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Abstract

These notes are based on Julius Shaneson's lectures for the course "Algebraic Topology, Part I" at UPenn. Any mistake in what follows is my own.

Contents

1 Background material

1.1 Lecture 1

Here are the topics for the course:

- fiber bundles over cell complexes,
- spectral sequences,
- characteristic classes, and
- cobordism theory.

To start, let's review some basic concepts from homology theory.

Definition 1.1.1. A *(finite) cell complex* is a (topological) space X that can be written as $\bigcup_{n=0}^{K} X^n$ for some $K \in \mathbb{N}$ (called the *dimension of* X) where

- X^0 is chosen to be finite,
- $X^n = \frac{X^{n-1} \coprod D_1^n \coprod \cdots \coprod D_{k_n}^n}{x \sim \varphi_i(x)}$,
- $D_i^n \cong D^n \coloneqq \{x \in \mathbb{R}^n : |x| \leq 1\}$ for each $i \in \{1, \ldots, k_n\}$, and
- $\varphi_i : \partial D_i^n = S^{n-1} \to X^{n-1}$, called an *attaching map.*

Terminology. Each D_i^n is called an *n-cell of X*.

Every attaching map $\varphi_i : \partial D_i^n \to X^{n-1}$ can be extended to a *characteristic map* given by the composite

$$D_i^n \hookrightarrow X^{n-1} \coprod D_1^n \coprod \cdots \coprod D_{k_n}^n \twoheadrightarrow X^n \hookrightarrow X.$$

Example 1.1.2. There are at least two ways of endowing S^2 with a cell structure.

1. $X^0 \equiv \{N, S\}$, $X^1 \equiv X^0 \cup_{\varphi_1} D_1^1 \cup_{\varphi_2} D_2^1$ where each φ_i is an embedding, and $X^2 \equiv X^1 \cup_{\varphi_1'} D_1^2 \cup_{\varphi_2'} D_2^2$ where each φ_i' is an embedding.
2. $\mathsf{pt} \cup_{\varphi} D^2$ where φ identifies the equator of the upper half-sphere with pt.

Definition 1.1.3. A cell complex X is *regular* if every characteristic map $D_i^n \to X$ is an embedding.

Definition 1.1.4. Given a family of functors $\left\{H_n : \mathbf{Top}^2 \to \mathbf{Ab}\right\}_{n \in \mathbb{N}}$ where $\mathbf{Top}^2$ denotes the category of (topological) pairs, we say that H_i is a *homology functor* if each of the following properties holds.

1. (LES) For any pair (X, A) of space, there is a natural long exact sequence

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \longrightarrow \cdots,$$

   where $H_i(Z) \coloneqq H_i(Z, \emptyset)$ for any space Z.

2. (Excision) If $\mathrm{cl}(A) \subset \underset{open}{U} \subset X$, then $H_i(X \setminus A, U \setminus A) \cong H_i(X, U)$.

3. (Dimension) $H_i(\mathsf{pt}) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases}$.

4. (Homotopy) If f and g are homotopic, then $f_* = g_*$, where $h_* := H_i(h)$ for any map $h : (X, A) \to (Y, B)$.

Theorem 1.1.5. *There exists a family of homology functors.*

Example 1.1.6. In singular homology theory, we have that $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$.

Let X be a cell complex. Let $C_n(X)$ denote the free abelian group on the set of all n-cells of X. Define $\partial : C_n(X) \to C_{n-1}(X)$ by $\partial[D_i^n] = \sum_{j=1}^{k_{n-1}} \lambda_{ij}[D_j^{n-1}]$ where λ_{ij} is defined, up to sign, as follows. Consider the map

$$\begin{tikzcd} S^{n-1} \arrow[r, equal] \arrow[rrrrrr, bend right, "\omega"'] & \partial D_i^n \arrow[r, "\varphi_i"] & X^{n-1} \arrow[r, two heads] & \frac{X^{n-1}}{X^{n-2} \cup (\text{all cells of dim. } n-1 \text{ except } D_j^{n-1})} \arrow[r, equal] & D^{n-1}/\partial D_j^{n-1} \arrow[r, equal] & S^{n-1} \end{tikzcd}.$$

Then let λ_{ij} satsify $\omega_*(x) = \lambda_{ij} x$ with x a chosen generator (i.e., orientation) of $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Terminology. The integer λ_{ij} is called the *degree of* ω, denoted by $\deg(\omega)$.

Theorem 1.1.7. $\partial_n \partial_{n+1} = 0$, *and* $H_n(X) \cong {}^{\ker \partial_n}\!/\!_{\operatorname{im} \partial_{n+1}}$, *which is independent of our choice of generator* x.

Example 1.1.8. Suppose that $f : S^n \to S^n$ is smooth. By Sard's theorem, we can find a regular value $x \in S^n$. There is some neighborhood U of x such that $f^{-1}(U) = U_1 \cup \cdots \cup U_n$ for some n. Using the inverse function theorem and the compactness of S^n, it follows that f^{-1} is of the form $\{x_1, \ldots, x_n\}$. Note that the differential $(df)_{x_i} : S^n_{x_i} \to S^n_x$ satisfies $\det (df)_{x_i} - \pm 1$. In fact,

$$\deg(f) = \sum_{i=1}^{n} \det (df)_{x_i}.$$

Exercise 1.1.9. *Prove that any finite cell complex $X = X^K$ is homotopy equivalent to a regular cell complex. (Hint: Consider the map $S^{n-1} \to X^{n-1} \times D^n$ given by $x \mapsto (\varphi(x), x)$ where φ denotes an attaching map of X.)*

Proof. Let us construct recursively a finite sequence $A^0, A^1, \ldots, A^K$ of spaces such that each A^i carries the stricture of a regular cell complex and is homotopy equivalent to X^i. For each $n \in \{1, \ldots, K\}$, let k_n denote the necessarily finite number of attaching maps $\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_{k_n}} : S^{n-1} \to X^{n-1}$ for the n-skeleton of X. Let

$$A^0 = X^0 \times D^1_{\alpha_1} \times \cdots D^1_{\alpha_{k_1}},$$

viewed as a product of finite cell-complexes. Note that the topology of A^0 is precisely the product topology. Thus, A^0 is homotopy equivalent to X^0 as D^1 is contractible. Now, suppose that $0 \leq n \leq K - 1$ and that we have constructed our desired space A^n. This means that there is some homotopy equivalence $\gamma_n : X^n \to A^n$. Form A^{n+1} by attaching finitely many $(n+1)$-cells $e^{n+1}_{\alpha_1}, \dots, e^{n+1}_{\alpha_{k_{n+1}}}$ to $Z_n \equiv A^n \times D^{n+1}_{\alpha_1} \times \cdots \times D^{n+1}_{\alpha_{k_{n+1}}}$ via the maps

$$\psi_{\alpha_i} : S^n \to A^n \times D^{n+1}_{\alpha_1} \times \cdots \times D^{n+1}_{\alpha_{k_{n+1}}}$$
$$x \mapsto \left(\gamma_n \circ \varphi_i(x), 0, \dots, 0, \underbrace{x}_{i\text{-th spot}}, 0, \dots, 0 \right)$$

where Z_n is viewed as a product of finite cell complexes (whose topology is precisely the product topology). It is easy to see that A^{n+1} is homotopy equivalent to X^{n+1}. Moreover, since each map ψ_{α_i} is an embedding and any n-disk has the structure of a regular cell complex, we see from our construction of $\left(A^i\right)$ that A^K has the structure of a regular cell complex. By design, this space is homotopy equivalent to X^K, thereby completing our proof. $\square$

1.2 Lecture 2

Example 1.2.1 (Real projective space). Recall that $\mathbb{RP}^n = {S^n}/{x \sim -x}$. Then $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\pi_{n-1}} D^n$ where $\pi_{n-1} : S^{n-1} \to \mathbb{RP}^{n-1}$ denotes the canonical projection. Thus, $\mathbb{RP}^n$ is an n-dimension cell complex with $(\mathbb{RP}^n)^m = \mathbb{RP}^m$ for each integer $0 \leq m \leq n$.

Now, for each $0 \leq m \leq n$, we have that $C_m(\mathbb{RP}^n) \cong \mathbb{Z}$ with generator $[D^m]$. To determine $\partial[D^m] \in C_{m-1}(\mathbb{RP}^m)$, we must find the degree of the map

$$\begin{tikzcd} S^{m-1} \arrow[r] \arrow[rrrr, bend right, "\varphi"'] & \mathbb{RP}^{m-1} \arrow[r, two heads] & {\mathbb{RP}^{m-1}}/{\mathbb{RP}^{m-1}} \arrow[r, equal] & {D^{m-1}}/{\partial D^{m-1}} \arrow[r, equal] & S^{m-1} \end{tikzcd}$$

Assume, for simplicity, that $m = 2$. Choose a regular value $p \in S^1$ so that $\varphi^{-1}(p) = \{N, S\}$. Let φ_T and φ_B denote the restrictions of φ to the top and bottom components of $S^1 \setminus \{(-1,0),(1,0)\}$, respectively. Note that both of these are homeomorphisms and thus have degrees equal to ± 1. If $a : S^{m-1} \to S^{m-1}$ denotes the antipodal map, we have that $\varphi_B \circ a = \varphi_T$. Hence $(d\varphi)_S \circ (da)_N = (d\varphi)_N$. Since $\deg(a) = \det(da) = (-1)^m$, it follows that

$$\deg(\varphi) = \begin{cases} \pm 2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}.$$

Thus, we get a chain complex

$$0 \longrightarrow C_n(\mathbb{RP}^n) \xrightarrow{\kappa_1} C_{n-1}(\mathbb{RP}^n) \xrightarrow{\kappa_2} \cdots \xrightarrow{0} C_2(\mathbb{RP}^n) \xrightarrow{\pm 2} C_1(\mathbb{RP}^n) \xrightarrow{0} C_0(\mathbb{RP}^n) \longrightarrow 0$$

where $\kappa_1 = \begin{cases} 0 & n \text{ odd} \\ \pm 2 & n \text{ even} \end{cases}$ and $\kappa_2 = \begin{cases} \pm 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$.

This proves that

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i = 1 \\ 0 & i = 2 \\ \mathbb{Z}_2 & \underset{odd}{i} < n \\ 0 & \underset{even}{i} < n \\ 0 & i > n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & i = n \text{ even} \end{cases}.$$

Example 1.2.2. $H_{2i}(\mathbb{CP}^n) \cong \mathbb{Z}$.

Next, let's introduce some fundamental concepts from homotopy theory.

Definition 1.2.3. Let $M(X, Y)$ denote the set of maps $X \to Y$.

1. For any compact $C \subset X$ and open $U \subset Y$, let

$$N(C, U) = \{f : X \to Y \mid f(C) \subset U\}.$$

The *compact-open topology on* $M(X, Y)$ consists of all unions of finite intersections of subsets of the form $N(C, U)$. Under this topology, $M(X, Y)$ is called a *mapping space*.

2. The *n-th loop space* of a pointed space (X, x) is

$$\Omega^{n-1}(X, x) \coloneqq M((D^{n-1}, \partial D^{n-1}), (X, x)),$$

which is a subset of $M(D^{n-1}, X)$.

Definition 1.2.4 (Higher homotopy groups). If $n \geq 2$, then the *n-th homotopy group* of (X, x) is

$$\pi_n(X, x) \coloneqq \pi_1(\Omega^{n-1}X, e_x).$$

Recall that $\pi_1(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Grp}$. Also, $\Omega^{n-1}(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Top}$ defined on morphisms $f : (X, x) \to (Y, y)$ by post-composition with f. Therefore, it's easy to see that $\pi_n(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Grp}$ as well.

Notation. Let $f_* = \pi_n(f)$ for any $f : (X, x) \to (Y, y)$.

Proposition 1.2.5. *There is a homeomorphism $M(X \times Y, Z) \cong M(X, M(Y, Z))$ so long as Y is locally compact and Hausdorff.*

In particular, we have a composite

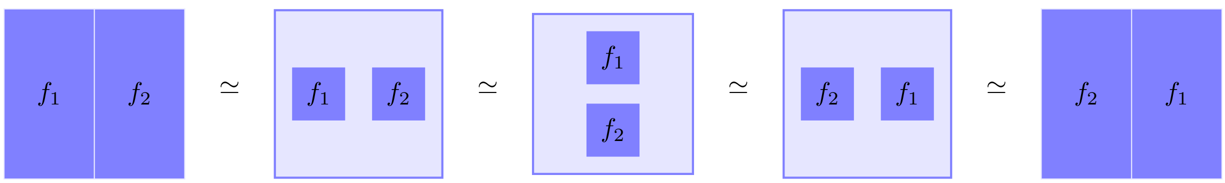
$$M(([0,1], \{0,1\}), (M((D^{n-1}, \partial), (X, x)), e_x)) \hookrightarrow M([0,1], M(D^{n-1}, X)) \xrightarrow{\cong} M([0,1] \times D^{n-1}, X),$$

whose image is precisely $M((D^n, \partial), (X, x)) \cong M((S^n, \mathsf{pt}), (X, x))$. This proves that $\pi_n(X, x)$ consists of all homotopy classes of maps $(I^n, \partial) \to (X, x)$ under the operation $[f] * [g] = [f * g]$ where

$$f * g(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \le t_1 \le \frac{1}{2} \\ f(2t_1 - 1, t_2, \ldots, t_n) & \frac{1}{2} \le t \le 1 \end{cases}.$$

Lemma 1.2.6. *If $n \ge 2$, then $\pi_n(X, x)$ is abelian.*

Proof. Let f_1 and f_2 be maps $(I^n, \partial) \to (X, x)$. We must find a homotopy $F : I^n \times I \to X$ between $f_1 * f_2$ and $f_2 * f_1$ such that $F(t_1, \ldots, t_n, s) = x$ for any $(t_1, \ldots, t_n) \in \partial I^n$ and $s \in I$. To this end, first shrink the domains of f_1 and f_2 to small n-cubes in I^n (thereby thickening the set of points mapped to x under F), then slide these small cubes past each other, and finally enlarge them to their original sizes as follows.



$\square$

Remark 1.2.7. A map $f : S^{n-1} \to X$ is homotopic to the constant map if and only if there is some g such that

$$\begin{tikzcd} D^n \arrow[dr, "g"] & \\ S^{n-1} \arrow[u, hook] \arrow[r, "f"'] & X \end{tikzcd}$$

commutes.

Theorem 1.2.8 (Whitehead). *If $\psi : X \to Y$ is a map of connected cell complexes, then f is a homotopy equivalence if and only if $\psi_* : \pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism for each $n \in \mathbb{N}$.*

A map $f : X \to Y$ of path connected spaces is a *weak homotopy equivalence* if it induces an isomorphism $\pi_n(X) \to \pi_n(Y)$ for each $n \in \mathbb{N}$. Since any cell complex is locally path connected, it is connected if and only if it is path connected. Hence Theorem 1.2.8 says that ψ is a homotopy equivalence if and only if it is a weak homotopy equivalence.

1.3 Lecture 3

Definition 1.3.1. If $x \in A \subset X$, then the *n-th relative homotopy group* $\pi_n(X, A, x)$ consists of all homotopy classes of maps $\left(D^n, S^{n-1}, x_0\right) \to (X, A, x)$.

We see that

$$M((D^n, S^{n-1}, x), (X, A, x_0)) \cong M((I^n, I^{n-1} \times \{1\}, \underbrace{\partial I^n \setminus \mathrm{Int}(I^{n-1} \times \{1\})}_{\partial_0 I^n}), (X, A, x_0))$$

by considering the homeomorphism $(I^n/\partial_0 I^n, \partial I^n/\partial_0 I^n) \cong \left(D^n, S^{n-1}\right)$. Therefore, $\pi_n(X, A, x)$ can be viewed as consisting of all homotopy classes of maps $(I^n, \partial I^n, \partial_0 I^n) \to (X, A, x)$.

Definition 1.3.2. In order to interpret an exact sequence involving objects in the category of pointed sets, we define the *kernel of a function $f : (X, x) \to (Y, y)$ of pointed sets* as $\ker f \equiv f^{-1}(y)$.

Proposition 1.3.3.

1. *If $n \geq 2$, then $\pi_n(X, A, x)$ is, in fact, a group.*

2. *If $n \geq 3$, then $\pi_n(X, A, x)$ is abelian.*

3. *We have a long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n(A,x) \arrow[r] & \pi_n(X,x) \arrow[r] & \pi_n(X,A,x) \arrow[r, "\partial"] & \pi_{n-1}(A,x) \arrow[dllll] \\ \pi_{n-1}(X,x) \arrow[r] & \cdots \arrow[r] & \pi_0(A,x) \arrow[r] & \pi_0(X,x) \arrow[r] & 0 \end{tikzcd}$$

with $\partial[f] = [f \restriction_{I^{n-1}}]$.

