

Abstract

We continue looking at higher Waldhausen K -theory by presenting several of its key theorems. At the end, we see an encoding of Waldhausen K -theory as the infinite loop space of a sort of spectrum. The main sources for this talk are the following.

- $n\text{Lab}$.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Ch. V.2.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

1 Extension and additivity

Let \mathcal{B} and \mathcal{C} be Waldhausen categories. We say that $F' \rightarrowtail F \twoheadrightarrow F''$ is a *short exact sequence* or *cofiber sequence of exact functors* $\mathcal{B} \rightarrow \mathcal{C}$ if

- (i) the sequence $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$ is a cofiber sequence for every $B \in \text{ob } \mathcal{B}$ and
- (ii) the map $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration in \mathcal{C} for every $A \rightarrowtail B$ in \mathcal{B} .

Let $\eta : A \rightarrowtail B \twoheadrightarrow C$ be an object in $S_2\mathcal{C}$. Define the source s , target t , and quotient q functors $S_2\mathcal{C} \rightarrow \mathcal{C}$ by $s(\eta) = A$, $t(\eta) = B$, and $q(\eta) = C$, respectively. Then $s \rightarrowtail t \twoheadrightarrow q$ is a cofiber sequence of functors.

Since defining a cofiber sequence of exact functors $\mathcal{B} \rightarrow \mathcal{C}$ is equivalent to defining an exact functor $\mathcal{B} \rightarrow S_2\mathcal{C}$, we may restrict our attention to $s \rightarrowtail t \twoheadrightarrow q$ when proving assertions about a given cofiber sequence of exact functors $\mathcal{B} \rightarrow \mathcal{C}$. (We say that $S_2\mathcal{C}$ is *universal* in this sense.)

Theorem 1.1 (Extension). *The exact functor $(s, q) : S_2\mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ induces a homotopy equivalence $K(S_2\mathcal{C}) \simeq K(\mathcal{C}) \times K(\mathcal{C})$. The functor $\coprod : (A, B) \rightarrow (A \rightarrowtail A \coprod B \twoheadrightarrow B)$ is a homotopy inverse.*

Proof sketch. Let \mathcal{C}_m^w denote the category of m -length sequences of weak equivalences. For each n , define $s_n\mathcal{C}_m^w$ as the commutative diagram

$$\begin{array}{ccccccc}
 X_1^0 & \rightarrowtail & X_2^0 & \rightarrowtail & \cdots & \rightarrowtail & X_n^0 \\
 \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\
 X_1^1 & \rightarrowtail & X_2^1 & \rightarrowtail & \cdots & \rightarrowtail & X_n^1 \\
 \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\
 \vdots & & \vdots & & & & \vdots \\
 \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\
 X_1^m & \rightarrowtail & X_2^m & \rightarrowtail & \cdots & \rightarrowtail & X_n^m
 \end{array} .$$

This is naturally isomorphic to an (m, n) -bisimplex in $N_\bullet wS_\bullet \mathcal{C}$, which is thus isomorphic to the bisimplicial set $s_\bullet \mathcal{C}_{(-)}^w$. One can show that the source s and quotient q functors $S_2 \mathcal{C} \rightarrow \mathcal{C}$ induce a homotopy equivalence $s \times q : s_\bullet S_2(\mathcal{C}_m^w) \rightarrow s_\bullet \mathcal{C}_m^w \times s_\bullet \mathcal{C}_m^w$ for each m . Thus, we get a homotopy equivalence

$$s_\bullet S_2(\mathcal{C}_{(-)}^w) \simeq s_\bullet \mathcal{C}_{(-)}^w \times s_\bullet \mathcal{C}_{(-)}^w$$

between bisimplicial sets. But we already have that $s_\bullet \mathcal{C}_{(-)}^w \cong N_\bullet wS_\bullet \mathcal{C}$, thereby completing our proof. \square

Recall that $|wS_\bullet \mathcal{C}|$ is an H -space via the map

$$\coprod : |wS_\bullet \mathcal{C}| \times |wS_\bullet \mathcal{C}| \cong |wS_\bullet \mathcal{C} \times wS_\bullet \mathcal{C}| \rightarrow |wS_\bullet \mathcal{C}|. \quad (\star)$$

This produces an H -space structure $(K(\mathcal{C}), +)$.

