

## Abstract

We introduce the concept of a universal property in category theory. The main sources for this talk are the following.

- nLab
- John Rognes’s *Lecture Notes on Algebraic K-Theory*, Ch. 4
- Peter Johnstone’s lecture notes for “Category Theory” (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4
- Steve Awodey’s *Category Theory*, Sect. 5.6

## 1 Universal arrows

**Definition 1.1.** Let  $\mathcal{C}$  be a category.

1. An object  $X$  of  $\mathcal{C}$  is *initial* if for each  $Y \in \text{ob } \mathcal{C}$ , there is a unique morphism  $f : X \rightarrow Y$ .
2. We say that  $X$  is *terminal* if for each  $Z \in \text{ob } \mathcal{C}$ , there is a unique morphism  $g : Z \rightarrow X$ .

Either condition is called a *universal property* of  $X$ .

Any property  $P$  of  $\mathcal{C}$  has a dual property  $P^{\text{op}}$  of  $\mathcal{C}^{\text{op}}$  obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing  $P$ . Then  $P$  is true of  $\mathcal{C}$  iff  $P^{\text{op}}$  is true of  $\mathcal{C}^{\text{op}}$ .

**Example 1.2.** Being initial and being terminal are dual properties.

**Lemma 1.3.** *Any two initial objects of  $\mathcal{C}$  are unique up to unique isomorphism. The same holds for any two terminal objects of  $\mathcal{C}$ .*

*Proof sketch.* Let  $X$  and  $X'$  be two initial objects. Compose the two unique morphisms  $X \rightarrow X'$  and  $X' \rightarrow X$  to get an isomorphism between  $X$  and  $X'$ . Apply duality to this argument for the case of terminal objects.  $\square$

We can think of a universal property as follows. Let  $F : \mathcal{D} \rightarrow \mathcal{C}$  be a functor and  $X \in \text{ob } \mathcal{C}$ . A *universal arrow from  $X$  to  $F$*  is an ordered pair  $(Y, f)$  with  $Y \in \text{ob } \mathcal{D}$  and  $f : X \rightarrow F(Y)$  a morphism of  $\mathcal{C}$  with the property that for any  $X' \in \text{ob } \mathcal{D}$  and morphism  $f' : X \rightarrow F(X')$  of  $\mathcal{C}$ , there exists a unique morphism  $\hat{f} : Y \rightarrow X'$  of  $\mathcal{D}$  such that  $F(\hat{f}) \circ f = f'$ .

$$\begin{array}{ccc} X & \xrightarrow{f} & F(Y) \\ & \searrow f' & \downarrow F(\hat{f}) \\ & & F(X') \end{array}$$

Dually, a *universal arrow from  $F$  to  $X$*  is an ordered pair  $(Y, f)$  with  $Y \in \text{ob } \mathcal{D}$  and  $f : F(Y) \rightarrow X$  of  $\mathcal{C}$  with the property that for any  $X' \in \text{ob } \mathcal{D}$  and morphism  $f' : F(X') \rightarrow X$ , there exists a unique morphism  $\hat{f} : X' \rightarrow Y$  such that  $f' = f \circ F(\hat{f})$ .

$$\begin{array}{ccc} F(X') & \xrightarrow{F(\hat{f})} & F(Y) \\ & \searrow f' & \downarrow f \\ & & X \end{array}$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

**Definition 1.4.**

1. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $Y \in \text{ob } \mathcal{D}$ . The *slice* or *left fiber category*, denoted by  $(F/Y)$  or  $(F \downarrow Y)$ , has as objects pairs  $(X, f)$  where  $f : F(X) \rightarrow Y$  and as morphisms from  $f : F(X) \rightarrow Y$  to  $f' : F(X') \rightarrow Y$  morphisms  $g : X \rightarrow X'$  such that  $f = f' \circ F(g)$ .
2. The *coslice* or *right fiber category*, denoted by  $(Y/F)$  or  $(Y \downarrow F)$ , has as objects pairs  $(X, f)$  where  $f : Y \rightarrow F(X)$  and as morphisms from  $f : Y \rightarrow F(X)$  to  $f' : Y \rightarrow F(X')$  morphisms  $g : X \rightarrow X'$  such that  $f' = F(g) \circ f$ .

