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Abstract

These notes are based on Julius Shaneson's lectures for the course "Algebraic Topology, Part I" given at UPenn. Any mistake in what follows is my own.

Contents

1 Background material

1.1 Lecture 1

Here are the topics for the course:

- fiber bundles over cell complexes,
- spectral sequences,
- characteristic classes, and
- cobordism theory.

To start, let's review some basic concepts from homology theory.

Definition 1.1.1. A *(finite) cell complex* is a (topological) space X that can be written as $\bigcup_{n=0}^{K} X^n$ for some $K \in \mathbb{N}$ (called the *dimension of* X) where

- X^0 is chosen to be finite,
- $X^n = \frac{X^{n-1} \coprod D_1^n \coprod \cdots \coprod D_{k_n}^n}{x \sim \varphi_i(x)}$,
- $D_i^n \cong D^n \coloneqq \{x \in \mathbb{R}^n : |x| \leq 1\}$ for each $i \in \{1, \ldots, k_n\}$, and
- $\varphi_i : \partial D_i^n = S^{n-1} \to X^{n-1}$, called an *attaching map.*

Terminology. Each D_i^n is called an *n-cell of X*.

Every attaching map $\varphi_i : \partial D_i^n \to X^{n-1}$ can be extended to a *characteristic map* given by the composition

$$D_i^n \hookrightarrow X^{n-1} \coprod D_1^n \coprod \cdots \coprod D_{k_n}^n \twoheadrightarrow X^n \hookrightarrow X.$$

Example 1.1.2. There are at least two ways of endowing S^2 with a cell structure.

1. $X^0 \equiv \{N, S\}$, $X^1 \equiv X^0 \cup_{\varphi_1} D_1^1 \cup_{\varphi_2} D_2^1$ where each φ_i is an embedding, and $X^2 \equiv X^1 \cup_{\varphi_1'} D_1^2 \cup_{\varphi_2'} D_2^2$ where each φ_i' is an embedding.
2. $\{\mathsf{pt}\} \cup_{\varphi} D^2$ where φ identifies the equator of the upper half-sphere with pt.

Definition 1.1.3. A cell complex X is *regular* if every characteristic map $D_i^n \to X$ is an embedding.

Definition 1.1.4. Given a family of functors $\{H_n : \mathbf{Top}^2 \to \mathbf{Ab}\}_{n \in \mathbb{N}}$ where $\mathbf{Top}^2$ denotes the category of (topological) pairs, we say that H_i is a *homology functor* if each of the following properties holds.

1. (LES) For any pair (X, A) of space, there is a natural long exact sequence

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \longrightarrow \cdots,$$

   where $H_i(Z) \coloneqq H_i(Z, \emptyset)$ for any space Z.

2. (Excision) If $\mathrm{cl}(A) \subset \underset{open}{U} \subset X$, then $H_i(X \setminus A, U \setminus A) \cong H_i(X, U)$.
3. (Dimension) $H_i(\mathsf{pt}) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases}$.
4. (Homotopy) If f and g are homotopic, then $f_* = g_*$, where $h_* \coloneqq H_i(h)$ for any map $h : (X, A) \to (Y, B)$.

Theorem 1.1.5. *There exists a family of homology functors.*

Example 1.1.6. In singular homology theory, we have that $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$.

Let X be a cell complex. Let $C_n(X)$ denote the free abelian group on the set of all n-cells of X. Define $\partial : C_n(X) \to C_{n-1}(X)$ by $\partial[D_i^n] = \sum_{j=1}^{k_{n-1}} \lambda_{ij}[D_j^{n-1}]$ where λ_{ij} is defined, up to sign, as follows. Consider the map

$$\begin{tikzcd} S^{n-1} \arrow[r, equal] \arrow[rrrrr, bend right, "\omega"'] & \partial D_i^n \arrow[r, "\varphi_i"] & X^{n-1} \arrow[r, two heads] & \frac{X^{n-1}}{X^{n-2} \cup (\text{all cells of dim. } n-1 \text{ except } D_j^{n-1})} \arrow[r, equal] & D^{n-1} / \partial D_j^{n-1} \arrow[r, equal] & S^{n-1} \end{tikzcd}.$$

Then let λ_{ij} satsify $\omega_*(x) = \lambda_{ij}x$ with x a chosen generator (i.e., orientation) of $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Terminology. The integer λ_{ij} is called the *degree of* ω, denoted by $\deg(\omega)$.

Theorem 1.1.7. *$\partial_n \partial_{n+1} = 0$, and $H_n(X) \cong {\ker \partial_n} / {\operatorname{im} \partial_{n+1}}$, which is independent of our choice of generator x.*

Example 1.1.8. Suppose that $f : S^n \to S^n$ is smooth. By Sard's theorem, we can find a regular value $x \in S^n$. There is some neighborhood U of x such that $f^{-1}(U) = U_1 \cup \cdots \cup U_n$ for some n. Using the inverse function theorem and the compactness of S^n, it follows that f^{-1} is of the form $\{x_1, \ldots, x_n\}$. Note that the differential $(df)_{x_i} : S^n_{x_i} \to S^n_x$ satisfies $\det (df)_{x_i} - \pm 1$. In fact,

$$\deg(f) = \sum_{i=1}^{n} \det (df)_{x_i}.$$

Exercise 1.1.9. *Prove that any cell complex $X = \bigcup_{n=0}^{K} X^n$ is homotopy equivalent to a regular cell complex.*[1]

Proof. ?? □

1.2 Lecture 2

Example 1.2.1 (Real projective space). Recall that $\mathbb{RP}^n = S^n / x \sim -x$. Then $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\pi_{n-1}} D^n$ where $\pi_{n-1} : S^{n-1} \to \mathbb{RP}^{n-1}$ denotes the canonical projection. Thus, $\mathbb{RP}^n$ is an n-dimension cell complex with $(\mathbb{RP}^n)^m = \mathbb{RP}^m$ for each integer $0 \le m \le n$.

