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K-theory reading seminar
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Abstract

We begin higher Waldhausen K-theory. The main sources for this talk are the following.

- nLab.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Ch. IV.8.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

For the original development, see Friedhelm Waldhausen's *Algebraic K-theory of spaces* (1985).

Our goal is to construct the K-theory $K(\mathscr{C})$ of a Waldhausen category $\mathscr{C}$ as a based loop space ΩY endowed with a loop completion map $\iota : |w\mathscr{C}| \to K(\mathscr{C})$ where $w\mathscr{C}$ denotes the subcategory of weak equivalences. This will produce a function $\operatorname{ob}\mathscr{C} \to |w\mathscr{C}| \to \Omega Y$. Further, we'll require of $K(\mathscr{C})$ certain limit and coherence properties, thereby making $K(\mathscr{C})$ the underlying infinite loop space of a spectrum $\mathbf{K}(\mathscr{C})$, called the *algebraic K-theory spectrum* of $\mathscr{C}$.

Definition 1. Let $\mathscr{C}$ be a category with cofibrations. Let the *extension category* $S_2\mathscr{C}$ have as objects the cofiber sequences in $(\mathscr{C}, \mathsf{co}\mathscr{C})$ and as morphisms the triples (f', f, f'') of maps in $\mathscr{C}$ such that

$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d, "f'"] & X \arrow[r, two heads] \arrow[d, "f"] & X'' \arrow[d, "f''"] \\ Y' \arrow[r, tail] & Y \arrow[r, two heads] & Y'' \end{tikzcd} \tag{$\star$}$$

commutes. This is pointed at $* \rightarrowtail * \twoheadrightarrow *$.

Definition 2. Suppose that $\mathscr{C}$ is Waldhausen. Consider any triple (f', f, f'') as in ($\star$) with the property that whenever f' and f'' are weak equivalences, then so is f. In this case, we say $\mathscr{C}$ is *extensional* or *closed under extensions*.

Say that the morphism (f', f, f'') is a cofibration if f', f'', and $Y' \cup_{X'} X \to Y$ are cofibrations in $\mathscr{C}$. Say that the same triple is a weak equivalence if f', f, and f'' are weak equivalences in $\mathscr{C}$. This makes $S_2\mathscr{C}$ into a Waldhausen category.

Definition 3. Let $q \geq 0$. Let the *arrow category* $\operatorname{Ar}[q]$ *on* $[q]$ have as objects ordered pairs (i, j) with $i \leq j \leq q$ and as morphisms commutative diagrams of the form

$$\begin{tikzcd} i \arrow[r, "\leq"] \arrow[d, "\leq"'] & j \arrow[d, "\leq"] \\ i' \arrow[r, "\leq"'] & j' \end{tikzcd} .$$

We view $[q]$ as a full subcategory of $\operatorname{Ar}[q]$ via the embedding $[q] \xrightarrow{k \mapsto (0,k)} \operatorname{Ar}[q]$.

Note 4.

1. Any triple $i \leq j \leq k$ determines the morphisms $(i,j) \to (i,k)$ and $(i,k) \to (j,k)$. Conversely, any morphism in the arrow category is a composite of such triples.

2. $\mathrm{Ar}[q] \cong \mathbf{Fun}([1],[q])$ with each pair (i,j) identified with the functor satisfying $0 \mapsto i$ and $1 \mapsto j$.

Example 5. The category $\mathrm{Ar}[2]$ is generated by the commutative diagram

$$\begin{tikzcd}
(0,0) \arrow[r] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\
& (1,1) \arrow[r] & (1,2) \arrow[d] \\
& & (2,2)
\end{tikzcd}.$$

Let $\mathscr{C}$ be a category with cofibrations and $q \in \mathbb{Z}_{\geq 0}$. Define $S_q\mathscr{C}$ as the full subcategory of $\mathbf{Fun}(\mathrm{Ar}[q],\mathscr{C})$ generated by $X : \mathrm{Ar}[q] \to \mathscr{C}$ such that

1. $X_{j,j} = *$ for each $j \in [q]$.

2. $X_{i,j} \rightarrowtail X_{i,k} \twoheadrightarrow X_{j,k}$ is a cofiber sequence for any $i < j < k$ in $[q]$. Equivalently, if $i \leq j \leq k$ in $[q]$, then the square

$$\begin{tikzcd}
X_{i,j} \arrow[r, tail] \arrow[d, two heads] & X_{i,k} \arrow[d, two heads] \\
X_{j,j} = * \arrow[r, tail] & X_{j,k}
\end{tikzcd}$$

is a pushout.

