

## Abstract

These notes are based on Julius Shaneson's lectures for the course "Algebraic Topology, Part I" at UPenn. Any mistake in what follows is my own.

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# 1 Background material

## 1.1 Lecture 1

Here are the topics for the course:

- fiber bundles over cell complexes,
- spectral sequences,
- characteristic classes, and
- cobordism theory.

To start, let's review some basic concepts from homology theory.

**Definition 1.1.1.** A (finite) cell complex is a (topological) space  $X$  that can be written as  $\bigcup_{n=0}^K X^n$  for some  $K \in \mathbb{N}$  (called the *dimension of  $X$* ) where

- $X^0$  is chosen to be finite,
- $X^n = \frac{X^{n-1} \amalg D_1^n \amalg \cdots \amalg D_{k_n}^n}{x \sim \varphi_i(x)}$ ,
- $D_i^n \cong D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$  for each  $i \in \{1, \dots, k_n\}$ , and
- $\varphi_i : \partial D_i^n = S^{n-1} \rightarrow X^{n-1}$ , called an *attaching map*.

*Terminology.* Each  $D_i^n$  is called an *n-cell of  $X$* .

Every attaching map  $\varphi_i : \partial D_i^n \rightarrow X^{n-1}$  can be extended to a *characteristic map* given by the composite

$$D_i^n \hookrightarrow X^{n-1} \amalg D_1^n \amalg \cdots \amalg D_{k_n}^n \rightarrow X^n \hookrightarrow X.$$

**Example 1.1.2.** There are at least two ways of endowing  $S^2$  with a cell structure.

1.  $X^0 \equiv \{N, S\}$ ,  $X^1 \equiv X^0 \cup_{\varphi_1} D_1^1 \cup_{\varphi_2} D_2^1$  where each  $\varphi_i$  is an embedding, and  $X^2 \equiv X^1 \cup_{\varphi'_1} D_1^2 \cup_{\varphi'_2} D_2^2$  where each  $\varphi'_i$  is an embedding.
2.  $\text{pt} \cup_{\varphi} D^2$  where  $\varphi$  identifies the equator of the upper half-sphere with  $\text{pt}$ .

**Definition 1.1.3.** A cell complex  $X$  is *regular* if every characteristic map  $D_i^n \rightarrow X$  is an embedding.

**Definition 1.1.4.** Given a family of functors  $\{H_n : \mathbf{Top}^2 \rightarrow \mathbf{Ab}\}_{n \in \mathbb{N}}$  where  $\mathbf{Top}^2$  denotes the category of (topological) pairs, we say that  $H_i$  is a *homology functor* if each of the following properties holds.

1. (LES) For any pair  $(X, A)$  of space, there is a natural long exact sequence

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \longrightarrow \cdots,$$

where  $H_i(Z) := H_i(Z, \emptyset)$  for any space  $Z$ .

2. (Excision) If  $\text{cl}(A) \subset \bigcup_{\text{open}} U \subset X$ , then  $H_i(X \setminus A, U \setminus A) \cong H_i(X, U)$ .

$$3. \text{ (Dimension) } H_i(\mathbf{pt}) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases}.$$

4. (Homotopy) If  $f$  and  $g$  are homotopic, then  $f_* = g_*$ , where  $h_* := H_i(h)$  for any map  $h : (X, A) \rightarrow (Y, B)$ .

**Theorem 1.1.5.** *There exists a family of homology functors.*

**Example 1.1.6.** In singular homology theory, we have that  $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}.$

Let  $X$  be a cell complex. Let  $C_n(X)$  denote the free abelian group on the set of all  $n$ -cells of  $X$ . Define  $\partial : C_n(X) \rightarrow C_{n-1}(X)$  by  $\partial[D_i^n] = \sum_{j=1}^{k_{n-1}} \lambda_{ij}[D_j^{n-1}]$  where  $\lambda_{ij}$  is defined, up to sign, as follows. Consider the map

$$S^{n-1} \xlongequal{\quad} \partial D_i^n \xrightarrow{\varphi_i} X^{n-1} \twoheadrightarrow \frac{X^{n-1}}{X^{n-2} \cup (\text{all cells of dim. } n-1 \text{ except } D_j^{n-1})} \xlongequal{\quad} D^{n-1} / \partial D_j^{n-1} \xlongequal{\quad} S^{n-1}.$$

$\omega$

Then let  $\lambda_{ij}$  satisfy  $\omega_*(x) = \lambda_{ij}x$  with  $x$  a chosen generator (i.e., orientation) of  $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ .

*Terminology.* The integer  $\lambda_{ij}$  is called the *degree* of  $\omega$ , denoted by  $\deg(\omega)$ .

**Theorem 1.1.7.**  $\partial_n \partial_{n+1} = 0$ , and  $H_n(X) \cong \ker \partial_n / \text{im } \partial_{n+1}$ , which is independent of our choice of generator  $x$ .

**Example 1.1.8.** Suppose that  $f : S^n \rightarrow S^n$  is smooth. By Sard's theorem, we can find a regular value  $x \in S^n$ . There is some neighborhood  $U$  of  $x$  such that  $f^{-1}(U) = U_1 \cup \dots \cup U_n$  for some  $n$ . Using the inverse function theorem and the compactness of  $S^n$ , it follows that  $f^{-1}$  is of the form  $\{x_1, \dots, x_n\}$ . Note that the differential  $(df)_{x_i} : S_{x_i}^n \rightarrow S_x^n$  satisfies  $\det(df)_{x_i} = \pm 1$ . In fact,

$$\deg(f) = \sum_{i=1}^n \det(df)_{x_i}.$$

**Exercise 1.1.9.** *Prove that any finite cell complex  $X = X^K$  is homotopy equivalent to a regular cell complex.*

(Hint: Consider the map  $S^{n-1} \rightarrow X^{n-1} \times D^n$  given by  $x \mapsto (\varphi(x), x)$  where  $\varphi$  denotes an attaching map of  $X$ .)

*Proof.* Let us construct recursively a finite sequence  $A^0, A^1, \dots, A^K$  of spaces such that each  $A^i$  carries the structure of a regular cell complex and is homotopy equivalent to  $X^i$ . For each  $n \in \{1, \dots, K\}$ , let  $k_n$  denote the necessarily finite number of attaching maps  $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_{k_n}} : S^{n-1} \rightarrow X^{n-1}$  for the  $n$ -skeleton of  $X$ . Let

$$A^0 = X^0 \times D_{\alpha_1}^1 \times \dots \times D_{\alpha_{k_1}}^1,$$

viewed as a product of finite cell-complexes. Note that the topology of  $A^0$  is precisely the product topology. Thus,  $A^0$  is homotopy equivalent to  $X^0$  as  $D^1$  is contractible. Now, suppose that  $0 \leq n \leq K-1$  and that we

have constructed our desired space  $A^n$ . This means that there is some homotopy equivalence  $\gamma_n : X^n \rightarrow A^n$ . Form  $A^{n+1}$  by attaching finitely many  $(n+1)$ -cells  $e_{\alpha_1}^{n+1}, \dots, e_{\alpha_{k_{n+1}}}^{n+1}$  to  $Z_n \equiv A^n \times D_{\alpha_1}^{n+1} \times \dots \times D_{\alpha_{k_{n+1}}}^{n+1}$  via the maps

$$\psi_{\alpha_i} : S^n \rightarrow A^n \times D_{\alpha_1}^{n+1} \times \dots \times D_{\alpha_{k_{n+1}}}^{n+1}$$

$$x \mapsto \left( \gamma_n \circ \varphi_i(x), 0, \dots, 0, \underbrace{x}_{i\text{-th spot}}, 0, \dots, 0 \right)$$

where  $Z_n$  is viewed as a product of finite cell complexes (whose topology is precisely the product topology). It is easy to see that  $A^{n+1}$  is homotopy equivalent to  $X^{n+1}$ . Moreover, since each map  $\psi_{\alpha_i}$  is an embedding and any  $n$ -disk has the structure of a regular cell complex, we see from our construction of  $(A^i)$  that  $A^K$  has the structure of a regular cell complex. By design, this space is homotopy equivalent to  $X^K$ , thereby completing our proof.  $\square$

## 1.2 Lecture 2

**Example 1.2.1 (Real projective space).** Recall that  $\mathbb{RP}^n = S^n / x \sim -x$ . Then  $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\pi_{n-1}} D^n$  where  $\pi_{n-1} : S^{n-1} \rightarrow \mathbb{RP}^{n-1}$  denotes the canonical projection. Thus,  $\mathbb{RP}^n$  is an  $n$ -dimension cell complex with  $(\mathbb{RP}^n)^m = \mathbb{RP}^m$  for each integer  $0 \leq m \leq n$ .

Now, for each  $0 \leq m \leq n$ , we have that  $C_m(\mathbb{RP}^n) \cong \mathbb{Z}$  with generator  $[D^m]$ . To determine  $\partial[D^m] \in C_{m-1}(\mathbb{RP}^n)$ , we must find the degree of the map

$$\begin{array}{ccccccc} S^{m-1} & \longrightarrow & \mathbb{RP}^{m-1} & \longrightarrow & \mathbb{RP}^{m-1} / \mathbb{RP}^{m-1} & \equiv & D^{m-1} / \partial D^{m-1} \equiv S^{m-1} \\ & & & & \searrow & & \nearrow \\ & & & & \varphi & & \end{array}$$

Assume, for simplicity, that  $m = 2$ . Choose a regular value  $p \in S^1$  so that  $\varphi^{-1}(p) = \{N, S\}$ . Let  $\varphi_T$  and  $\varphi_B$  denote the restrictions of  $\varphi$  to the top and bottom components of  $S^1 \setminus \{(-1, 0), (1, 0)\}$ , respectively. Note that both of these are homeomorphisms and thus have degrees equal to  $\pm 1$ . If  $a : S^{m-1} \rightarrow S^{m-1}$  denotes the antipodal map, we have that  $\varphi_B \circ a = \varphi_T$ . Hence  $(d\varphi)_S \circ (da)_N = (d\varphi)_N$ . Since  $\deg(a) = \det(da) = (-1)^m$ , it follows that

$$\deg(\varphi) = \begin{cases} \pm 2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}.$$

Thus, we get a chain complex

$$0 \longrightarrow C_n(\mathbb{RP}^n) \xrightarrow{\kappa_1} C_{n-1}(\mathbb{RP}^n) \xrightarrow{\kappa_2} \dots \xrightarrow{0} C_2(\mathbb{RP}^n) \xrightarrow{\pm 2} C_1(\mathbb{RP}^n) \xrightarrow{0} C_0(\mathbb{RP}^n) \longrightarrow 0$$

$$\text{where } \kappa_1 = \begin{cases} 0 & n \text{ odd} \\ \pm 2 & n \text{ even} \end{cases} \text{ and } \kappa_2 = \begin{cases} \pm 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

This proves that

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i = 1 \\ 0 & i = 2 \\ \mathbb{Z}_2 & i < n \\ 0 & i < n \\ 0 & i > n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & i = n \text{ even} \end{cases}.$$

**Example 1.2.2.**  $H_{2i}(\mathbb{CP}^n) \cong \mathbb{Z}$ .

Next, let's introduce some fundamental concepts from homotopy theory.

**Definition 1.2.3.** Let  $M(X, Y)$  denote the set of maps  $X \rightarrow Y$ .

1. For any compact  $C \subset X$  and open  $U \subset Y$ , let

$$N(C, U) = \{f : X \rightarrow Y \mid f(C) \subset U\}.$$

The *compact-open topology* on  $M(X, Y)$  consists of all unions of finite intersections of subsets of the form  $N(C, U)$ . Under this topology,  $M(X, Y)$  is called a *mapping space*.

2. The *n-th loop space* of a pointed space  $(X, x)$  is

$$\Omega^{n-1}(X, x) := M((D^{n-1}, \partial D^{n-1}), (X, x)),$$

which is a subset of  $M(D^{n-1}, X)$ .

**Definition 1.2.4 (Higher homotopy groups).** If  $n \geq 2$ , then the *n-th homotopy group* of  $(X, x)$  is

$$\pi_n(X, x) := \pi_1(\Omega^{n-1}X, e_x).$$

Recall that  $\pi_1(-)$  is a functor  $\mathbf{Top}_* \rightarrow \mathbf{Grp}$ . Also,  $\Omega^{n-1}(-)$  is a functor  $\mathbf{Top}_* \rightarrow \mathbf{Top}$  defined on morphisms  $f : (X, x) \rightarrow (Y, y)$  by post-composition with  $f$ . Therefore, it's easy to see that  $\pi_n(-)$  is a functor  $\mathbf{Top}_* \rightarrow \mathbf{Grp}$  as well.

*Notation.* Let  $f_* = \pi_n(f)$  for any  $f : (X, x) \rightarrow (Y, y)$ .

**Proposition 1.2.5.** *There is a homeomorphism  $M(X \times Y, Z) \cong M(X, M(Y, Z))$  so long as  $Y$  is locally compact and Hausdorff.*

In particular, we have a composite

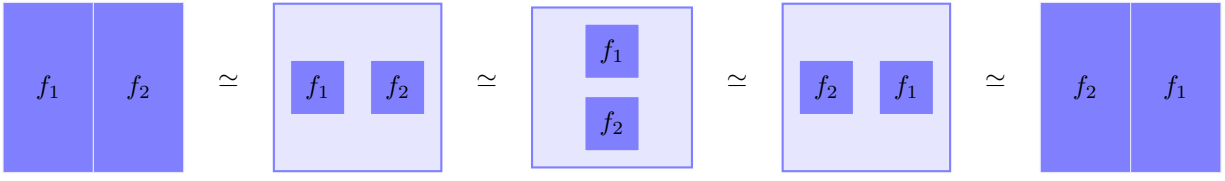
$$M([0, 1], \{0, 1\}), (M((D^{n-1}, \partial), (X, x)), e_x) \hookrightarrow M([0, 1], M(D^{n-1}, X)) \xrightarrow{\cong} M([0, 1] \times D^{n-1}, X),$$

whose image is precisely  $M((D^n, \partial), (X, x)) \cong M((S^n, \text{pt}), (X, x))$ . This proves that  $\pi_n(X, x)$  consists of all homotopy classes of maps  $(I^n, \partial) \rightarrow (X, x)$  under the operation  $[f] * [g] = [f * g]$  where

$$f * g(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \leq t_1 \leq \frac{1}{2} \\ f(2t_1 - 1, t_2, \dots, t_n) & \frac{1}{2} \leq t_1 \leq 1 \end{cases}.$$

**Lemma 1.2.6.** *If  $n \geq 2$ , then  $\pi_n(X, x)$  is abelian.*

*Proof.* Let  $f_1$  and  $f_2$  be maps  $(I^n, \partial) \rightarrow (X, x)$ . We must find a homotopy  $F : I^n \times I \rightarrow X$  between  $f_1 * f_2$  and  $f_2 * f_1$  such that  $F(t_1, \dots, t_n, s) = x$  for any  $(t_1, \dots, t_n) \in \partial I^n$  and  $s \in I$ . To this end, first shrink the domains of  $f_1$  and  $f_2$  to small  $n$ -cubes in  $I^n$  (thereby thickening the set of points mapped to  $x$  under  $F$ ), then slide these small cubes past each other, and finally enlarge them to their original sizes as follows.



□

*Remark 1.2.7.* A map  $f : S^{n-1} \rightarrow X$  is homotopic to the constant map if and only if there is some  $g$  such that

$$\begin{array}{ccc} & D^n & \\ \uparrow & \searrow g & \\ S^{n-1} & \xrightarrow{f} & X \end{array}$$

commutes.

**Theorem 1.2.8 (Whitehead).** *If  $\psi : X \rightarrow Y$  is a map of connected cell complexes, then  $f$  is a homotopy equivalence if and only if  $\psi_* : \pi_n(X, x) \rightarrow \pi_n(Y, y)$  is an isomorphism for each  $n \in \mathbb{N}$ .*

A map  $f : X \rightarrow Y$  of path connected spaces is a *weak homotopy equivalence* if it induces an isomorphism  $\pi_n(X) \rightarrow \pi_n(Y)$  for each  $n \in \mathbb{N}$ . Since any cell complex is locally path connected, it is connected if and only if it is path connected. Hence Theorem 1.2.8 says that  $\psi$  is a homotopy equivalence if and only if it is a weak homotopy equivalence.

### 1.3 Lecture 3

**Definition 1.3.1.** If  $x \in A \subset X$ , then the  $n$ -th relative homotopy group  $\pi_n(X, A, x)$  consists of all homotopy classes of maps  $(D^n, S^{n-1}, x_0) \rightarrow (X, A, x)$ .

