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Abstract

These notes are based on Ron Donagi's lectures for the course "Complex Algebraic Geometry" given at UPenn along with Daniel Huybrechts's *Complex Geometry*. Any mistake in what follows is my own.

Contents

1 A quick overview of algebraic geometry

1.1 Lectures 1-4

These lectures consisted of informal surveys of certain fundamental concepts of algebraic geometry. They were meant as previews of various topics that we will cover rigorously.

2 Complex analysis

2.1 Lecture 5

First, let's review some basic concepts about functions of a single complex variable.

Definition 2.1.1. Let $z_0 \in \mathbb{C}$. A function $f = u + iv : U \subset \mathbb{C} \to \mathbb{C}$ is *holomorphic* or *analytic* if at least one of the following equivalent conditions holds.

- Both u and v are C^1, and f satisfies the Cauchy-Riemann equations, i.e.,

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

- $\frac{\partial f}{\partial \bar{z}} = 0$, where $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.
- The Cauchy integral formula holds, i.e.,

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{\eta - w} d\eta$$

  for any closed circular path γ centered at w in U.
- f has a power series representation on U.

Definition 2.1.2. A bijective function $f : U \subset \mathbb{C} \to V \subset \mathbb{C}$ is *biholomorphic* if it is holomorphic and its inverse is holomorphic. In this case, we say that U is *biholomorphic to* V, written as $U \approx V$.

Fact 2.1.3.

(a) (The maximum modulus principle) If $U \subset \mathbb{C}$ is a domain, $f : U \to \mathbb{C}$ is holomorphic, and $|f|$ has a local maximum, then f is constant.

(b) (Liouville's theorem) Any bounded entire function is constant.

(c) (The Riemann extension theorem) If $\epsilon > 0$ and $f : B_\epsilon(z) \setminus \{z\} \subset \mathbb{C} \to \mathbb{C}$ is bounded and holomorphic, then f can be extended to a holomorphic function on $B_\epsilon(z)$.

(d) (The Riemann mapping theorem) If $U \subsetneq \mathbb{C}$ is simply connected and open, then $U \approx B_1(0)$.

(e) (The residue theorem) If $f : B_\epsilon(0) \setminus \{0\}$ is holomorphic, then f can be expanded in a Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ such that $a_{-1} = \frac{1}{2\pi i}\oint f(z)dz$.

Next, let's look at some basic concepts about functions of several complex variables.

Definition 2.1.4. A function $f = u + iv : U \subset \mathbb{C}^n \to \mathbb{C}$ is *holomorphic* if at least one of the following equivalent conditions holds.

- f is holomorphic in each variable individually.

- Both u and v are C^1, and f satisfies the Cauchy-Riemann equations,

$$\begin{aligned} u_{x_i} &= v_{y_i} \\ u_{y_i} &= -v_{x_i} \end{aligned}$$

for each $i = 1, \dots, n$.

- $\sum_{i=1}^{n} \frac{\partial f}{\partial \bar{z}_i} = 0$.
- f has a power series representation on U,

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,\dots,k_n} z_1^{k_1} \cdots z_n^{k_n}.$$

Note 2.1.5. Statements (a), (b), and (c) from Fact 2.1.3 generalize to functions of several variables, as does the Cauchy integral formula:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\eta_i - z_i| = \epsilon_i} \frac{f(\eta_1, \dots, \eta_n)}{(\eta_1 - z_1) \cdots (\eta_n - z_n)} d\eta_1 \cdots d\eta_n$$

where $\eta_i > 0$ for each $i = 1, \dots, n$.

Theorem 2.1.6 (Hartog). *If $n > 1$, then any holomorphic function $f : B_\epsilon(0) \setminus \{0\} \subset \mathbb{C}^n \to \mathbb{C}$ extends to a holomorphic function on $B_\epsilon(0)$.*

Definition 2.1.7. Let X be a (topological) space. A sheaf F on X is a presheaf on X such that for any open $U \subset X$ and any open cover $\{U_i\}_{i \in J}$ of U, there is an exact sequence

$$0 \longrightarrow F(U) \longrightarrow F(U_i) \longrightarrow F(U_{ij})$$

where $U_{ij} := U_i \cap U_j$.

Definition 2.1.8. A *ringed space* is a pair $(X, \mathcal{J})$ where X is a space and $\mathcal{J}$ is a sheaf of rings on X.

Remark 2.1.9. Given any standard object $(X, \mathcal{J})$, we can define a *geometric object* as a ringed space locally isomorphic to $(X, \mathcal{J})$.

