

### Abstract

More basic category theory. The main sources for this talk are the following.

- *nLab*.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 3.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 1.

**Definition 1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be functors. A *natural transformation*  $\phi : F \Rightarrow G$  is a function  $A \mapsto f_A$  from  $\text{ob } \mathcal{C}$  to  $\text{mor } \mathcal{D}$  such that  $f_A : F(A) \rightarrow G(A)$  and the following diagram commutes for any morphism  $f : A \rightarrow B$ .

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ f_A \downarrow & & \downarrow f_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

In symbols, this may be written as  $f_B f_* = f_* f_A$ , where  $f_A$  and  $f_B$  are called the *components* of  $\phi$ .

*Remark 1.* If every  $f_A$  is an isomorphism, then the  $(f_A)^{-1}$  define a natural transformation between the same two functors.

**Definition 2.** Let  $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$  be functors. The *identity natural transformation*  $\text{Id}_F : F \Rightarrow F$  is given by  $A \mapsto \text{Id}_{F(A)}$ . Moreover, given natural transformations  $\phi : F \rightarrow G$  and  $\psi : G \rightarrow H$ , define the *composite natural transformation*  $\psi \circ \phi$  by  $A \mapsto (\psi \circ \phi)_A := \psi_A \circ \phi_A$ .

**Definition 3.** If each  $f_A$  is an isomorphism, then we call  $\phi : F \cong G$  a *natural isomorphism*.

*Remark 2.* If  $\mathcal{D}$  is a groupoid, then  $\phi$  must be a natural isomorphism.

**Lemma 1.** A natural transformation  $\phi : F \Rightarrow G$  is a natural isomorphism iff it has an inverse  $\phi^{-1} : G \Rightarrow F$ .

*Proof.* This follows from Remark 1 and the definition of composite natural transformation. □

**Example 4.** Let  $R$  and  $S$  be commutative rings. Any ring homomorphism  $f : R \rightarrow S$  induces a ring homomorphism  $\text{GL}_n(f) : \text{GL}_n(R) \rightarrow \text{GL}_n(S)$  which satisfies  $f(\det(A)) = \det(\text{GL}_n(f)(A))$ . Viewing  $\text{GL}_n$  and  $R \mapsto R^*$  as functors from **Rng** to **Grp** and  $\det_R : \text{GL}_n(R) \rightarrow R^*$  as a morphism in **Grp**, we see that  $\det_R$  defines a natural transformation  $\phi : \text{GL}_n \Rightarrow f^*$ , where  $f^*$  denotes  $f \upharpoonright_{R^*} R^* \rightarrow S^*$ .

$$\begin{array}{ccc} \text{GL}_n(R) & \xrightarrow{\text{GL}_n(f)} & \text{GL}_n(S) \\ \det_R \downarrow & & \downarrow \det_S \\ R^* & \xrightarrow{f^*} & S^* \end{array}$$

**Example 5.** Recall the power set functor  $P : \mathbf{Set} \rightarrow \mathbf{Set}$  given by  $A \mapsto P(A)$  and  $Pg(S) = g(S)$  where  $g : A \rightarrow B$  is a function and  $S \subset A$ . Then the function  $f_A : A \rightarrow P(A)$  given by  $a \mapsto \{a\}$  defines a natural transformation  $\phi : \text{Id}_{\mathbf{Set}} \Rightarrow P$ .

**Example 6.** Set  $\mathcal{C} = \mathcal{D} = \mathbf{Grp}$ ,  $F = \text{Id}_{\mathcal{C}}$ , and  $G$  equal to the abelianization functor. Then given a group  $H$ , the homomorphism  $f : H \rightarrow H^{\text{ab}}$  defines a natural transformation  $\phi : F \Rightarrow G$ .

**Example 7.** Consider the preorders  $(P, \leq)$  and  $(Q, \leq)$  as small categories where functors  $F, G : P \rightarrow Q$  are order-preserving functions. Then there is a unique natural transformation  $\phi : F \Rightarrow G$  iff  $F(x) \leq G(x)$  for every  $x \in P$ .

**Example 8.** The inversion isomorphism from a group  $G$  to  $G^{\text{op}}$  defines a natural transformation  $\phi : \text{Id}_{\mathbf{Grp}} \Rightarrow (^{\text{op}} : \mathbf{Grp} \rightarrow \mathbf{Grp})$ . In other words,  $G$  is naturally isomorphic to  $G^{\text{op}}$ .

**Definition 9.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories with  $\mathcal{C}$  small. The *functor category*  $\mathbf{Fun}(\mathcal{C}, \mathcal{D}) := \mathcal{D}^{\mathcal{C}}$  has functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  as objects and natural transformations as morphisms.

*Remark 3.* Given functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ , why is the class of natural transformation  $\phi : F \Rightarrow G$  necessarily a set? A  $G$ -Universe models ZFC, in particular **Replacement**.