Now, for each $0 \le m \le n$, we have that $C_m(\mathbb{RP}^n) \cong \mathbb{Z}$ with generator $[D^m]$. To determine $\partial[D^m] \in C_{m-1}(\mathbb{RP}^m)$, we must find the degree of the map

$$\begin{tikzcd} S^{m-1} \arrow[r] \arrow[rrrr, bend right, "\varphi"'] & \mathbb{RP}^{m-1} \arrow[r, two heads] & \mathbb{RP}^{m-1} / \mathbb{RP}^{m-1} \arrow[r, equal] & D^{m-1} / \partial D^{m-1} \arrow[r, equal] & S^{m-1} \end{tikzcd}$$

Assume, for convenience, that $m = 2$. Choose a regular value $p \in S^1$ so that $\varphi^{-1}(p) = \{N, S\}$. Let φ_T and φ_B denote the restrictions of φ to the top and bottom components of $S^1 \setminus \{(-1, 0), (1, 0)\}$, respectively. Note that both of these are homeomorphisms and thus have degrees equal to ± 1. If $a : S^{m-1} \to S^{m-1}$ denotes the

[1]Hint: Consider the map $S^{n-1} \to X^{n-1} \times D^n$ given by $x \mapsto (\varphi(x), x)$.

antipodal map, we have that $\varphi_B \circ a = \varphi_T$. Hence $(d\varphi)_S \circ (da)_N = (d\varphi)_N$. Since $\deg(a) = \det(da) = (-1)^m$, it follows that

$$\deg(\varphi) = \begin{cases} \pm 2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}.$$

Thus, we get a chain complex

$$0 \longrightarrow C_n(\mathbb{RP}^n) \xrightarrow{\kappa_1} C_{n-1}(\mathbb{RP}^n) \xrightarrow{\kappa_2} \cdots \xrightarrow{0} C_2(\mathbb{RP}^n) \xrightarrow{\pm 2} C_1(\mathbb{RP}^n) \xrightarrow{0} C_0(\mathbb{RP}^n) \longrightarrow 0$$

where $\kappa_1 = \begin{cases} 0 & n \text{ odd} \\ \pm 2 & n \text{ even} \end{cases}$ and $\kappa_2 = \begin{cases} \pm 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$.

This proves that

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ 0 & i = 2 \\ \mathbb{Z}/2 & \underset{odd}{i} < n \\ 0 & \underset{even}{i} < n \\ 0 & i > n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & i = n \text{ even} \end{cases}.$$

Next, let's introduce some fundamental concepts from homotopy theory.

Definition 1.2.2. Let $M(X, Y)$ denote the set of maps $X \to Y$.

1. For any compact $C \subset X$ and open $U \subset Y$, let

$$N(C, U) = \{f : X \to Y \mid f(C) \subset U\}.$$

   The *compact-open topology on* $M(X, Y)$ consists of all unions of finite intersections of subsets of the form $N(C, U)$.

2. The *n-th loop space* of a pointed space (X, x) is

$$\Omega^{n-1}(X, x) \coloneqq M((D^{n-1}, \partial D^{n-1}), (X, x)),$$

   which is a subset of $M(D^{n-1}, X)$.

Definition 1.2.3 (Higher homotopy groups). If $n \geq 2$, then the *n-th homotopy group* of (X, x) is

$$\pi_n(X, x) \coloneqq \pi_1(\Omega^{n-1} X, e_x).$$

Recall that $\pi_1(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Grp}$. Also, $\Omega^{n-1}(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Top}$ defined on morphisms $f : (X, x) \to (Y, y)$ by post-composition with f. Therefore, it's easy to see that $\pi_n(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Grp}$ as well.

Notation. Let $f_* = \pi_n(f)$ for any $f : (X, x) \to (Y, y)$.

