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Abstract

We present solutions to all exercises from Scott Weinstein's "Model Theory" course lectures at UPenn. These are relatively self-contained and are meant to complement Weinstein's written memoirs of our class meetings. The official reference for the course is David Marker's *Model Theory: An Introduction*.

1. Let I be a countably infinite set. Let $\mathbb{D} \coloneqq \langle I, E \rangle$ be a structure where E is an equivalence relation for which there is exactly one equivalence class of size k for each $k \in \mathbb{Z}_{\geq 1}$.

(1) Show that the set Λ of (first-order) sentences expressing that E is an equivalence relation with exactly one equivalence class of size k for each $k \in \mathbb{Z}_{\geq 1}$ axiomatizes $\mathbb{D}$, i.e., $\mathsf{Th}(\mathbb{D}) = \mathsf{Cn}(\Lambda)$ where

$$\mathsf{Cn}(\Lambda) \coloneqq \{\varphi \in \mathsf{FO}_{\mathbb{D}} \mid \Lambda \models \varphi\}.$$

(2) Show that for every (first-order) formula $\theta(y, \overline{w})$ and every $\bar{a} \in I$, the set

$$\theta[\mathbb{D}, \bar{a}] \coloneqq \{x \in \mathrm{dom}(\mathbb{D}) \mid \mathbb{D} \models \theta[x, \bar{a}]\}$$

is either finite or cofinite.

(1) It suffices to prove that Λ is complete. For, in this case, any two models of Λ must be elementarily equivalent.

Claim 1. *Let $\mathbb{E}$ be any model of Λ of size $\kappa \geq \omega$. There exists an elementary extension $\mathbb{E}_\kappa \succeq \mathbb{E}$ of size κ such that $\mathbb{E}_\kappa$ has exactly κ equivalence classes each of size κ.*

Proof. Let λ denote the cardinality of the set of all equivalence classes in $\mathrm{dom}(\mathbb{E})$. Note that $\lambda \leq \kappa$. For every $\alpha, \beta \in \kappa$, adjoin to the language of $\mathbb{E}$ a new constant symbol $c(\alpha, \beta)$. Consider the theory

$$\Delta \coloneqq \Lambda \cup \{Ec(x,y)c(x,z) \mid x, y, z \in \kappa\} \cup \{\neg Ec(x,0)c(y,0) \mid x, y \in \kappa,\ x \neq y\}.$$

Any finite subset F of Δ is satisfiable by a suitable expansion $\mathbb{E}_F$ of $\mathbb{E}$. Then there exists an ultrafilter on the family of finite subsets of Δ such that the ultraproduct

$$\prod_{\substack{F \subset \Delta \\ \text{finite}}} \mathbb{E}_F / \mathcal{U}$$

satisfies Δ. Moreover, its reduct $\mathbb{A}$ to the language of $\mathbb{E}$ is an elementary extension of $\mathbb{E}$. By the downward Löwenheim-Skolem theorem, there exists a structure $\mathbb{E}_0$ of size κ such that $\mathbb{A} \succeq \mathbb{E}_0 \succeq \mathbb{E}$.

Now, repeat our preceding construction ω times to get an increasing chain

$$\mathbb{E} \preceq \mathbb{E}_0 \preceq \mathbb{E}_1 \preceq \mathbb{E}_2 \preceq \cdots$$

of structures such that each $\mathrm{dom}(\mathbb{E}_i)$ has cardinality κ. Note that $\mathbb{E}_\kappa$ is an elementary extension of $\mathbb{E}$. Further, the domain of the direct limit $\mathbb{E}_\kappa \coloneqq \bigcup_{i \in \omega} \mathbb{E}_i$ also has cardinality κ, so that $\mathbb{E}_\kappa$ has exactly κ equivalence classes. Finally, for any $x \in \mathbb{E}_\kappa$, x belongs to some $\mathbb{E}_n$. Hence the equivalence class $[x]$ has size κ in $\mathbb{E}_{n+1}$ and thus in $\mathbb{E}_\kappa$. It follows that every equivalence class in $\mathbb{E}_\kappa$ has size κ. $\square$

Suppose, toward a contradiction, that there is a sentence φ in the language of $\mathbb{D}$ such that neither φ nor $\neg\varphi$ belongs to $\mathsf{Cn}(\Lambda)$. Then there are models $\mathbb{A}^1$ and $\mathbb{A}^2$ of Λ a such that $\mathbb{A}^1 \models \neg\varphi$ and $\mathbb{A}^2 \models \varphi$. By the Löwenheim-Skolem theorem, we may assume that both of these have size $\kappa \geq \omega$. By Claim 1, we thus have two structures $\mathbb{A}^1_\kappa$ and $\mathbb{A}^2_\kappa$ such that $\mathbb{A}^1_\kappa \models \neg\varphi$ and $\mathbb{A}^2_\kappa \models \varphi$. But it's easy to see that $\mathbb{A}^1_\kappa$ and $\mathbb{A}^2_\kappa$ must be isomorphic, which yields a contradiction.