Theorem 1.3.4 (Hurewicz). *Let $n \in \mathbb{Z}_{\geq 2}$. If $\pi_i(X) = 0$ for each $i < n$, then $\pi_n(X) \cong H_n(X)$.*

Note 1.3.5. This result can't be improved in general. For example, $\pi_3(S^2) \cong \mathbb{Z}$, whereas $H_3(S^2) = 0$.

Let $A \subset X$ be a subcomplex. Recall that $H_i(X, A) \cong H_i(X/A.*)$ for each $i \geq 1$. But it is *not* the case that $\pi_i(X, A) \cong \pi_i(X/A.*)$, for otherwise $\pi_i(S^n) \cong \pi_i(D^n, S^{n-1}) \cong \pi_i(S^{n-1})$, which is known to be false exactly when $i > 2n - 2$.

Example 1.3.6. $\pi_4(S^3) \cong \mathbb{Z}_2 \ncong \pi_4(S^4)$.

Finally, let's review the notion of a fibration of spaces.

Consider any map $f : Z \to B$. Recall that if $p : E \to B$ is a covering projection, then TFAE.

1. There exists a unique $\hat{f} : Z \to E$ such that $p \circ \hat{f} = f$.

2. $f_*(\pi_1(Z)) \subset p_*(\pi_1(E))$.

The existence of $\hat{f}$ follows from the fact that any covering space satisfies the homotopy lifting property.

Definition 1.3.7 (Fibration). Suppose that $p : E \to B$ is any map. We say that p is a *(Serre) fibration* if it satisfies the homotopy lifting property, i.e., given a commutative square

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widehat{f_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "f"'] & B \end{tikzcd},$$

where X is a cell complex, there is some G such that

$$\begin{tikzcd} X \times \{0\} \arrow[r, "\widehat{f_0}"] \arrow[d, hook] & E \arrow[d, "p"] \\ X \times [0,1] \arrow[r, "f"'] \arrow[ur, "G"] & B \end{tikzcd}$$

commutes.

Theorem 1.3.8. *If $p : E \to B$ is a fibration with $e \in F \coloneqq p^{-1}(b)$, then*

$$p_* : \pi_i(E, F, e) \xrightarrow{\cong} \pi_i(B, b, b) = \pi_i(B, b).$$

Proof. Let $f : (I^n, \partial I^n) \to (B, b)$. To prove that p_* is surjective, it suffices to find some $G : (I^n, \partial I^n) \to (E, F)$ such that

$$\begin{tikzcd}
\partial_0 I^n \arrow[r, two heads] \arrow[d, hook] & \{e\} \arrow[r, hook] & F \arrow[r, hook] & E \arrow[d, "p"] \\
I^{n-1} \times [0,1] \arrow[rrr, "f"'] \arrow[rrru, "G"] & & & B
\end{tikzcd}$$

commutes, for in this case $[p \circ G'] = [f]$. Since p is a fibration, there is some G such that

$$\begin{tikzcd}
I^{n-1} \times \{0\} \arrow[r, two heads] \arrow[d, hook] & \{e\} \arrow[r, hook] & F \arrow[r, hook] & E \arrow[d, "p"] \\
I^{n-1} \times [0,1] \arrow[rrr, "f"'] \arrow[rrru, "G'"] & & & B
\end{tikzcd}$$

commutes. But $(I^n, \partial_0 I^n) \cong \big(I^n, I^{n-1} \times \{0\}\big)$, and thus such a G' is enough. $\square$

Corollary 1.3.9. *We have a long exact sequence*

$$\cdots \longrightarrow \pi_i(F, e) \longrightarrow \pi_i(E, e) \longrightarrow \pi_i(B, b) \xrightarrow{\partial} \pi_{i-1}(F, e) \longrightarrow \cdots .$$

Example 1.3.10.

1. Suppose that

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & B \times F \arrow[d, "\pi_B"] \\
X \times [0,1] \arrow[r, "f"] & B
\end{tikzcd}$$

   commutes. Then $\hat{f}(x, 0) = \Big(\hat{f}_1(x, 0), \hat{f}_2(x, 0)\Big)$ where $\hat{f}_1(x, 0) = f(x, 0)$. Let $G(X, t) = \Big(f(x, t), \hat{f}_2(x, 0)\Big)$. Then

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\widehat{f_0}"] \arrow[d, hook] & B \times F \arrow[d, "\pi_B"] \\
X \times [0,1] \arrow[r, "f"] \arrow[ru, "G"] & B
\end{tikzcd}$$

   commutes, so that π_B is a fibration. (Moreover, $\pi_n(B \times F) \cong \pi_n(B) \times \pi_n(F)$.)

2. Let $A \subset X$ be a subcomplex. The map $\varphi : M(X, Y) \to M(A, Y)$ defined by $f \mapsto f \restriction_A$ is a fibration.

3. Define the *Hopf fibration* as the quotient map

$$S^3 = \big\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1\overline{z_1} + z_2\overline{z_2} = 1\big\} \twoheadrightarrow {S^3}\big/_{x \sim -x} = \mathbb{CP}^1 = S^2.$$

Corollary 1.3.11. $\pi_3(S^3) \cong \pi_3(S^2)$.

Proof. Since we have an exact sequence

$$\pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) \ ,$$

it suffices to show that both $\pi_3(S^1)$ and $\pi_2(S^1)$ are trivial. To this end, note that since $\pi_1(S^k) = 0$ for every $k > 1$, we can always find, for any $f : S^k \to S^1$, a map $\hat{f}$ such that

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "e^{2\pi i x}"] \\ S^k \arrow[ur, "\hat{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

commutes. Thus, f is homotopic to the constant map. Since f was arbitrary, our proof is complete. $\square$

Definition 1.3.12. A map $p : E \to B$ is *locally trivial* if for any $b \in B$, there exist a neighborhood $U \ni b$ in B, a space F, and a homeomorphism $\varphi : p^{-1}(U) \xrightarrow{\cong} U \times F$ such that $\pi_U \circ \varphi = p \restriction_{p^{-1}(U)}$.

Theorem 1.3.13. *Any locally trivial map $p : E \to B$ is a fibration whenever B is a cell complex.*

Exercise 1.3.14. *Prove that the Hopf fibration is locally trivial.*

Proof. For each $k \in \{0, 1\}$, let $U_k = \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_k \neq 0\}$. Then U_0 and U_1 form an open cover of $\mathbb{CP}^1$. Note that the preimage of U_k under the Hopf fibration q is precisely $\{(z_0, z_1) \in S^3 \mid z_k \neq 0\}$. Define $f : q^{-1}(U_k) \to U_k \times S^1$ by

$$(z_0, z_1) \mapsto \left([z_0, z_1], \frac{z_k}{|z_k|} \right).$$

This is clearly continuous. Further, define the map $g : U_k \times S^1 \to q^{-1}(U_k)$ by

$$\left([z_0, z_1], e^{i\theta} \right) \mapsto \frac{e^{i\theta} |z_k|}{z_k |(z_0, z_1)|} (z_0, z_1).$$

Since U_k is a saturated open set, we have that the restriction of q to $q^{-1}(U_k)$ is a quotient map. But $g \circ q \restriction_{q^{-1}(U_k)}$ is continuous, so that g is also continuous by the characteristic property of quotient maps. Finally, it is easy to verify that g and f are inverses of each other and that $\pi_{U_I} \circ f = p \restriction_{q^{-1}(U_k)}$. $\square$

1.4 Lecture 4

Theorem 1.4.1. *Let $A \subset X$ be a subcomplex. Define $r : M(X, Y) \to M(A, Y)$ by $r(f) = f \restriction_A$. Then r is a fibration.*

Proof. We must fill any diagram of the form

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & M(X, Y) \arrow[d, "r"] \\ Z \times [0, 1] \arrow[r, "f"'] \arrow[ur, dashed, "F"] & M(A, Y) \end{tikzcd} .$$

It suffices to find a map $\overline{F}$ such that

$$\begin{tikzcd}
Z \times \{0\} \times X \arrow[r, "\hat{\hat{f}}"] \arrow[d, hook] & Y \arrow[d, equal] \\
Z \times [0,1] \times X \arrow[ur, "\overline{F}"] & Y \\
Z \times [0,1] \times A \arrow[u, hook] \arrow[ur, "\bar{f}"'] &
\end{tikzcd}$$

commutes for, in this case, we can set $F(z,t)(x) = \overline{F}(z,t,x)$.

Note 1.4.2. Suppose that such an $\overline{F}$ exists. Define $g : Z \times X \to Y$ by $g(z,x) = \hat{\hat{f}}(z,0,x)$. Define $h : Z \times X \times [0,1] \to Y$ by $H(z,x,t) = \overline{F}(z,t,x)$. Then

$$\begin{tikzcd}
Z \times X \times \{0\} \arrow[d, hook] \arrow[dr, "g"] & \\
Z \times X \times [0,1] \arrow[r, "H"] & Y \\
Z \times A \times [0,1] \arrow[u, hook] \arrow[ur, "K"'] &
\end{tikzcd}$$

commutes where $K(z,a,t) = \bar{f}(z,t,a)$. In the case where $Z = \mathsf{pt}$, this means that if $K : A \times [0,1] \to Y$ is a homotopy from a map $f : A \to Y$ and g extends f to X, then there exists a homotopy $H : X \times [0,1] \to Y$ such that $H \restriction_{A \times [0,1]} = K$. In other words, the extension problem for cell complexes is a homotopy problem.

Let's return to proving our theorem. By induction, it suffices to consider just the case where $X = A \cup_\varphi D^n$, with characteristic map $\chi : D^n \to X$. Thus, it suffices to find a map w such that

$$\begin{tikzcd}
Z \times D^n \times \{0\} \arrow[d, hook] \arrow[drr, "\mathrm{id}_Z \times (g \circ \chi)"] & & \\
Z \times D^n \times [0,1] \arrow[rr, "w"] & & Y \\
Z \times S^{n-1} \times [0,1] \arrow[u, hook] \arrow[r, "\mathrm{id}_Z \times \varphi \times \mathrm{id}_{[0,1]}"'] & Z \times A \times [0,1] \arrow[ur, "K"'] &
\end{tikzcd}$$

commutes for, in this case, we can set $H(z,x,t) = g \cup_\varphi w$, thereby making

$$\begin{tikzcd}
Z \times D^n \times \{0\} \arrow[d, hook] \arrow[drr, "\mathrm{id}_Z \times (g \circ \chi)"] & & \\
Z \times D^n \times [0,1] \arrow[r, "\mathrm{id}_Z \times \chi \times \mathrm{id}_{[0,1]}"'] \arrow[rr, bend right, "w"'] & Z \times X \times [0,1] \arrow[r, "H"'] & Y \\
Z \times S^{n-1} \times [0,1] \arrow[u, hook] \arrow[dr, "\mathrm{id}_Z \times \varphi \times \mathrm{id}_{[0,1]}"'] & & \\
& Z \times A \times [0,1] \arrow[uur, "K"'] &
\end{tikzcd}$$

commute. To this end, define the retraction $u : D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]$ by picking a point $*$ directly above the cylinder $D^n \times [0,1]$ and then sending any point x in the cylinder to the unique point where $D^n \times \{0\} \cup S^{n-1} \times [0,1]$ intersects the line containing $*$ and x. Now, define w so that

$$\begin{tikzcd}
Z \times (D^n \times [0,1]) \arrow[r, "w"] \arrow[d, "\mathrm{id}_Z \times u"'] & Y \\
Z \times (D^n \times \{0\} \cup S^{n-1} \times [0,1]) \arrow[ur, "\mathrm{id}_Z \times \left(g \circ \chi \cup K \circ (\varphi \times \mathrm{id}_{[0,1]})\right)"']
\end{tikzcd}$$

commutes. $\square$

Exercise 1.4.3. *Let $x \in X$. Consider the loop space $\Omega(X,x) \equiv M((S^1, \mathsf{pt}), (X,x))$. Prove that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$.*

Proof. Consider the *path space* $PX \equiv \{\gamma : [0,1] \to X \mid \gamma(0) = x\}$ *of* (X,x), equipped with the compact-open topology. We claim that PX is contractible. Indeed, define $K : PX \times [0,1] \to PX$ by

$$(\gamma, t) \mapsto (s \mapsto \gamma(s(1-t)))\,.$$

Then K is a homotopy from id_{PX} to the constant map at the constant path at x.

Define the map $p : PX \to X$ by $\gamma \mapsto \gamma(1)$. Then $p^{-1}(x) = \Omega(X)$. By Corollary 1.3.9, it suffices to show that p is a fibration. To this end, suppose that the square

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & PX \arrow[d, "p"] \\
Y \times [0,1] \arrow[r, "f"'] & X
\end{tikzcd}$$

commutes. Define $H : Y \times [0,1] \to PX$ by $(y,t) \mapsto H(y,t)$ where

$$H(y,t)(s) = \begin{cases} \hat{f}(y)\left((1+t)s\right) & 0 \le s \le \frac{1}{1+t} \\ f(y, (1+t)s - 1) & \frac{1}{1+t} \le s \le 1 \end{cases}.$$

We see that H is continuous when viewed as a function of (y,t,s) and thus is continuous. It is easy to check that

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & PX \arrow[d, "p"] \\
Y \times [0,1] \arrow[r, "f"'] \arrow[ur, "H"] & X
\end{tikzcd}$$

commutes, as desired. $\square$

Let $p : E \to B$ be a map. Recall that the pullback of p along $f : X \to B$ is given explicitly as

$$f^*E \equiv \{(x,e) \in X \times E \mid f(x) = p(e)\}\,.$$

Let f^*p denote the map $\pi_X \restriction_{f^*E}$.

Proposition 1.4.4. *If p is a fibration, then so is f^*p.*

Lemma 1.4.5. *If p is locally trivial, then so is f^*p.*

Proof. Let $a \in X$. Since p is locally trivial by assumption, we can find a neighborhood U of $f(a)$ in B and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$. Observe that

$$(f^*p)^{-1}(f^{-1}(U)) = \{(x,e) \mid f(x) = p(e),\ f(x) \in U\} \subset f^{-1}(U) \times p^{-1}(U).$$

Further, we have a map $\psi : f^{-1}(U) \to p^{-1}(U) \to f^{-1}(U) \times F$ given by $(x,e) \mapsto (x, \pi_F(\varphi(e)))$. Define $\lambda : f^{-1}(U) \times F \to (f^*p)^{-1}(f^{-1}(U))$ by $(x,y) \mapsto \big(x, \varphi^{-1}(f(x),y)\big)$. Using the fact that

$$\begin{tikzcd}
p^{-1}(U) \arrow[r, "\varphi"] \arrow[dr, "p"'] & U \times F \arrow[d, "\pi_U"] \\
& U
\end{tikzcd}$$

commutes, it is easy to check that ψ and λ are inverses of each other. $\square$

1.5 Lecture 5

Theorem 1.5.1. *Let B be a cell complex and let $p : E \to B$ be locally trivial. Then p is a fibration.*

Proof. It suffices to prove the following claim:

> If $h : Z \to X \times [0,1]$ is locally trivial, $X = \bigcup_{i=0}^{n} X^i$ is a cell complex, and $\sigma_0 : X \times \{0\} \to Z$ satisfies $h \circ \sigma_0 = \mathrm{id}_{X \times \{0\}}$, then there is some map $\sigma : X \times [0,1] \to Z$ such that $\sigma_{X \times \{0\}} = \sigma_0$ and $h \circ \sigma = \mathrm{id}_{X \times [0,1]}$.

For, in this case, Lemma 1.4.5 implies that given any commutative square

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\hat{f}"] \arrow[d, hook] & E \arrow[d, "p"] \\
X \times [0,1] \arrow[r, "f"'] & B
\end{tikzcd},$$

we can find some σ such that

$$\begin{tikzcd}
& f^*E \arrow[r] \arrow[d, shift right] & E \arrow[d, "p"] \\
X \times \{0\} \arrow[ur, "\sigma_0"] \arrow[r, hook] & X \times [0,1] \arrow[r, "f"'] \arrow[u, bend right, "\sigma"'] & B
\end{tikzcd}$$

commutes where $\sigma_0(x,0) = \Big(x, 0, \hat{f}(x,0)\Big)$.

For induction, let us assume that our claim is true for each $X^0, X^1, \ldots, X^{n-1}$. We may assume, wlog, that $X = D^n$. It suffices to find a map $\tau : S^{n-1} \times [0,1] \to Z$ such that $h \circ \tau = \mathrm{id}_{S^{n-1} \times [0,1]}$ and

$$\begin{tikzcd}
& Z \arrow[d, "h"] & \\
D^n \times \{0\} \arrow[ur, "\sigma_0"] \arrow[r, hook] & D^n \times [0,1] & S^{n-1} \times [0,1] \arrow[l, hook'] \arrow[ul, dashed, "\tau"'] \\
& S^{n-1} \times \{0\} \arrow[ul, hook] \arrow[ur, hook'] &
\end{tikzcd}$$

commutes as there is a retraction

$$r : D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]\,.$$

To this end, fix a positive integer m. For each $j \in \{0, 1, \ldots, m\}$, let $a_j = \frac{j}{m}$ and let $I_j = [a_j, a_{j+1}]$. Since $D^n \times [0,1]$ is compact, by making m large enough, we can ensure that $h \restriction_{h^{-1}(I_{j_1} \times \cdots \times I_{j_{n+1}})}$ is trivial.