Proposition 1.2.4. *There is a homeomorphism $M(X \times Y, Z) \cong M(X, M(Y, Z))$ so long as Y is locally compact and Hausdorff.*

In particular, we have a composite

$$M(([0,1], \{0,1\}), (M((D^{n-1}, \partial), (X, x)), e_x)) \hookrightarrow M([0,1], M(D^{n-1}, X)) \xrightarrow{\cong} M([0,1] \times D^{n-1}, X),$$

whose image is precisely $M((D^n, \partial), (X, x)) \cong M((S^n, \mathsf{pt}), (X, x))$. This proves that $\pi_n(X, x)$ consists of all homotopy classes of maps $(I^n, \partial) \to (X, x)$ under the operation $[f] * [g] = [f * g]$ where

$$f * g(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq \frac{1}{2} \\ f(2t_1 - 1, t_2, \ldots, t_n) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Proposition 1.2.5. *If $n \geq 2$, then $\pi_n(X, x)$ is abelian.*

Remark 1.2.6. A map $f : S^{n-1} \to X$ is homotopic to the constant map if and only if there is some g such that

$$\begin{tikzcd} D^n \arrow[dr, "g"] & \\ S^{n-1} \arrow[u, hook] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

Theorem 1.2.7 (Whitehead). *If $f : X \to Y$ is a map of connected cell complexes, then f is a homotopy equivalence if and only if $f_* : \pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism for each $n \in \mathbb{N}$.*

1.3 Lecture 3

Definition 1.3.1. If $x \in A \subset X$, then the *n-th relative homotopy group* $\pi_n(X, A, x)$ consists of all homotopy classes of maps $(D^n, S^{n-1}, x_0) \to (X, A, x)$.

We see that

$$M((D^n, S^{n-1}, x), (X, A, x_0)) \cong M((I^n, I^{n-1} \times \{1\}, \underbrace{\partial I^n \setminus \mathrm{Int}(I^{n-1} \times \{1\})}_{\partial_0 I^n}), (X, A, x_0))$$

by considering the homeomorphism $(I^n/\partial_0 I^n, \partial I^n/\partial_0 I^n) \cong (D^n, S^{n-1})$. Therefore, $\pi_n(X, A, x)$ can be viewed as consisting of all homotopy classes of maps $(I^n, \partial I^n, \partial_0 I^n) \to (X, A, x)$.

Proposition 1.3.2.

1. *If $n \geq 2$, then $\pi_n(X, A, x)$ is, in fact, a group.*
2. *If $n \geq 3$, then $\pi_n(X, A, x)$ is abelian.*
3. *The sequence*

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(A, x) \arrow[r] & \pi_n(X, x) \arrow[r] & \pi_n(X, A, x) \arrow[r, "\partial"] & \pi_{n-1}(A, x) \arrow[dlll] \\ & \pi_{n-1}(X, x) \arrow[r] & \cdots \arrow[r] & \pi_2(X, A, x) & \end{tikzcd}$$

with $\partial[f] = [f \restriction_{I^{n-1}}]$ is exact.

Theorem 1.3.3 (Hurewicz). *Let $n \in \mathbb{N}_{\geq 2}$. If $\pi_i(X) = 0$ for each $i < n$, then $\pi_n(X) \cong H_n(X)$.*

Note 1.3.4. This result can't be improved in general. For example, $\pi_3(S^2) \cong \mathbb{Z}$, whereas $H_3(S^2) = 0$.

Let $A \subset X$ be a subcomplex. Recall that $H_i(X, A) \cong H_i(X/A.*)$ for each $i \geq 1$. But it is *not* the case that $\pi_i(X, A) \cong \pi_i(X/A.*)$, for otherwise $\pi_i(S^n) \cong \pi_i(D^n, S^{n-1}) \cong \pi_i(S^{n-1})$, which is known to be false exactly when $i > 2n - 2$.

Example 1.3.5. $\pi_4(S^3) \cong \mathbb{Z}/2 \ncong \pi_4(S^4)$.

Finally, let's review the notion of a fibration of spaces.

Recall that if $p : E \to B$ is a covering projection, then TFAE.

1. For any $f : Z \to B$, there exists a unique $\hat{f} : Z \to E$ such that $p \circ \hat{f} = f$.
2. $f_*(\pi_1(Z)) \subset p_*(\pi_1(E))$.

The existence of $\hat{f}$ follows from the fact that any covering space satisfies the homotopy lifting property.

Definition 1.3.6 (Fibration). Suppose that $p : E \to B$ is any map. We say that p is a *fibration* if it satisfies the homotopy lifting property, i.e., given a commutative square

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widehat{f_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "f"'] & B \end{tikzcd},$$

where X is a cell complex, there is some G such that

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widehat{f_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "f"'] \arrow[ur, "G"] & B \end{tikzcd}$$

commutes.

Theorem 1.3.7. *If $p : E \to B$ is a fibration with $e \in F \coloneqq p^{-1}(b)$, then*

$$p_* : \pi_i(E, F, e) \xrightarrow{\cong} \pi_i(B, b, b) = \pi_i(B, b).$$

Proof. Let $f : (I^n, \partial I^n) \to (B, b)$. To prove that p_* is surjective, it suffices to find some $G : (I^n, \partial I^n) \to (E, F)$ such that

$$\begin{tikzcd} \partial_0 I^n \arrow[r, two heads] \arrow[d, hook] & \{e\} \arrow[r, hook] & F \arrow[r, hook] & E \arrow[d, "p"] \\ I^{n-1} \times [0,1] \arrow[rrr, "f"'] \arrow[urrr, "G"] & & & B \end{tikzcd}$$

commutes, for in this case $[p \circ G'] = [f]$. Since p is a fibration, there is some G such that

$$\begin{tikzcd} I^{n-1} \times \{0\} \arrow[r, two heads] \arrow[d, hook] & \{e\} \arrow[r, hook] & F \arrow[r, hook] & E \arrow[d, "p"] \\ I^{n-1} \times [0,1] \arrow[rrr, "f"'] \arrow[urrr, "G'"] & & & B \end{tikzcd}$$

commutes. But $(I^n, \partial_0 I^n) \cong (I^n, I^{n-1} \times \{0\})$, and thus such a G' is enough. $\square$

Corollary 1.3.8. *The sequence*

$$\cdots \longrightarrow \pi_i(F, e) \longrightarrow \pi_i(E, e) \longrightarrow \pi_i(B, b) \xrightarrow{\partial} \pi_{i-1}(F, e) \longrightarrow \cdots$$

is exact.

