

Abstract

We begin higher Waldhausen K -theory. The main sources for this talk are the following.

- $n\text{Lab}$.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Ch. IV.8.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

For the original development, see Friedhelm Waldhausen's *Algebraic K-theory of spaces* (1985).

Let \mathcal{C} be a Waldhausen category. Our goal is to construct the K -theory $K(\mathcal{C})$ of \mathcal{C} as a based loop space ΩY endowed with a loop completion map $\iota : |w\mathcal{C}| \rightarrow K(\mathcal{C})$ where $w\mathcal{C}$ denotes the subcategory of weak equivalences. This will produce a function $\text{ob } \mathcal{C} \rightarrow |w\mathcal{C}| \rightarrow \Omega Y$. Further, we'll require of $K(\mathcal{C})$ certain limit and coherence properties, thereby making $K(\mathcal{C})$ the underlying infinite loop space of a spectrum $\mathbf{K}(\mathcal{C})$, called the *algebraic K-theory spectrum* of \mathcal{C} .

Definition 1. Let \mathcal{C} be a category equipped with a subcategory $\text{co } \mathcal{C}$ of morphisms called *cofibrations*. The pair $(\mathcal{C}, \text{co } \mathcal{C})$ is a *category with cofibrations* if the following conditions hold.

1. (W0) Every isomorphism in \mathcal{C} is a cofibration.
2. (W1) There is a base point $*$ in \mathcal{C} such that the unique morphism $* \rightarrow A$ is a cofibration for any $A \in \text{ob } \mathcal{C}$.
3. (W2) We have a *cobase change*

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & B \cup_A C \end{array} .$$

We see that $B \amalg C$ always exists as the pushout $B \cup_* C$ and that the cokernel of any $i : A \rightarrow B$ exists as $B \cup_A *$ along $A \rightarrow *$. We call $A \rightarrow B \rightarrow B/A$ a *cofiber sequence*.

Definition 2. A *Waldhausen category* \mathcal{C} is a category with cofibrations together with a subcategory $w\mathcal{C}$ of morphisms called *weak equivalences* such that every isomorphism in \mathcal{C} is a weak equivalence and the following “Gluing axiom” holds.

1. (W3) For any diagram

$$\begin{array}{ccccc} C & \longleftarrow & A & \longrightarrow & B \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ C' & \longleftarrow & A' & \longrightarrow & B' \end{array} ,$$

the induced map $B \cup_A C \rightarrow B' \cup_{A'} C'$ is a weak equivalence.

Definition 3. A Waldhausen category (\mathcal{C}, w) is *saturated* if whenever fg makes sense and is a weak equivalence, then f is a weak equivalence iff g is.

Now, let us introduce the main concept to be generalized.

Definition 4. Let \mathcal{C} be a category with cofibrations. Let the *extension category* $S_2\mathcal{C}$ have as objects the cofiber sequences in $(\mathcal{C}, \text{co}\mathcal{C})$ and as morphisms the triples (f', f, f'') such that

$$\begin{array}{ccccc} X' & \twoheadrightarrow & X & \twoheadrightarrow & X'' \\ \downarrow f' & & \downarrow f & & \downarrow f'' \\ Y' & \twoheadrightarrow & Y & \twoheadrightarrow & Y'' \end{array}$$

commutes. This is pointed at $* \twoheadrightarrow * \twoheadrightarrow *$.

Definition 5. Suppose an arbitrary triple (f', f, f'') as above has the property that whenever f' and f'' are weak equivalences, then so is f . In this case, we say \mathcal{C} is *extensional* or *closed under extensions*.

Say that the morphism (f', f, f'') is a cofibration if f' , f'' , and $Y' \cup_{X'} X \rightarrow Y$ are cofibrations in \mathcal{C} . Say that the same triple is a weak equivalence if f' , f , and f'' are weak equivalences in \mathcal{C} . This makes $S_2\mathcal{C}$ into a Waldhausen category.

