

# Abstract

These notes are based on Davi Maximo's lectures for the course "Geometric Analysis and Topology I" given at UPenn along with John Lee's *Smooth Manifolds* and Michael Spivak's *A Comprehensive Introduction to Differential Geometry, Vol. 1*. Any mistake in what follows is my own.

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# 1 Smooth manifolds

## 1.1 Lecture 1

**Definition 1.1.1.** A space  $M$  is a (*topological*)  $n$ -dimensional manifold (or  $n$ -manifold) if it is

- Hausdorff,
- second-countable, and
- locally Euclidean of dimension  $n$ , i.e., for any  $x \in M$ , there exist an open  $U \ni x$  and a homeomorphism  $\phi : U \rightarrow V$  for some open  $V \subset \mathbb{R}^n$ .

**Definition 1.1.2.**

1. Let  $M$  be an  $n$ -manifold. A *coordinate chart* is a tuple  $(U, \phi)$  of an open subset  $U \subset M$  and a homeomorphism  $\phi : U \rightarrow \underset{\text{open}}{W} \subset \mathbb{R}^n$ .
2. If  $(U, \phi)$  is a coordinate chart and  $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$  denotes the  $i$ -th projection map, then we call elements of the set  $\{(\pi_1(\phi(p)), \dots, \pi_n(\phi(p))) \mid p \in U\}$  *local coordinates on  $U$* .

*Notation.* We will use the symbols  $x^i$  and  $x_i$  interchangeably for local coordinates.

**Definition 1.1.3.**

1. Given charts  $(U, \phi)$ ,  $(V, \psi)$  with  $U \cap V \neq \emptyset$ , we say that the two are  $C^k$ -compatible if the *transition map*  $\psi \circ \phi^{-1}$

$$\begin{array}{ccc} U & \xrightarrow{\phi} & \phi(U \cap V) \\ & \searrow \psi & \downarrow \psi \circ \phi^{-1} \\ & & \psi(U \cap V) \end{array}$$

is  $C^k$ .

2. A collection of charts  $(U_\alpha, \phi_\alpha)$  which covers a manifold  $M$  and is pairwise  $C^k$ -compatible is called a  $C^k$ -atlas for  $M$ .

**Example 1.1.4.** Consider  $(\mathbb{R}, x \mapsto x)$  and  $(\mathbb{R}, x \mapsto x^3)$ . Since  $x \mapsto x^{\frac{1}{3}}$  is not differentiable at 0, these charts do not form a  $C^1$ -atlas on  $\mathbb{R}$ .

**Definition 1.1.5.** An atlas  $A$  is *maximal* if it contains every chart that is  $C^\infty$ - (or smoothly) compatible with every chart in  $A$ .

**Lemma 1.1.6.**

1. Every smooth atlas is contained in a unique maximal atlas.
2. Two smooth atlases are contained in the same maximal atlas if and only if their union is also a smooth atlas.

This shows that an atlas is maximal if and only if it's maximal in the usual in the set-theoretic sense.

**Definition 1.1.7.** A manifold  $M$  is *smooth* if it admits a maximal smooth atlas, also known as a *smooth structure*.

Lemma 1.1.6 shows that it's enough to construct any smooth atlas for  $M$  to show it's a smooth manifold.

*Remark 1.1.8.* An open problem is whether there is more than one smooth structure on  $\mathbb{S}^4$ . This is known for each  $n \neq 4$ . Milnor (1958) gave an affirmative answer for  $\mathbb{S}^7$ .

## 1.2 Lecture 2

**Lemma 1.2.1.** *If  $M$  admits a smooth structure, then  $M$  admits uncountably many smooth structures.*

*Remark 1.2.2.*

1. There exists a 10-dimensional topological manifold that admits no smooth structure (Kervaire 1961).
2. Any 2- or 3-dimensional manifold admits a smooth structure.

**Example 1.2.3.** The following are examples of smooth structures on topological manifolds.

1. Any real vector space  $V$  where  $\dim(V) = n$  has a canonical smooth structure as follows. Endow  $V$  with any norm, since all norms on a finite-dimensional space are equivalent and hence generate the same topology. Pick any basis  $B := (b_1, \dots, b_n)$  of  $V$ . Define the isomorphism  $T : V \rightarrow \mathbb{R}^n$  by  $b_i \mapsto e_i$  where  $e_i$  denotes the standard basis. This is also a diffeomorphism, implying that  $V$  is a topological manifold and that  $(V, T)$  is an atlas on  $V$ . If  $B'$  is any other basis of  $V$  and  $T'$  the corresponding isomorphism, then the transition map  $T' \circ T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a linear isomorphism, hence a diffeomorphism. By Lemma 1.1.6(2), it follows that any two bases determine the same smooth structure on  $V$ .
2. The restriction of a smooth structure on a manifold  $M$  to an open subset  $U \subset M$  is called an open submanifold.
3. By our previous two examples,  $\text{GL}(n, \mathbb{F}) \subset \text{M}(n, \mathbb{F})$  is a smooth manifold.
4. Let  $U \subset \mathbb{R}^n$  be open and  $F : U \rightarrow \mathbb{R}^m$  be continuous. Let  $\Gamma(F)$  be the graph of  $F$  and  $\pi_1 \upharpoonright_{\Gamma(F)}$  be the restriction of the projection  $(x, y) \mapsto x$ . This is a homeomorphism between  $\Gamma(F)$  and  $U$ . Hence  $(\pi_1 \upharpoonright_{\Gamma(F)}, \Gamma(F))$  is a smooth atlas.
5. For each  $i = 1, 2, \dots, n+1$ , let  $U_i^+ := \{\vec{x} \in \mathbb{R}^{n+1} : x_i > 0\}$ . Define  $U_i^-$  similarly. The  $U_i^\pm$  cover  $\mathbb{S}^n$ . Define the map  $f : B_1(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f(\vec{u}) = \sqrt{1 - |\vec{u}|^2}$ . Define  $x_i : B_1(0) \rightarrow \mathbb{R}$  by  $f(x_1, \dots, \hat{x}_i, \dots, x_n)$ . Then  $\Gamma(x_i) = U_i^+ \cap \mathbb{S}^n$  and  $\Gamma(-x_i) = U_i^- \cap \mathbb{S}^n$ . By our previous example, these graphs with their corresponding projections form a smooth structure on  $\mathbb{S}^n$ .
6. Let  $f : U_{\text{open}} \subset \mathbb{R}^m \rightarrow \mathbb{R}$  be smooth. For  $c \in \mathbb{R}$ , let  $M_c := f^{-1}(c)$ . Assume that the total derivative  $\nabla f(a)$  is nonzero for each  $a \in M_c$ . Then  $f_{x_i}(a) \neq 0$  for some  $1 \leq i \leq m$ . Then, by the implicit function theorem, there is some smooth  $F : \mathbb{R}^{m-1} \rightarrow \mathbb{R}$  given by  $x_i = F(x_1, \dots, \hat{x}_i, \dots, x_m)$  on some neighborhood  $U_a \subset \mathbb{R}^m$  of  $a$  such that  $f^{-1}(c) \cap U_a$  is the graph of  $F$ . Then the  $f^{-1}(c) \cap U_a$  together with their graph coordinates define a smooth atlas on  $M_c$ .
7. For each  $i = 1, \dots, n+1$ , let  $\tilde{U}_i := \{\vec{x} \in \mathbb{R}^{n+1} : x_i \neq 0\}$ . Let  $\pi : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$  be the quotient map and  $U_i := \pi(\tilde{U}_i)$ . Since  $\tilde{U}_i$  is saturated and open, we know that  $\pi \upharpoonright_{\tilde{U}_i}$  is a quotient map.<sup>1</sup> Define  $f_i : U_i \rightarrow \mathbb{R}^n$  by  $[x_1, \dots, x_{n+1}] \mapsto (\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i})$ , which has inverse  $(x_1, \dots, x_n) \mapsto [x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n]$ . Since  $f_i \circ \pi$  is continuous, so is  $f_i$ .<sup>2</sup> Hence  $f_i$  is a homeomorphism. It's easy to check that each transition  $f_i \circ f_j^{-1}$  is smooth. Thus, the  $(U_i, f_i)$  define a smooth atlas on  $\mathbb{RP}^n$ .
8. Let  $M_1 \times \dots \times M_k$  be a product of  $n_i$ -dimensional smooth manifolds. Then this is a smooth manifold of dimension  $n_1 + \dots + n_k$ .

**Exercise 1.2.4.** *Show that  $\mathbb{RP}^n$  is second countable and Hausdorff.*

*Proof.* Recall that  $\mathbb{S}^n / \sim \cong \mathbb{RP}^n$  where  $x \sim y$  if  $y = -x$ . Thus it suffices to show these properties are true of  $P^n := \mathbb{S}^n / \sim$ .

Let  $B := \{V_n\}$  denote the usual countable basis of  $\mathbb{S}^n$  inherited from  $\mathbb{R}^{n+1}$ . If  $p \in U \subset P^n$  is open, then  $\pi^{-1}(U)$  is a neighborhood of  $\pi^{-1}(p)$ , which equals  $\{a, -a\}$  for some point  $a$  on the sphere. There is

<sup>1</sup>Munkres, James. Theorem 22.1. *Topology*.

<sup>2</sup>Munkres, James. Theorem 22.2. *Topology*.

some ball  $B \ni B_q(r) \cap \mathbb{S}^n \ni a$  with  $q \in \mathbb{Q}$  and  $r \in \mathbb{Q}^{n+1}$ . Then  $B \in B_q(-r) \cap \mathbb{S}^n \ni -a$ . Note that the union of these two balls is contained in  $\pi^{-1}(U)$  and is saturated, hence is mapped to a neighborhood  $N \subset U$  of  $p$ . Thus  $\pi(V_n)$  for  $n \in \mathbb{N}$  is a countable basis of  $P^n$ .

Proving that  $\mathbb{RP}^n$  is Hausdorff is pretty similar.  $\square$

**Lemma 1.2.5 (Smooth manifold construction).** *Let  $M$  be a set and  $\{U_\alpha\}$  a collection of subsets with injections  $\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$  such that*

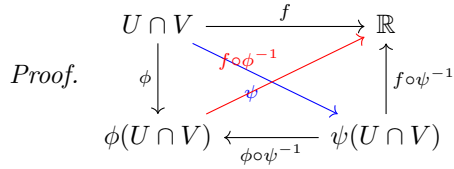
1. *Each  $\phi_\alpha(U_\alpha)$  is open.*
2. *Any  $\phi_\alpha(U_\alpha \cap U_\beta), \phi_\beta(U_\alpha \cap U_\beta)$  are open.*
3. *If  $U_\alpha \cap U_\beta \neq \emptyset$ , then  $\phi_\alpha \circ \phi_\beta^{-1}$  is smooth.*
4. *Countably many  $U_\alpha$  cover  $M$ .*
5. *If  $p, q \in M$  with  $p \neq q$ , then either both are in  $U_\alpha$  for some  $\alpha$  or they can be separated by sets in  $\{U_\alpha\}$*

*Then  $M$  has a unique smooth manifold structure with  $(U_\alpha, \phi_\alpha)$  as charts.*

*Proof.* We show that the  $U_\alpha$  give a topology on  $M$ .  $\square$

**Definition 1.2.6.** If  $M^n$  is an  $n$ -dimensional manifold and  $f : M^n \rightarrow \mathbb{R}$  is a function, we say that  $f$  is *differentiable at  $p$*  if there is some chart  $(U_\alpha, \phi_\alpha)$  such that the coordinate representation  $f \circ \phi_\alpha^{-1} : \phi(U_\alpha) \rightarrow \mathbb{R}$  is differentiable at  $p$ .

