

### Abstract

We begin low-dimensional  $K$ -theory, which consists of the groups  $K_0(-)$ ,  $K_1(-)$ , and  $K_2(-)$ . Specifically, we describe  $K_0$  for rings and for topological spaces. The main sources for this talk are the following.

- $n\text{Lab}$ .
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Chapters I and II.
- Eric M. Friedlander's *An Introduction to K-theory*, Chapter 1.

## 1 $K_0$ for rings

The forgetful functor  $U : \mathbf{Ab} \rightarrow \mathbf{CMon}$  admits a left adjoint  $K : \mathbf{CMon} \rightarrow \mathbf{Ab}$ , called the *group completion* functor. Specifically, for any commutative monoid  $(C, +)$ , we call the abelian group  $K(C)$  the *Grothendieck group of  $C$* , which is constructed as follows.

Consider  $S := C \times C / \sim$  where  $(a_1, b_1) \sim (a_2, b_2)$  if

$$a_1 + b_2 + k = b_1 + a_2 + k$$

for some  $k \in C$ . Note that  $\sim = \sim'$  where  $(a_1, b_1) \sim' (a_2, b_2)$  if

$$(a_1 + k_1, b_1 + k_1) = (a_2 + k_2, b_2 + k_2)$$

for some  $(k_1, k_2) \in C \times C$ . Now set  $K(C) = (S, +)$ , where  $+$  is inherited from  $C$  and acts componentwise on equivalence classes. Our definition of  $\sim'$  makes it clear that  $[a_1, b_1]^{-1} = [b_1, a_1]$ .

**Proposition 1.1.** *The inclusion  $C \hookrightarrow K(C)$  given by  $x \mapsto [x] := [x, 0]$  is injective iff  $C$  is a cancellation monoid.*

**Lemma 1.2 (Universal property of the  $K$ ).** *Let  $B$  be an abelian group and  $f : A \rightarrow B$  a monoid homomorphism. Then we have*

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow f & \\ K(A) & \dashrightarrow & B \end{array} \quad \begin{array}{c} \exists ! \tilde{f} \end{array}$$

*Proof.* Define  $\tilde{f}$  by  $[a_1, b_1] \mapsto f(a_1) - f(b_1)$ . □

**Lemma 1.3.**  $K(C_1 \times C_2) \cong K(C_1) \times K(C_2)$ .

**Definition 1.4.** A submonoid  $L$  of  $C$  is *cofinal* if for any  $c \in C$ , there is some  $c' \in C$  such that  $c + c' \in L$ .

**Proposition 1.5.** *Let  $L$  be cofinal in commutative  $C$ .*

1. Any element of  $K(C)$  can be written as  $[m] - [n]$  for some  $m, n \in C$ .

2.  $K(L) \leq K(C)$ .
3. Any element of  $K(C)$  can be written as  $[m] - [l]$  for some  $m \in C$  and  $l \in L$ .
4. If  $[m] = [m']$ , then  $m + l = m' + l$  for some  $l \in L$ .

**Example 1.6.**

1.  $K(\mathbb{N}) \cong \mathbb{Z}$  via the map  $[a_1, b_1] \mapsto a_1 - b_1$ .
2.  $K(\mathbb{Z}^\times) \cong \mathbb{Q}^\times$  via the map  $[a_1, b_1] \mapsto \frac{a_1}{b_1}$ .

Let  $R$  be a unital ring. Let  $(\mathbf{P}(R), \oplus, \otimes_R)$  denote the semiring of (isomorphism classes of) finitely generated projective  $R$ -modules. Let  $K_0(R) = K(\mathbf{P}(R))$ .

**Lemma 1.7.**  $\mathbf{P}(R_1 \times R_2) \cong \mathbf{P}(R_1) \times \mathbf{P}(R_2)$ . Therefore,  $K_0$  can be computed componentwise by Lemma 1.3.

$K_0(-)$  defines a functor from **Ring** to **Ab**. Let  $f : R \rightarrow S$  be a ring homomorphism and  $P$  be a finitely generated projective  $R$ -module. Define the group map  $K_0(f)$  as follows.

