

Abstract

This is an introduction to universal properties in category theory. The main sources for this talk are the following.

- nLab.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 4.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4.

1 Universal arrows

Definition 1.1. An object X of \mathcal{C} is *initial* if for each $Y \in \text{ob } \mathcal{C}$, there is a unique morphism $f : X \rightarrow Y$. Moreover, we say that X is *terminal* if for each $Z \in \text{ob } \mathcal{C}$, there is a unique morphism $g : Z \rightarrow X$. Either condition is called a *universal property* of X .

Any property P of \mathcal{C} has a dual property P^{op} of \mathcal{C}^{op} obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P . Then P is true of \mathcal{C} iff P^{op} is true of \mathcal{C}^{op} .

Example 1.2. Being initial and being terminal are dual properties.

Lemma 1.3. *Any two initial objects of \mathcal{C} are canonically isomorphic. The same holds for any two terminal objects of \mathcal{C} .*

Proof. Compose the two unique morphisms to get an isomorphism between the two initial objects. Apply duality to get the second claim. \square

We can think of a universal property as follows. Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and $X \in \text{ob } \mathcal{C}$. A *universal arrow from X to F* is an ordered pair (Y, f) with $Y \in \text{ob } \mathcal{D}$ and $f : X \rightarrow F(Y)$ a morphism of \mathcal{C} with the property that for any $X' \in \text{ob } \mathcal{D}$ and morphism $f' : X \rightarrow F(X')$ of \mathcal{C} , there exists a unique morphism $\hat{f} : Y \rightarrow X'$ of \mathcal{D} such that $F(\hat{f}) \circ f = f'$.

$$\begin{array}{ccc} X & \xrightarrow{f} & F(Y) \\ & \searrow f' & \downarrow F(\hat{f}) \\ & & F(X') \end{array}$$

Dually, a *universal arrow from F to X* is an ordered pair (Y, f) with $Y \in \text{ob } \mathcal{D}$ and $f : F(Y) \rightarrow X$ of \mathcal{C} with the property that for any $X' \in \text{ob } \mathcal{D}$ and morphism $f' : F(X') \rightarrow X$, there exists a unique morphism

$\hat{f} : X' \rightarrow Y$ such that $f' = f \circ F(\hat{f})$.

$$\begin{array}{ccc} F(X') & \xrightarrow{F(\hat{f})} & F(Y) \\ & \searrow f' & \downarrow f \\ & & X \end{array}$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

Definition 1.4.

1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $Y \in \text{ob } \mathcal{D}$. The *slice* or *left fiber category*, denoted by (F/Y) or $(F \downarrow Y)$, has as objects pairs (X, f) where $f : F(X) \rightarrow Y$ and as morphisms from $f : F(X) \rightarrow Y$ to $f' : F(X') \rightarrow Y$ morphisms $g : X \rightarrow X'$ such that $f = f' \circ F(g)$.
2. The *coslice* or *right fiber category*, denoted by (Y/F) or $(Y \downarrow F)$, has as objects pairs (X, f) where $f : Y \rightarrow F(X)$ and as morphisms from $f : Y \rightarrow F(X)$ to $f' : Y \rightarrow F(X')$ morphisms $g : X \rightarrow X'$ such that $f' = F(g) \circ f$.

If $F^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$ is opposite to the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $Y \in \text{ob } \mathcal{D}$, then $(Y/F)^{op} = F^{op}/Y$. Thus, the left and right fiber categories are dual in the sense that $P(Y, F)$ is true of any right fiber category Y/F iff $P^{op}(Y, F)$ is true of any left fiber category F/Y .

Proposition 1.5. *Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and $x \in \text{ob } \mathcal{C}$. Then $u : x \rightarrow Fr$ is a universal arrow from x to F iff it is initial object of the coslice $(x \downarrow F)$. Dually, $u' : Fr' \rightarrow x$ is a universal arrow from F to x iff it is a terminal object of the same category.*

Proof. Suppose that u is universal and $f : x \rightarrow Fy$ is another object of $(x \downarrow F)$. Then there is some unique $\hat{f} : r \rightarrow y$ such that $F(\hat{f}) \circ u = f$. Thus $F(\hat{f})$ is a unique morphism of the coslice.

