# Abstract

These notes are based on Julius Shaneson's lectures for the course "Algebraic Topology, Part I" given at UPenn. Any mistake in what follows is my own.

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# 1 Background material

### 1.1 Lecture 1

Here are the topics for the course:

- fiber bundles over cell complexes,
- spectral sequences,
- characteristic classes, and
- cobordism theory.

To start, let's review some basic concepts from homology theory.

**Definition 1.1.1.** A (finite) cell complex is a (topological) space X that can be written as  $\bigcup_{n=0}^{K} X^n$  for some  $K \in \mathbb{N}$  (called the *dimension of* X) where

- $X^0$  is chosen to be finite,
- $X^n = \frac{X^{n-1} \coprod D_1^n \coprod \cdots \coprod D_{k_n}^n}{x \sim \varphi_i(x)}$ ,
- $D_i^n \cong D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$  for each  $i \in \{1, \dots, k_n\}$ , and
- $\varphi_i: \partial D_i^n = S^{n-1} \to X^{n-1}$ , called an attaching map.

Terminology. Each  $D_i^n$  is called an n-cell of X.

Every attaching map  $\varphi_i:\partial D_i^n\to X^{n-1}$  can be extended to a *characteristic map* given by the composition

$$D_i^n \hookrightarrow X^{n-1} \coprod D_1^n \coprod \cdots \coprod D_{k_n}^n \twoheadrightarrow X^n \hookrightarrow X.$$

**Example 1.1.2.** There are at least two ways of endowing  $S^2$  with a cell structure.

- 1.  $X^0 \equiv \{N, S\}, \ X^1 \equiv X^0 \cup_{\varphi_1} D_1^1 \cup_{\varphi_2} D_2^1$  where each  $\varphi_i$  is an embedding, and  $X^2 \equiv X^1 \cup_{\varphi_1'} D_1^2 \cup_{\varphi_2'} D_2^2$  where each  $\varphi_i'$  is an embedding.
- 2.  $\{pt\} \cup_{\varphi} D^2$  where  $\varphi$  identifies the equator of the upper half-sphere with pt.

**Definition 1.1.3.** A cell complex X is regular if every characteristic map  $D_i^n \to X$  is an embedding.

**Definition 1.1.4.** Given a family of functors  $\{H_n : \mathbf{Top}^2 \to \mathbf{Ab}\}_{n \in \mathbb{N}}$  where  $\mathbf{Top}^2$  denotes the category of (topological) pairs, we say that  $H_i$  is a *homology functor* if each of the following properties holds.

1. (LES) For any pair (X, A) of space, there is a natural long exact sequence

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X,A) \xrightarrow{\partial} H_{i-1}(A) \longrightarrow \cdots,$$

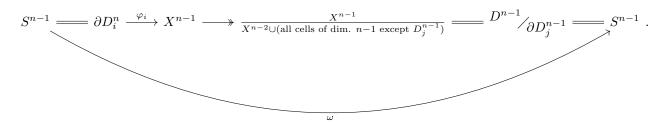
where  $H_i(Z) := H_i(Z, \emptyset)$  for any space Z.

- 2. (Excision) If  $\operatorname{cl}(A) \subset \underset{open}{U} \subset X$ , then  $H_i(X \setminus A, U \setminus A) \cong H_i(X, U)$ .
- 3. (Dimension)  $H_i(\mathsf{pt}) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases}$
- 4. (Homotopy) If f and g are homotopic, then  $f_* = g_*$ , where  $h_* := H_i(h)$  for any map  $h: (X, A) \to (Y, B)$ .

**Theorem 1.1.5.** There exists a family of homology functors.

**Example 1.1.6.** In singular homology theory, we have that  $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$ .

Let X be a cell complex. Let  $C_n(X)$  denote the free abelian group on the set of all n-cells of X. Define  $\partial: C_n(X) \to C_{n-1}(X)$  by  $\partial[D_i^n] = \sum_{j=1}^{k_{n-1}} \lambda_{ij}[D_j^{n-1}]$  where  $\lambda_{ij}$  is defined, up to sign, as follows. Consider the map



Then let  $\lambda_{ij}$  satisfy  $\omega_*(x) = \lambda_{ij}x$  with x a chosen generator (i.e., orientation) of  $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ . Terminology. The integer  $\lambda_{ij}$  is called the degree of  $\omega$ , denoted by  $\deg(\omega)$ .

**Theorem 1.1.7.**  $\partial_n \partial_{n+1} = 0$ , and  $H_n(X) \cong \ker \partial_n / \operatorname{im} \partial_{n+1}$ , which is independent of our choice of generator x.