Example 1.3.9.

1. Suppose that

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & B \times F \arrow[d, "\pi_B"] \\ X \times [0,1] \arrow[r, "f"] & B \end{tikzcd}$$

commutes. Then $\hat{f}(x,0) = (\hat{f}_1(x,0), \hat{f}_2(x,0))$ where $\hat{f}_1(x,0) = f(x,0)$. Let $G(X,t) = (f(x,t), \hat{f}_2(x,0))$. Then

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\hat{f}_0"] \arrow[d, hook] & B \times F \arrow[d, "\pi_B"] \\ X \times [0,1] \arrow[r, "f"] \arrow[ur, "G"] & B \end{tikzcd}$$

commutes, so that π_B is a fibration. (Moreover, $\pi_n(B \times F) \cong \pi_n(B) \times \pi_n(F)$.)

2. Let $A \subset X$ be a subcomplex. The map $\varphi : M(X,Y) \to M(A,Y)$ defined by $f \mapsto f\restriction_A$ is a fibration.

3. Define the *Hopf fibration* as the quotient map

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1\overline{z_1} + z_2\overline{z_2} = 1\} \twoheadrightarrow {S^3}\big/{x \sim -x} = \mathbb{CP}^1 = S^2.$$

Corollary 1.3.10. $\pi_3(S^3) \cong \pi_3(S^2)$.

Proof. Since we have an exact sequence

$$\pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) \ ,$$

it suffices to show that both $\pi_3(S^1)$ and $\pi_2(S^1)$ are trivial. To this end, note that since $\pi_1(S^k) = 0$ for every $k > 1$, we can always find, for any $f : S^k \to S^1$, a map $\hat{f}$ such that

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e^{2\pi i x}"] \\ S^k \arrow[r, "f"'] \arrow[ur, "\hat{f}"] & S^1 \end{tikzcd}$$

commutes. Thus, f is homotopic to the constant map. Since f was arbitrary, our proof is complete. $\square$

Definition 1.3.11. A map $p : E \to B$ is *locally trivial* if for any $b \in B$, there exist a neighborhood $U \ni b$ in B, a space F, and a homeomorphism $\varphi : p^{-1}(U) \xrightarrow{\cong} U \times F$ such that $\pi_U \circ \varphi = p\restriction_{p^{-1}(U)}$.

Theorem 1.3.12. *Any locally trivial map $p : E \to B$ is a fibration whenever B is a cell complex.*

Exercise 1.3.13. *Prove that the Hopf fibration is locally trivial.*

Proof. For each $k \in \{0,1\}$, let $U_k = \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_k \neq 0\}$. Then U_0 and U_1 form an open cover of $\mathbb{CP}^1$. Note that the preimage of U_k under the Hopf fibration q is precisely $\{(z_0, z_1) \in S^3 \mid z_k \neq 0\}$. Define $f : q^{-1}(U_k) \to U_k \times S^1$ by

$$(z_0, z_1) \mapsto \left([z_0, z_1], \frac{z_k}{|z_k|}\right).$$

This is clearly continuous. Further, define the map $g : U_k \times S^1 \to q^{-1}(U_k)$ by

$$\left([z_0, z_1], e^{i\theta}\right) \mapsto \frac{e^{i\theta}|z_k|}{z_k|(z_0, z_1)|}\,(z_0, z_1)\,.$$

Since U_k is a saturated open set, we have that the restriction of q to $q^{-1}(U_k)$ is a quotient map. But $g \circ q\restriction_{q^{-1}(U_k)}$ is continuous, so that g is also continuous by the characteristic property of quotient maps. Finally, it is easy to verify that g and f are inverses of each other and that $\pi_{U_I} \circ f = p\restriction_{q^{-1}(U_k)}$. $\square$

1.4 Lecture 4

Theorem 1.4.1. *Let $A \subset X$ be a subcomplex. Define $r : M(X,Y) \to M(A,Y)$ by $r(f) = f \restriction_A$. Then r is a fibration.*

Proof. We must fill any diagram of the form

$$\begin{tikzcd}
Z \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & M(X,Y) \arrow[d, "r"] \\
Z \times [0,1] \arrow[r, "f"'] \arrow[ur, dashed, "F"] & M(A,Y)
\end{tikzcd}.$$

It suffices to find a map $\overline{F}$ such that

$$\begin{tikzcd}
Z \times \{0\} \times X \arrow[r, "\hat{\hat{f}}"] \arrow[d, hook] & Y \arrow[d, equal] \\
Z \times [0,1] \times X \arrow[ur, "\overline{F}"] & Y \\
Z \times [0,1] \times A \arrow[u, hook] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

commutes for, in this case, we can set $F(z,t)(x) = \overline{F}(z,t,x)$.

