is precisely $\binom{4+2}{2} - 5 \cdot 3 = 0$. Since the square of C exists, it follows that our conjecture fails. $\square$

We can improve Conjecture 3.5.6 as follows.

Conjecture 3.5.7. *If the actual dimension is different from the expected dimension, then* $\Gamma\left(\mathcal{J}_{\sum_{m_ip_i}}(d)\right)$ *has a base curve.*

Answer. This is **unknown**. See the article “Linear Systems of Plane Curves” by Rick Miranda. ☐

Consider the map $|\mathcal{O}(d)| : \mathbb{P}^2 \hookrightarrow \mathbb{P}^{\binom{d+2}{2}-1}$. We also have a map

$$\begin{tikzcd} & \mathbb{P}^2 \arrow[r, dashed, "|\mathcal{J}_{\sum_{p_i}(d)}|"] & \mathbb{P}^{\dim -1} \\ \mathrm{Bl}_{p_1,\dots,p_n}\left(\mathbb{P}^2\right) \arrow[r, equal] & \widetilde{\mathbb{P}^2} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Proposition 3.5.8. *Consider the blow-up* $\pi : \underbrace{\widetilde{\mathbb{P}^2}}_{X} \to \mathbb{P}^2$. *We have that*

$$\mathrm{Pic}(X) \cong \mathbb{Z}\langle \pi^*\left(\mathcal{O}(1)\right), E_1, \dots, E_n\rangle$$

where E_i *denotes the divisor collapsing to* p_i.

Remark 3.5.9.

Good: $\pi^*\mathcal{O}(d) - \sum m_iE_i \longleftrightarrow \mathcal{J}_{\sum m_ip_i}(d)$.

Better: $\Gamma\left(X, ''\right) = \Gamma\left(\mathbb{P}^2, ''\right)$.

Best: $\pi_*\left(''\right) = ''$.

Conjecture 3.5.10. *Any line bundle* $L := \left(\pi^*\mathcal{O}(d) - \sum m_iE_i\right)$ *has the expected dimension of the space of sections unless* $\mathcal{BL}\left(L\right)$ *contains a* (-1)*-curve, i.e., a smooth curve* C *of genus zero such that* $C^2 = -1$.

Example 3.5.11 ((−1)-curve). Let $d = 1$, $n = 2$, and $m_1 = m_2 = 1$. If $C \in \mathcal{O}(1)(-p-q)$, then $C^2 = 1^2 - 1 - 1 = -1$. In general,

$$\mathcal{O}(d)\left(\left(-\sum m_IE_i\right)\left(\mathcal{O}(d) - \sum m_ip_i\right)\right) = dd' - \sum m_im_i'.$$

In $\mathbb{P}^2$, this means the number of intersections other than the p_i.

Space	C^2
$\mathcal{O}(1)$	1
$\mathcal{O}(1)(-p)$	0
$\mathcal{O}(1)(-p-q)$	−1
$\vdots$	
$\mathcal{O}(2)$	4
$\mathcal{O}(2)\left(-p_1\right)$	3
$\mathcal{O}(2)\left(-p_1-p_2\right)$	2
$\vdots$	
$\mathcal{O}(2)\left(-p_1-\cdots-p_4\right)$	0
$\mathcal{O}(2)\left(-\sum_{i=1}^5 p_i\right)$	−1

4 Kähler manifolds

4.1 Lecture 11

Consider the following Hasse diagram of subgroups:

$$\begin{tikzcd}[arrows=dash]
& \mathrm{GL}(2n,\mathbb{R}) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{GL}(n,\mathbb{C}) \arrow[dr] & \mathrm{SO}(2n,\mathbb{R}) \arrow[d] & \mathrm{Sp}(2n,\mathbb{R}) \arrow[dl] \\
& \mathrm{U}(n) &
\end{tikzcd}$$

where $\mathrm{Sp}(2n,\mathbb{R})$ denotes the group of real $2n \times 2n$ *symplectic matrices*, i.e., matrices M satisfying

$$M^t \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} M = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Similarly, we can view various areas of geometry as refinements of certain others. Specifically,

$$\begin{tikzcd}[arrows=dash]
& \text{Diff. geom.} \arrow[dl] \arrow[d] \arrow[dr] & \\
\text{Complex geom.} \arrow[dr] & \text{Riemannian geom.} \arrow[d] & \text{Symplectic geom.} \arrow[dl] \\
& \textbf{Kähler geom.} &
\end{tikzcd}$$

Before investigating Kähler geometry, we establish some basic geometric concepts.