**Example 1.1.8.** Suppose that  $f: S^n \to S^n$  is smooth. By Sard's theorem, we can find a regular value  $x \in S^n$ . There is some neighborhood U of x such that  $f^{-1}(U) = U_1 \cup \cdots \cup U_n$  for some n. Using the inverse function theorem and the compactness of  $S^n$ , it follows that  $f^{-1}$  is of the form  $\{x_1, \ldots, x_n\}$ . Note that the differential  $(df)_{x_i}: S^n_{x_i} \to S^n_x$  satisfies  $\det(df)_{x_i} - \pm 1$ . In fact,

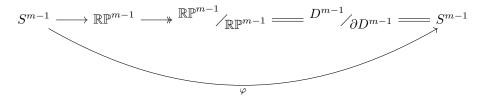
$$\deg(f) = \sum_{i=1}^{n} \det (df)_{x_i}.$$

**Exercise 1.1.9.** Prove that any cell complex  $X = \bigcup_{n=0}^{K} X^n$  is homotopy equivalent to a regular cell complex.  $\square$ 

# 1.2 Lecture 2

**Example 1.2.1 (Real projective space).** Recall that  $\mathbb{RP}^n = S^n/_x \sim -x$ . Then  $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\pi_{n-1}} D^n$  where  $\pi_{n-1}: S^{n-1} \to \mathbb{RP}^{n-1}$  denotes the canonical projection. Thus,  $\mathbb{RP}^n$  is an n-dimension cell complex with  $(\mathbb{RP}^n)^m = \mathbb{RP}^m$  for each integer  $0 \le m \le n$ .

Now, for each  $0 \leq m \leq n$ , we have that  $C_m(\mathbb{RP}^n) \cong \mathbb{Z}$  with generator  $[D^m]$ . To determine  $\partial[D^m] \in C_{m-1}(\mathbb{RP}^m)$ , we must find the degree of the map



Assume, for convenience, that m=2. Choose a regular value  $p \in S^1$  so that  $\varphi^{-1}(p) = \{N, S\}$ . Let  $\varphi_T$  and  $\varphi_B$  denote the restrictions of  $\varphi$  to the top and bottom components of  $S^1 \setminus \{(-1,0), (1,0)\}$ , respectively. Note that both of these are homeomorphisms and thus have degrees equal to  $\pm 1$ . If  $a: S^{m-1} \to S^{m-1}$  denotes the

Hint: Consider the map  $S^{n-1} \to X^{n-1} \times D^n$  given by  $x \mapsto (\varphi(x), x)$ .

antipodal map, we have that  $\varphi_B \circ a = \varphi_T$ . Hence  $(d\varphi)_S \circ (da)_N = (d\varphi)_N$ . Since  $\deg(a) = \det(da) = (-1)^m$ , it follows that

$$\deg(\varphi) = \begin{cases} \pm 2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}.$$

Thus, we get a chain complex

$$0 \longrightarrow C_n(\mathbb{RP}^n) \xrightarrow{\kappa_1} C_{n-1}(\mathbb{RP}^n) \xrightarrow{\kappa_2} \cdots \xrightarrow{0} C_2(\mathbb{RP}^n) \xrightarrow{\pm 2} C_1(\mathbb{RP}^n) \xrightarrow{0} C_0(\mathbb{RP}^n) \longrightarrow 0$$

where 
$$\kappa_1 = \begin{cases}
0 & n \text{ odd} \\
\pm 2 & n \text{ even}
\end{cases}$$
 and  $\kappa_2 = \begin{cases}
\pm 2 & n \text{ odd} \\
0 & n \text{ even}
\end{cases}$ .

This proves that

$$H_{i}(\mathbb{RP}^{n}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ 0 & i = 2 \\ \mathbb{Z}/2 & i < n \\ odd \\ 0 & i < n \\ 0 & i > n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & i = n \text{ even} \end{cases}.$$

Next, let's introduce some fundamental concepts from homotopy theory.

**Definition 1.2.2.** Let M(X,Y) denote the set of maps  $X \to Y$ .

1. For any compact  $C \subset X$  and open  $U \subset Y$ , let

$$N(C,U) = \{ f : X \to Y \mid f(C) \subset U \}.$$

The compact-open topology on M(X,Y) consists of all unions of finite intersections of subsets of the form N(C,U).

2. The *n*-th loop space of a pointed space (X, x) is

$$\Omega^{n-1}(X,x) \coloneqq M((D^{n-1},\partial D^{n-1}),(X,x)),$$

which is a subset of  $M(D^{n-1}, X)$ .

**Definition 1.2.3 (Higher homotopy groups).** If  $n \geq 2$ , then the *n*-th homotopy group of (X, x) is

$$\pi_n(X,x) := \pi_1(\Omega^{n-1}X, e_x).$$

Recall that  $\pi_1(-)$  is a functor  $\mathbf{Top}_* \to \mathbf{Grp}$ . Also,  $\Omega^{n-1}(-)$  is a functor  $\mathbf{Top}_* \to \mathbf{Top}$  defined on morphisms  $f: (X, x) \to (Y, y)$  by post-composition with f. Therefore, it's easy to see that  $\pi_n(-)$  is a functor  $\mathbf{Top}_* \to \mathbf{Grp}$  as well.

Notation. Let  $f_* = \pi_n(f)$  for any  $f: (X, x) \to (Y, y)$ .

**Proposition 1.2.4.** There is a homeomorphism  $M(X \times Y, Z) \cong M(X, M(Y, Z))$  so long as Y is locally compact and Hausdorff.