**Definition 10.** Given a category  $\mathcal{C}$ , the *arrow category*  $\text{Ar}(\mathcal{C})$  of  $\mathcal{C}$  has as objects morphisms  $f : X_0 \rightarrow X_1$  in  $\mathcal{C}$  and as morphisms  $M : (f : X_0 \rightarrow X_1) \rightarrow (g : Y_0 \rightarrow Y_1)$  the pairs  $M = (M_0, M_1)$  of morphisms  $M_0 : X_0 \rightarrow Y_0$  and  $M_1 : X_1 \rightarrow Y_1$  such that the following commutes.

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ M_0 \downarrow & & \downarrow M_1 \\ Y_0 & \xrightarrow{g} & Y_1 \end{array}$$

*Remark 4.*  $\text{Ar}(\mathcal{C}) \cong \mathbf{Fun}([1], \mathcal{C})$ .

**Lemma 2.**  $\mathbf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathbf{Fun}(\mathcal{C}, \mathbf{Fun}(\mathcal{D}, \mathcal{E}))$  via currying.

**Definition 11.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an *equivalence* if there is a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $F \circ G \cong \text{Id}_{\mathcal{D}}$  and  $G \circ F \cong \text{Id}_{\mathcal{C}}$ . In this case, we say that  $F$  and  $G$  are *equivalent categories*. Moreover, we say that a property of  $\mathcal{C}$  is *categorical* if it is invariant under such equivalence.

**Example 12.** Let  $k$  be a field. Let the category  $\mathbf{Mat}_k$  have natural numbers as objects and morphisms  $n \rightarrow p$  given by  $p \times n$  matrices over  $k$ . Let  $\mathbf{fdMod}$  denote the category of finite-dimensional vector spaces with linear maps as morphisms. These two categories are equivalent. Send  $\text{nat } n$  to  $k^n$  in one direction and the space  $V$  to  $\dim V$  in the other direction.

**Definition 13.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *essentially surjective* if for each object  $Z$  of  $\mathcal{D}$ , there is some object  $Y$  of  $\mathcal{C}$  such that  $F(Y) \cong Z$ .

**Theorem 3.** A functor is an equivalence iff it is full, faithful, and essentially surjective.

*Proof.* See Rognes, Theorem 3.2.10. □

**Definition 14.** A *skeleton* of  $\mathcal{C}$  is a full subcategory  $\mathcal{C}' \subset \mathcal{C}$  such that each element of  $\text{ob } \mathcal{C}$  is isomorphic to exactly one element of  $\text{ob } \mathcal{C}'$ .

**Lemma 4.** With notation as before,  $\mathcal{C}'$  and  $\mathcal{C}$  are equivalent categories via the inclusion functor.

*Proof.* Apply Theorem 3. □

**Lemma 5.** Any two skeleta  $\mathcal{C}', \mathcal{C}'' \subset \mathcal{C}$  are isomorphic.

*Proof.* Define  $F : \mathcal{C}' \rightarrow \mathcal{C}''$  by  $F(X) = Y$  where  $h_X : X \cong Y$  and  $F(f) = h_Y \circ f \circ (h_X)^{-1}$  for  $f \in \mathcal{C}(X, Y)$ . To get  $F^{-1}$ , similarly define  $G : \mathcal{C}'' \rightarrow \mathcal{C}'$  by choosing  $(h_X)^{-1}$ . □

*Remark 5.* The previous two lemmas are equivalent to the axiom of choice, as is the statement that every category admits a skeleton.

**Definition 15.** Fix  $X \in \text{ob } \mathcal{C}$ . Define the functor  $\mathcal{Y}^X : \mathcal{C} \rightarrow \mathbf{Set}$  by  $Y \mapsto \mathcal{C}(X, Y)$  and mapping each morphism  $g : Y \rightarrow Z$  to  $g_* : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$  given by  $f \mapsto gf$ . We call  $\mathcal{C}(X, -) := \mathcal{Y}^X$  the set-valued functor *corepresented* by  $X$  in  $\mathcal{C}$ .

**Definition 16.** Fix  $Z \in \text{ob } \mathcal{C}$ . Define the contravariant functor  $\mathcal{Y}_Z : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  by  $Y \mapsto \mathcal{C}(Y, Z)$  and mapping each morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  to  $f^* : \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$  given by  $g \mapsto gf$ . We call  $\mathcal{C}(-, Z) := \mathcal{Y}^Z$  the set-valued functor *represented* by  $Z$  in  $\mathcal{C}$ .

**Definition 17.** A functor  $F : \mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{D}$  is also called a *bifunctor*.

**Example 18.** Let  $\mathcal{C}$  be a category. Define  $\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$  by  $(X, X') \mapsto \mathcal{C}(X, X')$  and mapping each morphism  $(f, f') : (X, X') \rightarrow (Y, Y')$  to  $\mathcal{C}(f, f') : \mathcal{C}(X, X') \rightarrow \mathcal{C}(Y, Y')$  given by  $g \mapsto f'gf$ .