Definition 4.1.1. Let X be a real manifold. An *almost complex structure on* X is a bundle map $I : TX \to TX$ such that $I^2 = -1$.

Note that the eigenvalues of I are precisely i and $-i$.

Notation.

1. Let $T^{1,0}$ denote the eigenspace of i.
2. Let $T^{0,1}$ denote the eigenspace of $-i$.

Any complex manifold X has a natural almost complex structure. Indeed, given local coordinates x_i, y_i on X, define I by $\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}$. It follows that any manifold with an almost complex structure has even dimension.

Now, consider the complexification of our tangent bundle, $T^{\mathbb{C}}X \equiv TX \otimes_{\mathbb{R}} \mathbb{C}$.

Proposition 4.1.2.

1. *$T^{\mathbb{C}}X \cong T^{1,0} \oplus T^{0,1}$.*
2. *$T^{*\mathbb{C}}X \cong T^{*1,0} \oplus T^{*0,1}$*

Define, formally, the complex coordinates $z_j = x_j + iy_j$. Note that $T^{\mathbb{C}}X$ has as basis $\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\}$ and that $T^{*\mathbb{C}}X$ has as basis $\{dz_j, d\bar{z}_j\}$ where $dz_j \equiv dx_j + idy_j$.

Notation.

1. $\bigwedge^k X \coloneqq \bigwedge^k T^*X$.
2. $\bigwedge^{p,q} X \coloneqq \bigwedge^p T^{*1,0}X \otimes_{\mathbb{C}} \bigwedge^q T^{*0,1}X$.

Note 4.1.3. Let X be an n-dimensional complex manifold.

1. $\left(\bigwedge^k T^*X\right) \otimes \mathbb{C} = \bigwedge^k_{\mathbb{C}} \left(T^*X \otimes \mathbb{C}\right)$.

2. $\left(\bigwedge^k X\right) \otimes \mathbb{C} = \bigoplus_{p+q=k} \bigwedge^{p,q} X$.

Therefore, $\left(\bigwedge^k X\right) \otimes \mathbb{C}$ can be decomposed according to the counting equation $\binom{2n}{k} = \sum \binom{n}{p}\binom{n}{q}$.

Let U and V be open in $\mathbb{C}^n$. Let $f : U \to V$ be holomorphic. Then the map $df : TU \to TV$ extends to a map $df^{\mathbb{C}} : \mathcal{T}^{\mathbb{C}}U \to T^{\mathbb{C}}V$ that preserves both $T^{1,0}$ and $T^{0,1}$.

Let $\mathcal{A}^{p,q} = \Gamma\left(\bigwedge^{p,q}\right)$, i.e., $\mathcal{A}^{p,q}(U) = \Gamma\left(U, \bigwedge^{p,q}\right)$. Consider the exterior derivative $d : \mathcal{A}^k \to \mathcal{A}^{k+1}$. With π denoting the projection map, define the operators

$$\partial = \pi^{p+1,q} \circ d$$
$$\bar{\partial} = \pi^{p,q+1} \circ d$$

on $A^{p,q}$. Locally, we have that

$$df = \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i = \frac{\partial f}{\partial z_i} dz_i + \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

for any $f \in \mathcal{A}^{0,0}$. By the Cauchy-Riemann equations, it follows that f is holomorphic if and only if $\bar{\partial} f = 0$.

Remark 4.1.4. Any (p,q)-form locally looks like $f_{IJ} dz_I \wedge \bar{z}_J$.

Proposition 4.1.5.

1. $d = \partial + \bar{\partial}$.

2. $\partial^2 = 0 = \bar{\partial}^2$.

3. $\partial\bar{\partial} = -\bar{\partial}\partial$.

4. $\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \partial\beta$ *for any* $\alpha \in \mathcal{A}^{p,q}$ *and* $\beta \in \mathcal{A}^{r,s}$.

