

### Abstract

Even more basic category theory. The main sources for this talk are the following.

- *nLab*.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 4.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4.

**Definition 1.** An object  $X$  of  $\mathcal{C}$  is *initial* if for each  $Y \in \text{ob } \mathcal{C}$ , there is a unique morphism  $f : X \rightarrow Y$ . Moreover, we say that  $X$  is *terminal* if for each  $Z \in \text{ob } \mathcal{C}$ , there is a unique morphism  $g : Z \rightarrow X$ . Either condition is called a *universal property* of  $X$ .

Any property  $P$  of  $\mathcal{C}$  has a dual property  $P^{\text{op}}$  of  $\mathcal{C}^{\text{op}}$  obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing  $P$ . Then  $P$  is true of  $\mathcal{C}$  iff  $P^{\text{op}}$  is true of  $\mathcal{C}^{\text{op}}$ .

**Example 2.** Being initial and being terminal are dual properties.

**Lemma 1.** Any two initial objects of  $\mathcal{C}$  are canonically isomorphic. The same holds for any two terminal objects of  $\mathcal{C}$ .

*Proof.* Compose the two unique morphisms to get an isomorphism between the two initial objects. Apply duality to get the second claim.  $\square$

Think of a universal property as follows. Let  $F : \mathcal{D} \rightarrow \mathcal{C}$  be a functor and  $X \in \text{ob } \mathcal{C}$ . A *universal arrow from  $X$  to  $F$*  is an ordered pair  $(Y, f)$  with  $Y \in \text{ob } \mathcal{D}$  and  $f : X \rightarrow F(Y)$  a morphism of  $\mathcal{C}$  with the property that for any  $X' \in \text{ob } \mathcal{D}$  and morphism  $f' : X \rightarrow F(X')$  of  $\mathcal{C}$ , there exists a unique morphism  $\hat{f} : Y \rightarrow X'$  of  $\mathcal{D}$  such that  $F(\hat{f}) \circ f = f'$ .

$$\begin{array}{ccc} X & \xrightarrow{f} & F(Y) \\ & \searrow f' & \downarrow F(\hat{f}) \\ & & F(X') \end{array}$$

Dually, a *universal arrow from  $F$  to  $X$*  is an ordered pair  $(Y, f)$  with  $Y \in \text{ob } \mathcal{D}$  and  $f : F(Y) \rightarrow X$  of  $\mathcal{C}$  with the property that for any  $X' \in \text{ob } \mathcal{D}$  and morphism  $f' : F(X') \rightarrow X$ , there exists a unique morphism  $\hat{f} : X' \rightarrow Y$  such that  $f' = f \circ F(\hat{f})$ .

$$\begin{array}{ccc} F(X') & \xrightarrow{F(\hat{f})} & F(Y) \\ & \searrow f' & \downarrow f \\ & & X \end{array}$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

### Definition 3.

1. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $Y \in \text{ob } \mathcal{D}$ . The *slice* or *left fiber category*, denoted by  $(F/Y)$  or  $(F \downarrow Y)$ , has as objects pairs  $(X, f)$  where  $f : F(X) \rightarrow Y$  and as morphisms from  $f : F(X) \rightarrow Y$  to  $f' : F(X') \rightarrow Y$  morphisms  $g : X \rightarrow X'$  such that  $f = f' \circ F(g)$ .
2. The *coslice* or *right fiber category*, denoted by  $(Y/F)$  or  $(Y \downarrow F)$ , has as objects pairs  $(X, f)$  where  $f : Y \rightarrow F(X)$  and as morphisms from  $f : Y \rightarrow F(X)$  to  $f' : Y \rightarrow F(X')$  morphisms  $g : X \rightarrow X'$  such that  $f' = F(g) \circ f$ .

*Remark 1.* If  $F^{op} : C^{op} \rightarrow D^{op}$  is opposite to the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $Y \in \text{ob } \mathcal{D}$ , then  $(Y/F)^{op} = F^{op}/Y$ . Thus, the left and right fiber categories are dual in the sense that  $P(Y, F)$  is true for any right fiber category  $Y/F$  iff  $P^{op}(Y, F)$  is true for any left fiber category  $F/Y$ .