Definition 2.1.10 (Vector bundle). Let X and V be complex manifolds. Let $\pi : V \to X$ be holomorphic. We say that π is a *(holomorphic) vector bundle of rank n* if for any $x \in X$, there exist an open set $U \ni x$ in X and an isomorphism $\pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}^n$ such that the *transition maps* $U_{ij} \times \mathbb{C}^n \to U_{ij} \times \mathbb{C}^n$ are holomorphic and fiber linear.

Any vector bundle $\pi : V \to X$ induces a sheaf on X given by

$$F(U) = \Gamma\left(U, \pi^{-1}(U)\right).$$

Example 2.1.11.

1. The sheaf induced by the trivial bundle $\mathbf{1} := X \times \mathbb{C}$ is denoted by $\mathcal{O}_X$.
2. The tangent bundle TX of a smooth manifold X induces the sheaf of vector fields on X.
3. The cotangent bundle T^*X induces the sheaf $\Omega^1(X)$ of one-forms on X.
4. The alternating bundle $\bigwedge^p X$ of rank p induces the sheaf $\Omega^p(X)$ of p-forms on X.

3 Line bundles

3.1 Lecture 6

Definition 3.1.1. A *line bundle* is a vector bundle of rank 1.

Definition 3.1.2. Let X be a complex manifold. A *sheaf F of $\mathcal{O}_X$-modules* is a sheaf on X such that for any open set U in X,

- $F(U)$ is a module over $\mathcal{O}_X(U)$ and
- if $U \subset V \subset X$, then $(f \cdot a) \restriction_U = f \restriction_U \cdot a \restriction_U$.

Example 3.1.3 (Sheaf of sections). Let X be a complex manifold and J be a vector bundle over X. For any open $U \subset X$, let

$$\mathcal{L}_J(U) = \Gamma(U, L).$$

This inherits a vector space structure from the family of fibers of V. Also, any relation of the form $U_1 \subset U_2 \subset U$ induces a linear map $\Gamma(U_2, L) \to \Gamma(U_1, L)$ given by $\sigma \mapsto \sigma \restriction_{U_1}$. Thus, $\mathcal{L}_J(-)$ is a sheaf of vector spaces. Moreover, it is easily seen to be a sheaf of $\mathcal{O}_X$-modules.

Since any vector bundle is locally trivial, we see that $\mathcal{L}_J$ is *locally free*, i.e., for any $x \in X$, there exist an (open) neighborhood U of x in X and an isomorphism $\varphi : \mathcal{L}_J(U) \to \bigoplus_{i=1}^{\operatorname{rank}(J)} \mathcal{O}_X(U)$ such that for any open set $V \subset U$, the square

$$\begin{tikzcd}
\mathcal{L}_J(U) \arrow[r, "\cong"] \arrow[d] & \bigoplus_{i=1}^{\operatorname{rank}(J)} \mathcal{O}_X(U) \arrow[d] \\
\mathcal{L}_J(V) \arrow[r, "\cong"'] & \bigoplus_{i=1}^{\operatorname{rank}(J)} \mathcal{O}_X(V)
\end{tikzcd}$$

commutes. In other words, $\mathcal{L}_J$ is locally isomorphic to $(\mathcal{O}_X)^{\oplus \operatorname{rank}(J)}$.

Definition 3.1.4. A sheaf F on a complex manifold X is *invertible* if there exist an open cover $\{U_i\}$ of X and a family of holomorphic isomorphisms $\varphi_i : \mathcal{O}_{U_i} \to \mathcal{L}_J \restriction_{U_i}$.

Example 3.1.5. If J is a line bundle, then $\mathcal{L}_J$ is invertible.

Consider the composition

$$\mathcal{O}_{U_i \cap U_j} \xrightarrow{\varphi_i} \mathcal{L}_J \restriction_{U_i \cap U_j} \xrightarrow{\varphi_j^{-1}} \mathcal{O}_{U_i \cap U_j}, \qquad 1 \longmapsto g_{ij}.$$

From this, we can construct a line bundle L over X by defining the total space as

$$\coprod_i (U_i \times \mathbb{C}) \big/ \sim$$

where $(x, \lambda)_i \sim (y, \mu)$ if $x = y$ and $\mu = g_{ij}\lambda$.

Definition 3.1.6 (Divisor). A *divisor on a complex manifold* X is a locally finite $\mathbb{Z}$-combination of irreducible holomorphic hypersurfaces of X. Equivalently, it is a subset of X locally defined by the vanishing of a holomorphic function.