Claim. *$h \restriction_{h^{-1}(I_{j_1} \times \cdots \times I_{j_n} \times [0,1])}$ is also trivial.*

Proof. We may assume, wlog, that h is trivial on $h^{-1}(I_{j_1} \times \cdots \times I_{j_n} \times I_j)$ for any j. Let $k \in \{0, 1, \ldots, m-1\}$ and assume, for induction, that h is trivial on $h^{-1}(I_{j_1} \times \cdots \times I_{j_n} \times [0, a_k])$. Let $K = I_{j_1} \times \cdots \times I_{j_n}$ and $J = K \times [0, a_k]$ and $\tilde{J} = K \times [a_k, a_{k+1}]$. By assumption, there exists a homeomorphism $\varphi : h^{-1}(J) \xrightarrow{\cong} J \times F$ such that $\pi_J \circ \varphi = h$. Likewise, there exists a homeomorphism $\psi : h^{-1}(\tilde{J}) \xrightarrow{\cong} \tilde{J} \times F$ such that $\pi_{\tilde{J}} \circ \psi = h$. If $\psi = \varphi$ on $K \times \{a_k\}$, then $h^{-1}(J \cup \tilde{J}) \xrightarrow{\varphi \cup \psi} \left(J \cup \tilde{J}\right) \times F$ is a well-defined trivialization, in which case we're done. With this in mind, let

$$w = \left(\varphi \restriction_{K \times \{a_k\}} \circ \left(\psi \restriction_{K \times \{a_k\}}\right)^{-1}\right) \times \mathrm{id}_{[a_k, a_{k+1}]}\,.$$

Note that

$$\begin{array}{ccc} h^{-1}(\tilde{J}) & \xrightarrow{\psi} & \tilde{J} \times F \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle w \times \mathrm{id}_F} \\ \tilde{J} & \xleftarrow[\pi_{\tilde{J}}]{} & \tilde{J} \times F \end{array}$$

commutes and that $\gamma := ((w \times \mathrm{id}_F) \circ \psi)$ agrees with φ on $K \times \{a_k\}$. Hence we may take $\varphi \cup \gamma$ as our desired trivialization. $\square$

As a result, we may assume that h is trivial on its entire domain, i.e., that h is the projection

$$(D^n \times [0,1]) \times F \twoheadrightarrow D^n \times [0,1]\,.$$

Moreover, by induction, we can find a right inverse $\sigma : X^{n-1} \times [0,1] \to D^n \times [0,1]$ of $h : Z \to X^{n-1} \times [0,1]$ that extends $\sigma_0 \restriction_{X^{n-1} \times \{0\}}$. But X^{n-1} consists of all $(n-1)$-dimensional faces of the n-cube, and thus we have a map

$$\tau \equiv \sigma \restriction_{\partial I^n \times [0,1]} : S^{n-1} \times [0,1] \to (D^n \times [0,1]) \times F,$$

which has the form $(x,t) \mapsto (x, t, \tilde{\tau}(x,t))$. Further, σ_0 has the form $(x,0) \mapsto (x, 0, \tilde{\sigma}_0(x,0))$. Therefore, our desired map $\sigma : D^n \times [0,1] \to (D^n \times [0,1]) \times F$ is given by

$$\sigma(x,t) = (x, t, (\tilde{\sigma}_0 \cup \tilde{\tau})\,(r(x,t)))\,.$$

$\square$

2 Fiber bundles

Definition 2.0.1. A *topological group* G is a group such that both multiplication $G \times G \xrightarrow{\mu} G$ and inversion $G \xrightarrow{-^{-1}} G$ are continuous.

Definition 2.0.2 (Fiber bundle). Let G be a topological group.

1. A *fiber F of G* is a space equipped with a faithful (i.e., injective) group action $\rho : G \to \mathrm{Homeo}(F) \subset M(F,F)$.

2. An *atlas for the structure of a (fiber) bundle with group G and fiber F on a map $p : E \to B$* consists of
   (a) a family $(U_\alpha, h_\alpha)_{\alpha \in A}$ where each U_α is open and each h_α is a homeomorphism $p^{-1}(U_\alpha) \to U_\alpha \times F$ and
   (b) a family of continuous *transition functions* $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}_{\alpha,\beta \in A}$

   such that

   i $B = \bigcup_{\alpha \in A} U_\alpha$,
   ii $\pi_{U_\alpha} \circ h_\alpha = p \restriction_{p^{-1}(U_\alpha)}$, and
   iii $x \in U_\alpha \cap U_\beta \implies h_\beta \circ h_\alpha^{-1}(x, f) = (x, h_{\beta\alpha}(x) \cdot f)$
3. Two atlases are *compatible* if their union is an atlas.
4. A *bundle structure on B* is a maximal atlas on p.

Terminology. If B is equipped with a bundle structure, then we say that p is a (fiber) bundle.

Example 2.0.3.

1. The tangent bundle $\pi : TM \to M$ of a smooth n-manifold M is a bundle with group $\mathrm{GL}(n, \mathbb{R})$.

   Proof. Let (U, φ) be any coordinate chart for M with coordinate functions $\left(x^i\right)$. Define $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by
   $$v^i \frac{\partial}{\partial x^i}(p) \mapsto \left(p, \left(v^1, \dots, v^n\right)\right).$$
   It is clear that $\pi_U(h(p)) = \pi(c)$ for any $c \in \pi^{-1}(U)$. To see that h is a homeomorphism, note that the composite $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$ is given by
   $$v^i \frac{\partial}{\partial x^i}(p) \mapsto \left(x^1(p), \dots, x^n(p), v^1, \dots, v^n\right),$$
   the inverse of which is given by $\left(x^1, \dots, x^n, v^1, \dots, v^n\right) \mapsto v^i \frac{\partial}{\partial x^i}\left(\varphi^{-1}(x)\right)$. Therefore, $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h$ is given locally by
   $$\left(x^1, \dots, x^n, v^1, \dots, v^n\right) \mapsto \left(\tilde{x}^1(x), \dots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^j}(x) v^j, \dots, \frac{\partial \tilde{x}^n}{\partial x^j}(x) v^j\right),$$
   which is smooth. Thus, h is a diffeomorphism as the composite of two diffeomorphisms. In particular, h is a homeomorphism.

   It remains to describe the transition functions $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})\}$ for TM. Note that
   $$\begin{tikzcd} U_{\alpha\beta} \times \mathbb{R}^n \arrow[dr, "\pi_1"'] & \pi^{-1}(U_{\alpha\beta}) \arrow[l, "h_\alpha"'] \arrow[r, "h_\beta"] \arrow[d, "\pi"] & U_{\beta\alpha} \times \mathbb{R}^n \arrow[dl, "\pi_1"] \\ & U_{\alpha\beta} & \end{tikzcd}$$
   commutes. In particular, $\pi_1 \circ h_\beta \circ h_\alpha^{-1} = \pi_1$, which implies that $h_\beta \circ h_\alpha^{-1}(u, v) = (u, f(u, v))$ for some smooth map $f : U_{\alpha\beta} \times \mathbb{R}^n \to \mathbb{R}^n$. This must be a linear isomorphism when restricted to $\{u\} \times \mathbb{R}^n$ for any $u \in U_{\alpha\beta}$, which is uniquely determined by an element $h_{\beta\alpha}(u)$ of $\mathrm{GL}(n, \mathbb{R})$ (provided that we have fixed a basis of $\mathbb{R}^n$). Hence
   $$h_\beta \circ h_\alpha^{-1}(u, v) = (u, h_{\beta\alpha}(u) v).$$
   Since the map $h_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{R})$ is continuous, our proof is complete. □

2. Let $p : E \to B$ be any bundle with group $\{e\}$. Then p is the trivial bundle, i.e., is isomorphic to the projection map.

   Proof. We have that $h_\beta = h_\alpha$ on $p^{-1}(U_\alpha \cap U_\beta) = p^{-1}(U_\alpha) \cap p^{-1}(U_\beta)$, so that $h \equiv \bigcup_{\alpha\in A} h_\alpha$ is a well-defined homeomorphism $E \cong B \times F$. $\square$

2.1 Lecture 6

Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure with group G and fiber F on $p : E \to B$. Let $U = U_\alpha \cap U_\beta \cap U_\gamma$. Consider the commutative diagram

$$\begin{tikzcd}
& & p^{-1}(U) \arrow[drr, "h_\gamma"] & & \\
U \times F \arrow[urr, "h_\alpha^{-1}"] \arrow[r, "h_\alpha^{-1}"'] & p^{-1}(U) \arrow[r, "h_\beta"'] & U \times F \arrow[r, "h_\beta^{-1}"'] & p^{-1}(U) \arrow[r, "h_\gamma"'] & U \times F
\end{tikzcd}.$$

The bottom row is given by $(u, f) \mapsto (u, h_{\beta\alpha}(u) \cdot f) \mapsto (u, h_{\gamma\beta}(u) \cdot (h_{\beta\alpha}(u) \cdot f)) = (u, (h_{\gamma\beta}(u)h_{\beta\alpha}(u)) \cdot f)$, and the top composite is given by $(u, f) \mapsto (u, h_{\gamma\alpha}(u) \cdot f)$. It follows that

$$h_{\gamma\beta}(u)h_{\beta\alpha}(u) = h_{\gamma\alpha}(u)$$

for each $u \in U$. This property is known as the *cocycle condition.*

Theorem 2.1.1. *Let G be a topological group acting on a space F. Suppose that $\{U_\alpha\}$ is an open cover of B and $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is a family of continuous functions satisfying the cocycle condition. Then there exists a bundle $p : E \to B$ with group G, fiber F, and transition functions $h_{\beta\alpha}$.*

Proof sketch. Let $E = \coprod_\alpha U_\alpha \times F/\sim$ where $(u, f)_\alpha \sim (u, h_{\beta\alpha}(u) \cdot f)_\beta$. Define $p : E \to B$ by $(u, f) \mapsto u$. $\square$

Definition 2.1.2 (Bundle map). A *morphism of bundles p_1 and p_2 with group G and fiber F* is a commutative square of the form

$$\begin{tikzcd}
E_1 \arrow[r, "\hat{g}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\
B_1 \arrow[r, "g"'] & B_2
\end{tikzcd}.$$

Suppose that $(\hat{g}, g)$ is a bundle map $p_1 \to p_2$. Let $\{(U_\alpha, h_\alpha)\}$ and $\{(V_\beta, k_\beta)\}$ be bundle structures on B_2 and B_1, respectively. We have a commutative diagram

$$\begin{tikzcd}
\left(g^{-1}(U_\alpha) \cap V_\beta\right) \times F \arrow[r, "k_\beta^{-1}"] \arrow[dr, "\pi_1"'] \arrow[rrr, bend left, "d_{\alpha\beta}"] & p_1^{-1}\left(g^{-1}(U_\alpha) \cap V_\beta\right) \arrow[r, "\hat{g}"] \arrow[d] & p_2^{-1}(U_\alpha) \arrow[r, "h_\alpha"] \arrow[d] & U_\alpha \times F \arrow[dl, "\pi_1"] \\
& g^{-1}(U_\alpha) \cap V_\beta \arrow[r, "g"'] & U_\alpha &
\end{tikzcd},$$

so that $d_{\alpha\beta}(x, f) = (g(x), \lambda_{\alpha\beta}(x) \cdot f)$ for some continuous map $\lambda_{\alpha\beta} : g^{-1}(U_\alpha) \cap V_\beta \to G$. Letting $W = g^{-1}(U_\alpha \cap U_{\alpha'}) \cap (V_\beta \cap V_{\beta'})$, we have that

$$h_{\alpha'\alpha}(w)\lambda_{\alpha\beta}(w)k_{\beta\beta'}(w) = \lambda_{\alpha'\beta'}(w) \tag{†}$$

for every $w \in W$.

Exercise 2.1.3 (Pullback bundle). *Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure on $p : E \to B$ with group G and consider the pullback diagram*

$$\begin{tikzcd} g^*E \arrow[r] \arrow[d, "g^*p"'] & E \arrow[d, "p"] \\ X \arrow[r, "g"'] & B \end{tikzcd}.$$

Define $h'_{\beta\alpha} : g^{-1}(U_\alpha) \cap g^{-1}(U_\beta) \to G$ as the composite $h_{\beta\alpha} \circ g$ restricted to $g^{-1}(U_\alpha \cap U_\beta)$. Show that the family $\left\{h'_{\beta\alpha}\right\}$ induces a bundle structure on g^*p.

Theorem 2.1.4. *Every bundle map*

$$\begin{tikzcd} E_1 \arrow[r, "\hat{g}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"'] & B_2 \end{tikzcd}$$

factors as

$$\begin{tikzcd} E_1 \arrow[r, "\tau"] \arrow[d, "p_1"'] & g^*E_2 \arrow[r, "\bar{g}"] \arrow[d, "g^*p_2"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "\mathrm{id}_{B_1}"'] & B_1 \arrow[r, "g"'] & B_2 \end{tikzcd}$$

where $\tau(e) = (p_1(e), \hat{g}(e))$ for any $e \in E_1$.

2.2 Lecture 7

Note 2.2.1. If $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is a family of transition functions, then

$$h_{\alpha\beta}(x) = (h_{\beta\alpha}(x))^{-1}$$

for any $x \in U_\alpha \cap U_\beta$. In particular, $h_{\alpha\alpha}(x) = (h_{\alpha\alpha}(x))^{-1}$.

Theorem 2.2.2. *Any bundle map of the form*

$$\begin{tikzcd} E_1 \arrow[r, "\hat{g}"] \arrow[dr, "p_1"'] & E_2 \arrow[d, "p_2"] \\ & B \end{tikzcd}$$

is an isomorphism.

Proof. Note that

$$\begin{tikzcd} & p_2^{-1}(U_\alpha \cap U_\beta) \arrow[dl, "h_\beta"'] \arrow[dr, "h_\alpha"] & \\ (U_\alpha \cap U_\beta) \times F \arrow[dr] & p_1^{-1}(U_\alpha \cap U_\beta) \arrow[l, "k_\beta"'] \arrow[u, "\hat{g}"] \arrow[d] & (U_\alpha \cap U_\beta) \times F \arrow[l, "k_\alpha^{-1}"'] \arrow[dl] \\ & U_\alpha \cap U_\beta & \end{tikzcd}$$

commutes. We have that $h_\beta \circ \hat{g} \circ k_\alpha^{-1}(x,f) = (x, \lambda_{\beta\alpha}(x)\cdot f)$. Thus, if $h_\alpha(e) = (x,f)$, then $h_\alpha(\hat{g}(e)) = (x, \lambda_{\alpha\alpha}(x)\cdot d)$. Let

$$(\hat{g})^{-1}(e) = k_\alpha^{-1}\left(x, \lambda_{\alpha\alpha}(x)^{-1}\cdot f\right)$$

where $(x,f) = h_\alpha(e)$. If this is well-defined on E_2 (??), then it indeed equals the inverse of $\hat{g}$. Moreover, by Note 2.2.1, it is easy to check that $d_{\alpha'\beta'}(x)^{-1}$ satisfies (†), and thus it can be shown that $(\hat{g})^{-1}$ is a bundle map.

□

Corollary 2.2.3. *Every bundle $E \to X$ is isomorphic to the pullback of E by id_X.*

Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure with group G and fiber G on $p : E \to X$. In particular,

$$\begin{tikzcd} U_\alpha \times G \arrow[d, "\pi_1"'] & p^{-1}(U_\alpha) \arrow[l, "h_\alpha"'] \arrow[dl, "p"] \\ U_\alpha & \end{tikzcd}$$

commutes. Define the free action $E \times G \to E$ by

$$e\cdot g = h_\alpha^{-1}(h_\alpha(e)\cdot g).$$

where $p(e) \in U_\alpha$ and $(u,h)\cdot g \equiv (u, hg)$. This is well-defined because it does not depend on our choice of α. Indeed, suppose that $p(e)$ also belongs to U_β. We have that $h_\alpha(e) = (p(e), h)$ and $h_\beta(e) = (p(e), h')$ for some $h, h' \in G$. Then $e\cdot g = h_\alpha^{-1}(p(e), hg)$, and we must show that this equals $h_\beta^{-1}(p(e), h'g)$. Note that $h_\beta(e\cdot g) = (p(e), h_{\beta\alpha}(p(e))hg)$. But

$$(p(e), h_{\beta\alpha}(p(e))h) = h_\beta\left(h_\alpha^{-1}(p(e), h)\right) = (p(e), h'),$$

so that $h_{\beta\alpha}(p(e))h = h'$, and thus $h_\beta(e\cdot g) = (p(e), h'g)$, as desired.

Note 2.2.4. ${}^{E}\!/_{G} \cong \{p^{-1}(x) \mid x \in X\} \cong X$.

