

Abstract

We continue to look at low-dimensional K -theory, finishing our description of $K_0(-)$ and then defining $K_1(-)$, and $K_2(-)$ for rings. The main sources for this talk are the following.

- $n\text{Lab}$.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Chapters II and III.
- Eric M. Friedlander's *An Introduction to K-theory*, Chapter 1.

1 K_0 of a Waldhausen category

Definition 1.1. Let \mathcal{C} be a category equipped with a “subcategory” $\text{co}\mathcal{C}$ of morphisms called *cofibrations*. The pair (\mathcal{C}, co) is a *category with cofibrations* if the following conditions hold.

1. (W0) Every isomorphism in \mathcal{C} is a cofibration.
2. (W1) There is a base point 0 in \mathcal{C} such that the unique morphism $0 \rightarrowtail A$ for every $A \in \text{ob}\mathcal{C}$.
3. (W2) We have

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \cup_A C \end{array} .$$

Note that $B \amalg C$ always exists as the pushout $B \cup_0 C$ and that the cokernel of any $i : A \rightarrowtail B$ exists as $B \cup_A 0$ along $A \rightarrow 0$. We call $A \rightarrowtail B \rightarrowtail B/A$ a *cofibration sequence*.

Definition 1.2. A *Waldhausen category* \mathcal{C} is a category with cofibrations together with a subcategory $w(\mathcal{C})$ of morphisms called *weak equivalences* such that every isomorphism in \mathcal{C} is a w.e. and the following “glueing axiom” holds.

1. (W3) For any diagram

$$\begin{array}{ccccc} C & \longleftarrow & A & \rightarrowtail & B \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ C' & \longleftarrow & A' & \rightarrowtail & B' \end{array} ,$$

the induced map $B \cup_A C \rightarrow B' \cup_{A'} C'$ is a w.e.

Definition 1.3. Let \mathcal{C} be a Waldhausen category. Define $K_0(\mathcal{C})$ as the abelian group generated by $[C]$ for each object C of \mathcal{C} such that

1. $[C] = [C']$ if there some w.e. from C to C'
2. $[C] = [B] + [C/B]$ for every $B \rightarrowtail C \rightarrowtail C/B$

3. The weak equivalence classes of objects in \mathcal{C} is a set.

Proposition 1.4.

1. $[0] = 0$.
2. $[B] \coprod [C] = [B] + [C]$.
3. $[B \cup_A C] = [B] + [C] - [A]$.
4. $[C] = 0$ whenever $0 \simeq C$.

Example 1.5. Let $\mathcal{R}_f(*)$ denote the category of finite CW complexes. Here, cofibrations and weak equivalences correspond to cellular inclusions and weak homotopy equivalences, respectively. One can show that $K_0(\mathcal{R}_f) \cong \mathbb{Z}$.

Definition 1.6. if \mathcal{C} and \mathcal{D} are Waldhausen, then a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *exact* if

- (a) preserves base points, cofibrations, and weak equivalences and
- (b) for any $A \rightarrowtail B$, $FB \cup_{FA} FC \rightarrow F(B \cup_A C)$ is an isomorphism.

Note that F induces a group map $K_0(F) : K_0(\mathcal{C}) \rightarrow K_0(\mathcal{D})$.

Theorem 1.7. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor. Assume the following.

- (1) A morphism f is a w.e. iff $F(f)$ is a w.e.
- (2) For any morphism $b : FA \rightarrow B$ in \mathcal{B} , there is some $a : A \rightarrowtail A'$ in \mathcal{A} and a w.e. $b' : FA' \xrightarrow{\sim} B$ in \mathcal{B} such that $b = b' \circ F(a)$. Moreover, we may choose a to be a w.e. whenever b is a w.e.

Then F induces an isomorphism $K_0(\mathcal{A}) \cong K_0(\mathcal{B})$.

Proof. Apply condition (2) to any $0 \rightarrowtail B$ to get $FA' \xrightarrow{\sim} B$. If this is a w.e., then there is some $A \xrightarrow{\sim} A'$. Hence there is a bijection between the set W of w.e. classes of objects of \mathcal{A} and that in \mathcal{B} .