Example 3.1.7. If $X = \mathbb{A}^1$, then any divisor D on X is of the form

$$D = \sum m_i p_i, \quad p_i \in \mathbb{A}^1, \ m_i \in \mathbb{Z}.$$

Terminology. Each m_i is known as the *multiplicity of* p_i.

Any divisor D defines a line bundle $\mathcal{O}_X(D)$ on X and a holomorphic map $X \dashrightarrow \mathbb{P}(V^\vee)$ where $V \equiv \Gamma(X, \mathcal{O}_X(D))$. It is also true that any line bundle defines a divisor. It follows that

$$\text{(line bundles)} \xleftrightarrow{\sim} \text{(invertible sheaves)} \xleftrightarrow{\sim} \text{(divisors module linear equiv.)}\ . \tag{†}$$

Consider the case where $D = \mathsf{pt}$. Let $f \in \Gamma(U, \mathcal{O}_U)$ and let $U_i = X \setminus D$, which is a tubular neighborhood of D. Note that $U_i = f^{-1}(\mathbb{C} \setminus \text{hyperplane})$. Define $\mathcal{O}_X(D)$ as the line bundle with transition functions of the form $f\restriction_{U_i \cap U_j}$.

Alternatively, let

$$(\mathcal{O}_X(D))(U) = \{g : U \to \mathbb{C} \mid g \text{ is meromorphic, } \overbrace{fg}^{\text{product}} \text{ is holomorphic}\}.$$

For example, let $X = \mathbb{P}^1$ and D be a point p. Let (x_0, x_1) denote local coordinates on X near p. Let g be meromorphic in these coordinates and let $f(x_0, x_1) = \frac{x_1}{x_0}$. Then fg is holomorphic, i.e., g has a pole of order at most one at p.

Question.

1. What is $\Gamma\left(\mathbb{P}^1, \mathcal{O}_X\right)$?
2. What is $\Gamma\left(\mathbb{P}^1, \mathcal{O}_X(D)\right)$?

In fact, it can be shown that

$$\Gamma\left(\mathbb{P}^1, \mathcal{O}_X(m, p)\right) = \begin{cases} \mathbb{C}\langle 1, x, \ldots, x^m\rangle & m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In general, D is defined locally, and thus so is $\mathcal{O}_U(D)$. Specifically, $\Gamma(U, \mathcal{O}_U(D))$ consists of all holomorphic functions $f : U \setminus \operatorname{supp}(D) \to \mathbb{C}$ such that if $D = \sum m_i Y_i$ and $Y_i \cap U = \{f_i = 0\}$, then $g \prod_i f_i^{m_i}$ is holomorphic in U.

Example 3.1.8 (Veronese embedding). Let $X = \mathbb{P}^1$ and p be as before.

1. Let $D = \mathcal{O}(2p)$. Consider the space $V \coloneqq \Gamma\left(\mathbb{P}^1, \mathcal{O}(2p)\right) = \mathbb{C}\langle 1, x, x^2\rangle$. Define the map $\varphi_{\mathcal{O}(2p)} : \mathbb{P}^1 \to \mathbb{P}^2$ by
$$\varphi_{\mathcal{O}(2p)}(x) = \underbrace{\left(1, x, x^2\right)}_{(x,y,z)}.$$
This is an embedding. Its image is precisely the smooth curve given by $y^2 = xz$.
2. Let $D = \mathcal{O}(3p)$. Then the image of the map $\varphi_{\mathcal{O}(3p)} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $x \mapsto \left(1, x, x^2, x^3\right)$ is a so-called twisted cubic.

The line bundle L on X determines the map $X \dashrightarrow \mathbb{P}\left(\Gamma(X, L)^\vee\right)$ directly, as follows.

$$x \mapsto \ker\left(\Gamma(X, L) \xrightarrow{\text{eval}_x} L_p\right)$$

Definition 3.1.9. The *base locus of* L is $\mathcal{BL}(L) \equiv \{x \in X \mid s(x) = 0 \text{ for each } s \in \Gamma(X, L)\}$.

Note that we get a map $X \setminus \mathcal{BL}(L) \to \mathbb{P}\left(\Gamma(X, L)^\vee\right)$.