**Lemma 1.2.7.** *If  $f \circ \phi^{-1}$  is differentiable at  $\phi(p)$  and  $\psi : V \rightarrow \mathbb{R}^n$  is another coordinate neighborhood of  $p \in M^n$ , then  $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$  is also differentiable at  $\phi(p)$ . In particular, our previous definition is coordinate-independent.*



$\square$

## 2 Smooth maps

### 2.1 Lecture 3

**Definition 2.1.1.** Let  $M^n$  and  $N^k$  be smooth manifolds. We say that  $F : M \rightarrow N$  is *smooth at  $p \in M$*  if there are charts  $(V, \phi) \ni p$  and  $(V', \psi) \ni F(p)$  with  $F(V) \subset V'$  such that the coordinate representation  $\psi \circ F \circ \phi^{-1}$  is smooth.

$$\begin{array}{ccc} V & \xrightarrow{F} & V' \\ \phi \downarrow & & \downarrow \psi \\ \phi(V) & \xrightarrow{\psi \circ F \circ \phi^{-1}} & \psi(V') \end{array}$$

This definition is independent of coordinates. If  $(U, \bar{\phi})$  and  $(U', \bar{\psi})$  are other charts around  $p$  and  $F(p)$ , respectively, then

$$\begin{aligned} \bar{\psi} \circ F \circ \bar{\phi}^{-1} &= (\bar{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}) \\ \psi \circ F \circ \bar{\phi}^{-1} &= (\psi \circ F \circ \phi^{-1}) \circ (\phi \circ \bar{\phi}^{-1}), \end{aligned}$$

which are smooth at  $p$  as compositions of smooth maps.

**Lemma 2.1.2.** *Smoothness implies continuity.*

*Proof.* Using the notation from Definition 2.1.1, we see that for each  $p \in M$ , there is a neighborhood  $V$  of  $p$  such that  $F|_V = \psi^{-1} \circ \psi \circ F \circ \phi^{-1} \circ \phi$  is a composition of continuous maps (as we know smooth implies continuous for maps between Euclidean spaces) and thus itself continuous. We can glue these restrictions together to conclude that  $F$  is continuous.  $\square$

*Remark 2.1.3.*

1. Given  $F : M \rightarrow N$ , if every  $p \in M$  has a neighborhood  $U_p$  so that  $F|_{U_p}$  is smooth, then  $F$  is smooth.
2. Conversely, the restriction of any smooth map to an open subset is smooth.

**Example 2.1.4.** The natural projection  $\pi : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$  is smooth. Let  $v \in (\mathbb{R}^{n+1} \setminus \{0\}, \text{id})$ . Let  $(U_i, \phi_i) \in A_n$  be a neighborhood of  $\pi(p)$ . Since  $\pi$  is continuous,  $S := \pi^{-1}(U_i) \cap (\mathbb{R}^{n+1} \setminus \{0\})$  is a neighborhood of  $v$ . Further,  $\phi_i \circ \pi \circ \text{id} : S \rightarrow \phi_i(U_i)$  is given by  $x \mapsto \frac{(x_1, \dots, x_i, \dots, x_{n+1})}{x_i}$ , which is smooth.

**Definition 2.1.5.** A smooth map with smooth inverse is called a *diffeomorphism*.

**Note 2.1.6.**

1. This defines an equivalence relation  $\approx$  between smooth manifolds.
2. If  $M^n \approx N^k$ , then  $n = k$ .

**Example 2.1.7.** The following are diffeomorphisms.

1.  $(\mathbb{R}, \text{id}) \approx (\mathbb{R}, x \mapsto x^{\frac{1}{3}})$  via  $F : x \mapsto x^3$ .
2.  $F : \mathbb{B}^n \rightarrow \mathbb{R}^n$  given by  $F(x) = \frac{x}{\sqrt{1-|x|^2}}$  is a diffeomorphism with inverse  $G(y) = \frac{y}{\sqrt{1+|y|^2}}$ .
3.  $\mathbb{S}^n / \sim \approx \mathbb{RP}^n$ .
4. If  $M$  is a smooth manifold and  $(U, \phi)$  a chart, then  $\phi : U \rightarrow (\phi(U), \text{id})$  is a diffeomorphism.

**Definition 2.1.8.** If  $M$  is any topological space and  $f : M \rightarrow \mathbb{R}^n$  is continuous, then the *support* of  $f$  is

$$\text{supp } f := \text{cl}(\{x \in M : f(x) \neq 0\}).$$

**Lemma 2.1.9.** *Given any  $0 < r_1 < r_2$ , there is some smooth function  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $H = 1$  on  $\bar{B}_{r_1}(0)$ ,  $0 < H < 1$  on  $B_{r_2}(0) \setminus \bar{B}_{r_1}(0)$ , and  $H = 0$  elsewhere.*

*Proof.* We construct such an  $H$ . First recall that  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by  $e^{-\frac{1}{t}}$  for  $t > 0$  and 0 otherwise is smooth. Now define  $h : \mathbb{R} \rightarrow \mathbb{R}$  by  $h(t) = \frac{f(r_2-t)}{f(r_2-t)+f(t-r_1)}$ . Finally, define  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $H(x) = h(|x|)$ .  $\square$

## 2.2 Lecture 4

**Definition 2.2.1.** Let  $\mathcal{U}$  be an open cover of a topological space  $X$ . We say that

1. the open cover  $\mathcal{V}$  is a *refinement* of  $\mathcal{U}$  if for every  $V \in \mathcal{V}$ , there is some  $U \in \mathcal{U}$  such that  $V \subset U$ .
2.  $\mathcal{U}$  is *locally finite* if each  $x \in X$  has some neighborhood that intersects only finitely many  $U \in \mathcal{U}$ .
3.  $X$  is *paracompact* if every open cover of  $X$  admits a locally finite refinement.

**Definition 2.2.2.** Let  $M$  be a topological space and  $\mathbf{X} := (X_\alpha)_{\alpha \in A}$  be an open cover. A *partition of unity subordinate to  $\mathbf{X}$*  is a family  $(\psi_\alpha)_{\alpha \in A}$  of continuous functions  $\psi_\alpha : M \rightarrow \mathbb{R}$  such that

1.  $0 \leq \psi_\alpha(x) \leq 1$  for each  $\alpha$  and  $x$ .
2.  $\text{supp } \psi_\alpha \subset X_\alpha$  for each  $\alpha$ .

3. The family  $(\text{supp } \psi_\alpha)$  is locally finite, in that every point  $p \in M$  has a neighborhood  $V_p$  such that  $V_p \cap \text{supp } \psi_\alpha \neq \emptyset$  for at most finitely many  $\alpha$ . In particular,  $M$  is paracompact.
4.  $\sum_{\alpha \in A} \psi_\alpha(x) = \sup\{\sum_{\alpha \in F} \psi_\alpha(x) : F \subset A \text{ finite}\} = 1$  for each  $x$ .

**Lemma 2.2.3.** *Every topological manifold  $M$  is paracompact.*

*Proof.* Since  $M$  has a countable atlas, it has a countable basis  $\{B_n\}$  of precompact coordinate balls. (The continuous image of a precompact set into a Hausdorff space is also precompact.)

Step 1: By induction, we can build a countable covering  $\{U_n\}$  of precompact sets such that  $\text{cl}(U_{n-1}) \subset U_n$  and  $\bar{B}_n \subset U_n$  for each  $n$ .

Step 2: We build a countable locally finite open cover  $\{V_n\}$ . Set  $V_n = \text{cl}(U_n) \setminus U_{n-2}$  for  $n > 2$  and  $V_n = U_n$  otherwise. Note that every  $V_n$  intersects only finitely many other  $V_j$ , hence  $\{V_n\}$  is locally finite.

Step 3: Let  $\{X_\alpha\}$  be any open cover. For any  $p \in M$ , there is some  $\alpha$  with  $p \in X_\alpha$  and some neighborhood  $W_p$  that intersects  $V_j$  for only finitely many natural  $j$ . Set  $\widetilde{W}_p = W_p \cap X_\alpha$ . Then the  $\widetilde{W}_p$  cover  $M$ . Since each  $V_j$  is precompact by construction, we know  $V_j$  has a finite subcover  $\widetilde{W}_{p_{j_{k_1}}}, \dots, \widetilde{W}_{p_{j_{k_j}}}$ . Then  $V_j = (V_j \cap \widetilde{W}_{p_{j_{k_1}}}) \cup \dots \cup (V_j \cap \widetilde{W}_{p_{j_{k_j}}})$ . Therefore,  $\{(V_j \cap \widetilde{W}_{p_{j_{k_1}}}), \dots, (V_j \cap \widetilde{W}_{p_{j_{k_j}}})\}_{j \in \mathbb{N}}$  is a locally finite refinement of  $\{X_\alpha\}$ , as desired.  $\square$

*Remark 2.2.4.* If  $X$  is connected, then  $X$  is paracompact if and only if it is second-countable.

**Theorem 2.2.5 (Existence of partition of unity).** *If  $M$  is a smooth manifold, then any open cover  $\mathcal{X} := \{X_\alpha\}_{\alpha \in A}$  of  $M$  admits a partition of unity.*

*Proof.* For each  $\alpha \in A$ , we can find a countable basis  $\mathcal{C}_\alpha$  of precompact coordinate balls (centered at 0) for  $X_\alpha$ . Then  $\mathcal{C} := \bigcup_\alpha \mathcal{C}_\alpha$  is a basis for  $M$ . Since  $M$  is paracompact,  $\mathcal{X}$  admits a locally finite refinement  $\{C_i\}$  consisting of elements of  $\mathcal{C}$ . Note that the cover  $\{\text{cl}(B_i)\}$  is also locally finite. There are coordinate balls  $C'_i \subset X_{\alpha_i}$  such that  $C'_i \supset \text{cl}(C_i)$ . For each  $i$ , let  $\phi_i : C'_i \rightarrow \mathbb{R}^n$  be a smooth coordinate map so that  $\phi_i(C'_i) \supset \phi(C_i)$  and  $\phi(\text{cl}(C_i)) = \text{cl}(\phi(C_i))$ . Define  $f_i : M \rightarrow \mathbb{R}$  by

$$f_i(x) = \begin{cases} H_i \circ \phi_i & x \in C'_i \\ 0 & x \in M \setminus \text{cl}(C_i) \end{cases}$$

where  $H_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is a smooth function that is positive on  $\phi_i(C_i)$  and zero elsewhere, as in Section 2.1. Note that  $f_i$  is well-defined because  $f_i = 0$  on  $C'_i \setminus \text{cl}(C_i)$ . Also, it is smooth by the gluing lemma for open sets.

