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Homotopy and K-theory seminar
Talk #16
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Abstract

We continue doing higher Waldhausen K-theory. The main sources for this talk are the following.

- nLab.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Ch. V.2.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

Recall that $|wS_\bullet\mathscr{C}|$ is an H-space via the map

$$\coprod : |wS_\bullet\mathscr{C}| \times |wS_\bullet\mathscr{C}| \cong |wS_\bullet\mathscr{C} \times wS_\bullet\mathscr{C}| \to |wS_\bullet\mathscr{C}|.$$

This produces an H-space structure $(K(\mathscr{C}), +)$.

Definition 1. Let $\mathscr{B}$ and $\mathscr{C}$ be Waldhausen categories. We say that $F' \rightarrowtail F \twoheadrightarrow F''$ is a *short exact sequence* or *cofiber sequence of exact functors* if every $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$ is a cofiber sequence and $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration in $\mathscr{C}$ for every $A \rightarrowtail B$ in $\mathscr{B}$.

Let $\mathscr{C}$ be a Waldhausen category. Let $(\eta) : A \rightarrowtail B \twoheadrightarrow C$ be an object in $S_2\mathscr{C}$. Define the source s, target t, and quotient q functors $S_2\mathscr{C} \to \mathscr{C}$ by $s(\eta) = A$, $t(\eta) = B$, and $q(\eta) = C$. Then $s \rightarrowtail t \twoheadrightarrow q$ is a cofiber sequence of functors. Since defining a cofiber sequence of exact functors $\mathscr{B} \to \mathscr{C}$ is equivalent to defining an exact functor $\mathscr{B} \to S_2\mathscr{C}$, we may restrict our attention to $s \rightarrowtail t \twoheadrightarrow q$ when proving assertions about a given cofiber sequence of exact functors $\mathscr{B} \to \mathscr{C}$. We say that $S_2\mathscr{C}$ is universal in this sense.

Theorem 1 (Extension theorem). *Let $\mathscr{C}$ be Waldhausen. The exact functor $(s, q) : S_2\mathscr{C} \to \mathscr{C} \times \mathscr{C}$ induces a homotopy $K(S_2\mathscr{C}) \simeq K(\mathscr{C}) \times K(\mathscr{C})$. The functor $\coprod : (A, B) \to (A \rightarrowtail A \coprod B \twoheadrightarrow B)$ is a homotopy inverse.*

Proof. Let $\mathscr{C}_m^w$ denote the category of m-length sequences of weak equivalences. For each n, define $s_n\mathscr{C}_m^w$ as the commutative diagram

$$\begin{array}{ccccccc}
X_1^0 & \rightarrowtail & X_2^0 & \rightarrowtail & \cdots & \rightarrowtail & X_n^0 \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
X_1^1 & \rightarrowtail & X_2^1 & \rightarrowtail & \cdots & \rightarrowtail & X_n^1 \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
\vdots & & \vdots & & & & \vdots \\
\sim\downarrow & & \sim\downarrow & & & & \sim\downarrow \\
X_1^m & \rightarrowtail & X_2^m & \rightarrowtail & \cdots & \rightarrowtail & X_n^m
\end{array}.$$

This is naturally isomorphic to an (m, n)-bisimplex in $N_\bullet wS_\bullet\mathscr{C}$, which is thus isomorphic to the bisimplicial set $s_\bullet\mathscr{C}_{(-)}^w$. One can show that the source s and quotient q functors $S_2\mathscr{C} \to \mathscr{C}$ give a homotopy equivalence $s \times q : s_\bullet S_2(\mathscr{C}_m^w) \to s_\bullet\mathscr{C}_m^w \times s_\bullet\mathscr{C}_m^w$ for each m. Thus, we get a homotopy equivalence

$$s_\bullet S_2(\mathscr{C}_{(-)}^w) \simeq s_\bullet\mathscr{C}_{(-)}^w \times s_\bullet\mathscr{C}_{(-)}^w$$

between bisimplicial sets. But we already have that $s_\bullet\mathscr{C}_{(-)}^w \cong N_\bullet wS_\bullet\mathscr{C}$, completing the proof. $\square$