1. Construct  $S \otimes_R P$ , the base extension of  $P$ . This is the *unique*  $S$ -module compatible with the  $R$ -module structure on  $S$  induced by  $f$ , and its action is given by

$$(s', s \otimes p) \mapsto s' s \otimes p.$$

This is also an  $R$ -module with  $f(r) \cdot t := r \cdot t$  for  $t \in S \otimes_R P$ . We know that  $P \oplus Q$  is free for some  $R$ -module  $Q$ . Since  $S \otimes_R (P \oplus Q) \cong_S (S \otimes_R P) \oplus (S \otimes_R Q)$  and  $P \oplus Q$  is free over  $S$  via  $f$ , it follows that  $S \otimes_R P$  is a finitely generated projective  $S$ -module.

2. We've just defined a monoid homomorphism  $\tilde{f} : \mathbf{P}(R) \rightarrow \mathbf{P}(S)$ .
3. Apply the universal property of  $K$  to find the filler

$$\begin{array}{ccc} \mathbf{P}(R) & \xrightarrow{\tilde{f}} & \mathbf{P}(S) \\ \downarrow & & \downarrow \\ K(\mathbf{P}(R)) & \xrightarrow{f_*} & K(\mathbf{P}(S)) \end{array},$$

and set  $K_0(f) = f_*$ .

**Theorem 1.8 (Eilenberg swindle).** Suppose  $P \oplus Q = R^n$  as  $R$ -modules. Then

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong R^\infty.$$

Therefore, if we added  $R^\infty$  to  $\mathbf{P}(R)$ , then we would have  $[P] = 0$  for each finitely generated projective  $P$ .

**Example 1.9.** If  $R$  is a field, then  $\mathbf{P}(R) \cong \mathbb{N}$  and, by Example 1.6,  $K_0(R) \cong \mathbb{Z}$ .

We can generalize this phenomenon a bit.

**Definition 1.10.** A ring  $R$  has the *invariant basis property (IBP)* if  $R^n \not\cong R^m$  whenever  $n \neq m$ .

Note that any commutative ring has the IBP.

**Definition 1.11.** An  $R$ -module  $P$  is *stably free of rank  $n - m$*  if  $P \oplus R^m \cong R^n$ .

**Lemma 1.12.** The map  $f : \mathbb{N} \rightarrow \mathbf{P}(R)$  defined by  $n \mapsto R^n$  induces a homomorphism  $\phi : \mathbb{Z} \rightarrow K_0(R)$ .

1.  $\phi$  is injective iff  $R$  has the IBP.
2. Suppose  $R$  has IBP. Then  $K_0(R) \cong \mathbb{Z}$  iff every finitely generated projective  $R$ -module is stably free.

*Proof.*

1. By Proposition 1.5(4), we know that  $[P] = [Q]$  in  $K_0(R)$  iff  $P \oplus R^m \cong Q \oplus R^m$  for some  $m$ .
2.  $[P] = [R^n]$  iff  $P$  is stably free.

□

**Example 1.13.** Suppose that  $R$  is commutative. There is a ring homomorphism  $R \rightarrow F$  with  $F$  a field. Then the induced map  $K_0(R) \rightarrow K_0(F) \cong \mathbb{Z}$  sends  $[R]$  to 1. Also, the map  $\phi : \mathbb{Z} \rightarrow K_0(R)$  is injective by Lemma 1.12. With  $K := \ker(K_0(R) \rightarrow \mathbb{Z})$ , we get a split exact sequence of abelian groups

$$1 \longrightarrow K \longrightarrow K_0(R) \longrightarrow \mathbb{Z} \longrightarrow 1 ,$$

so that  $K_0(R) \cong \mathbb{Z} \oplus K$ .

**Example 1.14.** A ring  $R$  is a *flasque* if there exist an  $R$ -bimodule  $M$  which is also a finitely generated projective on one side and a bimodule isomorphism  $R \oplus M \cong M$ . In this case, since

$$P \oplus (P \otimes_R M) \cong P \otimes_R (R \oplus M) \cong P \otimes_R M,$$

we see that  $K_0(R) = 0$ .