Note 1.4.2. Suppose that such an $\overline{F}$ exists. Define $g : Z \times X \to Y$ by $g(z,x) = \hat{\hat{f}}(z,0,x)$. Define $h : Z \times X \times [0,1] \to Y$ by $H(z,x,t) = \overline{F}(z,t,x)$. Then

$$\begin{tikzcd}
Z \times X \times \{0\} \arrow[d, hook] \arrow[dr, "g"] & \\
Z \times X \times [0,1] \arrow[r, "H"] & Y \\
Z \times A \times [0,1] \arrow[u, hook] \arrow[ur, "K"'] &
\end{tikzcd}$$

commutes where $K(z,a,t) = \bar{f}(z,t,a)$. In the case where $Z = \{\mathsf{pt}\}$, this means that if $K : A \times [0,1] \to Y$ is a homotopy from a map $f : A \to Y$ and g extends f to X, then there exists a homotopy $H : X \times [0,1] \to Y$ such that $H \restriction_{A \times [0,1]} = K$. In other words, the extension problem for cell complexes is a homotopy problem.

Let's return to proving our theorem. By induction, it suffices to consider just the case where $X = A \cup_\varphi D^n$, with characteristic map $\chi : D^n \to X$. Thus, it suffices to find a map w such that

$$\begin{tikzcd}
Z \times D^n \times \{0\} \arrow[d, hook] \arrow[drr, "\mathrm{id}_Z \times (g \circ \chi)"] & & \\
Z \times D^n \times [0,1] \arrow[rr, "w"] & & Y \\
Z \times S^{n-1} \times [0,1] \arrow[u, hook] \arrow[r, "\mathrm{id}_Z \times \varphi \times \mathrm{id}_{[0,1]}"'] & Z \times A \times [0,1] \arrow[ur, "K"'] &
\end{tikzcd}$$

commutes for, in this case, we can set $H(z,x,t) = g \cup_\varphi w$, thereby making

$$\begin{tikzcd}
Z \times D^n \times \{0\} \arrow[d, hook] \arrow[drr, "\mathrm{id}_Z \times (g \circ \chi)"] & & \\
Z \times D^n \times [0,1] \arrow[r, "\mathrm{id}_Z \times \chi \times \mathrm{id}_{[0,1]}"'] \arrow[rr, bend right, "w"'] & Z \times X \times [0,1] \arrow[r, "H"'] & Y \\
Z \times S^{n-1} \times [0,1] \arrow[u, hook] \arrow[dr, "\mathrm{id}_Z \times \varphi \times \mathrm{id}_{[0,1]}"'] & & \\
 & Z \times A \times [0,1] \arrow[uur, "K"'] &
\end{tikzcd}$$

commute. To this end, define the retraction $u : D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]$ by picking a point $*$ directly above the cylinder $D^n \times [0,1]$ and then sending any point x in the cylinder to the unique point where $D^n \times \{0\} \cup S^{n-1} \times [0,1]$ intersects the line containing $*$ and x. Now, define w so that

$$\begin{tikzcd}
Z \times (D^n \times [0,1]) \arrow[r, "w"] \arrow[d, "\mathrm{id}_Z \times u"'] & Y \\
Z \times (D^n \times \{0\} \cup S^{n-1} \times [0,1]) \arrow[ur, "\mathrm{id}_Z \times \left(g \circ \chi \cup K \circ (\varphi \times \mathrm{id}_{[0,1]})\right)"'] &
\end{tikzcd}$$

commutes. $\square$

Exercise 1.4.3. *Let $x \in X$. Consider the loop space $\Omega(X,x) \equiv M((S^1, \mathsf{pt}), (X,x))$. Prove that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$.*

Proof. Consider the *path space* $PX \equiv \{\gamma : [0,1] \to X \mid \gamma(0) = x\}$ *of* (X,x), equipped with the compact-open topology. We claim that PX is contractible. Indeed, define $K : PX \times [0,1] \to PX$ by

$$(\gamma, t) \mapsto (s \mapsto \gamma(s(1-t)))\,.$$

Then K is a homotopy from id_{PX} to the constant map at the constant path at x.

Define the map $p : PX \to X$ by $\gamma \mapsto \gamma(1)$. Then $p^{-1}(x) = \Omega(X)$. By Corollary 1.3.8, it suffices to show that p is a fibration. To this end, suppose that the square

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & PX \arrow[d, "p"] \\
Y \times [0,1] \arrow[r, "f"'] & X
\end{tikzcd}$$

commutes. Define $H : Y \times [0,1] \to PX$ by $(y,t) \mapsto H(y,t)$ where

$$H(y,t)(s) = \begin{cases} \hat{f}(y)\left((1+t)s\right) & 0 \le s \le \frac{1}{1+t} \\ f(y, (1+t)s - 1) & \frac{1}{1+t} \le s \le 1 \end{cases}.$$

We see that H is continuous when viewed as a function of (y,t,s) and thus is continuous. It is easy to check that

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & PX \arrow[d, "p"] \\
Y \times [0,1] \arrow[r, "f"'] \arrow[ur, "H"] & X
\end{tikzcd}$$

commutes, as desired. $\square$

Let $p : E \to B$ be a map. Recall that the pullback of p along $f : X \to B$ is given explicitly as

$$f^*E \equiv \{(x, e) \in X \times E \mid f(x) = p(e)\}.$$

Let f^*p denote the map $\pi_X \restriction_{f^*E}$.