If  $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$  is opposite to the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $Y \in \text{ob } \mathcal{D}$ , then  $(Y/F)^{\text{op}} = F^{\text{op}}/Y$ . Thus, the left and right fiber categories are dual in the sense that  $P(Y, F)$  is true of any right fiber category  $Y/F$  iff  $P^{\text{op}}(Y, F)$  is true of any left fiber category  $F/Y$ .

**Proposition 1.5.** *Let  $F : \mathcal{D} \rightarrow \mathcal{C}$  be a functor and  $x \in \text{ob } \mathcal{C}$ . Then  $u : x \rightarrow Fr$  is a universal arrow from  $x$  to  $F$  iff it is an initial object of the coslice  $(x \downarrow F)$ . Dually,  $u' : Fr' \rightarrow x$  is a universal arrow from  $F$  to  $x$  iff it is a terminal object of the same category.*

*Proof.* Suppose that  $u$  is universal and  $f : x \rightarrow Fy$  is another object of  $(x \downarrow F)$ . Then there exists a unique  $\hat{f} : r \rightarrow y$  such that  $F(\hat{f}) \circ u = f$ . Thus  $F(\hat{f})$  is a unique morphism of the coslice.

Conversely, suppose that  $u$  is initial. Then for any object  $f : x \rightarrow Fy$  of  $(x \downarrow F)$ , there exists a unique arrow  $Sg : Fr \rightarrow Fy$  such that  $Sg \circ u = f$ . Hence setting  $\hat{f} = g$  makes  $u$  a universal arrow.  $\square$

**Corollary 1.6.** *Any two universal arrows from  $x$  to  $F$  can be canonically identified by Lemma 1.3.*

## 2 (Co)limits

**Definition 2.1.** A *zero object* of  $\mathcal{C}$  is an object that is both initial and terminal.

**Example 2.2.** The unique initial object of **Set** is  $\emptyset$ , and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in  $\text{iso}(\mathbf{Set})$ .

Given  $X \in \text{ob } \mathcal{C}$ , the *undercategory*  $X/\mathcal{C}$  has as objects morphisms in  $\mathcal{C}$  of the form  $i : X \rightarrow Y$  where  $X$  is fixed. Given  $i : X \rightarrow Y$  and  $i' : X \rightarrow Y'$  in  $\text{ob } X/\mathcal{C}$ , define the set of morphisms from  $i$  to  $i'$  as the morphisms  $f : Y \rightarrow Y'$  where

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow i' & \downarrow f \\ & & Y' \end{array}$$

commutes. (We call  $i$  the *structure morphism*.) Composition and identity carry over exactly from  $\mathcal{C}$ .

Likewise, given  $x \in \text{ob } \mathcal{C}$ , the *overcategory*  $\mathcal{C}/X$  has as objects morphisms in  $\mathcal{C}$  of the form  $i : Y \rightarrow X$  where  $X$  is fixed. Given  $i : Y \rightarrow X$  and  $i' : Y' \rightarrow X$  in  $\text{ob } \mathcal{C}/X$ , define the set of morphisms from  $i$  to  $i'$  as the morphisms  $f : Y \rightarrow Y'$  where

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & \searrow i & \downarrow i' \\ & & X \end{array}$$

commutes. Composition and identity carry over exactly from  $\mathcal{C}$ .

**Remark 2.3.** If  $X \in \text{ob } \mathcal{C}$ , then  $(X/\mathcal{C})^{\text{op}} = \mathcal{C}^{\text{op}}/X$ . Thus, the under- and overcategories are dual in the sense that  $P(X, \mathcal{C})$  is true of any undercategory  $X/\mathcal{C}$  iff  $P^{\text{op}}(X, \mathcal{C})$  is true of any overcategory  $\mathcal{C}/X$ .

**Lemma 2.4.** *For any  $X \in \mathcal{C}$ , the identity morphism on  $X$  is an initial object  $X/\mathcal{C}$ . Dually, it is a terminal object in  $\mathcal{C}/X$ .*

*Proof.* Any morphism  $i : X \rightarrow Y$  is itself the unique morphism from  $\text{Id}_X$  to  $i$ . □

**Lemma 2.5.** *Let  $X$  be an initial object of  $\mathcal{C}$ . The identity morphism on  $X$  is a zero object  $\mathcal{C}/X$ . Dually, if  $Y \in \text{ob } \mathcal{C}$  is terminal, then  $\text{Id}_Y$  is a zero object in  $Y/\mathcal{C}$ .*

*Proof.* We already know that  $\text{Id}_X$  is terminal. If  $p : Y \rightarrow X$  is an object in  $\mathcal{C}/X$ , then there is a unique morphism  $f : X \rightarrow Y$ . Then  $f \circ p$  must equal  $\text{Id}_X$ . □

**Example 2.6.** Let  $(X, x)$  be a pointed set with  $X = \{x\}$ . Let  $\mathbf{Set}_*$  denotes the category of pointed sets with base point preserving functions. Since  $\mathbf{Set}_* \cong X/\mathbf{Set}$ , it follows that  $X$  is a zero object in  $\mathbf{Set}_*$ .