**Proposition 2.** Let  $F : \mathcal{D} \rightarrow \mathcal{C}$  be a functor and  $x \in \text{ob } \mathcal{C}$ . Then  $u : x \rightarrow Fr$  is a universal arrow from  $x$  to  $F$  iff it is initial object of the coslice  $(x \downarrow F)$ . Dually,  $u' : Fr' \rightarrow x$  is a universal arrow from  $F$  to  $x$  iff it is a terminal object of the same category.

*Proof.* Suppose that  $u$  is universal and  $f : x \rightarrow Fy$  is another object of  $(x \downarrow F)$ . Then there is some unique  $\hat{f} : r \rightarrow y$  such that  $F(\hat{f}) \circ u = f$ . Thus  $F(\hat{f})$  is a unique morphism of the coslice.

Conversely, suppose that  $u$  is initial. Then for any object  $f : x \rightarrow Fy$  of  $(x \downarrow F)$ , there is some unique arrow  $Sg : Fr \rightarrow Fy$  such that  $Sg \circ u = f$ . Hence setting  $\hat{f} = g$  make  $u$  a universal arrow.  $\square$

**Corollary 3.** Any two universal arrows from  $x$  to  $F$  can be canonically identified by Lemma 1.

**Definition 4.** A *zero object* of  $\mathcal{C}$  is an object that is both initial and terminal. A *pointed category* is a category with a chosen zero object.

**Example 5.** The unique initial object of **Set** is  $\emptyset$ , and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in  $\text{iso}(\mathbf{Set})$ .

Given  $X \in \text{ob } \mathcal{C}$ , the *undercategory*  $X/\mathcal{C}$  has as objects morphisms in  $\mathcal{C}$  of the form  $i : X \rightarrow Y$  where  $X$  is fixed. Given  $i : X \rightarrow Y$  and  $i' : X \rightarrow Y'$  in  $\text{ob } X/\mathcal{C}$ , define the set of morphisms from  $i$  to  $i'$  as the morphisms  $f : Y \rightarrow Y'$  where

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow i' & \downarrow f \\ & & Y' \end{array}$$

commutes. (We call  $i$  the *structure morphism*.) Composition and identity carry over exactly from  $\mathcal{C}$ .

Likewise, given  $x \in \text{ob } \mathcal{C}$ , the *overcategory*  $\mathcal{C}/X$  has as objects morphisms in  $\mathcal{C}$  of the form  $i : Y \rightarrow X$  where  $X$  is fixed. Given  $i : Y \rightarrow X$  and  $i' : Y' \rightarrow X$  in  $\text{ob } \mathcal{C}/X$ , define the set of morphisms from  $i$  to  $i'$  as the morphisms  $f : Y \rightarrow Y'$  where

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & \searrow i & \downarrow i' \\ & & X \end{array}$$

commutes. Composition and identity carry over exactly from  $\mathcal{C}$ .

*Remark 2.* If  $X \in \text{ob } \mathcal{C}$ , then  $(X/\mathcal{C})^{op} = \mathcal{C}^{op}/X$ . Thus, the under- and overcategories are dual in the sense that  $P(X, \mathcal{C})$  is true for any undercategory  $X/\mathcal{C}$  iff  $P^{op}(X, \mathcal{C})$  is true for any overcategory  $\mathcal{C}/X$ .

**Lemma 4.** For any  $X \in \mathcal{C}$ , the identity morphism on  $X$  is an initial object  $X/\mathcal{C}$ . Dually, it is a terminal object in  $\mathcal{C}/X$ .

*Proof.* Any  $i : X \rightarrow Y$  is itself the unique morphism from  $\text{Id}_X$  to  $i$ .  $\square$

**Lemma 5.** Let  $X$  be an initial object of  $\mathcal{C}$ . The identity morphism on  $X$  is a zero object  $\mathcal{C}/X$ . Dually, if  $Y \in \text{ob } \mathcal{C}$  is terminal, then  $\text{Id}_Y$  is a zero object in  $Y/\mathcal{C}$ .