Conversely, suppose that u is initial. Then for any object $f : x \rightarrow Fy$ of $(x \downarrow F)$, there is some unique arrow $Sg : Fr \rightarrow Fy$ such that $Sg \circ u = f$. Hence setting $\hat{f} = g$ makes u a universal arrow. \square

Corollary 1.6. *Any two universal arrows from x to F can be canonically identified by Lemma 1.3.*

2 Colimits

Definition 2.1. A *zero object* of \mathcal{C} is an object that is both initial and terminal.

Example 2.2. The unique initial object of **Set** is \emptyset , and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in $\text{iso}(\mathbf{Set})$.

Given $X \in \text{ob } \mathcal{C}$, the *undercategory* X/\mathcal{C} has as objects morphisms in \mathcal{C} of the form $i : X \rightarrow Y$ where X is fixed. Given $i : X \rightarrow Y$ and $i' : X \rightarrow Y'$ in $\text{ob } X/\mathcal{C}$, define the set of morphisms from i to i' as the morphisms $f : Y \rightarrow Y'$ where

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow i' & \downarrow f \\ & & Y' \end{array}$$

commutes. (We call i the *structure morphism*.) Composition and identity carry over exactly from \mathcal{C} .

Likewise, given $x \in \text{ob } \mathcal{C}$, the *overcategory* \mathcal{C}/X has as objects morphisms in \mathcal{C} of the form $i : Y \rightarrow X$ where X is fixed. Given $i : Y \rightarrow X$ and $i' : Y' \rightarrow X$ in $\text{ob } \mathcal{C}/X$, define the set of morphisms from i to i' as the morphisms $f : Y \rightarrow Y'$ where

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & \searrow i & \downarrow i' \\ & & X \end{array}$$

commutes. Composition and identity carry over exactly from \mathcal{C} .

Remark 2.3. If $X \in \text{ob } \mathcal{C}$, then $(X/\mathcal{C})^{\text{op}} = \mathcal{C}^{\text{op}}/X$. Thus, the under- and overcategories are dual in the sense that $P(X, \mathcal{C})$ is true of any undercategory X/\mathcal{C} iff $P^{\text{op}}(X, \mathcal{C})$ is true of any overcategory \mathcal{C}/X .

Lemma 2.4. *For any $X \in \mathcal{C}$, the identity morphism on X is an initial object X/\mathcal{C} . Dually, it is a terminal object in \mathcal{C}/X .*

Proof. Any $i : X \rightarrow Y$ is itself the unique morphism from Id_X to i . □

Lemma 2.5. *Let X be an initial object of \mathcal{C} . The identity morphism on X is a zero object \mathcal{C}/X . Dually, if $Y \in \text{ob } \mathcal{C}$ is terminal, then Id_Y is a zero object in Y/\mathcal{C} .*

Proof. We already know that Id_X is terminal. If $p : Y \rightarrow X$ is an object in \mathcal{C}/X , then there is a unique morphism $f : X \rightarrow Y$. Then $f \circ p$ must equal Id_X . □

Example 2.6. Let (X, x) be a pointed set with $X = \{x\}$. Let \mathbf{Set}_* denotes the category of pointed sets with base point preserving functions. Since $\mathbf{Set}_* \cong X/\mathbf{Set}$, it follows that X is a zero object in \mathbf{Set}_* .

Given a morphism $\alpha : X \rightarrow Z$ in \mathcal{C} , define the *under-and-overcategory* $(X/\mathcal{C}/Z)_\alpha$ as having triples (Y, i, p) as objects where $i : X \rightarrow Y$ and $p : Y \rightarrow Z$ are morphisms in \mathcal{C} such that $p \circ i = \alpha$. Define the set of morphisms from (Y, i, p) to (Y', i', p') as the set of morphisms $f : Y \rightarrow Y'$ such that

$$\begin{array}{ccc} X & \xrightarrow{i'} & Y' \\ i \downarrow & \begin{array}{c} \text{blue } f \\ \text{red } \alpha \end{array} & \downarrow p' \\ Y & \xrightarrow{p} & Z \end{array}$$

commutes. If $\alpha = \text{Id}_X$, then we call $(X/\mathcal{C}/X)_{\text{Id}_X}$ the category of *retractive* objects over X , with each triple (Y, i, p) being a retraction of Y onto X .

Example 2.7. If $F : \mathcal{C} \rightarrow \mathcal{C}$ is the identity functor, then the undercategory Y/\mathcal{C} equals the right fiber category Y/F , and the overcategory \mathcal{C}/Y equals the left fiber category F/Y .