Definition 6. Let $q \geq 0$. Let the *arrow category* $\text{Ar}[q]$ on $[q]$ have as objects ordered pairs (i, j) with $i \leq j \leq q$ and as morphisms commutative diagrams of the form

$$\begin{array}{ccc} i & \xrightarrow{\leq} & j \\ \leq \downarrow & & \downarrow \leq \\ i' & \xrightarrow{\leq} & j' \end{array}$$

We view $[q]$ as a full subcategory of $\text{Ar}[q]$ via the embedding $[q] \xrightarrow{k \mapsto (0, k)} \text{Ar}[q]$.

Note 7.

1. Any triple $i \leq j \leq k$ determines the morphisms $(i, j) \rightarrow (i, k)$ and $(i, k) \rightarrow (j, k)$. Conversely, any morphism in the arrow category is a composite of such triples.
2. $\text{Ar}[q] \cong \mathbf{Fun}([1], [q])$ with each pair (i, j) identified with the functor satisfying $0 \mapsto i$ and $1 \mapsto j$.

Example 8. The category $\text{Ar}[2]$ is generated by the commutative diagram

$$\begin{array}{ccccc} (0, 0) & \longrightarrow & (0, 1) & \longrightarrow & (0, 2) \\ & & \downarrow & & \downarrow \\ & & (1, 1) & \longrightarrow & (1, 2) \\ & & & & \downarrow \\ & & & & (2, 2) \end{array}$$

Definition 9. Let \mathcal{C} be a category with cofibrations and $q \geq 0$. Define $S_q\mathcal{C}$ as the full subcategory of $\mathbf{Fun}(\text{Ar}[q], \mathcal{C})$ generated by $X : \text{Ar}[q] \rightarrow \mathcal{C}$ such that

1. $X_{j,j} = *$ for each $j \in [q]$.
2. $X_{i,j} \twoheadrightarrow X_{i,k} \twoheadrightarrow X_{j,k}$ is a cofiber sequence for any $i < j < k$ in $[q]$. Equivalently, if $i \leq j \leq k$ in $[q]$, then the square

$$\begin{array}{ccc} X_{i,j} & \twoheadrightarrow & X_{i,k} \\ \downarrow & & \downarrow \\ X_{j,j} = * & \twoheadrightarrow & X_{j,k} \end{array}$$

is a pushout.

This is pointed at the constant diagram at $*$.

Note 10. A generic object in $S_q\mathcal{C}$ looks like

$$\begin{array}{ccccccc} * & \twoheadrightarrow & X_1 & \twoheadrightarrow & \cdots & \twoheadrightarrow & X_{q-1} & \twoheadrightarrow & X_q \\ & & \downarrow & & & & \downarrow & & \downarrow \\ & & * & \twoheadrightarrow & \cdots & \twoheadrightarrow & X_{q-1}/X_1 & \twoheadrightarrow & X_q/X_1 \\ & & & & & & \downarrow & & \downarrow \\ & & & & \ddots & & \vdots & & \vdots \\ & & & & & & * & \twoheadrightarrow & X_q/X_{q-1} \\ & & & & & & & & \downarrow \\ & & & & & & & & * \end{array} \quad (\dagger)$$

where X_q corresponds to $X_{0,q}$ and X_j/X_i to $X_{i,j}$ for any $1 \leq i \leq j \leq q$.

Definition 11. Let $(\mathcal{C}, \text{co}\mathcal{C})$ be a category with cofibrations. Let $\text{co}S_q\mathcal{C} \subset S_q\mathcal{C}$ consist of the morphisms $f : X \twoheadrightarrow Y$ of $\text{Ar}[q]$ -shaped diagrams such that for each $1 \leq j \leq q$ we have

$$\begin{array}{ccccc} X_{0,j-1} & \twoheadrightarrow & X_{0,j} & & \\ f_{0,j-1} \downarrow & & \downarrow & \searrow f_{0,j} & \\ Y_{0,j-1} & \twoheadrightarrow & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} & & \\ & \searrow & & \swarrow & \\ & & & & Y_{0,j} \end{array}$$

Proposition 12. If $f : X \rightarrow Y$ is a cofibration of $S_q\mathcal{C}$, then

$$\begin{array}{ccc} X_{i,j} & \twoheadrightarrow & X_{i,k} \\ f_{i,j} \downarrow & & \downarrow f_{i,k} \\ Y_{i,j} & \twoheadrightarrow & Y_{i,k} \end{array}$$

for any $i \leq j \leq k$ in $[q]$.¹

¹Lemma 8.3.12 (Rognes).