Given a morphism  $\alpha : X \rightarrow Z$  in  $\mathcal{C}$ , define the *under-and-overcategory*  $(X/\mathcal{C}/Z)_\alpha$  as having triples  $(Y, i, p)$  as objects where  $i : X \rightarrow Y$  and  $p : Y \rightarrow Z$  are morphisms in  $\mathcal{C}$  such that  $p \circ i = \alpha$ . Define the set of morphisms from  $(Y, i, p)$  to  $(Y', i', p')$  as the set of morphisms  $f : Y \rightarrow Y'$  such that

$$\begin{array}{ccc} X & \xrightarrow{i'} & Y' \\ i \downarrow & \searrow f & \downarrow p' \\ Y & \xrightarrow{p} & Z \end{array}$$

$\alpha$

commutes. If  $\alpha = \text{Id}_X$ , then we call  $(X/\mathcal{C}/X)_{\text{Id}_X}$  the category of *retractive* objects over  $X$ , with each triple  $(Y, i, p)$  being a retraction of  $Y$  onto  $X$ .

**Example 2.7.** If  $F : \mathcal{C} \rightarrow \mathcal{C}$  is the identity functor, then the undercategory  $Y/\mathcal{C}$  equals the right fiber category  $Y/F$ , and the overcategory  $\mathcal{C}/Y$  equals the left fiber category  $F/Y$ .

Let  $\mathcal{J}$  be a category. A *diagram of shape  $\mathcal{J}$  in  $\mathcal{C}$*  is a functor  $F : \mathcal{J} \rightarrow \mathcal{C}$ .

**Definition 2.8.** Given a functor  $F : \mathcal{J} \rightarrow \mathcal{C}$  and  $X \in \text{ob } \mathcal{C}$ , a *cone over  $F$*  consists of an *apex*  $X \in \text{ob } \mathcal{C}$  and *legs*  $f_j : X \rightarrow F(j)$  for each  $j \in \text{ob } \mathcal{J}$  such that for any morphism  $\alpha : j \rightarrow j'$ , the triangle

$$\begin{array}{ccc} X & \xrightarrow{f_j} & F(j) \\ & \searrow f_{j'} & \downarrow F\alpha \\ & & F(j') \end{array}$$

commutes.

This is simply a natural transformation  $\Delta_{\mathcal{J}} X \Rightarrow F$  where  $\Delta_{\mathcal{J}} X$  denotes the constant functor on  $\mathcal{J}$  at  $X$ . If  $\mathcal{J}$  is small, then  $\Delta_{\mathcal{J}}$  is a functor from  $\mathcal{C}$  to  $\mathbf{Fun}(\mathcal{J}, \mathcal{C})$ .

**Definition 2.9.**

1. The *category of cones over  $F$*  is the right fiber category  $X/F$ .
2. The *category of cocones under  $F$*  is the left fiber category  $F/X$ .

**Definition 2.10 (Colimit).** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $g : Y \rightarrow Z$  a morphism in  $\mathcal{D}$ . Let  $\Delta_{\mathcal{C}} g : \Delta_{\mathcal{C}} Y \Rightarrow \Delta_{\mathcal{C}} Z$  be the natural transformation with components  $X \mapsto g$ .

1. A *colimit*  $\text{colim}_{\mathcal{C}} F$  of the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an object  $Y$  of  $\mathcal{D}$  together with a natural transformation  $i : F \Rightarrow \Delta_{\mathcal{C}} Y$  such that for any  $Z \in \text{ob } \mathcal{D}$  and any natural transformation  $j : F \Rightarrow \Delta_{\mathcal{C}} Z$ , there is a unique morphism  $g : Y \rightarrow Z$  such that  $j = \Delta_{\mathcal{C}} g \circ i$ .
2. We say that  $\mathcal{D}$  *admits/has  $\mathcal{C}$ -shaped colimits* if each functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  has a colimit.
3. We say that  $\mathcal{D}$  is *cocomplete* if each functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  with  $\mathcal{C}$  small has a colimit.