*Proof.* We already know that  $\text{Id}_X$  is terminal. If  $p : Y \rightarrow X$  is an object in  $\mathcal{C}/X$ , then there is a unique morphism  $f : X \rightarrow Y$ . Then  $f \circ p$  must equal  $\text{Id}_X$ .  $\square$

**Example 6.** Let  $(X, x)$  be a pointed set with  $X = \{x\}$ . Let  $\mathbf{Set}_*$  denotes the category of pointed sets with base point preserving functions. Then since  $\mathbf{Set}_* \cong X/\mathbf{Set}$ , it follows that  $X$  is a zero object in  $\mathbf{Set}_*$ .

Given a morphism  $\alpha : X \rightarrow Z$  in  $\mathcal{C}$ , define the *under-and-overcategory*  $(X/\mathcal{C}/Z)_\alpha$  as having triples  $(Y, i, p)$  as objects where  $i : X \rightarrow Y$  and  $p : Y \rightarrow Z$  are morphisms in  $\mathcal{C}$  such that  $p \circ i = \alpha$ . Define the set of morphisms from  $(Y, i, p)$  to  $(Y', u', p')$  as the morphisms  $f : Y \rightarrow Y'$  such that

$$\begin{array}{ccc} X & \xrightarrow{i'} & Y' \\ i \downarrow & \begin{array}{c} \textcolor{blue}{f} \nearrow \\ \textcolor{red}{\alpha} \searrow \end{array} & \downarrow p' \\ Y & \xrightarrow{p} & Z \end{array}$$

commutes. If  $\alpha = \text{Id}_X$ , then we call  $(X/\mathcal{C}/X)_{\text{Id}_X}$  the category of *retractive* objects over  $X$  as each triple  $(Y, i, p)$  is a retraction of  $Y$  onto  $X$ .

**Example 7.** If  $F : \mathcal{C} \rightarrow \mathcal{C}$  is the identity functor, then the undercategory  $Y/\mathcal{C}$  equals the right fiber category  $Y/F$  while the overcategory  $\mathcal{C}/Y$  equals the left fiber category  $F/Y$ .

**Definition 8.** Let  $\mathcal{J}$  be a category. A *diagram of shape  $\mathcal{J}$  in  $\mathcal{C}$*  is a functor  $F : \mathcal{J} \rightarrow \mathcal{C}$ .

**Definition 9.** Given a functor  $F : \mathcal{J} \rightarrow \mathcal{C}$  and  $X \in \text{ob } \mathcal{C}$ , a *cone over  $F$*  consists of an *apex*  $X \in \text{ob } \mathcal{C}$  and legs  $f_j : X \rightarrow F(j)$  for each  $J \in \mathcal{J}$  such that for any  $\alpha : j \rightarrow j'$ ,

$$\begin{array}{ccc} X & \xrightarrow{f_j} & F(j) \\ & \searrow f_{j'} & \downarrow F\alpha \\ & & F(j') \end{array}$$

commutes.

This is simply a natural transformation  $\Delta_{\mathcal{J}} X \Rightarrow F$  where  $\Delta_{\mathcal{J}} X$  denotes the constant functor on  $\mathcal{J}$  at  $X$ . If  $\mathcal{J}$  is small, then  $\Delta_{\mathcal{J}}$  is just a functor from  $\mathcal{C}$  to  $\mathbf{Fun}(\mathcal{J}, \mathcal{C})$ .

**Definition 10.** The *category of cones over  $F$*  is the right fiber category  $X/F$ . The *category of cones under  $F$*  is the left fiber category  $F/X$ .

**Definition 11.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $g : Y \rightarrow Z$  a morphism in  $\mathcal{D}$ . Let  $\Delta_{\mathcal{C}} g : \Delta_{\mathcal{C}} Y \Rightarrow \Delta_{\mathcal{C}} Z$  be the natural transformation with components  $X \mapsto g$ .