Define  $f : M \rightarrow \mathbb{R}$  by  $f(x) = \sum_i f_i(x)$ , which is a finite sum and hence well-defined. We see that  $f$  is a smooth function and that  $f(x) > 0$  for each  $x \in M$ . Then  $g_i(x) \equiv \frac{f_i(x)}{f(x)}$  defines a smooth function  $M \rightarrow \mathbb{R}$  for each  $i$ , so that  $\sum_i g_i(x) = 1$  and  $0 \leq g_i(x) \leq 1$  for each  $x \in M$ . Note that  $\text{supp}(g_i) = \text{cl}(C_i)$ .

For each  $\alpha \in A$ , define  $\psi_\alpha : M \rightarrow \mathbb{R}$  by

$$\psi_\alpha(x) = \sum_{i: \alpha_i = \alpha} g_i(x).$$

Interpret this as the zero function when there are no  $i$  such that  $\alpha_i = \alpha$ . Note that each  $\psi_\alpha$  is smooth as a finite sum of smooth functions and satisfies  $0 \leq \psi_\alpha \leq 1$ . Moreover, we have that

$$\text{supp}(\psi_\alpha) = \text{cl}\left(\bigcup_{i: \alpha_i = \alpha} C_i\right) = \bigcup_{i: \alpha_i = \alpha} \text{cl}(C_i).$$

Since  $\{\text{cl}(C_i)\}$  is locally finite, so is  $\{\text{supp}(\psi_\alpha)\}_{\alpha \in A}$ . Finally, the fact that  $\alpha_i \in A$  implies that

$$\sum_{\alpha} \psi_{\alpha}(x) = \sum_i g_i(x) = 1$$

for each  $x \in M$ . Therefore, we may take  $\{\psi_{\alpha}\}$  as our desired partition of unity.  $\square$

**Corollary 2.2.6.** *If  $A \subset U \subset M$  with  $A$  closed and  $U$  open in  $M$ , then there is a (smooth) bump function  $f : M \rightarrow \mathbb{R}$  such that  $f(x) = 1$  for each  $x \in A$  and  $f(x) = 0$  outside a neighborhood of  $A$ .*

*Proof.* Since  $\{U, M \setminus A\}$  is an open cover of  $M$ , there is a partition of unity  $\phi_1, \phi_2$  such that  $\text{supp } \phi_1 \subset U$ ,  $\text{supp } \phi_2 \subset M \setminus A$ , and  $\phi_1 + \phi_2 = 1$ . Hence  $\phi_1|_A = 1 - 0 = 1$ . And  $\phi_1|_{M \setminus U} = 0$ .  $\square$

## 2.3 Lecture 5

**Corollary 2.3.1 (Whitney).** *Let  $M$  be a smooth manifold and  $K \subset M$  be closed. Then there exists a non-negative smooth function  $f : M \rightarrow \mathbb{R}$  such that  $f^{-1}(0) = K$ .*

*Remark 2.3.2.* Hence closed subsets of smooth manifolds are completely characterized as the 0-level sets of smooth maps. To be the 0-level set of analytic maps, such as polynomials, is much more special (cf. algebraic geometry).

*Proof.* First assume  $M = \mathbb{R}^n$  for some  $n$ . We have  $M \setminus K$  open, which is thus the union of countably many balls  $B_{r_i}(x_i)$  with  $r_i \leq 1$ . Construct, as in Section 2.1, a smooth bump function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $h(x) = 1$  on  $\bar{B}_{\frac{1}{2}}(0)$  and that  $h$  is supported in  $B_1(0)$ . By our construction of  $h$ , we can verify that for each  $i \in \mathbb{N}$ , there is some  $C_i \geq 1$  that bounds any of the partials of  $h$  up through order  $i$ .

Define  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $\sum_{i=1}^{\infty} \frac{r_i^i}{2^i C_i} h(\frac{x-x_i}{r_i})$ . Each  $i$ -th term is bounded by  $\frac{1}{2^i}$ , implying by Weierstrass M-test that  $f$  is well-defined and continuous. Since  $h$  is zero outside  $B_1(0)$ , we see that  $f^{-1}(0) = K$ . To see that  $f$  is smooth, assume by induction that  $f$  is  $C^{k-1}$  for a given  $k \geq 1$ . By the chain rule and induction, we can write any  $k$ -th partial  $D_k$  of the  $i$ -th term of the series defining  $f$  as  $\frac{(r_i)^{i-k}}{2^i C_i} D_k h(\frac{x-x_i}{r_i})$ . As  $h$  is smooth, this expression is  $C^1$ . And since  $r_i \leq 1$  and  $C_i$  bounds all partials up to order  $i$ , it is eventually bounded by  $\frac{1}{2^i}$ . Hence the series of these expressions converges uniformly to a continuous function. By Theorem C.31 in Lee, it follows that  $D_k f$  exists and is continuous, completing the induction.

Now, assume  $M$  is arbitrary. Find a cover  $(B_{\alpha})$  of smooth coordinate balls for  $M$ . Let  $\{\phi_{\alpha}\}$  be a partition of unity subordinate to this cover. Note that each  $B_{\alpha}$  is diffeomorphic to  $\mathbb{R}^n$ . Since the property of admitting a non-negative smooth  $f : M \rightarrow \mathbb{R}$  with  $f^{-1}(0) = K$  can be stated in the language of smooth manifolds, it is invariant under diffeomorphism. Thus, there is some non-negative smooth  $f_{\alpha} : B_{\alpha} \rightarrow \mathbb{R}$  where  $f^{-1}(0) = K \cap B_{\alpha}$  for each  $\alpha$ . Then we can check that  $g = \sum_{\alpha} \phi_{\alpha} f_{\alpha}$  is the desired function.  $\square$

**Corollary 2.3.3.** *Let  $M$  be a smooth manifold and  $K \subset M$  be closed. Let  $c > 0$ . Then there exists a non-negative smooth  $f : M \rightarrow \mathbb{R}$  such that  $f^{-1}(c) = K$ .*

**Exercise 2.3.4.** *Prove that the restriction of a smooth map on  $\mathbb{R}^{n+1}$  to  $\mathbb{S}^n$  is smooth.*

## 3 Tangent vectors

### 3.1 Lecture 6

*Remark 3.1.1.* Imagine the tangent space of  $\mathbb{S}^n$  at a point  $p$  as all of the directions from  $p$  with respect to which I can find the rate of change of a smooth map  $f$  given that I'm only allowed to roam through  $\mathbb{S}^n$ .

**Definition 3.1.2.** Given  $a \in \mathbb{R}^n$ , a map  $\omega : C^{\infty}(\mathbb{R}^n) \rightarrow \mathbb{R}$  is called a *derivation at  $a$*  if it

- a. is linear over  $\mathbb{R}$

b. satisfies  $\omega(fg) = f(a)\omega(g) + g(a)\omega(f)$  for any  $f, g \in C^\infty(\mathbb{R}^n)$ .

*Remark 3.1.3.* If  $f$  is constant, then  $\omega f = 0$  for any derivation  $\omega$ .

**Example 3.1.4.** if  $u \in \mathbb{R}^n$ , recall the directional derivative of  $f \in C^\infty(\mathbb{R}^n)$  in the direction  $u$  at  $a$  is defined as

$$D_u f(a) = \lim_{h \rightarrow 0} \frac{1}{h} (f(a + hu) - f(a)) = \frac{d}{dh} \Big|_{h=0} f(a + hu).$$

Then this is a derivation of  $f$  at  $a$ .

**Theorem 3.1.5.** For  $a \in \mathbb{R}^n$ , define  $L_a : \mathbb{R}^n \rightarrow T_a \mathbb{R}^n$  by  $v_a \mapsto D_v|_a$ . This is an isomorphism.

*Proof.*  $L_a$  is clearly linear, so we just prove that it is injective and surjective.

Suppose  $u, v \in \mathbb{R}^n$  and  $L_a(u) = L_a(v)$ . Then by linearity  $L_a(u - v) = 0$ , implying  $\frac{d}{dt} \Big|_{t=0} f(a + t(u - v)) = 0$  for any smooth function  $f$ . But if  $u - v \neq 0$ , then this says that, for any  $f$ , the directional derivative of  $f$  at  $a$  in the direction of a certain nonzero vector vanishes, which is clearly false. Hence  $u = v$ , and  $L_a$  is injective.

Next, suppose  $\omega \in T_a \mathbb{R}^n$  and consider the coordinate projection  $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$  for each  $i = 1, \dots, n$ . Set  $v_i = \omega(x^i)$  and write  $v = v_i e_i$ . We show that  $L_a(v) = D_v|_a = \omega$ . By Taylor's Theorem, for any  $f \in C^\infty(\mathbb{R}^n)$  we can write

$$f(x) = f(a) + \sum_{i=1}^n f_{x_i}(a)(x_i - a_i) + c \sum_{i,j=1}^n (x_i - a_i)(x_j - a_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(a + t(x - a)) dt$$

for some  $c > 0$ . Each term of the second sum is the product of two smooth functions vanishing at  $a$ . We can apply the product rule and linearity of  $\omega$  to conclude that

$$\omega f = \omega \left( \sum_{i=1}^n f_{x_i}(a)(x_i - a_i) \right) = \sum_{i=1}^n \omega(f_{x_i}(a)(x_i - a_i)) = \sum_{i=1}^n f_{x_i}(a)(\omega(x_i) - \omega(a_i)) = \sum_{i=1}^n f_{x_i}(a)v_i = D_v|_a f.$$

□

**Corollary 3.1.6.** We have  $\dim(T_a \mathbb{R}^n) = n$ , where the partial derivatives  $\{\frac{\partial}{\partial x_i}|_a\}_{1 \leq i \leq n}$  form a basis for  $T_a \mathbb{R}^n$ .

**Definition 3.1.7.** If  $M$  is a smooth manifold and  $p \in M$ , an  $\mathbb{R}$ -linear map  $v : C^\infty(M) \rightarrow \mathbb{R}$  is called a *derivation at  $p$*  if  $v(fg) = f(p)v(g) + v(f)g(p)$  for any  $f$  and  $g$ .

**Definition 3.1.8.** The tangent space of  $M$  at  $p$  is defined as  $T_p M = \{v : C^\infty(M) \rightarrow \mathbb{R} : v \text{ is a derivation at } p\}$ .

**Definition 3.1.9.** Given smooth manifolds  $M$  and  $N$ , a smooth map  $F : M \rightarrow N$ , and  $p \in M$ , we define the *differential of  $F$  at  $p$*  as  $dF_p : T_p M \rightarrow T_{F(p)} N$  given by  $dF_p(v)(f) = v(f \circ F)$ , which is linear because  $v$  is linear. It's easy to verify that this satisfies the product rule. We call  $dF_p(v)$  the pushforward of  $v$  by  $dF$ .

