

Abstract

These notes are based on Scott Weinstein’s “Model Theory” lectures at UPenn along with David Marker’s *Model Theory: An Introduction*. Any mistake in what follows is my own.

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1 Introduction

1.1 Lecture 1

Recall the structure $\mathbb{N} := \langle \omega, \mathbf{S}, 0 \rangle$ where

- ω denotes the set of natural numbers $\{0, 1, 2, \dots\}$,
- \mathbf{S} is interpreted as the successor function $\omega \rightarrow \omega$, and
- the constant symbol 0 is interpreted as the natural number 0 .

The formal language \mathcal{L} for which \mathbb{N} is a structure consists of the first-order (FO) logical symbols

$$\forall, \exists, \wedge, \neg, \vee, \rightarrow, =$$

along with non-logical symbols such as 0 , $\mathbf{S}^n 0 := \underbrace{\mathbf{S} \cdots \mathbf{S}}_{n\text{-copies}} 0$, and $\mathbf{S}^n x$. Let FO denote the set of all (first-order) \mathcal{L} -sentences.

The *theory of \mathbb{N}* is

$$\text{Th}(\mathbb{N}) := \{\varphi \in \text{FO} \mid \mathbb{N} \models \varphi\},$$

which consists of all sentences satisfied by \mathbb{N} . Further, for any subset $\Sigma \subset \text{FO}$, consider the set

$$\text{Cn}(\Sigma) := \{\varphi \in \text{FO} \mid \Sigma \models \varphi\}$$

of consequences of Σ .

Question. Can we find a theory Σ (other than $\text{Th}(\mathbb{N})$) such that $\text{Cn}(\Sigma) = \text{Th}(\mathbb{N})$?

Let $\Delta = \{\forall x (\mathbf{S}x \neq 0), \forall x \forall y (\mathbf{S}x = \mathbf{S}y \rightarrow x = y), \forall x (x \neq 0 \rightarrow \exists y (\mathbf{S}y = x))\}$. Each element of Δ is clearly true in \mathbb{N} , i.e., $\mathbb{N} \models \Delta$. But is it the case that $\text{Cn}(\Delta) = \text{Th}(\mathbb{N})$? No, provided that we allow ourselves access to monadic second-order sentences. Specifically, consider the *induction axiom* **IA**:

$$\forall P ((P(0) \wedge \forall x (P(x) \rightarrow P(\mathbf{S}x)) \rightarrow \forall x (P(x))). \quad (*)$$

This is clearly true in \mathbb{N} . Consider, however, a new structure $\mathbb{A} := \langle \omega \cup \mathbb{Z}, \mathbf{S}, 0 \rangle$. Then $\Delta \subset \text{Th}(\mathbb{A})$, and we have a \mathbb{Z} -chain in \mathbb{A} (pretending, for the moment, that the universe $|\mathbb{A}|$ has the usual order $<$):

$$\dots \xrightarrow{\quad} -(n+1)^{\mathbb{A}} \xrightarrow{\quad} -n^{\mathbb{A}} \xrightarrow{\quad} \dots \xrightarrow{\quad} -1^{\mathbb{A}} \xrightarrow{\quad} 0^{\mathbb{A}} \xrightarrow{\quad} 1^{\mathbb{A}} \xrightarrow{\quad} \dots$$

The second-order sentence $(*)$ with P instantiated by the “initial segment” $\mathbb{Z}_{\geq -1}$ is not true in \mathbb{A} , so that $\mathbb{A} \not\models \text{IA}$. In this case, $\text{IA} \in \text{Th}(\mathbb{N}) \setminus \text{Cn}(\Delta)$.

Nevertheless, we want to restrict ourselves to FO. Recall that two structures \mathbb{B} and \mathbb{C} are *elementarily equivalent* if $\text{Th}(\mathbb{B}) = \text{Th}(\mathbb{C})$.

Question. Are \mathbb{A} and \mathbb{N} elementarily equivalent?

If we can find some sentence belonging to $\text{Th}(\mathbb{N}) \setminus \text{Th}(\mathbb{A})$, then $\text{Cn}(\Delta) \neq \text{Th}(\mathbb{N})$.

Definition 1.1.1. A theory Σ is *categorical* if for any structures \mathbb{B} and \mathbb{C} , if $\mathbb{B} \models \Sigma$ and $\mathbb{C} \models \Sigma$, then $\mathbb{B} \cong \mathbb{C}$.

Example 1.1.2. $\Delta' := \Delta + \text{IA}$ is categorical.

Perhaps exhibiting that the usual order $<$ on ω is definable in \mathbb{N} would reveal that $\mathbb{A} \neq \mathbb{N}$. For this, we must find a (well-formed) formula $\theta(x, y)$ such that for every $n, m \in \omega$,

$$m < n \iff \mathbb{N} \models \theta[n, m].$$

Thanks to Lagrange's four square theorem, we could define $<$ on the positive integers. But it's unclear how to proceed further.

Theorem 1.1.3. *If \mathbb{B} is infinite, then for every infinite cardinal κ , there is some \mathbb{C} such that $\mathbb{C} \equiv \mathbb{B}$ and $\text{card}(\mathbb{C}) = \kappa$.*

Corollary 1.1.4. *If \mathbb{B} is infinite, then there exists a \mathbb{C} such that $\mathbb{C} \equiv \mathbb{B}$ and $\mathbb{C} \not\equiv \mathbb{B}$.*

Therefore, Δ does *not* categorically describe \mathbb{N} . Now, consider the structure $\tilde{\mathbb{N}}$ obtained from \mathbb{N} by adding a single point \bullet fixed by \mathbf{S} . Then the sentence $\forall x (Sx \neq x)$ is true in \mathbb{N} but not in $\tilde{\mathbb{N}}$. Moreover, $\Delta \subset \text{Th}(\tilde{\mathbb{N}})$, which proves that

$$\text{Cn}(\Delta) \neq \text{Th}(\mathbb{N}).$$

With this in mind, let $\Sigma = \Delta \cup \{\forall x (S^n x \neq x) \mid n \in \omega\}$. To show that $\text{Cn}(\Sigma) = \text{Th}(\mathbb{N})$, it suffices to show that for any \mathbb{B} , if $\mathbb{B} \models \Sigma$, then $\mathbb{B} \equiv \mathbb{N}$. To this end, for any cardinal κ , let $\mathbb{A}_\kappa = \mathbb{N} \cup (\kappa \times \mathbb{Z})$, which is precisely the structure obtained from \mathbb{N} by adding κ many disjoint \mathbb{Z} -chains.

Question. How many structures that

- (a) satisfy Σ and
- (b) are of cardinality κ

are there up to isomorphism?

If $\kappa < \omega$, then $\text{card}(\mathbb{A}_\kappa) = \aleph_0$. Also, if $\kappa > \omega$, then $\text{card}(\mathbb{A}_\kappa) = \kappa$, so that Σ is κ -categorical, i.e., every structure satisfying both (a) and (b) is isomorphic to \mathbb{A}_κ .