Perry Hart
K-theory reading seminar
UPenn
September 26, 2018

Abstract

We introduce the concept of a natural transformation in category theory, leading to equivalences and adjunctions. The main sources for this talk are the following.

- nLab.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 3.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 1.

1 Natural transformations

Let $\mathscr{C}$ and $\mathscr{D}$ be categories and F and G be functors $\mathscr{C} \to \mathscr{D}$. A *natural transformation* $\phi : F \Rightarrow G$ is a function $A \mapsto f_A$ from $\mathrm{ob}\,\mathscr{C}$ to $\mathrm{mor}\,\mathscr{D}$ such that f_A is a map $F(A) \to G(A)$ and the following diagram commutes for any morphism $h : A \to B$ in $\mathscr{C}$.

$$\begin{tikzcd} FA \arrow[r, "Fh"] \arrow[d, "f_A"'] & FB \arrow[d, "f_B"] \\ GA \arrow[r, "Gh"'] & GB \end{tikzcd}$$

In symbols, this may be written as $f_B h_* = h_* f_A$, where f_A is called a *component* of ϕ.

Note 1.1. If every f_A is an isomorphism, then the maps $(f_A)^{-1}$ define a natural transformation $G \Rightarrow F$.

If each f_A is an isomorphism, then we say that ϕ is a *natural isomorphism*. Note that if $\mathscr{D}$ is a groupoid (i.e., a category in which every morphism is an isomorphism), then ϕ must be a natural isomorphism.

Let F, G, and H be functors $\mathscr{C} \to \mathscr{D}$. The *identity natural transformation* $\mathrm{Id}_F : F \Rightarrow F$ is given by $A \mapsto \mathrm{Id}_{F(A)}$. Moreover, given natural transformations $\phi : F \to G$ and $\psi : G \to H$, define the *composite natural transformation* $\psi \circ \phi$ by $A \mapsto (\psi \circ \phi)_A \coloneqq \psi_A \circ \phi_A$.

Lemma 1.2. *A natural transformation $\phi : F \Rightarrow G$ is a natural isomorphism iff it has an inverse $\phi^{-1} : G \Rightarrow F$.*

Proof. This follows from Note 1.1 along with our definition of a composite natural transformation. □

Example 1.3.

1. Let R and S be commutative rings. Any ring homomorphism $f : R \to S$ induces a ring homomorphism $\mathrm{GL}_n(f) : \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ satisfying

$$f(\det(A)) = \det\left(\underset{n}{\mathrm{GL}}(f)(A)\right).$$

By viewing GL_n and $R \mapsto R^*$ as functors from **Ring** to **Grp** and $\det_R : \mathrm{GL}_n(R) \to R^*$ as a morphism in **Grp**, we see that $\det_R$ defines a natural transformation $\phi : \mathrm{GL}_n \Rightarrow f^*$ where f^* denotes $f \restriction_{R^*}: R^* \to S^*$.

$$\begin{tikzcd}
\mathrm{GL}_n(R) \arrow[r, "\mathrm{GL}_n(f)"] \arrow[d, "\det_R"'] & \mathrm{GL}_n(S) \arrow[d, "\det_S"] \\
R^* \arrow[r, "f^*"'] & S^*
\end{tikzcd}$$

2. Consider the power set functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ defined on objects by $A \mapsto \mathcal{P}(A)$ and on morphisms g by $\mathcal{P}g(S) = g(S)$. Then the function $f_A : A \to \mathcal{P}(A)$ given by $a \mapsto \{a\}$ defines a natural transformation $\phi : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{P}$.

3. Set $\mathscr{C} = \mathscr{D} = \mathbf{Grp}$, $F = \mathrm{Id}_{\mathscr{C}}$, and $G = (-)^{\mathrm{ab}}$. Then given a group H, the natural projection $f : H \twoheadrightarrow H^{\mathrm{ab}}$ induces a natural transformation $\phi : F \Rightarrow G$.

4. We can view preorders $(P, \leq)$ and $(Q, \leq)$ as small categories and functors $F, G : P \to Q$ as order-preserving functions. Then there is a unique natural transformation $\phi : F \Rightarrow G$ iff $F(x) \leq G(x)$ for every $x \in P$.