This is pointed at the constant diagram at $*$.

Note 6. A generic object in $S_q\mathscr{C}$ looks like

$$\begin{tikzcd}
* \arrow[r, tail] & X_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & X_{q-1} \arrow[r, tail] \arrow[d, two heads] & X_q \arrow[d, two heads] \\
& * \arrow[r, tail] & \cdots \arrow[r, tail] & X_{q-1}/X_1 \arrow[r, tail] \arrow[d, two heads] & X_q/X_1 \arrow[d, two heads] \\
& & \ddots & \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
& & & * \arrow[r, tail] & X_q/X_{q-1} \arrow[d, two heads] \\
& & & & *
\end{tikzcd}. \qquad (\dagger)$$

where X_q corresponds to $X_{0,q}$ and X_j/X_i to $X_{i,j}$ for any $1 \leq i \leq j \leq q$.

Definition 7. Let $(\mathscr{C}, \mathsf{co}\,\mathscr{C})$ be a category with cofibrations. Let $\mathsf{co}\,S_q\mathscr{C} \subset S_q\mathscr{C}$ consist of the morphisms $f : X \rightarrowtail Y$ of $\mathrm{Ar}[q]$-shaped diagrams such that for each $1 \le j \le q$ we have

$$\begin{tikzcd}
X_{0,j-1} \arrow[r, tail] \arrow[d, "f_{0,j-1}"'] & X_{0,j} \arrow[d] \arrow[ddr, bend left, "f_{0,j}"] & \\
Y_{0,j-1} \arrow[r, tail] \arrow[ddrr, tail, bend right] & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} \arrow[dr, tail, dashed] & \\
& & Y_{0,j}
\end{tikzcd}.$$

Proposition 8. *If $f : X \to Y$ is a cofibration of $S_q\mathscr{C}$, then*

$$\begin{tikzcd}
X_{i,j} \arrow[r, tail] \arrow[d, tail, "f_{i,j}"'] & X_{i,k} \arrow[d, tail, "f_{i,k}"] \\
Y_{i,j} \arrow[r, tail] & Y_{i,k}
\end{tikzcd}$$

for any $i \le j \le k$ in $[q]$.[1]

Lemma 9. *$(S_q\mathscr{C}, \mathsf{co}\,S_1\mathscr{C})$ is a category with cofibrations.*

Proof. First notice that the composite of two cofibrations $g \circ f : X \to Y \to Z$ is a cofibration thanks to the commutative diagram

$$\begin{tikzcd}
X_{0,j-1} \arrow[r, tail] \arrow[d, "f_{0,j-1}"'] & X_{0,j} \arrow[d] \arrow[dr, bend left, "f_{0,j}"] & & \\
Y_{0,j-1} \arrow[r, tail] \arrow[d, "g_{0,j-1}"'] & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} \arrow[r, tail] \arrow[d] & Y_{0,j} \arrow[d] \arrow[dr, bend left, "f_{0,j}"] & \\
Z_{0,j-1} \arrow[r, tail] & X_{0,j} \cup_{X_{0,j-1}} Z_{0,j-1} \arrow[r, tail] & Y_{0,j} \cup_{Y_{0,j-1}} Z_{0,j-1} \arrow[r, tail] & Z_{0,j}
\end{tikzcd}.$$

It's clear that any isomorphism or initial morphism in $S_q\mathscr{C}$ is a cofibration.

To see that axiom W2 is satisfied, let $f : X \to Y$ and $g : X \to Z$ be morphisms in $S_q\mathscr{C}$. It's easy to verify that each component $f_{i,j} : X_{i,j} \to Y_{i,j}$ is a cofibration. Thus, each pushout of the form $W_{i,j} \coloneqq Y_{i,j} \cup_{X_{i,j}} Z_{i,j}$ exists. These form a functor $W : \mathrm{Ar}[q] \to \mathscr{C}$. If $i < j < k$, then we have a cofiber sequence $W_{i,j} \rightarrowtail W_{i,k} \twoheadrightarrow W_{j,k}$ because $W_{i,j} \rightarrowtail W_{i,k}$ factors as the composite of two cofibrations as follows.