**Definition 19.** This is due to Dan Kan. Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be functors. Consider the set-valued bifunctors  $\mathcal{D}(F(-), -), \mathcal{C}(-, G(-)) : \mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$ . An *adjunction* between  $F$  and  $G$  is a natural isomorphism  $\phi : \mathcal{D}(F(-), -) \Rightarrow \mathcal{C}(-, G(-))$ . If such  $\phi$  exists, then we say that  $(F, G)$  is an *adjoint pair* of functors. We also call  $F$  the *left adjoint* to  $G$  and  $G$  the *right adjoint* to  $F$ .

*Remark 6.* For each  $c : X' \rightarrow X$  and  $d : Y \rightarrow Y'$ , the following commutes.

$$\begin{array}{ccc} \mathcal{D}(F(X), Y) & \xrightarrow{\phi_{X,Y}} & \mathcal{C}(X, G(Y)) \\ c^* d_* \downarrow & & \downarrow c^* d_* \\ \mathcal{D}(F(X'), Y') & \xrightarrow{\phi_{X',Y'}} & \mathcal{C}(X', G(Y')) \end{array}$$

**Example 20.** The forgetful functor  $U : \mathbf{Grp} \rightarrow \mathbf{Set}$  admits a left adjoint  $F : \mathbf{Set} \rightarrow \mathbf{Grp}$  which maps a set to the free group generated by  $A$ . The adjunction is the natural bijection  $\mathbf{Set}(A, U(G)) \cong \mathbf{Grp}(F(A), G)$ .

**Example 21.** Let  $R$  be a ring. The forgetful functor  $U : R\text{-}\mathbf{Mod} \rightarrow \mathbf{Set}$  admits a left adjoint  $R(-)$  sending a set  $S$  to  $\bigoplus_{s \in S} R$ , the free  $R$ -module generated by  $S$ . The adjunction is the natural bijection  $\mathbf{Set}(S, U(M)) \cong R\text{-}\mathbf{Mod}(R(S), M)$ .

*Remark 7.*  $U$  does not admit a right adjoint in either of the previous two examples.

**Example 22.** The forgetful functor  $U : \mathbf{Top} \rightarrow \mathbf{Set}$  has left adjoint that sends a set to the same set equipped with the discrete topology. It also has a right adjoint via the functor sending a set to the same set equipped with the indiscrete topology.

**Example 23.** Let  $\mathbf{CMon}$  be the category of commutative monoids. Given  $M \in \text{ob } \mathbf{CMon}$ , we can construct the completion, or Grothendieck group,  $G(M)$  on  $M \times M$  as follows. Define addition on  $M \times M$  component-wise and say that  $(m_1, m_2) \sim (n_1, n_2)$  if  $m_1 + m_2 + k = m_2 + n_1 + k$  for some  $k \in M$ . Set  $G(M)$  as  $(M \times M / \sim, +)$ .

Then  $G : \mathbf{CMon} \rightarrow \mathbf{Ab}$  is a functor. This is left adjoint to the forgetful functor  $U : \mathbf{Ab} \rightarrow \mathbf{CMon}$ .

*Remark 8.* Read Rognes, Definition 3.4.8, where he constructs the group completion  $K(M)$  of non-commutative monoids  $M$ . It turns out that  $K(M)$  is realized as the fundamental group of an important classifying space.

**Definition 24.** A subcategory  $\mathcal{C} \subset \mathcal{D}$  is *reflective* if the inclusion functor is a right adjoint and is *coreflective* if the inclusion functor is a left adjoint.

**Example 25.**

1.  $\mathbf{Ab} \subset \mathbf{CMon}$  is reflective by Example 8.
2.  $\mathbf{Ab} \subset \mathbf{Grp}$  is reflective.
3. Let  $\mathbf{Ab}_T \subset \mathbf{Ab}$  denote the category of torsion groups. This is coreflective via the functor sending an abelian group to its torsion subgroup because any homomorphism  $f : A \rightarrow B$  where  $A$  is torsion has  $f(A) \subset B_T$ .

**Definition 26.** Given an adjunction  $\phi : \mathcal{D}(F(-), -) \Rightarrow \mathcal{C}(-, G(-))$ , define the *unit morphism*

$$\eta_X = \phi_{X, F(X)}(\text{Id}_{F(X)})$$

and the *counit morphism*

$$\epsilon_Y = \phi_{G(Y), Y}^{-1}(\text{Id}_{G(Y)}).$$

**Lemma 6.** *Given an adjunction  $\phi$ , the unit morphisms  $\eta_X$  define a natural transformation  $\eta : \text{Id}_{\mathcal{C}} \Rightarrow GF$  and the counit morphisms  $\epsilon_Y$  define a natural transformation  $\epsilon : FG \Rightarrow \text{Id}_{\mathcal{D}}$ .*