Theorem 1.2 (Additivity). *Let $F' \rightrightarrows F \rightrightarrows F''$ be a short exact sequence of exact functors $\mathcal{B} \rightarrow \mathcal{C}$. Then $F_* \simeq F'_* + F''_*$ as maps $K(\mathcal{B}) \rightarrow K(\mathcal{C})$, so that*

$$F_* = F'_* + F''_*$$

as maps $K_i(\mathcal{B}) \rightarrow K_i(\mathcal{C})$.

Proof. As $S_2 \mathcal{C}$ is universal, it suffices to prove that $t_* \simeq s_* + q_*$. Notice that the two composites

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\coprod} S_2 \mathcal{C} \xrightarrow[s \coprod q]{t} \mathcal{C}$$

are the same. Theorem 1.1 implies that $K(\coprod) : K(\mathcal{C}) \times K(\mathcal{C}) \rightarrow K(S_2 \mathcal{C})$ is a homotopy equivalence. Since the H -space structure on $K(\mathcal{C})$ is induced by \coprod , we conclude that $t_* \simeq s_* + q_*$. \square

Definition 1.3. We say that a sequence

$$* \rightarrow A_n \rightarrow \cdots \rightarrow A_0 \rightarrow *$$

is *admissibly exact* if each morphism in the sequences can be written as a cofiber sequence

$$A_{i+1} \twoheadrightarrow B_i \rightarrowtail A_i.$$

Corollary 1.4. *Suppose that*

$$* \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \rightarrow F^n \rightarrow *$$

is an admissibly exact sequence of exact functors $\mathcal{B} \rightarrow \mathcal{C}$. Then we have an equality

$$\sum_i (-1)^i F_*^i = 0$$

of maps $K_i(\mathcal{B}) \rightarrow K_i(\mathcal{C})$.

Corollary 1.5. *Let $F' \twoheadrightarrow F \twoheadrightarrow F''$ be a short exact sequence of exact functors $\mathcal{B} \rightarrow \mathcal{C}$. Then*

$$F''_* \simeq F_* - F_* \simeq 0.$$

This implies that the homotopy fiber of $F''_* : K(\mathcal{B}) \rightarrow K(\mathcal{C})$ is homotopy equivalent to $K(\mathcal{B}) \vee \Omega K(\mathcal{C})$.

Let \mathcal{C} be a Waldhausen category. Recall the arrow category $\text{Ar}(\mathcal{C})$ of \mathcal{C} consisting of morphisms in \mathcal{C} as objects and commutative squares as morphisms. Let s and t denote the source and target functors $\text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$, respectively.

Definition 1.6. A functor $T : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$ is a (*mapping*) *cylinder functor* on \mathcal{C} if it comes equipped with natural transformations $j_1 : s \Rightarrow T$, $j_2 : t \Rightarrow T$, and $p : T \Rightarrow t$ such that for any $f : A \rightarrow B$, we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j_1} & T(f) \xleftarrow{j_2} B \\ & \searrow f & \downarrow p \swarrow = \\ & & B \end{array} .$$

Moreover, T must satisfy the following axioms.

- (1) T sends every initial morphism $* \rightarrow A$ to A for any $A \in \text{ob } \mathcal{C}$.
- (2) The map $j_1 \amalg j_2 : A \amalg B \rightarrow T(f)$ is a cofibration for any $f : A \rightarrow B$.
- (3) Given a morphism $(a, b) : f \rightarrow f'$ in $\text{Ar}(\mathcal{C})$, if both a and b are weak equivalences in \mathcal{C} , then so is $T(f) \rightarrow T(f')$.
- (4) Given a morphism $(a, b) : f \rightarrow f'$ in $\text{Ar}(\mathcal{C})$, if both a and b are cofibrations in \mathcal{C} , then so is $T(f) \rightarrow T(f')$. Also, the map

$$A' \amalg_A T(f) \amalg_B B' \rightarrow T(f')$$

induced by axiom (2) is a cofibration in \mathcal{C} .

- (5) (*Cylinder axiom*) The map $p : T(f) \rightarrow B$ is a weak equivalence in \mathcal{C} .

Terminology. Let T be a cylinder functor on \mathcal{C} .

1. We call $T(A \rightarrow *)$ the *cone* of A , denoted by $\text{cone}(A)$.
2. We call $\text{cone}(A)/_A$ the *suspension* of A , denoted by ΣA .