$$\begin{tikzcd}
Z_{i,j} \arrow[r, tail] \arrow[d, "f_{i,j} \cup \mathrm{Id}"'] & Z_{i,k} \arrow[d, "f_{i,j} \cup \mathrm{Id}"] & \\
Y_{i,j} \cup_{X_{i,j}} Z_{i,j} \arrow[r, tail] & Y_{i,j} \cup_{X_{i,j}} Z_{i,k} \arrow[r, tail] & Y_{i,k} \cup_{X_{i,k}} Z_{i,k} \\
& Y_{i,j} \cup_{X_{i,j}} X_{i,k} \arrow[r, tail] \arrow[u, "\mathrm{Id} \cup g_{i,k}"] & Y_{i,k} \arrow[u, "\mathrm{Id} \cup g_{i,k}"']
\end{tikzcd}$$

[1] Lemma 8.3.12 (Rognes).

The fact that colimits commute with each other ensures that $W_{j,k} \cong {}^{W_{i,k}}\!/\!_{W_{i,j}}$. Hence W is the pushout of f and g. To verify that this is a cofibration, we must check that the pushout map $W_{0,j-1} \cup_{Z_{0,j-1}} Z_{0,j} \to W_{0,j}$ is a cofibration. But this follows from the pushout square

$$\begin{tikzcd} Y_{0,j-1} \cup_{X_{0,j-1}} X_{0,j} \arrow[r, tail] \arrow[d] & Y_{0,j} \arrow[d] \\ Y_{0,j-1} \cup_{X_{0,j-1}} Z_{0,j} \arrow[r, tail] & Y_{0,j} \cup_{X_{0,j}} Z_{0,j} \end{tikzcd}.$$

□

Definition 10. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Let $wS_q\mathscr{C} \subset S_q\mathscr{C}$ consist of the morphisms $f : X \xrightarrow{\sim} Y$ of $\mathrm{Ar}[q]$-shaped diagrams such that the component $f_{0,j} : X_{0,j} \to Y_{0,j}$ is a weak equivalence in $\mathscr{C}$ for each $1 \leq j \leq q$.

Proposition 11. *Let f be a weak equivalence in $S_q\mathscr{C}$. Each component $f_{i,j} : X_{i,j} \to Y_{i,j}$ is a weak equivalence in $\mathscr{C}$.*

Proof. Apply the Gluing axiom to the diagram

$$\begin{tikzcd} X_{0,j} \arrow[d, "\cong"'] & X_{0,i} \arrow[l, tail] \arrow[r] \arrow[d, "\cong"'] & * \arrow[d, "="'] \\ Y_{0,j} & Y_{0,i} \arrow[l, tail] \arrow[r] & * \end{tikzcd}.$$

Then $X_{i,j} \cong X_{0,j} \cup_{X_{0,i}} * \xrightarrow{\sim} Y_{0,j} \cup_{Y_{0,i}} * \cong Y_{i,j}$, as desired. □

Lemma 12. *$(S_q\mathscr{C}, wS_q\mathscr{C})$ is a Waldhausen category.*

Definition 13. Let $\mathscr{C}$ be a category with cofibrations. If $\alpha : [p] \to [q]$, then define $\alpha^* : S_q\mathscr{C} \to S_p\mathscr{C}$ by

$$\alpha^*(X : \mathrm{Ar}[q] \to \mathscr{C}) = X \circ \mathrm{Ar}(\alpha) : \mathrm{Ar}[p] \to \mathrm{Ar}[q] \to \mathscr{C}.$$

It's easy to check that this satisfies the two conditions of a diagram in $S_p\mathscr{C}$. Moreover, the face maps d_i are obtained by deleting the row $X_{i,-}$ and the column containing X_i in (†) and then reindexing as necessary. The degeneracy maps s_i are given by duplicating X_i and then reindexing such that $X_{i+1,i} = 0$.

Not sure that the s_i work.

Proposition 14. *Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Each functor $\alpha^* : S_q\mathscr{C} \to S_p\mathscr{C}$ is exact, so that $(S_\bullet\mathscr{C}, wS_\bullet\mathscr{C})$ is a simplicial Waldhausen category.*

The nerve $N_\bullet wS_\bullet\mathscr{C}$ is a bisimplicial set with (p,q)-bisimplices the diagrams of the form

$$\begin{tikzcd}
* \arrow[r, tail] & X_1^0 \arrow[r, tail] \arrow[d, "\sim"'] & X_2^0 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X_q^0 \arrow[d, "\sim"'] \\
* \arrow[r, tail] & X_1^1 \arrow[r, tail] \arrow[d, "\sim"'] & X_2^1 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X_q^1 \arrow[d, "\sim"'] \\
 & \vdots \arrow[d, "\sim"'] & \vdots \arrow[d, "\sim"'] & & \vdots \arrow[d, "\sim"'] \\
* \arrow[r, tail] & X_1^p \arrow[r, tail] & X_2^p \arrow[r, tail] & \cdots \arrow[r, tail] & X_q^p
\end{tikzcd}$$

such that $X_{i,j}^k \cong X_j^k / X_i^k$ for every $i \le j \le q$ and $k \in [p]$.