**Proposition 3.1.10.** Given  $M, N, P$  smooth manifolds,  $F : M \rightarrow N$ ,  $G : N \rightarrow P$  smooth maps, and  $p \in M$ , the following hold.

1.  $dF_p : T_p M \rightarrow T_{F(p)} N$  is linear.
2.  $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p M \rightarrow T_{G(F(p))} P$ .
3.  $d(\text{id}_M)_p = \text{id} : T_p M \rightarrow T_p M$ .
4. If  $F$  is a diffeomorphism, then  $dF_p$  is an isomorphism with inverse  $d(F^{-1})_{F(p)}$ .

*Aside.* This shows that mapping  $(M, p)$  to  $T_p M$  and  $F : (M, p) \rightarrow (N, F(p))$  to  $dF|_p$  defines a functor from  $\text{Diff}_*$  to  $\text{Vec}_{\mathbb{R}}$ , called the tangent space functor.



**Lemma 3.1.11.** *Let  $v \in T_p M$  and  $f, g \in C^\infty(M)$ . Then if  $f$  and  $g$  agree on some neighborhood  $N_p$  of  $p$ , then  $vg = vf$ .*

*Proof.* Set  $h = f - g$ , so that  $h$  vanishes on  $N_p$ . Find a smooth bump function  $\phi : M \rightarrow \mathbb{R}$  such that  $\phi \equiv 1$  on  $\text{supp}(h)$  and  $\text{supp}(\phi) \subset M \setminus \{p\}$ . Then  $\phi h(x) = h(x)$  for any  $x \in M$ . Since  $\phi$  and  $h$  vanish at  $p$ , it follows that  $vf - vg = v\phi h = v(\phi h) = 0$ .  $\square$

**Proposition 3.1.12.** *If  $M$  is an  $n$ -dimensional smooth manifold, then  $\dim(T_p M) = n$  for every  $p \in M$ . In particular, we identify the standard basis by  $e_i \leftrightarrow (0, \dots, 0, \frac{\partial}{\partial x_i}|_p, 0, \dots, 0)$ .*

## 3.2 Lecture 7

*Remark 3.2.1.* Given  $p \in M$ , find a chart  $(U, \phi) \ni p$ . Then  $d\phi_p : T_p M \cong T_p U \rightarrow T_{\phi(p)} \phi(U) \cong T_p \mathbb{R}^n$  is an isomorphism. This choice of chart gives a natural choice of basis for  $T_p M$ :

$$\left\{ \frac{\partial}{\partial x_i} |_p \right\}_{1 \leq i \leq n}$$

where

$$\frac{\partial}{\partial x_i} |_p := (d\phi_p)^{-1} \left( \frac{\partial}{\partial x_i} |_{\phi(p)} \right) = (d\phi^{-1})_{\phi(p)} \left( \frac{\partial}{\partial x_i} |_{\phi(p)} \right).$$

Let  $F : M \rightarrow N$  be smooth with  $M \subset \mathbb{R}^n$  and  $N \subset \mathbb{R}^m$  open. Then by the chain rule we get

$$dF_p \left( \frac{\partial}{\partial x_i} |_p \right) f = \frac{\partial}{\partial x_i} |_p (f \circ F) = \frac{\partial}{\partial x_i} |_p (f(F_1, \dots, F_m)) = \sum_{j=1}^m \frac{\partial f}{\partial F_j} (F(p)) \frac{\partial F_j}{\partial x_i} (p) = \sum_{j=1}^m \frac{\partial F_j}{\partial x_i} (p) \left( \frac{\partial}{\partial y_j} |_{F(p)} \right) f.$$

Therefore,  $dF_p$  can be represented by the familiar  $m \times n$  Jacobian matrix of  $F$  at  $p$ ,

$$DF(p) := \begin{bmatrix} \vdots & & \vdots \\ \frac{\partial F_j}{\partial x_1}(p) & \cdots & \frac{\partial F_j}{\partial x_n}(p) \\ \vdots & & \vdots \end{bmatrix},$$

which acts on  $\mathbb{R}^n \cong T_p M$ .

Now consider the general case  $F : M \rightarrow N$  smooth between manifolds. For  $p \in M$ , choose charts  $(U, \phi) \ni p$  and  $(V, \psi) \ni F(p)$ . Then the Euclidean map  $\hat{F} := \psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \rightarrow \psi(V)$  is smooth. If  $\hat{p} := \phi(p)$ , it follows from our previous remark that  $d\hat{F}_{\hat{p}}$  is represented by the Jacobian of  $\hat{F}$  at  $\hat{p}$ . Noting that  $F \circ \phi^{-1} = \psi^{-1} \circ \hat{F}$ , we compute

$$\begin{aligned} dF_p \left( \frac{\partial}{\partial x_i} |_p \right) &= dF_p (d(\phi^{-1})|_{\hat{p}} \left( \frac{\partial}{\partial x_i} |_{\hat{p}} \right)) = d(\psi^{-1})|_{\hat{F}(\hat{p})} (d\hat{F}_{\hat{p}} \left( \frac{\partial}{\partial x_i} |_{\hat{p}} \right)) \\ &= d(\psi^{-1})|_{\hat{F}(\hat{p})} \left( \sum_{j=1}^m \frac{\partial \hat{F}_j}{\partial x_i} (\hat{p}) \frac{\partial}{\partial y_j} |_{\hat{F}(\hat{p})} \right) = \sum_{j=1}^m \frac{\partial \hat{F}_j}{\partial x_i} (\hat{p}) \frac{\partial}{\partial y_j} |_{F(p)}. \end{aligned}$$

Therefore,  $dF_p$  can be represented by the Jacobian matrix of  $\hat{F}$  at  $\hat{p}$ .

**Note 3.2.2.** Given any two pairs of coordinates for  $p$  and  $F(p)$ , the respective Jacobian matrices are related by the familiar change-of-basis matrix. In particular, they are similar.

**Definition 3.2.3.** Given a smooth manifold  $M$ , we define a notion of a smoothly varying tangent space, called the *tangent bundle* of  $M$  by the set

$$TM = \coprod_{p \in M} T_p M$$

endowed with a natural topology induced by the projection  $\pi : TM \rightarrow M, (\phi, p) \mapsto p$ .

**Example 3.2.4.** As  $\mathbb{R}_a^n$  is canonically isomorphic to  $\mathbb{R}^n$ , we have  $T\mathbb{R}^n \cong \coprod_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ .

### 3.3 Lecture 8

**Proposition 3.3.1.** *For any smooth  $n$ -dimensional manifold  $M$ , the tangent bundle  $TM$  has a natural topology and smooth structure so that it's a  $2n$ -dimensional smooth manifold and the projection  $\pi : TM \rightarrow M$  is smooth.*

*Proof.* Given a chart  $(U, \phi)$ , define  $\tilde{\phi} : \pi^{-1}(U) \rightarrow \mathbb{R}^n$  by  $v_i \frac{\partial}{\partial x_i}|_p \mapsto (x^1(p), \dots, x^n(p), v_1, \dots, v_n)$  where  $\phi = (x^1, \dots, x^n)$ . We call the  $\tilde{\phi}((f, p))$  the natural coordinates on  $TM$ . This is continuous with  $\text{Im } \tilde{\phi} = \phi(U) \times \mathbb{R}^n$ , which is open. Further,  $\tilde{\phi}^{-1}$  on  $\phi(U) \times \mathbb{R}^n$  is given by  $(x_1, \dots, x_n, v_1, \dots, v_n) \mapsto v_i \frac{\partial}{\partial x_i}|_{\phi^{-1}(x)}$ . Define  $\{(\pi^{-1}(U), \tilde{\phi})\}$  as charts on  $TM$ . Given charts  $(\pi^{-1}(U), \tilde{\phi}), (\pi^{-1}(V), \tilde{\psi})$ , it's straightforward to check that  $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^n \rightarrow \psi(U \cap V) \times \mathbb{R}^n$  is smooth.

Next, notice that if we take a countable cover  $\{U_i\}$  of  $M$  by smooth coordinate domains, then  $\{\pi^{-1}(U_i)\}$  satisfies the conditions of Lemma 1.2.5.

Finally, to see that  $\pi : TM \rightarrow M$  is smooth, note that its coordinate representation at every point is given by the projection  $\pi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n, (x, v) \mapsto x$ .  $\square$

**Definition 3.3.2.** Given  $F : M \rightarrow N$  is smooth, define the *global differential*  $dF : TM \rightarrow TN$  of  $F$  by  $dF(\phi, p) = dF_p(\phi)$ .

**Proposition 3.3.3.** *The global differential  $dF : TM \rightarrow TN$  is smooth.*

*Aside.* This shows that mapping  $M$  to  $TM$  and  $F$  to  $dF$  defines a functor from  $\text{Diff}$  to itself, called the tangent functor.

*Remark 3.3.4.* If  $F$  is a diffeomorphism, then so is  $dF$  with  $d(F^{-1}) = (df)^{-1}$ .

**Definition 3.3.5.** Given a smooth curve  $\gamma : J \rightarrow M$  and  $t_0 \in J$ , the *velocity of  $\gamma$  at  $t_0$*  is defined as

$$\gamma'(t_0) = d\gamma \left( \frac{d}{dt} \Big|_{t_0} \right) \in T_{\gamma(t_0)}M.$$

*Remark 3.3.6.* Let  $(U, \phi) \ni \gamma(t_0)$  be chart on  $M$ . Then  $\gamma'(t_0) = \frac{d\gamma^i}{dt} \frac{\partial}{\partial x_i} \Big|_{\gamma(t_0)}$ .

**Proposition 3.3.7.** *Every  $v \in T_pM$  is the velocity of some smooth curve  $\gamma : J \rightarrow M$  at 0 such that  $\gamma(0) = p$ .*

*Proof.* Let  $(U, \phi)$  be a chart centered at  $p$ . Write  $v = v_i \frac{\partial}{\partial x_i} \Big|_p$ . For  $\epsilon > 0$  small, define  $\gamma : (-\epsilon, \epsilon) \rightarrow U$  by  $\gamma(t) = \phi^{-1}(tv_1, \dots, tv_n)$ . Our previous remark implies that  $\gamma'(0) = v$ .  $\square$

**Proposition 3.3.8.** *Let  $v \in T_pM$ . Then  $dF_p(v) = (F \circ \gamma)'(0)$  for any smooth  $\gamma : J \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma'(0) = v$ .*

*Aside.* A *smooth function element* on  $M$  is a pair  $(f, U)$  with  $U \subset M$  open and  $f : M \rightarrow \mathbb{R}$  smooth. Say that  $(f, U) \sim (g, V)$  if  $p \in U \cap V$  and  $f \equiv g$  on some neighborhood of  $p$ . The class  $[(f, U)] := [f]_p$  is called the *germ of  $f$  at  $p$* . The set of such classes is denoted by  $C_p^\infty(M)$ . This is an associative algebra over  $\mathbb{R}$ .