In particular, we have a composite

 $M(([0,1],\{0,1\}),(M((D^{n-1},\partial),(X,x)),e_x)) \hookrightarrow M([0,1],M(D^{n-1},X)) \xrightarrow{\cong} M([0,1] \times D^{n-1},X),$ 

whose image is precisely  $M((D^n, \partial), (X, x)) \cong M((S^n, \mathsf{pt}), (X, x))$ . This proves that  $\pi_n(X, x)$  consists of all homotopy classes of maps  $(I^n, \partial) \to (X, x)$  under the operation [f] \* [g] = [f \* g] where

$$f * g(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \le t_1 \le \frac{1}{2} \\ f(2t_1 - 1, t_2, \dots, t_n) & \frac{1}{2} \le t \le 1 \end{cases}.$$

**Proposition 1.2.5.** If  $n \geq 2$ , then  $\pi_n(X, x)$  is abelian.

Remark 1.2.6. A map  $f: S^{n-1} \to X$  is homotopic to the constant map if and only if there is some g such that

$$\int_{S^{n-1}}^{n} \xrightarrow{g} X$$

commutes.

**Theorem 1.2.7 (Whitehead).** If  $f: X \to Y$  is a map of connected cell complexes, then f is a homotopy equivalence if and only if  $f_*: \pi_n(X, x) \to \pi_n(Y, y)$  is an isomorphism for each  $n \in \mathbb{N}$ .

# 1.3 Lecture 3

**Definition 1.3.1.** If  $x \in A \subset X$ , then the *n*-th relative homotopy group  $\pi_n(X, A, x)$  consists of all homotopy classes of maps  $(D^n, S^{n-1}, x_0) \to (X, A, x)$ .

We see that

$$M((D^n, S^{n-1}, x), (X, A, x_0)) \cong M((I^n, I^{n-1} \times \{1\}, \underbrace{\partial I^n \setminus \operatorname{Int}(I^{n-1} \times \{1\})}_{\partial_0 I^n}), (X, A, x_0))$$

by considering the homeomorphism  $(I^n/\partial_0 I^n, \partial I^n/\partial_0 I^n) \cong (D^n, S^{n-1})$ . Therefore,  $\pi_n(X, A, x)$  can be viewed as consisting of all homotopy classes of maps  $(I^n, \partial I^n, \partial_0 I^n) \to (X, A, x)$ .

### Proposition 1.3.2.

- 1. If  $n \geq 2$ , then  $\pi_n(X, A, x)$  is, in fact, a group.
- 2. If  $n \geq 3$ , then  $\pi_n(X, A, x)$  is abelian.
- 3. The sequence

$$\cdots \longrightarrow \pi_n(A, x) \longrightarrow \pi_n(X, x) \longrightarrow \pi_n(X, A, x) \xrightarrow{\partial} \pi_{n-1}(A, x)$$

$$\pi_{n-1}(X, x) \longleftrightarrow \cdots \longrightarrow \pi_2(X, A, x)$$

with  $\partial[f] = [f \upharpoonright_{I^{n-1}}]$  is exact.

**Theorem 1.3.3 (Hurewicz).** Let  $n \in \mathbb{N}_{\geq 2}$ . If  $\pi_i(X) = 0$  for each i < n, then  $\pi_n(X) \cong H_n(X)$ .

Note 1.3.4. This result can't be improved in general. For example,  $\pi_3(S^2) \cong \mathbb{Z}$ , whereas  $H_3(S^2) = 0$ .

Let  $A \subset X$  be a subcomplex. Recall that  $H_i(X,A) \cong H_i(X/A.*)$  for each  $i \geq 1$ . But it is *not* the case that  $\pi_i(X,A) \cong \pi_i(X/A.*)$ , for otherwise  $\pi_i(S^n) \cong \pi_i(D^n,S^{n-1}) \cong \pi_i(S^{n-1})$ , which is known to be false exactly when i > 2n-2.

Example 1.3.5.  $\pi_4(S^3) \cong \mathbb{Z}/2 \ncong \pi_4(S^4)$ .

Finally, let's review the notion of a fibration of spaces.

Recall that if  $p: E \to B$  is a covering projection, then TFAE.

- 1. For any  $f: Z \to B$ , there exists a unique  $\hat{f}: Z \to E$  such that  $p \circ \hat{f} = f$ .
- 2.  $f_*(\pi_1(Z)) \subset p_*(\pi_1(E))$ .

The existence of  $\hat{f}$  follows from the fact that any covering space satisfies the homotopy lifting property.

**Definition 1.3.6 (Fibration).** Suppose that  $p: E \to B$  is any map. We say that p is a *fibration* if it satisfies the homotopy lifting property, i.e., given a commutative square

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\widehat{f_0}} & E \\ & & \downarrow^p, \\ X \times [0,1] & \xrightarrow{f} & B \end{array}$$

where X is a cell complex, there is some G such that

$$X \times \{0\} \xrightarrow{\widehat{f_0}} E$$

$$\downarrow \qquad \qquad \downarrow p$$

$$X \times [0,1] \xrightarrow{f} B$$

commutes.