1. A *colimit* for the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  consists of an object  $Y$  of  $\mathcal{D}$  and a natural transformation  $i : F \Rightarrow \Delta_{\mathcal{C}} Y$  such that for any  $Z \in \text{ob } \mathcal{D}$  and natural transformation  $j : F \Rightarrow \Delta_{\mathcal{C}} Z$ , there is a unique morphism  $g : Y \rightarrow Z$  such that  $j = \Delta_{\mathcal{C}} g \circ i$ . We write  $Y = \text{colim}_{\mathcal{C}} F$ .
2. We say that  $\mathcal{D}$  admits *all  $\mathcal{C}$ -shaped colimits* if each functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  has a colimit and that  $\mathcal{D}$  is *cocomplete* if each functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  with  $\mathcal{C}$  small has a colimit.

If  $\mathcal{C}$  is small, then a colimit of  $F : \mathcal{C} \rightarrow \mathcal{D}$  is just an initial object in the right fiber category  $F/\Delta_{\mathcal{C}}$ , which has as objects pairs  $(Z, j : F \rightarrow \Delta Z)$  and as morphisms from  $(Y, i)$  to  $(Z, j)$  the morphisms  $g : Y \rightarrow Z$  in  $\mathcal{D}$  such that  $\Delta g \circ i = j$ .

*Remark 3.* There is a natural bijection  $\mathcal{D}(Y, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta Z)$  iff  $Y = \text{colim}_{\mathcal{C}} F$ .

**Proposition 6.** Any two colimits are canonically isomorphic.

*Proof.* When  $\mathcal{C}$  is small, this is immediate from Lemma 1. But note that the proof of Lemma 1 does not require that  $\mathcal{C}$  be locally small (a property which Rognes stipulates of any category).  $\square$

**Lemma 7.** Assume that  $\mathcal{D}$  admits all  $\mathcal{C}$ -shaped colimits and that  $\mathcal{C}$  is small. Then a (possibly global) choice function  $\text{colim}_{\mathcal{C}} : \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$  given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor  $\Delta_{\mathcal{C}} : \mathcal{D} \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$ .

*Proof.* For any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , there is a bijection  $\mathcal{D}(\text{colim}_{\mathcal{C}} F, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta_{\mathcal{C}} Z)$ .  $\square$

**Definition 12.** A limit of the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is precisely the colimit of  $F^{op} : \mathcal{C}^{op} \rightarrow \mathcal{D}^{op}$ .

Explicitly, a limit for  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an object  $Z$  of  $\mathcal{D}$  and a natural transformation  $p : \Delta_{\mathcal{C}} Z \Rightarrow F$  such that for any  $Y \in \text{ob } \mathcal{D}$  and natural transformation  $q : \Delta_{\mathcal{C}} Y \Rightarrow F$ , there is a unique morphism  $g : Y \rightarrow Z$  such that  $q = p \circ \Delta_{\mathcal{C}} g$ .

*Remark 4.* The colimit of a functor  $F$  is the limit of  $F^{op}$ . Hence limit and colimit are dual properties, and the above results for colimits can be dualized.

**Example 13.** If  $\mathcal{C}$  is the empty category, then the empty functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  has  $F/\Delta_{\mathcal{C}} \cong \mathcal{D}$ , so that the colimit is an initial object of  $\mathcal{D}$ .

**Definition 14.** Let  $\mathcal{J}$  be a discrete small category. Consider a diagram  $\{A_i\}_{i \in J}$  of shape  $\mathcal{J}$ .

1. A limit for this diagram is the *product*  $\prod_i A_i$  equipped with projections  $\pi_i : \prod_i A_i \rightarrow A_i$  such that for every  $f_i : U \rightarrow A_i$  there is some unique  $f : U \rightarrow \prod_i A_i$  with  $\pi_i \circ f = f_i$ .
2. Dually, a colimit for the diagram is the *coproduct*  $\sum_i A_i$  equipped with inclusions  $u_i : A_i \rightarrow \sum_i A_i$  such that for any  $f_i : A_i \rightarrow Y$ , there is some unique  $f : \sum_i A_i \rightarrow Y$  with  $f_i = f \circ u_i$ .