(2) Suppose, toward a contradiction, that there exist a formula $\theta(y, w_1, \ldots, w_n)$ and an element $\bar{a} \in I$ such that $\theta[\mathbb{D}, \bar{a}]$ is both infinite and coinfinite. Adjoin to the language of $\mathbb{D}$ new constant symbols $\bar{e} := (e_1, \ldots, e_n)$, c, and d. For each $k \in \mathbb{Z}_{\geq 1}$, let $\lambda_k(x)$ denote the formula expressing that the equivalence class of x has cardinality $> k$. Now, consider the theory

$$\begin{aligned}\Gamma := \Lambda \cup \{\lambda_k(c) \mid k \geq 1\} &\cup \{\lambda_k(d) \mid k \geq 1\} \\ &\cup \{\neg E e_i c \mid 1 \leq i \leq n\} \\ &\cup \{\neg E e_i d \mid 1 \leq i \leq n\} \\ &\cup \{\theta(c, \bar{e}), \neg\theta(d, \bar{e})\}\end{aligned}$$

in our new language.

Let F be any finite subset of Γ. Since both $\theta[\mathbb{D}, \bar{a}]$ and $\neg\theta[\mathbb{D}, \bar{a}]$ are infinite by assumption, we can find an expansion of $\mathbb{D}$ that satisfies F by interpreting $\bar{e}$ as $\bar{a}$ and both c and d as members of large enough equivalence classes. By the compactness theorem, it follows that there is some model $\mathbb{C}$ of Γ, which must be infinite. Let $\mathbb{C}'$ denote the reduct of $\mathbb{C}$ to the language of $\mathbb{D}$. Thanks to the Löwenheim-Skolem theorem, we may assume that $\mathrm{dom}(\mathbb{C}')$ is countable. Thus, the equivalence classes $\left[c^{\mathbb{C}}\right]$ and $\left[d^{\mathbb{C}}\right]$ are countable. Note that $e_i^{\mathbb{C}} \notin \left[c^{\mathbb{C}}\right] \cup \left[d^{\mathbb{C}}\right]$ for each $1 \leq i \leq n$. Therefore, there is an automorphism of $\mathbb{C}'$ sending $c^{\mathbb{C}}$ to $d^{\mathbb{C}}$ and fixing each $e_i^{\mathbb{C}}$. But this contradicts the fact that $\mathbb{C}' \models \theta\left[c^{\mathbb{C}}, \bar{e}^{\mathbb{C}}\right] \wedge \neg\theta\left[d^{\mathbb{C}}, \bar{e}^{\mathbb{C}}\right]$.

■

Definition 1 (Categoricity). For any cardinal κ, we say that a theory T is *κ-categorical* if any two models of T of size κ are isomorphic.

2. Show that a $\mathcal{L}$-structure $\mathbb{A}$ is finite if and only if for any $\mathcal{L}$-structure $\mathbb{B}$,

$$\mathbb{A} \equiv \mathbb{B} \iff \mathbb{A} \cong \mathbb{B}.$$

Remark. This shows that any complete theory with a finite model is κ-categorical for *any* cardinal κ.

($\Longrightarrow$)

It is always true that any two isomorphic structures are elementarily equivalent. Thus, it remains to show that $\mathbb{A} \equiv \mathbb{B} \implies \mathbb{A} \cong \mathbb{B}$.

First, assume that $\mathcal{L}$ is finite. Consider the *atomic diagram* of $\mathbb{A}$, i.e., the set

$$\mathtt{D}(\mathbb{A}) := \{\varphi \mid \underline{\mathbb{A}} \models \varphi,\ \varphi \text{ is either atomic or the negation of an atomic formula}\}$$

where $\underline{\mathbb{A}}$ denotes the expansion of $\mathbb{A}$ obtained by adjoining a constant symbol c_a for each $a \in \mathrm{dom}(\mathbb{A})$. Since both $\mathcal{L}$ and $\mathrm{dom}(\mathbb{A})$ are finite, we can encode $\mathtt{D}(\mathbb{A})$ with a single sentence ψ. Therefore, the sentence

$$\psi_{\mathbb{A}} := \forall x \left(\bigvee_{a \in \mathrm{dom}(\mathbb{A})} x = c_a \right) \wedge \psi$$

has the property that $\underline{\mathbb{B}} \models \psi_{\mathbb{A}} \implies \mathbb{B} \cong \mathbb{A}$ for any other $\mathcal{L}$-structure $\mathbb{B}$. Now, if $\mathbb{A} \equiv \mathbb{B}$, then clearly both $\underline{\mathbb{A}}$ and $\underline{\mathbb{B}}$ satisfy $\psi_{\mathbb{A}}$, so that $\mathbb{B} \cong \mathbb{A}$.