If  $\mathcal{C}$  is small, then a colimit of  $F : \mathcal{C} \rightarrow \mathcal{D}$  is just an initial object in the right fiber category  $F/\Delta_{\mathcal{C}}$ , which has as objects pairs  $(Z, j : F \rightarrow \Delta Z)$  and as morphisms from  $(Y, i)$  to  $(Z, j)$  the morphisms  $g : Y \rightarrow Z$  in  $\mathcal{D}$  such that  $\Delta g \circ i = j$ .

**Example 2.11.** If  $\mathcal{C}$  is the empty category, then the empty functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  satisfies  $F/\Delta_{\mathcal{C}} \cong \mathcal{D}$ , so that the colimit of  $F$  is exactly the initial object of  $\mathcal{D}$ .

**Proposition 2.12.** *There is a natural bijection  $\mathcal{D}(Y, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta Z)$  if and only if  $Y = \text{colim}_{\mathcal{C}} F$ .*

**Lemma 2.13.** *Any two colimits are unique up to unique isomorphism.*

*Proof.* When  $\mathcal{C}$  is small, this follows immediately from Lemma 1.3. Notice, however, that our proof of Lemma 1.3 does *not* require that  $\mathcal{C}$  be locally small (a property which Rognes stipulates of any category).  $\square$

*Remark 2.14.* Assume that  $\mathcal{D}$  has  $\mathcal{C}$ -shaped colimits and that  $\mathcal{C}$  is small. Then a (possibly global) choice function  $\text{colim}_{\mathcal{C}} : \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$  given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor  $\Delta_{\mathcal{C}} : \mathcal{D} \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$ . Indeed, for any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , there is a bijection  $\mathcal{D}(\text{colim}_{\mathcal{C}} F, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta_{\mathcal{C}} Z)$ .

**Definition 2.15 (Limit).** A *limit* of the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a colimit of  $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ .

Explicitly, a limit for  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an object  $Z$  of  $\mathcal{D}$  along with a natural transformation  $p : \Delta_{\mathcal{C}} Z \Rightarrow F$  such that for any  $Y \in \text{ob } \mathcal{D}$  and any natural transformation  $q : \Delta_{\mathcal{C}} Y \Rightarrow F$ , there is a unique morphism  $g : Y \rightarrow Z$  such that  $q = p \circ \Delta_{\mathcal{C}} g$ .

Note that the colimit of a functor  $F$  is exactly the limit of  $F^{\text{op}}$ . Hence *limit* and *colimit* are dual properties, and our results so far for colimits can be dualized for limits.

**Definition 2.16 ((Co)product).** Let  $\mathcal{J}$  be a discrete small category. Consider a diagram  $\{A_i\}_{i \in \text{ob } \mathcal{J}}$  of shape  $\mathcal{J}$ .

1. The limit of  $\{A_i\}_i$  is called the *product*  $\prod_i A_i$ , equipped with projections  $\pi_i : \prod_i A_i \rightarrow A_i$  such that for every  $f_i : U \rightarrow A_i$  there exists a unique map  $f := (f_i) : U \rightarrow \prod_i A_i$  satisfying  $\pi_i \circ f = f_i$ .
2. The colimit of  $\{A_i\}_i$  is called the *coproduct*  $\coprod_i A_i$ , equipped with inclusions  $u_i : A_i \rightarrow \coprod_i A_i$  such that for any  $f_i : A_i \rightarrow Y$ , there exists a unique map  $f := (f_i) : \coprod_i A_i \rightarrow Y$  satisfying  $f_i = f \circ u_i$ .

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

**Example 2.17.**

- (1) Consider any small diagram  $F : \mathcal{J} \rightarrow \mathbf{Set}$ . On the one hand,

$$\text{colim}_{\mathcal{J}} F_j \cong \left( \coprod_{j \in \text{ob } \mathcal{J}} F_j \right) / \sim$$

where  $\sim$  is the smallest equivalence relation such that  $F_j \ni f_j \sim f_{j'} \in F_{j'}$ , whenever  $F(\psi)(f_j) = f_{j'}$  for some  $\psi : j \rightarrow j'$ .

On the other hand,

$$\lim_{\mathcal{J}} F_j \cong \left\{ (f_j)_j \in \prod_{j \in \text{ob } \mathcal{J}} F_j \mid \forall \psi : j \rightarrow j' \text{ in } \mathcal{J}, F(\psi)(f_j) = f_{j'} \right\}.$$

We have shown that  $\mathbf{Set}$  is both complete and cocomplete.

