Perry Hart
HoTT reading group
UMN
April 8, 2022

Abstract

This is a brief introduction to elementary toposes. These play a central role in categorical semantics of dependent type theory (along with other areas of categorical logic). We assume knowledge of basic category theory.

Let $\mathscr{C}$ be a category with finite limits. For any object $A \in \operatorname{ob} \mathscr{C}$, a *power object* of A is an object $\mathcal{P}(A)$ of $\mathscr{C}$ together with a monomorphism $\in_A \rightarrowtail A \times \mathcal{P}(A)$ such that for every monomorphism $f : C \to A \times D$ in $\mathscr{C}$, there is a unique pullback square of the form

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \in_A \arrow[d] \\ A \times D \arrow[r, "\mathrm{id}_A \times \chi_f"'] & A \times \mathcal{P}(A) \end{tikzcd} .$$

We call χ_f the *classifying map* of f. If $A = 1$, then a power object of A is called a *subobject classifier*.

A category $\mathscr{E}$ is an *elementary topos* if it

- has finite limits,
- is Cartesian closed, and
- has a subobject classifier $\texttt{true} : 1 \to \Omega$.

In this case, any global element $1 \to \Omega$ is called a *truth value*.

Proposition 0.1. *A category $\mathscr{C}$ with finite limits is a topos if and only if every object of $\mathscr{C}$ has a power object.*

In particular, for any topos $\mathscr{E}$ and $A \in \operatorname{ob} \mathscr{E}$, the exponential object Ω^A is a power object of A. In this case, the power object functor $\Omega^{(-)} : \mathscr{E}^{\mathrm{op}} \to \mathscr{E}$ sends a map $X \xrightarrow{f} Y$ in $\mathscr{E}$ to the transpose of the composite

$$\Omega^Y \times X \xrightarrow{\mathrm{id}_{\Omega^B} \times f} \Omega^Y \times Y \xrightarrow{\mathrm{ev}_{Y,\Omega}} \Omega$$

under the adjunction $- \times X \vdash -^X$. We have a chain of natural isomorphisms

$$\mathscr{E}(X, \Omega^Y) \cong \mathscr{E}(X \times Y, \Omega) \cong \mathscr{E}(Y \times X, \Omega) \cong \mathscr{E}(Y, \Omega^X) \cong \mathscr{E}^{\mathrm{op}}(\Omega^X, Y),$$

which gives us an adjunction $\left(\Omega^{(-)}\right)^{\mathrm{op}} \vdash \Omega^{(-)}$. By an argument due to Paré, this adjunction is *monadic* in the sense that $\Omega^{(-)}$ reflects isomorphisms and preserves reflexive coequalizers, which implies that $\Omega^{(-)}$ creates limits. Since $\mathscr{E}$ has finite limits as a topos, it follows that $\mathscr{E}^{\mathrm{op}}$ has finite limits, i.e., $\mathscr{E}$ has finite *colimits.*

Example 0.2.

1. The category **Set** is a *Boolean* topos, i.e., $\Omega \cong 1 \coprod 1$.

2. For any small category $\mathscr{C}$, the presheaf category $\widehat{\mathscr{C}} := [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ is a topos where the functor Ω sends $U \in \mathrm{ob}\,\mathscr{C}$ to the set $\mathtt{sieves}(U)$ of *sieves on* U, i.e., sets σ of morphisms over U such that for any morphisms $f : X \to Y$ and $g : Y \to U$ in $\mathscr{C}$,

$$Y \xrightarrow{g} U \;\in\; \sigma \quad\Longrightarrow\quad X \xrightarrow{f} Y \xrightarrow{g} U \;\in\; \sigma.$$

   The action of Ω on morphisms in $\mathscr{C}$ is defined by

$$V \xrightarrow{h} U \quad\mapsto\quad \sigma \mapsto \{f : X \to V \mid h \circ f \in \sigma,\ X \in \mathrm{ob}\,\mathscr{C}\}\,.$$

   The sieve on U generated by id_U is the top element $\mathtt{sieve}_{\mathtt{top}}(U)$ of $\mathtt{sieves}(U)$. We define $\mathtt{true} : 1 \to \Omega$ as the natural transformation with components

$$\begin{aligned} \mathtt{true}(U) : \{*\} &\to \mathtt{sieves}(U) \\ * &\mapsto \mathtt{sieve}_{\mathtt{top}}(U). \end{aligned}$$

   For any monomorphism $\varphi : F \hookrightarrow G$ in $\widehat{\mathscr{C}}$, the classifying map of φ has components

$$\begin{aligned} &\chi_\varphi(U) : G(U) \to \Omega(U) \\ &x \mapsto \{f : X \to U \mid G(f)(x) \in F(X),\ X \in \mathrm{ob}\,\mathscr{C}\}\,. \end{aligned}$$

Note 0.3. Let $\mathscr{C}$ be a small category.