Lemma 13. $(S_q\mathcal{C}, \text{co } S_1\mathcal{C})$ is a category with cofibrations.

Proof. First notice that the composite of two cofibrations $g \circ f : X \rightarrow Y \rightarrow Z$ is a cofibration because we have

$$\begin{array}{ccccc}
 X_{0,j-1} & \rightarrow & X_{0,j} & \xrightarrow{f_{0,j}} & Y_{0,j} \\
 f_{0,j-1} \downarrow & & \downarrow & & \downarrow \\
 Y_{0,j-1} & \rightarrow & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} & \rightarrow & Y_{0,j} \\
 g_{0,j-1} \downarrow & & \downarrow & & \downarrow \\
 Z_{0,j-1} & \rightarrow & X_{0,j} \cup_{X_{0,j-1}} Z_{0,j-1} & \rightarrow & Y_{0,j} \cup_{Y_{0,j-1}} Z_{0,j-1} \rightarrow Z_{0,j}
 \end{array}$$

It's clear that any isomorphism or initial morphism in $S_q\mathcal{C}$ is a cofibration.

To see that (W2) is satisfied, let $f : X \rightarrow Y$ and $g : X \rightarrow Z$ be morphisms in $S_q\mathcal{C}$. It's easy to verify that each component $f_{i,j} : X_{i,j} \rightarrow Y_{i,j}$ is a cofibration. Thus, each pushout $W_{i,j} := Y_{i,j} \cup_{X_{i,j}} Z_{i,j}$ exists. These form a functor $W : \text{Ar}[q] \rightarrow \mathcal{C}$. If $i < j < k$, then we have $W_{i,j} \rightarrow W_{i,k} \rightarrow W_{j,k}$ because the left morphism factors as the composite of two cofibrations

$$\begin{array}{ccccc}
 Z_{i,j} & \rightarrow & Z_{i,k} & & \\
 f_{i,j} \cup \text{Id} \downarrow & & \downarrow f_{i,j} \cup \text{Id} & & \\
 Y_{i,j} \cup_{X_{i,j}} Z_{i,j} & \rightarrow & Y_{i,j} \cup_{X_{i,j}} Z_{i,k} & \rightarrow & Y_{i,k} \cup_{X_{i,k}} Z_{i,k} \\
 & & \uparrow \text{Id} \cup g_{i,k} & & \uparrow \text{Id} \cup g_{i,k} \\
 & & Y_{i,j} \cup_{X_{i,j}} X_{i,k} & \rightarrow & Y_{i,k}
 \end{array}$$

The fact that colimits commute with each other ensures that $W_{j,k} \cong W_{i,k} / W_{i,j}$. Hence W is the pushout of f and g . To verify that this is a cofibration, we must check that the pushout map $W_{0,j-1} \cup_{Z_{0,j-1}} Z_{0,j} \rightarrow W_{0,j}$ is a cofibration. But this follows from the pushout square

$$\begin{array}{ccc}
 Y_{0,j-1} \cup_{X_{0,j-1}} X_{0,j} & \rightarrow & Y_{0,j} \\
 \downarrow & & \downarrow \\
 Y_{0,j-1} \cup_{X_{0,j-1}} Z_{0,j} & \rightarrow & Y_{0,j} \cup_{X_{0,j}} Z_{0,j}
 \end{array}$$

□

Definition 14. Let $(\mathcal{C}, w\mathcal{C})$ be a Waldhausen category. Let $wS_q\mathcal{C} \subset S_q\mathcal{C}$ consist of the morphisms $f : X \xrightarrow{\sim} Y$ of $\text{Ar}[q]$ -shaped diagrams such that the component $f_{0,j} : X_{0,j} \rightarrow Y_{0,j}$ is a weak equivalence in \mathcal{C} for each $1 \leq j \leq q$.