Lemma 4.1.6 (Single-variable Poincaré). *Consider the disk* $B_\epsilon \subset \overline{B_\epsilon} \subset U \subset \mathbb{C}$ *where* U *is open. Let* $\alpha = f d\bar{z} \in \mathcal{A}^{0,1}(U)$ *and*

$$g(z) = \frac{1}{2\pi i} \int_{\overline{B_\epsilon}} \frac{f(w)}{w - z} dw \wedge d\bar{w}.$$

Then $\bar{\partial} g = \alpha$.

Lemma 4.1.7 (Multi-variable Poincaré). *Consider the polydisk* $B_\epsilon \subset \overline{B_\epsilon} \subset U \subset \mathbb{C}^n$ *where* U *is open. Let* $\alpha \in \mathcal{A}^{p,q}$ *with* $q > 0$ *and* $\bar{\partial}\alpha = 0$. *Then there is some* $\beta \in \mathcal{A}^{p,q-1}(B_\epsilon)$ *such that* $\bar{\partial}\beta = \alpha$.

Remark 4.1.8. If U is contractible, then any differential form on U is closed if and only if it is exact.

Let $U \subset \mathbb{C}^n$ be open and let I denote the natural almost complex structure on U. Let g be a Riemannian metric on U.

Definition 4.1.9 (Hermitian metric).

1. We say that g is *compatible with* I or *(almost) Hermitian* if $g(u,v) = g(Iu, Iv)$.

2. If g is Hermitian, then the real $(1,1)$-form $\omega \in \mathcal{A}^{1,1}(U) \cap \mathcal{A}^2(U)$ defined by

$$\omega(u,v) = g(Iu, v)$$

   is called the *fundamental form of* g.

Notation. $h \coloneqq g - i\omega$.

Definition 4.1.10. A Hermitian matrix M is *positive-definite* if $z^*Mz > 0$ for every nonzero complex column vector z.

Note that h is a positive-definite form in the sense that, locally, its component functions define a positive-definite matrix at any given point.

Example 4.1.11. Let $g = \underbrace{dx^2}_{dx\otimes dx} + dy^2 = \sum_{i=1}^n dx_i^2 + dy_i^2 \in T^* \otimes T^* \subset (T^* \otimes T^*) \otimes_{\mathbb{R}} \mathbb{C}$. Since

$$dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy,$$

it follows that

$$\omega = dx \otimes dy - dy \otimes dx = \frac{i}{2} dz \wedge d\bar{z}.$$

Moreover, we see that

$$\begin{aligned} h &= z - i\omega \\ &= dx^2 - idxdy + idydx + dy^2 \\ &= dx\,(dx - idy) + idy\,(d_x + -idy) \\ &= (dx + idy)\,(dx - idy) \\ &= dz \otimes d\bar{z}. \end{aligned}$$

For each $z \in \mathbb{C}^n$, define the matrix $(h_{ij})\,(z)$ by

$$h_{ij}\,(z_1, \ldots, z_n) = h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Proposition 4.1.12. *Let I be an almost complex structure on $U \subset X$ and let g be compatible with I. Then $d\omega = 0$ if and only if for each $x \in X$, there exist a neighborhood U' of x and a holomorphic map $f : U' \to U$ such that f^*g oscialates the standard metric to the second order, i.e., $(h_{ij}) = \mathrm{id} + O\left(|z|^2\right)$.*

Notation. In this case, we write $h \approx \mathrm{id}$.

Definition 4.1.13 (Kähler manifold). Consider the four-tuple (X, I, g, ω). We say that X is a *Kähler manifold* if $d\omega = 0$. In this case, we call g a *Kähler metric on* X and ω a *Kähler form.*

Definition 4.1.14. Let (X, I, g, ω) be a Kähler structure with $\dim X = n$.