The group $K_0(\mathcal{B})$ is given by the free abelian group $\mathbb{Z}[W]$ modulo the relation

$$[C] = [B] + \left[\begin{array}{c} C \\ \diagdown \\ B \end{array} \right].$$

Let $FA \xrightarrow{\sim} B$. Then applying condition (2) yields the diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & FA & \twoheadrightarrow & FA' \\ & & \sim \downarrow & & \sim \downarrow \\ 0 & \longleftarrow & B & \twoheadrightarrow & C \end{array}.$$

Apply the glueing axiom to see that $F\left(\begin{array}{c} A' \\ \diagdown \\ A \end{array}\right) \rightarrow \begin{array}{c} C \\ \diagdown \\ B \end{array}$ is a w.e. Hence $[C] = [B] + \left[\begin{array}{c} C \\ \diagdown \\ B \end{array} \right]$ holds iff $[A'] = [A] + \left[\begin{array}{c} A' \\ \diagdown \\ A \end{array} \right]$ holds. \square

2 K_1 for rings

Let R be a unital ring. Recall that direct limits in \mathbf{Mod}_R always exist. Let

$$K_1 = \mathrm{GL}(R)^{\mathrm{ab}}$$

where $\mathrm{GL}(R) \equiv \mathrm{colim}_n \mathrm{GL}(n, R)$.

Note 2.1 (Universal property of K_1). The universal property of $\mathrm{ab} : \mathbf{Grp} \rightarrow \mathbf{Ab}$ induces the universal property of K_1 that any homomorphism $f : \mathrm{GL}(R) \rightarrow H$ with H abelian has $f = g \circ \pi$ for some unique $g : K_1(R) \rightarrow H$.

Proposition 2.2. *Any ring map $f : R \rightarrow S$ induces a natural map $\mathrm{GL}(R) \rightarrow \mathrm{GL}(S)$. Hence K_1 is a functor $\mathbf{Rng} \rightarrow \mathbf{Ab}$.*

Thanks to Whitehead, we know that the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$ is equal to $E(R) = \bigcup_n E_n(R)$, the group of elementary matrices $E_{i,j}(r)$ where $r \in R$ and $i \neq j$. Thus, $K_1(R)$ can be viewed as the “stabilized” group of automorphisms of the trivial projective module modulo trivial automorphisms.

Example 2.3. If F is a field, then $K_1(F) = F^\times$.

Proof. It is each to check that $E_n(F) \cong \mathrm{SL}_n(F)$ for any $n \in \mathbb{N}$. Therefore, $E(F) \cong \mathrm{SL}(F)$. \square

Proposition 2.4. *Suppose R is commutative. Consider the sequence $R^\times \cong \mathrm{GL}(1, R) \rightarrow \mathrm{GL}(R) \rightarrow K_1(R)$. This induces a natural split exact sequence.*

$$1 \longrightarrow SK_1(R) \hookrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1,$$

where $SK_1(R)$ denotes $\ker(\det)$.

This means that $K_1(R) \cong R^\times \times SK_1(R)$.

Example 2.5. Suppose R is a Euclidean domain. Then $SK_1(R) = 1$, so that $K_1(R) \cong R^\times$.

Lemma 2.6. *Let D be a division ring. Then $K_1(D) \cong \mathrm{GL}_n(D) / E_n(D)$ for any $n \geq 3$.*

Proof. Any invertible matrix over D is reducible (a la Gaussian elimination) to a diagonal matrix of the form $(r, 1, \dots, 1)$. Moreover, $E_n(D) \trianglelefteq \mathrm{GL}_n(D)$ for each n . In particular, Dieudonné (1943) showed that $\mathrm{GL}_n(D) / E_n(D) \cong D^\times / (D^\times)'$ for any $n \neq 2$. \square

Now, suppose that R is Noetherian of dimension d , so that $E_n(R) \trianglelefteq \mathrm{GL}_n(R)$ for any $n \geq d + 2$.