We see that

$$M((D^n, S^{n-1}, x), (X, A, x_0)) \cong M((I^n, I^{n-1} \times \{1\}, \underbrace{\partial I^n \setminus \text{Int}(I^{n-1} \times \{1\})}_{\partial_0 I^n}), (X, A, x_0))$$

by considering the homeomorphism  $(I^n / \partial_0 I^n, \partial I^n / \partial_0 I^n) \cong (D^n, S^{n-1})$ . Therefore,  $\pi_n(X, A, x)$  can be viewed as consisting of all homotopy classes of maps  $(I^n, \partial I^n, \partial_0 I^n) \rightarrow (X, A, x)$ .

**Definition 1.3.2.** In order to interpret an exact sequence involving objects in the category of pointed sets, we define the *kernel of a function*  $f : (X, x) \rightarrow (Y, y)$  of pointed sets as  $\ker f \equiv f^{-1}(y)$ .

**Proposition 1.3.3.**

1. If  $n \geq 2$ , then  $\pi_n(X, A, x)$  is, in fact, a group.
2. If  $n \geq 3$ , then  $\pi_n(X, A, x)$  is abelian.
3. We have a long exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n(A, x) & \longrightarrow & \pi_n(X, x) & \longrightarrow & \pi_n(X, A, x) \xrightarrow{\partial} \pi_{n-1}(A, x) \\ & & & & & & \swarrow \\ & & \pi_{n-1}(X, x) & \longleftarrow & \cdots & \longrightarrow & \pi_0(A, x) \longrightarrow \pi_0(X, x) \longrightarrow 0 \end{array}$$

with  $\partial[f] = [f \downarrow_{I^{n-1}}]$ .

**Theorem 1.3.4 (Hurewicz).** Let  $n \in \mathbb{Z}_{\geq 2}$ . If  $\pi_i(X) = 0$  for each  $i < n$ , then  $\pi_n(X) \cong H_n(X)$ .

**Note 1.3.5.** This result can't be improved in general. For example,  $\pi_3(S^2) \cong \mathbb{Z}$ , whereas  $H_3(S^2) = 0$ .

Let  $A \subset X$  be a subcomplex. Recall that  $H_i(X, A) \cong H_i(X/A, *)$  for each  $i \geq 1$ . But it is *not* the case that  $\pi_i(X, A) \cong \pi_i(X/A, *)$ , for otherwise  $\pi_i(S^n) \cong \pi_i(D^n, S^{n-1}) \cong \pi_i(S^{n-1})$ , which is known to be false exactly when  $i > 2n - 2$ .

**Example 1.3.6.**  $\pi_4(S^3) \cong \mathbb{Z}_2 \not\cong \pi_4(S^4)$ .

Finally, let's review the notion of a fibration of spaces.

Recall that if  $p : E \rightarrow B$  is a covering projection, then TFAE.

1. For any  $f : Z \rightarrow B$ , there exists a unique  $\hat{f} : Z \rightarrow E$  such that  $p \circ \hat{f} = f$ .
2.  $f_*(\pi_1(Z)) \subset p_*(\pi_1(E))$ .

The existence of  $\hat{f}$  follows from the fact that any covering space satisfies the homotopy lifting property.

**Definition 1.3.7 (Fibration).** Suppose that  $p : E \rightarrow B$  is any map. We say that  $p$  is a *fibration* if it satisfies the homotopy lifting property, i.e., given a commutative square

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}_0} & E \\ \downarrow & & \downarrow p \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

where  $X$  is a cell complex, there is some  $G$  such that

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}_0} & E \\ \downarrow & \nearrow G & \downarrow p \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

commutes.

**Theorem 1.3.8.** *If  $p : E \rightarrow B$  is a fibration with  $e \in F := p^{-1}(b)$ , then*

$$p_* : \pi_i(E, F, e) \xrightarrow{\cong} \pi_i(B, b, b) = \pi_i(B, b).$$

*Proof.* Let  $f : (I^n, \partial I^n) \rightarrow (B, b)$ . To prove that  $p_*$  is surjective, it suffices to find some  $G : (I^n, \partial I^n) \rightarrow (E, F)$  such that

$$\begin{array}{ccccc} \partial_0 I^n & \longrightarrow & \{e\} & \hookrightarrow & F \hookrightarrow E \\ \downarrow & & \nearrow G & & \downarrow p \\ I^{n-1} \times [0, 1] & \xrightarrow{\quad f \quad} & & & B \end{array}$$

commutes, for in this case  $[p \circ G'] = [f]$ . Since  $p$  is a fibration, there is some  $G$  such that

$$\begin{array}{ccccc} I^{n-1} \times \{0\} & \longrightarrow & \{e\} & \hookrightarrow & F \hookrightarrow E \\ \downarrow & & \nearrow G' & & \downarrow p \\ I^{n-1} \times [0, 1] & \xrightarrow{\quad f \quad} & & & B \end{array}$$

commutes. But  $(I^n, \partial_0 I^n) \cong (I^n, I^{n-1} \times \{0\})$ , and thus such a  $G'$  is enough.  $\square$

**Corollary 1.3.9.** *We have a long exact sequence*

$$\cdots \longrightarrow \pi_i(F, e) \longrightarrow \pi_i(E, e) \longrightarrow \pi_i(B, b) \xrightarrow{\partial} \pi_{i-1}(F, e) \longrightarrow \cdots$$

**Example 1.3.10.**

1. Suppose that

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}} & B \times F \\ \downarrow & & \downarrow \pi_B \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

commutes. Then  $\hat{f}(x, 0) = (\hat{f}_1(x, 0), \hat{f}_2(x, 0))$  where  $\hat{f}_1(x, 0) = f(x, 0)$ . Let  $G(X, t) = (f(x, t), \hat{f}_2(x, 0))$ . Then

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}_0} & B \times F \\ \downarrow & \nearrow G & \downarrow \pi_B \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

commutes, so that  $\pi_B$  is a fibration. (Moreover,  $\pi_n(B \times F) \cong \pi_n(B) \times \pi_n(F)$ .)

2. Let  $A \subset X$  be a subcomplex. The map  $\varphi : M(X, Y) \rightarrow M(A, Y)$  defined by  $f \mapsto f|_A$  is a fibration.

3. Define the *Hopf fibration* as the quotient map

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1\} \twoheadrightarrow S^3 / \sim_x = \mathbb{CP}^1 = S^2.$$

**Corollary 1.3.11.**  $\pi_3(S^3) \cong \pi_3(S^2)$ .



*Proof.* Since we have an exact sequence

$$\pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) ,$$

it suffices to show that both  $\pi_3(S^1)$  and  $\pi_2(S^1)$  are trivial. To this end, note that since  $\pi_1(S^k) = 0$  for every  $k > 1$ , we can always find, for any  $f : S^k \rightarrow S^1$ , a map  $\hat{f}$  such that

$$\begin{array}{ccc} & & \mathbb{R} \\ & \nearrow \hat{f} & \downarrow e^{2\pi i x} \\ S^k & \xrightarrow{f} & S^1 \end{array}$$

commutes. Thus,  $f$  is homotopic to the constant map. Since  $f$  was arbitrary, our proof is complete.  $\square$

**Definition 1.3.12.** A map  $p : E \rightarrow B$  is *locally trivial* if for any  $b \in B$ , there exist a neighborhood  $U \ni b$  in  $B$ , a space  $F$ , and a homeomorphism  $\varphi : p^{-1}(U) \xrightarrow{\cong} U \times F$  such that  $\pi_U \circ \varphi = p \upharpoonright_{p^{-1}(U)}$ .

**Theorem 1.3.13.** Any locally trivial map  $p : E \rightarrow B$  is a fibration whenever  $B$  is a cell complex.

**Exercise 1.3.14.** Prove that the Hopf fibration is locally trivial.

*Proof.* For each  $k \in \{0, 1\}$ , let  $U_k = \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_k \neq 0\}$ . Then  $U_0$  and  $U_1$  form an open cover of  $\mathbb{CP}^1$ . Note that the preimage of  $U_k$  under the Hopf fibration  $q$  is precisely  $\{(z_0, z_1) \in S^3 \mid z_k \neq 0\}$ . Define  $f : q^{-1}(U_k) \rightarrow U_k \times S^1$  by

$$(z_0, z_1) \mapsto \left( [z_0, z_1], \frac{z_k}{|z_k|} \right).$$

This is clearly continuous. Further, define the map  $g : U_k \times S^1 \rightarrow q^{-1}(U_k)$  by

$$([z_0, z_1], e^{i\theta}) \mapsto \frac{e^{i\theta}|z_k|}{z_k|(z_0, z_1)|} (z_0, z_1).$$

Since  $U_k$  is a saturated open set, we have that the restriction of  $q$  to  $q^{-1}(U_k)$  is a quotient map. But  $g \circ q \upharpoonright_{q^{-1}(U_k)}$  is continuous, so that  $g$  is also continuous by the characteristic property of quotient maps. Finally, it is easy to verify that  $g$  and  $f$  are inverses of each other and that  $\pi_{U_I} \circ f = p \upharpoonright_{q^{-1}(U_k)}$ .  $\square$

## 1.4 Lecture 4

**Theorem 1.4.1.** Let  $A \subset X$  be a subcomplex. Define  $r : M(X, Y) \rightarrow M(A, Y)$  by  $r(f) = f \upharpoonright_A$ . Then  $r$  is a fibration.

*Proof.* We must fill any diagram of the form

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\hat{f}} & M(X, Y) \\ \downarrow & \nearrow F & \downarrow r \\ Z \times [0, 1] & \xrightarrow{f} & M(A, Y) \end{array} .$$

It suffices to find a map  $\bar{F}$  such that

$$\begin{array}{ccc} Z \times \{0\} \times X & \xrightarrow{\hat{f}} & Y \\ \downarrow & \nearrow \bar{F} & \parallel \\ Z \times [0, 1] \times X & & Y \\ \uparrow & \nearrow \bar{f} & \\ Z \times [0, 1] \times A & & \end{array}$$

commutes for, in this case, we can set  $F(z, t)(x) = \bar{F}(z, t, x)$ .

**Note 1.4.2.** Suppose that such an  $\bar{F}$  exists. Define  $g : Z \times X \rightarrow Y$  by  $g(z, x) = \hat{f}(z, 0, x)$ . Define  $h : Z \times X \times [0, 1] \rightarrow Y$  by  $H(z, x, t) = \bar{F}(z, t, x)$ . Then

$$\begin{array}{ccc} Z \times X \times \{0\} & & \\ \downarrow & \searrow g & \\ Z \times X \times [0, 1] & \xrightarrow{H} & Y \\ \uparrow & \nearrow K & \\ Z \times A \times [0, 1] & & \end{array}$$

commutes where  $K(z, a, t) = \bar{f}(z, t, a)$ . In the case where  $Z = \mathbf{pt}$ , this means that if  $K : A \times [0, 1] \rightarrow Y$  is a homotopy from a map  $f : A \rightarrow Y$  and  $g$  extends  $f$  to  $X$ , then there exists a homotopy  $H : X \times [0, 1] \rightarrow Y$  such that  $H \upharpoonright_{A \times [0, 1]} = K$ . In other words, the extension problem for cell complexes is a homotopy problem.

Let's return to proving our theorem. By induction, it suffices to consider just the case where  $X = A \cup_{\varphi} D^n$ , with characteristic map  $\chi : D^n \rightarrow X$ . Thus, it suffices to find a map  $w$  such that

$$\begin{array}{ccc} Z \times D^n \times \{0\} & & \\ \downarrow & \searrow \text{id}_Z \times (g \circ \chi) & \\ Z \times D^n \times [0, 1] & \xrightarrow{w} & Y \\ \uparrow & \nearrow K & \\ Z \times S^{n-1} \times [0, 1] & \xrightarrow{\text{id}_Z \times \varphi \times \text{id}_{[0, 1]}} & Z \times A \times [0, 1] \end{array}$$

commutes for, in this case, we can set  $H(z, x, t) = g \cup_{\varphi} w$ , thereby making

$$\begin{array}{ccccc} Z \times D^n \times \{0\} & & & \searrow \text{id}_Z \times (g \circ \chi) & \\ \downarrow & & & & \\ Z \times D^n \times [0, 1] & \xrightarrow{\text{id}_Z \times \chi \times \text{id}_{[0, 1]}} & Z \times X \times [0, 1] & \xrightarrow{H} & Y \\ \uparrow & & \searrow w & \nearrow K & \\ Z \times S^{n-1} \times [0, 1] & & & & \\ & \searrow \text{id}_Z \times \varphi \times \text{id}_{[0, 1]} & & & \\ & & Z \times A \times [0, 1] & & \end{array}$$

commute. To this end, define the retraction  $u : D^n \times [0, 1] \rightarrow D^n \times \{0\} \cup S^{n-1} \times [0, 1]$  by picking a point  $*$  directly above the cylinder  $D^n \times [0, 1]$  and then sending any point  $x$  in the cylinder to the unique point where  $D^n \times \{0\} \cup S^{n-1} \times [0, 1]$  intersects the line containing  $*$  and  $x$ . Now, define  $w$  so that

$$\begin{array}{ccc} Z \times (D^n \times [0, 1]) & \xrightarrow{w} & Y \\ \text{id}_Z \times u \downarrow & \nearrow \text{id}_Z \times (g \circ \chi \cup K \circ (\varphi \times \text{id}_{[0, 1]})) & \\ Z \times (D^n \times \{0\} \cup S^{n-1} \times [0, 1]) & & \end{array}$$

commutes. □

**Exercise 1.4.3.** Let  $x \in X$ . Consider the loop space  $\Omega(X, x) \equiv M((S^1, \mathbf{pt}), (X, x))$ . Prove that  $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ .

*Proof.* Consider the *path space*  $PX \equiv \{\gamma : [0, 1] \rightarrow X \mid \gamma(0) = x\}$  of  $(X, x)$ , equipped with the compact-open topology. We claim that  $PX$  is contractible. Indeed, define  $K : PX \times [0, 1] \rightarrow PX$  by

$$(\gamma, t) \mapsto (s \mapsto \gamma(s(1-t))).$$

Then  $K$  is a homotopy from  $\text{id}_{PX}$  to the constant map at the constant path at  $x$ .

Define the map  $p : PX \rightarrow X$  by  $\gamma \mapsto \gamma(1)$ . Then  $p^{-1}(x) = \Omega(X)$ . By Corollary 1.3.9, it suffices to show that  $p$  is a fibration. To this end, suppose that the square

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\hat{f}} & PX \\ \downarrow & & \downarrow p \\ Y \times [0, 1] & \xrightarrow{f} & X \end{array}$$

commutes. Define  $H : Y \times [0, 1] \rightarrow PX$  by  $(y, t) \mapsto H(y, t)$  where

$$H(y, t)(s) = \begin{cases} \hat{f}(y)((1+t)s) & 0 \leq s \leq \frac{1}{1+t} \\ f(y, (1+t)s - 1) & \frac{1}{1+t} \leq s \leq 1 \end{cases}.$$

We see that  $H$  is continuous when viewed as a function of  $(y, t, s)$  and thus is continuous. It is easy to check that

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\hat{f}} & PX \\ \downarrow & \nearrow H & \downarrow p \\ Y \times [0, 1] & \xrightarrow{f} & X \end{array}$$

commutes, as desired. □

Let  $p : E \rightarrow B$  be a map. Recall that the pullback of  $p$  along  $f : X \rightarrow B$  is given explicitly as

$$f^*E \equiv \{(x, e) \in X \times E \mid f(x) = p(e)\}.$$

Let  $f^*p$  denote the map  $\pi_X \upharpoonright_{f^*E}$ .

**Proposition 1.4.4.** *If  $p$  is a fibration, then so is  $f^*p$ .*

**Lemma 1.4.5.** *If  $p$  is locally trivial, then so is  $f^*p$ .*

*Proof.* Let  $a \in X$ . Since  $p$  is locally trivial by assumption, we can find a neighborhood  $U$  of  $f(a)$  in  $B$  and a homeomorphism  $\varphi : p^{-1}(U) \rightarrow U \times F$ . Observe that

$$(f^*p)^{-1}(f^{-1}(U)) = \{(x, e) \mid f(x) = p(e), f(x) \in U\} \subset f^{-1}(U) \times p^{-1}(U).$$

Further, we have a map  $\psi : f^{-1}(U) \rightarrow p^{-1}(U) \rightarrow f^{-1}(U) \times F$  given by  $(x, e) \mapsto (x, \pi_F(\varphi(e)))$ . Define  $\lambda : f^{-1}(U) \times F \rightarrow (f^*p)^{-1}(f^{-1}(U))$  by  $(x, y) \mapsto (x, \varphi^{-1}(f(x), y))$ . Using the fact that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & \searrow p & \downarrow \pi_U \\ & & U \end{array}$$

commutes, it is easy to check that  $\psi$  and  $\lambda$  are inverses of each other. □

## 1.5 Lecture 5

**Theorem 1.5.1.** *Let  $B$  be a cell complex and let  $p : E \rightarrow B$  be locally trivial. Then  $p$  is a fibration.*

*Proof.* It suffices to prove the following claim:

If  $h : Z \rightarrow X \times [0, 1]$  is locally trivial,  $X = \bigcup_{i=0}^n X^i$  is a cell complex, and  $\sigma_0 : X \times \{0\} \rightarrow Z$  satisfies  $h \circ \sigma_0 = \text{id}_{X \times \{0\}}$ , then there is some map  $\sigma : X \times [0, 1] \rightarrow Z$  such that  $\sigma_{X \times \{0\}} = \sigma_0$  and  $h \circ \sigma = \text{id}_{X \times [0, 1]}$ .