Theorem 2 (Additivity theorem). *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors $\mathscr{B} \to \mathscr{C}$. Then $F_* \simeq F'_* + F''_*$ as maps $K(\mathscr{B}) \to K(\mathscr{C})$. Hence $F_* = F'_* + F''_*$ as maps $K_i(\mathscr{B}) \to K_i(\mathscr{C})$.*

Proof. As $S_2\mathscr{C}$ is universal, it suffices to prove that $t_* \simeq s_* + q_*$. Notice that the two composites

$$\mathscr{C} \times \mathscr{C} \xrightarrow{\coprod} S_2\mathscr{C} \;\overset{t}{\underset{s \coprod q}{\rightrightarrows}}\; \mathscr{C}$$

are the same. The extension theorem implies that $K(\coprod) : K(\mathscr{C}) \times K(\mathscr{C}) \to K(S_2\mathscr{C})$ is a homotopy equivalence. Since the H-space structure on $K(\mathscr{C})$ is induced by $\coprod$, we get $t_* \simeq s_* + q_*$. $\square$

Definition 2. Let $\mathscr{C}$ be Waldhausen. We say that a sequence $* \to A_n \to \cdots \to A_0 \to *$ is *admissibly exact* if each morphism in the sequences can be written as a cofiber sequence $A_{i+1} \twoheadrightarrow B_i \rightarrowtail A_i$.

Corollary 3. *Suppose that $* \to F^0 \to F^1 \to \cdots \to F^n \to *$ is an admissibly exact sequence of exact functors $\mathscr{B} \to \mathscr{C}$. Then $\sum_i (-1)^i F^i_* = 0$ as maps $K_i(\mathscr{B}) \to K_i(\mathscr{C})$.*

Proof. Induct on n. $\square$

Corollary 4. *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors $\mathscr{B} \to \mathscr{C}$. Then*

$$F''_* \simeq F_* - F_* \simeq 0.$$

This implies that the homotopy fiber of $F''_* : K(\mathscr{B}) \to K(\mathscr{C})$ is homotopy equivalent to $K(\mathscr{B}) \vee \Omega K(\mathscr{C})$.

Let $\mathscr{C}$ be a Waldhausen category. Recall the arrow category $\mathrm{Ar}(\mathscr{C})$ of $\mathscr{C}$ consisting of morphisms in $\mathscr{C}$ as objects and commutative squares as morphisms. Let s and t denote the source and target functors $\mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$, respectively.

Definition 3. A functor $T : \mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$ is a *(mapping) cylinder functor* on $\mathscr{C}$ if it comes equipped with natrual transformations $j_1 : s \Rightarrow T$, $j_2 : t \Rightarrow T$, and $p : T \Rightarrow t$ such that for any $f : A \to B$, we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "j_2"'] \arrow[dl, "="] \\ & B & \end{tikzcd} .$$

Moreover, T must satisfy the following axioms.

1. T sends every initial morphism $* \to A$ to A for any $A \in \mathrm{ob}\,\mathscr{C}$.
2. $j_1 \coprod j_2 : A \coprod B \rightarrowtail T(f)$ is a cofibration for any $f : A \to B$.
3. Given a morphism $(a, b) : f \to f'$ in $\mathrm{Ar}(\mathscr{C})$, if both a and b are w.e. in $\mathscr{C}$, then so is $T(f) \to T(f')$.
4. Given a morphism $(a, b) : f \to f'$ in $\mathrm{Ar}(\mathscr{C})$, if both a and b are cofibrations in $\mathscr{C}$, then so is $T(f) \to T(f')$. Also, the map $A' \coprod_A T(f) \coprod_B B' \to T(f')$ induced by axiom 2 is a cofibration in $\mathscr{C}$.
5. (Cylinder Axiom) The map $p : T(f) \to B$ is a w.e. in $\mathscr{C}$.

Terminology. Let T be a cylinder functor on $\mathscr{C}$.

1. We call $T(A \to *)$ the *cone* of A, denoted by $\mathrm{cone}(A)$.
2. We call ${}^{\mathrm{cone}(A)}\!/_{A}$ the *suspension* of A, denoted by ΣA.