Proposition 2.7 (Vaserstein). $K_1(R) \cong \mathrm{GL}_n(R) / E_n(R)$ for any $n \geq d + 2$.

Let D be a d -dimensional division algebra over the field $F := Z(D)$. We know that $d = n^2$ for some integer n . By Zorn’s lemma, there is some maximal subfield $E \subset D$ such that $[E : F] = n$. Then $D \otimes_F E \cong M_n(E)$, where M_n denotes the n -dimensional matrix ring over E . Any field with this property is called a *splitting field* for D .

Let E' be a splitting field for D . For any $r \in \mathbb{N}$, the inclusions $D \hookrightarrow M_n(E')$ and $M_r(D) \hookrightarrow M_{nr}(E')$ induce maps $D^\times \subset \mathrm{GL}_n(E') \xrightarrow{\det} (E')^\times$ and $\mathrm{GL}_r(D) \rightarrow \mathrm{GL}_{nr}(E') \xrightarrow{\det} (E')^\times$ whose images are contained in F^* . The induced maps are called the *reduced norms* N_{red} for D .

Example 2.8. If $D = \mathbb{H}$, then N_{red} is the square of the usual norm. It induces an isomorphism $K_1(\mathbb{H}) \cong \mathbb{R}_+^\times$.

Let R be a commutative Banach algebra over \mathbb{R} or \mathbb{C} . Recall that $\text{GL}_n(R)$ and $\text{SL}_n(R)$ are topological groups as subspaces of \mathbb{R}^{n^2} .

Proposition 2.9. *We have that $E_n(R)$ is the path component of the identity matrix I_n for any $n \geq 2$.*

Corollary 2.10. *We may identify $SK_1(R)$ with the set $\pi_0 \text{SL}(R)$.*

Proof. Note that $E(R) \leq \text{SL}(R)$. By the third isomorphism theorem, we get

$$\text{GL}(R)/_{E(R)}\text{SL}(R)/_{E(R)} \cong \text{GL}(R)/_{\text{SL}(R)}.$$

Thus, we get the short exact sequence

$$1 \longrightarrow \text{SL}(R)/_{E(R)} \longrightarrow \text{GL}(R)/_{E(R)} \cong K_1(R) \longrightarrow \text{GL}(R)/_{\text{SL}(R)} \cong R^\times \longrightarrow 1$$

By Proposition 2.9, we know that $\text{SL}(R)/_{E(R)} \cong \pi_0 \text{SL}(R)$, yielding a short exact sequence.

$$1 \longrightarrow \pi_0 \text{SL}(R) \longrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1.$$

□

Example 2.11. If X is compact, then

$$\begin{aligned} SK_1(\mathbb{R}^X) &\leftrightarrow [X, \text{SL}(\mathbb{R})] \cong [X, \text{SO}] \\ SK_1(\mathbb{C}^X) &\leftrightarrow [X, \text{SL}(\mathbb{C})] \cong [X, \text{SU}]. \end{aligned}$$

In particular, $SK_1(\mathbb{R}^{S^1}) \leftrightarrow \pi_1 \text{SO} \cong C_2$.

Let P be a finitely generated projective R -module. Each isomorphism $P \oplus Q \cong R^n$ induces a group map

$$\text{Aut}(P) \rightarrow \text{Aut}(P) \oplus \text{Aut}(Q) \cong \text{Aut}(R^n) \cong \text{GL}(n, R).$$

The group map $\text{Aut}(P) \rightarrow \text{GL}(R)$ is independent of the choice of isomorphism up to inner automorphism of $\text{GL}(R)$. Therefore, there is a well-defined homomorphism $\Phi : \text{Aut}(P) \rightarrow K_1(R)$.