**Theorem 1.3.7.** If  $p: E \to B$  is a fibration with  $e \in F := p^{-1}(b)$ , then

$$p_*: \pi_i(E, F, e) \xrightarrow{\cong} \pi_i(B, b, b) = \pi_i(B, b).$$

*Proof.* Let  $f:(I^n,\partial I^n)\to (B,b)$ . To prove that  $p_*$  is surjective, it suffices to find some  $G:(I^n,\partial I^n)\to (E,F)$  such that

commutes, for in this case  $[p \circ G'] = [f]$ . Since p is a fibration, there is some G such that

$$I^{n-1} \times \{0\} \longrightarrow \{e\} \hookrightarrow F \hookrightarrow E$$

$$\downarrow p$$

$$I^{n-1} \times [0,1] \longrightarrow B$$

commutes. But  $(I^n, \partial_0 I^n) \cong (I^n, I^{n-1} \times \{0\})$ , and thus such a G' is enough.

Corollary 1.3.8. The sequence

$$\cdots \longrightarrow \pi_i(F,e) \longrightarrow \pi_i(E,e) \longrightarrow \pi_i(B,b) \xrightarrow{\partial} \pi_{i-1}(F,e) \longrightarrow \cdots$$

is exact.

# Example 1.3.9.

1. Suppose that

$$\begin{array}{ccc} X \times \{0\} & \stackrel{\hat{f}}{\longrightarrow} & B \times F \\ & \downarrow & \downarrow^{\pi_B} \\ X \times [0,1] & \stackrel{f}{\longrightarrow} & B \end{array}$$

commutes. Then  $\hat{f}(x,0) = (\hat{f}_1(x,0), \hat{f}_2(x,0))$  where  $\hat{f}_1(x,0) = f(x,0)$ . Let  $G(X,t) = (f(x,t), \hat{f}_2(x,0))$ . Then

$$\begin{array}{c} X \times \{0\} \xrightarrow{\widehat{f_0}} B \times F \\ \downarrow & \downarrow^{\pi_B} \\ X \times [0,1] \xrightarrow{f} B \end{array}$$

commutes, so that  $\pi_B$  is a fibration. (Moreover,  $\pi_n(B \times F) \cong \pi_n(B) \times \pi_n(F)$ .)

- 2. Let  $A \subset X$  be a subcomplex. The map  $\varphi: M(X,Y) \to M(A,Y)$  defined by  $f \mapsto f \upharpoonright_A$  is a fibration.
- 3. Define the *Hopf fibration* as the quotient map

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \overline{z_1} + z_2 \overline{z_2} = 1\} \twoheadrightarrow S^3 /_{x \sim -x} = \mathbb{CP}^1 = S^2.$$

Corollary 1.3.10.  $\pi_3(S^3) \cong \pi_3(S^2)$ .

*Proof.* Since we have an exact sequence

$$\pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1)$$
,

it suffices to show that both  $\pi_3(S^1)$  and  $\pi_2(S^1)$  are trivial. To this end, note that since  $\pi_1(S^k) = 0$  for every k > 1, we can always find, for any  $f: S^k \to S^1$ , a map  $\hat{f}$  such that

$$\begin{array}{ccc}
& & \mathbb{R} \\
& \downarrow e^{2\pi i x} \\
S^k & \xrightarrow{f} & S^1
\end{array}$$

commutes. Thus, f is homotopic to the constant map. Since f was arbitrary, our proof is complete.  $\Box$ 

**Definition 1.3.11.** A map  $p: E \to B$  is locally trivial if for any  $b \in B$ , there exist a neighborhood  $U \ni b$  in B, a space F, and a homeomorphism  $\varphi: p^{-1}(U) \xrightarrow{\cong} U \times F$  such that  $\pi_U \circ \varphi = p \upharpoonright_{p^{-1}(U)}$ .

**Theorem 1.3.12.** Any locally trivial map  $p: E \to B$  is a fibration whenever B is a cell complex.

Exercise 1.3.13. Prove that the Hopf fibration is locally trivial.

*Proof.* For each  $k \in \{0,1\}$ , let  $U_k = \{[z_0,z_1] \in \mathbb{CP}^1 \mid z_k \neq 0\}$ . Then  $U_0$  and  $U_1$  form an open cover of  $\mathbb{CP}^1$ . Note that the preimage of  $U_k$  under the Hopf fibration q is precisely  $\{(z_0,z_1) \in S^3 \mid z_k \neq 0\}$ . Define  $f:q^{-1}(U_k) \to U_k \times S^1$  by

$$(z_0, z_1) \mapsto \left( [z_0, z_1], \frac{z_k}{|z_k|} \right).$$

This is clearly continuous. Further, define the map  $g: U_k \times S^1 \to q^{-1}(U_k)$  by

$$([z_0, z_1], e^{i\theta}) \mapsto \frac{e^{i\theta}|z_k|}{z_k|(z_0, z_1)|} (z_0, z_1).$$

Since  $U_k$  is a saturated open set, we have that the restriction of q to  $q^{-1}(U_k)$  is a quotient map. But  $g \circ q \upharpoonright_{q^{-1}(U_k)}$  is continuous, so that g is also continuous by the characteristic property of quotient maps. Finally, it is easy to verify that g and f are inverses of each other and that  $\pi_{U_I} \circ f = p \upharpoonright_{q^{-1}(U_k)}$ .

## 1.4 Lecture 4

**Theorem 1.4.1.** Let  $A \subset X$  be a subcomplex. Define  $r: M(X,Y) \to M(A,Y)$  by  $r(f) = f \upharpoonright_A$ . Then r is a fibration.