Corollary 5. *The induced suspension map $\Sigma : K(\mathscr{C}) \to K(\mathscr{C})$ is a homotopy inverse for the H-space $K(\mathscr{C})$.*

Proof. Note that axiom 3 gives us a cofiber sequence $A \rightarrowtail \mathrm{cone}(A) \twoheadrightarrow \Sigma A$. Therefore, $1 \rightarrowtail \mathrm{cone} \twoheadrightarrow \Sigma$ is an exact sequence of functors. By the cylinder axiom, we know that cone is null-homotopic. It follows by the additivity theorem that $\Sigma_* + 1 = \mathrm{cone}_* = *$. □

Let $\mathscr{C}$ be a category with cofibrations. Equip it with two Waldhausen subcategories $v(\mathscr{C})$ and $w(\mathscr{C})$ of weak equivalences such that $v(\mathscr{C}) \subset w(\mathscr{C})$. Assume that $(\mathscr{C}, w)$ admits a cylinder functor. Suppose that $w(\mathscr{C})$ is saturated and closed under extensions.

Notation. Let $\mathscr{C}^w$ denote the Waldhausen subcategory of $(\mathscr{C}, v)$ consisting of any A where $* \to A$ is in $w(\mathscr{C})$.

Are the initial morphisms the only w.e.?

Theorem 6 (Waldhausen localization theorem). *The sequence*

$$K(A^w) \to K(\mathscr{C}, v) \to K(\mathscr{C}, w)$$

is a homotopy fibration sequence.

Proof. Recall that a small bicategory is a bisimplicial set such that each row/column is the nerve of a category. Note that $v_{(-)}w_{(-)}\mathscr{C}$ is a bicategory whose bimorphisms are commutative squares of the form

$$\begin{tikzcd} (-) \arrow[r, "w'"] \arrow[d, "v"'] & (-) \arrow[d, "v'"] \\ (-) \arrow[r, "w"] & (-) \end{tikzcd} .$$

It turns out that treating $w\mathscr{C}$ as a bicategory with a single vertical morphism proves that $w\mathscr{C} \simeq v_{(-)}w_{(-)}\mathscr{C}$. This gives $wS_n\mathscr{C} \simeq v_{(-)}w_{(-)}S_n\mathscr{C}$ for each n.

Now, let $v_{(-)} \operatorname{co} w_{(-)}\mathscr{C}$ denote the subcategory of the above squares where the horizontal maps are also cofibrations. One can show that the inclusion $v_{(-)} \operatorname{co} w_{(-)}\mathscr{C} \subset v_{(-)}w_{(-)}\mathscr{C}$ is a homotopy equivalence. Since each $S_n\mathscr{C}$ inherits a cylinder functor from $\mathscr{C}$, we simplicial bi-subcategory $v_{(-)} \operatorname{co} w_{(-)}S_\bullet\mathscr{C}$ such that the inclusion intro $v_{(-)}w_{(-)}S_\bullet\mathscr{C}$ is a homotopy equivalence. We have now obtained the following diagram.

$$\begin{tikzcd} vS_\bullet C^w \arrow[r] & vS_\bullet C \arrow[r] \arrow[d] & v_{(-)} \operatorname{co} w_{(-)} S_\bullet C \arrow[d, "\simeq"] \\ & wS_\bullet C \arrow[r, "\simeq"] & v_{(-)}w_{(-)}S_\bullet C \end{tikzcd}$$

What about the left vertical morphism?

It therefore suffices to show that the top row is a fibration. One can do this by using the relative K-theory space construction. See IV.8.5.3 and V.2.1 (Weibel). □

Let $\mathscr{A}$ be an exact category embedded in an abelian category $\mathscr{B}$ and let $\mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$ denote the category of bounded chain complexes in $\mathscr{A}$. One can verify that $\mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$ is Waldhausen where the cofibrations $A_\bullet \rightarrowtail B_\bullet$ are precisely the degree-wise admissible monomorphisms (i.e., those giving a short exact sequence $A_n \to B_n \to {}^{B_n}\!/_{A_n}$ in $\mathscr{A}$ for each n) and the w.e. are precisely the chain maps which are quasi-isomorphisms of complexes in $\mathbf{Ch}(\mathscr{B})$.