Definition 2.2.5 (Balanced product). Let F be a space. The *balanced product $E \times_G F$ of E and F* is the quotient space ${}^{E\times F}\!/_{\sim}$ where

$$(e,f) \sim (eg, g^{-1}f)$$

for any $e \in E$ and $f \in F$.

By the universal property of the quotient space, there is a unique map $\bar{p}$ such that

$$\begin{tikzcd} E \times F \arrow[r, two heads] \arrow[d, "p\circ\pi_E"'] & E \times_G F \arrow[dl, "\bar{p}"] \\ X & \end{tikzcd} . \qquad (\star)$$

Notation. Let $\mathcal{B}(X, G, \rho, F)$ denote the set of all isomorphism classes of bundles over X with group G and fiber F.

Lemma 2.2.6. *$\bar{p}$ is a bundle with group G and fiber F.*

Proof. As $(g,f) \sim (e_G, gf)$, we see that $(U \times G) \times_G F \cong U \times F$. Thus, we can endow $\bar{p}$ with local trivializations and transition functions that are exactly similar to those for p. □

Proposition 2.2.7. *The function $p \mapsto \bar{p}$ defines a set isomorphism $\mathcal{B}(X, G, \rho, G) \xrightarrow{\cong} \mathcal{B}(X, G, \rho, F)$.*

Let $p_1 : E \to B_1$ and $p_2 : E \to B_2$ be bundles. Let $e_1 \in E_1$, $e_2 \in E_2$, and $b_1 \in B_1$.

Question. Can we find a bundle map

$$\begin{tikzcd} E_1 \arrow[r, dashed] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, dashed] & B_2 \end{tikzcd}$$

such that $e_1 \mapsto e_2$ and $e_1 \mapsto b_1$?

Define the action $G \times E_2 \to E_2$ by $g * e_2 = e_2 \cdot g^{-1}$. From this, we obtain a bundle

$$\psi : \underbrace{E_1 \times_G E_2}_{(E_1 \times E_2)/G} \to E_1 \times_G \mathsf{pt} \cong B_1$$

with fiber E_2.

Lemma 2.2.8. *There is a one-to-one correspondence between bundle maps $p_1 \to p_2$ and sections of ψ.*

Proof. Suppose that σ is a section of ψ. As G acts freely on $E_1 \times E_2$, we see that for any $e \in E_1$, there exists a unique $\tilde{e}$ such that $\sigma(p(e)) = [(e, \tilde{e})]$. Define $\hat{g} : E_1 \to E_2$ by $e \mapsto \tilde{e}$. This respects the action of G and thus must be a bundle map. □

Now, let $A \subset B_1$ and suppose that

$$\begin{tikzcd} p_1^{-1}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha"] & E_2 \arrow[d, "p_2"] \\ A \arrow[r] & B_2 \end{tikzcd}$$

is a bundle map. Then α extends when ??. Also, the corresponding section

$$\sigma : A \to p^{-1}(A) \times_G E_2 \subset E_1 \times_G E_2$$

extends.

Definition 2.2.9 (Principal bundle). Let G be a topological group. A *principal G-bundle* is a fiber bundle with group G and fiber G with G acting on itself by left translation.

Theorem 2.2.10. *Let f and g be homotopic maps $X \to Y$. Let $p : E \to Y$ be any bundle with group G and fiber F. Then $f^* p \cong g^* p$.*

2.3 Lecture 8

Before proving this, we wish to determine when, given any two bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ and any map $g : B_1 \to B_2$, we can find a map $\hat{g}$ such that

$$\begin{tikzcd} E_1 \arrow[r, "\hat{g}"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "g"'] & B_2 \end{tikzcd}$$

commutes.

Define the *diagonal action* ΔG of G on $E_1 \times E_2$ by

$$(e_1, e_2) \cdot h = (e_1 \cdot h, e_2 \cdot h)\,,$$

so that $E_1 \times_G E_2 = {E_1 \times E_2}/{\Delta G}$. By ($\star$), we can find a unique map τ such that

$$\begin{tikzcd} E_1 \times_G E_2 \arrow[d] \arrow[dr, "\tau"] & \\ B_1 & B_1 \times B_2 \arrow[l, "\pi_1"] \end{tikzcd}$$

commutes.

Exercise 2.3.1. *Show that $\hat{g}$ exists if and only if there is some $\lambda : B_1 \to E_1 \times_G E_2$ such that $\tau(\lambda(b_1)) = (b_1, g(b_1))$.*

Proof.
($\Longleftarrow$) As G acts freely on $E_1 \times E_2$, we see that $(e, e') \sim (e, e'') \implies e' = e''$ for any $e', e'' \in E_2$. Hence for any $e \in E_1$, there exists a unique $\hat{e} \in E_2$ such that $\lambda(p_1(e)) = [(e, \hat{e})]$. Let $\hat{g}(e) = \hat{e}$. Then $\hat{g}$ is clearly continuous and G-equivariant, and thus $(\hat{g}, g)$ is a bundle map.

($\Longrightarrow$) Consider the homeomorphism $\varphi : B_1 \xrightarrow{\cong} {E_1}/{G}$ with $\varphi(b) = p_1^{-1}(b)$. Let $b \in B_1$. Let $\varphi(b) = [e]$. Define $\lambda : B_1 \to E_1 \times_G E_2$ by $\lambda(b) = [(e, \hat{g}(e))]$. Since $\hat{g}$ is G-equivariant, we see that λ is well-defined. Further, λ is continuous as the quotient of the map

$$f : E_1 \to E_1 \times E_2, \quad f(x) = (x, \hat{g}(x))$$

by G. Finally, it is easy to check that $\tau(\lambda(b_1)) = (b_1, g(b_1))$ for any $b_1 \in B_1$. $\square$

Lemma 2.3.2. *τ is locally trivial, hence a fibration.*

Proof. Locally, we have that $E_1 \cong U \times G$ and $E_2 \cong V \times G$, so that $E_1 \times E_2 \cong U \times V \times G \times G$. It follows that, locally, $E_1 \times_G E_2 \cong U_1 \times U_2 \times {G \times G}/{\Delta G}$ where $\Delta G \equiv \{(g, g) \mid g \in G\}$. $\square$

Remark 2.3.3. In fact, τ is a bundle with fiber ${G \times G}/{\Delta G} \cong G$.

Proof of Theorem 2.2.10. Due to Proposition 2.2.7, we may assume that p is a principal G-bundle. By assumption, there is some homotopy $H : X \times I \to Y$ from f to g. Let $\omega = H^* p$. Then

$$f^* p = \omega \restriction_{\omega^{-1}(X \times \{0\})} : \omega^{-1}(X \times \{0\}) \to X \times \{0\} \cong X$$
$$g^* p = \omega \restriction_{\omega^{-1}(X \times \{1\})} : \omega^{-1}(X \times \{1\}) \to X \times \{1\} \cong X.$$

Therefore, it suffices to show that $f^* p \times \mathrm{id}_I \cong \omega$ such that the diagram

$$\begin{tikzcd} f^* E \times I \arrow[r, "\cong"] \arrow[d, "f^* p \times \mathrm{id}_I"'] & H^* E \arrow[r] \arrow[d, "\omega"] & E \arrow[d, "p"] \\ X \times I \arrow[r, equal] & X \times I \arrow[r, "H"'] & Y \end{tikzcd}$$

commutes. For, in this case, our isomorphism restricts over $X \times \{1\}$, i.e., $g^* p = \omega \restriction_{X \times \{1\}} \cong f^* p$. It thus suffices to exhibit a bundle map $f^* p \times I \to \omega$ over $\mathrm{id}_{X \times I}$ that equals the identity over $\omega \restriction_{X \times \{0\}} = f^* p$.

Remark 2.3.4. It is easy to show that there is some bundle map $f^*p \times \mathrm{id}_I \to \omega$. Indeed, by the homotopy lifting property, we obtain a section σ fitting into the commutative diagram

$$\begin{tikzcd} & (f^*E \times I) \times_G H^*E \arrow[d, shift right] \\ X \times \{0\} \arrow[ur, "\lambda_0"] \arrow[r] & X \times I \arrow[u, dashed, shift right, "\sigma"'] \end{tikzcd},$$

in which case we obtain our desired map by Lemma 2.2.8. As mentioned, however, we want a bundle map that equals the identity over f^*p.

To get such a map, we must find a section λ such that

$$\begin{tikzcd} & (f^*E \times I) \times_G H^*E \arrow[d, shift right] \arrow[dr, "\tau"] & \\ X \times \{0\} \arrow[ur, "\lambda_0"] \arrow[r, hook] & X \times I \arrow[u, dashed, shift right, "\lambda"'] \arrow[r, "\Delta"'] & (X \times I) \times (X \times I) \end{tikzcd}$$

commutes. But λ must exist since τ is a fibration by virtue of Lemma 2.3.2.

□

Corollary 2.3.5. *Any bundle over a contractible space B is trivial.*

Proof. Let $i : \mathsf{pt} \to B$ and $\pi : B \to \mathsf{pt}$ denote inclusion and projection, respectively. Then

$$\begin{aligned} p &\cong (\mathrm{id})^* p \\ &\cong (i\pi)^* p \\ &\cong \pi^* \underbrace{i^* p}_{\text{trivial}}, \end{aligned}$$

which is trivial since the pullback of a trivial bundle is trivial. □

Corollary 2.3.6. *Every bundle p over $X \times I$ is isomorphic to $\left(p \restriction_{p^{-1}(X \times \{0\})}\right) \times \mathrm{id}_I$.*

Example 2.3.7. Consider $S^1 \subset \mathbb{R}^2$ with center the origin. Let $p : E \to S^1$ be a bundle with group G and fiber F. Cover S^1 with the open intervals $I_1 \coloneqq S^1 \setminus \{-1\}$ and $I_2 \coloneqq S^1 \setminus \{1\}$. We may assume that $F = p^{-1}(-1)$. Then $E = E_1 \cup E_2$ where $E_i \cong I_i \times F$ via, say, φ_i for each $i = 1, 2$. By Corollary 2.3.6, we see that

$$\varphi_1 \restriction_{\varphi_1^{-1}(\{1\} \times F)} = \varphi_2 \restriction_{\varphi_2^{-1}(\{-1\} \times F)} = \mathrm{id}_F .$$

Moreover, the transition function $\varphi_2^{-1} \circ \varphi_1 \restriction_{p^{-1}(1)} : F \to F$ is given by multiplication by some $g \in G$. Hence the map $G \to \mathcal{B}\left(S^1, G, F\right)$ is surjective. In fact, it can be shown that this maps descends to an isomorphism

$$\pi_0(G) \cong {}^{G}\!/_{G_0} \xrightarrow{\cong} \mathcal{B}\left(S^1, G, F\right)$$

where G_0 denotes the connected component of e_G.

For example, if $G = F = \mathrm{GL}(n, \mathbb{R})$, then $\pi_0(G)$ consists of the set of matrices with positive determinant and the set of matrices with negative determinant, so that $\mathcal{B}\left(S^1, G, F\right) \cong \mathbb{Z}_2$.

Example 2.3.8. The set $\mathcal{B}\left(S^2, G, F\right)$ is isomorphic to the set of homotopy classes of maps $S^1 \to G$, As it turns out, we can ignore base points, so that $\mathcal{B}\left(S^2, G, F\right) \cong \pi_1(G)$.

For example, if $G = F = \mathrm{SO}(2)$, then $G \cong S^1$, so that $\mathcal{B}\left(S^2, G, F\right) \cong \mathbb{Z}$.

2.4 Lecture 9

Theorem 2.4.1. *Let X be a cell complex with $\dim X \leq n$. Let $A \subset X$ be a subcomplex. Let $p : E \to X$ be a bundle with fiber F such that $\pi_i(F, f) = 0$ for each $i \leq n-1$. Suppose that $\sigma_0 : A \to E$ satisfies $p \circ \sigma_0(a) = a$ for each $a \in A$. Then σ_0 extends to a section $\sigma : X \to E$ of p.*

$$\begin{tikzcd}
 & E \arrow[d, "p"'] \\
A \arrow[ur, "\sigma_0"] \arrow[r, hook] & X \arrow[u, dashed, bend right, "\sigma"']
\end{tikzcd}$$

Proof. First, assume that X is a regular complex. Since X is finite, we may assume that $X = A \cup_{S^{k-1}} D^k$ where $k \leq n$. Further, we may assume, wlog, that $X = D^k$. Thus, we must find a section σ such that

$$\begin{tikzcd}
 & E \arrow[d, "p"'] \\
S^{k-1} \arrow[ur, "\sigma_0\restriction_{S^{k-1}}"] \arrow[r, hook] & D^k \arrow[u, dashed, bend right, "\sigma"']
\end{tikzcd}$$

commutes. Since D^k is contractible, we have that $E \cong D^k \times F$. Then $\sigma_0(x) = (x, \tilde{\sigma}_0(x))$ for each $x \in S^{k-1}$. But $\tilde{\sigma}_0(x) : S^{k-1} \to F$ extends to a map $\tilde{\sigma} : D^k \to F$ because $\pi_{k-1}(F) = 0$. Hence we can take σ to be the map defined by $x \mapsto (x, \tilde{\sigma}(x))$.

Next, drop the assumption that X is regular. Using Exercise 1.1.9, we get a homotopy equivalence

$$\begin{tikzcd}
(X, A) \arrow[r, bend left, "h"] & \underbrace{(\overline{X}, \overline{A})}_{\text{regular}} \arrow[l, bend left, "g"]
\end{tikzcd}$$

of pairs. Define $\overline{A} \to g^*E$ by $\bar{\sigma}_0(a) = (a, \sigma_0(g(a)))$. By our preceding discussion, this extends to a section $\bar{\sigma}$ on $\overline{X}$.[1] We wish to find σ such that

$$\begin{tikzcd}
g^*E \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
\overline{X} \arrow[u, bend right, "\bar{\sigma}"'] \arrow[r, "g"'] & X \arrow[u, dashed, bend left, "\sigma"] \\
\overline{A} \arrow[u, hook] \arrow[r, "g"'] \arrow[uu, bend left=60, "\bar{\sigma}_0"] & A \arrow[u, hook] \arrow[uu, bend right=60, "\sigma_0"']
\end{tikzcd}$$

commutes. But since $p \cong h^*g^*p$, we have a commutative diagram

$$\begin{tikzcd}
g^*E \arrow[d, "g^*p"] & h^*g^*E \arrow[l] \arrow[r, "\cong"] \arrow[d, "h^*g^*p"'] & E \arrow[dl, "p"] \\
\overline{X} \arrow[u, bend left, "\bar{\sigma}"] & X \arrow[l, "h"] &
\end{tikzcd},$$

from which we obtain our desired section σ. ∎

Notation. $[X, Y] \coloneqq$ (homotopy classes of maps $X \to Y$).

Corollary 2.4.2. *Let $p : E \to B$ be a principal G-bundle and suppose that $\pi_i(E) = 0$ for any $i \leq n-1$. The function $\chi_X : [X, B] \to \mathcal{B}(X, G, G)$ given by $f \mapsto f^*p$ is bijective.*

[1] As $\dim \overline{X} > \dim X$, we tacitly rely on the fact that $\pi_i(F)$ is trivial for large enough i.

Proof.

Surjective: Let $p_1 : E_1 \to X$ be a bundle. Due to Theorem 2.1.4, it suffices to find a bundle map $\left(\hat{f}, f\right)$ such that

$$\begin{tikzcd} E_1 \arrow[r, dashed, "\hat{f}"] \arrow[d, "p_1"'] & E \arrow[d] \\ X \arrow[r, dashed, "f"'] & B \end{tikzcd}$$

commutes. Such a map can be found precisely when there exists a section of the bundle $E_1 \times_G E \to X$, which holds by applying Theorem 2.4.1 to the case where $A = \emptyset$.

Injective: Suppose that $\chi_X(f) = \chi_X(g)$. We must show that $f \simeq g$, i.e., that there is some bundle map $\left(\hat{H}, H\right)$ such that

$$\begin{tikzcd} & f^*p \times \{0,1\} \arrow[r, hook] \arrow[dl, "\cong"'] \arrow[rrr, bend left] & f^*p \times I \arrow[r, dashed, "\hat{H}"'] \arrow[d] & E \arrow[d, "p"] \\ f^*p \cup g^*p \arrow[dr] & & X \times I \arrow[r, dashed, "H"'] & B \\ & X \times \{0,1\} \arrow[ur] & & \end{tikzcd}$$

commutes. This is equivalent to finding a section λ such that

$$\begin{tikzcd} (X \times \{0,1\}) \times B & (f^*p \times I) \times_G E \arrow[l, "\tau"'] \arrow[d] \\ X \times \{0,1\} \arrow[u, "\gamma"] \arrow[ur, "\lambda_0"] \arrow[r, hook] & X \times I \arrow[u, dashed, bend right, "\lambda"'] \end{tikzcd}$$

commutes where

$$\gamma\left(x,t\right) = \begin{cases} (x, t, f(x)) & t = 0 \\ (x, t, g(x)) & t = 1 \end{cases}.$$

But this exists by Theorem 2.4.1 because $\pi_i(E) = 0$ by assumption. $\square$

Definition 2.4.3 (Classifying space). A *classifying space for principal G-bundles* is a space B such that χ_X is bijective for every cell complex X.