*Proof.* We must fill any diagram of the form

$$Z \times \{0\} \xrightarrow{\hat{f}} M(X,Y)$$

$$\downarrow \qquad \qquad \downarrow r \qquad \qquad \downarrow r$$

$$Z \times [0,1] \xrightarrow{f} M(A,Y)$$

It suffices to find a map  $\overline{F}$  such that

$$Z \times \{0\} \times X \xrightarrow{\hat{f}} Y$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

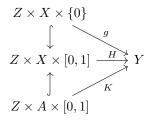
$$Z \times [0,1] \times X \qquad Y$$

$$\uparrow \qquad \qquad \downarrow$$

$$Z \times [0,1] \times A$$

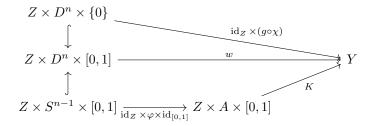
commutes for, in this case, we can set  $F(z,t)(x) = \overline{F}(z,t,x)$ .

**Note 1.4.2.** Suppose that such an  $\overline{F}$  exists. Define  $g: Z \times X \to Y$  by  $g(z,x) = \hat{f}(z,0,x)$ . Define  $h: Z \times X \times [0,1] \to Y$  by  $H(z,x,t) = \overline{F}(z,t,x)$ . Then

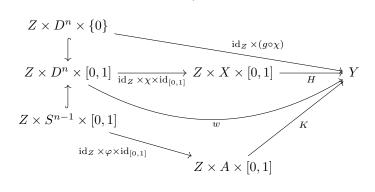


commutes where  $K(z, a, t) = \bar{f}(z, t, a)$ . In the case where  $Z = \{\mathsf{pt}\}$ , this means that if  $K : A \times [0, 1] \to Y$  is a homotopy from a map  $f : A \to Y$  and g extends f to X, then there exists a homotopy  $H : X \times [0, 1] \to Y$  such that  $H \upharpoonright_{A \times [0, 1]} = K$ . In other words, the extension problem for cell complexes is a homotopy problem.

Let's return to proving our theorem. By induction, it suffices to consider just the case where  $X = A \cup_{\varphi} D^n$ , with characteristic map  $\chi: D^n \to X$ . Thus, it suffices to find a map w such that



commutes for, in this case, we can set  $H(z,x,t)=g\cup_{\varphi} w$ , thereby making



commute. To this end, define the retraction  $u: D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]$  by picking a point \* directly above the cylinder  $D^n \times [0,1]$  and then sending any point x in the cylinder to the unique point where  $D^n \times \{0\} \cup S^{n-1} \times [0,1]$  intersects the line containing \* and x. Now, define w so that

$$Z\times (D^n\times [0,1]) \xrightarrow{w} Y$$
 
$$\mathrm{id}_Z\times u \Big\downarrow \qquad \qquad \mathrm{id}_Z\times \left(g\circ\chi\cup K\circ(\varphi\times\mathrm{id}_{[0,1]})\right)$$
 
$$Z\times (D^n\times \{0\}\cup S^{n-1}\times [0,1])$$

commutes.

**Exercise 1.4.3.** Let  $x \in X$ . Consider the loop space  $\Omega(X,x) \equiv M((S^1,\mathsf{pt}),(X,x))$ . Prove that  $\pi_n(\Omega X) \cong \pi_{n+1}(X)$ .

*Proof.* Consider the path space  $PX \equiv \{\gamma : [0,1] \to X \mid \gamma(0) = x\}$  of (X,x), equipped with the compact-open topology. We claim that PX is contractible. Indeed, define  $K : PX \times [0,1] \to PX$  by

$$(\gamma, t) \mapsto (s \mapsto \gamma(s(1-t)))$$
.

Then K is a homotopy from  $id_{PX}$  to the constant map at the constant path at x.

Define the map  $p: PX \to X$  by  $\gamma \mapsto \gamma(1)$ . Then  $p^{-1}(x) = \Omega(X)$ . By Corollary 1.3.8, it suffices to show that p is a fibration. To this end, suppose that the square

$$Y \times \{0\} \xrightarrow{\hat{f}} PX$$

$$\downarrow p$$

$$Y \times [0,1] \xrightarrow{f} X$$

commutes. Define  $H: Y \times [0,1] \to PX$  by  $(y,t) \mapsto H(y,t)$  where

$$H(y,t)(s) = \begin{cases} \hat{f}(y) ((1+t)s) & 0 \le s \le \frac{1}{1+t} \\ f(y,(1+t)s-1) & \frac{1}{1+t} \le s \le 1 \end{cases}.$$

We see that H is continuous when viewed as a function of (y, t, s) and thus is continuous. It is easy to check that

$$Y \times \{0\} \xrightarrow{\hat{f}} PX$$

$$\downarrow p$$

$$Y \times [0,1] \xrightarrow{f} X$$

commutes, as desired.

Let  $p: E \to B$  be a map. Recall that the pullback of p along  $f: X \to B$  is given explicitly as

$$f^*E \equiv \{(x, e) \in X \times E \mid f(x) = p(e)\}.$$

Let  $f^*p$  denote the map  $\pi_X \upharpoonright_{f^*E}$ .