Proposition 15. Let f be a weak equivalence in $S_q\mathcal{C}$. Each component $f_{i,j} : X_{i,j} \rightarrow Y_{i,j}$ is a weak equivalence in \mathcal{C} .

Proof. Apply the Gluing axiom to the diagram

$$\begin{array}{ccccc}
 X_{0,j} & \leftarrow & X_{0,i} & \rightarrow & * \\
 \cong \downarrow & & \cong \downarrow & & = \downarrow \\
 Y_{0,j} & \leftarrow & Y_{0,i} & \rightarrow & *
 \end{array}$$

Then $X_{i,j} \cong X_{0,j} \cup_{X_{0,i}} * \xrightarrow{\sim} Y_{0,j} \cup_{Y_{0,i}} * \cong Y_{i,j}$, as desired. \square

Lemma 16. $(S_q\mathcal{C}, wS_q\mathcal{C})$ is a Waldhausen category.

Definition 17. Let \mathcal{C} be a category with cofibrations. If $\alpha : [p] \rightarrow [q]$, then define $\alpha^* : S_q\mathcal{C} \rightarrow S_p\mathcal{C}$ by

$$\alpha^*(X : \text{Ar}[q] \rightarrow \mathcal{C}) = X \circ \text{Ar}(\alpha) : \text{Ar}[p] \rightarrow \text{Ar}[q] \rightarrow \mathcal{C}.$$

It's easy to check that this satisfies the two conditions of a diagram in $S_p\mathcal{C}$. Moreover, the face maps d_i are obtained by deleting the row $X_{i,-}$ and the column containing X_i in (\dagger) and then reindexing as necessary. The degeneracy maps s_i are given by duplicating X_i and then reindexing such that $X_{i+1,i} = 0$.

Proposition 18. Let $(\mathcal{C}, w\mathcal{C})$ be a Waldhausen category. Each functor $\alpha^* : S_q\mathcal{C} \rightarrow S_p\mathcal{C}$ is exact, so that $(S_\bullet\mathcal{C}, wS_\bullet\mathcal{C})$ is a simplicial Waldhausen category.

The nerve $N_\bullet wS_\bullet\mathcal{C}$ is a bisimplicial set with (p, q) -bisimplices the diagrams of the form

$$\begin{array}{ccccccc} * & \longrightarrow & X_1^0 & \longrightarrow & X_2^0 & \longrightarrow & \cdots \longrightarrow X_q^0 \\ & & \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ * & \longrightarrow & X_1^1 & \longrightarrow & X_2^1 & \longrightarrow & \cdots \longrightarrow X_q^1 \\ & & \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ & & \vdots & & \vdots & & \vdots \\ & & \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ * & \longrightarrow & X_1^p & \longrightarrow & X_2^p & \longrightarrow & \cdots \longrightarrow X_q^p \end{array}$$

such that $X_{i,j}^k \cong X_j^k / X_i^k$ for every $i \leq j \leq q$ and $k \in [p]$.

Lemma 19. There is a natural map $N_\bullet w\mathcal{C} \wedge \Delta_\bullet^1 \rightarrow N_\bullet wS_\bullet\mathcal{C}$, which automatically induces a based map $\sigma : \Sigma |w\mathcal{C}| \rightarrow |wS_\bullet\mathcal{C}|$ of classifying spaces.

Proof. We can treat $N_\bullet wS_\bullet\mathcal{C}$ as the simplicial set $[q] \mapsto N_\bullet wS_q\mathcal{C}$. This defines a right skeletal structure on $N_\bullet wS_\bullet\mathcal{C}$.