1. The *Lefschetz operator* $L : \bigwedge^k X \to \bigwedge^{k+2} X$ is defined by $\alpha \mapsto \alpha \wedge \omega$.
2. The *Hodge $*$-operator* $* : \bigwedge^k X \to \bigwedge^{2n-k} X$ is defined by the property

   $$\alpha \wedge *\beta = \hat{g}\,(\alpha, \beta)\,\omega^n$$

   where $\hat{g}$ is induced by g and ω^n denotes the (positively oriented) volume form on X.
3. The *dual Lefschetz operator* $\Lambda : \bigwedge^k X \to \bigwedge^{k-2} X$ is defined as the composite $*^{-1} \circ L \circ *$.

Note 4.1.15.

1. In coordinates in which $h \approx \mathrm{id}$, we have that $*dx^I = dx^\partial$ where $\partial := I^{\mathbb{C}}$??.
2. Λ is $\mathcal{O}$-linear.

4.2 Lecture 12

Proposition 4.2.1. *Let X be a complex manifold. Let ω be a closed real positive-definite form of type $(1,1)$, i.e., locally, $\omega = \frac{i}{2}\sum h_{ij} d_{z_i} \wedge d\bar{z}_j$ such that the matrix $(h_{ij}(p))$ is positive-definite for each p. Then there exists a Kähler metric g on X such that ω equals the fundamental form of g.*

Since every Kähler form is positive-definite, it follows that the set $\mathbb{K}_X$ of all Kähler forms on X is precisely the set of all closed real positive-definite forms of type $(1,1)$.

Definition 4.2.2. Let V be a vector space over $\mathbb{R}$. A subset $C \subset V$ is a *convex cone* if $av_1 + bv_2 \in C$ for any $v_1, v_2 \in C$ and any $a, b \in \mathbb{R}_{>0}$.

Corollary 4.2.3. *Suppose that X is compact. Then $\mathbb{K}_X$ is an open convex cone in the infinite-dimensional real vector space $S := \{\omega \in \mathcal{A}^{(1,1)}(X) \cap \mathcal{A}^2(X) \mid d\omega = 0\}$.*

Idea. The fact that $\mathbb{K}_X$ is a convex cone follows from the fact that the set of all positive-definite matrices is a convex cone. It remains to show that $\mathbb{K}_X$ is open. Since X is compact, it has a finite open cover $\{U_i\}$. The set $P_{U_i} \subset S$ of all forms that are positive-definite on U_i is open. Thus, $\bigcap_i P_{U_i} = \mathbb{K}_X$ is also open. □

Remark 4.2.4. It turns out that $S \cong H^2(X, \mathbb{R})$.

Example 4.2.5.

1. The form $\omega \equiv \frac{i}{2} dz \wedge d\bar{z}$ is Kähler on $\mathbb{C}$ and is exact.
2. The same form descends to a Kähler form on the torus $\mathbb{C}/\Lambda$, which is not exact.
3. Consider the inclusion $i : X \to Y$ of a closed submanifold. If ω is Kähler on Y, then $i^*\omega$ is Kähler on X.

Note 4.2.6. Let $f : X \to Y$ be holomorphic and let ω be a Kähler form on Y. It is *not* necessarily true that $f^*\omega$ is Kähler on X. For example, if $f(x) = \mathsf{pt}$ for all $x \in X$, then $f^*\omega$ is the zero form and thus not positive. In general, f must be injective. For example, if $f : C \to \mathbb{C}$ is a double cover where C is a Riemann surface, then C inherits a Kähler form only outside the *ramification of* f, i.e., the set

$$\{c \in C \mid \text{there is no neighborhood } U \text{ of } c \text{ such that } f\restriction_U \text{ is injective}\}.$$

This is precisely the set of points at which df is nonzero.

Example 4.2.7.

1. Consider the open cover $\{U_i\}_{1 \le i \le n}$ of $\mathbb{P}^n$ where $U_i \equiv \{z \in \mathbb{P}^n \mid z_i \neq 0\}$. Define $\varphi_i : U_i \xrightarrow{\cong} \mathbb{C}^n$ by

$$(z_0, \ldots, z_n) \mapsto \underbrace{\left(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_n}{z_i}\right)}_{(w_1, \ldots, w_n)}.$$

Then $\{(U_i, \varphi_i)\}$ is a holomorphic atlas on $\mathbb{P}^n$. For each i, let

$$\omega_i = \frac{i}{2\pi}\partial\bar{\partial}\log\left(\sum_{l=0}^{n} |\frac{z_l}{z_i}|^2\right).$$

By way of φ_i, this becomes

$$\frac{i}{2\pi}\partial\bar{\partial}\log\left(1 + \sum_{k=1}^{n} |w_k|^2\right).$$

Exercise 4.2.8. *Show that $\omega_i\restriction_{U_i \cap U_j} = \omega_j\restriction_{U_i \cap U_j}$.*

Therefore, the ω_i patch together to form a metric ω on $\mathbb{P}^n$, known as the *Fubini-Study metric*.