Define a *derivation* of  $C_p^\infty(M)$  as a linear map  $v : C_p^\infty(M) \rightarrow \mathbb{R}$  such that  $v[fg]_p = f(p)v[g]_p + g(p)v[f]_p$ .

The tangent space  $\mathcal{D}_pM$  of such derivations serves as an equivalent (via isomorphism) definition of the tangent space of  $M$  at  $p$ .

### 3.4 Lecture 9

**Theorem 3.4.1 (Inverse function theorem).** *If  $F : M \rightarrow N$  is smooth and  $dF_p$  is invertible, then there are connected neighborhoods  $U_0$  of  $p$  and  $V_0$  of  $F(p)$  such that  $F|_{U_0} : U_0 \rightarrow V_0$  is a diffeomorphism.*

*Proof.* Notice that  $M$  and  $N$  have equal dimension (say  $n$ ) because  $dF_p$  is invertible. Choose charts  $(U, f)$  centered at  $p$  and  $(V, g)$  centered at  $F(p)$  such that  $F(U) \subset V$ . Then  $\widehat{F} := g \circ F \circ f^{-1}$  is smooth map from  $\widehat{U} := f(U) \subset \mathbb{R}^n$  to  $\widehat{V} := g(V) \subset \mathbb{R}^n$  with  $\widehat{F}(0) = 0$ . Now  $d\widehat{F}_0$  is invertible as the composition of three invertible maps. The Euclidean inverse function theorem implies that there are open balls  $B_r(0)$  and  $B_s(0)$  such that  $\widehat{F} : B_r(0) \rightarrow B_s(0)$  is a diffeomorphism. Then  $F : f^{-1}(B_r(0)) \rightarrow g^{-1}(B_s(0))$  is a diffeomorphism.  $\square$

**Corollary 3.4.2.** *If  $dF_p$  is nonsingular at each  $p \in M$ , then  $F$  is a local diffeomorphism.*

**Proposition 3.4.3.**

1. *The finite product of local diffeomorphisms is a local diffeomorphism.*
2. *The composition of two local diffeomorphisms is a local diffeomorphism.*
3. *Any bijective local diffeomorphism is a diffeomorphism.*
4. *A map  $F$  is a local diffeomorphism if and only if each point in  $\text{dom}(F)$  has a neighborhood where  $F$ 's coordinate representation is a local diffeomorphism.*

**Definition 3.4.4.** The rank of a smooth map  $F$  at a point  $p$  is the rank of  $dF_p$ . If the rank of  $F$  is equal at each point, then we say  $F$  has constant rank.

**Theorem 3.4.5 (Constant rank).** *Let  $F : M^m \rightarrow N^n$  be smooth with constant rank  $r \leq m, n$ . Then for each  $p \in M$ , there are charts  $(U, f)$  centered at  $p$  and  $(V, g)$  centered at  $F(p)$  with  $F(U) \subset V$  where the coordinate representation of  $F$  is given by*

$$\widehat{F}(x_1, \dots, x_r, x_{r+1}, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0).$$

**Note 3.4.6.**

- If  $m = n = r$ , then this follows immediately from the inverse function theorem.
- The global condition on the rank of  $F$  cannot be weakened, as the space of  $n \times m$  matrices of rank  $r$  need not be open. For example, consider  $A(t) = \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$ , which has rank 2 for  $t \neq 1$  and rank 1 otherwise.

*Proof. (Constant rank theorem)* Since our statement is local, we may assume that  $M \subset \mathbb{R}^m$  and  $N \subset \mathbb{R}^n$  are open subsets. Since  $DF(p)$  has rank  $r$ , it has some invertible  $r \times r$  sub-matrix, which we may assume is the upper left sub-matrix  $\left(\frac{\partial F^i}{\partial x^j}\right)_{i,j \in [r]}$ . Write  $(x, y) = (x^1, \dots, x^r, y^1, \dots, y^{m-r})$  and  $(v, w) = (v^1, \dots, v^r, w^1, \dots, w^{n-r})$  for the standard coordinates of  $\mathbb{R}^m$  and  $\mathbb{R}^n$ , respectively. By applying translations, we may assume that  $p = (0, 0)$  and  $F(p) = (0, 0)$ . Let  $F(x, y) = (Q(x, y), R(x, y))$  for some smooth  $Q : M \rightarrow \mathbb{R}^r$  and  $R : M \rightarrow \mathbb{R}^{n-r}$ . Then the Jacobian matrix  $\left(\frac{\partial Q^i}{\partial x^j}\right)$  is invertible at  $(0, 0)$  by hypothesis.

Define  $f : M \rightarrow \mathbb{R}^m$  by  $(x, y) \mapsto (Q(x, y), y)$ . Define the Kronecker delta symbol  $\delta_i^j$  by  $\delta_i^j = 1$  when  $i = j$  and 0 otherwise. Then

$$D[f](0, 0) \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}(0, 0) & \frac{\partial Q^i}{\partial y^j}(0, 0) \\ 0 & \delta_j^i \end{bmatrix}.$$

Since

$$\det(D[f](0, 0)) = \det\left(\frac{\partial Q^i}{\partial x^j}(0, 0)\right) \cdot \det(\delta_j^i) = \det\left(\frac{\partial Q^i}{\partial x^j}(0, 0)\right) \neq 0,$$

it follows that  $D[f]$  is invertible at  $(0, 0)$ .

Thus, we can apply the inverse function theorem to get a connected open set  $U_0 \ni (0,0)$  and an open cube  $\tilde{U}_0 \ni f(0,0) = (0,0)$  such that  $f : U_0 \rightarrow \tilde{U}_0$  is a diffeomorphism. Let  $f^{-1}(x,y) = (A(x,y), B(x,y))$ . Then  $(x,y) = f(A(x,y), B(x,y)) = (Q(A(x,y), B(x,y)), B(x,y))$ , so that  $y = B(x,y)$ . Hence

$$f^{-1}(x,y) = (A(x,y), y).$$

Additionally,  $Q(A(x,y), y) = x$  since  $f \circ f^{-1} = \text{id}_{\tilde{U}_0}$ . If  $\tilde{R} : \tilde{U}_0 \rightarrow \mathbb{R}^{n-r}$  is defined by  $(x,y) \mapsto R(A(x,y), y)$ , then

$$F \circ f^{-1}(x,y) = (x, \tilde{R}(x,y)).$$

Therefore,

$$D[F \circ f^{-1}](x,y) = \begin{bmatrix} \delta_j^i & 0 \\ \frac{\partial \tilde{R}^i}{\partial x^j}(x,y) & \frac{\partial \tilde{R}^i}{\partial y^j}(x,y) \end{bmatrix}$$

for any  $(x,y) \in \tilde{U}_0$ . It's clear that the first  $r$  columns of this matrix are linearly independent. But since  $f^{-1}$  is a diffeomorphism, it has rank  $r$  on  $\tilde{U}_0$ . It follows that  $\frac{\partial \tilde{R}^i}{\partial y^j}(x,y) = 0$  for each  $(x,y) \in \tilde{U}_0$ . But  $\tilde{U}_0$  was chosen to be an open cube, so that  $\tilde{R}(x,y) = \tilde{R}(x,0)$ . If  $S(x) := \tilde{R}(x,0)$ , then  $F \circ f^{-1}(x,y) = (x, S(x))$ .

Now, let  $V_0 = \{(v,w) \in N \mid (v,0) \in \tilde{U}_0\}$ , which is a neighborhood of  $(0,0)$  in  $N$ . Since  $\tilde{U}_0$  is a cube, we see that  $F \circ f^{-1}(\tilde{U}_0) \subset V_0$ . Hence  $F(U_0) \subset V_0$ . Define  $g : V_0 \rightarrow \mathbb{R}^n$  by  $(v,w) \mapsto (v, w - S(v))$ , which is smooth with inverse  $g^{-1}(s,t) = (s, t + S(s))$ . Then

$$\hat{F}(x,y) = g \circ F \circ f^{-1}(x,y) = (x, S(x) - S(x)) = (x, 0),$$

as desired. □

### 3.5 Lecture 10

**Definition 3.5.1.** A smooth map  $F : M \rightarrow N$  is a (*smooth*) *submersion* if it has constant rank equal to  $\dim(N)$ . It is a (*smooth*) *immersion* if it has constant rank equal to  $\dim(M)$ .

**Definition 3.5.2.** A *topological embedding* is a continuous map  $F : M \rightarrow N$  which is a homeomorphism onto  $F(M)$ .

**Example 3.5.3.**

1. The map  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  defined by  $t \mapsto (t^3, 0)$  is a smooth topological embedding but not an immersion, since  $\gamma'(0) = 0$ .
2. The curve  $f : (-\pi, \pi) \rightarrow \mathbb{R}^2$  defined by  $f(t) = (\sin 2t, \sin t)$  is known as a *lemniscate*, describing a figure-eight curve. This is not a topological embedding, because the figure-eight is compact whereas  $(-\pi, \pi)$  is not. But it is a smooth immersion as  $f'$  never vanishes.

**Definition 3.5.4.** A map is a *smooth embedding* if it both a topological embedding and a smooth immersion.

**Example 3.5.5.**

1. There is a smooth embedding of  $\mathbb{RP}^2$  into  $\mathbb{R}^4$  but not into  $\mathbb{R}^3$ .
2. If  $U \subset M$  is open, then the inclusion  $U \hookrightarrow M$  is a smooth embedding.

**Definition 3.5.6.** A manifold  $S \subset M$  in the subspace topology is an *embedded* (or *regular*) *submanifold* if it has a smooth structure such that the inclusion  $S \hookrightarrow M$  is smooth.

*Remark 3.5.7.* The image of a smooth embedding is an embedded submanifold.

**Definition 3.5.8.** If  $S \subset M$  is an embedded submanifold, then  $\dim M - \dim S$  is called the *codimension* of  $S$  in  $M$ .

**Proposition 3.5.9.** Let  $U \subset M$  be open and  $f : U \rightarrow N$  be smooth where  $\dim M = m$  and  $\dim N = n$ . If  $\Gamma(f)$  denotes the graph of  $f$ , then it is an embedded  $m$ -dimensional manifold of  $M \times N$ .

*Proof.* Define  $\gamma_f(x) : U \rightarrow M \times N$  by  $\gamma_f(x) = (x, f(x))$ . It's easy to check this is a smooth embedding.  $\square$

**Definition 3.5.10.** We say  $S$  has the *local  $k$ -slice condition* if for each  $p \in S$ , there is a chart  $(U, f) \ni p$  for  $M$  such that  $f(U \cap S) = \{x \in \mathbb{R}^n : x^{k+1} = \dots = x^m = 0\}$ , where  $m = \dim M$ .

**Theorem 3.5.11.** Let  $M^n$  be a smooth manifold. If  $S \subset M$  is an embedded manifold with  $\dim S = k$ , then  $S$  has the local  $k$ -slice condition. Conversely, if  $S \subset M$  has the local  $k$ -slice condition, then  $S$  is a manifold in the subspace topology and has a smooth structure making it an embedded submanifold of dimension  $k$ .