For, in this case, Lemma 1.4.5 implies that given any commutative square

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}} & E \\ \downarrow & & \downarrow p, \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

we can find some  $\sigma$  such that

$$\begin{array}{ccccc} & & f^*E & \longrightarrow & E \\ & \nearrow \sigma_0 & \downarrow \sigma & & \downarrow p \\ X \times \{0\} & \hookrightarrow & X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

commutes where  $\sigma_0(x, 0) = (x, 0, \hat{f}(x, 0))$ .

For induction, let us assume that our claim is true for each  $X^0, X^1, \dots, X^{n-1}$ . We may assume, wlog, that  $X = D^n$ . It suffices to find a map  $\tau : S^{n-1} \times [0, 1] \rightarrow Z$  such that  $h \circ \tau = \text{id}_{S^{n-1} \times [0, 1]}$  and

$$\begin{array}{ccccc} & & Z & & \\ & \nearrow \sigma_0 & \downarrow h & \nwarrow \tau & \\ D^n \times \{0\} & \hookrightarrow & D^n \times [0, 1] & \longleftrightarrow & S^{n-1} \times [0, 1] \\ & \nwarrow & \uparrow & \nearrow & \\ & & S^{n-1} \times \{0\} & & \end{array}$$

commutes as there is a retraction

$$r : D^n \times [0, 1] \rightarrow D^n \times \{0\} \cup S^{n-1} \times [0, 1].$$

To this end, fix a positive integer  $m$ . For each  $j \in \{0, 1, \dots, m\}$ , let  $a_j = \frac{j}{m}$  and let  $I_j = [a_j, a_{j+1}]$ . Since  $D^n \times [0, 1]$  is compact, by making  $m$  large enough, we can ensure that  $h \upharpoonright_{h^{-1}(I_{j_1} \times \dots \times I_{j_n})}$  is trivial.

**Claim.**  $h \upharpoonright_{h^{-1}(I_{j_1} \times \dots \times I_{j_n} \times [0, 1])}$  is also trivial.

*Proof.* We may assume, wlog, that  $h$  is trivial on  $h^{-1}(I_{j_1} \times \dots \times I_{j_n} \times I_j)$  for any  $j$ . Let  $k \in \{0, 1, \dots, m-1\}$  and assume, for induction, that  $h$  is trivial on  $h^{-1}(I_{j_1} \times \dots \times I_{j_n} \times [0, a_k])$ . Let  $K = I_{j_1} \times \dots \times I_{j_n}$  and  $J = K \times [0, a_k]$  and  $\tilde{J} = K \times [a_k, a_{k+1}]$ . By assumption, there exists a homeomorphism  $\varphi : h^{-1}(J) \xrightarrow{\cong} J \times F$  such that  $\pi_J \circ \varphi = h$ . Likewise, there exists a homeomorphism  $\psi : h^{-1}(\tilde{J}) \xrightarrow{\cong} \tilde{J} \times F$  such that  $\pi_{\tilde{J}} \circ \psi = h$ . If  $\psi = \varphi$  on  $K \times \{a_k\}$ , then  $h^{-1}(J \cup \tilde{J}) \xrightarrow{\varphi \cup \psi} (J \cup \tilde{J}) \times F$  is a well-defined trivialization, in which case we're done. With this in mind, let

$$w = \left( \varphi \upharpoonright_{K \times \{a_k\}} \circ (\psi \upharpoonright_{K \times \{a_k\}})^{-1} \right) \times \text{id}_{[a_k, a_{k+1}]}.$$

Note that

$$\begin{array}{ccc} h^{-1}(\tilde{J}) & \xrightarrow{\psi} & \tilde{J} \times F \\ h \downarrow & & \downarrow w \times \text{id}_F \\ \tilde{J} & \xleftarrow{\pi_{\tilde{J}}} & \tilde{J} \times F \end{array}$$

commutes and that  $\gamma := ((w \times \text{id}_F) \circ \psi)$  agrees with  $\varphi$  on  $K \times \{a_k\}$ . Hence we may take  $\varphi \cup \gamma$  as our desired trivialization.  $\square$

As a result, we may assume that  $h$  is trivial on its entire domain, i.e., that  $h$  is the projection

$$(D^n \times [0, 1]) \times F \twoheadrightarrow D^n \times [0, 1].$$

Moreover, by induction, we can find a right inverse  $\sigma : X^{n-1} \times [0, 1] \rightarrow D^n \times [0, 1]$  of  $h : Z \rightarrow X^{n-1} \times [0, 1]$  that extends  $\sigma_0 \upharpoonright_{X^{n-1} \times \{0\}}$ . But  $X^{n-1}$  consists of all  $(n-1)$ -dimensional faces of the  $n$ -cube, and thus we have a map

$$\tau \equiv \sigma \upharpoonright_{\partial I^n \times [0, 1]} : S^{n-1} \times [0, 1] \rightarrow (D^n \times [0, 1]) \times F,$$

which has the form  $(x, t) \mapsto (x, t, \tilde{\tau}(x, t))$ . Further,  $\sigma_0$  has the form  $(x, 0) \mapsto (x, 0, \tilde{\sigma}_0(x, 0))$ . Therefore, our desired map  $\sigma : D^n \times [0, 1] \rightarrow (D^n \times [0, 1]) \times F$  is given by

$$\sigma(x, t) = (x, t, (\tilde{\sigma}_0 \cup \tilde{\tau})(r(x, t))).$$

$\square$

## 2 Fiber bundles

**Definition 2.0.1.** A *topological group*  $G$  is a group such that both multiplication  $G \times G \xrightarrow{\mu} G$  and inversion  $G \xrightarrow{\iota^{-1}} G$  are continuous.

**Definition 2.0.2 (Fiber bundle).** Let  $G$  be a topological group.

1. A *fiber*  $F$  of  $G$  is a space equipped with a faithful (i.e., injective) group action  $\rho : G \rightarrow \text{Homeo}(F) \subset M(F, F)$ .
2. An *atlas for the structure of a (fiber) bundle with group  $G$  and fiber  $F$  on a map  $p : E \rightarrow B$*  consists of
  - (a) a family  $(U_\alpha, h_\alpha)_{\alpha \in A}$  where each  $U_\alpha$  is open and each  $h_\alpha$  is a homeomorphism  $p^{-1}(U_\alpha) \rightarrow U_\alpha \times F$  and
  - (b) a family of continuous *transition functions*  $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow G\}_{\alpha, \beta \in A}$

such that

- i  $B = \bigcup_{\alpha \in A} U_\alpha$ ,
- ii  $\pi_{U_\alpha} \circ h_\alpha = p \upharpoonright_{p^{-1}(U_\alpha)}$ , and
- iii  $x \in U_\alpha \cap U_\beta \implies h_\beta \circ h_\alpha^{-1}(x, f) = (x, h_{\beta\alpha}(x) \cdot f)$

3. Two atlases are *compatible* if their union is an atlas.

4. A bundle structure on  $B$  is a maximal atlas on  $p$ .

*Terminology.* If  $B$  is equipped with a bundle structure, then we say that  $p$  is a (fiber) bundle.

**Example 2.0.3.**

1. The tangent bundle  $\pi : TM \rightarrow M$  of a smooth  $n$ -manifold  $M$  is a bundle with group  $\mathrm{GL}(n, \mathbb{R})$ .

*Proof.* Let  $(U, \varphi)$  be any coordinate chart for  $M$  with coordinate functions  $(x^i)$ . Define  $h : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  by

$$v^i \frac{\partial}{\partial x^i} (p) \mapsto (p, (v^1, \dots, v^n)).$$

It is clear that  $\pi_U(h(p)) = \pi(c)$  for any  $c \in \pi^{-1}(U)$ . To see that  $h$  is a homeomorphism, note that the composite  $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h : \pi^{-1}(U) \rightarrow \varphi(U) \times \mathbb{R}^n$  is given by

$$v^i \frac{\partial}{\partial x^i} (p) \mapsto (x^1(p), \dots, x^n(p), v^1, \dots, v^n),$$

the inverse of which is given by  $(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto v^i \frac{\partial}{\partial x^i} (\varphi^{-1}(x))$ . Therefore,  $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h$  is given locally by

$$(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto \left( \tilde{x}^1(x), \dots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^j}(x) v^j, \dots, \frac{\partial \tilde{x}^n}{\partial x^j}(x) v^j \right),$$

which is smooth. Thus,  $h$  is a diffeomorphism as the composite of two diffeomorphisms. In particular,  $h$  is a homeomorphism.

It remains to describe the transition functions  $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \mathrm{GL}(n, \mathbb{R})\}$  for  $TM$ . Note that

$$\begin{array}{ccccc} U_{\alpha\beta} \times \mathbb{R}^n & \xleftarrow{h_\alpha} & \pi^{-1}(U_{\alpha\beta}) & \xrightarrow{h_\beta} & U_{\beta\alpha} \times \mathbb{R}^n \\ & \searrow \pi_1 & \downarrow \pi & \swarrow \pi_1 & \\ & & U_{\alpha\beta} & & \end{array}$$

commutes. In particular,  $\pi_1 \circ h_\beta \circ h_\alpha^{-1} = \pi_1$ , which implies that  $h_\beta \circ h_\alpha^{-1}(u, v) = (u, f(u, v))$  for some smooth map  $f : U_{\alpha\beta} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ . This must be a linear isomorphism when restricted to  $\{u\} \times \mathbb{R}^n$  for any  $u \in U_{\alpha\beta}$ , which is uniquely determined by an element  $h_{\beta\alpha}(u)$  of  $\mathrm{GL}(n, \mathbb{R})$  (provided that we have fixed a basis of  $\mathbb{R}^n$ ). Hence

$$h_\beta \circ h_\alpha^{-1}(u, v) = (u, h_{\beta\alpha}(u)v).$$

Since the map  $h_{\beta\alpha} : U_{\alpha\beta} \rightarrow \mathrm{GL}(n, \mathbb{R})$  is continuous, our proof is complete.  $\square$

2. Let  $p : E \rightarrow B$  be any bundle with group  $\{e\}$ . Then  $p$  is the trivial bundle, i.e., is isomorphic to the projection map.

*Proof.* We have that  $h_\beta = h_\alpha$  on  $p^{-1}(U_\alpha \cap U_\beta) = p^{-1}(U_\alpha) \cap p^{-1}(U_\beta)$ , so that  $h \equiv \bigcup_{\alpha \in A} h_\alpha$  is a well-defined homeomorphism  $E \cong B \times F$ .  $\square$

## 2.1 Lecture 6

Let  $\{(U_\alpha, h_\alpha)\}$  be a bundle structure with group  $G$  and fiber  $F$  on  $p : E \rightarrow B$ . Let  $U = U_\alpha \cap U_\beta \cap U_\gamma$ . Consider the commutative diagram

$$\begin{array}{ccccccc} & & & p^{-1}(U) & & & \\ & \nearrow h_\alpha^{-1} & & & \nwarrow h_\gamma & & \\ U \times F & \xrightarrow{h_\alpha^{-1}} & p^{-1}(U) & \xrightarrow{h_\beta} & U \times F & \xrightarrow{h_\beta^{-1}} & p^{-1}(U) & \xrightarrow{h_\gamma} & U \times F \end{array} .$$

The bottom row is given by  $(u, f) \mapsto (u, h_{\beta\alpha}(u) \cdot f) \mapsto (u, h_{\gamma\beta}(u) \cdot (h_{\beta\alpha}(u) \cdot f)) = (u, (h_{\gamma\beta}(u)h_{\beta\alpha}(u)) \cdot f)$ , and the top composite is given by  $(u, f) \mapsto (u, h_{\gamma\alpha}(u) \cdot f)$ . It follows that

$$h_{\gamma\beta}(u)h_{\beta\alpha}(u) = h_{\gamma\alpha}(u)$$

for each  $u \in U$ . This property is known as the *cocycle condition*.

**Theorem 2.1.1.** *Let  $G$  be a topological group acting on a space  $F$ . Suppose that  $\{U_\alpha\}$  is an open cover of  $B$  and  $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow G\}$  is a family of continuous functions satisfying the cocycle condition. Then there exists a bundle  $p : E \rightarrow B$  with group  $G$ , fiber  $F$ , and transition functions  $h_{\beta\alpha}$ .*

*Proof sketch.* Let  $E = \coprod_\alpha U_\alpha \times F / \sim$  where  $(u, f)_\alpha \sim (u, h_{\beta\alpha}(u) \cdot f)_\beta$ . Define  $p : E \rightarrow B$  by  $(u, f) \mapsto u$ .  $\square$

**Definition 2.1.2 (Bundle map).** A morphism of bundles  $p_1$  and  $p_2$  with group  $G$  and fiber  $F$  is a commutative square of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{g}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{g} & B_2 \end{array} .$$

Suppose that  $(\hat{g}, g)$  is a bundle map  $p_1 \rightarrow p_2$ . Let  $\{(U_\alpha, h_\alpha)\}$  and  $\{(V_\beta, k_\beta)\}$  be bundle structures on  $B_2$  and  $B_1$ , respectively. We have a commutative diagram

$$\begin{array}{ccccccc} & & & d_{\alpha\beta} & & & \\ & \searrow & & & \swarrow & & \\ (g^{-1}(U_\alpha) \cap V_\beta) \times F & \xrightarrow{k_\beta^{-1}} & p_1^{-1}(g^{-1}(U_\alpha) \cap V_\beta) & \xrightarrow{\hat{g}} & p_2^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times F \\ & \searrow \pi_1 & \downarrow & & \downarrow & \swarrow \pi_1 & \\ & & g^{-1}(U_\alpha) \cap V_\beta & \xrightarrow{g} & U_\alpha & & \end{array} ,$$

so that  $d_{\alpha\beta}(x, f) = (g(x), \lambda_{\alpha\beta}(x) \cdot f)$  for some continuous map  $\lambda_{\alpha\beta} : g^{-1}(U_\alpha) \cap V_\beta \rightarrow G$ . Letting  $W = g^{-1}(U_\alpha \cap U_{\alpha'}) \cap (V_\beta \cap V_{\beta'})$ , we have that

$$h_{\alpha'\alpha}(w)\lambda_{\alpha\beta}(w)k_{\beta\beta'}(w) = \lambda_{\alpha'\beta'}(w) \quad (\dagger)$$

for every  $w \in W$ .

**Exercise 2.1.3 (Pullback bundle).** Let  $\{(U_\alpha, h_\alpha)\}$  be a bundle structure on  $p : E \rightarrow B$  with group  $G$  and consider the pullback diagram

$$\begin{array}{ccc} g^*E & \longrightarrow & E \\ g^*p \downarrow & & \downarrow p \\ X & \xrightarrow{g} & B \end{array}$$

Define  $h'_{\beta\alpha} : g^{-1}(U_\alpha) \cap g^{-1}(U_\beta) \rightarrow G$  as the composite  $h_{\beta\alpha} \circ g$  restricted to  $g^{-1}(U_\alpha \cap U_\beta)$ . Show that the family  $\{h'_{\beta\alpha}\}$  induces a bundle structure on  $g^*p$ .

**Theorem 2.1.4.** Every bundle map

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{g}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

factors as

$$\begin{array}{ccccc} E_1 & \xrightarrow{\tau} & g^*E_2 & \xrightarrow{\bar{g}} & E_2 \\ p_1 \downarrow & & \downarrow g^*p_2 & & \downarrow p_2 \\ B_1 & \xrightarrow{\text{id}_{B_1}} & B_1 & \xrightarrow{g} & B_2 \end{array}$$

where  $\tau(e) = (p_1(e), \hat{g}(e))$  for any  $e \in E_1$ .

## 2.2 Lecture 7

**Note 2.2.1.** If  $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow G\}$  is a family of transition functions, then

$$h_{\alpha\beta}(x) = (h_{\beta\alpha}(x))^{-1}$$

for any  $x \in U_\alpha \cap U_\beta$ . In particular,  $h_{\alpha\alpha}(x) = (h_{\alpha\alpha}(x))^{-1}$ .