If $q = 0$, then $wS_0\mathcal{C} = S_0\mathcal{C} = *$, so that $N_\bullet wS_0\mathcal{C} = *$ as well. If $q = 1$, then $wS_1\mathcal{C} \cong w\mathcal{C}$. Thus, the right 1-skeleton is equal to $N_\bullet w\mathcal{C} \wedge \Delta_\bullet^1$, which in turn must be equal to the image I of the canonical map

$$\coprod_{q \leq 1} N_\bullet wS_q\mathcal{C} \times \Delta_\bullet^q \rightarrow N_\bullet wS_\bullet\mathcal{C}.$$

Now, the degeneracy map s_0 collapses $\{*\} \times \Delta_\bullet^1$, and the face maps d_0 and d_1 collapse $N_\bullet w\mathcal{C} \times \partial\Delta_\bullet^1$. Therefore, I must equal

$$N_\bullet w\mathcal{C} \wedge \Delta_\bullet^1 = \frac{N_\bullet w\mathcal{C} \times \Delta_\bullet^1}{\{*\} \times \Delta_\bullet^1 \cup N_\bullet w\mathcal{C} \times \partial\Delta_\bullet^1}.$$

We have defined a natural inclusion map $\lambda : N_{\bullet}w\mathcal{C} \wedge \Delta_{\bullet}^1 \rightarrow N_{\bullet}wS_{\bullet}\mathcal{C}$.

Since Δ_{\bullet}^1 is isomorphic to the unit interval and the map λ agrees on the endpoints, we can pass to S^1 during the suspension. Hence λ induces the desired map σ .² \square

Note 20. The axiom (W3) implies that $w\mathcal{C}$ is closed under coproducts, making $|wS_{\bullet}\mathcal{C}|$ into an H -space via the map

$$\coprod : |wS_{\bullet}\mathcal{C}| \times |wS_{\bullet}\mathcal{C}| \cong |wS_{\bullet}\mathcal{C} \times wS_{\bullet}\mathcal{C}| \rightarrow |wS_{\bullet}\mathcal{C}|.$$

Definition 21. Let $(\mathcal{C}, w\mathcal{C})$ be a Waldhausen category. Define the *algebraic K -theory space*

$$K(\mathcal{C}, w) = \Omega |N_{\bullet}wS_{\bullet}\mathcal{C}|.$$

Then we have a right adjoint $\iota : |w\mathcal{C}| \rightarrow K(\mathcal{C}, w)$ to the based map σ .

Moreover, let $F : (\mathcal{C}, w\mathcal{C}) \rightarrow (\mathcal{D}, w\mathcal{D})$ be an exact functor. Let

$$K(F) = \Omega |wS_{\bullet}F| : K(\mathcal{C}, w) \rightarrow K(\mathcal{D}, w).$$

This is the *algebraic K -theory functor* $K : \mathbf{Wald} \rightarrow \mathbf{Top}_*$.

Note that any exact category \mathcal{A} is a Waldhausen category with cofibrations the admissible exact sequences and weak equivalences the isomorphisms. Waldhausen showed that $|iS_{\bullet}\mathcal{A}|$ (where i denotes the iso. category) and $BQ\mathcal{A}$ are homotopy equivalent. Therefore, our current definition of higher algebraic K -theory agrees with Quillen's.

Example 22. Let R be a ring. Define the *algebraic K -theory space of R* as

$$K(R) = K(\mathbf{P}(R), i)$$

where the weak equivalences are precisely the injective R -linear maps with projective cokernel and the cofibrations are precisely the R -linear maps.

Example 23. Assume that \mathcal{C} is a small Waldhausen category where $w\mathcal{C}$ consists of the isomorphisms in \mathcal{C} . If $s_n\mathcal{C}$ denotes the set of objects of $S_n\mathcal{C}$, then we get a simplicial set $s_{\bullet}\mathcal{C}$. Waldhausen showed that the inclusion map $|s_{\bullet}\mathcal{C}| \hookrightarrow |iS_{\bullet}\mathcal{C}|$ is a homotopy equivalence. This makes $\Omega |s_{\bullet}\mathcal{C}|$ into a so-called simplicial model for $K(\mathcal{C}, w)$.