**Example 1.15.** A module is *semisimple* if it is the direct sum of simple modules. A ring  $R$  is *semisimple* if it is a semisimple  $R$ -module. Notice that any semisimple module is both Noetherian and Artinian and that any module over a semisimple ring is semisimple.

Suppose  $R$  is semisimple with summands  $V_1, \dots, V_m$ . Then any finitely generated  $R$ -module has the form  $\bigoplus_{i=1}^m V_i^{l_i}$ , where each integer  $l_i$  is uniquely determined thanks to Krull-Remak-Schmidt. Hence  $\mathbf{P}(R) \cong \mathbb{N}^m$ , and  $K_0(R) \cong \mathbb{Z}^m$ .

**Example 1.16.** A ring  $R$  is *von Neumann regular* if

$$(\forall r \in R)(\exists x_r \in R)(rx_r r = r).$$

It turns out that any one-sided ideal in  $R$  is generated by an idempotent element. Let  $E/\sim$  denote the set of idempotent elements in  $R$  modulo the equivalence relation where  $e_1 \sim e_2$  if the two generate the same ideal. Then  $E/\sim$  forms a lattice where the join and meet correspond to the addition and intersection of ideals, respectively.

Kaplansky (1998) proved that any projective  $R$ -module is some direct sum of  $(e)$  with  $e$  idempotent. It follows that  $E/\sim$  determines  $K_0(R)$ .

**Proposition 1.17.** *Let  $R$  be commutative. TFAE*

1.  $R_{\text{red}}$  is a commutative von Neumann regular ring.
2.  $R$  has (Krull) dimension 0.
3.  $\text{Spec}(R)$  is compact, Hausdorff, and totally disconnected.

**Lemma 1.18.** *If  $I \subset R$  is nilpotent, then it's not hard to show that  $\mathbf{P}(R/I) \cong \mathbf{P}(R)$ , hence  $K_0(R) \cong K_0(R/I)$ .*

**Definition 1.19.** Let  $R$  be a commutative ring. The *rank* of a finitely generated projective  $R$ -module  $P$  at a prime ideal  $\mathfrak{p}$  is the function

$$\text{rk} : \text{Spec}(R) \rightarrow \mathbb{N}, \quad \mathfrak{p} \mapsto \dim_{R_{\mathfrak{p}}}(P \otimes R_{\mathfrak{p}}).$$

**Proposition 1.20.** *The rank of a finitely generated projective module is*

1. continuous and
2. a semiring homomorphism.

**Definition 1.21.** An  $R$ -module  $M$  is a *componentwise free module* if we have  $R = \prod_{i=1}^n R_i$  and  $M \cong \prod_{i=1}^n R_i^{c_i}$  for some integers  $c_i$ .

Note that  $M$  must be projective in this case.

**Lemma 1.22.** *Let  $R$  be commutative. The monoid  $L$  of finitely generated componentwise free  $R$ -modules has is isomorphic to  $[\text{Spec}(R), \mathbb{N}]$ .*

*Proof.* Let  $f : \text{Spec}(R) \rightarrow \mathbb{N}$  be continuous. By some point-set topology, we see that  $\text{im } f$  is finite, say  $\{n_1, \dots, n_c\}$ . It's also possible to write  $R = R_1 \times \dots \times R_c$ . Then  $R^f := R_1^{n_1} \times \dots \times R_c^{n_c}$  is a finitely generated componentwise free  $R$ -module. Moreover,  $f \mapsto R^f$  has inverse  $\text{rk}$  restricted to componentwise free modules.  $\square$

**Theorem 1.23 (Pierce).** *If  $R$  is a 0-dimensional commutative ring, then*

$$K_0(R) \cong [\text{Spec}(R), \mathbb{Z}],$$

where  $[X, Y]$  denotes the semiring of continuous maps  $f : X \rightarrow Y$ .