- (2) Let  $A$  be any set. Define the *cumulative hierarchy*  $V_n(A)$  of rank  $n < \omega$  over  $A$  along with a countable sequence

$$V_0 \xrightarrow{v_0} V_1 \xrightarrow{v_1} V_2 \longrightarrow \cdots \longrightarrow V_n \xrightarrow{v_n} V_{n+1} \longrightarrow \cdots$$

of maps recursively by

$$\begin{aligned} V_0(A) &= A \\ V_{n+1}(A) &= A \coprod \mathcal{P}(V_n(A)) \\ v_0 : A &\hookrightarrow A \coprod \mathcal{P}(A), & a &\mapsto a \\ v_{n+1} : A \coprod \mathcal{P}(V_n(A)) &\rightarrow A \coprod \mathcal{P}(V_{n+1}(A)), & (\text{Id}_A, \mathcal{P}(V_n(A))) & \end{aligned}$$

Let  $V_\omega(A) = \text{colim}_{n < \omega} V_n(A)$ , which exists by part (1). Then  $V_\omega(\emptyset)$  is exactly the set of all hereditarily finite sets. To see that  $V_\omega(-)$  is a functor  $\mathbf{Set} \rightarrow \mathbf{Set}$ , let  $f : A \rightarrow B$  be a function. Then we can build a cocone

$$\begin{array}{ccccccc} V_0(A) & \xrightarrow{v_0} & V_1(A) & \longrightarrow & \cdots & \longrightarrow & V_n(A) & \xrightarrow{v_n} & V_{n+1}(A) \\ f_0 \equiv f \downarrow & & \downarrow (f, \mathcal{P}(f)) & & & & \downarrow (f, \mathcal{P}(f_{n-1})) & & \downarrow (f, \mathcal{P}(f_n)) \\ V_0(B) & \longrightarrow & V_1(B) & \longrightarrow & \cdots & \longrightarrow & V_n(B) & \longrightarrow & V_{n+1}(B) & \longrightarrow & V_\omega(B) \end{array}$$

under  $\{V_n(A)\}_n$  recursively. By the universal property of colimits, there exists a unique map  $V_\omega(A) \rightarrow V_\omega(B)$ , so that  $V_\omega(-)$  is functorial.

Let  $\mathcal{J}$  be a category of the form  $\bullet \rightrightarrows \bullet$ . Then a diagram  $D$  of shape  $\mathcal{J}$  looks like  $A \begin{smallmatrix} f \\ \rightrightarrows \\ g \end{smallmatrix} B$ . A cone over  $D$  with apex  $C$  and legs  $f_1 : C \rightarrow A$  and  $f_2 : C \rightarrow B$  satisfies  $f \circ f_1 = f_2 \circ g_1$ .

**Definition 2.18 ((Co)equalizer).**

1. If such an object  $C$  together with  $f_1$  is the limit of  $D$ , then we say it is the *equalizer* of  $f$  and  $g$ .
2. The colimit of  $D$  is called the *coequalizer* of  $f$  and  $g$ .

**Example 2.19.** The equalizer in **Set** of  $f, g : X \rightarrow Y$  is the subset  $X' := \{x \in X : f(x) = g(x)\}$  together with the inclusion function  $X' \hookrightarrow X$ .

The coequalizer of  $(f, g)$  is precisely  $Y/\sim$  together with the quotient map on  $B$  where  $\sim$  is the smallest equivalence relation under which  $f(x) \sim g(x)$  for every  $x$ .

It is easy to check that any equalizer  $f : C \rightarrow A$  must be monic. Further, if  $f$  is split epic, i.e., has a section  $g : A \rightarrow C$ , as well, then  $f$  is an isomorphism. For, in this case,  $f \circ (g \circ f) = \text{Id}_A \circ f = f = f \circ \text{Id}_C$ . As  $f$  is monic, we have that  $g \circ f = \text{Id}_C$ , so that  $g$  is an inverse of  $f$ .