Now, let's consider a slight generalization of our preceding discussion. Let $V \subset \Gamma(X, L)$. This induces a map

$$\begin{tikzcd} X \arrow[r, dashed] & \mathbb{P}(V^\vee) \\ X \setminus \mathcal{BL}(V) \arrow[u, hook] \arrow[ur] & \end{tikzcd}.$$

Let $X = \mathbb{P}^1$ and $p = \{x = 0\}$. Then $V \subset \Gamma\left(\mathbb{P}^1, \mathcal{O}(2)\right) = \mathbb{C}\langle 1, x, x^2\rangle$, and

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\varphi_{\mathcal{O}(2)}"] \arrow[dr, dashed, "\varphi_V"'] & \mathbb{P}^2 \arrow[d, dashed, "\rho"] \\ & \mathbb{P}^1 \end{tikzcd}$$

commutes where ρ denotes the linear projection. Note that φ_V is a morphism so long as the center of ρ is not in the image of $\varphi_{\mathcal{O}(2)}$. In this case, we have that

$$\varphi_{\mathcal{O}(2)}(x) = \frac{a + by + cx^2}{d + ex + fx^2}$$
$$\rho(x) = \frac{a + bx}{c + dx}.$$

3.2 Lecture 7

Let L_1 and L_2 be line bundles over X with transition functions $\{g_1^{kl} : U_{kl} \to \mathbb{C}^*\}$ and $\{g_2^{ij} : U_{ij} \to \mathbb{C}^*\}$, respectively. We can take a refinement $\{U_i \cap U_k\}$ where both L_1 and L_2 are trivial. Define $L^1 \otimes L^2$ as the line bundle with transition functions $\{g_1^{kl} g_2^{ij} : U_{ij} \cap U_{kl} \to \mathbb{C}^*\}$. Further, define $\left(L^1\right)^{-1}$ as the line bundle with transition functions $\{\left(g_1^{kl}\right)^{-1} : U_{kl} \to \mathbb{C}^*\}$. Note that, locally, $L^1 \otimes \left(L^1\right)^{-1} \cong \mathcal{O}_X$.

Definition 3.2.1. We say that a divisor $D = \sum_i m_i Y_i$ is effective if $m_i \geq 0$ for each i.

Let $V = \Gamma\left(X, \mathcal{O}_X(D)\right)$ and let D be effective. Note that $\mathbb{C}\langle D\rangle \subset V$. We have that $\operatorname{supp}(D) = \varphi^{-1}$ (hyperplane) where $\left(\mathbb{C}\langle 0\rangle\right)^{\perp}$ is precisely the hyperplane in $\mathbb{P}\left(V^\vee\right)$.

Example 3.2.2. Let $X = \mathbb{P}^1$.

1. Let $x = \frac{x_1}{x_0}$ and $D = p := \{x = 0\}$. Then $V = \mathbb{C}\langle 1, x\rangle$, and the map $\varphi_V : \mathbb{P}^1 \to \mathbb{P}\left(V^\vee\right)$ is given by $c \mapsto y := \frac{x}{1}$.
2. Let $D = m\left(\infty\right)$ with $m > 0$. Then $V = \mathbb{C}\langle x_1, \ldots, x_m^m\rangle$, and the map $\varphi_{m\infty} : \mathbb{P}^1 \to \mathbb{P}^m$ is given by

$$\begin{aligned} (x_0, x_1) &\mapsto \left(x_0^m, x_0^{m-1}x_1, \ldots, x_0 x_1^{m-1}, x_1^m\right) \\ x &\mapsto \left(1, x, \ldots, x^m\right). \end{aligned}$$

3. Let $D = p_1 + \cdots + p_m$ where $p_i = [1 : t_i]$. Let $x = \frac{x_1}{x_0}$, so that ∞ is given by $x_0 = 0$. Then $V = \mathbb{C}\langle \underbrace{1, \frac{1}{x - t_1}, \ldots, \frac{1}{x - t_m}}_{a_0, \frac{a_1}{x - t_1}, \ldots, \frac{a_m}{x - t_m}}\rangle$. This can be viewed as the space of all regular meromorphic functions on open subsets of $\mathbb{P}^1$ having poles of order at most m. The image of $\varphi : \mathbb{P}^1 \to \mathbb{P}^m$ is precisely the hyperplane $\{a_0 = 0\}$.

Example 3.2.3. Let X be an elliptic curve, i.e., a space of the form $\mathbb{C}/\Lambda$. Let p be the image of 0 and let $D = mp$.

1. Let $m = 1$. Then $V = \Gamma\left(X, \mathcal{O}_X(D)\right)$, which consists of all maps $f : X \to \mathbb{P}^1$ such that $f^{-1}\left(\infty\right) = \{0\}$. These are precisely the constant maps, so that $V \cong \mathbb{C}\langle s\rangle$ where s is a holomorphic section of $\mathcal{O}_X(D)$ vanishing at p and is meromorphic on $\mathcal{O}_X$.

$$\begin{tikzcd} X \arrow[r, dashed] & \mathbb{P}^0 \\ X \setminus p \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

   It follows that $\mathcal{BL}\left(\mathcal{O}_X(D)\right) = p$.