Next, let $\mathcal{L}$ be arbitrary and let $\mathbb{A} \equiv \mathbb{B}$. Suppose, toward a contradiction, that $\mathbb{A} \ncong \mathbb{B}$. Then for any bijection $f : \mathrm{dom}(\mathbb{A}) \to \mathrm{dom}(\mathbb{B})$, there is some finite sublanguage $\mathcal{L}_f$ of $\mathcal{L}$ such that f is *not* an isomorphism $\mathbb{A}^{\mathcal{L}_f} \to \mathbb{B}^{\mathcal{L}_f}$ of reducts to $\mathcal{L}_f$. Consider the language

$$\mathcal{L}' := \bigcup_{\substack{f : \mathrm{dom}(\mathbb{A}) \to \mathrm{dom}(\mathbb{B}) \\ \text{bijection}}} \mathcal{L}_f,$$

which is finite as the finite union of finite sets. Thanks to our preceding discussion, we obtain an isomorphism $g : \mathbb{A}^{\mathcal{L}'} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}'}$. But $\mathcal{L}_g \subset \mathcal{L}'$ by our construction of $\mathcal{L}'$, and thus g induces an isomorphism $\mathbb{A}^{\mathcal{L}_g} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}_g}$, contrary to our choice of $\mathcal{L}_g$.

($\Longleftarrow$)

Suppose that $\mathbb{A}$ is infinite. We must find a structure $\mathbb{B}$ such that $\mathbb{A} \equiv \mathbb{B}$ but $\mathbb{A} \not\cong \mathbb{B}$. But this follows at once from the the Löwenheim-Skolem theorem, which implies that $\mathsf{Th}(\mathbb{A})$ has a model of any infinite size.

∎

Definition 2 (Ehrenfeucht-Fraïssé game). Suppose that $\mathcal{L}$ is a finite language without function symbols. Let $\mathbb{D}$ and $\mathbb{E}$ be two $\mathcal{L}$-structures. Let $n \in \omega$. The *Ehrenfeucht-Fraïssé game* $\mathrm{EF}_n(\mathbb{D}, \mathbb{E})$ *of length* n *on* $\mathbb{D}$ *and* $\mathbb{E}$ is a game of perfect information played as follows.

(a) There are exactly two players, the *spoiler* and the *duplicator*.

(b) There are exactly n rounds.

(c) The spoiler begins round $k \leq n$ by picking an element (sometimes called a pebble) of either $\mathrm{dom}(\mathbb{D})$ or $\mathrm{dom}(\mathbb{E})$. Next, the duplicator picks an element of the other domain.

(d) This yields two sequences $(d_1, \ldots, d_n)$ and $(e_1, \ldots, e_n)$ such that $d_i \in \mathrm{dom}(\mathbb{D})$ and $e_i \in \mathrm{dom}(\mathbb{E})$ for each $i = 1, \ldots, n$. If the mapping $d_i \mapsto e_i$ defines an isomorphism of finite substructures, then we say that the duplicator has won $\mathrm{EF}_n(\mathbb{D}, \mathbb{E})$. Otherwise, we say that the spoiler has won.

Theorem 3 (Fraïssé). *The duplicator has a winning strategy in* $\mathrm{EF}_n(\mathbb{D}, \mathbb{E})$ *for each* $n \in \omega$ *if and only if* $\mathbb{D} \equiv \mathbb{E}$.

3. Let $\mathbb{N}^* = \langle \omega, < \rangle$. Show that for any infinite cardinal κ, $\mathsf{Th}(\mathbb{N}^*)$ is *not* κ-categorical.

Expand the language of $\mathbb{N}^*$ by adjoining countably many constants $\{c_i\}_{i \in \mathbb{Z}}$. Consider the theory

$$T := \mathsf{Th}(\mathbb{N}^*) \cup \{c_i > c_{i+1} \mid i \in \mathbb{Z}\}. \tag{$\star$}$$

in our new language. Any finite subset of T is satisfied by an expansion of $\mathbb{N}^*$ suitably interpreting the c_i since $\mathbb{N}^*$ has descending chains of all finite lengths. By the compactness theorem, it follows that there is some model $\mathbb{A}$ of T, which must be infinite. If $|\mathbb{A}| > \aleph_0$, then apply the Löwenheim-Skolem theorem to get a model $\mathbb{B}$ of T such that $|\mathbb{B}| = \aleph_0$. Let

$$\mathbb{A}' = \begin{cases} \mathbb{B} & |\mathbb{A}| > \aleph_0 \\ \mathbb{A} & |\mathbb{A}| = \aleph_0 \end{cases}.$$

Note that $\mathbb{A}' \models T$. Since the property of being a linearly ordered set is expressible by a first-order sentence, we see that $\mathbb{A}'$ is linearly ordered by $<$. Further, we see that $\mathbb{A}'$ has an infinite descending chain, which means that $\mathbb{A}'$ is not well-ordered by $<$. But $(\omega, <)$ is a well-ordered set, and thus the reduct of $\mathbb{A}'$ to the language of $\mathbb{N}^*$ is not isomorphic to $\mathbb{N}^*$. It does, however, satisfy $\mathsf{Th}(\mathbb{N}^*)$. This shows that $\mathsf{Th}(\mathbb{N}^*)$ is not $\aleph_0$-categorical.

Unfortunately, it's unclear that this method can be adapted to show that $\mathsf{Th}(\mathbb{N}^*)$ is not κ-categorical when κ is uncountable. In this case, we instead shall employ two binary operations on the class of all linear orderings. Let L_1 and L_2 be linearly ordered sets.

- L_1^{op} refers to L_1 equipped with the inverse order.
- $L_1 \otimes L_2$ refers to $L_1 \times L_2$ equipped with the lexicographic order.
- $L_1 \oplus L_2$ refers to L_1 with its ordering followed by L_2 with its ordering.

