Perry Hart
Homotopy and K-theory seminar
Talk #3
September 26, 2018

Abstract

More basic category theory. The main sources for this talk are the following.

- nLab.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 3.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 1.

Definition 1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F, G : \mathscr{C} \to \mathscr{D}$ be functors. A *natural transformation* $\phi : F \Rightarrow G$ is a function $A \mapsto f_A$ from $\mathrm{ob}\,\mathscr{C}$ to $\mathrm{mor}\,\mathscr{D}$ such that $f_A : F(A) \to G(A)$ and the following diagram commutes for any morphism $f : A \to B$.

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "f_A"'] & FB \arrow[d, "f_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

In symbols, this may be written as $f_B f_* = f_* f_A$, where f_A and f_B are called the *components* of ϕ.

Remark 1. If every f_A is an isomorphism, then the $(f_A)^{-1}$ define a natural transformation between the same two functors.

Definition 2. Let $F, G, H : \mathscr{C} \to \mathscr{D}$ be functors. The *identity natural transformation* $\mathrm{Id}_F : F \Rightarrow F$ is given by $A \mapsto \mathrm{Id}_{F(A)}$. Moreover, given natural transformations $\phi : F \to G$ and $\psi : G \to H$, define the *composite natural transformation* $\psi \circ \phi$ by $A \mapsto (\psi \circ \phi)_A \coloneqq \psi_A \circ \phi_A$.

Definition 3. If each f_A is an isomorphism, then we call $\phi : F \cong G$ a *natural isomorphism*.

Remark 2. If $\mathscr{D}$ is a groupoid, then ϕ must be a natural isomorphism.

Lemma 1. *A natural transformation $\phi : F \Rightarrow G$ is a natural isomorphism iff it has an inverse $\phi^{-1} : G \Rightarrow F$.*

Proof. This follows from Remark 1 and the definition of composite natural transformation. $\square$

Example 4. Let R and S be commutative rings. Any ring homomorphism $f : R \to S$ induces a ring homomorphism $\mathrm{GL}_n(f) : \mathrm{GL}_n(R) \to \mathrm{GL}_n(S)$ which satisfies $f(\det(A)) = \det(\mathrm{GL}_n(f)(A))$. Viewing GL_n and $R \mapsto R^*$ as functors from **Rng** to **Grp** and $\det_R : \mathrm{GL}_n(R) \to R^*$ as a morphism in **Grp**, we see that $\det_R$ defines a natural transformation $\phi : \mathrm{GL}_n \Rightarrow f^*$, where f^* denotes $f \restriction_{R^*} R^* \to S^*$.

$$\begin{tikzcd} \mathrm{GL}_n(R) \arrow[r, "\mathrm{GL}_n(f)"] \arrow[d, "\det_R"'] & \mathrm{GL}_n(S) \arrow[d, "\det_S"] \\ R^* \arrow[r, "f^*"] & S^* \end{tikzcd}$$

Example 5. Recall the power set functor $P : \mathbf{Set} \to \mathbf{Set}$ given by $A \mapsto P(A)$ and $Pg(S) = g(S)$ where $g : A \to B$ is a function and $S \subset A$. Then the function $f_A : A \to P(A)$ given by $a \mapsto \{a\}$ defines a natural transformation $\phi : \mathrm{Id}_{\mathbf{Set}} \Rightarrow P$.

Example 6. Set $\mathscr{C} = \mathscr{D} = \mathbf{Grp}$, $F = \mathrm{Id}_{\mathscr{C}}$, and G equal to the abelianization functor. Then given a group H, the homomorphism $f : H \to H^{\mathrm{ab}}$ defines a natural transformation $\phi : F \Rightarrow G$.

Example 7. Consider the preorders $(P, \leq)$ and $(Q, \leq)$ as small categories where functors $F, G : P \to Q$ are order-preserving functions. Then there is a unique natural transformation $\phi : F \Rightarrow G$ iff $F(x) \leq G(x)$ for every $x \in P$.

Example 8. The inversion isomorphism from a group G to G^{op} defines a natural transformation $\phi : \mathrm{Id}_{\mathbf{Grp}} \Rightarrow ({}^{\mathrm{op}} : \mathbf{Grp} \to \mathbf{Grp})$. In other words, G is naturally isomorphic to G^{op}.

Definition 9. Let $\mathscr{C}$ and $\mathscr{D}$ be categories with $\mathscr{C}$ small. The *functor category* $\mathbf{Fun}(\mathscr{C}, \mathscr{D}) := \mathscr{D}^{\mathscr{C}}$ has functors $F : \mathscr{C} \to \mathscr{D}$ as objects and natural transformations as morphisms.

Remark 3. Given functors $F, G : \mathscr{C} \to \mathscr{D}$, why is the class of natural transformation $\phi : F \Rightarrow G$ necessarily a set? A G-Universe models ZFC, in particular $\mathsf{Replacement}$.