5. The inversion isomorphism from a group G to its opposite group G^{op} defines a natural transformation $\phi : \mathrm{Id}_{\mathbf{Grp}} \Rightarrow ((-)^{\mathrm{op}} : \mathbf{Grp} \to \mathbf{Grp})$. In this sense, G is naturally isomorphic to G^{op}.

Definition 1.4. Let $\mathscr{C}$ and $\mathscr{D}$ be categories with $\mathscr{C}$ small. The *functor category* $\mathbf{Fun}(\mathscr{C}, \mathscr{D}) \coloneqq \mathscr{D}^{\mathscr{C}}$ has functors $F : \mathscr{C} \to \mathscr{D}$ as objects and natural transformations as morphisms.

Remark 1.5. Any Grothendieck universe models ZFC, in particular Replacement. This ensure that for any two functors $F, G : \mathscr{C} \to \mathscr{D}$, the class of natural transformation $\phi : F \Rightarrow G$ is a set. This means that $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$ is locally small, a condition of our definition of a category.

Definition 1.6. Given a category $\mathscr{C}$, the *arrow category* $\mathrm{Ar}(\mathscr{C})$ of $\mathscr{C}$ has as objects morphisms $f : X_0 \to X_1$ in $\mathscr{C}$ and as morphisms $M : (f : X_0 \to X_1) \to (g : Y_0 \to Y_1)$ the pairs (M_0, M_1) of morphisms $M_0 : X_0 \to Y_0$ and $M_1 : X_1 \to Y_1$ such that

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d, "M_0"'] & X_1 \arrow[d, "M_1"] \\
Y_0 \arrow[r, "g"'] & Y_1
\end{tikzcd}$$

commutes.

Note 1.7.

1. $\mathrm{Ar}(\mathscr{C}) \cong \mathbf{Fun}([1], \mathscr{C})$.

2. $\mathbf{Fun}(\mathscr{C} \times \mathscr{D}, \mathscr{E}) \cong \mathbf{Fun}(\mathscr{C}, \mathbf{Fun}(\mathscr{D}, \mathscr{E}))$.

2 Equivalences

Usually, it is useful to make our notion of *sameness* between categories weaker than *isomorphism*.

Definition 2.1. A functor $F : \mathscr{C} \to \mathscr{D}$ is an *equivalence* if there is a functor $G : \mathscr{D} \to \mathscr{C}$, called the *quasi-inverse of* F, such that $F \circ G \cong \mathrm{Id}_{\mathscr{C}}$ and $G \circ F \cong \mathrm{Id}_{\mathscr{D}}$. In this case, we say that F and G are *equivalent categories*. Moreover, we say that a property of $\mathscr{C}$ is *categorical* if it is invariant under categorical equivalence.

Example 2.2. Let k be a field. Let the category $\mathbf{Mat}_k$ have natural numbers as objects and morphisms $n \to p$ given by $p \times n$ matrices over k. Let **fdMod** denote the category of finite-dimensional vector spaces with linear maps as morphisms. These two categories are equivalent. Indeed, send the natural number n to k^n in one direction and the space V to $\dim V$ in the other direction.

Definition 2.3. A functor $F : \mathscr{C} \to \mathscr{D}$ is *essentially surjective* if for each object Z of $\mathscr{D}$, there is some object Y of $\mathscr{C}$ such that $F(Y) \cong Z$.

Theorem 2.4. *A functor is an equivalence iff it is full, faithful, and essentially surjective.*[1]

Definition 2.5. A *skeleton* of $\mathscr{C}$ is a full subcategory $\mathscr{C}' \subset \mathscr{C}$ such that each element of $\mathrm{ob}\,\mathscr{C}$ is isomorphic to exactly one element of $\mathrm{ob}\,\mathscr{C}'$.

An application of Theorem 2.4 yields the following result.