Now, consider the following linearly ordered structures:

$$\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa)$$
$$\mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa)),$$

both of which have cardinality κ. These orderings possess bottom elements and are *discrete* in the sense that both structures satisfy the sentences

$$\begin{aligned} &\forall x \exists y \left(x < y \wedge \neg \exists z \left(x < z \wedge z < y \right) \right) \\ &\forall x \left(\exists w \left(w < x \right) \rightarrow \exists y \left(y < x \wedge \neg \exists z \left(y < z \wedge z < x \right) \right) \right). \end{aligned} \tag{1}$$

(Informally, we can view y here as the *successor/predecessor* of x.) Note that, on the one hand, $\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa)$ cannot possess an descending chain of length ω^2, for otherwise κ, which is well-ordered, would possess an infinite descending chain. On the other hand, $\mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa))$ does possess such a chain since ω^* (the order type of $\mathbb{Z}_{<0}$) can be embedded in $\mathbb{Q}$. Therefore,

$$\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa) \ncong \mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa)).$$

Claim 2. *Suppose that* $(\mathbb{E}, <)$ *is a discrete linear ordering with a bottom element but no top element. Then* $\mathbb{E} \equiv \mathbb{N}^*$.

Proof sketch. Consider the Ehrenfeucht-Fraïssé game $\mathrm{EF}_n(\mathbb{E}, \mathbb{N}^*)$. The duplicator has a winning strategy in $\mathrm{EF}_n(\mathbb{E}, \mathbb{N}^*)$ by adhering to the following rules.

(i) If, in round m, the spoiler chooses an element of one of the structures that is connected to a previously chosen element or the bottom element by a path of successors of length $k < \infty$, then choose the corresponding element of the other structure in round m.

(ii) Otherwise, make sure that any chosen element of $\mathrm{dom}(\mathbb{N}^*)$ is always separated by at least $n+1$ elements from any previously chosen element of $\mathrm{dom}(\mathbb{N}^*)$ while preserving the required order of your choices.

In this case, choose first a natural number separated by more than 3^n elements from the greatest previously chosen clement of $\mathrm{dom}(\mathbb{N}^*)$.

□

Thanks to Theorem 3, it follows that both $\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa)$ and $\mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa))$ are elementarily equivalent to $\mathbb{N}^*$ and thus models of $\mathsf{Th}(\mathbb{N}^*)$. Hence $\mathsf{Th}(\mathbb{N}^*)$ is not κ-categorical. ■

4. Show that any set definable over $\mathbb{N}^*$ is either finite or cofinite.

Remark. This shows that $\mathbb{N}^*$ is *o-minimal* in the sense that every definable set over $\mathbb{N}^*$ is a finite union of points and intervals in ω.

Note that any set definable over $\mathbb{N}^*$ is 0-definable because any natural number n is uniquely determined by the first-order property

$$\begin{cases} \text{“}n \text{ is less than any other element”} & n = 0 \\ \text{“there are exactly } n-1 \text{ elements between 0 (the bottom element) and } n\text{”} & n > 1 \end{cases}.$$

Suppose, toward a contradiction, that there exist a formula $\theta(y)$ such that $\theta\left[\mathbb{N}^*\right]$ is both infinite and coinfinite. Consider, again, the theory ($\star$). Let

$$T' = T \cup \{\theta(c_0), \neg\theta(c_1)\}.$$

Since both $\theta\left[\mathbb{N}^*\right]$ and $\neg\theta\left[\mathbb{N}^*\right]$ are infinite by assumption, we can find an expansion of $\mathbb{N}^*$ that satisfies any finite subset of T', By the compactness theorem together with the Löwenheim-Skolem theorem, we thus can find a countable model $\mathbb{D}$ of T' and take its reduct $\mathbb{C}$ to the language of $\mathbb{N}^*$. Note that $(\mathrm{dom}(\mathbb{C}), <)$ is a countable linear ordering with an infinite descending and ascending chain χ on which both $c_0^{\mathbb{D}}$ and $c_1^{\mathbb{D}}$ lie. Moreover, this ordering is discrete in the sense of (1). Therefore, we may assume that χ has the form

$$\cdots < x_{m-1} < x_m < x_{m+1} < \cdots$$

where x_{m+1} denotes the immediate successor of x_m. There is an automorphism of $\mathbb{C}$ mapping $c_0^{\mathbb{D}}$ to $c_1^{\mathbb{D}}$ by suitably shifting χ finitely many places to the left and fixing all elements outside χ. But this contradicts the fact that $\mathbb{C} \models \theta\left[c_0^{\mathbb{D}}\right] \wedge \neg\theta\left[c_1^{\mathbb{D}}\right]$. ■

5. Consider the theory DLO of the dense linear ordering without endpoints. For any uncountable cardinal κ, show that there are 2^κ many models of DLO up to isomorphism.

Remark. This shows that DLO is *not* κ-categorical even though it is $\aleph_0$-categorical.