Proposition 1.4.4. *If p is a fibration, then so is f^*p.*

Lemma 1.4.5. *If p is locally trivial, then so is f^*p.*

Proof. Let $a \in X$. Since p is locally trivial by assumption, we can find a neighborhood U of $f(a)$ in B and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$. Observe that

$$(f^*p)^{-1}(f^{-1}(U)) = \{(x, e) \mid f(x) = p(e),\ f(x) \in U\} \subset f^{-1}(U) \times p^{-1}(U).$$

Further, we have a map $\psi : f^{-1}(U) \to p^{-1}(U) \to f^{-1}(U) \times F$ given by $(x, e) \mapsto (x, \pi_F(\varphi(e)))$. Define $\lambda : f^{-1}(U) \times F \to (f^*p)^{-1}(f^{-1}(U))$ by $(x, y) \mapsto (x, \varphi^{-1}(f(x), y))$. Using the fact that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "p"'] & U \times F \arrow[d, "\pi_U"] \\ & U \end{tikzcd}$$

commutes, it is easy to check that ψ and λ are inverses of each other. $\square$

1.5 Lecture 5

Theorem 1.5.1. *Let B be a cell complex and let $p : E \to B$ be locally trivial. Then p is a fibration.*

Proof. It suffices to prove the following claim:

> If $h : Z \to X \times [0, 1]$ is locally trivial, $X = \bigcup_{i=0}^n X^i$ is a cell complex, and $\sigma_0 : X \times \{0\} \to Z$ satisfies $h \circ \sigma_0 = \mathrm{id}_{X\times\{0\}}$, then there is some map $\sigma : X \times [0, 1] \to Z$ such that $\sigma_{X\times\{0\}} = \sigma_0$ and $h \circ \sigma = \mathrm{id}_{X\times[0,1]}$.

For, in this case, Lemma 1.4.5 implies that given any commutative square

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0, 1] \arrow[r, "f"'] & B \end{tikzcd},$$

we can find some σ such that

$$\begin{tikzcd} & f^*E \arrow[r] \arrow[d, shift right] & E \arrow[d, "p"] \\ X \times \{0\} \arrow[ur, "\sigma_0"] \arrow[r, hook] & X \times [0, 1] \arrow[r, "f"'] \arrow[u, bend right, "\sigma"'] & B \end{tikzcd}$$

commutes where $\sigma_0(x, 0) = (x, 0, \hat{f}(x, 0))$.

For induction, we will assume that our claim is true for each $X^0, X^1, \ldots, X^{n-1}$. We may assume, wlog, that $X = D^n$. It suffices to find a map $\tau : S^{n-1} \times [0, 1] \to Z$ such that $h \circ \tau = \mathrm{id}_{S^{n-1}\times[0,1]}$ and

$$\begin{tikzcd} & Z \arrow[d, "h"] & \\ D^n \times \{0\} \arrow[ur, "\sigma_0"] \arrow[r, hook] & D^n \times [0, 1] & S^{n-1} \times [0, 1] \arrow[l, hook'] \arrow[ul, "\tau"'] \\ & S^{n-1} \times \{0\} \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

commutes since there is a retraction $D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]$. Fix a positive integer m. For any $i \in \mathbb{N}$, let $a_i = \frac{i}{m}$ and let $I_j = [a_j, a_{j+1}]$. By making m large enough, we can ensure that $p \restriction_{p^{-1}(I_{j_1} \times \cdots I_{j_{n+1}})}$ is trivial.

Claim. *$p \restriction_{p^{-1}(I_{j_1} \times I_{j_n} \times \cdots [0,1])}$ is also trivial.*

Proof. ?? $\square$

?? $\square$

2 Fiber bundles

Definition 2.0.1. A *topological group* G is a group such that both multiplication $G \times G \xrightarrow{\mu} G$ and inversion $G \xrightarrow{(-)^{-1}} G$ are continuous.

Definition 2.0.2 (Fiber bundle). Let G be a topological group.

1. A *fiber F of G* is a space equipped with a faithful (i.e., injective) group action $\rho : G \to \mathrm{Homeo}(F) \subset M(F,F)$.

2. An *atlas for the structure of a (fiber) bundle with group G and fiber F on a map $p : E \to B$* consists of

   (a) a family $(U_\alpha, h_\alpha)_{\alpha \in A}$ where each U_α is open and each h_α is a homeomorphism $p^{-1}(U_\alpha) \to U_\alpha \times F$ and

   (b) a family of continuous *transition functions* $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}_{\alpha,\beta \in A}$

   such that

   i $B = \bigcup_{\alpha \in A} U_\alpha$,

   ii $\pi_{U_\alpha} \circ h_\alpha = p \restriction_{p^{-1}(U_\alpha)}$, and

   iii $x \in U_\alpha \cap U_\beta \implies h_\beta \circ h_\alpha^{-1}(x,f) = (x, h_{\beta\alpha}(x) \cdot f)$

3. Two atlases are *compatible* if their union is an atlas.

4. A *bundle structure on B* is a maximal atlas on p.

Terminology. If B is equipped with a bundle structure, then we say that p is a (fiber) bundle.