Definition 10. Given a category $\mathscr{C}$, the *arrow category* $\mathrm{Ar}(\mathscr{C})$ of $\mathscr{C}$ has as objects morphisms $f : X_0 \to X_1$ in $\mathscr{C}$ and as morphisms $M : (f : X_0 \to X_1) \to (g : Y_0 \to Y_1)$ the pairs (M_0, M_1) of morphisms $M_0 : X_0 \to Y_0$ and $M_1 : X_1 \to Y_1$ such that

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "M_0"'] & X_1 \arrow[d, "M_1"] \\ Y_0 \arrow[r, "g"] & Y_1 \end{tikzcd}$$

commutes.

Note that $\mathrm{Ar}(\mathscr{C}) \cong \mathbf{Fun}([1], \mathscr{C})$.

Lemma 2. $\mathbf{Fun}(\mathscr{C} \times \mathscr{D}, \mathscr{E}) \cong \mathbf{Fun}(\mathscr{C}, \mathbf{Fun}(\mathscr{D}, \mathscr{E}))$ *via currying.*

Definition 11. A functor $F : \mathscr{C} \to \mathscr{D}$ is an *equivalence* if there is a functor $G : \mathscr{D} \to \mathscr{C}$ such that $F \circ G \cong \mathrm{Id}_{\mathscr{C}}$ and $G \circ F \cong \mathrm{Id}_{\mathscr{D}}$. In this case, we say that F and G are *equivalent categories.* Moreover, we say that a property of $\mathscr{C}$ is *categorical* if it is invariant under such equivalence.

Example 12. Let k be a field. Let the category $\mathbf{Mat}_k$ have natural numbers as objects and morphisms $n \to p$ given by $p \times n$ matrices over k. Let **fdMod** denote the category of finite-dimensional vector spaces with linear maps as morphisms. These two categories are equivalent. Send nat n to k^n in one direction and the space V to $\dim V$ in the other direction.

Definition 13. A functor $F : \mathscr{C} \to \mathscr{D}$ is *essentially surjective* if for each object Z of $\mathscr{D}$, there is some object Y of $\mathscr{C}$ such that $F(Y) \cong Z$.

Theorem 3. *A functor is an equivalence iff it is full, faithful, and essentially surjective.*

Proof. See Rognes, Theorem 3.2.10. □

Definition 14. A *skeleton* of $\mathscr{C}$ is a full subcategory $\mathscr{C}' \subset \mathscr{C}$ such that each element of $\mathrm{ob}\,\mathscr{C}$ is isomorphic to exactly one element of $\mathrm{ob}\,\mathscr{C}'$.

Lemma 4. *With notation as before,* $\mathscr{C}'$ *and* $\mathscr{C}$ *are equivalent categories via the inclusion functor.*

Proof. Apply Theorem 3. □

Lemma 5. *Any two skeleta* $\mathscr{C}', \mathscr{C}'' \subset \mathscr{C}$ *are isomorphic.*

Proof. Define $F : \mathscr{C}' \to \mathscr{C}''$ by $F(X) = Y$ where $h_X : X \cong Y$ and $F(f) = h_Y \circ f \circ (h_X)^{-1}$ for $f \in \mathscr{C}(X, Y)$. To get F^{-1}, similarly define $G : \mathscr{C}'' \to \mathscr{C}'$ by choosing $(h_X)^{-1}$. □

Remark 4. The previous two lemmas are equivalent to the axiom of choice, as is the statement that every category admits a skeleton.

Definition 15. Fix $X \in \mathrm{ob}\,\mathscr{C}$. Define the functor $\mathscr{Y}^X : \mathscr{C} \to \mathbf{Set}$ by $Y \mapsto \mathscr{C}(X, Y)$ and mapping each morphism $g : Y \to Z$ to $g_* : \mathscr{C}(X, Y) \to \mathscr{C}(X, Z)$ given by $f \mapsto gf$. We call $\mathscr{C}(X, -) := \mathscr{Y}^X$ the set-valued functor *corepresented* by X in $\mathscr{C}$.

Definition 16. Fix $Z \in \operatorname{ob}\mathscr{C}$. Define the contravariant functor $\mathscr{Y}_Z : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ by $Y \mapsto \mathscr{C}(Y,Z)$ and mapping each morphism $f : X \to Y$ in $\mathscr{C}$ to $f^* : \mathscr{C}(Y,Z) \to \mathscr{C}(X,Z)$ given by $g \mapsto gf$. We call $\mathscr{C}(-,Z) := \mathscr{Y}^Z$ the set-valued functor *represented* by Z in $\mathscr{C}$.