**Theorem 2.2.2.** Any bundle map of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{g}} & E_2 \\ & \searrow p_1 & \downarrow p_2 \\ & & B \end{array}$$

is an isomorphism.

*Proof.* Note that

$$\begin{array}{ccccc} & & p_2^{-1}(U_\alpha \cap U_\beta) & & \\ & \swarrow h_\beta & \uparrow \hat{g} & \searrow h_\alpha & \\ (U_\alpha \cap U_\beta) \times F & \xleftarrow{k_\beta} & p_1^{-1}(U_\alpha \cap U_\beta) & \xleftarrow{k_\alpha^{-1}} & (U_\alpha \cap U_\beta) \times F \\ & \searrow & \downarrow & \swarrow & \\ & & U_\alpha \cap U_\beta & & \end{array}$$

commutes. We have that  $h_\beta \circ \hat{g} \circ k_\alpha^{-1}(x, f) = (x, \lambda_{\beta\alpha}(x) \cdot f)$ . Thus, if  $h_\alpha(e) = (x, f)$ , then  $h_\alpha(\hat{g}(e)) = (x, \lambda_{\alpha\alpha}(x) \cdot d)$ . Let

$$(\hat{g})^{-1}(e) = k_\alpha^{-1}(x, \lambda_{\alpha\alpha}(x)^{-1} \cdot f)$$



where  $(x, f) = h_\alpha(e)$ . If this is well-defined on  $E_2$  (??), then it indeed equals the inverse of  $\hat{g}$ . Moreover, by Note 2.2.1, it is easy to check that  $d_{\alpha'\beta'}(x)^{-1}$  satisfies  $(\dagger)$ , and thus it can be shown that  $(\hat{g})^{-1}$  is a bundle map. □

**Corollary 2.2.3.** *Every bundle  $E \rightarrow X$  is isomorphic to the pullback of  $E$  by  $\text{id}_X$ .*

Let  $\{(U_\alpha, h_\alpha)\}$  be a bundle structure with group  $G$  and fiber  $G$  on  $p : E \rightarrow X$ . In particular,

$$\begin{array}{ccc} U_\alpha \times G & \xleftarrow{h_\alpha} & p^{-1}(U_\alpha) \\ \pi_1 \downarrow & \swarrow p & \\ U_\alpha & & \end{array}$$

commutes. Define the free action  $E \times G \rightarrow E$  by

$$e \cdot g = h_\alpha^{-1}(h_\alpha(e) \cdot g).$$

where  $p(e) \in U_\alpha$  and  $(u, h) \cdot g \equiv (u, hg)$ . This is well-defined because it does not depend on our choice of  $\alpha$ . Indeed, suppose that  $p(e)$  also belongs to  $U_\beta$ . We have that  $h_\alpha(e) = (p(e), h)$  and  $h_\beta(e) = (p(e), h')$  for some  $h, h' \in G$ . Then  $e \cdot g = h_\alpha^{-1}(p(e), hg)$ , and we must show that this equals  $h_\beta^{-1}(p(e), h'g)$ . Note that  $h_\beta(e \cdot g) = (p(e), h_{\beta\alpha}(p(e))hg)$ . But

$$(p(e), h_{\beta\alpha}(p(e))h) = h_\beta(h_\alpha^{-1}(p(e), h)) = (p(e), h'),$$

so that  $h_{\beta\alpha}(p(e))h = h'$ , and thus  $h_\beta(e \cdot g) = (p(e), h'g)$ , as desired.

**Note 2.2.4.**  $E/G \cong \{p^{-1}(x) \mid x \in X\} \cong X$ .

**Definition 2.2.5 (Balanced product).** Let  $F$  be a space. The *balanced product*  $E \times_G F$  of  $E$  and  $F$  is the quotient space  $E \times F / \sim$  where

$$(e, f) \sim (eg, g^{-1}f)$$

for any  $e \in E$  and  $f \in F$ .

By the universal property of the quotient space, there is a unique map  $\bar{p}$  such that

$$\begin{array}{ccc} E \times F & \twoheadrightarrow & E \times_G F \\ p \circ \pi_E \downarrow & \swarrow \bar{p} & \\ X & & \end{array} \quad (\star)$$

*Notation.* Let  $\mathcal{B}(X, G, \rho, F)$  denote the set of all isomorphism classes of bundles over  $X$  with group  $G$  and fiber  $F$ .

**Lemma 2.2.6.**  $\bar{p}$  is a bundle with group  $G$  and fiber  $F$ .

*Proof.* As  $(g, f) \sim (e_g, gf)$ , we see that  $(U \times G) \times_G F \cong U \times F$ . Thus, we can endow  $\bar{p}$  with local trivializations and transition functions that are exactly similar to those for  $p$ . □

**Proposition 2.2.7.** *The function  $p \mapsto \bar{p}$  defines a set isomorphism  $\mathcal{B}(X, G, \rho, G) \xrightarrow{\cong} \mathcal{B}(X, G, \rho, F)$ .*

Let  $p_1 : E \rightarrow B_1$  and  $p_2 : E \rightarrow B_2$  be bundles. Let  $e_1 \in E_1$ ,  $e_2 \in E_2$ , and  $b_1 \in B_1$ .

*Question.* Can we find a bundle map

$$\begin{array}{ccc} E_1 & \dashrightarrow & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \dashrightarrow & B_2 \end{array}$$

such that  $e_1 \mapsto e_2$  and  $e_1 \mapsto b_1$ ?

Define the action  $G \times E_2 \rightarrow E_2$  by  $g * e_2 = e_2 \cdot g^{-1}$ . From this, we obtain a bundle

$$\psi : \underbrace{E_1 \times_G E_2}_{(E_1 \times E_2)/G} \rightarrow E_1 \times_G \mathbf{pt} \cong B_1$$

with fiber  $E_2$ .

**Lemma 2.2.8.** *There is a one-to-one correspondence between bundle maps  $p_1 \rightarrow p_2$  and sections of  $\psi$ .*

*Proof.* Suppose that  $\sigma$  is a section of  $\psi$ . As  $G$  acts freely on  $E_1 \times E_2$ , we see that for any  $e \in E_1$ , there exists a unique  $\tilde{e}$  such that  $\sigma(p(e)) = [(e, \tilde{e})]$ . Define  $\hat{g} : E_1 \rightarrow E_2$  by  $e \mapsto \tilde{e}$ . This respects the action of  $G$  and thus must be a bundle map.  $\square$

Now, let  $A \subset B_1$  and suppose that

$$\begin{array}{ccc} p_1^{-1}(A) & \longrightarrow & E_2 \\ \downarrow & \alpha & \downarrow p_2 \\ A & \longrightarrow & B_2 \end{array}$$

is a bundle map. Then  $\alpha$  extends when ???. Also, the corresponding section

$$\sigma : A \rightarrow p^{-1}(A) \times_G E_2 \subset E_1 \times_G E_2$$

extends.

**Definition 2.2.9 (Principal bundle).** Let  $G$  be a topological group. A *principal  $G$ -bundle* is a fiber bundle with group  $G$  and fiber  $G$  with  $G$  acting on itself by left translation.

**Theorem 2.2.10.** *Let  $f$  and  $g$  be homotopic maps  $X \rightarrow Y$ . Let  $p : E \rightarrow Y$  be any bundle with group  $G$  and fiber  $F$ . Then  $f^*p \cong g^*p$ .*

## 2.3 Lecture 8

Before proving this, we wish to determine when, given any two bundles  $p_1 : E_1 \rightarrow B_1$  and  $p_2 : E_2 \rightarrow B_2$  and any map  $g : B_1 \rightarrow B_2$ , we can find a map  $\hat{g}$  such that

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{g}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

commutes.

Define the *diagonal action*  $\Delta G$  of  $G$  on  $E_1 \times E_2$  by

$$(e_1, e_2) \cdot h = (e_1 \cdot h, e_2 \cdot h),$$

so that  $E_1 \times_G E_2 = E_1 \times E_2 / \Delta G$ . By  $(\star)$ , we can find a unique map  $\tau$  such that

$$\begin{array}{ccc} E_1 \times_G E_2 & & \\ \downarrow & \searrow \tau & \\ B_1 & \xleftarrow{\pi_1} & B_1 \times B_2 \end{array}$$

commutes.

**Exercise 2.3.1.** Show that  $\hat{g}$  exists if and only if there is some  $\lambda : B_1 \rightarrow E_1 \times_G E_2$  such that  $\tau(\lambda(b_1)) = (b_1, g(b_1))$ .

*Proof.*

( $\Leftarrow$ ) As  $G$  acts freely on  $E_1 \times E_2$ , we see that  $(e, e') \sim (e, e'') \implies e' = e''$  for any  $e', e'' \in E_2$ . Hence for any  $e \in E_1$ , there exists a unique  $\hat{e} \in E_2$  such that  $\lambda(p_1(e)) = [(e, \hat{e})]$ . Let  $\hat{g}(e) = \hat{e}$ . Then  $\hat{g}$  is clearly continuous and  $G$ -equivariant, and thus  $(\hat{g}, g)$  is a bundle map.

( $\Rightarrow$ ) Consider the homeomorphism  $\varphi : B_1 \xrightarrow{\cong} E_1 / G$  with  $\varphi(b) = p_1^{-1}(b)$ . Let  $b \in B_1$ . Let  $\varphi(b) = [e]$ . Define  $\lambda : B_1 \rightarrow E_1 \times_G E_2$  by  $\lambda(b) = [(e, \hat{g}(e))]$ . Since  $\hat{g}$  is  $G$ -equivariant, we see that  $\lambda$  is well-defined. Further,  $\lambda$  is continuous as the quotient of the map

$$f : E_1 \rightarrow E_1 \times E_2, \quad f(x) = (x, \hat{g}(x))$$

by  $G$ . Finally, it is easy to check that  $\tau(\lambda(b_1)) = (b_1, g(b_1))$  for any  $b_1 \in B_1$ .  $\square$

**Lemma 2.3.2.**  $\tau$  is locally trivial, hence a fibration.

*Proof.* Locally, we have that  $E_1 \cong U \times G$  and  $E_2 \cong V \times G$ , so that  $E_1 \times E_2 \cong U \times V \times G \times G$ . It follows that, locally,  $E_1 \times_G E_2 \cong U_1 \times U_2 \times G \times G / \Delta G$  where  $\Delta G \equiv \{(g, g) \mid g \in G\}$ .  $\square$

*Remark 2.3.3.* In fact,  $\tau$  is a bundle with fiber  $G \times G / \Delta G \cong G$ .

*Proof of Theorem 2.2.10.* Due to Proposition 2.2.7, we may assume that  $p$  is a principal  $G$ -bundle. By assumption, there is some homotopy  $H : X \times I \rightarrow Y$  from  $f$  to  $g$ . Let  $\omega = H^*p$ . Then

$$\begin{aligned} f^*p &= \omega \upharpoonright_{\omega^{-1}(X \times \{0\})} : \omega^{-1}(X \times \{0\}) \rightarrow X \times \{0\} \cong X \\ g^*p &= \omega \upharpoonright_{\omega^{-1}(X \times \{1\})} : \omega^{-1}(X \times \{1\}) \rightarrow X \times \{1\} \cong X. \end{aligned}$$

Therefore, it suffices to show that  $f^*p \times \text{id}_I \cong \omega$  such that the diagram

$$\begin{array}{ccccc} f^*E \times I & \xrightarrow{\cong} & H^*E & \longrightarrow & E \\ f^*p \times \text{id}_I \downarrow & & \downarrow \omega & & \downarrow p \\ X \times I & \xlongequal{\quad} & X \times I & \xrightarrow{H} & Y \end{array}$$

commutes. For, in this case, our isomorphism restricts over  $X \times \{1\}$ , i.e.,  $g^*p = \omega \upharpoonright_{X \times \{1\}} \cong f^*p$ . It thus suffices to exhibit a bundle map  $f^*p \times I \rightarrow \omega$  over  $\text{id}_{X \times I}$  that equals the identity over  $\omega \upharpoonright_{X \times \{0\}} = f^*p$ .

*Remark 2.3.4.* It is easy to show that there is some bundle map  $f^*p \times \text{id}_I \rightarrow \omega$ . Indeed, by the homotopy lifting property, we obtain a section  $\sigma$  fitting into the commutative diagram

$$\begin{array}{ccc} & (f^*E \times I) \times_G H^*E & \\ \lambda_0 \nearrow & \downarrow \tilde{\lambda} \sigma & \\ X \times \{0\} & \longrightarrow & X \times I \end{array},$$

in which case we obtain our desired map by Lemma 2.2.8. As mentioned, however, we want a bundle map that equals the identity over  $f^*p$ .

To get such a map, we must find a section  $\lambda$  such that

$$\begin{array}{ccccc} & (f^*E \times I) \times_G H^*E & & & \\ \lambda_0 \nearrow & \downarrow \tilde{\lambda} \lambda & \searrow \tau & & \\ X \times \{0\} & \longrightarrow & X \times I & \xrightarrow{\Delta} & (X \times I) \times (X \times I) \end{array}$$

commutes. But  $\lambda$  must exist since  $\tau$  is a fibration by virtue of Lemma 2.3.2. □

**Corollary 2.3.5.** *Any bundle over a contractible space  $B$  is trivial.*

*Proof.* Let  $i : \text{pt} \rightarrow B$  and  $\pi : B \rightarrow \text{pt}$  denote inclusion and projection, respectively. Then

$$\begin{aligned} p &\cong (\text{id})^* p \\ &\cong (i\pi)^* p \\ &\cong \pi^* \underbrace{i^* p}_{\text{trivial}}, \end{aligned}$$

which is trivial since the pullback of a trivial bundle is trivial. □

**Corollary 2.3.6.** *Every bundle  $p$  over  $X \times I$  is isomorphic to  $(p \upharpoonright_{p^{-1}(X \times \{0\})}) \times \text{id}_I$ .*

**Example 2.3.7.** Consider  $S^1 \subset \mathbb{R}^2$  with center the origin. Let  $p : E \rightarrow S^1$  be a bundle with group  $G$  and fiber  $F$ . Cover  $S^1$  with the open intervals  $I_1 := S^1 \setminus \{-1\}$  and  $I_2 := S^1 \setminus \{1\}$ . We may assume that  $F = p^{-1}(-1)$ . Then  $E = E_1 \cup E_2$  where  $E_i \cong I_i \times F$  via, say,  $\varphi_i$  for each  $i = 1, 2$ . By Corollary 2.3.6, we see that

$$\varphi_1 \upharpoonright_{\varphi_1^{-1}(\{1\} \times F)} = \varphi_2 \upharpoonright_{\varphi_2^{-1}(\{-1\} \times F)} = \text{id}_F.$$

Moreover, the transition function  $\varphi_2^{-1} \circ \varphi_1 \upharpoonright_{p^{-1}(1)} : F \rightarrow F$  is given by multiplication by some  $g \in G$ . Hence the map  $G \rightarrow \mathcal{B}(S^1, G, F)$  is surjective. In fact, it can be shown that this maps descends to an isomorphism

$$\pi_0(G) \cong G/G_0 \xrightarrow{\cong} \mathcal{B}(S^1, G, F)$$

where  $G_0$  denotes the connected component of  $e_G$ .

For example, if  $G = F = \text{GL}(n, \mathbb{R})$ , then  $\pi_0(G)$  consists of the set of matrices with positive determinant and the set of matrices with negative determinant, so that  $\mathcal{B}(S^1, G, F) \cong \mathbb{Z}_2$ .

**Example 2.3.8.** The set  $\mathcal{B}(S^2, G, F)$  is isomorphic to the set of homotopy classes of maps  $S^1 \rightarrow G$ . As it turns out, we can ignore base points, so that  $\mathcal{B}(S^2, G, F) \cong \pi_1(G)$ .

For example, if  $G = F = \text{SO}(2)$ , then  $G \cong S^1$ , so that  $\mathcal{B}(S^2, G, F) \cong \mathbb{Z}$ .