Lemma 2.6. *Let $\mathscr{C}'$ be a skeleton of $\mathscr{C}$. Then the inclusion functor $\mathscr{C}' \hookrightarrow \mathscr{C}$ is an equivalence.*

Lemma 2.7. *Any two skeleta $\mathscr{C}', \mathscr{C}'' \subset \mathscr{C}$ are isomorphic.*

Proof. Define $F : \mathscr{C}' \to \mathscr{C}''$ on objects by $F(X) = Y$ where $X \cong Y$ via a chosen isomorphism h_X and on morphisms $f \in \mathscr{C}(X, Y)$ by $F(f) = h_Y \circ f \circ (h_X)^{-1}$. To get F^{-1}, define $G : \mathscr{C}'' \to \mathscr{C}'$ by similarly choosing an isomorphism $(h_X)^{-1}$ for each $X \in \mathrm{ob}\,\mathscr{C}''$. □

Remark 2.8. Both Lemma 2.6 and Lemma 2.7 are logically equivalent to the axiom of choice, as is the statement that every category admits a skeleton.

3 Adjunctions

Definition 3.1.

1. Let $Z \in \mathrm{ob}\,\mathscr{C}$. Define the contravariant functor $\mathscr{Y}_Z : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ on objects by $Y \mapsto \mathscr{C}(Y, Z)$ and on morphisms by sending $f : X \to Y$ in $\mathscr{C}$ to the map $f^* : \mathscr{C}(Y, Z) \to \mathscr{C}(X, Z)$ given by $g \mapsto gf$.

   We call $\mathscr{C}(-, Z) \coloneqq \mathscr{Y}^Z$ the set-valued functor *represented by* Z in $\mathscr{C}$.

2. Let $X \in \mathrm{ob}\,\mathscr{C}$. Define the functor $\mathscr{Y}^X : \mathscr{C} \to \mathbf{Set}$ on objects by $Y \mapsto \mathscr{C}(X, Y)$ and on morphisms by sending $g : Y \to Z$ to the map $g_* : \mathscr{C}(X, Y) \to \mathscr{C}(X, Z)$ given by $f \mapsto gf$.

   We call $\mathscr{C}(X, -) \coloneqq \mathscr{Y}^X$ the set-valued functor *corepresented by* X in $\mathscr{C}$.

[1] Theorem 3.2.10 (Rognes).

A functor of the form $\mathscr{C} \times \mathscr{C}' \to \mathscr{D}$ is called a *bifunctor*. In particular, define $\mathscr{C}(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ on objects by $(X, X') \to \mathscr{C}(X, X')$ and on morphisms by sending $(f, f') : (X, X') \to (Y, Y')$ to the map $\mathscr{C}(f, f') : \mathscr{C}(X, X') \to \mathscr{C}(Y, Y')$ given by $g \mapsto f'gf$.

Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors.

Definition 3.2 (Kan). Consider the set-valued bifunctors $\mathscr{D}(F(-), -), \mathscr{C}(-, G(-)) : \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$. An *adjunction from F to G* is a natural isomorphism

$$\phi : \mathscr{D}(F(-), -) \Rightarrow \mathscr{C}(-, G(-)).$$

If such a ϕ exists, then we say that (F, G) is an *adjoint pair (of functors)*.

Note that ϕ is natural in the sense that for any map $c : X' \to X$ in $\mathscr{C}$ and $d : Y \to Y'$ in $\mathscr{D}$, the square

$$\begin{tikzcd}
\mathscr{D}(FX, Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "c^* d_*"'] & \mathscr{C}(X, GY) \arrow[d, "c^* d_*"] \\
\mathscr{D}(FX', Y') \arrow[r, "\phi_{X',Y'}"'] & \mathscr{C}(X', GY')
\end{tikzcd}$$

commutes in $\mathbf{Set}$.

Proposition 3.3. *Left and right adjoints are both unique up to unique isomorphism.*

Terminology. We call F the *left adjoint* to G and G the *right adjoint* to F. In symbols, $F \dashv G$.