Example 2.4.4. Let $G = \{\pm 1\}$. Then any principal G-bundle over X is a two-fold covering space of X, i.e., a subgroup of index two in $\pi(X)$, i.e., a nontrivial homomorphism $\pi_1 X \to G$.

For example, let $\{U_i\}$ denote the usual open covering of $\mathbb{RP}^n = {S^n}/{G}$. Let $\pi : S^n \to \mathbb{RP}^n$ denote the projection map. We have that $\pi^{-1}(U_i) \cong U_i \times G$. Indeed, define $h_i : \pi^{-1}(U_i) \to U_i \times G$ by

$$(x_0, \ldots, x_n) \mapsto \left([x_0, \ldots, x_n], \frac{x_i}{|x_i|}\right),$$

the inverse of which is given by

$$(y_0, \ldots y_n) \leftarrow\!\shortmid ([x_0, \ldots, x_n], \epsilon)$$

$$y_k \equiv \epsilon x_k \cdot \frac{|x_i|}{x_i}.$$

Note that any transition function $h_{ji} : U_i \cap U_j \to G$ is given by $h_{ji}(x) = -1$.

Using the fact that π_1 is the abelianization of H_1 along with the universal coefficient theorem for cohomology, one can prove the following.

Proposition 2.4.5. $\mathcal{B}(X, \mathbb{Z}_2, F) \cong [X, \mathbb{RP}^n] \cong \mathrm{Hom}(\pi_1(X), \mathbb{Z}_2) \cong H^1(X; \mathbb{Z}_2)$.

Let $w_1 \in H^1(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ be nonzero. Let $p_1 : E \to X$ be a $\mathbb{Z}_2$-bundle. We call $w_1(p_1) := f^* w_1 \in H^1(X; \mathbb{Z}_2)$ the *first Stiefel-Whitney class of p.*

2.5 Lecture 10

Example 2.5.1. Let $n \in \mathbb{N}$. Recall that $\mathbb{CP}^n$, by definition, consists of all the complex lines in $\mathbb{C}^{n+1}$. Let $G = S^1$. Then G acts on $\mathbb{C}^{n+1}$ by $g \cdot (z_0, \dots, z_n) = (gz_0, \dots, gz_n)$. We have that $\mathbb{CP}^n \cong {S^{2n+1}}/{\sim}$ where $z \sim \zeta \cdot z$ for any $\zeta \in S^1$. Consider the projection map $\pi : S^{2n+1} \twoheadrightarrow \mathbb{CP}^n$. For each $i \in \{0, \dots, n\}$, let $H_i = \{z \in \mathbb{CP}^n \mid z_i = 0\} \cong \mathbb{CP}^{n-1}$ and let $U_i = \mathbb{CP}^n \setminus H_i$. Then the U_i form an open cover of $\mathbb{CP}^n$. Define $h_i : \pi^{-1}(U_i) \to U_i \times S^1$ by $(z_0, \dots, z_n) \mapsto \left([z_0, \dots, z_n], \frac{z_i}{|z_i|}\right)$.

Exercise 2.5.2.

1. *Prove that h_i is a homeomorphism.*
2. *Find the transition functions $h_{ij} : U_j \cap U_i \to S^1$.*

Proof.

1. It is obvious that h_i is continuous. Define $g_i : U_i \times S^1 \to \pi^{-1}(U_i)$ by
$$([z_0, \dots, z_n], \epsilon) \mapsto (y_0, \dots, y_n)$$
$$y_k \equiv \epsilon z_k \cdot \frac{|z_i|}{z_i}, \; k = 0, \dots, n.$$
It is easy to check that this is well-defined and that g_i is the inverse of h_i. It remains to show that g_i is continuous. Consider the quotient map $q := \pi \times \mathrm{id}_{S^1} : S^{2n+1} \times S^1 \to \mathbb{CP}^n \times S^1$. Let $\widetilde{U}_i = \left\{z \in S^{2n+1} \mid z_i \neq 0\right\}$. Note that $g_i \circ q \restriction_{\widetilde{U}_i \times S^1}$ is clearly continuous. But $\widetilde{U}_i \times S^1$ is both open in $S^{2n+1} \times S^1$ and saturated with respect to q. Hence $\restriction_{\widetilde{U}_i \times S^1}$ is a quotient map, so that g_i is continuous.

2. Note that
$$h_i \circ h_j^{-1}([z_0, \dots, z_n], \epsilon) = \left([z_0, \dots, z_n], \epsilon \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}\right)$$
for any $[z_0, \dots, z_n] \in U_i \cap U_j$. This implies that
$$h_{ij}([z_0, \dots, z_n]) = \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}.$$

$\square$

It follows that π is a principal S^1-bundle. Since each homotopy group $\pi_i\left(S^{2n+1}\right)$ is trivial, Corollary 2.4.2 implies that
$$\mathcal{B}\left(X, S^1, F\right) \cong [X, \mathbb{CP}^n],$$
which for large enough n, is isomorphic to $[X, \mathbb{CP}^\infty]$ where X denotes and any cell complex and
$$\mathbb{CP}^\infty \equiv \bigcup_{k \in \mathbb{N}} \mathbb{CP}^k$$
equipped with the weak topology.

Definition 2.5.3. An *Eilenberg-MacLane space of type* $K(G,n)$ is a space satisfying

$$\begin{cases} \pi_i K = 0 & i \neq n \\ \pi_i K \cong G & i = n \end{cases}.$$

Theorem 2.5.4. *If X is a cell complex, then $[X, K(G,n)] \cong H^n(X;G)$.*

Example 2.5.5. By inspecting the long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \pi_2\left(S^{2n+1}\right) \arrow[r] & \pi_2\left(\mathbb{CP}^n\right) \arrow[dll] & \\ \underbrace{\pi_1\left(S^1\right)}_{\mathbb{Z}} \arrow[r] & \pi_1\left(S^{2n+1}\right) \arrow[r] & \cdots \end{tikzcd},$$

we see that $\mathbb{CP}^n$ is an Eilenberg-MacLane space of type $K(\mathbb{Z},2)$. Moreover, there is a commutative triangle

$$\begin{tikzcd} \mathbb{CP}^\infty & \mathbb{CP}^n \arrow[l, hook'] \\ S^i \arrow[u] \arrow[ur] & \end{tikzcd}$$

for any $i \in \mathbb{N}$. Thus, $\pi_i(\mathbb{CP}^\infty) = \pi_i(\mathbb{CP}^n)$ when n is large enough. This means that $\mathbb{CP}^\infty$ is also an Eilenberg-MacLane space of type $K(\mathbb{Z},2)$. By Theorem 2.5.4, we have that

$$\mathcal{B}\left(X, S^1, F\right) \cong H^2(X;\mathbb{Z})$$

whenever X is a cell complex.

For us, a CW complex refers to a cell complex X for which there may be infinitely many attaching maps of any dimension. In this name, "C" stands for the property *closure-finite*, i.e., every open cell e^i is contained in a finite subcomplex of X. Further, "W" stands for the weak topology, with which X is equipped.

Remark 2.5.6. Each of our results holds even if we assume that a certain space is merely a CW complex rather than a cell complex.

We want to ensure that a classifying space for any topological group G exists. There are at least two famous ways of finding a classifying space for G.

Theorem 2.5.7 (Milnor construction). *There is some functor* **TopGrp** $\to$ **PrinBund** *that maps each topological group G to a principal G-bundle*

$$E_G \xrightarrow{p_G} B_G$$

such that B_G is a CW complex and $\pi_i(E_G) = 0$.

This means that B_G is a classifying space for principal G-bundles. Moreover, by applying our LES on homotopy groups to p_G, we see that $\pi_i(B_G) \cong \pi_{i-1}(G)$.

Our next method is slightly less powerful than Theorem 2.5.7 in that it produces a classifying space for principal G-bundles only over pointed connected (equivalently, path connected) CW complexes. At the same time, it produces "classifying" objects in settings other than that of principal G-bundles.

Theorem 2.5.8 (Brown representability). *Consider the homotopy category* $\mathrm{Ho}(\mathbf{CW}_*^{conn})$ *of pointed connected CW complexes. Let F be a functor* $\mathrm{Ho}(\mathbf{CW}_*^{conn})^{\mathrm{op}} \to \mathbf{Set}_*$ *with the following properties.*

(i) (Wedge axiom) F takes coproducts in $\mathrm{Ho}(\mathbf{CW}_*^{conn})$ *to products in* $\mathbf{Set}_*$*, i.e.,*

$$F\left(\bigvee_\alpha X_\alpha\right) \cong \prod_\alpha F(X_\alpha).$$

(ii) (Mayer-Vietoris axiom) F takes weak pushouts to weak pullbacks, i.e., the universal morphism

$$F(B \cup_A C) \to F(B) \times_{F(A)} F(C)$$

is a surjection (or split epimorphism) for any two cofibrations $A \to B$ and $A \to C$ in $\mathbf{CW}_*^{conn}$*.*

Then F is representable.[2]

By the Yoneda lemma, this means that there exists a pointed connected CW complex B along with an element $b \in F(B)$ such that the set map $[X, B] \to F(X)$ given by $g \mapsto F(g)(b)$ is a natural bijection in X. In particular, Theorem 2.5.8 applied to the functor taking any pointed connected CW complex to the set of all pointed principal G-bundles over X makes the object (B'_G, b) representing F a classifying space for G.

Remark 2.5.9. The converse of Theorem 2.5.8 is true because any representable contravariant functor takes colimits to limits.

Proof sketch of Theorem 2.5.8.
We say that a pointed connected CW complex B along with an element $b \in F(B)$ *spherically represents* F if the set map $\nu_X : [X, B] \to F(X)$ given by $g \mapsto F(g)(b)$ is a natural bijection in $X \in \{S^n \mid n \in \mathbb{Z}_{\geq 1}\}$.

Suppose that (B, b) and (B', b') are two objects spherically representing F. Let $f : B \to B'$ be a map such that $F(f)(b') = b$. Then f induces a weak homotopy equivalence. By Theorem 1.2.8, f must be a homotopy equivalence.

Now, let X be a pointed CW complex and $x \in F(X)$. It is known that there exist an object (B, b) spherically representing F and a map

$$\varphi : X \to B, \qquad F(\varphi)(b) = x. \tag{$\bullet$}$$

Therefore, it suffices to prove the following assertion.

Claim. *Any object (B, b) spherically representing F represents F.*

Proof. We must show that ν_X is a bijection. In the interest of space, let us prove just that it is surjective. Let $x \in F(X)$ and consider the coproduct

$$X \xrightarrow{i_1} X \vee B \xleftarrow{i_2} B\ .$$

By the wedge axiom, we have that $F(X \vee B) \cong F(X) \times F(B)$ with $F(i_1) \cong \pi_1$ and $F(i_2) \cong \pi_2$. Thus, we have an element $(x, b) \in F(X \vee B)$ such that $F(i_1)(x, b) = x$ and $F(i_2)(x, b) = b$. Thanks to ($\bullet$), we can find

[2] According to a certain MathOverflow answer, Theorem 2.5.8 holds when F is instead a functor $\mathbf{CW} \to \mathbf{Ab}$.

an object $\left(\widehat{B},\hat{b}\right)$ spherically representing F along with a map $\varphi : X \vee B \to \widehat{B}$ such that $F(\varphi)\left(\hat{b}\right) = (x,b)$. It follows that

$$F(\varphi \circ i_1)\left(\hat{b}\right) = x$$
$$F(\varphi \circ i_2)\left(\hat{b}\right) = b.$$

Hence $\varphi \circ i_2$ is a homotopy equivalence $B \to \widehat{B}$, with homotopy inverse, say, η. This implies that

$$F(\eta \circ \varphi \circ i_2)(b) = x,$$

so that ν_X is surjective. □

□

Let us turn to the question of uniqueness of a classifying space, having just considered the question of existence.

Lemma 2.5.10. *Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be classifying spaces for principal G-bundles. Then $B_1 \simeq B_2$.*

Proof. By Corollary 2.4.2, there is some map $f : B_1 \to B_2$ such that $f^*p_2 \cong p_1$. Likewise, there is some map $g : B_2 \to B_1$ such that $g^*p_1 \cong p_2$. Therefore,

$$\begin{aligned}(f \circ g)^* p_2 &\cong g^* f^* p_2 \\ &\cong g^* p_1 \\ &\cong p_2 \\ &\cong \mathrm{id}_{B_2}^* p_2.\end{aligned}$$

Therefore, $f \circ g \simeq \mathrm{id}_{B_2}$. Similarly, $g \circ f \simeq \mathrm{id}_{B_1}$. □

In particular, $B_G \simeq B'_G$.

Example 2.5.11. $B_{S^1} = \mathbb{CP}^\infty$.

Let $H \leq G$. Consider the commutative square

$$\begin{tikzcd} E_G \arrow[r, two heads, "q"] \arrow[d, "p_G"'] & E_G/H \arrow[d, "r"] \\ B_G \arrow[r, equal] & E_G/G \end{tikzcd}.$$

Note that, locally, r looks like the trivial map with fiber G/H. Thus, q locally looks like the map

$$U \times G \to U \times G/H.$$

This shows that if the natural projection $G \to G/H$ is a principal H-bundle, then so is q. In this case, we have that $B_H \simeq E_G/H$ by Corollary 2.4.2 together with Lemma 2.5.10.

Theorem 2.5.12. *If G is a Lie group and H is a closed subgroup of G, then the natural projection $G \to {}^{G}\!/_{H}$ is a principal H-bundle.*

Definition 2.5.13. The *orthogonal group* $\mathrm{O}(n, \mathbb{R})$ is the group of $n \times n$ real matrices A such that $AA^t = A^tA = I_n$, equivalently, $Av \bullet Aw = v \bullet w$ for any $v, w \in \mathbb{R}^n$. We call such an A *orthogonal*.

In particular, if A is orthogonal, then $\|Av\| = \|v\|$ for any $v \in \mathbb{R}^n$.

Example 2.5.14. The orthogonal group $\mathrm{O}(n, \mathbb{R})$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ because $\mathrm{O}(n, \mathbb{R}) = f^{-1}(I_n)$ where $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is given by $X \mapsto XX^t$. Let $\gamma : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{O}(n, \mathbb{R})$ denote the map given by the Gram-Schmidt procedure. Let $i : \mathrm{O}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ denote the inclusion map. Then γ and i are homotopy inverses of each other, so that

$$\mathrm{GL}(n, \mathbb{R}) \simeq \mathrm{O}(n, \mathbb{R}).$$

Since $\pi : \mathrm{GL}(n, \mathbb{R}) \to \underbrace{{}^{\mathrm{GL}(n, \mathbb{R})}\!/_{\mathrm{O}(n, \mathbb{R})}}_{M}$ is an $\mathrm{O}(n, \mathbb{R})$-bundle by Theorem 2.5.12, our LES on homotopy groups applied to π shows that $\pi_i(M) = 0$ for each $i \in \mathbb{N}$. Further, our LES applied to the M-bundle $r : B_{\mathrm{O}(n,\mathbb{R})} \to B_{\mathrm{GL}(n,\mathbb{R})}$ shows that

$$\pi_i\left(B_{\mathrm{O}(n,\mathbb{R})}\right) \cong \pi_i\left(B_{\mathrm{GL}(n,\mathbb{R})}\right)$$

for each i. By Theorem 1.2.8, it follows that

$$B_{\mathrm{O}(n,\mathbb{R})} \simeq B_{\mathrm{GL}(n,\mathbb{R})}.$$

An exactly similar argument proves that $B_{\mathrm{U}(n,\mathbb{C})} \simeq B_{\mathrm{GL}(n,\mathbb{C})}$.

Eventually, we want to describe $H^*(B_G)$. This will lead us to the notion of a spectral sequence.

2.6 Lecture 11

Before moving to spectral sequences, let us look at a couple more examples of fiber bundles.

Example 2.6.1. Let $\{e_i\}_{1 \leq i \leq n}$ denote the standard basis of $\mathbb{R}^n$. Consider the map $\rho : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^n \setminus \{0\}$ given by $A \mapsto Ae_n$ and its restriction $\tau : \mathrm{O}(n, \mathbb{R}) \to S^{n-1}$. Note that $\rho^{-1}(e_n)$ consists of all $n \times n$ matrices of the form

$$\left(\begin{array}{c|c} B & 0 \\ \hline * & 1 \end{array}\right)$$

where B denotes an invertible $(n-1) \times (n-1)$ matrix. This means that $\rho^{-1}(e_n) \simeq \mathrm{GL}(n-1, \mathbb{R})$. Similarly, we see that $\tau^{-1}(e_n) \simeq \mathrm{O}(n-1, \mathbb{R})$. Moreover, both ρ and τ are locally trivial. In particular, this yields a LES

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_{i+1}(S^{n=1}) \arrow[dll] \\
\pi_i(\mathrm{O}(n-1)) \arrow[r] & \pi_i(\mathrm{O}(n)) \arrow[r] & \pi_i(S^{n-1}) \arrow[dll] \\
\pi_{i-1}(\mathrm{O}(n-1)) \arrow[r] & \cdots &
\end{tikzcd} .$$

Since $\pi_i(S^{n-1})$ is trivial for any $0 \le i \le n-2$, we see that the map $\pi_i(\mathrm{O}(n-1)) \to \pi_i(\mathrm{O}(n))$ is an isomorphism for any $i \le n-3$ and an epimorphism when $i = n-2$. The same result holds with $\mathrm{O}(n)$ replaced by $\mathrm{GL}(n, \mathbb{R})$.