Consider the linear orderings

$$L_1 \coloneqq \mathbb{Q} \otimes (\omega_1^{\mathrm{op}} \oplus \omega_1)$$
$$L_2 \coloneqq \mathbb{Q} \otimes (1 \oplus \omega_1^{\mathrm{op}} \oplus \omega_1)\,.$$

Now, by replacing each $\alpha \in \kappa$ with a choice of L_1 or L_2, we obtain 2^κ many dense linear orderings $\{P_\beta\}_{\beta<2^\kappa}$ without endpoints such that $|P_\beta| = \kappa$ for ever β. It remains to show that these are pairwise non-isomorphic.

To this end, suppose that there is an isomorphism $f : P_\beta \xrightarrow{\cong} P_{\beta'}$. By construction, both P_β and $P_{\beta'}$ consist of κ-sequences

$$L_{i_0} < L_{i_1} < \cdots < L_{i_\alpha} < \cdots$$
$$L_{i'_0} < L_{i'_1} < \cdots < L_{i'_\alpha} < \cdots\,,$$

respectively, where $i_\alpha, i'_\alpha \in \{1,2\}$ Since any isomorphism of well-ordered sets is unique, we see that the function $f\restriction_{L_{i_\alpha}}$ is an isomorphism $L_{i_\alpha} \xrightarrow{\cong} L_{i'_\alpha}$ for any $\alpha \in \kappa$.

Claim 3. $L_1 \ncong L_2$.

Proof. On the one hand, L_1 has a suborder isomorphic to ω_1^{op} with no lower bound in L_1. On the other hand, any such suborder of L_2 has a lower bound in L_2. Hence there is no isomorphism from L_1 to L_2. □

It follows that $L_{i_\alpha} = L_{i'_\alpha}$ for every $\alpha \in \kappa$, which completes our proof. ■

Definition 4. Let T be a theory and let $\Gamma(\bar{x})$ be a set of formulas in free variables $x_1, \ldots, x_n$. We say that Γ is an *n-type over T* if for any finite subset $\Delta \subset \Gamma$, the expanded theory

$$T \cup \left\{(\exists\bar{x}) \bigwedge \Delta\right\}$$

is satisfiable.

Notation. Let $\mathbb{M}$ be an $\mathcal{L}$-structure and let $A \subset \mathrm{dom}(\mathbb{M})$. Let $\mathcal{L}_A = \mathcal{L} \cup \{c_a \mid a \in A\}$ and let $\mathbb{M}_A$ denote the $\mathcal{L}_A$-structure induced by $\mathbb{M}$. Then $\mathbb{S}_n^{\mathbb{M}}(A)$ refers to the set of all complete n-types over $\mathsf{Th}_A(\mathbb{M}) \coloneqq \mathsf{Th}(\mathbb{M}_A)$.

Definition 5 (Stability). Let T be a complete theory in $\mathcal{L}$ and let κ be an infinite cardinal. We say that T is *κ-stable* if whenever $\mathbb{M} \models T$, $A \subset \mathrm{dom}(\mathbb{M})$, and $|A| = \kappa$, we have that $\left|\mathbb{S}_n^{\mathbb{M}}(A)\right| = \kappa$.

6. Let $\mathbb{A}$ be a structure and $\theta(x,y)$ be a formula in the language of $\mathbb{A}$. Suppose that $B \subset \mathrm{dom}(\mathbb{A})$ is an infinite set on which $\theta[\mathbb{A}]$ is a linear order $\prec$. Show that $\mathsf{Th}(\mathbb{A})$ is *not* ω-stable (i.e., $\aleph_0$-stable).

Thanks to the axiom of dependent choice, we can find a countably infinite chain of at least one of the following two forms.

$$a_0 \prec b_0 \prec a_1 \prec b_1 \prec a_2 \prec b_2 \prec \cdots$$
$$\cdots \prec b_2 \prec a_2 \prec b_1 \prec a_1 \prec b_0 \prec a_0$$

with $a_i, b_i \in B$ for each $i = 0, 1, 2, \ldots$. Without loss of generality, assume that we can find the former kind of chain and that θ has the form $x \prec y$. In this case,

$$\mathbb{A} \models \theta[a_i, b_j] \iff i \leq j. \tag{$*$}$$

Claim 4. *There exist sequences $(a_x)_{x \in 2^{\aleph_0}}$ and $(b_x)_{x \in 2^{\aleph_0}}$ along with a model $\mathbb{A}'$ of $\mathsf{Th}(\mathbb{A})$ such that*

$$\mathbb{A}' \models \theta[a_x, b_y] \iff x \leq y.$$

Proof. Adjoin to the language of $\mathbb{A}$ two new constant symbols c_x and d_y for every $x, y \in 2^{\aleph_0}$. Consider the theory