Note 3.4. It is straightforward to check that any adjoint triple $F \dashv G \dashv H$ yields two new adjunctions:

$$GF \dashv GH$$
$$FG \dashv HG$$

Definition 3.5. Given an adjunction $\phi : \mathscr{D}(F(-), -) \Rightarrow \mathscr{C}(-, G(-))$, define the *unit morphism*

$$\eta_X = \phi_{X,FX}\,(\mathrm{Id}_{FX}) \in \mathscr{C}(X, GF(X))$$

and the *counit morphism*

$$\epsilon_Y = \phi^{-1}_{GY,Y}\,(\mathrm{Id}_{GY}) \in \mathscr{D}(FG(Y), Y).$$

Lemma 3.6. *The unit morphisms $(\eta_X)_{X \in \mathrm{ob}\,\mathscr{C}}$ define a natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow GF$, and the counit morphisms $(\epsilon_Y)_{Y \in \mathrm{ob}\,\mathscr{D}}$ define a natural transformation $\epsilon : FG \Rightarrow \mathrm{Id}_{\mathscr{D}}$.*

Proof. For simplicity, let us just prove that ϵ is a natural transformation. We must check that

$$\begin{tikzcd}
FG(Y) \arrow[r, "FG(y)"] \arrow[d, "\epsilon_Y"'] & FG(Y') \arrow[d, "\epsilon_{Y'}"] \\
Y \arrow[r, "y"'] & Y'
\end{tikzcd}$$

commutes for any map $y : Y \to Y'$ in $\mathscr{D}$. By the naturality of ϕ, we have that

$$\begin{aligned} y \circ \epsilon_Y &= y \circ \phi^{-1}\left(\mathrm{Id}_{GY}\right) \\ &= \phi^{-1}\left(Gy \circ \mathrm{Id}_{GY}\right) \\ &= \phi^{-1}\left(\mathrm{Id}_{GY'} \circ Gy\right) \\ &= \phi^{-1}\left(\mathrm{Id}_{GY'}\right) \circ FG(y) \\ &= \epsilon_{Y'} \circ FG(y), \end{aligned}$$

as required. □

Moreover, one can verify that the unit and counit of ϕ satisfy the *triangle identities*,

$$\epsilon_{FX} \circ F\eta_X = 1_{FX} \tag{$\triangle_1$}$$

$$G\epsilon_Y \circ \eta_{GY} = 1_{GY}, \tag{$\triangle_2$}$$

for any $X \in \operatorname{ob}\mathscr{C}$ and $Y \in \operatorname{ob}\mathscr{D}$.

Conversely, suppose that F and G come equipped with two natural transformations

$$\begin{aligned} \eta &: \mathrm{Id}_{\mathscr{C}} \Rightarrow GF \\ \epsilon &: FG \Rightarrow \mathrm{Id}_{\mathscr{D}} \end{aligned}$$

satisfying the triangle identities. Then we get an adjunction ϕ from F to G with component

$$\phi_{X,Y} : \mathscr{D}(FX, Y) \to \mathscr{C}(X, GY), \qquad f \mapsto Gf \circ \eta_X.$$

Indeed, define $\psi_{X,Y} : \mathscr{C}(X, GY) \to \mathscr{D}(FX, Y)$ by $g \mapsto \epsilon_Y \circ Fg$. We have that

$$\begin{aligned} \psi_{X,Y}\left(\phi_{X,Y}(f)\right) &= \psi_{X,Y}\left(Gf \circ \eta_X\right) \\ &= \epsilon_Y \circ F\left(Gf \circ \eta_X\right) \\ &= \epsilon_Y \circ F(Gf) \circ F\eta_X \\ &= f \circ \epsilon_{FX} \circ F\eta_X && \text{(naturality of } \epsilon\text{)} \\ &= f. && ((\triangle_1)) \end{aligned}$$

Likewise, we have that $\phi_{X,Y}\left(\psi_{X,Y}(g)\right) = g$. Hence $\phi_{X,Y}$ is a natural isomorphism in both X and Y with inverse $\psi_{X,Y}$.

Even so, $\mathscr{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{D}$ need *not* be an equivalence of categories, as η and ϵ may not be isomorphisms. Further, a given equivalence $\mathscr{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{D}$ of categories need *not* be an adjunction, as its associated natural transformations

$$\begin{aligned} \eta' &: \mathrm{Id}_{\mathscr{C}} \Rightarrow RL \\ \epsilon' &: LR \Rightarrow \mathrm{Id}_{\mathscr{D}} \end{aligned}$$

may not satisfy the triangle inequalities. Nevertheless, (L, R) *is* an adjoint pair with unit η' and counit another natural transformation defined in terms of η' and ϵ'. By symmetry, (R, L) is also an adjoint pair.