## 2.4 Lecture 9

**Theorem 2.4.1.** *Let  $X$  be a cell complex with  $\dim X \leq n$ . Let  $A \subset X$  be a subcomplex. Let  $p : E \rightarrow X$  be a bundle with fiber  $F$  such that  $\pi_i(F, f) = 0$  for each  $i \leq n-1$ . Suppose that  $\sigma_0 : A \rightarrow E$  satisfies  $p \circ \sigma_0(a) = a$  for each  $a \in A$ . Then  $\sigma_0$  extends to a section  $\sigma : X \rightarrow E$  of  $p$ .*

$$\begin{array}{ccc} & E & \\ \sigma_0 \nearrow & \downarrow p & \nwarrow \sigma \\ A & \hookrightarrow & X \end{array}$$

*Proof.* First, assume that  $X$  is a regular complex. Since  $X$  is finite, we may assume that  $X = A \cup_{S^{k-1}} D^k$  where  $k \leq n$ . Further, we may assume, wlog, that  $X = D^k$ . Thus, we must find a section  $\sigma$  such that

$$\begin{array}{ccc} & E & \\ \sigma_0|_{S^{k-1}} \nearrow & \downarrow p & \nwarrow \sigma \\ S^{k-1} & \hookrightarrow & D^k \end{array}$$

commutes. Since  $D^k$  is contractible, we have that  $E \cong D^k \times F$ . Then  $\sigma_0(x) = (x, \tilde{\sigma}_0(x))$  for each  $x \in S^{k-1}$ . But  $\tilde{\sigma}_0(x) : S^{k-1} \rightarrow F$  extends to a map  $\tilde{\sigma} : D^k \rightarrow F$  because  $\pi_{k-1}(F) = 0$ . Hence we can take  $\sigma$  to be the map defined by  $x \mapsto (x, \tilde{\sigma}(x))$ .

Next, drop the assumption that  $X$  is regular. Using Exercise 1.1.9, we get a homotopy equivalence

$$\begin{array}{ccc} & h \nearrow & \\ (X, A) & & (\overline{X}, \overline{A}) \\ & g \nwarrow & \\ & \text{regular} & \end{array}$$

of pairs. Define  $\overline{A} \rightarrow g^*E$  by  $\bar{\sigma}_0(a) = (a, \sigma_0(g(a)))$ . By our preceding discussion, this extends to a section  $\bar{\sigma}$  on  $\overline{X}$ .<sup>1</sup> We wish to find  $\sigma$  such that

$$\begin{array}{ccc} g^*E & \longrightarrow & E \\ \bar{\sigma} \uparrow \left( \begin{array}{c} \downarrow p \\ \downarrow \sigma \end{array} \right) \bar{\sigma} & & \uparrow \left( \begin{array}{c} \downarrow p \\ \downarrow \sigma \end{array} \right) \sigma \\ \overline{X} & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ \overline{A} & \xrightarrow{g} & A \end{array}$$

commutes. But since  $p \cong h^*g^*p$ , we have a commutative diagram

$$\begin{array}{ccccc} g^*E & \longleftarrow & h^*g^*E & \xrightarrow{\cong} & E \\ \bar{\sigma} \uparrow \left( \begin{array}{c} \downarrow g^*p \\ \downarrow h^*g^*p \end{array} \right) \bar{\sigma} & & \downarrow & \nearrow p & \\ \overline{X} & \xleftarrow{h} & X & & \end{array},$$

from which we obtain our desired section  $\sigma$ . □

*Notation.*  $[X, Y] := (\text{homotopy classes of maps } X \rightarrow Y)$ .

**Corollary 2.4.2.** *Let  $p : E \rightarrow B$  be a principal  $G$ -bundle and suppose that  $\pi_i(E) = 0$  for any  $i \leq n-1$ . The function  $\chi_X : [X, B] \rightarrow \mathcal{B}(X, G, G)$  given by  $f \mapsto f^*p$  is bijective.*

<sup>1</sup>As  $\dim \overline{X} > \dim X$ , we tacitly rely on the fact that  $\pi_i(F)$  is trivial for large enough  $i$ .

*Proof.*

Surjective: Let  $p_1 : E_1 \rightarrow X$  be a bundle. Due to Theorem 2.1.4, it suffices to find a bundle map  $(\hat{f}, f)$  such that

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E \\ p_1 \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{array}$$

commutes. Such a map can be found precisely when there exists a section of the bundle  $E_1 \times_G E \rightarrow X$ , which holds by applying Theorem 2.4.1 to the case where  $A = \emptyset$ .

Injective: Suppose that  $\chi_X(f) = \chi_X(g)$ . We must show that  $f \simeq g$ , i.e., that there is some bundle map  $(\hat{H}, H)$  such that

$$\begin{array}{ccccc} & & & & \text{curved arrow} \\ & & & & \nearrow \\ f^*p \times \{0, 1\} & \hookrightarrow & f^*p \times I & \xrightarrow{\hat{H}} & E \\ \nwarrow \cong & & \downarrow & & \downarrow p \\ f^*p \cup g^*p & & X \times I & \xrightarrow{H} & B \\ & \nearrow & & & \\ & X \times \{0, 1\} & & & \end{array}$$

commutes. This is equivalent to finding a section  $\lambda$  such that

$$\begin{array}{ccc} (X \times \{0, 1\}) \times B & \xleftarrow{\tau} & (f^*p \times I) \times_G E \\ \gamma \uparrow & \nearrow \lambda_0 & \downarrow \lambda \\ X \times \{0, 1\} & \hookrightarrow & X \times I \end{array}$$

commutes where

$$\gamma(x, t) = \begin{cases} (x, t, f(x)) & t = 0 \\ (x, t, g(x)) & t = 1 \end{cases}.$$

But this exists by Theorem 2.4.1 because  $\pi_i(E) = 0$  by assumption.  $\square$

**Definition 2.4.3 (Classifying space).** A *classifying space for principal  $G$ -bundles* is a space  $B$  such that  $\chi_X$  is bijective for every cell complex  $X$ .

**Example 2.4.4.** Let  $G = \{\pm 1\}$ . Then any principal  $G$ -bundle over  $X$  is a two-fold covering space of  $X$ , i.e., a subgroup of index two in  $\pi_1(X)$ , i.e., a nontrivial homomorphism  $\pi_1 X \rightarrow G$ .

For example, let  $\{U_i\}$  denote the usual open covering of  $\mathbb{RP}^n = S^n/G$ . Let  $\pi : S^n \rightarrow \mathbb{RP}^n$  denote the projection map. We have that  $\pi^{-1}(U_i) \cong U_i \times G$ . Indeed, define  $h_i : \pi^{-1}(U_i) \rightarrow U_i \times G$  by

$$(x_0, \dots, x_n) \mapsto \left( [x_0, \dots, x_n], \frac{x_i}{|x_i|} \right),$$

the inverse of which is given by

$$\begin{aligned} (y_0, \dots, y_n) &\mapsto ([x_0, \dots, x_n], \epsilon) \\ y_k &\equiv \epsilon x_k \cdot \frac{|x_i|}{x_i}. \end{aligned}$$

Note that any transition function  $h_{ji} : U_i \cap U_j \rightarrow G$  is given by  $h_{ji}(x) = -1$ .

Using the fact that  $\pi_1$  is the abelianization of  $H_1$  along with the universal coefficient theorem for cohomology, one can prove the following.

**Proposition 2.4.5.**  $\mathcal{B}(X, \mathbb{Z}_2, F) \cong [X, \mathbb{RP}^n] \cong \text{Hom}(\pi_1(X), \mathbb{Z}_2) \cong H^1(X; \mathbb{Z}_2)$ .

Let  $w_1 \in H^1(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$  be nonzero. Let  $p_1 : E \rightarrow X$  be a  $\mathbb{Z}_2$ -bundle. We call  $w_1(p_1) := f^*w_1 \in H^1(X; \mathbb{Z}_2)$  the *first Stiefel-Whitney class* of  $p$ .

## 2.5 Lecture 10

**Example 2.5.1.** Let  $n \in \mathbb{N}$ . Recall that  $\mathbb{CP}^n$ , by definition, consists of all the complex lines in  $\mathbb{C}^{n+1}$ . Let  $G = S^1$ . Then  $G$  acts on  $\mathbb{C}^{n+1}$  by  $g \cdot (z_0, \dots, z_n) = (gz_0, \dots, gz_n)$ . We have that  $\mathbb{CP}^n \cong S^{2n+1}/\sim$  where  $z \sim \zeta \cdot z$  for any  $\zeta \in S^1$ . Consider the projection map  $\pi : S^{2n+1} \rightarrow \mathbb{CP}^n$ . For each  $i \in \{0, \dots, n\}$ , let  $H_i = \{z \in \mathbb{CP}^n \mid z_i = 0\} \cong \mathbb{CP}^{n-1}$  and let  $U_i = \mathbb{CP}^n \setminus H_i$ . Then the  $U_i$  form an open cover of  $\mathbb{CP}^n$ . Define  $h_i : \pi^{-1}(U_i) \rightarrow U_i \times S^1$  by  $(z_0, \dots, z_n) \mapsto ([z_0, \dots, z_n], \frac{z_i}{|z_i|})$ .

**Exercise 2.5.2.**

1. Prove that  $h_i$  is a homeomorphism.
2. Find the transition functions  $h_{ij} : U_j \cap U_i \rightarrow S^1$ .

*Proof.*

1. It is obvious that  $h_i$  is continuous. Define  $g_i : U_i \times S^1 \rightarrow \pi^{-1}(U_i)$  by

$$([z_0, \dots, z_n], \epsilon) \mapsto (y_0, \dots, y_n)$$

$$y_k \equiv \epsilon z_k \cdot \frac{|z_i|}{z_i}, \quad k = 0, \dots, n.$$

It is easy to check that this is well-defined and that  $g_i$  is the inverse of  $h_i$ . It remains to show that  $g_i$  is continuous. Consider the quotient map  $q := \pi \times \text{id}_{S^1} : S^{2n+1} \times S^1 \rightarrow \mathbb{CP}^n \times S^1$ . Let  $\tilde{U}_i = \{z \in S^{2n+1} \mid z_i \neq 0\}$ . Note that  $g_i \circ q|_{\tilde{U}_i \times S^1}$  is clearly continuous. But  $\tilde{U}_i \times S^1$  is both open in  $S^{2n+1} \times S^1$  and saturated with respect to  $q$ . Hence  $q|_{\tilde{U}_i \times S^1}$  is a quotient map, so that  $g_i$  is continuous.

2. Note that

$$h_i \circ h_j^{-1}([z_0, \dots, z_n], \epsilon) = \left([z_0, \dots, z_n], \epsilon \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}\right)$$

for any  $[z_0, \dots, z_n] \in U_i \cap U_j$ . This implies that

$$h_{ij}([z_0, \dots, z_n]) = \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}.$$

□

It follows that  $\pi$  is a principal  $S^1$ -bundle. Since each homotopy group  $\pi_i(S^{2n+1})$  is trivial, Corollary 2.4.2 implies that

$$\mathcal{B}(X, S^1, F) \cong [X, \mathbb{CP}^n],$$

which for large enough  $n$ , is isomorphic to  $[X, \mathbb{CP}^\infty]$  where  $X$  denotes any cell complex and

$$\mathbb{CP}^\infty \equiv \bigcup_{k \in \mathbb{N}} \mathbb{CP}^k$$

equipped with the weak topology.

**Definition 2.5.3.** An *Eilenberg-MacLane space of type  $K(G, n)$*  is a space satisfying

$$\begin{cases} \pi_i K = 0 & i \neq n \\ \pi_i K \cong G & i = n \end{cases}.$$

**Theorem 2.5.4.** If  $X$  is a cell complex, then  $[X, K(G, n)] \cong H^n(X; G)$ .

**Example 2.5.5.** By inspecting the long exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_2(S^{2n+1}) & \longrightarrow & \pi_2(\mathbb{CP}^n) & & \\ & & \searrow & & \swarrow & & \\ & & \pi_1(S^1) & \longrightarrow & \pi_1(S^{2n+1}) & \longrightarrow & \cdots \end{array},$$

$\underbrace{\pi_1(S^1)}_{\mathbb{Z}}$

we see that  $\mathbb{CP}^n$  is an Eilenberg-MacLane space of type  $K(\mathbb{Z}, 2)$ . Moreover, there is a commutative triangle

$$\begin{array}{ccc} \mathbb{CP}^\infty & \longleftrightarrow & \mathbb{CP}^n \\ \uparrow & \nearrow & \\ S^i & & \end{array}$$

for any  $i \in \mathbb{N}$ . Thus,  $\pi_i(\mathbb{CP}^\infty) = \pi_i(\mathbb{CP}^n)$  when  $n$  is large enough. This means that  $\mathbb{CP}^\infty$  is also an Eilenberg-MacLane space of type  $K(\mathbb{Z}, 2)$ . By Theorem 2.5.4, we have that

$$\mathcal{B}(X, S^1, F) \cong H^2(X; \mathbb{Z})$$

whenever  $X$  is a cell complex.

For us, a CW complex refers to a cell complex  $X$  for which there may be infinitely many attaching maps of any dimension. In this name, “C” stands for the property *closure-finite*, i.e., every open cell  $e^i$  is contained in a finite subcomplex of  $X$ . Further, “W” stands for the weak topology, with which  $X$  is equipped.

*Remark 2.5.6.* Each of our results holds even if we assume that a certain space is merely a CW complex rather than a cell complex.

We want to ensure that a classifying space for a topological group  $G$  exists. There are at least two famous ways of finding a classifying space for  $G$ .

**Theorem 2.5.7 (Milnor construction).** *There is some functor  $\mathbf{TopGrp} \rightarrow \mathbf{PrinBund}$  that maps each topological group  $G$  to a principal  $G$ -bundle*

$$E_G \xrightarrow{p_G} B_G$$

*such that  $B_G$  is a CW complex and  $\pi_i(E_G) = 0$ .*

This means that  $B_G$  is a classifying space for principal  $G$ -bundles. Moreover, by applying our LES on homotopy groups to  $p_G$ , we see that  $\pi_i(B_G) \cong \pi_{i-1}(G)$ .

Our next method is slightly less powerful than Theorem 2.5.7 in that it produces a classifying space for principal  $G$ -bundles only over pointed connected (equivalently, path connected) CW complexes. At the same time, it produces “classifying” objects in settings other than that of principal  $G$ -bundles.



**Theorem 2.5.8 (Brown representability).** *Consider the homotopy category  $\mathrm{Ho}(\mathbf{CW}_*^{\mathrm{conn}})$  of pointed connected CW complexes. Let  $F$  be a functor  $\mathrm{Ho}(\mathbf{CW}_*^{\mathrm{conn}})^{\mathrm{op}} \rightarrow \mathbf{Set}_*$  with the following properties.*

(i) (Wedge axiom)  $F$  takes coproducts in  $\mathrm{Ho}(\mathbf{CW}_*^{\mathrm{conn}})$  to products in  $\mathbf{Set}_*$ , i.e.,

$$F\left(\bigvee_{\alpha} X_{\alpha}\right) \cong \prod_{\alpha} F(X_{\alpha}).$$

(ii) (Mayer-Vietoris axiom)  $F$  takes weak pushouts to weak pullbacks, i.e., the universal morphism

$$F(B \cup_A C) \rightarrow F(B) \times_{F(A)} F(C)$$

is a surjection (or split epimorphism) for any two cofibrations  $A \rightarrow B$  and  $A \rightarrow C$  in  $\mathbf{CW}_*^{\mathrm{conn}}$ .

Then  $F$  is representable.<sup>2</sup>

By the Yoneda lemma, this means that there exists a pointed connected CW complex  $B$  along with an element  $b \in F(B)$  such that the set map  $[X, B] \rightarrow F(X)$  given by  $g \mapsto F(g)(b)$  is a natural bijection in  $X$ . In particular, Theorem 2.5.8 applied to the functor taking any pointed connected CW complex to the set of all pointed principal  $G$ -bundles over  $X$  makes the object  $(B'_G, b)$  representing  $F$  a classifying space for  $G$ .

*Remark 2.5.9.* The converse of Theorem 2.5.8 is true because any representable contravariant functor takes colimits to limits.

*Proof sketch of Theorem 2.5.8.*

We say that a pointed connected CW complex  $B$  along with an element  $b \in F(B)$  *spherically represents*  $F$  if the set map  $\nu_X : [X, B] \rightarrow F(X)$  given by  $g \mapsto F(g)(b)$  is a natural bijection in  $X \in \{S^n \mid n \in \mathbb{Z}_{\geq 1}\}$ .

Suppose that  $(B, b)$  and  $(B', b')$  are two objects spherically representing  $F$ . Let  $f : B \rightarrow B'$  be a map such that  $F(f)(b') = b$ . Then  $f$  induces a weak homotopy equivalence. By Theorem 1.2.8,  $f$  must be a homotopy equivalence.

Now, let  $X$  be a pointed CW complex and  $x \in F(X)$ . It is known that there exist an object  $(B, b)$  spherically representing  $F$  and a map

$$\varphi : X \rightarrow B, \quad F(\varphi)(b) = x. \quad (\bullet)$$

Therefore, it suffices to prove the following assertion.