*Proof.* We have that  $R_{\text{red}}$  is a commutative von Neumann regular ring by Proposition 1.17. Any ideal  $(d)$  in  $R_{\text{red}}$  where  $d$  is idempotent is componentwise free. By Kaplansky, every object  $X$  of  $\mathbf{P}(R)$  is therefore componentwise free. Therefore,  $\mathbf{P}(R_{\text{red}}) \cong [\text{Spec}(R_{\text{red}}), \mathbb{N}]$ , giving  $K_0(R_{\text{red}}) \cong [\text{Spec}(R_{\text{red}}), \mathbb{Z}]$ . By Lemma 1.18 and the fact that  $\text{Spec}(R_{\text{red}})$  is homeomorphic to  $\text{Spec}(R)$ , it follows that  $K_0(R) \cong [\text{Spec}(R_{\text{red}}), \mathbb{Z}] \cong [\text{Spec}(R), \mathbb{Z}]$ .  $\square$

When  $R$  is commutative, let  $H_0(R) := [\text{Spec}(R), \mathbb{Z}]$ . If  $R$  is Noetherian, then  $H_0(R) \cong \mathbb{Z}^c$  where  $c < \infty$  denotes the number of components of  $H_0(R)$ . If  $R$  is a domain, then  $H_0(R)$  is connected, implying  $H_0(R) \cong \mathbb{Z}$ .

The submonoid  $L \subset \mathbf{P}(R)$  of componentwise free modules is cofinal, so that  $K(L) \leq K_0(R)$ . Moreover,  $K(L) \cong H_0(R)$  by Lemma 1.22.

The rank of a projective module induces a homomorphism  $\text{rank} : K_0(R) \rightarrow H_0(R)$ . Since  $\text{rank}(R^f) = f$  for any  $R^f \in L$ , we see that

$$1 \longrightarrow H_0(R) \cong K(L) \hookrightarrow K_0(R) \xrightarrow{\text{rank}} H_0(R) \longrightarrow 1$$

splits. This implies that

$$K_0(R) \cong H_0(R) \oplus \tilde{K}_0(R),$$

where  $\tilde{K}_0(R)$  denotes  $\ker(\text{rank})$ .

**Example 1.24.** The *Whitehead group* of a group  $G$  is the quotient

$$\text{Wh}_0(G) \equiv K_0(\mathbb{Z}[G]) / \mathbb{Z},$$

where  $\mathbb{Z}[G]$  denotes the group ring of  $G$  over  $\mathbb{Z}$ . The augmentation map  $f : \mathbb{Z}[G] \rightarrow \mathbb{Z}$  induces a split exact sequence

$$1 \longrightarrow \text{Wh}_0(G) \longrightarrow K_0(\mathbb{Z}[G]) \longrightarrow \underbrace{K_0(\mathbb{Z})}_{\mathbb{Z}} \longrightarrow 1.$$

Hence  $K_0(\mathbb{Z}[G]) \cong \mathbb{Z} \oplus \text{Wh}_0(G)$ . Due to Theorem 2.10, if  $G$  is finite, then  $\text{Wh}_0(G) \cong \tilde{K}_0(\mathbb{Z}[G])$  and  $\mathbb{Z} \cong H_0(\mathbb{Z})$ .

**Definition 1.25.**

1. A category  $\mathcal{C}$  is *preadditive* if each of its hom-sets is an abelian group.
2. A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  of preadditive categories is *additive* if  $F : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$  is a group homomorphism for any  $X, Y \in \text{ob } \mathcal{C}$ .

**Definition 1.26.** The rings  $R$  and  $S$  are *Morita equivalent* if there exists an additive equivalence between  $\mathbf{Mod}_R$  and  $\mathbf{Mod}_S$ .

**Theorem 1.27.** *If  $R$  and  $S$  are Morita equivalent, then  $K_0(R) \cong K_0(S)$ .*

Our results thus far can be extended to symmetric monoidal categories because these come equipped with a notion of direct sum that enabled our Grothendieck construction.