Example 2.6.2. Consider the *Stiefel manifold* $V_{n+k,k}$ consisting of orthonormal k-frames (i.e., k-tuples) in $\mathbb{R}^{n+k}$. If we view the standard basis of $\mathbb{R}^k$ as the "zero element" of $V_{n+k,k}$, then we have a "short exact sequence"

$$0 \longrightarrow \mathrm{O}(n) \xhookrightarrow{i} \mathrm{O}(n+k) \xrightarrow{p_1} V_{n+k,k} \longrightarrow 0$$

where i is given by $A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array}\right)$ and p_1 is given by $A \mapsto (Ae_{n+1}, \ldots, Ae_{n+k})$. In this case,

$$V_{n+k,k} \cong \frac{\mathrm{O}(n+k)}{\mathrm{O}(n)},$$

a coset space. Note that i induces an isomorphism $\pi_i(\mathrm{O}(n)) \xrightarrow{\cong} \mathrm{O}(n+k)$ for each $i \le n-2$ and an epimorphism when $i = n-1$.

Claim. *The map p_1 is a fiber bundle.*

Proof. Let $F \in V_{n+k,k}$ and choose any orthonormal basis B of the n-plane orthogonal to F. For any n-plane near B, take the orthogonal projection of B onto B' and then apply the Gram-Schmidt process to the new basis to obtain an orthonormal basis $\underline{B'}$ of B'. The assignment $B \mapsto \underline{B'}$ is continuous, and the space of all n-planes orthogonal to any $(n+k)$-plane near F is identifiable with $V_n(\mathbb{R}^n) \cong \mathrm{O}(n)$. Therefore, we get a trivialization around F, which was arbitrary. $\square$

Using the LES obtained from Corollary 1.3.9, we see that $\pi_i(V_{n+k,k}) = 0$ for each $i \le n-1$. Consider now the *Grassmann manifold*

$$G_{n+k,k} \equiv \frac{\mathrm{O}(n+k)}{\mathrm{O}(n) \times \mathrm{O}(k)}$$

where each pair $(A, B) \in \mathrm{O}(n) \times \mathrm{O}(k)$ is identified with the orthogonal $(n+k) \times (n+k)$ matrix $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$. Note that $G_{n_k,k}$ may be viewed as the space of all k-dimensional planes in $\mathbb{R}^{n+k}$.

Proposition 2.6.3.

1. *The space $E_{\mathrm{O}(k)}$ consists of all orthonormal k-frames in $\mathbb{R}^\infty$.*
2. *The Grassmannian $G_{\infty,k} \equiv B_{\mathrm{O}(k)} = B_{\mathrm{GL}(k)}$ consists of all k-planes in $\mathbb{R}^\infty$.*
3. *Similarly, the space $B_{U(k)}$ consists of all k-planes in $\mathbb{C}^\infty$.*

Define $p_2 : V_{n+k,k} \to G_{n+k,k}$ by sending each $v \in V_{n+k,k}$ to the subspace of $\mathbb{R}^{n+k}$ spanned by v.

Claim. *The map p_2 is a principal $\mathrm{O}(k)$-bundle.*

Proof. This follows from the fact that $\mathrm{O}(n+k) \to G_{n+k,l}$ is a principal $\mathrm{O}(n) \times \mathrm{O}(k)$-bundle. $\square$

As a result, $\pi_i(G_{n+k,k}) = 0$ for each $i \le n-2$.

3 Spectral sequences

We are given a fibration:

$$\begin{tikzcd} F \arrow[d] \\ E \arrow[d, "\pi"] \\ X \end{tikzcd}$$

where X is a connected cell complex and $F = \pi^{-1}(x)$ for some distinguished point x.

Question. What is $H_n(E)$ if we know $H_n(F)$ and $H_n(X)$?

Recall that $H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$ where ∂_n is defined as the composite

$$\begin{tikzcd} \overbrace{H_n(X^n, X^{n-1})}^{C_n(X)} \arrow[r] \arrow[rr, bend right, "\partial_n"'] & H_{n-1}(X^{n-1}) \arrow[r] & \overbrace{H_{n-1}(X^{n-1}, X^{n-2})}^{C_{n-1}}(X) \end{tikzcd} ,$$

where $H_i(X^n, X^{n-1}) = 0$ for any $i \neq n$. Furthermore, letting $E_n = \pi^{-1}(X_n)$, we have that $H_*(E_n, E_{n-1}) = C_*(X) \otimes H_*(F)$.

At this point, it is useful to generalize our situation by developing the theory of spectral sequences. For each $r \in \mathbb{Z}_{\geq 0}$, let $\left\{E^r_{p,q}\right\}_{p,q\in\mathbb{Z}}$ be a family of abelian groups and let $\left\{d^{p,q}_r : E^r_{p,q} \to E^r_{p-r,q+r-1}\right\}_{p,q\in\mathbb{Z}}$ be a family of maps (called *differentials*) such that

(a) $d^{p,q}_r \circ d^{p+r,q-r+1}_r = 0$ and

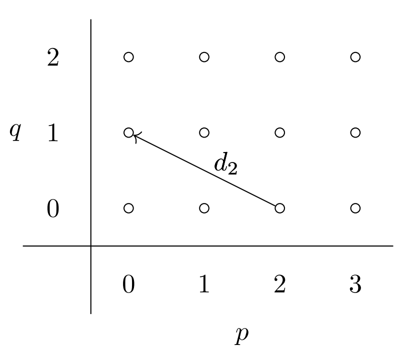
(b) $E^{r+1}_{p,q} = \frac{\ker d^{p,q}_r}{\operatorname{im} d^{p+r,q-r+1}_r}$.

Such a sequence $(E^r, d_r)_{r\in\mathbb{Z}_{\geq 0}}$ of pairs is called a *homological spectral sequence*, and each double complex (E^r, d_r) is called the *r-th page* of the sequence.

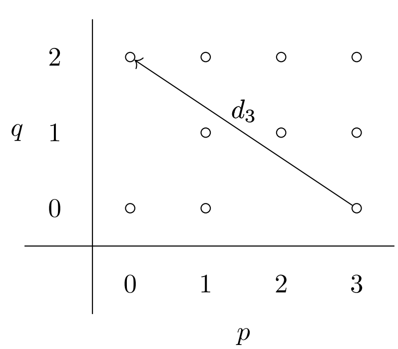
Note 3.0.1. $E^{r+1} = H_*(E^r, d_r)$.

We shall consider only *first-quadrant* spectral sequences, i.e., those for which $E^r_{p,q} = 0$ unless $p, q \geq 0$.

Page 2



Page 3



As a result, there is some $k \in \mathbb{N}$ such that $E^r = E^{r+1}$ for any $r \geq k$.

Notation. $E^\infty \coloneqq E^k$.

Definition 3.0.2 (Convergence). We say that a spectral sequence $E^* \coloneqq (E^r, d_r)$ *converges* to a sequence of abelian groups $\{A_n\}_{n\in\mathbb{Z}_{\geq 0}}$, written as

$$E^* \Rrightarrow \{A_n\},$$

if for each n, there exists a filtration

$$\cdots \subset A_{-1,n+1} = \{0\} \subset A_{0,n} \subset \cdots \subset A_{n-1,1} \subset A_{n,0} = A_n$$

of A_n such that $\frac{A_{p,q}}{A_{p-1,q+1}} \cong E^\infty_{p,q}$.

Theorem 3.0.3. *Let B be a simply connected, path connected cell complex with n-skeleton B^n and suppose that $\pi : E \to B$ is a fibration with fiber F. There exists a (first-quadrant) spectral sequence (E^r, d_r) that*

(a) converges to $\{H_n(E)\}_{n\in\mathbb{Z}_{\geq 0}}$ and

(b) satisfies $E^2_{p,q} \cong H_p(B; H_q(F))$.

The filtration $D_{p,q} \coloneqq (H_n(E))_{p+q=n}$ witnessing this convergence is given by $\operatorname{im}(H_n(\pi^{-1}(B^p)) \to H_n(E))$.

Remark 3.0.4. This holds without the hypothesis that B is a cell complex.

Example 3.0.5. Consider the path space fibration

$$\begin{array}{c} \Omega X \\ \downarrow \\ PX \\ \downarrow \\ X \end{array}.$$

Recall that PX is contractible. Let $n \geq 2$ and $X = S^n$. Then

$$E^2_{p,q} \cong H_p(S^n; H_q(\Omega S^n)) = \begin{cases} H_q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise} \end{cases},$$

and $(E^r, d_r) \Rrightarrow \{\mathbb{Z}, 0, 0, \ldots\}$. This means that $d_k = 0$ for any $k \neq n$, so that

$$E^2 = E^3 = \cdots = E^n$$
$$E^{n+1} = E^{n+2} = \cdots = E^\infty.$$

As a result, each differential $d_n^{p,q}$ is an isomorphism provided that $(p,q) \neq (n, 1-n)$ for, otherwise, $E^{n+1}_{p,q}$ is nontrivial, which is impossible. Hence the n-th page looks like

$$
\begin{tikzcd}
3(n-1) & H_{3n-3}(\Omega S^n) & H_{3n-3}(\Omega S^n) \\
2(n-1) & H_{2n-2}(\Omega S^n) & H_{2n-2}(\Omega S^n) \arrow[ul, "iso."'] \\
n-1 & H_{n-1}(\Omega S^n) & H_{n-1}(\Omega S^n) \arrow[ul, "iso."'] \\
0 & \mathbb{Z} & \mathbb{Z} \arrow[ul, "iso."'] \\
 & 0 & n
\end{tikzcd}
$$

This implies that $H_q(\Omega S^n) \cong H_{q+n-1}(\Omega X)$ for any $q \in \mathbb{Z}_{\geq 0}$. But ΩS^n is path connected since S^n is simply connected. By induction, it follows that

$$
H_q(\Omega X) \cong \begin{cases} \mathbb{Z} & q \equiv 0 \mod (n-1) \\ 0 & \text{otherwise} \end{cases}.
$$

3.1 Lecture 12

Suppose that

$$
\begin{tikzcd}
F \arrow[d] \\
E \arrow[d, "\pi"] \\
B
\end{tikzcd}
$$

is a fibration with B simply connected and F path connected. Thanks to Theorem 3.2.5, we have the inclusion

$$
E^{\infty}_{0,n} \cong \frac{D_{0,n}}{D_{-1,n+1}} = D_{0,n} \subset H_n(E)
$$

as well as a commutative diagram

$$
\begin{tikzcd}
\overbrace{H_p(B; H_0(F))}^{H_p(B)} \arrow[r, equal] & E^2_{p,0} & E^3_{p,0} \arrow[l, hook'] & & \\
 & E^4_{p,0} \arrow[ur, hook] & \cdots \arrow[l, hook'] & E^p_{p,0} \arrow[l, hook'] & \\
 & & E^{\infty}_{p,0} \arrow[ur, hook] & H_p(E) \arrow[l] \arrow[r, equal] \arrow[llluu, bend left, "\pi_*"] & D_{p,0}
\end{tikzcd}
$$

of abelian groups. Let i denote the inclusion map $i : F \cong p^{-1}(b) \to E$ where b is any chosen element of B. This induces a map i_* in homology

$$\begin{tikzcd}
\underbrace{H_q(F)}_{H_0(B,H_q(F))} \arrow[r, two heads] \arrow[rrrr, bend left, "i_*"] & E^3_{0,q} \arrow[r, two heads] & \cdots \arrow[r, two heads] & E^\infty_{0,q} \arrow[r, hook] & H_q(E) .
\end{tikzcd}$$

Now, consider the commutative diagram

$$\begin{tikzcd}
\pi_n(B) \arrow[r, "\partial"] \arrow[d, "h_n"'] & \pi_{n-1}(F) \arrow[d, "h_{n-1}"] \\
H_n(B) \arrow[r, "\partial"] & H_{n-1}(F) \arrow[d, two heads] \\
E^2_{n,0} \arrow[u, "\cong"] & E^3_{0,n-1} \arrow[d, two heads] \\
\vdots \arrow[u, hook] & \vdots \arrow[d, two heads] \\
E^n_{n,0} \arrow[u, hook] \arrow[r, "d_n"'] & E^n_{0,n-1}
\end{tikzcd}$$

where h_n denotes the *Hurewicz homomorphism*, defined for an arbitrary path connected space X as follows. Let $\gamma \coloneqq [f]$ be any element of $\pi_n(X, x)$, so that f is a map $(S^n, x_0) \to (X, x)$. Choose any generator $\tau \in H_n(S^n) \cong \mathbb{Z}$ and let

$$h(\gamma) = f_*(\tau) \in H_n(X).$$

Likewise, we can define the *relative Hurewicz homomorphism* $\tilde{h} : \pi_n(X, A) \to H_n(X, A)$ by

$$\left[f : \left(D^n, S^{n-1}, x_0\right) \to (X, A, \mathsf{pt})\right] \mapsto f_*(\sigma)$$

where σ is any chosen generator of $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$.

Theorem 3.1.1 (Hurewicz). *Let $n \in \mathbb{Z}_{\geq 2}$. If $\pi_i(X) = 0$ for each $1 \leq i \leq n-1$, then h_n is an isomorphism and h_{n+1} is surjective.*

Theorem 3.1.2 (Relative Hurewicz). *Let $n \in \mathbb{Z}_{\geq 2}$. If both X and A are simply connected and $\pi_i(X, A) = 0$ for each $i \leq n-1$, then $\tilde{h}_n$ is an isomorphism and $\tilde{h}_{n+1}$ is surjective.*

Proof of Theorem 3.1.1. Suppose that $\pi_i(X) = 0$ for each $1 \leq i \leq n-1$. For induction, assume that h_{n-1} is an isomorphism for any path connected space. From Example 3.0.5, we gather that the n-th page of the spectral sequence induced by the path space fibration $\Omega X \to PX \to X$ looks like

$$\begin{tikzcd}
n-1 & H_{n-1}(\Omega X) & \bullet & H_{n-1}(\Omega X) \\
0 & \mathbb{Z} & & H_n(X) \arrow[ull, "d_n"'] \\
 & 0 & & n
\end{tikzcd}$$

where d_n is an isomorphism. Thanks to our inductive hypothesis together with Exercise 1.4.3, we have now a commutative square of the form

$$\begin{tikzcd} \pi_n(X) \arrow[r, "\partial", "\cong"'] \arrow[d, "h_n"'] & \pi_{n-1}(\Omega X) \arrow[d, "h_{n-1}", "\cong"'] \\ H_n(X) \arrow[r, "\cong", "d_n"'] & H_{n-1}(\Omega X) \end{tikzcd} . \tag{$*$}$$

This implies that h_n is an isomorphism. It remains to verify our base case. Note that $\pi_1(\Omega X)$ is isomorphic to $\pi_2(X)$ and thus abelian. It can be shown directly that h_1 factors as a composite

$$\begin{tikzcd} \pi_1(\Omega X) \arrow[r, two heads, "\cong"] \arrow[rr, bend right, "h_1"'] & \pi_1(\Omega X)^{\mathrm{ab}} \arrow[r, "\cong"] & H_1(\Omega X) \end{tikzcd}$$

of isomorphisms. Hence h_2 must be an isomorphism in light of ($*$). $\square$

Question. Does a similar argument work for Theorem 3.1.2?

Answer. Yes, in the sense that there is a spectral sequence proof of it. Specifically, there is a relative version of the spectral sequence. Suppose that B' is a simply connected subspace of B and let $E' = \pi^{-1}(B')$, yielding a fibration $F \to E' \xrightarrow{\pi\restriction_{E'}} B'$. Then there exists a spectral sequence E^* converging to $H_*(E, E')$ such that

$$E^2_{p,q} \cong H_p(B, B'; H_q(F)).$$

One can use this to deduce Theorem 3.1.2 from Theorem 3.1.1. $\square$

Corollary 3.1.3. *Let X be path connected.*

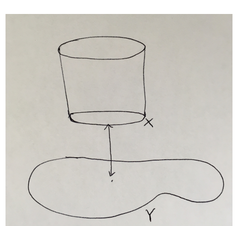
1. $H_1(X) \cong \pi_1^{\mathrm{ab}}(X)$.
2. *If X is simply connected and $H_i(X) = 0$ for every $1 \leq i \leq n-1$, then $\pi_i(X) = 0$ for every $1 \leq i \leq n-1$.*
3. *If $\pi_i(X) = 0$ for each $0 \leq i \leq n-1$, then $\widetilde{H}_i(X) = 0$ for each $0 \leq i \leq n-1$.*

Let $n \geq 2$ and pick any generator $[f]$ of $\pi_{n-1}(\Omega S^n) \cong \pi_n(S^n) \cong \mathbb{Z}$. By Theorem 3.1.1, the induced map $f_* : H_{n-1}(S^{n-1}) \to H_{n-1}(\Omega S^n)$ is an isomorphism.