Lemma 2.12. *Suppose that R is commutative and T is an R -algebra. Then $K_1(T)$ has a natural module structure over $K_0(R)$.*

Proof. For any $P \in \mathbf{P}(R)$ and $m \in \mathbb{N}$, consider the homomorphism $\Phi : \text{Aut}(P \otimes T^m) \rightarrow K_1(R \otimes T)$. For any $\beta \in \text{GL}_m(T)$, let

$$[P] \cdot \beta = \Phi(1_P \otimes \beta).$$

This action factors through $K_0(R)$ and $K_1(T)$, inducing an operation $K_0(R) \times K_1(T) \rightarrow K_1(R \otimes T)$. Now, since T is an R -algebra, there is a ring map $R \otimes T \rightarrow T$. The induced composite $K_0(R) \times K_1(T) \rightarrow K_1(R \otimes T) \rightarrow K_1(T)$ is the desired module structure. □

As it turns out, $K_1(R)$ is completely determined by the category $\mathbf{P}(R)$. This means that K_1 is invariant under Morita equivalence, just as K_0 is.

Theorem 2.13. *if R and S are Morita equivalent, then $K_1(R) \cong K_1(S)$.*

Let π be a finitely generated group. Define the *first Whitehead group* $\text{Wh}_1(\pi)$ of π as the cokernel of the map $\pi \times \{\pm 1\} \rightarrow K_1(\mathbb{Z}[\pi])$ given by $(g, \pm 1) \mapsto (\pm g)$.

Theorem 2.14. *A homotopy equivalence of finite CW-complexes with fundamental group π is a simple homotopy equivalence iff it vanishes under the so-called Whitehead torsion τ (which is a certain function from continuous maps to $\text{Wh}_1(\pi)$).*

Theorem 2.15 (The s -cobordism theorem). *Suppose that W , M , and N are compact PL-manifolds and that W is a cobordism of M and N . If $\dim(M) \geq 5$, then $(W, M, N) \cong (M \times [0, 1], M \times 0, M \times 1)$ iff $\tau = 0$.*

Corollary 2.16. *The generalized Poincaré conjecture.*

Definition 2.17. Let I is an ideal in R . Define $\text{GL}(I)$ as the kernel of the map $\text{GL}(R) \rightarrow \text{GL}(R/I)$. Moreover, define $E(R, I)$ as the smallest normal subgroup of $E(R)$ that contains $E_{i,j}(x)$ for $r \in I$ and $i \neq j$.

Proposition 2.18. $[\text{GL}(I), \text{GL}(I)] \subset E(R, I) \trianglelefteq \text{GL}(I)$

Definition 2.19. The *relative group* $K_1(R, I)$ is the the abelian group $\text{GL}(I)/E(R, I)$.

Remark 2.20. Swan has shown that a ring homomorphism $f : R \rightarrow S$ mapping the ideal I isomorphically to the ideal J need *not* induce an isomorphism $K_1(R, I) \rightarrow K_1(S, J)$.

Proposition 2.21. *We have an exact sequence*

$$K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \longrightarrow K_0(I) \longrightarrow K_0(R) \longrightarrow K_0(R/I) .^1$$

3 K_2 for rings

Definition 3.1. Let $n \geq 3$ and R be a ring. The *Steinberg group* $\text{St}_n(R)$ is the group generated by the symbols $x_{ij}(r)$ with $1 \leq i \neq j \leq n$ and $r \in R$ that satisfy the following relations.

1.

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r + s)$$

2.

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & j \neq k, \quad i \neq l \\ x_{il}(rs) & j = k, \quad i \neq l \\ x_{kj}(-sr) & j \neq k, \quad i = l \end{cases}$$

We have a natural group surjection $\phi_n : \text{St}_n(R) \rightarrow E_n(R)$ given by $x_{ij}(r) \mapsto E_{ij}(r)$. Moreover, there is a group map $\text{St}_n(R) \hookrightarrow \text{St}_{n+1}(R)$. Note that $\text{St}(R) := \text{colim}_n \text{St}_n(R)$ exists. The ϕ_n thus form a group epimorphism $\phi : \text{St}(R) \rightarrow E(R)$. Let

$$K_2(n, R) = \ker \phi_n$$

$$K_2(R) = \ker \phi.$$

¹Section III.2.3 (Weibel).