**Definition 1.28.** A *symmetric monoidal category*  $S$  is equipped with a functor  $\square : S \times S \rightarrow S$ , a base object  $e$ , and four natural isomorphisms expressing commutativity, associativity, and that  $e$  acts as an identity. These four isomorphisms must also satisfy certain coherence properties.

**Example 1.29.** The following are examples of symmetric monoidal category .

1.  $k$ -vector spaces with  $\otimes_k$ .
2. Any category with finite coproducts where  $s \square t := s \amalg t$ .
3. The category of pointed topological spaces where  $s \square t := s \wedge t$  and  $e := S^0$ .

Suppose that the class of isomorphism classes of objects of a category  $S$  is a set and denote it by  $S^{\text{iso}}$ . If  $S$  is symmetric monoidal, then  $(S^{\text{iso}}, \square)$  is an abelian monoid with identity element  $e$ . In this case, we define the *Grothendieck group* of  $S$  as  $K_0(S)$ .

## 2 Topological $K$ -theory

**Definition 2.1.** Let  $f : F \rightarrow X$  and  $g : G \rightarrow X$  be vector bundles. The *Whitney sum* of  $f$  and  $g$  is the vector bundle  $F \oplus G$  on  $X$  whose fiber at  $x \in X$  is  $F_x \oplus G_x$ . The *tensor product bundle*  $F \otimes G$  is defined similarly.

**Definition 2.2.** A *vector bundle homomorphism* between  $\phi : E_1 \rightarrow X_1$  and  $\psi : E_2 \rightarrow X_2$  is a pair of maps  $f : E_1 \rightarrow E_2$  and  $g : X_1 \rightarrow X_2$  such that

(i) the square

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \phi \downarrow & & \downarrow \psi \\ X_1 & \xrightarrow{g} & X_2 \end{array}$$

commutes and

(ii) for each  $x \in X_1$ , the map  $f|_{\phi^{-1}(x)} : \phi^{-1}(x) \rightarrow \psi^{-1}(g(x))$  is linear.

**Definition 2.3 (Topological  $K$ -groups).** Let  $(\mathbf{Vect}_{\mathbb{F}}(X), \oplus)$  denote the abelian monoid of (isomorphism classes of)  $\mathbb{F}$ -vector bundles on a paracompact space  $X$ .

- $KU(X) \equiv K(\mathbf{Vect}_{\mathbb{C}}(X))$
- $KO(X) \equiv K(\mathbf{Vect}_{\mathbb{R}}(X))$ .

Note that these are commutative rings with identity.

**We apply the notation  $K_{\text{top}}(-)$  to topological spaces when we wish to omit the base field.**

Both  $KU(-)$  and  $KO(-)$  define contravariant functors  $\mathbf{Top} \rightarrow \mathbf{Ab}$ . Let  $f : Y \rightarrow X$  be a map of spaces and  $\phi : E \rightarrow X$  be a vector bundle. Recall the pullback  $f^*E = \{(y, e) \in Y \times E : f(y) = \phi(e)\}$  of  $E$  in  $\mathbf{Top}$ . Define the vector bundle  $f^*(\phi) : f^*E \rightarrow Y$  as the appropriate restriction of the projection map  $\pi : Y \times E \rightarrow E$ . The assignment  $\phi \mapsto f^*(\phi)$  defines a morphism  $\mathbf{Vect}_{\mathbb{F}}(X) \rightarrow \mathbf{Vect}_{\mathbb{F}}(Y)$  of monoids. In turn, the universal property of  $K$  induces a unique morphism  $f^* : K_{\text{top}}(X) \rightarrow K_{\text{top}}(Y)$ .