**Claim.** *Any object  $(B, b)$  spherically representing  $F$  represents  $F$ .*

*Proof.* We must show that  $\nu_X$  is a bijection. In the interest of space, let us prove just that it is surjective. Let  $x \in F(X)$  and consider the coproduct

$$X \xrightarrow{i_1} X \vee B \xleftarrow{i_2} B.$$

By the wedge axiom, we have that  $F(X \vee B) \cong F(X) \times F(B)$  with  $F(i_1) \cong \pi_1$  and  $F(i_2) \cong \pi_2$ . Thus, we have an element  $(x, b) \in F(X \vee B)$  such that  $F(i_1)(x, b) = x$  and  $F(i_2)(x, b) = b$ . Thanks to  $(\bullet)$ , we can find

<sup>2</sup>According to a certain [MathOverflow answer](#), Theorem 2.5.8 holds when  $F$  is instead a functor  $\mathbf{CW} \rightarrow \mathbf{Ab}$ .

an object  $(\widehat{B}, \hat{b})$  spherically representing  $F$  along with a map  $\varphi : X \vee B \rightarrow \widehat{B}$  such that  $F(\varphi)(\hat{b}) = (x, b)$ . It follows that

$$\begin{aligned} F(\varphi \circ i_1)(\hat{b}) &= x \\ F(\varphi \circ i_2)(\hat{b}) &= b. \end{aligned}$$

Hence  $\varphi \circ i_2$  is a homotopy equivalence  $B \rightarrow \widehat{B}$ , with homotopy inverse, say,  $\eta$ . This implies that

$$F(\eta \circ \varphi \circ i_2)(b) = x,$$

so that  $\nu_X$  is surjective. □

□

Let us turn to the question of uniqueness of a classifying space, having just considered the question of existence.

**Lemma 2.5.10.** *Let  $p_1 : E_1 \rightarrow B_1$  and  $p_2 : E_2 \rightarrow B_2$  be classifying spaces for principal  $G$ -bundles. Then  $B_1 \simeq B_2$ .*

*Proof.* By Corollary 2.4.2, there is some map  $f : B_1 \rightarrow B_2$  such that  $f^*p_2 \cong p_1$ . Likewise, there is some map  $g : B_2 \rightarrow B_1$  such that  $g^*p_1 \cong p_2$ . Therefore,

$$\begin{aligned} (f \circ g)^*p_2 &\cong g^*f^*p_2 \\ &\cong g^*p_1 \\ &\cong p_2 \\ &\cong \text{id}_{B_2}^*p_2. \end{aligned}$$

Therefore,  $f \circ g \simeq \text{id}_{B_2}$ . Similarly,  $g \circ f \simeq \text{id}_{B_1}$ . □

In particular,  $B_G \simeq B'_G$ .

**Example 2.5.11.**  $B_{S^1} = \mathbb{CP}^\infty$ .

Let  $H \leq G$ . Consider the commutative square

$$\begin{array}{ccc} E_G & \xrightarrow{q} & E_{G/H} \\ p_G \downarrow & & \downarrow r \\ B_G & \xlongequal{\quad} & E_{G/H} \end{array} \quad .$$

Note that, locally,  $r$  looks like the trivial map with fiber  $G/H$ . Thus,  $q$  locally looks like the map

$$U \times G \rightarrow U \times G/H.$$

This shows that if the natural projection  $G \rightarrow G/H$  is a principal  $H$ -bundle, then so is  $q$ . In this case, we have that  $B_H \simeq E_{G/H}$  by Corollary 2.4.2 together with Lemma 2.5.10.

**Theorem 2.5.12.** *If  $G$  is a Lie group and  $H$  is a closed subgroup of  $G$ , then the natural projection  $G \rightarrow G/H$  is a principal  $H$ -bundle.*

**Definition 2.5.13.** The *orthogonal group*  $O(n, \mathbb{R})$  is the group of  $n \times n$  real matrices  $A$  such that  $AA^t = A^t A = I_n$ , equivalently,  $Av \bullet Aw = v \bullet w$  for any  $v, w \in \mathbb{R}^n$ . We call such an  $A$  *orthogonal*.

In particular, if  $A$  is orthogonal, then  $\|Av\| = \|v\|$  for any  $v \in \mathbb{R}^n$ .

**Example 2.5.14.** The orthogonal group  $O(n, \mathbb{R})$  is a closed subgroup of  $GL(n, \mathbb{R})$  because  $O(n, \mathbb{R}) = f^{-1}(I_n)$  where  $f : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$  is given by  $X \mapsto XX^t$ . Let  $\gamma : GL(n, \mathbb{R}) \rightarrow O(n, \mathbb{R})$  denote the map given by the Gram-Schmidt procedure. Let  $i : O(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$  denote the inclusion map. Then  $\gamma$  and  $i$  are homotopy inverses of each other, so that

$$GL(n, \mathbb{R}) \simeq O(n, \mathbb{R}).$$

Since  $\pi : GL(n, \mathbb{R}) \rightarrow \underbrace{GL(n, \mathbb{R})/O(n, \mathbb{R})}_M$  is an  $O(n, \mathbb{R})$ -bundle by Theorem 2.5.12, our LES on homotopy groups applied to  $\pi$  shows that  $\pi_i(M) = 0$  for each  $i \in \mathbb{N}$ . Further, our LES applied to the  $M$ -bundle  $r : B_{O(n, \mathbb{R})} \rightarrow B_{GL(n, \mathbb{R})}$  shows that

$$\pi_i(B_{O(n, \mathbb{R})}) \cong \pi_i(B_{GL(n, \mathbb{R})})$$

for each  $i$ . By Theorem 1.2.8, it follows that

$$B_{O(n, \mathbb{R})} \simeq B_{GL(n, \mathbb{R})}.$$

An exactly similar argument proves that  $B_{U(n, \mathbb{C})} \simeq B_{GL(n, \mathbb{C})}$ .

Eventually, we want to describe  $H^*(B_G)$ . This will lead us to the notion of a spectral sequence.

## 2.6 Lecture 11

Before moving to spectral sequences, let us look at a couple more examples of fiber bundles.

**Example 2.6.1.** Let  $\{e_i\}_{1 \leq i \leq n}$  denote the standard basis of  $\mathbb{R}^n$ . Consider the map  $\rho : GL(n, \mathbb{R}) \rightarrow \mathbb{R}^n \setminus \{0\}$  given by  $A \mapsto Ae_n$  and its restriction  $\tau : O(n, \mathbb{R}) \rightarrow S^{n-1}$ . Note that  $\rho^{-1}(e_n)$  consists of all  $n \times n$  matrices of the form

$$\left( \begin{array}{c|c} B & 0 \\ \hline * & 1 \end{array} \right)$$

where  $B$  denotes an invertible  $(n-1) \times (n-1)$  matrix. This means that  $\rho^{-1}(e_n) \simeq GL(n-1, \mathbb{R})$ . Similarly, we see that  $\tau^{-1}(e_n) \simeq O(n-1, \mathbb{R})$ . Moreover, both  $\rho$  and  $\tau$  are locally trivial. In particular, this yields a LES

$$\begin{array}{ccccccc} & & & \cdots & \longrightarrow & \pi_{i+1}(S^{n-1}) & \\ & & & & \swarrow & & \\ \pi_i(O(n-1)) & \longrightarrow & \pi_i(O(n)) & \longrightarrow & \pi_i(S^{n-1}) & \cdot & \\ & & & \nwarrow & & & \\ \pi_{i-1}(O(n-1)) & \longrightarrow & \cdots & & & & \end{array}$$

Since  $\pi_i(S^{n-1})$  is trivial for any  $0 \leq i \leq n-2$ , we see that the map  $\pi_i(O(n-1)) \rightarrow \pi_i(O(n))$  is an isomorphism for any  $i \leq n-3$  and an epimorphism when  $i = n-2$ . The same result holds with  $O(n)$  replaced by  $GL(n, \mathbb{R})$ .

**Example 2.6.2.** Consider the *Stiefel manifold*  $V_{n+k,k}$  consisting of orthonormal  $k$ -frames (i.e.,  $k$ -tuples) in  $\mathbb{R}^{n+k}$ . If we view the standard basis of  $\mathbb{R}^k$  as the “zero element” of  $V_{n+k,k}$ , then we have a “short exact sequence”

$$0 \longrightarrow O(n) \xhookrightarrow{i} O(n+k) \xrightarrow{p_1} V_{n+k,k} \longrightarrow 0$$

where  $i$  is given by  $A \mapsto \left( \begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array} \right)$  and  $p_1$  is given by  $A \mapsto (Ae_{n+1}, \dots, Ae_{n+k})$ . In this case,

$$V_{n+k,k} \cong \frac{O(n+k)}{O(n)},$$

a coset space. Note that  $i$  induces an isomorphism  $\pi_i(O(n)) \xrightarrow{\cong} O(n+k)$  for each  $i \leq n-2$  and an epimorphism when  $i = n-1$ .

**Claim.** *The map  $p_1$  is a fiber bundle.*

*Proof.* Let  $F \in V_{n+k,k}$  and choose any orthonormal basis  $B$  of the  $n$ -plane orthogonal to  $F$ . For any  $n$ -plane near  $B$ , take the orthogonal projection of  $B$  onto  $B'$  and then apply the Gram-Schmidt process to the new basis to obtain an orthonormal basis  $\underline{B}'$  of  $B'$ . The assignment  $B \mapsto \underline{B}'$  is continuous, and the space of all  $n$ -planes orthogonal to any  $(n+k)$ -plane near  $F$  is identifiable with  $V_n(\mathbb{R}^n) \cong O(n)$ . Therefore, we get a trivialization around  $F$ , which was arbitrary.  $\square$

Using the LES obtained from Corollary 1.3.9, we see that  $\pi_i(V_{n+k,k}) = 0$  for each  $i \leq n-1$ . Consider now the *Grassmann manifold*

$$G_{n+k,k} \equiv \frac{O(n+k)}{O(n) \times O(k)}$$

where each pair  $(A, B) \in O(n) \times O(k)$  is identified with the orthogonal  $(n+k) \times (n+k)$  matrix  $\left( \begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right)$ .

Note that  $G_{n+k,k}$  may be viewed as the space of all  $k$ -dimensional planes in  $\mathbb{R}^{n+k}$ .

**Proposition 2.6.3.**

1. *The space  $E_{O(k)}$  consists of all orthonormal  $k$ -frames in  $\mathbb{R}^\infty$ .*
2. *The Grassmannian  $G_{\infty,k} \equiv B_{O(k)} = B_{GL(k)}$  consists of all  $k$ -planes in  $\mathbb{R}^\infty$ .*
3. *Similarly, the space  $B_{U(k)}$  consists of all  $k$ -planes in  $\mathbb{C}^\infty$ .*

Define  $p_2 : V_{n+k,k} \rightarrow G_{n+k,k}$  by sending each  $v \in V_{n+k,k}$  to the subspace of  $\mathbb{R}^{n+k}$  spanned by  $v$ .

**Claim.** *The map  $p_2$  is a principal  $O(k)$ -bundle.*

*Proof.* This follows from the fact that  $O(n+k) \rightarrow G_{n+k,k}$  is a principal  $O(n) \times O(k)$ -bundle.  $\square$

As a result,  $\pi_i(G_{n+k,k}) = 0$  for each  $i \leq n-2$ .

### 3 Spectral sequences

We are given a fibration:

$$\begin{array}{c} F \\ \downarrow \\ E \\ \downarrow \pi \\ X \end{array}$$

where  $X$  is a connected cell complex and  $F = \pi^{-1}(x)$  for some distinguished point  $x$ .

*Question.* What is  $H_n(E)$  if we know  $H_n(F)$  and  $H_n(X)$ ?

Recall that  $H_n(X) = \frac{\ker \partial_n}{\text{im } \partial_{n+1}}$  where  $\partial_n$  is defined as the composite

$$\begin{array}{ccccc} \overbrace{H_n(X^n, X^{n-1})}^{C_n(X)} & \longrightarrow & H_{n-1}(X^{n-1}) & \longrightarrow & \overbrace{H_{n-1}(X^{n-1}, X^{n-2})(X)}^{C_{n-1}} \\ & & & \searrow \partial_n & \nearrow \end{array}$$

where  $H_i(X^n, X^{n-1}) = 0$  for any  $i \neq n$ . Furthermore, letting  $E_n = \pi^{-1}(X_n)$ , we have that  $H_*(E_n, E_{n-1}) = C_*(X) \otimes H_*(F)$ .

At this point, it is useful to generalize our situation by developing the theory of spectral sequences. For each  $r \in \mathbb{Z}_{\geq 0}$ , let  $\{E_{p,q}^r\}_{p,q \in \mathbb{Z}}$  be a family of abelian groups and let  $\{d_r^{p,q} : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r\}_{p,q \in \mathbb{Z}}$  be a family of maps (called *differentials*) such that

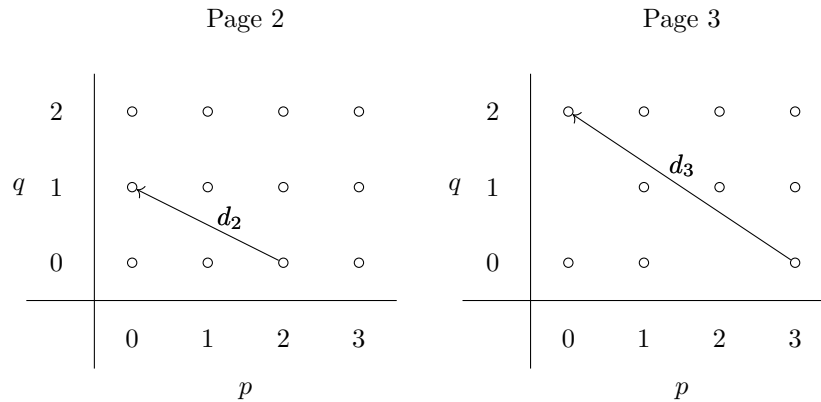
(a)  $d_r^{p,q} \circ d_r^{p+r,q-r+1} = 0$  and

(b)  $E_{p,q}^{r+1} = \frac{\ker d_r^{p,q}}{\text{im } d_r^{p+r,q-r+1}}$ .

Such a sequence  $(E^r, d_r)_{r \in \mathbb{Z}_{\geq 0}}$  of pairs is called a *homological spectral sequence*, and each double complex  $(E^r, d_r)$  is called the *r-th page* of the sequence.

**Note 3.0.1.**  $E^{r+1} = H_*(E^r, d_r)$ .

We shall consider only *first-quadrant* spectral sequences, i.e., those for which  $E_{p,q}^r = 0$  unless  $p, q \geq 0$ .



As a result, there is some  $k \in \mathbb{N}$  such that  $E^r = E^{r+1}$  for any  $r \geq k$ .

*Notation.*  $E^\infty := E^k$ .

**Definition 3.0.2 (Convergence).** We say that a spectral sequence  $E^* := (E^r, d_r)$  *converges* to a sequence of abelian groups  $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$ , written as

$$E^* \Rightarrow \{A_n\},$$

if for each  $n$ , there exists a filtration

$$\cdots \subset A_{-1,n+1} = \{0\} \subset A_{0,n} \subset \cdots \subset A_{n-1,1} \subset A_{n,0} = A_n$$

of  $A_n$  such that  $\frac{A_{p,q}}{A_{p-1,q+1}} \cong E_{p,q}^\infty$ .

**Theorem 3.0.3.** *Let  $B$  be a simply connected, path connected cell complex with  $n$ -skeleton  $B^n$  and suppose that  $\pi : E \rightarrow B$  is a fibration with fiber  $F$ . There exists a (first-quadrant) spectral sequence  $(E^r, d_r)$  that*

(a) *converges to  $\{H_n(E)\}_{n \in \mathbb{Z}_{\geq 0}}$  and*

(b) *satisfies  $E_{p,q}^2 \cong H_p(B; H_q(F))$ .*

*The filtration  $D_{p,q} := (H_n(E))_{p+q=n}$  witnessing this convergence is given by  $\text{im}(H_n(\pi^{-1}(B^p)) \rightarrow H_n(E))$ .*

*Remark 3.0.4.* This holds without the hypothesis that  $B$  is a cell complex.