Example 2.0.3.

1. The tangent bundle $\pi : TM \to M$ of a smooth n-manifold M is a bundle with group $\mathrm{GL}(n, \mathbb{R})$.

   Proof. Let (U, φ) be any coordinate chart for M with coordinate functions (x^i). Define $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by
   $$v^i \frac{\partial}{\partial x^i}(p) \mapsto \left(p, \left(v^1, \ldots, v^n\right)\right).$$
   It is clear that $\pi_U(h(p)) = \pi(c)$ for any $c \in \pi^{-1}(U)$. To see that h is a homeomorphism, note that the composite $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$ is given by
   $$v^i \frac{\partial}{\partial x^i}(p) \mapsto \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\right),$$
   the inverse of which is given by $\left(x^1, \ldots, x^n, v^1, \ldots, v^n\right) \mapsto v^i \frac{\partial}{\partial x^i}\left(\varphi^{-1}(x)\right)$. Therefore, $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h$ is given locally by
   $$\left(x^1, \ldots, x^n, v^1, \ldots, v^n\right) \mapsto \left(\tilde{x}^1(x), \ldots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^j}(x) v^j, \ldots, \frac{\partial \tilde{x}^n}{\partial x^j}(x) v^j\right),$$

which is smooth. Thus, h is a diffeomorphism as the composition of two diffeomorphisms. In particular, h is a homeomorphism.

It remains to describe the transition functions $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})\}$ for TM. Note that

$$\begin{tikzcd} U_{\alpha\beta} \times \mathbb{R}^n \arrow[dr, "\pi_1"'] & \pi^{-1}(U_{\alpha\beta}) \arrow[l, "h_\alpha"'] \arrow[r, "h_\beta"] \arrow[d, "\pi"] & U_{\beta\alpha} \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ & U_{\alpha\beta} & \end{tikzcd}$$

commutes. In particular, $\pi_1 \circ h_\beta \circ h_\alpha^{-1} = \pi_1$, which implies that $h_\beta \circ h_\alpha^{-1}(u, v) = (u, f(u, v))$ for some smooth map $f : U_{\alpha\beta} \times \mathbb{R}^n \to \mathbb{R}^n$. This must be a linear isomorphism when restricted to $\{u\} \times \mathbb{R}^n$ for any $u \in U_{\alpha\beta}$, which is uniquely determined by an element $h_{\beta\alpha}(u)$ of $\mathrm{GL}(n, \mathbb{R})$ (provided that we have fixed a basis of $\mathbb{R}^n$). Hence

$$h_\beta \circ h_\alpha^{-1}(u, v) = (u, h_{\beta\alpha}(u)v)\,.$$

Since the map $h_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{R})$ is continuous, our proof is complete. □

2. Let $p : E \to B$ be a bundle with group $\{e\}$. Then p is the trivial bundle, i.e., is isomorphic to the projection map.

   Proof. We have that $h_\beta = h_\alpha$ on $U_\alpha \cap U_\beta$, so that $h \equiv \bigcup_{\alpha \in A} h_\alpha$ is a well-defined homeomorphism $E \cong B \times F$. □

2.1 Lecture 6

Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure with group G and fiber F on $p : E \to B$. Let $U = U_\alpha \cap U_\beta \cap U_\gamma$. Consider the commutative diagram

$$\begin{tikzcd} & & p^{-1}(U) \arrow[rrrd, "h_\gamma"] & & & \\ U \times F \arrow[rru, "h_\alpha^{-1}"] \arrow[r, "h_\alpha^{-1}"'] & p^{-1}(U) \arrow[r, "h_\beta"'] & U \times F \arrow[r, "h_\beta^{-1}"'] & p^{-1}(U) \arrow[r, "h_\gamma"'] & U \times F \end{tikzcd}.$$

The bottom row is given by $(u, f) \mapsto (u, h_{\beta\alpha}(u) \cdot f) \mapsto (u, h_{\gamma\beta}(u) \cdot (h_{\beta\alpha}(u) \cdot f)) = (u, (h_{\gamma\beta}(u)h_{\beta\alpha}(u)) \cdot f)$, and the top composite is given by $(u, f) \mapsto (u, h_{\gamma\alpha}(u) \cdot f)$. It follows that

$$h_{\gamma\beta}(u)h_{\beta\alpha}(u) = h_{\gamma\alpha}(u)$$

for each $u \in U$. This property is known as the *cocycle condition.*

Theorem 2.1.1. *Let G be a topological group acting on a space F. Suppose that $\{U_\alpha\}$ is an open cover of B and $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is a family of continuous functions satisfying the cocycle condition. Then there exists a bundle $p : E \to B$ with group G, fiber F, and transition functions $h_{\beta\alpha}$.*

Proof sketch. Let $E = \coprod_\alpha U_\alpha \times F/\sim$ where $(u, f)_\alpha \sim (u, h_{\beta\alpha}(u) \cdot f)_\beta$. Define $p : E \to B$ by $(u, f) \mapsto u$. □