Exercise 4.2.9. *Show that ω is closed, real, positive, and of type $(1,1)$.*

It follows that ω is a Kähler metric.

2. Any branched cover of $\mathbb{P}^n$ admits a Kähler metric (which must be different from the pullback of a Kähler metric on $\mathbb{P}^n$). For example, consider an elliptic curve $E \to \mathbb{P}^1$, which fits into a commutative sqaure

$$\begin{tikzcd} E \arrow[r] \arrow[d, equal] & \mathbb{P}^1 \\ E \arrow[r, hook] & \mathbb{P}^2 \arrow[u, dashed] \end{tikzcd}.$$

Definition 4.2.10. A complex manifold is *projective* if it is isomorphic to a closed submanifold of projective space.

Proposition 4.2.11. *Any projective complex manifold is Kähler.*

Proof. This follows from Example 4.2.5(3) together with Example 4.2.7(1). $\square$

Definition 4.2.12. Let X be a complex manifold. Let D be a first-order operator on $\mathcal{A}^*(X)$.

1. The *adjoint of D* is
$$D^* \equiv -* \circ D \circ *$$
2. The *Laplacian associated to D* is
$$\Delta_D \equiv DD^* + D^*D.$$

Definition 4.2.13. The *Laplace operator* is $\Delta \equiv dd^* + d^*d$.

Example 4.2.14.

1. Let $D = \partial$. Then $\partial^*\left(f_{IJ}dz^I \wedge dz^J\right) = \sum_{i\in I} f_{IJ}dz^{I-i} \wedge d\bar{z}^J$.
2. Let $D = d$. Let $(x_1, \ldots, x_n)$ be local coordinates on X. Then
$$d\left(f dx^I\right) = \sum_{i\notin I} \frac{\partial f}{\partial x_i} dx^n \wedge dx^I$$
$$d^*\left(f dx^I\right) = \sum_{i\in I} \frac{\partial f}{\partial x_i} dx^{I-i}.$$

Therefore,
$$\begin{aligned} d \circ d^*\left(f dx^I\right) &= \frac{\partial^2}{\partial x_i \partial x_j} dx^{I-i\cup j} \\ &= \sum_{\substack{i\in I \\ j\notin I}} \cdots + \sum_{i=j\in I} \cdots . \\ d^* \circ d\left(f dx^I\right) &= 0 + \sum_{i=j\notin I} \cdots , \end{aligned}$$

so that $\Delta_D = \sum \frac{\partial^2 f}{\partial x_i^2}$.

Theorem 4.2.15 (Kähler identities). *Let (X, I, g, ω) be a Kähler manifold.*

1. $[\bar{\partial}, L] = 0 = [\partial, L]$.
2. $[\partial^*, \Lambda] = 0 = [\bar{\partial}^*, \Lambda]$.
3. $[\bar{\partial}^*, L] = i\partial$ *and* $[\partial^*, L] = -i\bar{\partial}$.
4. $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$, *and Δ commutes with $*$, ∂, $\bar{\partial}$, ∂^*, $\bar{\partial}^*$, L, and Λ.*

5 Lie algebras

Let G be any Lie group. For any $g \in G$, $\ell_g : G \to G$ is an isomorphism of $\mathbb{C}$-manifolds. Thus, if V is a vector field on G, then so is $(\ell_g)_* V$.

Definition 5.0.1. We say that V is *left-invariant* if $(\ell_g)_* V = V$ for every $g \in G$.

Definition 5.0.2 (Lie algebra). The *Lie algebra* $\mathfrak{G}_G$ *of* G is the space of left-invariant vector fields on G under the Lie bracket.