*Proof.* See Lee, Theorem 5.8.  $\square$

**Example 3.5.12.** For any  $n$ ,  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$  is an embedded hypersurface because it is locally the graph of smooth map and thus has the local  $n$ -slice condition.

**Theorem 3.5.13.** Let  $F : M^m \rightarrow N^n$  be smooth with constant rank  $r$ . Each level set of  $F$  is an embedded submanifold of codimension  $r$  in  $M$ .

*Proof.* Set  $k = m - r$ . Let  $c \in N$  and  $p \in F^{-1}(c)$ . By the constant rank theorem, there are charts  $(U, f)$  centered at  $p$  and  $(V, g)$  centered at  $F(p) = c$  for which  $F$  has coordinate representation  $(x_1, \dots, x_r, x_{r+1}, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$ , which must send each point in  $f(F^{-1}(c) \cap U)$  to 0. Thus,  $f(F^{-1}(c) \cap U)$  equals the  $k$ -slice  $\{x \in \mathbb{R}^m : x_1 = \dots = x_r = 0\}$ . By Theorem 3.5.11,  $S$  is an embedded submanifold of dimension  $k$ .  $\square$

### 3.6 Lecture 11

*Question.* Can  $M^n$  with  $n \geq 1$  be homeo-/diffeomorphic to  $M \setminus \{p\}$ ?

*Remark 3.6.1.* We can generalize Theorem 3.5.13 to maps that are not necessarily of constant rank.

**Definition 3.6.2.** Let  $\phi : M \rightarrow N$  be smooth. We say that  $p \in M$  is a

1. *regular point* if  $d\phi_p$  is surjective.
2. *critical point* otherwise.

**Definition 3.6.3.** With notation as before, we say that  $c \in N$  is a

1. *regular value* if each point in  $\phi^{-1}(c)$  is regular.
2. *critical value* otherwise.

**Theorem 3.6.4.** Every regular level set of a smooth map  $F : M^m \rightarrow N^n$  is an embedded submanifold of codimension  $n$ .

*Proof.* Let  $c \in N$ . Note that since the subspace of full-rank matrices is open, the set  $U$  of points  $p \in M$  where  $dF_p$  is surjective is open in  $M$ . Hence  $F|_U : U \rightarrow N$  is a smooth submersion. In particular it has constant rank  $n$ , so that  $F^{-1}(c)$  is an embedded submanifold with codimension  $n$  of  $U$ , which itself is an open submanifold of  $M$ .  $\square$

**Example 3.6.5.**  $\mathbb{S}^n$  is the regular level set of the smooth function  $x \mapsto |x|^2$ .

**Theorem 3.6.6 (Sard).** If  $F : M \rightarrow N$  is smooth, then the set of all critical values of  $F$  has measure zero in  $N$ .

**Proposition 3.6.7.** Suppose  $M$  is smooth and  $S \subset M$  is embedded. Then for any  $f \in C^\infty(S)$ , then there is some neighborhood  $U$  of  $S$  in  $M$  and  $\hat{f} \in C^\infty(U)$  such that  $\hat{f}|_S = f$ .

**Proposition 3.6.8.** *The tangent space of a submanifold  $S \subset M$  at  $p \in S$  is just the image of the injective canonical map  $di_p : T_p S \rightarrow T_p M$  where  $i$  denotes inclusion. More concretely, this is equal to*

$$A := \{\gamma'(0) \in T_p M : \gamma : (-\epsilon, \epsilon) \rightarrow S \text{ and } \gamma(0) = p\}.$$

*Proof.* Let  $v \in T_p S$ . We know that  $v = \gamma'(0)$  for some curve  $\gamma$  in  $S$ . Then  $i \circ \gamma$  is a curve in  $M$  with  $(i \circ \gamma)' = di_p(v)$ . Conversely, let  $v := w'(0) \in A$ . We have  $w = j \circ w$  where  $j : i(S) \rightarrow S$  is the reverse inclusion. Since  $(j \circ w)'(0) = dj_p(v) \in T_p S$ , it follows that  $d_i((j \circ w)'(0)) = v$ .  $\square$

**Remark 3.6.9.** The gradient  $\nabla F$  has two main properties.

1. It is orthogonal to the level sets of  $F$ .
2.  $dF_p(v) = \langle \nabla F_p, v \rangle$ .

But we don't have an inner product on  $M$  unless  $M$  is a Riemannian manifold, which by definition has a smoothly varying inner product.

### 3.7 Lecture 12

**Definition 3.7.1.** If  $\pi : M \rightarrow N$  is a continuous map, a *section* of  $\pi$  is a continuous right inverse for  $\pi$ .

**Definition 3.7.2.** A (smooth) *vector field*  $X$  is a smooth section of the projection  $\pi : TM \rightarrow M$ , i.e.,  $X_p := F(p) \in T_p M$  for each  $p \in M$ . Let  $\mathcal{X}(M)$  denote the space of smooth vector fields in  $M$ .

**Remark 3.7.3.** Given a chart  $U$  on  $M$ , if  $p \in U$ , then we can write  $X_p = \sum_i r_i \frac{\partial}{\partial x_i} |_p$  for some unique real coefficients  $r_i$ . Define each  $X^i : U \rightarrow \mathbb{R}$  by  $X_i(p) = r_i$ . Then  $X_p = \sum_i X_i(p) \frac{\partial}{\partial x_i} |_p$ .

**Definition 3.7.4.** We call such  $X_i$  the *component functions* of  $X$  for the chart  $U$ .

**Proposition 3.7.5.** *A vector field  $X$  is smooth if and only if each component function in any given chart is smooth.*

**Remark 3.7.6.**  $\mathcal{X}(M)$  is a module over  $C^\infty(M)$  by the action  $f \cdot X = (p \mapsto f(p)X_p)$ .

**Lemma 3.7.7.** *If  $S$  is a closed subset of  $M$  and  $X$  a smooth vector field along  $S$ , then there is an extension of  $X$  to a smooth vector field on  $M$ .*

**Definition 3.7.8.** Let  $U \subset M^n$  be open and  $X_1, \dots, X_k \in \mathcal{X}(M)$ .

1.  $X_1, \dots, X_k$  are *linearly independent* if for any  $p \in U$ , we have  $\{X_1(p), \dots, X_k(p)\}$  linearly independent in  $T_p M$ .
2. If  $k = n$  and  $X_1, \dots, X_k$  are linearly independent, then  $\{X_1, \dots, X_k\}$  is a *local frame* in  $U$ .

**Example 3.7.9.** The basis vectors  $p \mapsto \frac{\partial}{\partial x_i} |_p$  form a local frame for a given chart  $U$  around  $p$ , called the *coordinate frame*.

**Definition 3.7.10.** A local frame for  $U$  is called a *global frame* if  $U = M$ . If such a frame exists, then  $M$  is called *parallelizable*.

**Example 3.7.11.**  $\mathbb{R}^n$  is parallelizable via the standard coordinate vector fields.

**Lemma 3.7.12.**  *$M$  is parallelizable if and only if  $TM \approx M \times \mathbb{R}^n$ .*

**Theorem 3.7.13 (Kervaire).**  *$\mathbb{S}^n$  is parallelizable if and only if  $n \in \{0, 1, 3, 7\}$ .*

**Example 3.7.14.** Any Lie group is parallelizable.

*Remark 3.7.15.* Note that  $\mathcal{X}(M)$  acts on  $C^\infty(U)$  for  $U \subset M$  via  $X \cdot f = (p \mapsto X_p(f))$ . Given  $X \in \mathcal{X}(M)$  fixed, this induces a linear map  $X : C^\infty(U) \rightarrow C^\infty(U)$  satisfying the product rule  $X(fg) = fXg + gXf$ . We call such a map a *derivation* of  $C^\infty(U)$ .

Moreover, if  $F : M \rightarrow N$  is smooth, then we have  $dF_p X(p) \in T_{F(p)}N$  for each  $p \in M$ . But this may not define a vector field on  $N$ , since  $F$  may not be surjective.

**Example 3.7.16.** Note that for  $X, Y \in \mathcal{X}(M)$ ,  $X(Yf)$  need not be a derivation. Indeed, let  $M = \mathbb{R}^2$ ,  $X = \frac{\partial}{\partial x}$ , and  $Y = x \frac{\partial}{\partial y}$ . If  $f(x, y) = x$  and  $g(x, y) = y$ , then  $XY(fg) = 2x$  whereas  $fXY(g) + gXY(f) = x$ , so that  $XY(fg)$  is not a derivation.

**Definition 3.7.17.** Let  $X, Y \in \mathcal{X}(M)$ . Then  $[X, Y] := XY - YX : C^\infty(M) \rightarrow C^\infty(M)$  is called the *Lie bracket* of  $X$  and  $Y$ .

**Proposition 3.7.18 (Clairaut).** If  $X_i = \frac{\partial}{\partial x_i} \in \mathcal{X}(M)$ , then  $[X_i, X_j] = 0$  for any  $1 \leq i, j \leq n$ .

**Proposition 3.7.19.** A map  $D : C^\infty(M) \rightarrow C^\infty(M)$  is a derivation if and only if there is some  $X \in \mathcal{X}(M)$  such that  $Df = Xf$  for any  $f$ .

*Proof.* We've established the backward implication. Conversely, assume that  $D$  is a derivation. Define  $X : M \rightarrow TM$  by  $X_p(f) = (Df)(p)$ . Since  $Df = Xf$  is smooth for each  $X$ , it follows that  $X$  is smooth by Lee, Proposition 8.14.  $\square$

**Proposition 3.7.20.** Any Lie bracket  $[X, Y]$  is a smooth vector field.

*Proof.* By our last proposition, it suffices to show that  $[X, Y]$  is a derivation. Let  $f, g$  be smooth functions on  $M$ . Then

$$\begin{aligned} [X, Y](fg) &= X(Y(fg)) - Y(X(fg)) = X(fYg + gYf) - Y(fXg + gXf) = XfYg + XgYf - YfXg - YgXf \\ &= fXYg + YgXf + gXYf + YfXg - fYXg - XgYf - gYXf - XfYg = fXYg + gXYf - fYXg - gYXf \\ &= f[X, Y]g + g[X, Y]f. \end{aligned}$$

$\square$

### 3.8 Lecture 13

**Definition 3.8.1.** The function  $[X, Y] : M \rightarrow TM$  is given by  $p \mapsto (f \mapsto X_p(Yf) - Y_p(Xf))$ .