**Lemma 2.4.** *If  $X$  and  $Y$  are homotopy equivalent, then  $K(X) \cong K(Y)$ .*

*Proof.* Apply the homotopy invariance theorem (HIT), which states that if  $Y$  is paracompact and  $f, g : Y \rightarrow X$  are homotopic, then  $f^*E \cong g^*E$  for any vector bundle  $E$  over  $X$ .  $\square$

**Example 2.5.**

1.  $K_{\text{top}}(*) = \mathbb{Z}$ .
2. If  $X$  is contractible, then the HIT implies that  $KO(X) = KU(X) = \mathbb{Z}$

3. According to I.4.9 of *The K-book*, we have

$$\begin{aligned} KO(S^1) &\cong \mathbb{Z} \times C_2 \\ KU(S^1) &\cong \mathbb{Z} \\ KO(S^2) &\cong \mathbb{Z} \times C_2 \\ KU(S^2) &\cong \mathbb{Z} \times \mathbb{Z} \\ KO(S^3) &\cong KU(S^3) \cong \mathbb{Z} \\ KO(S^4) &\cong KU(S^4) \cong \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

**Definition 2.6.** The *dimension* of bundle  $E$  over  $X$  is the continuous homomorphism  $\widehat{\dim}(E) : X \rightarrow \mathbb{N}$  given by  $x \mapsto \dim(E_x)$ .

**Definition 2.7.** A vector bundle  $p : E \rightarrow X$  is a *componentwise trivial bundle* if we can write  $X = \coprod X_i$  such that each  $X_i$  is a component of  $X$  and  $p|_{p^{-1}(X_i)}$  is trivial.

**Lemma 2.8.** *The submonoid of componentwise trivial bundles over  $X$  is isomorphic to  $[X, \mathbb{N}]$ .*

*Proof.* Send a given map  $f : X \rightarrow \mathbb{N}$  to  $T^f := \coprod_{i \in \mathbb{N}} (f^{-1}(i) \times \mathbb{F})$ . Conversely, if  $E$  is a componentwise trivial bundle, then  $E \cong T^{\widehat{\dim}(E)}$ .  $\square$

Thus, the sub-monoid of trivial bundles and the sub-monoid of componentwise trivial bundles are naturally isomorphic to  $\mathbb{N}$  and  $[X, \mathbb{N}]$ , respectively. When  $X$  is compact, these are cofinal in  $\mathbf{Vect}_{\mathbb{F}}(X)$  by the subbundle theorem (proven using Riemannian geometry), yielding the relations

$$\mathbb{Z} \leq [X, \mathbb{Z}] \leq K_{\text{top}}(X).$$

**Note 2.9.**

1. We have a split exact sequence.

$$1 \longrightarrow \widetilde{K}_{\text{top}}(X) \longrightarrow K_{\text{top}}(X) \xrightarrow[\widehat{\dim}]{\quad \quad} [X, \mathbb{Z}] \longrightarrow 1,$$

where  $\widetilde{K}_{\text{top}}(X)$  denotes  $\ker(\widehat{\dim})$ .

2. The map of monoids  $\mathbf{Vect}_{\mathbb{R}}(X) \rightarrow \mathbf{Vect}_{\mathbb{C}}(X)$  given by  $[E] \mapsto [E \otimes \mathbb{C}]$  extends by universality to a homomorphism  $KO(X) \rightarrow KU(X)$ . Likewise, the forgetful functor  $\mathbf{Vect}_{\mathbb{C}}(X) \rightarrow \mathbf{Vect}_{\mathbb{R}}(X)$  extends to a homomorphism  $KU(X) \rightarrow KO(X)$ .

To state a nice early connection between algebraic and topological  $K$ -theory, let  $X$  be a compact Hausdorff space and  $\mathcal{C}(X, \mathbb{F})$  denote the ring of continuous functions  $X \rightarrow \mathbb{F}$ . For any vector bundle  $p : E \rightarrow X$  over  $\mathbb{F}$ , set

$$\Gamma(X, E) = \{s : X \rightarrow E : p \circ s = \text{Id}_X\},$$

the vector space of global sections of  $E$ .

**Theorem 2.10 (Swan).** *The map  $E \mapsto \Gamma(X, E)$  induces isomorphisms  $KO(X) \cong K_0(\mathcal{C}(X, \mathbb{R}))$  and  $KU(X) \cong K_0(\mathcal{C}(X, \mathbb{C}))$ .*