Definition 2.8. Let \mathcal{J} be a category. A *diagram of shape \mathcal{J} in \mathcal{C}* is a functor $F : \mathcal{J} \rightarrow \mathcal{C}$.

Definition 2.9. Given a functor $F : \mathcal{J} \rightarrow \mathcal{C}$ and $X \in \text{ob } \mathcal{C}$, a *cone over F* consists of an *apex* $X \in \text{ob } \mathcal{C}$ and *legs* $f_j : X \rightarrow F(j)$ for each $j \in \text{ob } \mathcal{J}$ such that for any $\alpha : j \rightarrow j'$,

$$\begin{array}{ccc} X & \xrightarrow{f_j} & F(j) \\ & \searrow f_{j'} & \downarrow F\alpha \\ & & F(j') \end{array}$$

commutes.

This is simply a natural transformation $\Delta_{\mathcal{J}} X \Rightarrow F$ where $\Delta_{\mathcal{J}} X$ denotes the constant functor on \mathcal{J} at X . If \mathcal{J} is small, then $\Delta_{\mathcal{J}}$ is just a functor from \mathcal{C} to $\mathbf{Fun}(\mathcal{J}, \mathcal{C})$.

Definition 2.10. The *category of cones over F* is the right fiber category X/F . The *category of cones under F* is the left fiber category F/X .

Definition 2.11. Let \mathcal{C} and \mathcal{D} be categories and $g : Y \rightarrow Z$ a morphism in \mathcal{D} . Let $\Delta_{\mathcal{C}} g : \Delta_{\mathcal{C}} Y \Rightarrow \Delta_{\mathcal{C}} Z$ be the natural transformation with components $X \mapsto g$.

1. A *colimit* for the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ consists of an object Y of \mathcal{D} and a natural transformation $i : F \Rightarrow \Delta_{\mathcal{C}} Y$ such that for any $Z \in \text{ob } \mathcal{D}$ and natural transformation $j : F \Rightarrow \Delta_{\mathcal{C}} Z$, there is a unique morphism $g : Y \rightarrow Z$ such that $j = \Delta_{\mathcal{C}} g \circ i$. We write $Y = \text{colim}_{\mathcal{C}} F$.
2. We say that \mathcal{D} admits *all \mathcal{C} -shaped colimits* if each functor $G : \mathcal{C} \rightarrow \mathcal{D}$ has a colimit.
3. We say that \mathcal{D} is *cocomplete* if each functor $G : \mathcal{C} \rightarrow \mathcal{D}$ with \mathcal{C} small has a colimit.

If \mathcal{C} is small, then a colimit of $F : \mathcal{C} \rightarrow \mathcal{D}$ is just an initial object in the right fiber category $F/\Delta_{\mathcal{C}}$, which has as objects pairs $(Z, j : F \rightarrow \Delta Z)$ and as morphisms from (Y, i) to (Z, j) the morphisms $g : Y \rightarrow Z$ in \mathcal{D} such that $\Delta g \circ i = j$.

Remark 2.12. There is a natural bijection $\mathcal{D}(Y, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta Z)$ iff $Y = \text{colim}_{\mathcal{C}} F$.

Proposition 2.13. *Any two colimits are canonically isomorphic.*

Proof. When \mathcal{C} is small, this is immediate from Lemma 1.3. But note that the proof of Lemma 1.3 does not require that \mathcal{C} be locally small (a property which Rognes stipulates of any category). \square

Assume that \mathcal{D} admits all \mathcal{C} -shaped colimits and that \mathcal{C} is small. Then a (possibly global) choice function $\text{colim}_{\mathcal{C}} : \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$ given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor $\Delta_{\mathcal{C}} : \mathcal{D} \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$. Indeed, for any functor $F : \mathcal{C} \rightarrow \mathcal{D}$, there is a bijection $\mathcal{D}(\text{colim}_{\mathcal{C}} F, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta_{\mathcal{C}} Z)$.

Definition 2.14. A *limit* of the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is the colimit of $F^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$.

Explicitly, a limit for $F : \mathcal{C} \rightarrow \mathcal{D}$ is an object Z of \mathcal{D} along with a natural transformation $p : \Delta_{\mathcal{C}} Z \Rightarrow F$ such that for any $Y \in \text{ob } \mathcal{D}$ and natural transformation $q : \Delta_{\mathcal{C}} Y \Rightarrow F$, there is a unique morphism $g : Y \rightarrow Z$ such that $q = p \circ \Delta_{\mathcal{C}} g$.