Corollary 1.7. *The induced suspension map $\Sigma : K(\mathcal{C}) \rightarrow K(\mathcal{C})$ is a homotopy inverse for the H -space structure (\star) .*

Proof. Note that axiom (3) gives us a cofiber sequence $A \rightarrow \text{cone}(A) \rightarrow \Sigma A$. Therefore, $1 \rightarrow \text{cone} \rightarrow \Sigma$ is an exact sequence of functors. By the cylinder axiom, we know that cone is null-homotopic. It follows by Theorem 1.2 that $\Sigma_* + 1 = \text{cone}_* = *$. \square

2 Localization

Let \mathcal{C} be a category with cofibrations. Equip it with two Waldhausen subcategories $v(\mathcal{C})$ and $w(\mathcal{C})$ of weak equivalences such that $v(\mathcal{C}) \subset w(\mathcal{C})$. Assume that (\mathcal{C}, w) admits a cylinder functor. Suppose that $w(\mathcal{C})$ is saturated and closed under extensions.

Let \mathcal{C}^w denote the Waldhausen subcategory of (\mathcal{C}, v) consisting of all A such that $* \rightarrow A$ belongs to $w(\mathcal{C})$.

Theorem 2.1 (Waldhausen localization). *The sequence*

$$K(\mathcal{C}^w) \rightarrow K(\mathcal{C}, v) \rightarrow K(\mathcal{C}, w)$$

is a homotopy fibration sequence.

Proof sketch. Recall that a small bicategory is a bisimplicial set such that each row/column is the nerve of a category. Note that $v_{(-)}w_{(-)}\mathcal{C}$ is a bicategory whose bimorphisms are commutative squares of the form

$$\begin{array}{ccc} (-) & \xrightarrow{w'} & (-) \\ v \downarrow & & \downarrow v' \\ (-) & \xrightarrow{w} & (-) \end{array} \quad (\star)$$

Treating $w\mathcal{C}$ as a bicategory with a single vertical morphism reveals that

$$w\mathcal{C} \simeq v_{(-)}w_{(-)}\mathcal{C}.$$

This yields $wS_n\mathcal{C} \simeq v_{(-)}w_{(-)}S_n\mathcal{C}$ for each n .

Now, let $v_{(-)}\operatorname{co} w_{(-)}\mathcal{C}$ denote the subcategory of all squares like (\star) where the horizontal maps are also cofibrations. One can show that the inclusion $v_{(-)}\operatorname{co} w_{(-)}\mathcal{C} \subset v_{(-)}w_{(-)}\mathcal{C}$ is a homotopy equivalence. Since each $S_n\mathcal{C}$ inherits a cylinder functor from \mathcal{C} , we obtain a simplicial bi-subcategory $v_{(-)}\operatorname{co} w_{(-)}S_{\bullet}\mathcal{C}$ such that the inclusion into $v_{(-)}w_{(-)}S_{\bullet}\mathcal{C}$ is a homotopy equivalence. This yields a commutative diagram

$$\begin{array}{ccccc} vS_{\bullet}C^w & \longrightarrow & vS_{\bullet}C & \longrightarrow & v_{(-)}\operatorname{co} w_{(-)}S_{\bullet}C \\ & & \downarrow & & \downarrow \simeq \\ & & wS_{\bullet}C & \xrightarrow{\simeq} & v_{(-)}w_{(-)}S_{\bullet}C \end{array}.$$

It therefore suffices to show that the top row is a fibration. One can do this by using the relative K -theory space construction. See IV.8.5.3 and V.2.1 (Weibel). \square

Now, let \mathcal{A} be an exact category embedded in an abelian category \mathcal{B} and let $\mathbf{Ch}^b(\mathcal{A})$ denote the category of bounded chain complexes in \mathcal{A} . One can verify that $\mathbf{Ch}^b(\mathcal{A})$ is Waldhausen where the cofibrations $A_{\bullet} \rightarrow B_{\bullet}$ are precisely the degree-wise admissible monomorphisms (i.e., those admitting a short exact sequence $A_n \rightarrow B_n \rightarrow B_n/A_n$ in \mathcal{A} for each n) and the weak equivalences are precisely the chain maps which are quasi-isomorphisms of complexes in $\mathbf{Ch}(\mathcal{B})$.