Remark 3.1.4. Let $g : X \to Y$ be any map of spaces. Recall the mapping cylinder

$$\mathrm{Cyl}(g) \equiv \frac{(X \times I) \coprod Y}{(x, 0) \sim g(x)}$$

of g.



This is precisely the pushout of the span $X \times I \xleftarrow{\sigma_0} X \xrightarrow{g} Y$. As it turns out, g factors as

$$\begin{tikzcd} X \arrow[r, hook, "\iota"'] \arrow[rr, bend left, "g"] & \mathrm{Cyl}(g) \arrow[r, "h"'] & Y \end{tikzcd}$$

for some deformation retraction h. Further, ι is a cofibration, the dual notion to a fibration.

Consider the subspace of ΩS^n consisting of all great circles passing through, say, the north pole. This is clearly homeomorphic to S^{n-1}. Thus, we get a LES in homology

$$\begin{tikzcd}
\cdots \arrow[r] & \overbrace{H_{2n-2}(S^{n-1})}^{0} \arrow[r] & \overbrace{H_{2n-2}(\Omega S^n)}^{\mathbb{Z}} \arrow[r, "\cong"] & H_{2n-2}(\Omega S^n, S^{n-1}) \arrow[dll] & \\
& \underbrace{H_{2n-3}(S^{n-1})}_{0} \arrow[r] & H_{2n-3}(\Omega S^n) \arrow[r, "\cong"] & H_{2n-3}(\Omega S^n, S^{n-1}) \arrow[r] & \cdots \arrow[dll] \\
& & \underbrace{H_{n+1}(S^{n-1})}_{0} \arrow[r] & H_{n+1}(\Omega S^n) \arrow[r, "\cong"] & H_{n+1}(\Omega S^n, S^{n-1}) \arrow[dll] \\
& & \underbrace{H_{n}(S^{n-1})}_{0} \arrow[r] & H_{n}(\Omega S^n) \arrow[r, "\cong"] & H_{n}(\Omega S^n, S^{n-1}) \arrow[dll, "0"'] \\
& & H_{n-1}(S^{n-1}) \arrow[r, "\cong", "f_*"'] & H_{n-1}(\Omega S^n) \arrow[r, "0"'] & H_{n-1}(\Omega S^n, S^{n-1}) \arrow[dl] \\
& & & \underbrace{H_{n-2}(S^{n-1})}_{0} \arrow[r] & \cdots
\end{tikzcd} .$$

From this, we deduce that

$$H_i(\Omega S^n, S^{n-1}) = \begin{cases} 0 & i \leq 2n-3 \\ \mathbb{Z} & i = 2n-2 \end{cases}.$$

By Corollary 3.1.3(2), this means that

$$\pi_i(\Omega S^n, S^{n-1}) = \begin{cases} 0 & i \leq 2n-3 \\ \mathbb{Z} & i = 2n-2 \end{cases}.$$

This yields a LES in homotopy

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{2n-2}(\Omega S^n) \arrow[r] & \pi_{2n-2}(\Omega S^n, S^{n-1}) \arrow[dl] & & \\
\pi_{2n-3}(S^{n-1}) \arrow[r] & \pi_{2n-3}(\Omega S^n) \arrow[r] & \overbrace{\pi_{2n-3}(\Omega S^n, S^{n-1})}^{0} \arrow[dl] & & \\
& \pi_{2n-4}(S^{n-1}) \arrow[r, "\cong"] & \underbrace{\pi_{2n-4}(\Omega S^n)}_{\pi_{2n-3}(S^n)} \arrow[r] & \underbrace{\pi_{2n-4}(\Omega S^n, S^{n-1})}_{0} \arrow[r] & \cdots
\end{tikzcd} ,$$

which proves the following statement.

Theorem 3.1.5 (Suspension). *If $0 \leq i \leq 2n-4$, then $\pi_i(S^{n-1}) \cong \pi_{i+1}(S^n)$.*

This can be generalized as follows.

Exercise 3.1.6 (Freudenthal suspension). *Let $i \in \mathbb{Z}_{\geq 1}$ and suppose that the space X satisfies $\pi_n(X) = 0$ for each $0 \leq n \leq i-1$. Show that*

$$\pi_i(X) \cong \pi_{i+1}(SX)$$

through around dimension $2i-3$ (figure this out exactly), where $S(-)$ denotes the suspension functor.

Proof. We could use a spectral sequence argument together with the relative Hurewicz theorem. Rather, let us show that it follows from the following famous theorem in homotopy theory:

Theorem 3.1.7 (Blakers-Massey). *Suppose that*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, hook] & B \arrow[d] \\ A \arrow[r] & C \end{tikzcd}$$

is a pushout diagram. Also, suppose that $\pi_n(B,X) = 0$ and $\pi_m(A,X) = 0$ for each $0 \leq n \leq k_1$ and each $0 \leq m \leq k_2$. The map $(A,X) \to (C,B)$ of pairs induces an isomorphism $\pi_n(A,X) \xrightarrow{\cong} \pi_n(C,B)$ for each $0 \leq n \leq k_1 + k_2 - 1$.

Now, let us prove the Freudenthal suspension theorem with an upper bound of $2i-2$ rather than $2i-3$. To start, decompose SX into the union of two cones $\underset{\text{top}}{C_+X}$ and $\underset{\text{bottom}}{C_-X}$ that meet at a copy of X. Note that SX is precisely the pushout of the diagram

$$C_-X \hookleftarrow X \hookrightarrow C_+X.$$

As C^+X is contractible, the LES of homotopy groups for the pair (C_+X, X) shows that the map $\partial : \pi_{n+1}(C_+X, X) \to \pi_n(X)$ is an isomorphism. Similarly, we see that the map $\iota : \pi_{n+1}(SX) \to \pi_{n+1}(SX, C_-X)$ induced by inclusion is an isomorphism. Consider the sequence of homomorphisms

$$\pi_n(X) \xrightarrow{\partial^{-1}} \pi_{n+1}(C_+X, X) \xrightarrow{\psi} \pi_{n+1}(SX, C_-X) \xrightarrow{\iota^{-1}} \pi_{n+1}(SX)$$

where ψ is induced by pullback. Since $\pi_n(X) = 0$ for each $0 \leq n \leq i-1$, the LES for the pair $(C_\pm X, X)$ also shows that $\pi_n(C_\pm X) = 0$ for any $0 \leq n \leq i$. By Theorem 3.1.7, it follows that ψ is an isomorphism so long as $n+1 \leq 2i-1$. Hence $\pi_n(X) \cong \pi_{n+1}(SX)$ for any $n \leq 2i-2$, as desired.

Furthermore, the upper bound of $2i-2$ is sharp. Indeed, we have that

- $\pi_0(S^2) = \pi_1(S^2) = 0$,
- $\pi_3(S^2) \cong \mathbb{Z}$, and
- $\pi_4(S^3) \cong \mathbb{Z}/2$.

If we could increase our upper bound to $2i-1$, then we would have an isomorphism $\pi_3(S^2) \cong \pi_4(S^3)$, which is impossible. □

3.2 Lecture 13

As expected, spectral sequences have exact analogues in cohomology. Before introducing them, let us review a bit of singular cohomology theory. Let X be a cell complex and let $n \in \mathbb{Z}_{\geq 0}$. Recall that $C_n(X)$ the free abelian group on the set of all n-cells of X and the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$. Let

$$C^n(X) = \operatorname{Hom}(C_n(X), \mathbb{Z})$$

and define the homomorphism $\delta^n : C^n(X) \to C^{n+1}(X)$ by

$$\delta^n(\varphi) = \varphi \circ \partial_n.$$

Theorem 3.2.1. $H^n(X;\mathbb{Z}) \cong \frac{\ker \delta^{n+1}}{\operatorname{im} \delta^n}$.

Example 3.2.2. $H^*(\mathbb{CP}^n;\mathbb{Z}) \cong \mathbb{Z}[x] \big/ (x^{n+1})$ with $\underset{\text{dim.}}{|x|} = 2$.

Theorem 3.2.3 (Poincaré duality). *If M is a connected orientable n-manifold, then $H_i(M) \cong H^{n-i}(M)$.*

Now, a cohomological spectral sequence consists of the following data:

- for each $r \in \mathbb{Z}_{\geq 0}$, a family of abelian groups $\{E_r^{p,q}\}_{p,q\in\mathbb{Z}}$ and
- a family of maps $\left\{d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\right\}_{p,q\in\mathbb{Z}}$ (called *differentials*) such that
- $d_r^{p,q} \circ d_r^{p-r,q+r-1} = 0$ and
- $E_{r+1}^{p,q} = \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}$.

Again, we shall consider only *first-quadrant* spectral sequences, i.e., those for which $E_r^{p,q} = 0$ unless $p, q \geq 0$. As a result, there is some $k \in \mathbb{N}$ such that $E_r = E_{r+1}$ for any $r \geq k$.

Notation. $E_\infty \coloneqq E_k$.

Definition 3.2.4 (Convergence). We say that a spectral sequence $E_* \coloneqq (E_r, d_r)$ *converges* to a sequence of abelian groups $\{D^n\}_{n\in\mathbb{Z}_{\geq 0}}$, written as

$$E_* \Rrightarrow \{D^n\},$$

if for each n, there exists a filtration

$$\cdots \subset D^{n+1,-1} = \{0\} \subset D^{n,0} \subset \cdots \subset D^{1,n-1} \subset D^{0,n} = D^n$$

of D^n such that $\frac{D^{p,q}}{D^{p+1,q-1}} \cong E_\infty^{p,q}$.

Theorem 3.2.5. *Let B be simply connected and path connected and suppose that $\pi : E \to B$ is a fibration with fiber F. There exists a (first-quadrant) spectral sequence (E^r, d_r) that*

(a) converges to $\{H^n(E)\}_{n\in\mathbb{Z}_{\geq 0}}$ and

(b) satisfies $E_2^{p,q} \cong H^p(B; H^q(F))$.

In pictures, we have

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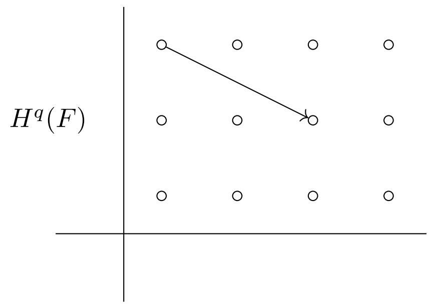

$H^p(B) \coloneqq H^p(B; H^0(F))$

$$\begin{tikzcd}
H^q(E) \arrow[r, equal] \arrow[rrrrr, bend right, "i^*"'] & D^{0,q} \arrow[r, two heads] & E_\infty^{0,q} \arrow[r, hook] & \cdots \arrow[r, hook] & E_2^{0,q} \arrow[r, equal] & H^q(F)
\end{tikzcd}.$$

$$\begin{tikzcd}
H^p(B) \arrow[r, equal] \arrow[rrrrr, bend right, "\pi^*"'] & E_2^{p,0} \arrow[r, two heads] & E_3^{p,0} \arrow[r, two heads] & \cdots \arrow[r, two heads] & E_\infty^{p,0} \arrow[r, hook] & H^p(E)
\end{tikzcd}$$

Let X be a cell complex. Recall the *cup product* operation $H^i(X) \times H^j(X) \xrightarrow{\smile} H^{i+j}(X)$ on cohomology, which is both bilinear and *anti-commutative* in the sense that

$$x \smile y = (-1)^{ij} y \smile x.$$

Consider the constant map $C_0(X) \to \mathbb{Z}$ given by $D^0 \mapsto 1$, which corresponds to an element $\mathbf{1}$ of $H^0(X)$ via Theorem 3.2.1. We have that

$$-\mathbf{1} \smile x = x \smile \mathbf{1} = \mathbf{1}.$$

Suppose that Y is another cell complex. Let $x \in H^i(X)$ and $y \in H^j(X)$ and let f denote a map $Y \to X$. Then

$$f^*(x \smile y) = f^*(x) \smile f^*(y),$$

i.e., f^* is a graded ring homomorphism. Now, $X \times Y$ carries a cell complex structure with n-cells of the form

$$D^i \times D^j, \quad i + j = n$$

and n-skeleton

$$(X \times Y)^n \equiv \bigcup_{i+j=n} X^i \times Y^j.$$

We have that

$$C_n(X \times Y) \cong C_n(X) \otimes_{\mathbb{Z}} C_n(Y)$$

and, in light of the fact that $\partial(D^i \times D^j) = \big(\partial D^i \times D^j\big) \cup \big(D^i \times \partial D^j\big)$, that

$$\partial[D^i \times D^j] = \partial[D^i] \otimes D^j + (-1)^i [D^i] \otimes \partial[D^j].$$

Consider any two maps $f : C_i(X) \to \mathbb{Z}$ and $g : C_j(X) \to \mathbb{Z}$, extending them both by 0 to the entire graded abelian group $C_*(X)$. Define $f \otimes g : C_m(X \times Y) \cong C_m(X) \otimes C_m(Y) \to \mathbb{Z}$ by

$$(f \otimes g)\,(u \otimes v) = f(u) \cdot g(v).$$

Proposition 3.2.6. $\delta(f \otimes g) = \delta f \otimes g + (-1)^i f \otimes \delta g.$

As it turns out, this means that the map $(f, g) \mapsto (f \otimes g)$ induces an operation $H^i(X) \times H^j(Y) \xrightarrow{\times} H^{i+j}(X \times Y)$ on cohomology known as the *cross product*. The relation between the cup and cross product has the form $\Delta^*(x \times y) = x \smile y$, where $\Delta : X \to X \times X$ denotes the diagonal map.

In general, let R_1, R_2, and R_3 be commutative rings and let $\mu : R_1 \times R_2 \to R_3$ denote "multiplication." This induces the cup product on cohomology

$$\begin{gathered} H^i(X; R_1) \times H^j(X; R_2) \xrightarrow{\smile} H^{i+j}(X; R_3) \\ \Downarrow \\ H^p(B, H^q(F)) \times H^{p'}(B, H^{q'}(F)) \xrightarrow{\smile} H^{p+p'}(B, H^{q+q'}(F)) \\ E_2^{p,q} \times E_2^{p',q'} \xrightarrow{\smile} E_2^{p+p',q+q'}. \end{gathered}$$

Proposition 3.2.7. *For any $r \in \mathbb{Z}_{\geq 2}$, there is a certain operation $\smile_r$: $E_r^{p,q} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$ such that*

$$d_r(x \smile y) = d_r(x) \smile y + (-1)^{p+q} x \smile d_r(y).$$

Construction. Let $r \in \mathbb{Z}_{\geq 2}$ and suppose, for induction, that we have already constructed $\smile_r$. Let $x \in E_r^{p,q}$ and $y \in E_r^{p',q'}$. Suppose that $d_r x = d_r y = 0$, so that $d_r(x \smile y) = 0$. If $y = d_r(z)$, then

$$x \smile y = x \smile d_r(z) = d(x \smile z) \pm \underbrace{d_r(x)}_{0} \smile z.$$

by induction. This means that $\smile_r$ induces a pairing $\smile_{r+1}$ on E_{r+1}. To complete our induction on r, simply take the ordinary cup product on cohomology to be $\smile_2$. ☐

Now, given the filtration

$$\{0\} \subset D^{n,0} \subset \cdots \subset D^{0,n} \subset H^n(E),$$

the operation $\smile_r$ on E_r carries $D^{p,q} \times D^{p',q'}$ to $D^{p+p',q+q'}$ where $p + q = p' + q' = n$, thereby inducing a pairing

$$\smile_\infty\colon E_\infty^{p,q} \times E_\infty^{p',q'} \to E_\infty^{p+p',q+q'}$$

on E_∞.