**Proposition 3.8.2.** Write  $X = X^i \frac{\partial}{\partial x_i}$  and  $Y = Y^j \frac{\partial}{\partial x_j}$  in local coordinates. Then

$$[X, Y] = \sum_{i,j} (X^i \frac{\partial Y^j}{\partial x_i} - Y^j \frac{\partial X^i}{\partial x_j}) \frac{\partial}{\partial x_j}.$$

*Proof.* Since  $[X, Y]$  is a vector field,  $([X, Y]f) \upharpoonright_U = [X, Y](f \upharpoonright_U)$  for any open  $U \subset M$ . Therefore, we can compute, say,  $Xf$  in a coordinate expression for  $X$ . We can apply the product rule and Clairaut to get

$$\begin{aligned} [X, Y]f &= X^i \frac{\partial}{\partial x_i} (Y^j \frac{\partial f}{\partial x_j}) - Y^j \frac{\partial}{\partial x_j} (X^i \frac{\partial f}{\partial x_i}) = X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial f}{\partial x_j} + X^i Y^j \frac{\partial^2 f}{\partial x_i \partial x_j} - Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial f}{\partial x_i} - Y^j X^i \frac{\partial^2 f}{\partial x_j \partial x_i} \\ &= X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial f}{\partial x_j} - Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial f}{\partial x_i} = \sum_{i,j} (X^i \frac{\partial Y^j}{\partial x_i} - Y^j \frac{\partial X^i}{\partial x_j}) \frac{\partial f}{\partial x_j}. \end{aligned}$$

$\square$

*Remark 3.8.3.* If  $X_1, \dots, X_n \in \mathcal{X}(U)$  such that  $[X_i, X_j] = 0$ , then there are local coordinates  $x^i : V \rightarrow \mathbb{R}$  such that  $X_i = \frac{\partial}{\partial x^i}$ . This is a converse to Clairaut.

**Proposition 3.8.4.**

1. (Bilinearity) For any  $a, b \in \mathbb{R}$ ,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

2. (Antisymmetry)

$$[X, Y] = -[Y, X].$$

3. (Jacobi Identity)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

4. For any  $f, g \in C^\infty(M)$ ,

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X,$$

where  $fX$  denotes the module action  $f \cdot X$ .

*Proof.* Compute directly. □

**Definition 3.8.5.** Let  $X \in \mathcal{X}(M)$  and  $Y \in \mathcal{X}(N)$ . Let  $F : M \rightarrow N$  be a diffeomorphism. The *pushforward of  $X$  by  $F$* , denoted by  $F_*X$ , is the vector field on  $N$  given by  $q \mapsto dF_{F^{-1}(q)}(X_{F^{-1}(q)})$ .

**Definition 3.8.6.** Let  $X \in \mathcal{X}(M)$  and  $Y \in \mathcal{X}(N)$ . If  $F : M \rightarrow N$  is a diffeomorphism, then  $X$  and  $Y$  are  *$F$ -related* if  $Y = F_*X$ .

*Remark 3.8.7.*  $X(f \circ F) = (Yf) \circ F$  if and only if  $X$  and  $Y$  are  $F$ -related.

**Theorem 3.8.8 (Naturality of the Lie bracket).** Suppose  $F : M \rightarrow N$  is a diffeomorphism and  $X, Y \in \mathcal{X}(M)$ . Then  $F_*[X, Y] = [F_*X, F_*Y]$ .

*Proof.* Let  $f \in C^\infty(M)$ . By our previous remark, we see that  $XY(f \circ F) = X(F_*Yf \circ F) = F_*X(F_*Yf) \circ F$ , and likewise  $YX(f \circ F) = F_*Y(F_*Xf) \circ F$ . Thus,

$$[X, Y](f \circ F) = F_*X(F_*Yf) \circ F - F_*Y(F_*Xf) \circ F = ([F_*X, F_*Y]f) \circ F.$$

We conclude by again applying our previous remark. □

**Corollary 3.8.9.** Let  $S \subset M$  be a submanifold. If  $X, Y \in \mathcal{X}(M)$  have  $X_p, Y_p \in T_p(S)$  for each  $p \in S$ , then  $[X, Y]_p \in T_p(S)$  as well.

*Proof.* Let  $i : S \rightarrow M$  denote inclusion. Then there are  $X', Y' \in \mathcal{X}(S)$  with  $X'$   $i$ -related to  $X|_S$  and  $Y'$  to  $Y|_S$ . This implies  $[X', Y']$  is  $i$ -related to  $[X, Y]|_S$ , which in turn implies that  $[X, Y]_p \in T_p(S)$  for any  $p \in S$ . □

## 4 Vector bundles

**Definition 4.0.1.** Let  $M$  be a topological space. A (real) *vector bundle of rank  $k$  over  $M$*  is a topological space  $E$  endowed with the following structure.

1. A surjective continuous map  $\pi : E \rightarrow M$ .
2. For each  $p \in M$ ,  $E_p := \pi^{-1}(p)$  is endowed with the structure of a  $k$  dimensional real vector space.
3. For each  $p \in M$ , there is a neighborhood  $U_p$  in  $M$  and a homeomorphism  $\phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$  such that

(a)  $\pi_U \circ \phi = \pi|_{\pi^{-1}(U)}$ , where  $\pi_U : U \times \mathbb{R}^k \rightarrow U$  is the projection.

(b) For each  $q \in U$ ,  $\phi|_{E_q}$  is a linear isomorphism  $E_q \cong \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$ .

If  $M$  and  $E$  are smooth manifolds and  $\pi$  and the  $\psi$  are smooth, then  $E$  is called a *smooth vector bundle*.

**Example 4.0.2.** The Mobius strip and  $\mathbb{S}^1 \times \mathbb{R}$  are different vector bundles over  $\mathbb{S}^1$ .

*Remark 4.0.3.* We can always construct a global section for a smooth vector bundle by using partition of unity. But we cannot always ensure that it is non-vanishing, as shown by the hairy ball theorem for bundles over  $\mathbb{S}^2$ .



## 4.1 Lecture 14

**Lemma 4.1.1 (Vector bundle construction).** Let  $M^n$  be a smooth manifold and suppose that for any  $p \in M$ , there is some vector space  $E_p$  of some fixed dimension  $k$ . Let  $E := \coprod_{p \in M} E_p$  and  $\pi : E \rightarrow M$  be the projection map. Further, suppose we have the following data:

1. an open cover  $\{U_\alpha\}$ .
2. for each  $\alpha$ , a bijective  $\phi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$  whose restriction to each  $E_p$  is a linear isomorphism to  $\{p\} \times \mathbb{R}^k$ .
3. for each  $U_\alpha \cap U_\beta \neq \emptyset$ , a smooth map  $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(k, \mathbb{R})$  such that  $\phi_\alpha \circ \phi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$ .

Then  $E$  has a unique topology and smooth structure making it into a smooth vector bundle of rank  $k$  over  $M$ .

**Remark 4.1.2.** The matrices  $\tau_{\alpha\beta}(p)$  are called the *transition functions* of the vector bundle  $E$ . They satisfy the so-called cocycle condition:

$$\tau_{\alpha\alpha}(p) = I_k \quad \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)\tau_{\gamma\alpha}(p) = I_k.$$

**Definition 4.1.3.** If  $V$  is a real vector space, then define the *dual space*  $V^* = \text{Hom}(V, \mathbb{R})$ .

**Proposition 4.1.4.**

1. If  $\dim(V) = n$ , then  $\dim(V^*) = n$ .
2. There is a canonical isomorphism  $V \cong (V^*)^*$  via  $v \mapsto (\phi \mapsto \phi(v))$ .

**Definition 4.1.5.** Let  $v_1, \dots, v_n$  be a basis for  $V$ . Then the *dual basis* consists of  $\phi_i : V \rightarrow \mathbb{R}$  given by  $\phi_i(v_j) = 1$  when  $i = j$  and  $\phi_i(v_j) = 0$  otherwise.

*Notation.* If  $A : V \rightarrow W$  is linear, then let  $A^*$  denote the linear map  $W^* \rightarrow V^*$  defined by  $w \mapsto (v \mapsto w(Av))$ .

**Definition 4.1.6.** Let  $M^n$  be a smooth manifold. Define the *cotangent space* at  $p$  as  $T_p^*M$ . Define the *cotangent bundle* of  $M$  as  $T^*M = \coprod_p T_p^*M$ .

**Lemma 4.1.7.**  $T^*M$  is a smooth  $n$ -vector bundle over  $M$ .

*Proof.* Let  $(U, \phi)$  be a smooth chart for  $M$ . Define  $\phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  by  $a_i \lambda^i|_p \mapsto (p, a_1, \dots, a_n)$ , where  $\{\lambda^i|_p\}$  is a dual basis for  $T_p^*M$ . Now we apply the vector bundle construction lemma, the details of which can be found in Lee, Proposition 11.9.  $\square$

**Remark 4.1.8.** Let  $(U, x^i)$  be smooth coordinates for  $M^n$ . Then  $\psi : a_i \lambda^i|_p \mapsto (x^1(p), \dots, x^n(p), a_1, \dots, a_n)$  is a local chart  $(\pi^{-1}(U), \psi)$  for  $T^*M$ .

**Definition 4.1.9.** A section of  $T^*M$  is called a *covector field* or a *(differential/smooth) 1-form*.

## 4.2 Lecture 15

**Definition 4.2.1.** Define  $Df : C^\infty(M) \rightarrow \Gamma(T^*M)$  by  $f \mapsto (p \mapsto df_p)$  where  $df_p(v) = vf$  for every  $v \in T_pM$ . We call  $df$  the *differential* of  $f$ .

**Remark 4.2.2.** Let  $(U, x^i)$  be local coordinates for  $M$ . Let  $(dx^i)$  denote the corresponding coordinate coframe on  $U$ . Write  $df_p = A_i(p)dx^i|_p$  for some functions  $A_i : U \rightarrow \mathbb{R}$ . Then  $A_i(p) = df_p(\frac{\partial}{\partial x^i}|_p) = \frac{\partial f}{\partial x^i}(p)$ , so that  $df_p = \frac{\partial f}{\partial x^i}(p)dx^i|_p$ . In this way, the differential of  $f$  generalizes the gradient of a smooth function on  $\mathbb{R}^n$ .

**Proposition 4.2.3.** If  $M$  is connected, then  $f$  is constant if and only if  $df = 0$ .

*Proof.* Since  $vf = 0$  for any derivation  $v$  and constant  $f$ , the forward direction is clear. Conversely, suppose that  $df = 0$  and let  $p \in M$ . Set  $C = \{q \in M : f(q) = f(p)\}$ . We want  $C = M$ . It suffices to show that  $C$  is clopen. For any  $q \in C$ , choose a coordinate ball  $U \ni p$ . Then since  $0 = df = \frac{\partial f}{\partial x^i}dx^i$ , it follows that  $\frac{\partial f}{\partial x^i} = 0$  for each  $i$ . Elementary calculus implies that  $f$  must be constant on  $U$ . Hence  $C$  is open. Since  $C = f^{-1}(f(p))$ , it is also closed.  $\square$

**Note 4.2.4 (Transition functions for changing coordinates).** ??

**Definition 4.2.5.** Let  $F : M \rightarrow N$  be smooth. Let  $\omega \in \Gamma(T^*N)$ . Define the *pullback*  $F^*\omega$  of  $\omega$  as the element of  $\Gamma(T^*M)$  given by

$$F^*\omega|_p(X|_p) = \omega|_{F(p)}(F_*|_p X_p).$$

Note that unlike the pushforward, the pullback requires just that  $F$  be smooth.