Our next result is a consequence of Theorem 2.1 and can be found in V.2.2 (Weibel).

Theorem 2.2 (Gillet-Waldhausen). *Let \mathcal{A} be an exact category closed under kernels of surjections. Then the exact inclusion $\mathcal{A} \rightarrow \mathbf{Ch}^b(\mathcal{A})$ induces a homotopy equivalence $K(\mathcal{A}) \simeq K \mathbf{Ch}^b(\mathcal{A})$. Hence*

$$K_i(\mathcal{A}) = K_i \mathbf{Ch}^b(\mathcal{A})$$

for every i .

Definition 2.3. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor between Waldhausen categories. We say that F satisfies the *approximate lifting property* if for any map $b : F(A) \rightarrow B$ in \mathcal{B} , there exists a map $a : A \rightarrow A'$ in \mathcal{A} along with a weak equivalence $b' : F(A') \simeq B$ in \mathcal{B} such that

$$\begin{array}{ccc} F(A') & \xrightarrow{\sim} & B \\ \uparrow F(a) & \nearrow b & \\ F(A) & & \end{array} .$$

commutes.

This means that F has the approximate lifting property just in case we can always lift it to a weak equivalence.

Proposition 2.4. *Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor between Waldhausen categories with the following properties.*

1. *F satisfies the approximate lifting property.*
2. *\mathcal{A} admits a cylinder functor.*
3. *A morphism f in \mathcal{A} is a weak equivalence iff $F(f)$ is a weak equivalence in \mathcal{B} .*

Then $wF : w\mathcal{A} \rightarrow w\mathcal{B}$ is a homotopy equivalence.

Corollary 2.5 (Waldhausen approximation). *With the same hypotheses as in Proposition 2.4, we have*

$$K(\mathcal{A}) \simeq K(\mathcal{B}).$$

Proof sketch. One can show that each functor $S_n \mathcal{A} \rightarrow S_n \mathcal{B}$ is exact and also has the approximate lifting property. Proposition 2.4 thus gives rise to degree-wise homotopy equivalence between the bisimplicial map $wS_\bullet \mathcal{A} \rightarrow wS_\bullet \mathcal{B}$, which is enough. \square

Definition 2.6. Let \mathcal{A} be an abelian category $\mathbf{Ch}(\mathcal{A})$ denote the category of chain complexes over \mathcal{A} . We say that a complex C_\bullet is *homologically bounded* if only finitely many $H_i(C_j)$ are nonzero.

Notation. Let $\mathbf{Ch}_\pm^{\text{hb}}$ denote the subcategory of bounded below (respectively, bounded above) complexes.

Example 2.7. Let \mathcal{A} be an abelian category. One can show that the inclusions $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}_-^{\text{hb}}(\mathcal{A})$ and $\mathbf{Ch}_+^{\text{hb}}(\mathcal{A}) \subset \mathbf{Ch}^b(\mathcal{A})$ have the approximate lifting property. Also, the inclusions $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}_+^{\text{hb}}(\mathcal{A})$ and $\mathbf{Ch}_-^{\text{hb}}(\mathcal{A}) \subset \mathbf{Ch}^b(\mathcal{A})$ satisfy the dual of the approximate lifting property. Thus, we can apply Corollary 2.5 along with Theorem 2.2 to find that

$$K(\mathcal{A}) \simeq K \mathbf{Ch}^b(\mathcal{A}) \simeq K \mathbf{Ch}_-^{\text{hb}}(\mathcal{A}) \simeq K \mathbf{Ch}_+^{\text{hb}}(\mathcal{A}) \simeq K \mathbf{Ch}^{\text{hb}}(\mathcal{A}).$$

3 K -theory spectrum

Definition 3.1. A *symmetric spectrum* \mathbf{X} in topological spaces in a sequence of based Σ_n -spaces (X_n) endowed with structure maps $\sigma : X_n \wedge S^1 \rightarrow X_{n+1}$ such that $\sigma^k : X_n \wedge S^k \rightarrow X_{n+k}$ is $(\Sigma_n \times \Sigma_k)$ -equivariant for any $n, k \geq 0$, where $S^k \equiv \underbrace{S^1 \wedge \cdots \wedge S^1}_{k \text{ times}}$.