Theorem 7 (Gillet-Waldhausen). *Let $\mathscr{A}$ be an exact category closed under kernels of surjections. Then the exact inclusion $\mathscr{A} \to \mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$ induces a homotopy equivalence $K(\mathscr{A}) \simeq K\,\mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$. Hence*

$$K_i(\mathscr{A}) = K_i\,\mathbf{Ch}^{\mathrm{b}}(\mathscr{A})$$

for every i.

Proof. Apply the localization theorem. See V.2.2 (Weibel). $\square$

Definition 4. Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between Waldhausen categories. We say that F satisfies the *approximate lifting property* if for any map $b : F(A) \to B$ in $\mathscr{B}$, there is some map $a : A \to A'$ in $\mathscr{A}$ and some w.e. $b' : F(A') \simeq B$ in $\mathscr{B}$ so that

$$\begin{tikzcd} F(A') \arrow[r, dashed, "\sim"] & B \\ F(A) \arrow[u, dashed, "F(a)"] \arrow[ur, "b"'] & \end{tikzcd}.$$

commutes. In this way, we can lift to a w.e.

Proposition 8. *Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between Waldhausen categories such that the following hold.*

1. *F satisfies the approximate lifting property.*
2. *$\mathscr{A}$ admits a cylinder functor.*
3. *A morphism f in $\mathscr{A}$ is a w.e. iff $F(f)$ is a w.e. in $\mathscr{B}$.*

Then $wF : w\mathscr{A} \to w\mathscr{B}$ is a homotopy equivalence.

Corollary 9 (Waldhausen approximation theorem). *With the same conditions as before, we have*

$$K(\mathscr{A}) \simeq K(\mathscr{B}).$$

Proof. One can show that each functor $S_n\mathscr{A} \to S_n\mathscr{B}$ is exact and also has the approximate lifting property. The previous proposition thus gives degree-wise homotopy equivalence between the bisimplicial map $wS_\bullet\mathscr{A} \to wS_\bullet\mathscr{B}$, which is enough. $\square$

Definition 5. Let $\mathscr{A}$ be an abelian category $\mathbf{Ch}(\mathscr{A})$ denote the category of chain complexes over $\mathscr{A}$. We say that a complex $C_\bullet$ is *homologically bounded* if only finitely many $H_i(C_j)$ are nonzero.

Notation. Let $\mathbf{Ch}^{\mathrm{hb}}_{\pm}$ denote the subcategory of bounded below (respectively, bounded above) complexes.

Example 6. Let $\mathscr{A}$ be an abelian category. One can show that $\mathbf{Ch}^{\mathrm{b}}(\mathscr{A}) \subset \mathbf{Ch}^{\mathrm{hb}}_{-}(\mathscr{A})$ and $\mathbf{Ch}^{\mathrm{hb}}_{+}(\mathscr{A}) \subset \mathbf{Ch}^{\mathrm{hb}}(\mathscr{A})$ have the approximate lifting property. We also have that $\mathbf{Ch}^{\mathrm{b}}(\mathscr{A}) \subset \mathbf{Ch}^{\mathrm{hb}}_{+}(\mathscr{A})$ and $\mathbf{Ch}^{\mathrm{hb}}_{+}(\mathscr{A}) \subset \mathbf{Ch}^{\mathrm{hb}}(\mathscr{A})$ satisfy the dual of the approximate lifting property. Thus, we can apply the approximation theorem and Gillet-Waldhausen to see that

$$K(\mathscr{A}) \simeq K\,\mathbf{Ch}^{\mathrm{b}}(\mathscr{A}) \simeq K\,\mathbf{Ch}^{\mathrm{hb}}_{-} \simeq K\,\mathbf{Ch}^{\mathrm{hb}}_{+}(\mathscr{A}) \simeq K\,\mathbf{Ch}^{\mathrm{hb}}(\mathscr{A}).$$

Definition 7. A *symmetric spectrum* $\mathbf{X}$ in topological spaces in a sequence of based Σ_n-spaces (X_n) endowed with structure maps $\sigma : X_n \wedge S^1 \to X_{n+1}$ such that $\sigma^k : X_n \wedge S^k \to X_{n+k}$ is $(\Sigma_n \times \Sigma_k)$-equivariant for any $n, k \geq 0$, where $S^k \equiv \underbrace{S^1 \wedge \cdots \wedge S^1}_{k\text{-times}}$. A map $\mathbf{f} : \vec{x} \to \mathbf{Y}$ of symmetric spectra is a sequence $(f_n : X_n \to Y_n)$ of based Σ_n-equivariant maps such that for each $n \geq 0$, the square

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "f_n \wedge \mathrm{Id}"] \arrow[d, "\sigma"'] & Y_n \wedge S^1 \arrow[d, "\sigma"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

commutes. Let Sp^{Σ} denote the category of symmetric spectra in topological spaces.