Remark 2.15. The colimit of a functor F is the limit of F^{op} . Hence *limit* and *colimit* are dual properties, and our results so far for colimits can be dualized to work for limits.

Example 2.16. If \mathcal{C} is the empty category, then the empty functor $F : \mathcal{C} \rightarrow \mathcal{D}$ has $F/\Delta_{\mathcal{C}} \cong \mathcal{D}$, so that the colimit is an initial object of \mathcal{D} .

Definition 2.17. Let \mathcal{J} be a discrete small category. Consider a diagram $\{A_i\}_{i \in J}$ of shape \mathcal{J} .

1. The limit of this diagram is the *product* $\prod_i A_i$, equipped with projections $\pi_i : \prod_i A_i \rightarrow A_i$ such that for every $f_i : U \rightarrow A_i$ there is some unique $f : U \rightarrow \prod_i A_i$ with $\pi_i \circ f = f_i$.

2. Dually, the colimit of the diagram is the *coproduct* $\sum_i A_i$, equipped with inclusions $u_i : A_i \rightarrow \sum_i A_i$ such that for any $f_i : A_i \rightarrow Y$, there is some unique $f : \sum_i A_i \rightarrow Y$ with $f_i = f \circ u_i$.

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Let \mathcal{J} be the category $\bullet \rightrightarrows \bullet$. Then a diagram of shape \mathcal{J} looks like $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$. A cone over this with apex C and legs $f_1 : C \rightarrow A$ and $f_2 : C \rightarrow B$ satisfies $f f_1 = f_2 g f_1$.

Definition 2.18. If such an object C together with f_1 is the limit of the diagram, then we say it is the *equalizer* of f and g . Dually, the colimit is called the *coequalizer*.

Example 2.19. The equalizer in **Set** of $f, g : X \rightarrow Y$ is the subset $X' := \{x \in X : f(x) = g(x)\}$ together with the inclusion function $X' \hookrightarrow X$. The coequalizer of (f, g) is Y/\sim together with the quotient map on B where \sim is the smallest equivalence relation under which $f(x) \sim g(x)$ for every x .

Example 2.20. The equalizer in **Grp** is defined as in Example 2.19 except that the relation \sim becomes a certain minimal normal subgroup.

Now, let \mathcal{J} be the category $\bullet \rightarrow \bullet \leftarrow \bullet$. Then a diagram of this shape looks like $B \xrightarrow{f} D \xleftarrow{g} A$, while a cone over this diagram looks like

$$\begin{array}{ccc} C & \xrightarrow{j} & A \\ \downarrow i & \searrow \alpha & \downarrow g \\ B & \xrightarrow{f} & D \end{array}$$

Definition 2.21. If such an object C together with i and j is the limit of this diagram, then we call it the *pullback* of f and g , denoted by $B \times_D A$.

We can perform an analogous construction for \mathcal{J}^{op} . Then the colimit of the resulting diagram is called the *pushout*, denoted by $B \cup_D A$.

Example 2.22. The pullback in **Set** of $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ is the subset $\{(x, y) \in X \times Y : f(x) = g(y)\}$, called the *fiber product* of X and Y over Z .

Theorem 2.23 (Freyd).

- (i) If \mathcal{C} has equalizers and all small (resp. finite) products, then it has all small (resp. finite) limits.
- (ii) If \mathcal{C} has pullbacks and a terminal object, then it has all finite limits.

Proof.

1. See Theorem 4.9 (Johnstone).
2. Thanks to part (i), it suffices to show that \mathcal{C} has equalizers and all finite products. By assumption, there is some terminal object 1 . Then any product $A_1 \times A_2$ can be realized as the pullback of $A_1 \rightarrow 1 \leftarrow A_2$. By induction, \mathcal{C} has all finite products. Moreover, for morphisms $f, g : A \rightarrow B$, note that any cone over the diagram

$$A \xrightarrow{(1_A, g)} A \times B \xleftarrow{(1_A, f)} A$$

yields morphisms $h : A \rightarrow C$ and $k : C \rightarrow A$ such that $h = k$ and $fk = gh$. Thus the pullback for this diagram is an equalizer for (f, g) , and thus our proof is complete. □

Corollary 2.24. *Both **Set** and **Grp** are complete and cocomplete (or bicomplete).*

It turns out that adjoints interact nicely with (co)limits under mild conditions.