A map $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ of symmetric spectra is a sequence $(f_n : X_n \rightarrow Y_n)$ of based Σ_n -equivariant maps such that for each $n \geq 0$, the square

$$\begin{array}{ccc} X_n \wedge S^1 & \xrightarrow{f_n \wedge \text{Id}} & Y_n \wedge S^1 \\ \sigma \downarrow & & \downarrow \sigma \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

commutes. Let Sp^Σ denote the category of symmetric spectra in topological spaces.

Definition 3.2. Let $(\mathcal{C}, w\mathcal{C})$ be a Waldhausen category. The *external n -fold S_\bullet -construction* on \mathcal{C} is the n -multisimplicial Waldhausen category

$$(S_\bullet \cdots S_\bullet \mathcal{C}, wS_\bullet \cdots S_\bullet \mathcal{C}).$$

In multidegree (q_1, \dots, q_n) , it has as objects the diagrams $X : \text{Ar}[q_1] \times \cdots \times \text{Ar}[q_n] \rightarrow \mathcal{C}$ such that

- (i) $X((i_1, j_1), \dots, (i_n, j_n)) = *$ when $i_k = j_k$ for some $1 \leq k \leq n$ and
- (ii) $X(\dots, (i_t, j_t), \dots) \rightrightarrows X(\dots, (i_t, k_t), \dots) \rightarrow X(\dots, (j_t, k_t), \dots)$ is a cofiber sequence in the $(n-1)$ -fold iterated S_\bullet -construction for any $i_t \leq j_t \leq k_t$ in $[q_t]$.

Definition 3.3. Let $(\mathcal{C}, w\mathcal{C})$ be a Waldhausen category. The *internal n -fold S_\bullet -construction* on \mathcal{C} is the simplicial Waldhausen category

$$(S_\bullet^{(n)} \mathcal{C}, wS_\bullet^{(n)} \mathcal{C}).$$

It has as q -simplices the functor categories $(S_q \cdots S_q \mathcal{C}, wS_q \cdots S_q \mathcal{C})$ whose objects are precisely the $(\text{Ar}[q])^n$ -shaped diagrams $X : (\text{Ar}[q])^n \rightarrow \mathcal{C}$ such that

- (i) $X((i_1, j_1), \dots, (i_n, j_n)) = *$ when $i_k = j_k$ for some $1 \leq k \leq n$.
- (ii) $X(\dots, (i_t, j_t), \dots) \rightrightarrows X(\dots, (i_t, k_t), \dots) \rightarrow X(\dots, (j_t, k_t), \dots)$ is a cofiber sequence in the $(n-1)$ -fold iterated S_\bullet -construction for any $i_t \leq j_t \leq k_t$ in $[q]$.

Note that Σ_n acts on $S_\bullet^{(n)} \mathcal{C}$ by the relation $(\pi \cdot X)(\dots, (i_t, j_t), \dots) = X(\dots, (i_{\pi^{-1}(t)}, j_{\pi^{-1}(t)}), \dots)$.

The *(symmetric) algebraic K -theory spectrum* $\mathbf{K}(\mathcal{C}, w)$ of a small Waldhausen category $(\mathcal{C}, w\mathcal{C})$ has n -th space

$$K(\mathcal{C}, w)_n \equiv \left| wS_\bullet^{(n)} \mathcal{C} \right|$$

based at $*$. There is a Σ_n -action on $K(\mathcal{C}, w)_n$ induced by permuting the order of the internal S_\bullet -constructions. Moreover, we have that

$$\left| wS_\bullet^{(n)}\mathcal{C} \right| \wedge S^1 \cong \left| wS_\bullet^{(n)}S_\bullet\mathcal{C} \right|^{(1)} \subset \left| wS_\bullet^{(n)}S_\bullet\mathcal{C} \right| \cong \left| wS_\bullet^{(n+1)}\mathcal{C} \right|$$

where $^{(1)}$ denotes the 1-skeleton with respect to the last simplicial direction. This determines the structure map σ .

Note 3.4. σ^k is $(\Sigma_n \times \Sigma_k)$ -invariant.

Theorem 3.5. *For any $i \geq 0$, we have that $K_i(\mathcal{C}, w) = \pi_{i+1}K(\mathcal{C}, w)_1 \cong \pi_i\mathbf{K}(\mathcal{C}, w)$.*¹

This enables us to encode our algebraic K -theory in an infinite loop space.

¹Lemma 8.7.4 (Rognes).