**Example 3.0.5.** Consider the path space fibration

$$\begin{array}{c} \Omega X \\ \downarrow \\ PX \\ \downarrow \\ X \end{array}.$$

Recall that  $PX$  is contractible. Let  $n \geq 2$  and  $X = S^n$ . Then

$$E_{p,q}^2 \cong H_p(S^n; H_q(\Omega S^n)) = \begin{cases} H_q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise} \end{cases},$$

and  $(E^r, d_r) \Rightarrow \{\mathbb{Z}, 0, 0, \dots\}$ . This means that  $d_k = 0$  for any  $k \neq n$ , so that

$$\begin{aligned} E^2 &= E^3 = \cdots = E^n \\ E^{n+1} &= E^{n+2} = \cdots = E^\infty. \end{aligned}$$

As a result, each differential  $d_n^{p,q}$  is an isomorphism provided that  $(p, q) \neq (n, 1 - n)$  for, otherwise,  $E_{p,q}^{n+1}$  is

nontrivial, which is impossible. Hence the  $n$ -th page looks like

$$\begin{array}{c|ccc}
 3(n-1) & H_{3n-3}(\Omega S^n) & & H_{3n-3}(\Omega S^n) \\
 & & \swarrow \text{iso.} & \\
 2(n-1) & H_{2n-2}(\Omega S^n) & & H_{2n-2}(\Omega S^n) \\
 & & \swarrow \text{iso.} & \\
 n-1 & H_{n-1}(\Omega S^n) & & H_{n-1}(\Omega S^n) \\
 & & \swarrow \text{iso.} & \\
 0 & \mathbb{Z} & & \mathbb{Z} \\
 & 0 & & n
 \end{array}$$

This implies that  $H_q(\Omega S^n) \cong H_{q+n-1}(\Omega X)$  for any  $q \in \mathbb{Z}_{\geq 0}$ . But  $\Omega S^n$  is path connected since  $S^n$  is simply connected. By induction, it follows that

$$H_q(\Omega X) \cong \begin{cases} \mathbb{Z} & q \equiv 0 \pmod{n-1} \\ 0 & \text{otherwise} \end{cases}.$$

### 3.1 Lecture 12

Suppose that

$$\begin{array}{c}
 F \\
 \downarrow \\
 E \\
 \downarrow \pi \\
 B
 \end{array}$$

is a fibration with  $B$  simply connected and  $F$  path connected. Thanks to Theorem 3.2.5, we have the inclusion

$$E_{0,n}^\infty \cong \frac{D_{0,n}}{D_{-1,n+1}} = D_{0,n} \subset H_n(E)$$

as well as a commutative diagram

$$\begin{array}{c}
 \overbrace{H_p(B; H_0(F))}^{H_p(B)} = E_{p,0}^2 \longleftrightarrow E_{p,0}^3 \\
 \quad \quad \quad \nearrow \\
 E_{p,0}^4 \longleftrightarrow \cdots \longleftrightarrow E_{p,0}^p \\
 \quad \quad \quad \nearrow \\
 E_{p,0}^\infty \longleftarrow H_p(E) = D_{p,0} \\
 \quad \quad \quad \nwarrow \pi_* \\
 \quad \quad \quad \nearrow
 \end{array}$$

of abelian groups. Let  $i$  denote the inclusion map  $i : F \cong p^{-1}(b) \rightarrow E$  where  $b$  is any chosen element of  $B$ . This induces a map  $i_*$  in homology

$$\begin{array}{c} \underbrace{H_q(F)}_{H_0(B, H_q(F))} \longrightarrow \cdots \longrightarrow E_{0,q}^3 \longrightarrow \cdots \longrightarrow E_{0,q}^\infty \hookrightarrow H_q(E) \end{array} \quad \begin{array}{c} \xrightarrow{i_*} \end{array}$$

Now, consider the commutative diagram

$$\begin{array}{ccc} \pi_n(B) & \xrightarrow{\partial} & \pi_{n-1}(F) \\ h_n \downarrow & & \downarrow h_{n-1} \\ H_n(B) & \xrightarrow{\partial} & H_{n-1}(F) \\ \cong \uparrow & & \downarrow \\ E_{n,0}^2 & & E_{0,n-1}^3 \\ \uparrow & & \downarrow \\ \vdots & & \vdots \\ \uparrow & & \downarrow \\ E_{n,0}^n & \xrightarrow{d_n} & E_{0,n-1}^n \end{array}$$

where  $h_n$  denotes the *Hurewicz homomorphism*, defined for an arbitrary path connected space  $X$  as follows. Let  $\gamma := [f]$  be any element of  $\pi_n(X, x)$ , so that  $f$  is a map  $(S^n, x_0) \rightarrow (X, x)$ . Choose any generator  $\tau \in H_n(S^n) \cong \mathbb{Z}$  and let

$$h(\gamma) = f_*(\tau) \in H_n(X).$$

Likewise, we can define the *relative Hurewicz homomorphism*  $\tilde{h} : \pi_n(X, A) \rightarrow H_n(X, A)$  by

$$[f : (D^n, S^{n-1}, x_0) \rightarrow (X, A, \text{pt})] \mapsto f_*(\sigma)$$

where  $\sigma$  is any chosen generator of  $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ .

**Theorem 3.1.1 (Hurewicz).** *Let  $n \in \mathbb{Z}_{\geq 2}$ . If  $\pi_i(X) = 0$  for each  $1 \leq i \leq n-1$ , then  $h_n$  is an isomorphism and  $h_{n+1}$  is surjective.*

**Theorem 3.1.2 (Relative Hurewicz).** *Let  $n \in \mathbb{Z}_{\geq 2}$ . If both  $X$  and  $A$  are simply connected and  $\pi_i(X, A) = 0$  for each  $i \leq n-1$ , then  $\tilde{h}_n$  is an isomorphism and  $\tilde{h}_{n+1}$  is surjective.*

*Proof of Theorem 3.1.1.* Suppose that  $\pi_i(X) = 0$  for each  $1 \leq i \leq n-1$ . For induction, assume that  $h_{n-1}$  is an isomorphism for any path connected space. From Example 3.0.5, we gather that the  $n$ -th page of the spectral sequence induced by the path space fibration  $\Omega X \rightarrow PX \rightarrow X$  looks like

$$\begin{array}{ccc} n-1 & H_{n-1}(\Omega X) & \bullet & H_{n-1}(\Omega X) \\ & \nwarrow d_n & & \\ 0 & \mathbb{Z} & & H_n(X) \\ & 0 & & n \end{array}$$



where  $d_n$  is an isomorphism. Thanks to our inductive hypothesis together with Exercise 1.4.3, we have now a commutative square of the form

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\cong} & \pi_{n-1}(\Omega X) \\ h_n \downarrow & & \downarrow h_{n-1} \cong \\ H_n(X) & \xrightarrow[\cong]{d_n} & H_{n-1}(\Omega X) \end{array} \quad (*)$$

This implies that  $h_n$  is an isomorphism. It remains to verify our base case. Note that  $\pi_1(\Omega X)$  is isomorphic to  $\pi_2(X)$  and thus abelian. It can be shown directly that  $h_1$  factors as a composite

$$\begin{array}{ccccc} \pi_1(\Omega X) & \xrightarrow{\cong} & \pi_1(\Omega X)^{\text{ab}} & \xrightarrow{\cong} & H_1(\Omega X) \\ & \searrow & & \nearrow & \\ & & h_1 & & \end{array}$$

of isomorphisms. Hence  $h_2$  must be an isomorphism in light of (\*).  $\square$

*Question.* Does a similar argument work for Theorem 3.1.2?

*Answer.* Yes, in the sense that there is a spectral sequence proof of it. Specifically, there is a relative version of the spectral sequence. Suppose that  $B'$  is a simply connected subspace of  $B$  and let  $E' = \pi^{-1}(B')$ , yielding a fibration  $F \rightarrow E' \xrightarrow{\pi|_{E'}} B'$ . Then there exists a spectral sequence  $E^*$  converging to  $H_*(E, E')$  such that

$$E_{p,q}^2 \cong H_p(B, B'; H_q(F)).$$

One can use this to deduce Theorem 3.1.2 from Theorem 3.1.1.  $\square$

**Corollary 3.1.3.** *Let  $X$  be path connected.*

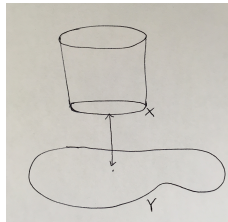
1.  $H_1(X) \cong \pi_1^{\text{ab}}(X)$ .
2. If  $X$  is simply connected and  $H_i(X) = 0$  for every  $1 \leq i \leq n-1$ , then  $\pi_i(X) = 0$  for every  $1 \leq i \leq n-1$ .
3. If  $\pi_i(X) = 0$  for each  $0 \leq i \leq n-1$ , then  $\tilde{H}_i(X) = 0$  for each  $0 \leq i \leq n-1$ .

Let  $n \geq 2$  and pick any generator  $[f]$  of  $\pi_{n-1}(\Omega S^n) \cong \pi_n(S^n) \cong \mathbb{Z}$ . By Theorem 3.1.1, the induced map  $f_* : H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(\Omega S^n)$  is an isomorphism.

*Remark 3.1.4.* Let  $g : X \rightarrow Y$  be any map of spaces. Recall the mapping cylinder

$$\text{Cyl}(g) \equiv \frac{(X \times I) \amalg Y}{(x, 0) \sim g(x)}$$

of  $g$ .



This is precisely the pushout of the span  $X \times I \xleftarrow{\sigma_0} X \xrightarrow{g} Y$ . As it turns out,  $g$  factors as

$$\begin{array}{ccccc} & & g & & \\ & \searrow & & \nearrow & \\ X & \xleftarrow{\iota} & \text{Cyl}(g) & \xrightarrow{h} & Y \end{array}$$

for some deformation retraction  $h$ . Further,  $\iota$  is a cofibration, the dual notion to a fibration.

Consider the subspace of  $\Omega S^n$  consisting of all great circles passing through, say, the north pole. This is clearly homeomorphic to  $S^{n-1}$ . Thus, we get a LES in homology

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \underbrace{H_{2n-2}(S^{n-1})}_0 & \longrightarrow & \underbrace{H_{2n-2}(\Omega S^n)}_{\mathbb{Z}} & \xrightarrow{\cong} & H_{2n-2}(\Omega S^n, S^{n-1}) \\ & & \swarrow & & & & \searrow \\ & & \underbrace{H_{2n-3}(S^{n-1})}_0 & \longrightarrow & H_{2n-3}(\Omega S^n) & \xrightarrow{\cong} & H_{2n-3}(\Omega S^n, S^{n-1}) \longrightarrow \cdots \\ & & & & \swarrow & & \searrow \\ & & & & \underbrace{H_{n+1}(S^{n-1})}_0 & \longrightarrow & H_{n+1}(\Omega S^n) \xrightarrow{\cong} H_{n+1}(\Omega S^n, S^{n-1}) \\ & & & & \swarrow & & \searrow \\ & & & & \underbrace{H_n(S^{n-1})}_0 & \longrightarrow & H_n(\Omega S^n) \xrightarrow{\cong} H_n(\Omega S^n, S^{n-1}) \\ & & & & \swarrow & & \searrow \\ & & & & H_{n-1}(S^{n-1}) & \xrightarrow[\cong]{f_*} H_{n-1}(\Omega S^n) \xrightarrow{0} H_{n-1}(\Omega S^n, S^{n-1}) \\ & & & & & \swarrow & \searrow \\ & & & & & \underbrace{H_{n-2}(S^{n-1})}_0 & \longrightarrow \cdots \end{array}$$

From this, we deduce that

$$H_i(\Omega S^n, S^{n-1}) = \begin{cases} 0 & i \leq 2n-3 \\ \mathbb{Z} & i = 2n-2 \end{cases}.$$

By Corollary 3.1.3(2), this means that

$$\pi_i(\Omega S^n, S^{n-1}) = \begin{cases} 0 & i \leq 2n-3 \\ \mathbb{Z} & i = 2n-2 \end{cases}.$$

This yields a LES in homotopy

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{2n-2}(\Omega S^n) & \longrightarrow & \pi_{2n-2}(\Omega S^n, S^{n-1}) & & \\ & & \swarrow & & & & \\ & & \pi_{2n-3}(S^{n-1}) & \longrightarrow & \pi_{2n-3}(\Omega S^n) & \longrightarrow & \underbrace{\pi_{2n-3}(\Omega S^n, S^{n-1})}_0 \\ & & & & \swarrow & & \searrow \\ & & & & \pi_{2n-4}(S^{n-1}) & \xrightarrow{\cong} & \underbrace{\pi_{2n-4}(\Omega S^n)}_{\pi_{2n-3}(S^n)} \longrightarrow \underbrace{\pi_{2n-4}(\Omega S^n, S^{n-1})}_0 \longrightarrow \cdots \end{array},$$

which proves the following statement.

**Theorem 3.1.5 (Suspension).** *If  $0 \leq i \leq 2n - 4$ , then  $\pi_i(S^{n-1}) \cong \pi_{i+1}(S^n)$ .*

This can be generalized as follows.

**Exercise 3.1.6 (Freudenthal suspension).** *Let  $i \in \mathbb{Z}_{\geq 1}$  and suppose that the space  $X$  satisfies  $\pi_n(X) = 0$  for each  $0 \leq n \leq i - 1$ . Show that*

$$\pi_i(X) \cong \pi_{i+1}(SX)$$

*through around dimension  $2i - 3$  (figure this out exactly), where  $S(-)$  denotes the suspension functor.*

*Proof.* We could use a spectral sequence argument together with the relative Hurewicz theorem. Rather, let us show that it follows from the following famous theorem in homotopy theory:

**Theorem 3.1.7 (Blakers-Massey).** *Suppose that*

$$\begin{array}{ccc} X & \hookrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

*is a pushout diagram. Also, suppose that  $\pi_n(B, X) = 0$  and  $\pi_m(A, X) = 0$  for each  $0 \leq n \leq k_1$  and each  $0 \leq m \leq k_2$ . The map  $(A, X) \rightarrow (C, B)$  of pairs induces an isomorphism  $\pi_n(A, X) \xrightarrow{\cong} \pi_n(C, B)$  for each  $0 \leq n \leq k_1 + k_2 - 1$ .*

Now, let us prove the Freudenthal suspension theorem with an upper bound of  $2i - 2$  rather than  $2i - 3$ . To start, decompose  $SX$  into the union of two cones  $C_+X$  and  $C_-X$  that meet at a copy of  $X$ . Note that  $SX$  is precisely the pushout of the diagram

$$C_-X \longleftarrow X \longrightarrow C_+X.$$

As  $C_+X$  is contractible, the LES of homotopy groups for the pair  $(C_+X, X)$  shows that the map  $\partial : \pi_{n+1}(C_+X, X) \rightarrow \pi_n(X)$  is an isomorphism. Similarly, we see that the map  $\iota : \pi_{n+1}(SX) \rightarrow \pi_{n+1}(SX, C_-X)$  induced by inclusion is an isomorphism. Consider the sequence of homomorphisms

$$\pi_n(X) \xrightarrow{\partial^{-1}} \pi_{n+1}(C_+X, X) \xrightarrow{\psi} \pi_{n+1}(SX, C_-X) \xrightarrow{\iota^{-1}} \pi_{n+1}(SX)$$

where  $\psi$  is induced by pullback. Since  $\pi_n(X) = 0$  for each  $0 \leq n \leq i - 1$ , the LES for the pair  $(C_{\pm}X, X)$  also shows that  $\pi_n(C_{\pm}X) = 0$  for any  $0 \leq n \leq i$ . By Theorem 3.1.7, it follows that  $\psi$  is an isomorphism so long as  $n + 1 \leq 2i - 1$ . Hence  $\pi_n(X) \cong \pi_{n+1}(SX)$  for any  $n \leq 2i - 2$ , as desired.

Furthermore, the upper bound of  $2i - 2$  is sharp. Indeed, we have that

- $\pi_0(S^2) = \pi_1(S^2) = 0$ ,
- $\pi_3(S^2) \cong \mathbb{Z}$ , and
- $\pi_4(S^3) \cong \mathbb{Z}/2$ .

If we could increase our upper bound to  $2i - 1$ , then we would have an isomorphism  $\pi_3(S^2) \cong \pi_4(S^3)$ , which is impossible.  $\square$

### 3.2 Lecture 13

As expected, spectral sequences have exact analogues in cohomology. Before introducing them, let us review a bit of singular cohomology theory. Let  $X$  be a cell complex and let  $n \in \mathbb{Z}_{\geq 0}$ . Recall that  $C_n(X)$  the free abelian group on the set of all  $n$ -cells of  $X$  and the boundary map  $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$ . Let

$$C^n(X) = \text{Hom}(C_n(X), \mathbb{Z})$$

and define the homomorphism  $\delta^n : C^n(X) \rightarrow C^{n+1}(X)$  by

$$\delta^n(\varphi) = \varphi \circ \partial_n.$$

**Theorem 3.2.1.**  $H^n(X; \mathbb{Z}) \cong \frac{\ker \delta^{n+1}}{\text{im } \delta^n}.$

**Example 3.2.2.**  $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x] / (x^{n+1})$  with  $|x| = 2$ .  
dim.

**Theorem 3.2.3 (Poincaré duality).** *If  $M$  is a connected orientable  $n$ -manifold, then  $H_i(M) \cong H^{n-i}(M)$ .*

Now, a cohomological spectral sequence consists of the following data:

- for each  $r \in \mathbb{Z}_{\geq 0}$ , a family of abelian groups  $\{E_r^{p,q}\}_{p,q \in \mathbb{Z}}$  and
- a family of maps  $\{d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}\}_{p,q \in \mathbb{Z}}$  (called *differentials*) such that
- $d_r^{p,q} \circ d_r^{p-r, q+r-1} = 0$  and
- $E_{r+1}^{p,q} = \frac{\ker d_r^{p,q}}{\text{im } d_r^{p-r, q+r-1}}.$

Again, we shall consider only *first-quadrant* spectral sequences, i.e., those for which  $E_r^{p,q} = 0$  unless  $p, q \geq 0$ .