1. The subobject $\Omega_{\mathtt{dec}} \hookrightarrow \Omega$ of decidable sieves classifies all monomorphisms $F \xrightarrow{\psi} G$ in $\widehat{\mathscr{C}}$ such that $\psi_A : F(A) \to G(A)$ has decidable image for every $A \in \mathrm{ob}\,\mathscr{C}$. Here, for any set T, a subset $S \subset T$ is decidable if and only if for any $x \in T$, the disjunction $x \in S \vee x \notin S$ is provable. If our metatheory includes LEM, then $\Omega_{\mathtt{dec}} = \Omega$.

2. Let $\mathcal{Y} : \mathscr{C} \to \widehat{\mathscr{C}}$ denote the Yoneda embedding. Let $U \in \mathrm{ob}\,\mathscr{C}$. For any sieve σ, define the subfunctor $F_\sigma \hookrightarrow \mathcal{Y}_U$ by

$$A \mapsto \mathcal{Y}_U(A) \cap \sigma$$

   for all $A \in \mathrm{ob}\,\mathscr{C}$. Conversely, for every subfunctor F of $\mathcal{Y}_U$, define the sieve

$$\sigma_F \equiv \coprod_{X \in \mathrm{ob}\,\mathscr{C}} F(X)$$

   on U. Then $F_- : \mathtt{sieves}(U) \to \mathtt{Sub}(\mathcal{Y}_U)$ is a bijection with inverse σ_-.

Definition 0.4 (Heyting algebra). Let L be a bounded lattice. We say that L is a *Heyting algebra* if it has a binary operation $\Rightarrow : L \times L \to L$, called *implication*, such that

$$\begin{aligned} p \Rightarrow p &= 1 \\ p \wedge (p \Rightarrow q) &= p \wedge q \\ q \wedge (p \Rightarrow q) &= q \\ p \Rightarrow (q \wedge r) &= (p \Rightarrow q) \wedge (p \Rightarrow r)\,. \end{aligned}$$

For any topos $\mathscr{E}$ and $A \in \operatorname{ob}\mathscr{E}$, the poset $\mathtt{Sub}(A)$ is a Heyting algebra. As a result, $\mathtt{Sub}(A)$ is a model of intuitionistic propositional calculus. For example, the meet $\cap$ and join $\cup$ operation for $\mathtt{Sub}(A)$ are precisely the binary product and binary coproduct in $\mathtt{Sub}(A)$, respectively.

Proposition 0.5. *Let U_1 and U_2 be subobjects of A.*

1. *We have a pullback square*

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U_2 \arrow[d] \\ U_1 \arrow[r] & A \end{tikzcd}$$

   in $\mathscr{E}$ consisting of monomorphisms.

2. *We have a pushout square*

$$\begin{tikzcd} U_1 \cap U_2 \arrow[r] \arrow[d] & U_2 \arrow[d] \arrow[ddr, bend left] & \\ U_1 \arrow[r] \arrow[drr, bend right] & U_1 \cup U_2 \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\alpha"] & \\ & & A \end{tikzcd}$$

   in $\mathscr{E}$ where α is a monomorphism.

Remark 0.6. A *Boolean algebra* is a Heyting algebra L where every $x \in L$ has a complement, i.e., an element $c_x \in L$ such that $x \vee c_x = 1$ and $x \wedge c_x = 0$. A topos $\mathscr{E}$ is Boolean if and only if $\mathtt{Sub}(A)$ is a Boolean algebra for all $A \in \operatorname{ob}\mathscr{E}$. In this case, $\mathtt{Sub}(A)$ satisfies LEM.

Let $\mathscr{E}$ be a topos and consider a map $\mathtt{El} : \widehat{U} \to U$ in $\mathscr{E}$. We say that a map $f : X \to Y$ in $\mathscr{E}$ is *U-small* if there exists a pullback square (not necessarily unique) of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \widehat{U} \arrow[d, "\mathtt{El}"] \\ Y \arrow[r] & U \end{tikzcd}.$$

Note that the class of U-small maps is closed under pullbacks.

We say that $\mathtt{El}$ is a *universe in $\mathscr{E}$* if the class of U-small maps

(a) is closed under

- products,
- dependent sums,
- dependent products, and
- pullbacks of $1 \xrightarrow{\mathtt{true}} \Omega$ and

(b) contains the unique map $\Omega \to 1$.

Condition (b) expresses that U is *impredicative*. The subobject classifier is a *predicative* universe as long as $\Omega \neq 1$, and the Ω-small maps are precisely the monomorphisms.

Remark 0.7. Closure under dependent sums is sometimes used as an alternative definition of *impredicative*, in which case Ω is impredicative. Unfortunately, both definitions appear in the type theory literature.