Next, let  $\mathcal{J}$  be a category of the form  $\bullet \rightarrow \bullet \leftarrow \bullet$ . Then a diagram of this shape looks like  $B \xrightarrow{f} C \xleftarrow{g} A$ , and a cone over this diagram looks like

$$\begin{array}{ccc} E & \xrightarrow{j} & A \\ i \downarrow & \searrow \alpha & \downarrow g \\ B & \xrightarrow{f} & C \end{array}$$

**Definition 2.20 (Pullback).** If such an object  $E$  together with  $i$  and  $j$  is the limit of this diagram, then we call it the *pullback* of  $f$  and  $g$ , denoted by  $B \times_C A$ .

The universal property of a pullback square states that for any commutative diagram of the form

$$\begin{array}{ccc} & & \\ & & \\ Z & \searrow & \\ & & B \times_C A \xrightarrow{\pi_A} A \\ & \searrow \pi_B \downarrow & \downarrow f \\ & B \xrightarrow{g} C & \end{array}$$

there is a unique *mediating map*  $Z \rightarrow B \times_C A$  fitting into it.

If we perform a dual construction for  $\mathcal{J}^{\text{op}}$ , then the colimit of the resulting diagram is called the *pushout*, denoted by  $B \cup_C A$ . The universal property of a pullback square states that for any commutative diagram of the form

$$\begin{array}{ccc} B \times_C A & \xrightarrow{\pi_A} & A \\ \pi_B \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array} \quad , \quad \begin{array}{ccc} & & Z \\ & \nearrow & \\ & & \end{array}$$

there is a unique mediating map  $B \cup_C A \rightarrow Z$  fitting into it.

**Example 2.21.**

1. The pullback in **Set** of  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  is precisely  $\{(x, y) \in X \times Y : f(x) = g(y)\}$ , called the *fibered product* of  $X$  and  $Y$  over  $Z$ .
2. The pushout in **Set** of  $f : Z \rightarrow X$  and  $g : Z \rightarrow Y$  is precisely the quotient of  $X \amalg Y$  by the equivalence relation  $\sim$  generated by the formula  $(\forall z \in Z) (f(z) \sim g(z))$ . We call  $X \amalg Y / \sim$  the *fibered sum* of  $X$  and  $Y$  under  $Z$ .

**Example 2.22.** Let  $\mathbf{FinSet}_{\text{mono}}$  denote the category of all finite sets with injective functions as arrows. The category of *nominal sets* consists of all pullback-preserving functors  $\mathbf{FinSet}_{\text{mono}} \rightarrow \mathbf{Set}$  with natural transformations as arrows. These correctly encode the syntax of functional programming languages modulo renaming of bound variables (which is necessary for substitution).

**Proposition 2.23.** *The pullback of a monomorphism in a category  $\mathcal{C}$  is again a monomorphism in  $\mathcal{C}$ .*

*Proof.* Consider any pullback square

$$\begin{array}{ccc} B \times_C A & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

in  $\mathcal{C}$  where  $f$  is monic. We must show that  $\pi_1$  is monic. Let  $h_1, h_2 : B' \rightarrow B \times_C A$  be morphisms in  $\mathcal{C}$  such that

$$\begin{aligned} \pi_1 \circ h_1 &= \pi_1 \circ h_2 \\ \Downarrow \\ f \circ \pi_2 \circ h_1 &= g \circ \pi_1 \circ h_1 = g \circ \pi_1 \circ h_2 = f \circ \pi_2 \circ h_2. \end{aligned}$$

Since  $f$  is monic by assumption, it follows that  $\pi_2 \circ h_1 = \pi_2 \circ h_2$ . As a result, the universal property of pullbacks implies that  $h_1 = h_2$ , as required.  $\square$

Our next two results are quite useful and follow directly from the universal property of pullback (dually, pushout) squares.

**Proposition 2.24.** *Let  $f : X \rightarrow Y$  be a morphism in a category  $\mathcal{C}$ .*

1. *The commutative square*

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \parallel & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

*is a pullback if and only if  $f$  is a monomorphism.*

2. *The commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f \downarrow & & \parallel \\ Y & \xlongequal{\quad} & Y \end{array}$$

*is a pushout if and only if  $f$  is an epimorphism.*

**Proposition 2.25 (Pasting law).** *Consider a commutative diagram of the form*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

*in a category  $\mathcal{C}$ .*

1. *Suppose that the righthand square is a pullback. Then the total rectangle is a pullback if and only if the lefthand square is one.*
2. *Suppose that the lefthand square is a pushout. Then the total rectangle is a pushout if and only if the righthand square is one.*

**Corollary 2.26.** *The operations of forming pullbacks and forming pushouts are associative up to isomorphism.*

All coequalizers  $A \begin{smallmatrix} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{smallmatrix} B \xrightarrow{h} C$  can be obtained from taking binary coproducts and pushouts as follows.

$$\begin{array}{ccc} A \amalg A & \xrightarrow{(f,g)} & B \\ (\text{Id}_A, \text{Id}_A) \downarrow & \lrcorner & \downarrow h \\ A & \longrightarrow & C \end{array}$$

Therefore, any category with binary coproducts and pushouts has coequalizers.