Example 3.7 (Monad). Let $(\mathscr{C}, \otimes, 1)$ be a monoidal category. A *monoid* in $\mathscr{C}$ is an object M equipped with a *multiplication* map $\mu : M \otimes M \to M$ and a *unit* map $\eta : 1 \to M$ that satisfy certain coherence properties expressing that μ is associative and that η is a two-sided identity. Given two monoids (M, μ, η) and (M', μ', η') in $\mathscr{C}$, a map $f : M \to M'$ in $\mathscr{C}$ is a *morphism of monoids* if it satisfies

$$f \circ \mu = \mu' \circ (f \otimes f) \qquad f \circ \eta = \eta'.$$

A *comonoid* N in $\mathscr{C}$ is a monoid in $\mathscr{C}^{\mathrm{op}}$, equipped with a *comultiplication* map $\delta : N \to N^2$ and a *counit* map $\epsilon : N \to 1$.

For example, a monoid in the monoidal category $(\mathsf{End}(\mathscr{C}), \circ, \mathrm{Id}_{\mathscr{C}})$ of endofunctors of $\mathscr{C}$ is called a *monad on* $\mathscr{C}$. A comonoid in $\mathsf{End}(\mathscr{C})$ is called a *comonad on* $\mathscr{C}$.

Explicitly, a monad on $\mathscr{C}$ consists of an endofunctor $T : \mathscr{C} \to \mathscr{C}$ together with two natural transformations $\eta : \mathrm{Id}_{\mathscr{C}} \to Y$ and $\mu : T^2 \to T$ such that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} .$$

These are precisely the *associativity* and *unit* laws, respectively. Now, let $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ be an adjoint pair with unit $\eta : \mathrm{Id}_{\mathscr{C}} : G \circ F$ and counit $\epsilon : F \circ G \to \mathrm{Id}_{\mathscr{D}}$. We then have a natural transformation $(G \circ F)^2 \to G \circ F$ given componentwise by

$$G(\epsilon_{FX}) : GFGFX \to GFX$$

One can check that $(G \circ F, \eta, G\epsilon_F)$ is a monad on $\mathscr{C}$.

Dually, a comonad $R : \mathscr{C} \to \mathscr{C}$ on $\mathscr{C}$ satisfies the relations

$$\delta_R \circ \delta = R\delta \circ \delta$$
$$\epsilon_R \circ \delta = \mathrm{Id}_R = R\epsilon \circ \delta.$$

Moreover, any adjoint pair $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ with unit η and counit ϵ induces a comonad (G, ϵ, δ) on $\mathscr{D}$ where

$$G \equiv F \circ G : \mathscr{D} \to \mathscr{D}$$
$$\delta \equiv F\eta_G : G \to G^2.$$

Theorem 3.8. *The category of monoids in $\mathscr{C}$ is equivalent to the category of $\mathscr{C}$-enriched categories with one object.*

Example 3.9.

(1) The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ has a left adjoint $F : \mathbf{Set} \to \mathbf{Grp}$ sending a set to the free group generated by A.

(2) Let R be a ring. The forgetful functor $U : R{-}\mathbf{Mod} \to \mathbf{Set}$ has a left adjoint $R(-)$ sending a set S to $\bigoplus_{s \in S} R$, the free R-module generated by S.

The forgetful functor has no right adjoint in either Example 3.9(1) or Example 3.9(2). It does, however, have one in the following setting.

Example 3.10. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has a left adjoint that sends a set to the same set equipped with the discrete topology. It also has a right adjoint that sends a set to the same set equipped with the indiscrete topology.

Definition 3.11. A subcategory $\mathscr{C} \subset \mathscr{D}$ is *reflective* if the inclusion functor has a left adjoint and is *coreflective* if the inclusion functor has a right adjoint.

Example 3.12.

1. The full subcategory $\mathbf{Ab} \subset \mathbf{Grp}$ is reflexive as the inclusion functor is right adjoint to $(-)^{\mathrm{ab}}$.
2. Let $\mathbf{Ab}_T \subset \mathbf{Ab}$ denote the subcategory of torsion groups. This is coreflective as the inclusion functor is right adjoint to the functor sending an abelian group to its torsion subgroup.