Example 3.2.8. Consider the fiber bundle $S^1 \to S^{2n+1} \twoheadrightarrow \mathbb{CP}^n$, so that

$$E_2^{p,q} \cong H^p(\mathbb{CP}^n; H^q(S^1)).$$

Pick a generator x of the group $H^1(S^1) \cong \mathbb{Z}$. Then the cohomology ring $H^*(S^1)$ is isomorphic to $\mathbb{Z}[x] / (x^2)$, and

$$H^i(S^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases}.$$

Moreover, recall that

$$H^i(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, \ i \equiv 0 \mod 2 \\ 0 & \text{otherwise} \end{cases},$$

which yields

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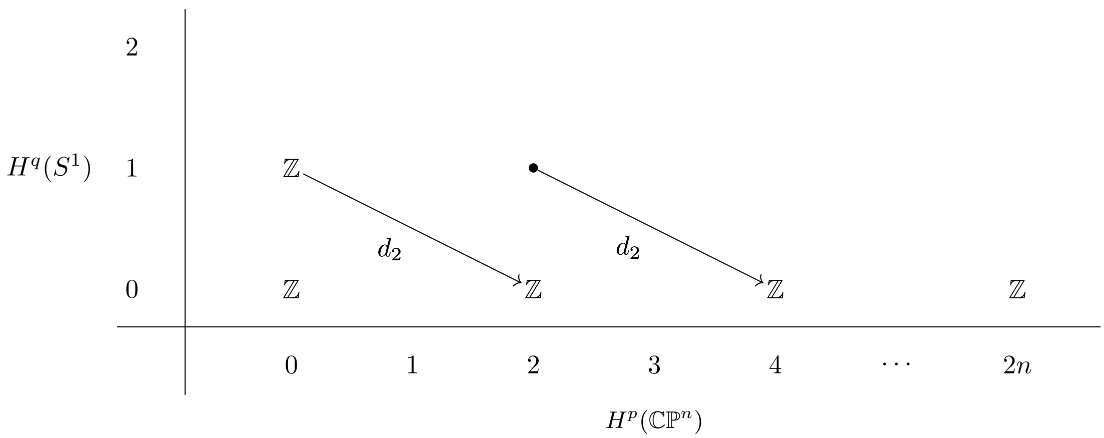

where each d_2 is an isomorphism. Suppose that x is a generator of $H^1(S^1)$ and let $c = d_2(x)$. Then

$$d_2(c \smile x) = c \smile d_2(x) = c^2,$$

which is a generator of $H^4(\mathbb{CP}^n)$. Similarly, c^i is a generator of $H^{2i}(\mathbb{CP}^n)$ for each $i \in \mathbb{Z}_{\geq 0}$.

By letting $c^0 = 1$ and making n large enough, we have determined the ring structure of $H^*(\mathbb{CP}^\infty)$.

Theorem 3.2.9. *If c_1 is a generator of $H^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$, then* $\underbrace{H^*(B_{S^1}) = H^*(\mathbb{CP}^\infty)}_{\textit{Example } 2.5.11} \cong \mathbb{Z}[c_1]$.

4 Characteristic classes

4.1 Lecture 14

Recall the space $B_{\mathrm{U}(n)} = B_{\mathrm{GL}(n,\mathbb{C})}$ of n-planes in $\mathbb{C}^\infty$ as well as the space $B_{\mathrm{O}(n)} = B_{\mathrm{GL}(n,\mathbb{R})}$ of n-planes in $\mathbb{R}^\infty$. We want to classify the graded rings $H^*(B_{\mathrm{U}(n)}; \mathbb{Z})$ and $H^*(B_{\mathrm{O}(n)}; \mathbb{Z}_2)$. Let's begin with the former.

Consider the embedding $\mathrm{U}(n-1) \hookrightarrow \mathrm{U}(n)$ given by $A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$. Consider also the mapping $\mathrm{U}(n) \to S^{2n-1}$ given by $A \mapsto Ae_n$. These fit into a "short exact sequence"

$$0 \to \mathrm{U}(n-1) \to \mathrm{U}(n) \to S^{2n-1} \to 0 \tag{1}$$

of spaces, and thus the bundle

$$\frac{\mathrm{U}(n)}{\mathrm{U}(n-1)} \to B\mathrm{U}(n-1) \xrightarrow{\pi} B\mathrm{U}(n) \tag{2}$$

has fiber S^{2n-1}.

Remark 4.1.1. Let us make (1) precise. Note that $\mathrm{U}(n)$ acts transitively on S^{2n-1} because any point in S^{2n-1} belongs to at least one orthonormal bases of $\mathbb{C}^n$.This means that S^{2n-1} is a *homogenous* $\mathrm{U}(n)$-space.

Proposition 4.1.2. *Any homogenous G-space X is homeomorphic to the coset space ${}^{G}\!/_{\mathrm{Stab}_G(\vec{o})}$ where $\vec{o}$ is any choice of "identity."*

In our case, let $\vec{o} = e_n$, so that $\mathrm{Stab}_{\mathrm{U}(n)}(\vec{o}) = \mathrm{U}(n-1)$.

Now, (2) induces a spectral sequence E_* converging to $H^*(B\mathrm{U}(n-1))$ such that

$$E_2^{p,q} = H^p(B\mathrm{U}(n); H^q(S^{2n-1}))$$

and

$$\begin{tikzcd} H^p(B\mathrm{U}(n)) \arrow[r] \arrow[rrrrr, bend right, "\pi^*"] & E_2^{p,0} \arrow[r, two heads] & E_3^{p,0} \arrow[r, two heads] & \cdots \arrow[r, two heads] & E_\infty^{p,0} \arrow[r, hook] & H^p(B\mathrm{U}(n-1)) \end{tikzcd}$$

commutes. If $p < 2n$, then $H^p(B\mathrm{U}(n)) \cong E_\infty^{p,0}$, in which case π^* is injective. Also, we have that $E_2^{p-k,k} = E_\infty^{p-k,k}$ whenever $0 < p-k < 2n+1$, so that π^* is surjective whenever $p < 2n$. It follows that π^* is an isomorphism

$$H^p(B\mathrm{U}(n)) \cong H^p(B\mathrm{U}(n-1)) \tag{3}$$

when $p \leq 2n-1$.

Consider the differential $d_2 : \underbrace{H^{2n-1}(S^{2n-1})}_{\mathbb{Z}} \to H^{2n}(B\mathrm{U}(n))$ and pick a generator g_n of $H^{2n-1}(S^{2n-1})$, i.e., an orientation of S^{2n-1}. Let

$$c_n = d_2(g_n).$$

In light of (3), we see that

$$c_i \in H^{2i}(B\mathrm{U}(n)) \cong \cdots \cong H^{2i}(B\mathrm{U}(i+1)) \cong H^{2i}(B\mathrm{U}(i))$$

for each $i \leq n$. Abusing notation, we shall write $c_i \in H^{2i}(B\mathrm{U}(i))$.

Theorem 4.1.3. $H^*(B\mathrm{U}(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_n]$

Proof. Proceed by induction on $n \in \mathbb{N}$. Our base case holds by virtue of Theorem 3.2.9. Assume that

$$H^*(B\mathrm{U}(n-1)) \cong \mathbb{Z}[c_1, \ldots, c_{n-1}].$$

Note that $E_{2n} = E_\infty$ and $E_{2n-1} = E_2$, yielding

Page $2n-1$

$$
H^q(S^{2n-1}) \quad
\begin{tikzcd}
2n-1 & g_n \arrow[rrrrrrdd, "d_{2n-1}"] & & & & & & \\
\vdots & & & & & & & \\
0 & \mathbb{Z} & 0 & c_1 & 0 & c_2 & \cdots & c_n \\
 & 0 & 1 & 2 & 3 & 4 & \cdots & 2n
\end{tikzcd}
$$

$$H^p(B\mathrm{U}(n))$$

.

?? □

We call c_i the i-th *Chern class* of $B\mathrm{U}(n)$. The following may be seen as a real analogue to Theorem 4.1.3.

Theorem 4.1.4. $H^*(B\mathrm{O}(n), \mathbb{Z}_2) \cong \mathbb{Z}[w_1, \ldots, w_n]$ *for certain cohomology classes* $w_i \in H^i(B\mathrm{O}(n), \mathbb{Z}_2)$.

We call w_i the i-th *Stiefel-Whitney* class of $B\mathrm{O}(n)$.

Example 4.1.5. Let M be a smooth n-manifold. Consider the tangent bundle TM with group $\mathrm{O}(n)$ and fiber $\mathbb{R}^n$. Let $f : M \to B\mathrm{O}(n)$ denote the classifying map for $TM \xrightarrow{\pi} M$, so that $f^*\gamma_n = \pi$ where γ_n denotes the universal $\mathrm{O}(n)$-bundle. For each $i \in \{1, 2, \ldots, n\}$, the i-th Stiefel-Whitney class for M is exactly the element

$$w_i(M) \equiv f^* w_i\left(E_{\mathrm{O}(n)}\right)$$

of $H^i(M, \mathbb{Z}_2)$. Note that w_i is natural in the sense that $w_i\left(f^* E_{\mathrm{O}(n)}\right) = f^* w_i\left(E_{\mathrm{O}(n)}\right)$. Now, the complexification of π is the vector bundle $T_{\mathbb{C}}M$, with classifying map $f : M \to B\mathrm{U}(n)$. In this case, the i-th Chern class for M is exactly the element

$$c_i(M) \equiv f^* c_i\left(E_{\mathrm{U}(n)}\right)$$

of $H^{2i}(M, \mathbb{Z})$. In general, any vector bundle ξ with group $\mathrm{O}(n) \subset \mathrm{U}(n)$ and fiber $\mathbb{R}^n$ has a complexification $\xi \otimes \mathbb{C}$ with group $\mathrm{U}(n)$ and fiber $\mathbb{C}^n$. Every fiber E_x of ξ is converted to a fiber $E_x \oplus E_x$ of $\xi \otimes \mathbb{C}$ with scalar multiplication given by $i \cdot (x, y) \equiv (-y, x)$. The *$i$-th Pontryagin class* for M is exactly

$$p_i(M) \equiv (-1)^i c_{2i}(TM \otimes \mathbb{C}).$$

For any two complex vector bundles ξ and η over X with groups $\mathrm{U}(n)$ and $\mathrm{U}(m)$, respectively, we can form a new vector bundle $\xi \oplus \eta$ with group $\mathrm{U}(n+m)$ and fiber $\mathbb{C}^n \times \mathbb{C}^m$ such that

$$h_{\alpha\beta}(\xi \oplus \eta) \equiv \begin{bmatrix} h_{\alpha\beta}(\xi) & 0 \\ 0 & h_{\alpha\beta}(\eta) \end{bmatrix}.$$

Terminology. The sum $c(\xi) \coloneqq 1 + c_1 + c_2 + \cdots + c_n \in H^*(X, \mathbb{Z})$ is called the *total Chern class* for η. If ξ is instead a real vector bundle, then we have the *total Stiefel-Whitney class* $w(\xi) \coloneqq 1 + w_1 + w_2 + \cdots + w_n \in H^*(X, \mathbb{Z}_2)$.

Theorem 4.1.6 (Whitney sum formulas).

1. $c(\xi \oplus \eta) = c(\xi) \smile c(\eta)$, *i.e.*, $c_k(\xi \oplus \eta) = \sum_{i+j=k} c_i(\xi) \smile c_j(\eta), \quad c_0 \equiv 1.$
2. $w(\xi \oplus \eta) = w(\xi) \smile c(\eta)$, *i.e.*, $w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) \smile w_j(\eta), \quad w_0 \equiv 1.$

Exercise 4.1.7. *Let ξ and η be fiber bundles over a CW complex X with group $U(1) \cong S^1$ and fiber $\mathbb{C}$, i.e., (complex) line bundles over X. Consider the tensor bundle $\xi \otimes \eta$.*

(a) Show that the set $\mathcal{LB}(X)$ of isomorphism classes of line bundles over X is an abelian group under $\otimes$.

(b) Compute $c_1(\xi \otimes \eta)$ where c_1 denotes the first Chern class.

Proof.

(a) It is easy to see that $\otimes$ is commutative and associative. It is also easy to see that the natural isomorphism $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ extends to an isomorphism between $\xi \otimes \eta$ and the trivial line bundle $X \times \mathbb{C}$, which is thus the identity element of $\mathcal{LB}(X)$.

It remains to show that $\mathcal{LB}(X)$ has all inverses. To this end, consider the conjugate line bundle $\bar{\xi}$. If $h_{\beta\alpha} : U_\alpha \cap U_\beta \to S^1$ is any transition function for ξ, then the corresponding transition function for $\xi \otimes \bar{\xi}$ is given by

$$\tilde{h}_{\beta\alpha} \coloneqq h_{\beta\alpha} \otimes \overline{h_{\beta\alpha}} : U_\alpha \cap U_\beta \to S^1, \qquad x \mapsto h_{\beta\alpha}(x) \cdot \overline{h_{\beta\alpha}(x)}.$$

But this means that every transition function for $\xi \otimes \bar{\xi}$ is the constant map at 1. As a result,

$$\tilde{h}_\beta \circ \tilde{h}_\alpha^{-1}(x, f) = (x, f)\,, \tag{$\star$}$$

i.e., the local trivializations $\tilde{h}_\alpha$ and $\tilde{h}_\beta$ for $\xi \otimes \bar{\xi}$ agree on $U_\alpha \cap U_\beta$. By gluing together these local trivializations, we get a well-defined bundle isomorphism

$$\tilde{h} \equiv \bigcup_\alpha h_\alpha : \xi \otimes \bar{\xi} \xrightarrow{\cong} X \times \mathbb{C}.$$

This proves that $\bar{\xi}$ is the inverse of ξ.

(b) Since $U(1)$ is a topological group, we can apply the Milton construction to it to obtain the universal bundle

$$p_{U(1)} : L \to BU(1) = \mathbb{CP}^\infty.$$

For each $i \in \{1, 2\}$, let $\pi_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$ denote the i-th projection. Consider now the tensor bundle

$$\tilde{L} \coloneqq \underbrace{\pi_1^*(L)}_{L_1} \otimes \underbrace{\pi_2^*(L)}_{L_2}$$

over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$. Then $c_1(\tilde{L})$ belongs to $H := H^2(\mathbb{CP}^n \times \mathbb{CP}^n; \mathbb{Z})$. From lecture, we know that $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ is precisely the infinite cyclic group generated by $c_1(L)$. In particular, it is a finitely-generated abelian group. Further, $\mathbb{CP}^\infty$ is a CW complex as the colimit of CW complexes. Since $H^0(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}$ and $H^1(\mathbb{CP}^\infty; \mathbb{Z}) = 0$, it follows from the Künneth theorem that

$$H \cong \big(H^2(\mathbb{CP}^\infty; \mathbb{Z}) \otimes \mathbb{Z}\big) \oplus \big(\mathbb{Z} \otimes H^2(\mathbb{CP}^\infty; \mathbb{Z})\big) \cong H^2(\mathbb{CP}^\infty; \mathbb{Z}) \oplus H^2(\mathbb{CP}^\infty; \mathbb{Z}). \tag{1}$$

Under this sequence of isomorphisms, $c_1(\tilde{L}) \in H$ corresponds uniquely to $(nc_1(L), mc_1(L))$ for some $n, m \in \mathbb{Z}$. Choose any point $P \in \mathbb{CP}^\infty$ and let $i : \{P\} \times \mathbb{CP}^\infty \hookrightarrow \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ denote the inclusion map. Taking the pullback of L along i yields a line bundle i^*L over $\{P\} \times \mathbb{CP}^\infty \cong \mathbb{CP}^\infty$ that is isomorphic to L_2 viewed over $\mathbb{CP}^\infty$. Hence $c_1(i^*L) = c_1(L)$. But $i^*(c_1(L)) = mc_1(L)$, so that $c_1(L) = mc_1(L)$ by naturality of c_1. It follows that $m = 1$. Similarly, we see that $n = 1$.

By inspection, $c_1(L_1) + c_1(L_2)$ also corresponds uniquely to $(c_1(L), c_1(L))$ under (1). This means that

$$c_1(L_1) + c_1(L_2) = c_1(\tilde{L}). \tag{2}$$

From (2), we can derive that $c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$. Indeed, there must be maps $g_1, g_2 : X \to BU(1)$ such that $\xi \cong f_1^* L$ and $\eta \cong f_2^* L$. Note that the map $F := (f_1, f_2) : X \to BU(1) \times BU(1)$ satisfies

$$\begin{gathered} F^*(L_1) = F^*(\pi_1^*(L)) \cong (\pi_1 \circ F)^*(L) = f_1^* L \cong \xi \\ \\ F^*(L_2) = F^*(\pi_2^*(L)) \cong (\pi_2 \circ F)^*(L) = f_2^* L \cong \eta. \end{gathered} \tag{3}$$

Now, recalling that for any line bundle E over X, the local trivializations and transition functions of $F^*(E)$ are given by precomposition with F, we easily see that

$$c_1(F^*(L_1) \otimes F^*(L_2)) = c_1(F^*(L_1 \otimes L_2)). \tag{4}$$

Finally, by combining (2), (3), (4), and the naturality of c_1, we get

$$\begin{aligned} c_1(\xi \otimes \eta) &= c_1(F^*(L_1) \otimes F^*(L_2)) \\ &= c_1(F^*(L_1 \otimes L_2)) \\ &= F^*(c_1(L_1 \otimes L_2)) \\ &= F^*(c_1(L_1) + c_1(L_2)) \\ &= F^*(c_1(L_1)) + F^*(c_1(L_2)) \\ &= c_1(F^*(L_1)) + c_1(F^*(L_2)) \\ &= c_1(\xi) + c_1(\eta). \end{aligned}$$

□