Moreover, any colimit of a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \quad (*)$$

is precisely the coequalizer of

$$\prod_n X_n \begin{array}{c} \xrightarrow{\text{Id}} \\ \xrightarrow{(u_{n+1} \circ f_n)_n} \end{array} \prod_n X_n.$$

Therefore, any category with coequalizers and small coproducts has colimits of diagrams like  $(*)$ . This fact can be generalized as follows.

**Theorem 2.27 (Freyd).**

- (i) *If  $\mathcal{C}$  has equalizers and small (resp. finite) products, then it has small (resp. finite) limits.*
- (ii) *If  $\mathcal{C}$  has pullbacks and a terminal object, then it has finite limits.*

*Proof sketch.*

1. Let  $F : \mathcal{J} \rightarrow \mathcal{C}$  be any diagram with  $\mathcal{J}$  small. Consider the following two morphisms in  $\mathcal{C}$ :

$$\begin{aligned} f, g : \prod_{j \in \text{ob } \mathcal{J}} F_j &\rightarrow \prod_{\alpha : i \rightarrow j} F_j \\ \pi_{\alpha : i \rightarrow j} \circ f &\equiv \pi_j \\ \pi_{\alpha : i \rightarrow j} \circ g &\equiv F(\alpha) \circ \pi_i. \end{aligned}$$

Then  $\lim_{\mathcal{J}} F$  is precisely the equalizer of  $f$  and  $g$ .



2. Thanks to part (i), it suffices to show that  $\mathcal{C}$  has equalizers and finite products. By assumption, there is some terminal object  $1$ . Then any product  $A_1 \times A_2$  can be realized as the pullback of  $A_1 \rightarrow 1 \leftarrow A_2$ . By induction, it follows that  $\mathcal{C}$  has finite products. Moreover, for any morphisms  $f, g : A \rightarrow B$ , note that any cone over the diagram

$$A \xrightarrow{(\text{Id}_A, g)} A \times B \xleftarrow{(\text{Id}_A, f)} A$$

yields morphisms  $h : A \rightarrow C$  and  $k : C \rightarrow A$  such that  $h = k$  and  $fk = gh$ . As a result, the pullback for this cospan is an equalizer of  $f$  and  $g$ , and thus our proof is complete.  $\square$

We may view Example 2.17(1) as an instance of Theorem 2.27.

Next, let us show that adjoints interact nicely with (co)limits under mild conditions.

**Proposition 2.28 (Left adjoints preserve colimits).** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be functors such that  $(F, G)$  is an adjoint pair. Let  $\mathcal{E}$  be small category. If  $X : \mathcal{E} \rightarrow \mathcal{C}$  is a functor whose colimit exists, then*

$$\text{colim}_{\mathcal{E}}(F \circ X) \cong F \left( \text{colim}_{\mathcal{E}} X \right).$$

*Dually, if  $Y : \mathcal{E} \rightarrow \mathcal{D}$  is a functor whose limit exists, then*

$$\lim_{\mathcal{E}}(G \circ Y) \cong G \left( \lim_{\mathcal{E}} Y \right).$$

*Proof.* We have the following chain of natural bijections in  $Y \in \text{ob } \mathcal{D}$ :

$$\begin{aligned} \mathcal{D} \left( F \left( \text{colim}_{\mathcal{E}} X \right), Y \right) &\cong \mathcal{C} \left( \text{colim}_{\mathcal{E}} X, G(Y) \right) \\ &\cong \lim_{\mathcal{E}} \mathcal{C}(X(-), G(Y)) \\ &\cong \lim_{\mathcal{E}} \mathcal{D}(F(X(-)), Y) \\ &\cong \mathbf{Fun}(\mathcal{E}, \mathcal{D})(F \circ X, \Delta Y). \end{aligned}$$

The second bijection exists because both sets can be identified with the components of all natural transformations  $X \Rightarrow \Delta G(Y)$ .  $\square$

### 3 Fibers and Fibrations

**Definition 3.1.** Suppose  $\mathcal{C}$  has a terminal object  $1$ . Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{C}$ .