**Proposition 1.4.4.** If p is a fibration, then so is  $f^*p$ .

**Lemma 1.4.5.** If p is locally trivial, then so is  $f^*p$ .

*Proof.* Let  $a \in X$ . Since p is locally trivial by assumption, we can find a neighborhood U of f(a) in B and a homeomorphism  $\varphi: p^{-1}(U) \to U \times F$ . Observe that

$$(f^*p)^{-1}(f^{-1}(U)) = \{(x,e) \mid f(x) = p(e), \ f(x) \in U\} \subset f^{-1}(U) \times p^{-1}(U).$$

Further, we have a map  $\psi: f^{-1}(U) \to p^{-1}(U) \to f^{-1}(U) \times F$  given by  $(x, e) \mapsto (x, \pi_F(\varphi(e)))$ . Define  $\lambda: f^{-1}(U) \times F \to (f^*p)^{-1}(f^{-1}(U))$  by  $(x, y) \mapsto (x, \varphi^{-1}(f(x), y))$ . Using the fact that

$$p^{-1}(U) \xrightarrow{\varphi} U \times F$$

$$\downarrow^{\pi_U}$$

$$U$$

commutes, it is easy to check that  $\psi$  and  $\lambda$  are inverses of each other.

### 1.5 Lecture 5

**Theorem 1.5.1.** Let B be a cell complex and let  $p: E \to B$  be locally trivial. Then p is a fibration.

*Proof.* It suffices to prove the following claim:

If  $h: Z \to X \times [0,1]$  is locally trivial,  $X = \bigcup_{i=0}^n X^i$  is a cell complex, and  $\sigma_0: X \times \{0\} \to Z$  satisfies  $h \circ \sigma_0 = \mathrm{id}_{X \times \{0\}}$ , then there is some map  $\sigma: X \times [0,1] \to Z$  such that  $\sigma_{X \times \{0\}} = \sigma_0$  and  $h \circ \sigma = \mathrm{id}_{X \times [0,1]}$ .

For, in this case, Lemma 1.4.5 implies that given any commutative square

$$\begin{array}{ccc} X \times \{0\} & \stackrel{\hat{f}}{\longrightarrow} & E \\ & & \downarrow^p, \\ X \times [0,1] & \stackrel{f}{\longrightarrow} & B \end{array}$$

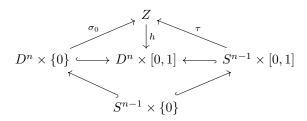
we can find some  $\sigma$  such that

$$f^*E \xrightarrow{\sigma_0} F \xrightarrow{f^*E} \downarrow^p$$

$$X \times \{0\} \xrightarrow{G} X \times [0,1] \xrightarrow{f} B$$

commutes where  $\sigma_0(x,0) = (x,0,\hat{f}(x,0)).$ 

For induction, we will assume that our claim is true for each  $X^0, X^1, \ldots, X^{n-1}$ . We may assume, wlog, that  $X = D^n$ . It suffices to find a map  $\tau : S^{n-1} \times [0,1] \to Z$  such that  $h \circ \tau = \mathrm{id}_{S^{n-1} \times [0,1]}$  and



commutes since there is a retraction  $D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]$ . Fix a positive integer m. For any  $i \in \mathbb{N}$ , let  $a_i = \frac{i}{m}$  and let  $I_j = [a_j, a_{j+1}]$ . By making m large enough, we can ensure that  $p \upharpoonright_{p^{-1}(I_{j_1} \times \cdots I_{j_{n+1}})}$  is trivial.

Claim.  $p \upharpoonright_{p^{-1}(I_{j_1} \times I_{j_n} \times \cdots [0,1])}$  is also trivial.

??

# 2 Fiber bundles

**Definition 2.0.1.** A topological group G is a group such that both multiplication  $G \times G \xrightarrow{\mu} G$  and inversion  $G \xrightarrow{(-)^{-1}} G$  are continuous.

**Definition 2.0.2 (Fiber bundle).** Let G be a topological group.

- 1. A fiber F of G is a space equipped with a faithful (i.e., injective) group action  $\rho: G \to \operatorname{Homeo}(F) \subset M(F,F)$ .
- 2. An atlas for the structure of a (fiber) bundle with group G and fiber F on a map  $p:E\to B$  consists of
  - (a) a family  $(U_{\alpha}, h_{\alpha})_{\alpha \in A}$  where each  $U_{\alpha}$  is open and each  $h_{\alpha}$  is a homeomorphism  $p^{-1}(U_{\alpha}) \to U_{\alpha} \times F$  and
  - (b) a family of continuous transition functions  $\{h_{\beta\alpha}: U_{\alpha} \cap U_{\beta} \to G\}_{\alpha,\beta \in A}$

such that

i 
$$B = \bigcup_{\alpha \in A} U_{\alpha}$$
,  
ii  $\pi_{U_{\alpha}} \circ h_{\alpha} = p \upharpoonright_{p^{-1}(U_{\alpha})}$ , and

iii 
$$x \in U_{\alpha} \cap U_{\beta} \implies h_{\beta} \circ h_{\alpha}^{-1}(x, f) = (x, h_{\beta\alpha}(x) \cdot f)$$

- 3. Two atlases are *compatible* if their union is an atlas.
- 4. A bundle structure on B is a maximal atlas on p.