**Example 15.** Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Let  $\mathcal{J}$  be the category  $\bullet \rightrightarrows \bullet$ . Then a diagram of shape  $\mathcal{J}$  looks like  $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$ . A cone over this with apex  $C$  and legs  $f_1 : C \rightarrow A$  and  $f_2 : C \rightarrow B$  satisfies  $f f_1 = f_2 g$ .

**Definition 16.** If such an object  $C$  together with  $f_1$  is the limit of the diagram, then we say it is the *equalizer* of  $f$  and  $g$ . Dually, a colimit is called the *coequalizer*.

**Example 17.** The equalizer in **Set** of  $f, g : X \rightarrow Y$  is the subset  $X' := \{x \in X : f(x) = g(x)\}$  together with the inclusion  $X' \hookrightarrow X$ . The coequalizer of  $(f, g)$  is  $Y/\sim$  together with the quotient map on  $B$  where  $\sim$  is the smallest equivalence relation under which  $f(x) \sim g(x)$  for every  $x$ .

*Remark 5.* The same idea applies to **Grp**. The relation  $\sim$  just becomes a certain minimal normal subgroup.

Let  $\mathcal{J}$  be the category  $\bullet \rightarrow \bullet \leftarrow \bullet$ . Then a diagram of this shape looks like  $B \xrightarrow{f} D \xleftarrow{g} A$ , while a cone over this diagram looks like

$$\begin{array}{ccc} C & \xrightarrow{j} & A \\ i \downarrow & \searrow \alpha & \downarrow g \\ B & \xrightarrow{f} & D \end{array}$$

**Definition 18.** If such an object  $C$  together with  $i$  and  $j$  is the limit of this diagram, then we call it the *pullback* of  $f$  and  $g$ , denoted by  $B \times_D A$ .

**Definition 19.** We can perform an analogous construction for  $\mathcal{J}^{op}$ . Then the colimit of the resulting diagram is called the *pushout*, denoted by  $B \cup_D A$ .

**Example 20.** The pullback in **Set** of  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  is the subset  $\{(x, y) \in X \times Y : f(x) = g(y)\}$ , called the *fiber product* of  $X$  and  $Y$  over  $Z$ .

**Theorem 8 (Freyd).**

- (i) If  $\mathcal{C}$  has equalizers and all small (resp. finite) products, then it has all small (resp. finite) limits.
- (ii) If  $\mathcal{C}$  has pullbacks and a terminal object, then it has all finite limits.

*Proof.*

1. See Theorem 4.9 (Johnstone).
2. Thanks to part (i), it suffices to show that  $\mathcal{C}$  has equalizers and all finite products. By assumption there is some terminal object  $1$ . Then any product  $A_1 \times A_2$  can be realized as the pullback of  $A_1 \rightarrow 1 \leftarrow A_2$ . By induction  $\mathcal{C}$  has all finite products. Moreover, for morphisms  $f, g : A \rightarrow B$ , note that any cone over the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & A \times B & \xleftarrow{\quad} & A \\ & \scriptstyle (1_A, g) & & \scriptstyle (1_A, f) & \end{array}$$

admits morphisms  $h : A \rightarrow C$  and  $k : C \rightarrow A$  such that  $h = k$  and  $fk = gh$ . Thus the pullback for this diagram is an equalizer for  $(f, g)$ , completing the proof. □

**Corollary 9.** *Both **Set** and **Grp** are complete and cocomplete (or bicomplete).*

It turns out that adjoints interact nicely with (co)limits under mild conditions.