1. Given a global element  $p : 1 \rightarrow Y$  of  $Y$ , the *fiber*  $f^{-1}(p)$  of  $f$  at  $p$  is the pullback of the cospan  $1 \rightarrow Y \leftarrow X$ .
2. The *cofiber*  $Y/X$  of  $f$  is the pushout of the span  $1 \leftarrow X \rightarrow Y$ .

For any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the *fiber category*  $F^{-1}(Y)$  is the full subcategory of  $\mathcal{C}$  generated by those objects  $X$  such that  $F(X) = Y$ .

For each  $Y \in \text{ob } \mathcal{D}$ , there is a full and faithful functor  $F^{-1}(Y) \rightarrow F/Y$  given by  $X \mapsto (X, \text{Id}_Y)$ . We say that  $\mathcal{C}$  is a *precofibered category* over  $\mathcal{D}$  if  $F$  has a left adjoint given by

$$(Z, g : F(Z) \rightarrow Y) \mapsto g_*(Z).$$

Further, there is a full and faithful functor  $F^{-1}(Y) \rightarrow Y/F$ . We say that  $\mathcal{C}$  is a *prefibered category* over  $\mathcal{D}$  if this functor has a right adjoint given by  $(Z, g : Y \rightarrow F(Z)) \mapsto g_*(Z)$ .

**Definition 3.2.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. Let  $f : c' \rightarrow c$  be a morphism in  $\mathcal{C}$ . We say  $f$  is *cartesian* if for any morphism  $f' : c'' \rightarrow c$  in  $\mathcal{C}$  and any morphism  $g : F(c'') \rightarrow F(c')$  in  $\mathcal{D}$  such that  $Ff \circ g = Ff'$ , there exists a unique morphism  $\phi : c'' \rightarrow c$  such that  $f' = f \circ \phi$  and  $F\phi = g$ .

In pictures,

$$\begin{array}{ccc} F(c'') & \xrightarrow{g} & F(c') \\ & \searrow Ff' & \downarrow Ff \\ & & F(c) \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} c'' & \xrightarrow{\exists! \phi} & c' \\ & \searrow f' & \downarrow f \\ & & c \end{array}, \quad \phi \xrightarrow{F} g.$$

2. We say that  $F$  is a *fibration* if for any  $c \in \text{ob } \mathcal{C}$  and morphism  $f : d \rightarrow Fc$  in  $\mathcal{D}$ , there is a cartesian morphism  $\phi_f : c' \rightarrow c$  such that  $F\phi_f = f$ . Such a  $\phi_f$  is called a *cartesian lifting* of  $f$  to  $c$ .

In this case, assuming the axiom of choice, we obtain a mapping  $f \mapsto \phi_f$ , known as a *cleavage* of  $F$ . If this respects the identity map and composition, then we call  $F$  a *normal* and *split* fibration, respectively.

Intuitively, if  $F$  is a fibration, then the fibers  $F^{-1}(Y)$  depend functorially on  $Y \in \text{ob } \mathcal{D}$ .

**Example 3.3.**

1. Let the category **Mod** consist of pairs  $(R, M)$  as objects where  $R$  is a ring and  $M$  is a left  $R$ -module and pairs  $(f, \tilde{f})$  as morphisms where  $f : R \rightarrow R'$  is a ring homomorphism and  $\tilde{f} : M \rightarrow M'$  is an  $R$ -linear map with  $M'$  viewed as an  $R$ -module via  $f$ . Then the forgetful functor  $U : \mathbf{Mod} \rightarrow \mathbf{Ring}$  is a fibration.
2. For any category  $\mathcal{C}$  with pullbacks, consider the arrow category  $\text{Ar}(\mathcal{C})$  along with the codomain functor  $\text{cod} : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$  defined by

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ a' & \xrightarrow{\quad} & b' \end{array} \quad \mapsto \quad b \rightarrow b'.$$

This is a fibration. Indeed, for any object  $x \rightarrow y$  in  $\text{Ar}(\mathcal{C})$  and any morphism  $z \rightarrow y$  in  $\mathcal{C}$ , the cartesian lifting of  $z \rightarrow y$  to  $x \rightarrow y$  is given by the pullback square

$$\begin{array}{ccc} z \times_y x & \longrightarrow & x \\ \downarrow & & \downarrow \\ z & \longrightarrow & y \end{array}.$$