Definition 8. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. The *external n-fold $S_\bullet$-construction* on $\mathscr{C}$ is the n-multisimplicial Waldhausen category

$$(S_\bullet \cdots S_\bullet \mathscr{C}, wS_\bullet \cdots S_\bullet \mathscr{C}).$$

It multidegree $(q_1, \ldots, q_n)$, it has as objects the diagrams $X : \mathrm{Ar}[q_1] \times \cdots \times \mathrm{Ar}[q_n] \to \mathscr{C}$ such that

1. $X((i_1, j_1), \ldots, (i_n, j_n)) = *$ if $i_k = j_k$ for some $1 \leq k \leq n$.
2. $X(\ldots, (i_t, j_t), \ldots) \rightarrowtail X(\ldots, (i_t, k_t), \ldots) \twoheadrightarrow X(\ldots, (j_t, k_t), \ldots)$ is a cofiber sequence in the $(n-1)$-fold iterated $S_\bullet$-construction for any $i_t \leq j_t \leq k_t$ in $[q_t]$.

Definition 9. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. The *internal n-fold $S_\bullet$-construction* on $\mathscr{C}$ is the simplicial Waldhausen category

$$(S_\bullet^{(n)}\mathscr{C}, wS_\bullet^{(n)}\mathscr{C}).$$

It has as q-simplices the functor categories $(S_q \cdots S_q \mathscr{C}, wS_q \cdots S_q \mathscr{C})$ whose objects are the $(\mathrm{Ar}[q])^n$-shaped diagrams $X : (\mathrm{Ar}[q])^n \to \mathscr{C}$ such that

1. $X((i_1, j_1), \ldots, (i_n, j_n)) = *$ if $i_k = j_k$ for some $1 \leq k \leq n$.
2. $X(\ldots, (i_t, j_t), \ldots) \rightarrowtail X(\ldots, (i_t, k_t), \ldots) \twoheadrightarrow X(\ldots, (j_t, k_t), \ldots)$ is a cofiber sequence in the $(n-1)$-fold iterated $S_\bullet$-construction for any $i_t \leq j_t \leq k_t$ in $[q]$.

Note that Σ_n acts on $S_\bullet^{(n)}\mathscr{C}$ by $(\pi \cdot X)(\ldots, (i_t, j_t), \ldots) = X(\ldots, (i_{\pi^{-1}(t)}, j_{\pi^{-1}(t)}), \ldots)$.

Definition 10. The *(symmetric) algebraic K-theory spectrum* $\mathbf{K}(\mathscr{C}, w)$ of a small Waldhausen category $(\mathscr{C}, w\mathscr{C})$ has n-th space $K(\mathscr{C}, w)_n = |wS_\bullet^{(n)}\mathscr{C}|$ based at $*$. There is a Σ_n-action on $K(\mathscr{C}, w)_n$ induced by permuting the order of the internal $S_\bullet$-constructions. Moreover, we have

$$|wS_\bullet^{(n)}\mathscr{C}| \wedge S^1 \cong |wS_\bullet^{(n)} S_\bullet \mathscr{C}|^{(1)} \subset |wS_\bullet^{(n)} S_\bullet \mathscr{C}| \cong |wS_\bullet^{(n+1)}\mathscr{C}|$$

, where $^{(1)}$ denotes the 1-skeleton with respect to the last simplicial direction. This determines the structure map σ. Then σ^k is $(\Sigma_n \times \Sigma_k)$-invariant.

Theorem 10. *For any $i \geq 0$, we have that $K_i(\mathscr{C}, w) = \pi_{i+1} K(\mathscr{C}, w)_1 \cong \pi_i \mathbf{K}(\mathscr{C}, w)$.*[1]

Thanks to Theorem 10, we can encode our algebraic K-theory in an infinite loop space.

[1] Lemma 8.7.4 (Rognes).