Lemma 15. *There is a natural map $N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 \to N_\bullet wS_\bullet\mathscr{C}$, which automatically induces a based map $\sigma : \Sigma\,|w\mathscr{C}| \to |wS_\bullet\mathscr{C}|$ of classifying spaces.*

Proof. We can treat $N_\bullet wS_\bullet\mathscr{C}$ as the simplicial set $[q] \mapsto N_\bullet wS_q\mathscr{C}$. This defines a right skeletal structure on $N_\bullet wS_\bullet\mathscr{C}$.

If $q = 0$, then $wS_0\mathscr{C} = S_0\mathscr{C} = *$, so that $N_\bullet wS_0\mathscr{C} = *$ as well. If $q = 1$, then $wS_1\mathscr{C} \cong w\mathscr{C}$. Thus, the right 1-skeleton is equal to $N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1$, which in turn must be equal to the image I of the canonical map

$$\coprod_{q\le 1} N_\bullet wS_q\mathscr{C} \times \Delta_\bullet^q \to N_\bullet wS_\bullet\mathscr{C}.$$

Now, the degeneracy map s_0 collapses $\{*\} \times \Delta_\bullet^1$, and the face maps d_0 and d_1 collapse $N_\bullet w\mathscr{C} \times \partial\Delta_\bullet^1$. Therefore, I must equal

$$N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 = \frac{N_\bullet w\mathscr{C} \times \Delta_\bullet^1}{\{*\} \times \Delta_\bullet^1 \cup N_\bullet w\mathscr{C} \times \partial\Delta_\bullet^1}.$$

We have defined a natural inclusion map $\lambda : N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 \to N_\bullet wS_\bullet\mathscr{C}$.

Since $\Delta_\bullet^1$ is isomorphic to the unit interval and the map λ agrees on the endpoints, we can pass to S^1 during the suspension. Hence λ induces the desired map σ.[2] $\square$

Note 16. Axiom W3 implies that $w\mathscr{C}$ is closed under coproducts, making $|wS_\bullet\mathscr{C}|$ into an H-space via the map

$$\coprod : |wS_\bullet\mathscr{C}| \times |wS_\bullet\mathscr{C}| \cong |wS_\bullet\mathscr{C} \times wS_\bullet\mathscr{C}| \to |wS_\bullet\mathscr{C}|\,.$$

Definition 17. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Define the *algebraic K-theory space*

$$K(\mathscr{C}, w) = \Omega\,|N_\bullet wS_\bullet\mathscr{C}|\,.$$

Note 18. We have a right adjoint $\iota : |w\mathscr{C}| \to K(\mathscr{C}, w)$ to the based map σ.

[2] This is a tentative explanation due to Thomas Brazelton.

Let $F : (\mathscr{C}, w\mathscr{C}) \to (\mathscr{D}, w\mathscr{D})$ be an exact functor. Let

$$K(F) = \Omega\, |wS_\bullet F| : K(\mathscr{C}, w) \to K(\mathscr{D}, w).$$

This is the *algebraic K-theory functor* $K : \mathbf{Wald} \to \mathbf{Top}_*$.

Note that any exact category $\mathscr{A}$ is a Waldhausen category with cofibrations the admissible exact sequences and weak equivalences the isomorphisms. Waldhausen showed that $|iS_\bullet \mathscr{A}|$ (where $i(-)$ denotes the isomorphism category) and $BQ\mathscr{A}$ are homotopy equivalent. Therefore, our current definition of higher algebraic K-theory agrees with Quillen's.

Example 19. Let R be a ring. Define the *algebraic K-theory space of R* as

$$K(R) = K(\mathbf{P}(R), i)$$

where the weak equivalences are precisely the injective R-linear maps with projective cokernel and the cofibrations are precisely the R-linear maps.