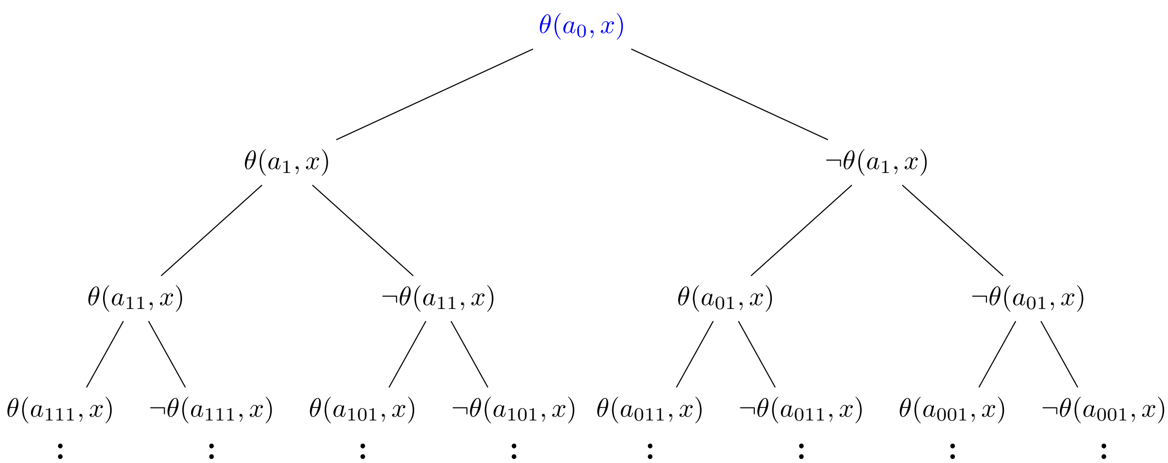
$$\Gamma := \mathsf{Th}(\mathbb{A}) \cup \left\{\theta(c_x, d_y) \mid x, y \in 2^{\aleph_0},\ x \leq y\right\} \cup \left\{\neg\theta(c_x, d_y) \mid x, y \in 2^{\aleph_0},\ x > y\right\}.$$

in our expanded language. In light of ($*$), any finite subset of Γ is satisfiable by a suitable expansion of $\mathbb{A}$. Thus, by the compactness theorem, Γ has a model $\mathbb{B}$. Finally, let $\mathbb{A}'$ denote the reduct of $\mathbb{B}$ to the language of $\mathbb{A}$. $\square$

Instead of indexing the sequences (a_x) and (b_x) by $\left(2^{\aleph_0}, \in\right)$, let us index them by the set of all $2^{\aleph_0}$-indexed binary strings σ under the string order $<$. We have that

$$\mathbb{A}' \models \theta[a_\sigma, b_{\sigma'}] \iff \sigma \leq \sigma'.$$

Consider the countably infinite subset $X := \left\{a_\sigma \mid \sigma \in 2^{\aleph_0}\right\}$ of $\mathrm{dom}(\mathbb{A}')$. By recursion, we can build a binary tree of the form



($\Cup$)

with height ω. We call nodes of the form $\theta(a_\sigma, x)$ *positive* and those of the form $\neg\theta(a_\sigma, x)$ *negative*. Let U denote any branch of ($\Cup$). Let U_p denote the set of all strings $\sigma \in 2^{\aleph_0}$ such that a_σ occurs in a positive node of U. Since U_p is countable, it has an upper bound in $\left(2^{\aleph_0}, <\right)$. Since $\left(2^{\aleph_0}, <\right)$ is a complete order and $2^{\aleph_0}$ is a limit ordinal, it follows that U_p has a supremum τ in $2^{\aleph_0}$. By construction of ($\Cup$), if $\theta(a_\sigma, x)$ is a positive node of U and $\neg\theta(a_{\sigma'}, x)$ is a negative one, then $\sigma' > \sigma$. Hence $\tau \leq \sigma'$ for any σ' occurring in a negative node of U. As a result, we see that $\mathbb{A}' \models \varphi[a_\sigma, b_\tau]$ for any node φ of U.

Therefore, every branch of ($\Cup$) determines a unique 1-type over $\mathsf{Th}_Y(\mathbb{A}')$ where

$$Y := \{x \in X \mid x \text{ occurs in a node of } (\Cup)\}.$$

This shows that $\left|\mathbb{S}_1^{\mathbb{A}'}(Y)\right| = 2^{\aleph_0} > \aleph_0$. But ($\Cap$) has exactly

$$\left|\bigcup_{n\in\omega} 2^n\right| = \aleph_0$$

many nodes, so that $|Y| = \aleph_0$. Hence $\mathsf{Th}(\mathbb{A})$ is not ω-stable.

■

Informally, an *abstract logic* L consists of a set of L-sentences together with a satisfaction relation $\models_L$ between structures and L-sentences.

Definition 6 (Löwenheim-Skolem property). We say that L has the *Löwenheim-Skolem property* if any countable satisfiable L-theory has a model of size $\leq \aleph_0$.

7. Consider the extension $L(Q_0)$ of first-order logic by the *generalized quantifier* $\exists^{<\omega}$ signifying "there are finitely many." Formally,

$$\mathbb{A} \models (Q_0 x)\, \varphi(x) \iff |\{a \in \mathrm{dom}(\mathbb{A}) \mid \mathbb{A} \models \varphi[a]\}| < \aleph_0.$$

Show that $L(Q_0)$ has the Löwenheim-Skolem property.

Without loss of generality, consider $L(Q_0)$ with $\exists^{<\omega}$ replaced by $\exists^{\infty} := \neg\exists^{<\omega}$. We have the following version of the Tarski-Vaught elementary submodel criterion.