Proposition 2.25. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors such that (F, G) is an adjoint pair. Let \mathcal{E} be small category. If $X : \mathcal{E} \rightarrow \mathcal{C}$ is a functor whose colimit exists, then*

$$\operatorname{colim}_{\mathcal{E}}(F \circ X) = F(\operatorname{colim}_{\mathcal{E}} X).$$

Dually, if $Y : \mathcal{E} \rightarrow \mathcal{D}$ is a functor whose limit exists, then

$$\lim_{\mathcal{E}}(G \circ Y) = G(\lim_{\mathcal{E}} Y).$$

Proof. We have the following chain of bijections natural in $Y \in \mathcal{D}$:

$$\begin{aligned} \mathcal{D}(F(\operatorname{colim}_{\mathcal{E}} X), Y) &\cong \mathcal{C}(\operatorname{colim}_{\mathcal{E}} X, G(Y)) \\ &\cong \lim_{\mathcal{E}} \mathcal{C}(X(-), G(Y)) \\ &\cong \lim_{\mathcal{E}} \mathcal{D}(F(X(-)), Y) \\ &\cong \mathbf{Fun}(\mathcal{E}, \mathcal{D})(F \circ X, \Delta Y). \end{aligned}$$

The second bijection exists because both sets can be identified with the components of all natural transformations $X \Rightarrow \Delta G(Y)$. □

3 Fibrations

Definition 3.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. The *fiber category* $F^{-1}(Y)$ is the full subcategory of \mathcal{C} generated by the objects X with $F(X) = Y$.

Definition 3.2. Suppose \mathcal{C} has a terminal object 1 .

1. Given a morphism $p : 1 \rightarrow Y$, the *fiber* of f at p is the pullback $f^{-1}(p)$ of $1 \rightarrow Y \leftarrow X$.
2. The *cofiber* of a morphism $f : X \rightarrow Y$ is the pushout Y/X of $1 \leftarrow X \rightarrow Y$.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. For each $Y \in \operatorname{ob} \mathcal{D}$, there is a full and faithful functor $F^{-1}(Y) \rightarrow F/Y$ given by $X \mapsto (X, \operatorname{Id}_Y)$.

Definition 3.3. We say that \mathcal{C} is a *precofibered category* over \mathcal{D} if F admits a left adjoint given by $(Z, g : F(Z) \rightarrow Y) \mapsto g_*(Z)$.

Likewise, there is a full and faithful functor $F^{-1}(Y) \rightarrow Y/F$. We say that \mathcal{C} is a *prefibered category* over \mathcal{D} if this functor admits a right adjoint given by $(Z, g : Y \rightarrow F(Z)) \mapsto g_*(Z)$.

Definition 3.4. Let $F; \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

1. Let $f : c' \rightarrow c$ be a morphism in \mathcal{C} . We say f is *cartesian* if for any morphism $f' : c'' \rightarrow c$ in \mathcal{C} and any morphism $g : F(c'') \rightarrow F(c')$ in \mathcal{D} such that $Ff \circ g = Ff'$, there exists a unique $\phi : c'' \rightarrow c$ such that $f' = f \circ \phi$ and $F\phi = g$.

In other words, any filler of

$$\begin{array}{ccc} c'' & \overset{\exists!}{\dashrightarrow} & c' \\ & \searrow f' & \downarrow f \\ & & c \end{array}$$

can be lifted to a filler in \mathcal{D} .

2. We say that F is a *fibration* if for any $c \in \mathcal{C}$ and morphism $f : d \rightarrow Fc$, there is a cartesian $\phi : c' \rightarrow c$ such that $F\phi = f$. Such an ϕ is called a *cartesian lifting* of f to c .

Example 3.5. Let the category **Mod** consist of pairs (R, M) as objects where R is a ring and M is a left R -module and pairs (f, \tilde{f}) as morphisms where $f : R \rightarrow R'$ is a ring homomorphism and $\tilde{f} : M \rightarrow M'$ is an R -linear map with M' viewed as an R -module via f . Then the forgetful functor $U : \mathbf{Mod} \rightarrow \mathbf{Ring}$ is a fibration.