Remark 24. Since $wS_0\mathcal{C} = *$ and every simplex of degree $n > 0$ is attached to $*$, it follows that the classifying space $|wS_{\bullet}\mathcal{C}|$ is connected. Therefore, we preserve any homotopical information when passing to the loop space.

Definition 25. The *i -th algebraic K -group* is $K_i(\mathcal{C}, w) \equiv \pi_i K(\mathcal{C}, w)$ for each $i \geq 0$.

Proposition 26. $\pi_1 |wS_{\bullet}\mathcal{C}| \cong K_0(\mathcal{C}, w)$.

²This is a tentative explanation due to Thomas Brazelton.

Lemma 27. *The group $K_0(\mathcal{C}, w)$ is generated by all elements $[X]$ such that*

- $[X'] + [X''] = [X]$ for every cofiber sequence $X' \rightarrow X \rightarrow X''$ and
- $[X] = [Y]$ for every weak equivalence $X \xrightarrow{\sim} Y$.

Proof. In light of Proposition 26, it suffices to compute $\pi_1 |N_\bullet wS_\bullet \mathcal{C}|$ based at the $(0, 0)$ -bisimplex $*$. For this, just notice the CW structure of $|N_\bullet wS_\bullet \mathcal{C}|$, with 1-cells the $(0, 1)$ -bisimplices and 2-cells the $(0, 2)$ -bisimplices $X' \rightarrow X \rightarrow X''$ and the $(1, 1)$ -bisimplices $X \xrightarrow{\sim} Y$, which are attached to the 1-cells X and Y . Any cell of dimension $n > 2$ is irrelevant to computing π_1 . \square

As a result, we obtain functors

$$K_i : \mathbf{Wald} \rightarrow \mathbf{Top}_* \rightarrow \mathbf{Ab}$$

known as the *algebraic K-group functors*. Indeed, thanks to Proposition 26, we know that

$$K_i(\mathcal{C}, w) = \pi_{i+1} |wS_\bullet \mathcal{C}|,$$

which is abelian for $i \geq 1$. Moreover, note that if $X' \rightarrow X' \vee X'' \rightarrow X''$ and $X'' \rightarrow X' \vee X'' \rightarrow X'$ are cofiber sequences, then Lemma 27 implies that

$$[X'] + [X''] = [X' \vee X''] = [X'' + X'].$$

Hence $K_0(\mathcal{C}, w)$ is also abelian.

Example 28. Let X be a CW complex and $\mathcal{R}(X)$ denote the category of CW complexes Y obtained by attaching at least one cell to X so that X is a retract of Y . Equip this with cofibrations in the form of cellular inclusions fixing X and weak equivalence in the form of homotopy equivalences. This makes $\mathcal{R}(X)$ into a Waldhausen category.

If $\mathcal{R}_f(X)$ denotes the subcategory of those Y obtained by attaching finitely many cells, then we denote $K(\mathcal{R}_f(X))$ by $A(X)$.

Proposition 29. $A_0(X) \cong \mathbb{Z}$.

Definition 30. If \mathcal{B} is a Waldhausen subcategory of \mathcal{C} , then it is *cofinal in \mathcal{C}* if for any $X \in \text{ob } \mathcal{C}$, there is some $X' \in \text{ob } \mathcal{B}$ such that $X \coprod X' \in \text{ob } \mathcal{B}$.

Theorem 31. Let (\mathcal{B}, w) be cofinal in (\mathcal{C}, w) and closed under extensions. Assume that $K_0(\mathcal{B}) = K_0(\mathcal{C})$. Then $wS_\bullet \mathcal{B} \rightarrow wS_\bullet \mathcal{C}$ is a homotopy equivalence.

It follows that $K_i(\mathcal{B}) \cong K_i(\mathcal{C})$ for every $i \geq 0$.