Definition 2.1.2 (Bundle map). A *morphism of bundles p_1 and p_2 with group G and fiber F* is a commutative square of the form

$$\begin{tikzcd} E_1 \arrow[r, "\hat{g}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"'] & B_2 \end{tikzcd}$$

such that

Suppose that $(\hat{g}, g)$ is a bundle map $p_1 \to p_2$. Let $\{(U_\alpha, h_\alpha)\}$ and $\{(V_\beta, k_\beta)\}$ be bundle structures on B_2 and B_1, respectively. We have a commutative diagram

$$\begin{tikzcd}
\left(g^{-1}(U_\alpha) \cap V_\beta\right) \times F \arrow[r, "k_\beta^{-1}"] \arrow[dr, "\pi_1"'] \arrow[rrr, bend left, "d_{\alpha\beta}"] & p_1^{-1}\left(g^{-1}(U_\alpha) \cap V_\beta\right) \arrow[r, "\hat{g}"] \arrow[d] & p_2^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d] & U_\alpha \times F \arrow[dl, "\pi_1"] \\
& g^{-1}(U_\alpha) \cap V_\beta \arrow[r, "g"'] & U_\alpha &
\end{tikzcd},$$

so that $d_{\alpha\beta}(x, f) = (g(x), \lambda_{\alpha\beta}(x) \cdot f)$ for some continuous map $\lambda_{\alpha\beta} : g^{-1}(U_\alpha) \cap V_\beta \to G$. Letting $W = g^{-1}(U_\alpha \cap U_{\alpha'}) \cap (V_\beta \cap V_{\beta'})$, we have that

$$h_{\alpha'\alpha}(w)\lambda_{\alpha\beta}(w)k_{\beta\beta'}(w) = \lambda_{\alpha'\beta'}(w) \tag{$\dagger$}$$

for every $w \in W$.

Exercise 2.1.3 (Pullback bundle). *Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure on $p : E \to B$ with group G and consider the pullback diagram*

$$\begin{tikzcd}
g^*E \arrow[r] \arrow[d, "g^*p"'] & E \arrow[d, "p"] \\
X \arrow[r, "g"'] & B
\end{tikzcd}.$$

Define $h'_{\beta\alpha} : g^{-1}(U_\alpha) \cap g^{-1}(U_\beta) \to G$ as the composite $h_{\beta\alpha} \circ g$ restricted to $g^{-1}(U_\alpha \cap U_\beta)$. Show that the family $\{h'_{\beta\alpha}\}$ induces a bundle structure on g^*p.

Theorem 2.1.4. *Any bundle map*

$$\begin{tikzcd}
E_1 \arrow[r, "\hat{g}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\
B_1 \arrow[r, "g"'] & B_2
\end{tikzcd}$$

factors as

$$\begin{tikzcd}
E_1 \arrow[r, "\tau"] \arrow[d, "p_2"] & g^*E_2 \arrow[r, "\bar{g}"] \arrow[d, "g^*p_2"] & E_2 \arrow[d, "p_2"] \\
B_1 \arrow[r, "\mathrm{id}_{B_1}"'] & B_1 \arrow[r, "g"'] & B_2
\end{tikzcd}$$

where $\tau(e) = (p_1(e), \hat{g}(e))$ for any $e \in E_1$.

2.2 Lecture 7

Note 2.2.1. If $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is a family of transition functions, then

$$h_{\alpha\beta}(x) = (h_{\beta\alpha}(x))^{-1}$$

for any $x \in U_\alpha \cap U_\beta$.

Theorem 2.2.2. *Any bundle map of the form*

$$\begin{tikzcd}
E_1 \arrow[r, "\hat{g}"] \arrow[dr, "p_1"'] & E_2 \arrow[d, "p_2"] \\
& B
\end{tikzcd}$$

is an isomorphism.

Proof. Note that

$$\begin{tikzcd}
 & p_2^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "h_\beta"'] \arrow[dr, "h_\alpha"] & \\
(U_\alpha \cap U_\beta) \times F \arrow[dr] & p_1^{-1}(U_\alpha \cap U_\beta) \arrow[l, "k_\beta"'] \arrow[u, "\hat{g}"] \arrow[d] & (U_\alpha \cap U_\beta) \times F \arrow[l, "k_\alpha^{-1}"'] \arrow[dl] \\
 & U_\alpha \cap U_\beta &
\end{tikzcd}$$

commutes. We have that $h_\beta \circ \hat{g} \circ k_\alpha^{-1}(x, f) = (x, \lambda_{\beta\alpha}(x) \cdot f)$. Thus, if $h_\alpha(e) = (x, f)$, then $h_\alpha(\hat{g}(e)) = (x, \lambda_{\alpha\alpha}(x) \cdot d)$. Let

$$(\hat{g})^{-1}(e) = k_\alpha^{-1}\left(x, \lambda_{\alpha\alpha}(x)^{-1} \cdot f\right)$$

where $(x, f) = h_\alpha(e)$. If this is well-defined, then it equals the inverse of g. Moreover, it is a bundle map because of (†). ??

□

Corollary 2.2.3. *Every bundle over a space E is isomorphic to the pullback bundle over E.*