Claim 5. *Let $\mathbb{B}$ be a structure for $L(Q_0)$ and $\mathbb{A}$ be a submodel of $\mathbb{B}$. Suppose that for any formula $\varphi(\bar{x}, y)$ and any $\bar{a} \in \mathrm{dom}(\mathbb{A})$,*

$$\begin{aligned} \{b \in \mathrm{dom}(\mathbb{B}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\} \neq \emptyset &\implies \{b \in \mathrm{dom}(\mathbb{A}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\} \neq \emptyset \\ |\{b \in \mathrm{dom}(\mathbb{B}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\}| \geq \aleph_0 &\implies |\{b \in \mathrm{dom}(\mathbb{A}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\}| \geq \aleph_0 \end{aligned}$$

Then $\mathbb{A} \preceq_{L(Q_0)} \mathbb{B}$.

Proof sketch. This is easily proved by induction on the complexity of formulas just as it is for first-order logic. □

Now, suppose that Γ is a countable $L(Q_0)$-theory with an infinite model $\mathbb{M}$. It suffices to show that for any $X \subset \mathrm{dom}(\mathbb{M})$, there is an elementary submodel $\mathbb{M}'$ of $\mathbb{M}$ such that $X \subset \mathrm{dom}(\mathbb{M}')$ and $|\mathbb{M}'| = |X| + \aleph_0$. To this end, inductively construct an ω-sequence

$$X := X_0 \subset X_1 \subset X_2 \subset \cdots$$

of subsets of $\mathrm{dom}(\mathbb{M})$ such that $|X_i| = |X| + \aleph_0$ for every $i \in \omega$ as follows. Suppose that we have already defined X_i as desired. For every formula $\varphi(\bar{x}, y)$ and any $\bar{a} \in X_i$, consider the set

$$F_{\varphi,\bar{a}} := \{b \in \mathrm{dom}(\mathbb{M}) \mid \mathbb{M} \models \varphi\,[\bar{a}, b]\}\,.$$

By the axiom of choice, we can find a set of the form

$$\widetilde{F}_{\varphi,\bar{a}} := \begin{cases} F_{\varphi,\bar{a}} & F_{\varphi,\bar{a}} \text{ is finite} \\ \text{a chosen countably infinite subset of } F_{\varphi,\bar{a}} & \text{otherwise} \end{cases}.$$

Now, let

$$X_{i+1} = X_i \cup \bigcup_{\varphi,\bar{a}} \widetilde{F}_{\varphi,\bar{a}}.$$

Since there are countably many formulas and, by induction, $|X| + \aleph_0$ many $\bar{a} \in X_i$, we deduce that X_{i+1} has cardinality $|X| + \aleph_0$.

It is easy to see that the union $\mathbb{M}' := \bigcup_{i\in\omega} X_i$ forms an elementary submodel of $\mathbb{M}$ thanks to Claim 5. Further, we have that $|\mathbb{M}'| = |X| + \aleph_0 + \aleph_0 = |X| + \aleph_0$, as desired. ■

Let $X \subset \omega$. We write $\phi_e^X : W_e^X \to \{0,1\}$ for the partial function computed by the Turing machine with index e and access to an oracle for X.

Definition 7 (Computability). Let $Y \subset \omega$.

1. We say that Y is *computable/recursive relative to* X if the characteristic function χ_Y equals ϕ_e^X for some index e.
2. We say that Y is *computably/recursively enumerable relative to* X if $Y = W_e^X$ for some index e.

If $Y = \emptyset$, then we omit "relative to X."

Let $\mathcal{C}$ denote any collection of computably enumerable sets. We say that a set $B \subset \{0,1\}^*$ of binary strings is a *weak index set for* $\mathcal{C}$ if $\mathcal{C} = \{W_e \mid e \in B\}$.

8. Let REC denote the collection of all recursive sets. Show that REC has a computably enumerable weak index set.

By mapping all invalid encodings of Turing machines to a distinguished trivial Turing machine, we may assume that our binary representation scheme

$$\langle - \rangle : \{a \mid a \text{ is a } \mathsf{TM}\} \to \{0,1\}^*$$

of Turing machines is surjective, i.e., every binary string represents a Turing machine. Therefore, we may computably enumerate all Turing machines

$$M_1 < M_2 < M_3 < \cdots$$

according to the string order $<$. With this in mind, construct an enumerator E that prints, for each Turing machine M_i, the binary representation of a new Turing machine M_i' given as follows.

Algorithm 1: pseudocode describing M_i'

Input: the binary string x
run U_{TM} on $\langle M_i, x \rangle$;
if U_{TM} *rejects* **then**
  reject
else
  run U_{TM} on $\langle M_i, y \rangle$ for all strings y such that $|y| \le |x|$;
  if U_{TM} *halts for each such* y **then**
    accept
  else
    reject
  end
end

Here, U_{TM} denotes a universal Turing machine. It is easy to see that

$$L(M_i') = \begin{cases} L(M_i) & M_i \text{ is total} \\ \text{a finite set} & \text{otherwise} \end{cases}.$$

We claim that the language $\{\langle M_i' \rangle \mid i \in \mathbb{Z}_{\ge 1}\}$ enumerated by E is a weak index set for REC, i.e.,

$$\mathsf{REC} = \{L(M_i') \mid i \in \mathbb{Z}_{\ge 1}\}.$$

Indeed, if Y is recursive, then there is some Turing machine M_k deciding it, in which case $Y = L(M_k) = L(M_k')$. Conversely, for any language of the form $L(M_k')$, M_k is either total or non-total. If it is total, then $L(M_k') = L(M_k)$ and $L(M_k)$ is recursive. If it is non-total, then $L(M_k')$ is finite and thus recursive.