Consider the commutative diagram

$$\begin{tikzcd} \mathfrak{G}_G \arrow[r, hook] \arrow[dr, "\alpha"'] & (\mathscr{X}(G), [-,-]) \arrow[d, "\mathrm{eval}_1"] \\ & T_1(G) \end{tikzcd} .$$

Proposition 5.0.3. *α is an isomorphism of vector spaces.*

Example 5.0.4. Let $G = \mathrm{GL}(n, \mathbb{C})$, which is a complex Lie group. We have that $\mathrm{GL}(n, \mathbb{C})$ is an open submanifold of the vector space $M_n(\mathbb{C})$. Hence $\mathfrak{G}_G$ is isomorphic to $M_n(\mathbb{C})$ under the *commutator bracket*, which is given by $[A, B] = AB - BA$.

Definition 5.0.5 (Matrix exponential). Define the map $e^{(\cdot)} : M_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

This is well-defined. Indeed, letting $\|\cdot\|$ denote the operator norm, we see that $\frac{\|A^n\|}{n!} \leq \frac{\|A\|^n}{n!}$ on any bounded subset $S \subset \mathbb{C}^n$. But $\sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} = e^{\|A\|}$ on S, and thus e^A converges uniformly on S. Moreover, one can show that its limit must be invertible.

Exercise 5.0.6. *Let $G = \mathrm{SL}_2(\mathbb{C})$, which is complex Lie group. Show that*

$$\mathfrak{G}_G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}) \mid a + d = 0 \right\}.$$

Proof. Any element X of $\mathfrak{G}_G$ generates a local flow $\theta : D \subset \mathbb{R} \times G \to G$. Since X is left-invariant, it is complete. In particular, the maximal integral curve θ^1 is defined on $\mathbb{R}$. Left-invariance also implies that for any $s \in \mathbb{R}$, $L_{\theta^1(s)} \circ \theta^1$ is an integral curve starting at $\theta^1(s)$. But the curve given by $t \mapsto \theta^1(s+t)$ is also an integral curve starting at $\theta^1(s)$. Hence $\theta^1(s+t) = \theta^1(s)\theta^1(s)$. By the uniqueness of maximal integral curves, this proves that $\theta^1(s)$ is a smooth group homomorphism $\mathbb{R} \to G$, known as a *one-parameter subgroup of G*. Moreover, any one-parameter subgroup γ of G has the form $\gamma(t) = e^{tA}$ where $A = \gamma'(0) \in T_1(G) \subset T_1(\mathrm{GL}_2(\mathbb{C})) \cong M_2(\mathbb{C})$. It follows that

$$\begin{aligned} X \in T_1(G) &\iff \forall t \in \mathbb{R},\ e^{tX} \in G \\ &\iff \forall t \in \mathbb{R},\ \det\left(e^{tX}\right) = 1 \\ &\iff \forall t \in \mathbb{R},\ e^{t\,\mathrm{tr}(X)} = 1 \\ &\iff \forall t \in \mathbb{R},\ t\,\mathrm{tr}(X) = 0 \\ &\iff \mathrm{tr}(X) = 0. \end{aligned}$$

□

Intuitively, Theorem 4.2.15 means that the space $\mathcal{A}^{p,q}(X)$ has a symmetry encoded in the $\mathrm{SL}_2(\mathbb{C})$-action.

5.1 Lecture 13

Definition 5.1.1. Let V be a vector space endowed with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. The *orthogonal group* $\mathrm{O}\,(V, \langle \cdot, \cdot \rangle)$ is the group of all linear maps $f : V \to V$ such that $\langle fx, fy \rangle = \langle x, y \rangle$ for any $x, y \in V$.

Example 5.1.2. Consider the Lie group $G \coloneqq \mathrm{O}(\mathbb{R}^n)$. Define the smooth map $\varphi : \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ by $A \mapsto AA^t$, which as constant rank. Then $G = \varphi^{-1}(I_n)$, so that $T_{I_n} G = \ker d\varphi_{I_n}$. Since $d\varphi_{I_n}(A) = A^t + A$ for any $A \in M_n(\mathbb{R})$, it follows that $\mathfrak{G}_G$ consists of all $n \times n$ skew-symmetric matrices.