2. Let $m=2$. Then $V=\mathbb{C}\langle 1,p\rangle$, and $\varphi_{2p}:X\to\mathbb{P}^1$ is precisely the D-th Weierstrass function. Moreover, we have an exact sequence

$$0\longrightarrow\mathcal{O}_X(p)\longrightarrow\mathcal{O}_X(2p)\longrightarrow\mathcal{O}_p\longrightarrow\cdots.$$

3. Let $m=3$. Then $V=\langle 1,p,p'\rangle$, and the image of $\varphi_{3p}:X\to\mathbb{P}^2$ is given by $y^2=x^3+ax+b$.

Example 3.2.4. Let $X=\mathbb{P}^2$. Let $D=m\underbrace{(\text{line at }\infty)}_{\{z=0\}}$.

1. Let $m=0$. Then $V=\mathbb{C}\langle 1\rangle$, and $\mathcal{BL}=\emptyset$.
2. Let $m=1$. Then $C=\mathbb{C}\langle\frac{x}{z},\frac{y}{z},\frac{z}{z}\rangle\cong\mathbb{C}\langle 1,X,Y\rangle$, and $\mathcal{BL}=\emptyset$. The map $\varphi_D:\mathbb{P}^2\to\mathbb{P}^2$ is precisely the identity.
3. Let $m=2$. Then $V=\langle\frac{x^2}{z^2},\frac{x^4}{z^2},\frac{y^2}{z^2}.\frac{x}{z},\frac{y}{z},\frac{z}{z}\rangle$, and the map $\varphi_D:\mathbb{P}^2\to\mathbb{P}^5$ is an embedding given by $(x,y,z)\mapsto\langle x^2,xy,y^2,xz,yz,z^2\rangle$.

In general, if $H\subset\mathbb{P}^n$ is a hyperplane, then $\varphi_{\mathcal{O}(dH)}:\mathbb{P}^n\to\mathbb{P}^{\binom{d+n}{n}-1}$ is given by

$$(x_0,\ldots,x_n)\mapsto(d\text{-th order homogenous polynomials}),$$

known as the d-th order Veronese embedding on $\mathbb{P}^n$.

Example 3.2.5. Let $X=\mathbb{P}^2$ with coordinates (x,y,z). Let H denote the hyperplane given by $z=0$ and let $D=2H$. Then $V=\{s\in\Gamma(\mathcal{O}(2H))\mid s(0,0,1)=0\}$, and

$$\begin{tikzcd}
V \arrow[r, hook] & \Gamma(\mathcal{O}(2H)) \\
\mathbb{C}\langle x^2,xy,y^2,xz,yz\rangle \arrow[u, "\cong"] \arrow[r, hook] & \mathbb{C}\langle x^2,xy,y^2,xz,yz,z^2\rangle \arrow[u, "\cong"']
\end{tikzcd}$$

commutes. Further, $\mathcal{BL}(V)=\{0\}=[0,0,1]$, and φ_V is a map $\mathbb{P}^2\setminus\{0\}\to\mathbb{P}^4$ but does not extend to $\mathbb{P}^2$. Indeed, we have that

$$\begin{aligned}
\lim_{\substack{(0,y,1)\\ y\to0}}\varphi_V &= \lim_{y\to0}\left(0,0,y^2,0,y\right)=(0,0,0,0,1)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\not\parallel\\
\lim_{\substack{(x,0,1)\\ x\to0}}\varphi_V &= \lim_{x\to0}\left(x^2,0,0,x,0\right)=(0,0,0,1,0).
\end{aligned}$$

Note that for any $p\in X$, there exist $\widetilde{X}$ and $\pi:\widetilde{X}\to X$ such that π restricted to $\pi^{-1}(X\setminus p)$ is an isomorphism and $\pi^{-1}(p)$ is a divisor on $\widetilde{X}$ that is isomorphic to $\mathbb{P}^1$.

Proposition 3.2.6. *Let $Y\subset X$ be a submanifold of codimension $k\geq2$. Let $\varphi:X\setminus Y\to Z$. Then there exist $\widetilde{X}$ and $\pi:\widetilde{X}\to X$ such that π restricted to $\pi^{-1}(X\setminus Y)$ is an isomorphism and restricted to $\underbrace{\pi^{-1}(Y)}_{\text{divisor on }X}$ is a bundle with each fiber isomorphic to $\mathbb{P}^{k-1}$.*

3.3 Lecture 8

3.4 Lecture 9