Remark. We could *not* have used the set $S \coloneqq \{D_1, D_2, D_3, \ldots\}$ of all deciders as our weak index set for Rec, for S is not computably enumerable. Indeed, suppose, toward a contradiction, that S is computably enumerable. Enumerate all binary strings

$$w_1 < w_2 < w_3 < \cdots .$$

We now can construct a Turing machine N such that for each integer $i \geq 1$, N accepts w_i if D_i rejects it and rejects w_i if D_i accepts it. Then $L(N)$ is a decidable language, but $N \notin S$ by construction, a contradiction.

■

Let T be a countable complete theory. For any $n \in \mathbb{Z}_{\geq 1}$, consider the set $S_n(T)$ of all complete n-types of T endowed with the topology generated by all sets of the form

$$[\theta(x_1, \ldots, x_n)] \coloneqq \{\tau \in S_n(T) \mid \theta(\bar{x}) \in \tau\}, \qquad \theta(\bar{x}) \text{ a formula in the language of } T.$$

This is known as the *n-th Stone space of T*. It is clearly Hausdorff. It is also totally disconnected in the sense that every point in $S_n(T)$ has a clopen neighborhood.

Next, consider the Boolean algebra

$$B_n(T) \coloneqq \{[\theta(x_1, \ldots, x_n)]\}_{\theta(\bar{x})}$$

with meet $\cap$, join $\cup$, and complement $(-)^c$. This is isomorphic to the Boolean algebra of all equivalence classes of the form

$$\{\varphi(\bar{x}) \mid T \models \varphi \leftrightarrow \theta\}$$

with meet $\wedge$, join $\vee$, and complement $\neg$.

Theorem 8. *The space $S_n(T)$ is compact.*

Proof. Recall that a topological space X is compact if and only if every family $\{C_i \mid i \in I\}$ of closed sets in X with the finite intersection property satisfies $\bigcap_{i \in I} C_i \neq \emptyset$. Suppose that $U \coloneqq \{\Gamma_i(\bar{x}) \mid i \in I\}$ is any family of closed sets in $S_n(T)$ with the finite intersection property. As all basic open sets in $S_n(T)$ are clopen, each n-type $\Gamma_i(\bar{x})$ has the form $[\neg\theta_i(\bar{x})]$. We see that $\{\neg\theta_i(\bar{x}) \mid i \in I\}$ is an n-type over T because U has the finite intersection property.

Claim 6. *Every n-type over T is contained in a complete n-type over T.*

Proof. Let $\Delta(\bar{x})$ be an n-type over T. Let $\bar{c}$ be an n-tuple of new constant symbols added to the language of T. By definition of an n-type over T, the theory $T \cup \Delta(\bar{c})$ in our expanded language is finitely satisfiable. By the compactness theorem, there is some model $\mathbb{M}$ of $T \cup \Delta(\bar{c})$. Take the reduct $\mathbb{M}'$ of $\mathbb{M}$ to the language of L and let $\bar{a} = \bar{c}^{\mathbb{M}}$. Then $\mathbb{M}' \models T \cup \Delta(\bar{a})$, so that

$$\Delta(\bar{x}) \subset \{\psi(\bar{x}) \mid \mathbb{M}' \models \psi(\bar{a})\},$$

which is a complete n-type over T. □

By Claim 6, we can find a complete n-type τ over T that contains $\{\neg\theta_i(\bar{x}) \mid i \in I\}$. Then τ must belong to the intersection $\bigcap_{i \in I} \Gamma_i(\bar{x})$. □

Remark. Our proof of Theorem 8 reveals why the compactness theorem is so named.

8. Prove that $S_n(T)$ is finite if and only if $B_n(T)$ is finite.

First, suppose that $S_n(T)$ is finite. Every element of $B_n(T)$ is a subset of $S_n(T)$, and thus

$$|B_n(T)| \leq 2^{|S_n(T)|},$$

which is finite.

Conversely, suppose that $B_n(T)$ is finite. Consider the function $h : S_n(T) \to \mathcal{P}(B_n(T))$ defined by

$$\Gamma(\bar{x}) \mapsto \{[\psi(\bar{x})] \mid \psi \in \Gamma\}.$$

If $\Gamma(\bar{x})$ and $\Gamma'(\bar{x})$ are distinct complete n-types over T, then there is some formula $\theta(\bar{x})$ in the language of T such that $\theta \in \Gamma$ and $\neg\theta \in \Gamma'$. Hence $h(\Gamma) \neq h(\Gamma')$, so that h is injective. This implies that

$$|S_n(T)| \leq 2^{|B_n(T)|},$$

which is finite. ■