Terminology. If B is equipped with a bundle structure, then we say that p is a (fiber) bundle.

#### Example 2.0.3.

1. The tangent bundle  $\pi: TM \to M$  of a smooth n-manifold M is a bundle with group  $GL(n, \mathbb{R})$ .

*Proof.* Let  $(U, \varphi)$  be any coordinate chart for M with coordinate functions  $(x^i)$ . Define  $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$  by

$$v^{i}\frac{\partial}{\partial x^{i}}(p) \mapsto (p, (v^{1}, \dots, v^{n})).$$

It is clear that  $\pi_U(h(p)) = \pi(c)$  for any  $c \in \pi^{-1}(U)$ . To see that h is a homeomorphism, note that the composite  $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$  is given by

$$v^{i} \frac{\partial}{\partial x^{i}}(p) \mapsto (x^{1}(p), \dots, x^{n}(p), v^{1}, \dots, v^{n}),$$

the inverse of which is given by  $(x^1, \ldots, x^n, v^1, \ldots, v^n) \mapsto v^i \frac{\partial}{\partial x^i} (\varphi^{-1}(x))$ . Therefore,  $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h$  is given locally by

$$(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto (\tilde{x}^1(x), \dots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^j}(x)v^j, \dots, \frac{\partial \tilde{x}^n}{\partial x^j}(x)v^j),$$

which is smooth. Thus, h is a diffeomorphism as the composition of two diffeomorphisms. In particular, h is a homeomorphism.

It remains to describe the transition functions  $\{h_{\beta\alpha}: U_{\alpha} \cap U_{\beta} \to \mathrm{GL}(n,\mathbb{R})\}$  for TM. Note that

$$U_{\alpha\beta} \times \mathbb{R}^n \xleftarrow{h_{\alpha}} \pi^{-1}(U_{\alpha\beta}) \xrightarrow{h_{\beta}} U_{\beta\alpha} \times \mathbb{R}^n$$

$$\downarrow^{\pi} \qquad \downarrow^{\pi}$$

$$U_{\alpha\beta}$$

commutes. In particular,  $\pi_1 \circ h_\beta \circ h_\alpha^{-1} = \pi_1$ , which implies that  $h_\beta \circ h_\alpha^{-1}(u,v) = (u, f(u,v))$  for some smooth map  $f: U_{\alpha\beta} \times \mathbb{R}^n \to \mathbb{R}^n$ . This must be a linear isomorphism when restricted to  $\{u\} \times \mathbb{R}^n$  for any  $u \in U_{\alpha\beta}$ , which is uniquely determined by an element  $h_{\beta\alpha}(u)$  of  $GL(n,\mathbb{R})$  (provided that we have fixed a basis of  $\mathbb{R}^n$ ). Hence

$$h_{\beta} \circ h_{\alpha}^{-1}(u,v) = (u, h_{\beta\alpha}(u)v).$$

Since the map  $h_{\beta\alpha}: U_{\alpha\beta} \to \mathrm{GL}(n,\mathbb{R})$  is continuous, our proof is complete.

2. Let  $p: E \to B$  be a bundle with group  $\{e\}$ . Then p is the trivial bundle, i.e., is isomorphic to the projection map.

*Proof.* We have that  $h_{\beta} = h_{\alpha}$  on  $U_{\alpha} \cap U_{\beta}$ , so that  $h \equiv \bigcup_{\alpha \in A} h_{\alpha}$  is a well-defined homeomorphism  $E \cong B \times F$ .

#### 2.1 Lecture 6

Let  $\{(U_{\alpha}, h_{\alpha})\}$  be a bundle structure with group G and fiber F on  $p: E \to B$ . Let  $U = U_{\alpha} \cap U_{\beta} \cap U_{\gamma}$ . Consider the commutative diagram

$$U \times F \xrightarrow[h_{\alpha}^{-1}]{h_{\alpha}^{-1}} p^{-1}(U) \xrightarrow[h_{\beta}]{h_{\gamma}} U \times F \xrightarrow[h_{\beta}^{-1}]{h_{\gamma}} p^{-1}(U) \xrightarrow[h_{\gamma}]{h_{\gamma}} U \times F$$

The bottom row is given by  $(u, f) \mapsto (u, h_{\beta\alpha}(u) \cdot f) \mapsto (u, h_{\gamma\beta}(u) \cdot (h_{\beta\alpha}(u) \cdot f)) = (u, (h_{\gamma\beta}(u)h_{\beta\alpha}(u)) \cdot f)$ , and the top composite is given by  $(u, f) \mapsto (u, h_{\gamma\alpha}(u) \cdot f)$ . It follows that

$$h_{\gamma\beta}(u)h_{\beta\alpha}(u) = h_{\gamma\alpha}(u)$$

for each  $u \in U$ . This property is known as the *cocycle condition*.

**Theorem 2.1.1.** Let G be a topological group acting on a space F. Suppose that  $\{U_{\alpha}\}$  is an open cover of B and  $\{h_{\beta\alpha}: U_{\alpha} \cap U_{\beta} \to G\}$  is a family of continuous functions satisfying the cocycle condition. Then there exists a bundle  $p: E \to B$  with group G, fiber F, and transition functions  $h_{\beta\alpha}$ .