**Proposition 10.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be an adjoint pair and  $\mathcal{E}$  small. If  $X : \mathcal{E} \rightarrow \mathcal{C}$  is a functor with  $\text{colim}_{\mathcal{E}} X$ , then*

$$\text{colim}_{\mathcal{E}}(F \circ X) = F(\text{colim}_{\mathcal{E}} X).$$

*Dually, if  $Y : \mathcal{E} \rightarrow \mathcal{D}$  is a functor with  $\lim_{\mathcal{E}} Y$ , then*

$$\lim_{\mathcal{E}}(G \circ Y) = G(\lim_{\mathcal{E}} Y).$$

*Proof.* We have the following chain of natural bijections for each  $Y \in \mathcal{D}$ :

$$\begin{aligned} \mathcal{D}(F(\text{colim}_{\mathcal{E}} X), Y) &\cong \mathcal{C}(\text{colim}_{\mathcal{E}} X, G(Y)) \\ &\cong \lim_{\mathcal{E}} \mathcal{C}(X(-), G(Y)) \\ &\cong \lim_{\mathcal{E}} \mathcal{D}(F(X(-)), Y) \\ &\cong \mathbf{Fun}(\mathcal{E}, \mathcal{D})(F \circ X, \Delta Y). \end{aligned}$$

The second bijection follows from the fact that both sets can be identified with the components of the natural transformations from  $X$  to  $\Delta G(Y)$ . □

**Definition 21.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. The *fiber category*  $F^{-1}(Y)$  is the full subcategory of  $\mathcal{C}$  generated by the objects  $X$  with  $F(X) = Y$ .

**Definition 22.** Suppose  $\mathcal{C}$  has terminal object  $1$ . A *cofiber* of a morphism  $f : X \rightarrow Y$  is a pushout of the diagram  $1 \leftarrow X \rightarrow Y$ . We write  $Y/X$ . Further, given a morphism  $p : 1 \rightarrow Y$ , the *fiber* of  $f$  at  $p$  is the pullback of  $1 \rightarrow Y \leftarrow X$ , written as  $f^{-1}(p)$ .

Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. For each  $Y \in \text{ob } \mathcal{D}$ , there is a full and faithful functor  $F^{-1}(Y) \rightarrow F/Y$  given by  $X \mapsto (X, \text{Id}_Y)$ .

**Definition 23.** We say that  $\mathcal{C}$  is a *precofibered category* over  $\mathcal{D}$  if this functor admits a left adjoint given by  $(Z, g : F(Z) \rightarrow Y) \mapsto g_*(Z)$ .

Moreover, there is a full and faithful functor  $F^{-1}(Y) \rightarrow Y/F$  defined similarly. We say that  $\mathcal{C}$  is a *prefibered category* over  $\mathcal{D}$  if this functor admits a right adjoint given by  $(Z, g : Y \rightarrow F(Z)) \mapsto g_*(Z)$ .

**Definition 24.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. Let  $f : c' \rightarrow c$  be a morphism in  $\mathcal{C}$ . We say  $f$  is *cartesian* if for any morphism  $f' : c'' \rightarrow c$  in  $\mathcal{C}$  and  $g : F(c'') \rightarrow F(c')$  in  $\mathcal{D}$  such that  $Ff \circ g = Ff'$ , there exists a unique  $\phi : c'' \rightarrow c$  such that  $f' = f \circ \phi$  and  $F\phi = g$ . In other words, any filling of the following diagram can be lifted to a filling in  $\mathcal{D}$ .

$$\begin{array}{ccc} c'' & \overset{\exists!}{\dashrightarrow} & c' \\ & \searrow f' & \downarrow f \\ & & c \end{array}$$

2. We say that  $F$  is a *fibration* if for any  $c \in \mathcal{C}$  and morphism  $f : d \rightarrow Fc$ , there is a cartesian  $\phi : c' \rightarrow c$  such that  $F\phi = f$ . Such  $\phi$  is called a *cartesian lifting* of  $f$  to  $c$ .

**Example 25.** Let **Mod** denote the category of left  $R$ -modules where  $R$  is a ring. Then the forgetful functor  $U : \mathbf{Mod} \rightarrow \mathbf{Ring}$  is a fibration.