As a result, there is some  $k \in \mathbb{N}$  such that  $E_r = E_{r+1}$  for any  $r \geq k$ .

*Notation.*  $E_\infty := E_k$ .

**Definition 3.2.4 (Convergence).** We say that a spectral sequence  $E_* := (E_r, d_r)$  *converges* to a sequence of abelian groups  $\{D^n\}_{n \in \mathbb{Z}_{\geq 0}}$ , written as

$$E_* \Rightarrow \{D^n\},$$

if for each  $n$ , there exists a filtration

$$\dots \subset D^{n+1, -1} = \{0\} \subset D^{n, 0} \subset \dots \subset D^{1, n-1} \subset D^{0, n} = D^n$$

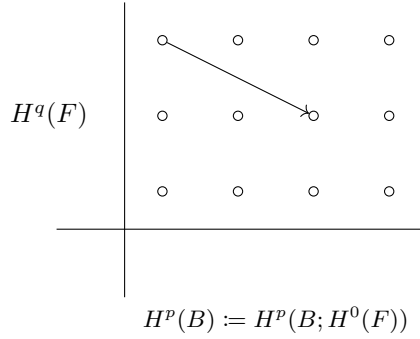
of  $D^n$  such that  $\frac{D^{p,q}}{D^{p+1, q-1}} \cong E_\infty^{p,q}.$

**Theorem 3.2.5.** *Let  $B$  be simply connected and path connected and suppose that  $\pi : E \rightarrow B$  is a fibration with fiber  $F$ . There exists a (first-quadrant) spectral sequence  $(E^r, d_r)$  that*

- (a) *converges to  $\{H^n(E)\}_{n \in \mathbb{Z}_{\geq 0}}$  and*
- (b) *satisfies  $E_2^{p,q} \cong H^p(B; H^q(F))$ .*

In pictures, we have

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$$\begin{array}{c}
 H^q(E) \cong D^{0,q} \twoheadrightarrow E_{\infty}^{0,q} \hookrightarrow \cdots \hookrightarrow E_2^{0,q} \cong H^q(F) \\
 \searrow \quad \quad \quad \nearrow \\
 \quad \quad \quad i^* \quad \quad \quad \\
 \hline
 H^p(B) \cong E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{\infty}^{p,0} \hookrightarrow H^p(E) \\
 \searrow \quad \quad \quad \nearrow \\
 \quad \quad \quad \pi^* \quad \quad \quad
 \end{array}$$

Let  $X$  be a cell complex. Recall the *cup product* operation  $H^i(X) \times H^j(X) \xrightarrow{\smile} H^{i+j}(X)$  on cohomology, which is both bilinear and *anti-commutative* in the sense that

$$x \smile y = (-1)^{ij} y \smile x.$$

Consider the constant map  $C_0(X) \rightarrow \mathbb{Z}$  given by  $D^0 \mapsto 1$ , which corresponds to an element  $\mathbf{1}$  of  $H^0(X)$  via Theorem 3.2.1. We have that

$$-\mathbf{1} \smile x = x \smile \mathbf{1} = \mathbf{1}.$$

Suppose that  $Y$  is another cell complex. Let  $x \in H^i(X)$  and  $y \in H^j(X)$  and let  $f$  denote a map  $Y \rightarrow X$ . Then

$$f^*(x \smile y) = f^*(x) \smile f^*(y),$$

i.e.,  $f^*$  is a graded ring homomorphism. Now,  $X \times Y$  carries a cell complex structure with  $n$ -cells of the form

$$D^i \times D^j, \quad i + j = n$$

and  $n$ -skeleton

$$(X \times Y)^n \equiv \bigcup_{i+j=n} X^i \times Y^j.$$

We have that

$$C_n(X \times Y) \cong C_n(X) \otimes_{\mathbb{Z}} C_n(Y)$$

and, in light of the fact that  $\partial(D^i \times D^j) = (\partial D^i \times D^j) \cup (D^i \times \partial D^j)$ , that

$$\partial[D^i \times D^j] = \partial[D^i] \otimes D^j + (-1)^i [D^i] \otimes \partial[D^j].$$

Consider any two maps  $f : C_i(X) \rightarrow \mathbb{Z}$  and  $g : C_j(X) \rightarrow \mathbb{Z}$ , extending them both by 0 to the entire graded abelian group  $C_*(X)$ . Define  $f \otimes g : C_m(X \times Y) \cong C_m(X) \otimes C_m(Y) \rightarrow \mathbb{Z}$  by

$$(f \otimes g)(u \otimes v) = f(u) \cdot g(v).$$

**Proposition 3.2.6.**  $\delta(f \otimes g) = \delta f \otimes g + (-1)^i f \otimes \delta g$ .

As it turns out, this means that the map  $(f, g) \mapsto (f \otimes g)$  induces an operation  $H^i(X) \times H^j(Y) \xrightarrow{\times} H^{i+j}(X \times Y)$  on cohomology known as the *cross product*. The relation between the cup and cross product has the form  $\Delta^*(x \times y) = x \smile y$ , where  $\Delta : X \rightarrow X \times X$  denotes the diagonal map.

In general, let  $R_1, R_2$ , and  $R_3$  be commutative rings and let  $\mu : R_1 \times R_2 \rightarrow R_3$  denote “multiplication.” This induces the cup product on cohomology

$$\begin{aligned} H^i(X; R_1) \times H^j(X; R_2) &\xrightarrow{\smile} H^{i+j}(X; R_3) \\ \Downarrow \\ H^p(B, H^q(F)) \times H^{p'}(B, H^{q'}(F)) &\xrightarrow{\smile} H^{p+p'}(B, H^{q+q'}(F)) \\ E_2^{p,q} \times E_2^{p',q'} &\xrightarrow{\smile} E_2^{p+p',q+q'}. \end{aligned}$$

**Proposition 3.2.7.** For any  $r \in \mathbb{Z}_{\geq 2}$ , there is a certain operation  $\smile_r : E_r^{p,q} \times E_r^{p',q'} \rightarrow E_r^{p+p',q+q'}$  such that

$$d_r(x \smile y) = d_r(x) \smile y + (-1)^{p+q} x \smile d_r(y).$$

*Construction.* Let  $r \in \mathbb{Z}_{\geq 2}$  and suppose, for induction, that we have already constructed  $\smile_r$ . Let  $x \in E_r^{p,q}$  and  $y \in E_r^{p',q'}$ . Suppose that  $d_r x = d_r y = 0$ , so that  $d_r(x \smile y) = 0$ . If  $y = d_r(z)$ , then

$$x \smile y = x \smile d_r(z) = d(x \smile z) \pm \underbrace{d_r(x)}_0 \smile z.$$

by induction. This means that  $\smile_r$  induces a pairing  $\smile_{r+1}$  on  $E_{r+1}$ . To complete our induction on  $r$ , simply take the ordinary cup product on cohomology to be  $\smile_2$ .  $\square$

Now, given the filtration

$$\{0\} \subset D^{n,0} \subset \dots \subset D^{0,n} \subset H^n(E),$$

the operation  $\smile_r$  on  $E_r$  carries  $D^{p,q} \times D^{p',q'}$  to  $D^{p+p',q+q'}$  where  $p+q = p'+q' = n$ , thereby inducing a pairing

$$\smile_\infty : E_\infty^{p,q} \times E_\infty^{p',q'} \rightarrow E_\infty^{p+p',q+q'}$$

on  $E_\infty$ .

**Example 3.2.8.** Consider the fiber bundle  $S^1 \rightarrow S^{2n+1} \twoheadrightarrow \mathbb{CP}^n$ , so that

$$E_2^{p,q} \cong H^p(\mathbb{CP}^n; H^q(S^1)).$$

Pick a generator  $x$  of the group  $H^1(S^1) \cong \mathbb{Z}$ . Then the cohomology ring  $H^*(S^1)$  is isomorphic to  $\mathbb{Z}[x]/(x^2)$ , and

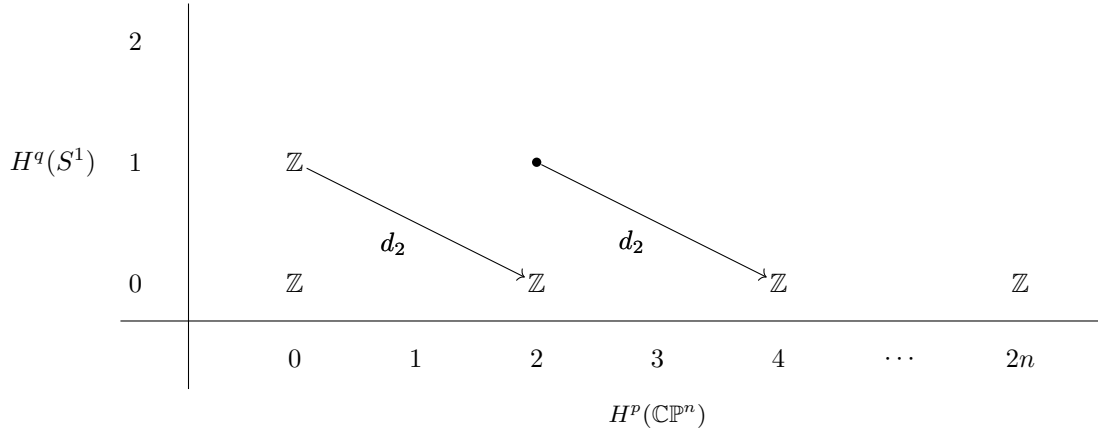
$$H^i(S^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases}.$$

Moreover, recall that

$$H^i(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, i \equiv 0 \pmod{2} \\ 0 & \text{otherwise} \end{cases},$$

which yields

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where each  $d_2$  is an isomorphism. Suppose that  $x$  is a generator of  $H^1(S^1)$  and let  $c = d_2(x)$ . Then

$$d_2(c \smile x) = c \smile d_2(x) = c^2,$$

which is a generator of  $H^4(\mathbb{CP}^n)$ . Similarly,  $c^i$  is a generator of  $H^{2i}(\mathbb{CP}^n)$  for each  $i \in \mathbb{Z}_{\geq 0}$ .

By letting  $c^0 = 1$  and making  $n$  large enough, we have determined the ring structure of  $H^*(\mathbb{CP}^\infty)$ .

**Theorem 3.2.9.** *If  $c_1$  is a generator of  $H^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$ , then  $H^*(B_{S^1}) = H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[c_1]$ .*

*Example 2.5.11*

## 4 Characteristic classes

### 4.1 Lecture 14

Recall the space  $B_{U(n)} = B_{GL(n, \mathbb{C})}$  of  $n$ -planes in  $\mathbb{C}^\infty$  as well as the space  $B_{O(n)} = B_{GL(n, \mathbb{R})}$  of  $n$ -planes in  $\mathbb{R}^\infty$ . We want to classify the graded rings  $H^*(B_{U(n)}; \mathbb{Z})$  and  $H^*(B_{O(n)}; \mathbb{Z}_2)$ . Let's begin with the former.

Consider the embedding  $U(n-1) \hookrightarrow U(n)$  given by  $A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$ . Consider also the mapping  $U(n) \rightarrow S^{2n-1}$  given by  $A \mapsto Ae_n$ . These fit into a “short exact sequence”

$$0 \rightarrow U(n-1) \rightarrow U(n) \rightarrow S^{2n-1} \rightarrow 0 \quad (1)$$

of spaces, and thus the bundle

$$\frac{U(n)}{U(n-1)} \rightarrow BU(n-1) \xrightarrow{\pi} BU(n) \quad (2)$$

has fiber  $S^{2n-1}$ .

*Remark 4.1.1.* Let us make (1) precise. Note that  $U(n)$  acts transitively on  $S^{2n-1}$  because any point in  $S^{2n-1}$  belongs to at least one orthonormal bases of  $\mathbb{C}^n$ . This means that  $S^{2n-1}$  is a *homogenous*  $U(n)$ -space.

*Proposition 4.1.2.* Any homogenous  $G$ -space  $X$  is homeomorphic to the coset space  $G/\text{Stab}_G(\vec{o})$  where  $\vec{o}$  is any choice of “identity.”

In our case, let  $\vec{o} = e_n$ , so that  $\text{Stab}_{U(n)}(\vec{o}) = U(n-1)$ .

Now, (2) induces a spectral sequence  $E_*$  converging to  $H^*(BU(n-1))$  such that

$$E_2^{p,q} = H^p(BU(n); H^q(S^{2n-1}))$$

and

$$\begin{array}{ccccccc} H^p(BU(n)) & \longrightarrow & E_2^{p,0} & \longrightarrow & E_3^{p,0} & \longrightarrow & \cdots \longrightarrow E_\infty^{p,0} \hookrightarrow H^p(BU(n-1)) \\ & & & & & \searrow & \nearrow \\ & & & & & \pi^* & \end{array}$$

commutes. If  $p < 2n$ , then  $H^p(BU(n)) \cong E_\infty^{p,0}$ , in which case  $\pi^*$  is injective. Also, we have that  $E_2^{p-k,k} = E_\infty^{p-k,k}$  whenever  $0 < p-k < 2n+1$ , so that  $\pi^*$  is surjective whenever  $p < 2n$ . It follows that  $\pi^*$  is an isomorphism

$$H^p(BU(n)) \cong H^p(BU(n-1)) \quad (3)$$

when  $p \leq 2n-1$ .

Consider the differential  $d_2 : \underbrace{H^{2n-1}(S^{2n-1})}_{\mathbb{Z}} \rightarrow H^{2n}(BU(n))$  and pick a generator  $g_n$  of  $H^{2n-1}(S^{2n-1})$ , i.e., an orientation of  $S^{2n-1}$ . Let

$$c_n = d_2(g_n).$$

In light of (3), we see that

$$c_i \in H^{2i}(BU(n)) \cong \cdots \cong H^{2i}(BU(i+1)) \cong H^{2i}(BU(i))$$

for each  $i \leq n$ . Abusing notation, we shall write  $c_i \in H^{2i}(BU(i))$ .

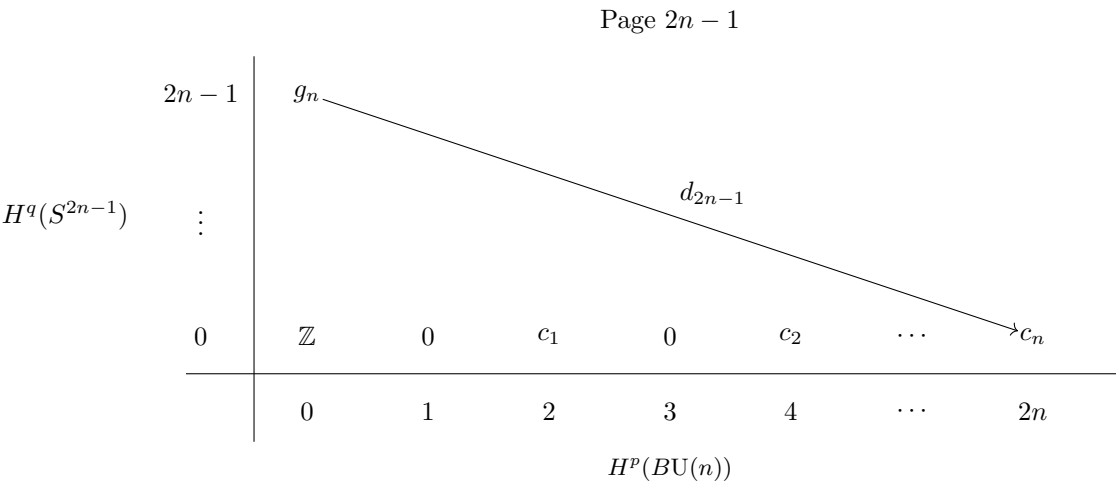
**Theorem 4.1.3.**  $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$

*Proof.* Proceed by induction on  $n \in \mathbb{N}$ . Our base case holds by virtue of Theorem 3.2.9. Assume that

$$H^*(BU(n-1)) \cong \mathbb{Z}[c_1, \dots, c_{n-1}].$$



Note that  $E_{2n} = E_\infty$  and  $E_{2n-1} = E_2$ , yielding



□

⋮

5
Cobordism theory

*To do.*