*Proof sketch.* Let  $E = \coprod_{\alpha} U_{\alpha} \times F_{\alpha}$  where  $(u, f)_{\alpha} \sim (u, h_{\beta\alpha}(u) \cdot f)_{\beta}$ . Define  $p : E \to B$  by  $(u, f) \mapsto u$ .  $\square$ 

**Definition 2.1.2 (Bundle map).** A morphism of bundles  $p_1$  and  $p_2$  with group G and fiber F is a commutative square of the form

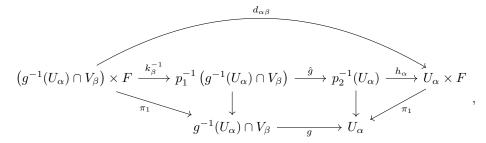
$$E_{1} \xrightarrow{\hat{g}} E_{2}$$

$$\downarrow^{p_{1}} \qquad \qquad \downarrow^{p_{2}}$$

$$B_{1} \xrightarrow{g} B_{2}$$

such that

Suppose that  $(\hat{g}, g)$  is a bundle map  $p_1 \to p_2$ . Let  $\{(U_\alpha, h_\alpha)\}$  and  $\{(V_\beta, k_\beta)\}$  be bundle structures on  $B_2$  and  $B_1$ , respectively. We have a commutative diagram



so that  $d_{\alpha\beta}(x,f) = (g(x), \lambda_{\alpha\beta}(x) \cdot f)$  for some continuous map  $\lambda_{\alpha\beta} : g^{-1}(U_{\alpha}) \cap V_{\beta} \to G$ . Letting  $W = g^{-1}(U_{\alpha} \cap U_{\alpha'}) \cap (V_{\beta} \cap V_{\beta'})$ , we have that

$$h_{\alpha'\alpha}(w)\lambda_{\alpha\beta}(w)k_{\beta\beta'}(w) = \lambda_{\alpha'\beta'}(w) \tag{\dagger}$$

for every  $w \in W$ .

**Exercise 2.1.3 (Pullback bundle).** Let  $\{(U_{\alpha}, h_{\alpha})\}$  be a bundle structure on  $p : E \to B$  with group G and consider the pullback diagram

$$\begin{array}{ccc} g^*E & \longrightarrow & E \\ g^*{}_p \Big| & & & \Big|_p \,. \\ X & \xrightarrow{g} & B \end{array}$$

Define  $h'_{\beta\alpha}: g^{-1}(U_{\alpha}) \cap g^{-1}(U_{\beta}) \to G$  as the composite  $h_{\beta\alpha} \circ g$  restricted to  $g^{-1}(U_{\alpha} \cap U_{\beta})$ . Show that the family  $\{h'_{\beta\alpha}\}$  induces a bundle structure on  $g^*p$ .

Theorem 2.1.4. Any bundle map

$$E_1 \xrightarrow{\hat{g}} E_2$$

$$\downarrow^{p_1} \qquad \qquad \downarrow^{p_2}$$

$$B_1 \xrightarrow{g} B_2$$

factors as

$$E_{1} \xrightarrow{\tau} g^{*}E_{2} \xrightarrow{\bar{g}} E_{2}$$

$$\downarrow^{p_{2}} \qquad \downarrow^{g^{*}p_{2}} \qquad \downarrow^{p_{2}}$$

$$B_{1} \xrightarrow{\operatorname{id}_{B_{1}}} B_{1} \xrightarrow{g} B_{2}$$

where  $\tau(e) = (p_1(e), \hat{g}(e))$  for any  $e \in E_1$ .

#### 2.2 Lecture 7

**Note 2.2.1.** If  $\{h_{\beta\alpha}: U_{\alpha} \cap U_{\beta} \to G\}$  is a family of transition functions, then

$$h_{\alpha\beta}(x) = (h_{\beta\alpha}(x))^{-1}$$

for any  $x \in U_{\alpha} \cap U_{\beta}$ .

**Theorem 2.2.2.** Any bundle map of the form

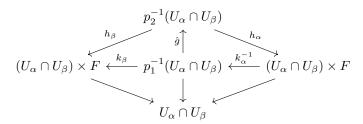
$$E_1 \xrightarrow{\hat{g}} E_2$$

$$\downarrow^{p_2}$$

$$B$$

is an isomorphism.

Proof. Note that



commutes. We have that  $h_{\beta} \circ \hat{g} \circ k_{\alpha}^{-1}(x, f) = (x, \lambda_{\beta\alpha}(x) \cdot f)$ . Thus, if  $h_{\alpha}(e) = (x, f)$ , then  $h_{\alpha}(\hat{g}(e)) = (x, \lambda_{\alpha\alpha}(x) \cdot d)$ . Let

$$(\hat{g})^{-1}(e) = k_{\alpha}^{-1}(x, \lambda_{\alpha\alpha}(x)^{-1} \cdot f)$$

where  $(x, f) = h_{\alpha}(e)$ . If this is well-defined, then it equals the inverse of g. Moreover, it is a bundle map because of  $(\dagger)$ . ??

Corollary 2.2.3. Every bundle over a space E is isomorphic to the pullback bundle over E.