Example 20. Assume that $\mathscr{C}$is a small Waldhausen category where $w\mathscr{C}$ consists of the isomorphisms in $\mathscr{C}$. If $s_n\mathscr{C}$ denotes the set of objects of $S_n\mathscr{C}$, then we get a simplicial set $s_\bullet\mathscr{C}$. Waldhausen showed that the inclusion map $|s_\bullet\mathscr{C}| \hookrightarrow |iS_\bullet\mathscr{C}|$ is a homotopy equivalence. This makes $\Omega|s_\bullet\mathscr{C}|$ into a so-called simplicial model for $K(\mathscr{C}, w)$.

Remark 21. Since $wS_0\mathscr{C} = *$ and every simplex of degree $n > 0$ is attached to $*$, it follows that the classifying space $|wS_\bullet\mathscr{C}|$ is connected. Therefore, we preserve any homotopical information when passing to the loop space.

Definition 22. The *i-th algebraic K-group* is $K_i(\mathscr{C}, w) \equiv \pi_i K(\mathscr{C}, w)$ for each $i \geq 0$.

Proposition 23. *$\pi_1\, |wS_\bullet\mathscr{C}| \cong K_0(\mathscr{C}, w)$.*

Lemma 24. *The group $K_0(\mathscr{C}, w)$ is generated by all elements $[X]$ such that*

- *$[X'] + [X''] = [X]$ for every cofiber sequence $X' \rightarrowtail X \twoheadrightarrow X''$ and*
- *$[X] = [Y]$ for every weak equivalence $X \xrightarrow{\sim} Y$.*

Proof. In light of Proposition 23, it suffices to compute $\pi_1\, |N_\bullet wS_\bullet\mathscr{C}|$ based at the $(0,0)$-bisimplex $*$. For this, just notice the CW structure of $|N_\bullet wS_\bullet\mathscr{C}|$, with 1-cells the $(0,1)$-bisimplices and 2-cells the $(0,2)$-bisimplices $X' \rightarrowtail X \twoheadrightarrow X''$ and the $(1,1)$-bisimplices $X \xrightarrow{\sim} Y$, which are attached to the 1-cells X and Y. Any cell of dimension $n > 2$ is irrelevant to computing π_1. □

As a result, we obtain functors

$$K_i : \mathbf{Wald} \to \mathbf{Top}_* \to \mathbf{Ab}$$

known as the *algebraic K-group functors.* Indeed, thanks to Proposition 23, we know that

$$K_i(\mathscr{C}, w) = \pi_{i+1}\, |wS_\bullet\mathscr{C}|\,,$$

which is abelian for $i \geq 1$. Moreover, note that if $X' \rightarrowtail X' \vee X'' \twoheadrightarrow X''$ and $X'' \rightarrowtail X' \vee X'' \twoheadrightarrow X'$ are cofiber sequences, then Lemma 24 implies that

$$[X'] + [X''] = [X' \vee X''] = [X'' + X'] .$$

Hence $K_0(\mathscr{C}, w)$ is also abelian.

Example 25. Let X be a CW complex and $\mathcal{R}(X)$ denote the category of CW complexes Y obtained by attaching at least one cell to X so that X is a retract of Y. Equip this with cofibrations in the form of cellular inclusions fixing X and weak equivalence in the form of homotopy equivalences. This makes $\mathcal{R}(X)$ into a Waldhausen category.

If $\mathcal{R}_f(X)$ denotes the subcategory of those Y obtained by attaching finitely many cells, then we denote $K(\mathcal{R}_f(X))$ by $A(X)$.

Proposition 26. *$A_0(X) \cong \mathbb{Z}$.*

Definition 27. If $\mathscr{B}$ is a Waldhausen subcategory of $\mathscr{C}$, then it is *cofinal in $\mathscr{C}$* if for any $X \in \operatorname{ob}\mathscr{C}$, there is some $X' \in \operatorname{ob}\mathscr{C}$ such that $X \coprod X' \in \operatorname{ob}\mathscr{B}$.

Theorem 28. *Let $(\mathscr{B}, w)$ be cofinal in $(\mathscr{C}, w)$ and closed under extensions. Assume that $K_0(\mathscr{B}) = K_0(\mathscr{C})$. Then $wS_\bullet\mathscr{B} \to wS_\bullet\mathscr{C}$ is a homotopy equivalence.*

It follows that $K_i(\mathscr{B}) \cong K_i(\mathscr{C})$ for every $i \geq 0$.