Note that $K_2(-)$ is a functor $\mathbf{Rng} \rightarrow \mathbf{Ab}$. Furthermore, we have an exact sequence

$$1 \longrightarrow K_2(R) \longrightarrow \mathrm{St}(R) \xrightarrow{\phi} \mathrm{GL}(R) \longrightarrow K_1(R) \longrightarrow 1 .$$

Lemma 3.2. $K_2(R) \cong Z(\mathrm{St}(R))$.

Proof. The fact that $K_2(R) \supset Z(\mathrm{St}(R))$ follows from the fact that $Z(E(R))$ is trivial. The reverse containment is easy but more tedious to prove. See III.5.2.1 (Weibel). \square

Example 3.3. A certain sort of Euclidean algorithm yields the following computations.

1. $K_2(\mathbb{Z}) \cong C_2$
2. $K_2(\mathbb{Z}[i]) = 1$
3. $K_2(F) \cong K_2(F[t])$ when F is a field

Theorem 3.4. Suppose that R is Noetherian of dimension d . Then $K_2(n, R) \cong K_2(R)$ for any $n \geq d + 3$.

Theorem 3.5. If R and S are Morita equivalent, then $K_2(R) \cong K_2(S)$.

Example 3.6. Let $n \in \mathbb{Z}_{\geq 1}$. Let R be any ring and let $S = M_n(R)$. These are Morita equivalent, so that

$$K_i(R) \cong K_i(M_n(R))$$

for each $i = 0, 1, 2$. Indeed, in one direction, define $F : M \mapsto M^n$. In the other direction, define $G : M \mapsto e_{11}M$ where e_{11} denotes the matrix with 1 in position $(1, 1)$ and 0 elsewhere. Define the natural isomorphism $\mathrm{Id}_{\mathrm{Mod}_R} \Rightarrow G \circ F$ by the components $f_M : M \rightarrow \{(m, 0, \dots, 0) : m \in M\}$. Further, define the natural isomorphism $\mathrm{Id}_{\mathrm{Mod}_S} \Rightarrow F \circ G$ by the components $g_M : M \rightarrow (e_{11}M)^n$ given by $m \mapsto (e_{11}m, \dots, e_{1n}m)$. Hence Mod_R and Mod_S are equivalent, hence Morita equivalence as they are preadditive.

Lemma 3.7. Let R be a commutative Banach algebra. Then there is a surjection from $K_2(R)$ onto $\pi_1 \mathrm{SL}(R)$.²

Example 3.8. There is a surjection $K_2(\mathbb{R}) \rightarrow \pi_1 \mathrm{SL}(\mathbb{R}) \cong \pi_1 \mathrm{SO} \cong C_2$. Hence $K_2(\mathbb{R})$ is nontrivial.

Theorem 3.9 (Matsumoto 1969). Let F be a field. Then $K_2(F)$ is isomorphic to the free abelian group with system of generators $\{a, b\}$ satisfying the following relations.

- (i) $\{ac, b\} = \{a, b\}\{c, b\}$
- (ii) $\{a, bd\} = \{a, b\}\{a, d\}$
- (iii) $\{a, 1 - a\} = 1$ when $a \neq 1 \neq 1 - a$.

Terminology. The $\{a, b\}$ are called *Steinberg symbols*.

²III.5.9 (Weibel).

Suppose that $A, B \in E(F)$ commute. Write $\phi(a) = A$ and $\phi(b) = B$. Then define

$$A \star B = [a, b] \in K_2(R).$$

If $a, b \in F$, then we can alternatively define the Steinberg symbol

$$\{a, b\} = \begin{bmatrix} r & & \\ & r^{-1} & \\ & & 1 \end{bmatrix} \star \begin{bmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{bmatrix}.$$

Corollary 3.10. $K_2(\mathbb{F}_p^n) = 1$ for any prime p and any integer $n \geq 1$.

Proof. The proof is entirely computational. See III.6.1.1 (Weibel). □

Proposition 3.11. If $F \supset \mathbb{Q}(t)$, then $|K_2(F)| = |F|$.