Definition 17. A functor $F : \mathscr{C} \times \mathscr{C}' \to \mathscr{D}$ is also called a *bifunctor*.

Example 18. Let $\mathscr{C}$ be a category. Define $\mathscr{C}(-,-) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$ by $(X,X') \to \mathscr{C}(X,X')$ and mapping each morphism $(f,f') : (X,X') \to (Y,Y')$ to $\mathscr{C}(f,f') : \mathscr{C}(X,X') \to \mathscr{C}(Y,Y')$ given by $g \mapsto f'gf$.

Definition 19. This is due to Dan Kan. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors. Consider the set-valued bifunctors $\mathscr{D}(F(-),-), \mathscr{C}(-,G(-)) : \mathscr{C}^{\mathrm{op}} \times \mathscr{D} \to \mathbf{Set}$. An *adjunction* between F and G is a natural isomorphism $\phi : \mathscr{D}(F(-),-) \Rightarrow \mathscr{C}(-,G(-))$. If such ϕ exists, then we say that (F,G) is an *adjoint pair* or functors. We also call F the *left adjoint* to G and G the *right adjoint* to F.

Remark 5. For each $c : X' \to X$ and $d : Y \to Y'$, the following commutes.

$$\begin{tikzcd}
\mathscr{D}(F(X),Y) \arrow[r, "\phi_{X,Y}"] \arrow[d, "c^*d_*"'] & \mathscr{C}(X,G(Y)) \arrow[d, "c^*d_*"] \\
\mathscr{D}(F(X'),Y') \arrow[r, "\phi_{X',Y'}"] & \mathscr{C}(X',G(Y'))
\end{tikzcd}$$

Example 20. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ admits a left adjoint $F : \mathbf{Set} \to \mathbf{Grp}$ which maps a set to the free group generated by A. The adjunction is the natural bijection $\mathbf{Set}(A, U(G)) \cong \mathbf{Grp}(F(A), G)$.

Example 21. Let R be a ring. The forgetful functor $U : R - \mathbf{Mod} \to \mathbf{Set}$ admits a left adjoint $R(-)$ sending a set S to $\bigoplus_{s\in S} R$, the free R-module generated by S. The adjunction is the natural bijection $\mathbf{Set}(S, U(M)) \cong R - \mathbf{Mod}(R(S), M)$.

Remark 6. U does not admit a right adjoint in either of the previous two examples.

Example 22. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ has left adjoint that sends a set to the same set equipped with the discrete topology. It also has a right adjoint via the functor sending a set to the same set equipped with the indiscrete topology.

Example 23. Let $\mathbf{CMon}$ be the category of commutative monoids. Given $M \in \operatorname{ob}\mathbf{CMon}$, we can construct the completion, or Grothendieck group, $G(M)$ on $M \times M$ as follows. Define addition on $M \times M$ component-wise and say that $(m_1, m_2) \sim (n_1, n_2)$ if $m_1 + m_2 + k = m_2 + n_1 + k$ for some $k \in M$. Set $G(M)$ as $\left(M \times M/\sim, +\right)$.

Then $G : \mathbf{CMon} \to \mathbf{Ab}$ is a functor. This is left adjoint to the forgetful functor $U : \mathbf{Ab} \to \mathbf{CMon}$.

Remark 7. Read Rognes, Definition 3.4.8, where he constructs the group completion $K(M)$ of non-commutative monoids M. It turns out that $K(M)$ is realized as the fundamental group of an important classifying space.

Definition 24. A subcategory $\mathscr{C} \subset \mathscr{D}$ is *reflective* if the inclusion functor is a right adjoint and is *coreflective* if the inclusion functor is a left adjoint.

Example 25.

1. $\mathbf{Ab} \subset \mathbf{CMond}$ is reflective by Example 8.
2. $\mathbf{Ab} \subset \mathbf{Grp}$ is reflective.
3. Let $\mathbf{Ab}_T \subset \mathbf{Ab}$ denote the category of torsion groups. This is coreflective via the functor sending an abelian group to its torsion subgroup because any homomorphism $f : A \to B$ where A is torsion has $f(A) \subset B_T$.

Definition 26. Given an adjunction $\phi : \mathscr{D}(F(-), -) \Rightarrow \mathscr{C}(-, G(-))$, define the *unit morphism*

$$\eta_X = \phi_{X,F(X)}(\mathrm{Id}_{F(X)})$$

and the *counit morphism*

$$\epsilon_Y = \phi^{-1}_{G(Y),Y}(\mathrm{Id}_{G(Y)}).$$

Lemma 6. *The unit morphisms η_X define a natural transformation $\eta : \mathrm{Id}_{\mathscr{C}} \Rightarrow GF$, and the counit morphisms η_Y define a natural transformation $\epsilon : FG \Rightarrow \mathrm{Id}_{\mathscr{D}}$.*