**Lemma 4.2.6.** Let  $F : M \rightarrow N$  be smooth,  $\alpha, \beta \in \Gamma(T^*N)$  and  $f, g \in C^\infty(N)$ . Then

$$F^*(f\alpha + g\beta) = (f \circ F)F^*\alpha + (g \circ F)F^*\beta.$$

*Proof.* Let  $X \in \mathcal{X}(M)$ .

$$\begin{aligned} F^*(f\alpha + g\beta)|_p(X_p) &= (f\alpha + g\beta)|_{F(p)}(F_*|_p X_p) = f(F(p))\alpha_{F(p)}(F_*|_p X_p) + g(F(p))\beta_{F(p)}(F_*|_p X_p) \\ &= [(f \circ F)F^*\alpha]_p(X_p) + [(g \circ F)F^*\beta]_p(X_p). \end{aligned}$$

□

**Remark 4.2.7.** Let  $\gamma : J \subset \mathbb{R} \rightarrow M$  be a curve in  $M$ . Note that  $\Gamma(T^*\mathbb{R}) = \{f(t)dt : f : T \rightarrow \mathbb{R}\}$ . Then

$$\omega \in \Gamma(T^*M) \implies \gamma^*\omega \in \Gamma(T^*\mathbb{R}) \implies \gamma^*\omega = f(t)dt$$

for some curve  $f$  along  $J$ .

**Definition 4.2.8.** We define the *integral of  $\omega$  along  $\gamma$*  as

$$\int_\gamma \omega = \int_J \gamma^*\omega.$$

**Proposition 4.2.9.** Suppose  $\phi$  is a positive reparameterization of  $\gamma$ . Then  $\int_\gamma \omega = \int_{\gamma \circ \phi} \omega$ .

*Proof.* See Lee, Proposition 11.31. □

**Definition 4.2.10.** A differential 1-form is *closed* if  $\frac{\partial w_i}{\partial x^j} - \frac{\partial w_j}{\partial x^i} = 0$  for any  $i, j$  where  $w = w_i dx^i$ .

**Exercise 4.2.11.** Show that being closed is a well-defined property.

**Example 4.2.12.** By Clairaut,  $df$  is closed for any  $f \in C^\infty(M)$ .

## 5 Differential forms

### 5.1 Lecture 16

**Theorem 5.1.1 (Universal property of the tensor product).** Let  $V_1, \dots, V_k$  be (real) vector spaces. There exists a vector space  $V_1 \otimes \dots \otimes V_k$  (called the *tensor product of the  $V_i$* ) and map  $\otimes : V_1 \times \dots \times V_k$  so that for any multilinear map  $T : V_1 \times \dots \times V_k \rightarrow W$ , there is some unique linear  $\tilde{T} : V_1 \otimes \dots \otimes V_k \rightarrow W$  such that the following commutes.

$$\begin{array}{ccc} V_1 \times \dots \times V_k & \xrightarrow{T} & W \\ \otimes \downarrow & \nearrow \exists! \tilde{T} & \\ V_1 \otimes \dots \otimes V_k & & \end{array}$$

*Proof.* If we prove it when  $k = 2$ , then we're done by induction. Let  $\mathbb{R}\langle V_1 \times V_2 \rangle$  denote the free vector space on  $V_1 \times V_2$ , i.e., the set of all finite formal linear combinations of  $V_1 \times V_2$ . Set

$$G = \langle (av_1, v_2) - a(v_1, v_2), (v_1, av_2) - a(v_1, v_2), (v_1 + w_1, v_2) - (v_1, v_2) - (w_1, v_2), (v_1, w_2 + v_2) - (v_1, w_2) - (v_1, v_2) \rangle.$$

Given  $T : V_1 \times V_2 \rightarrow W$  multilinear, define  $\tilde{T} : \mathbb{R}\langle V_1 \times V_2 \rangle \rightarrow W$  by  $\sum a_{(v_1, v_2)}(v_1, v_2) \mapsto \sum a_{(v_1, v_2)}T(v_1, v_2)$ . Since  $T$  is multilinear,  $G \subset \ker \tilde{T}$ . Therefore, if  $V_1 \otimes V_2 := \mathbb{R}\langle V_1 \times V_2 \rangle / G$ , then we get

$$\begin{array}{ccc} \mathbb{R}\langle V_1 \times V_2 \rangle & \xrightarrow{\tilde{T}} & W \\ \pi \downarrow & \nearrow \tilde{T} & \\ V_1 \otimes V_2 & & \end{array}.$$

Thus, if  $i : V_1 \times V_2 \rightarrow \mathbb{R}\langle V_1 \times V_2 \rangle$  denotes inclusion, then  $\tilde{T} \circ \pi \circ i = \tilde{T} \circ i$ , which gives the desired diagram. We see that  $\tilde{T}$  is unique because it is uniquely determined by elements of the form  $v_1 \otimes v_2 := [(v_1, v_2)]$  by  $T$  and every element of  $V_1 \otimes V_2$  can be written as some linear combination of such elements.  $\square$

**Proposition 5.1.2.** *If  $a, b \in \mathbb{R}$ , then  $(av_1 + bw_1) \otimes v_2 = a(v_1 \otimes v_2) + b(w_1 \otimes v_2)$ .*

**Proposition 5.1.3.**

1.  $(\mathbf{Vect}_{\mathbb{R}}, \oplus, \otimes)$  is a semiring.
2.  $V \otimes W \cong W \otimes V$ .
3.  $V \otimes \mathbb{R} \cong V$ .
4.  $(V \otimes W)^* \cong V^* \otimes W^*$ .

**Proposition 5.1.4.**  $V^* \otimes W^* \cong B(V, W)$  canonically where  $B(V, W)$  denotes the space of bilinear maps  $V \times W \rightarrow \mathbb{R}$ .

*Proof.* Define  $\Phi : V^* \times W^* \rightarrow B(V, W)$  by  $(\omega, \eta) \mapsto ((v, w) \mapsto \omega(v)\eta(w))$ . This is linear, hence induces

$$\begin{array}{ccc} V^* \times W^* & \xrightarrow{\Phi} & B(V, W) \\ \pi \downarrow & \nearrow \tilde{\Phi} & \\ V^* \otimes W^* & & \end{array}.$$

One can show that  $\tilde{\Phi}$  is a natural isomorphism.  $\square$

*Remark 5.1.5.* Our previous result can be generalized.

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \dots, V_k; \mathbb{R}).$$

**Definition 5.1.6.** (tensor type) We say that an element of  $V_l^k := \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{l \text{ times}}$  is a  $(k, l)$ -tensor. A  $(k, 0)$  tensor is called *covariant*. A  $(0, l)$ -tensor is called *contravariant*.

**Definition 5.1.7.** Let  $M$  be a manifold. Then define the  $(k, l)$ -tensor bundle

$$T_l^k M = \coprod_{p \in M} (T_p)_l^k M.$$

**Exercise 5.1.8.** Find the dimension of  $T_l^k M$ .

**Example 5.1.9.**  $T^1 M = T^* M$ , and  $T_1 M = TM$ .

*Remark 5.1.10.* Suppose  $(x^i)$  and  $(y^i)$  are two local coordinate systems for  $p \in M$ . Then

$$\begin{aligned} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} &= \left( \frac{\partial x^{i_1}}{\partial y^{l_1}} dy^{p_1} \right) \otimes \cdots \otimes \left( \frac{\partial x^{i_k}}{\partial y^{l_k}} dy^{p_k} \right) \\ &= \sum_{p_1, \dots, p_k} \frac{\partial x^{i_1}}{\partial y^{l_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{l_k}} \otimes dy^{p_1} \otimes \cdots \otimes dy^{p_k}. \end{aligned}$$

**Definition 5.1.11.** A  $(k, l)$ -tensor field is a (smooth) section of  $T_l^k M$ . Let  $\mathcal{T}_l^k(M) := \Gamma(T_l^k M)$ .

## 5.2 Lecture 17

**Remark 5.2.1.** Let  $(U, x^i)$  be local coordinates for  $M$ . Then  $A \in \mathcal{T}_k^l(M)$  can be written as

$$A|_p = A_{i_1 \dots i_k}^{j_1 \dots j_l} dx^{i_1}|_p \otimes \dots \otimes dx^{i_k}|_p \otimes \frac{\partial}{\partial x^{j_1}}|_p \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}}|_p$$

summed over  $n^k \cdot n^l$  tensors.

**Example 5.2.2.** Define  $\sigma = \delta_j^i dx^j \otimes \frac{\partial}{\partial x^i}$ ,  $X = X^k \frac{\partial}{\partial x^k}$ , and  $w = w_l dx^l$ . Then

$$\sigma(X, w) = \delta_j^i dx^j \otimes \frac{\partial}{\partial x^i} (X^k \frac{\partial}{\partial x^k} w_l dx^l) = \delta_j^i dx^j (X^k \frac{\partial}{\partial x^k}) \frac{\partial}{\partial x^i} w_l dx^l = \delta_j^i \delta_k^j X^k w_l \delta_i^l = \underbrace{w_k X^k}_{[[\text{Why?}]]} = w(X).$$

We say that  $\sigma$  is *invariant* in this case.

**Example 5.2.3.** Show that  $\delta_i^j dx^i \otimes dx^j$  is not an invariant tensor.

**Proposition 5.2.4.**

1. Any  $\sigma \in \mathcal{T}_l^k(M)$  induces a  $C^\infty(M)$ -multilinear map

$$\hat{\sigma} : \underbrace{\mathcal{X}(M) \times \dots \times \mathcal{X}(M)}_{k \text{ times}} \times \underbrace{\mathcal{X}^*(M) \times \dots \times \mathcal{X}^*(M)}_{l \text{ times}} \rightarrow C^\infty(M)$$

$$(X_1, \dots, X_k, w_1, \dots, w_l) \mapsto (p \mapsto \sigma(X_1|_p, \dots, X_k|_p, w_1|_p, \dots, w_l|_p)).$$

2. Any multilinear map over  $C^\infty(M)$  is of the above form for some  $(k, l)$ -tensor field.

**Example 5.2.5.** The Lie bracket is not multilinear over  $C^\infty(M)$ , as  $[fX + gY, Z] = f[X, Y] + g[Y, Z] - Z(f)X - Z(g)Y$ .

**Remark 5.2.6.** The smooth function  $\hat{\sigma}_p$  above is determined completely by the values  $X_1(p), \dots, X_k(p), w_1(p), \dots, w_l(p)$ .

**Definition 5.2.7.** A covariant  $k$ -tensor  $T$  is *alternating* if for any vectors  $Y, X_1, \dots, X_{k-1}$ , it follows that

$$T(X_1, X_2, \dots, Y, \dots, Y, \dots, X_{k-1}) = 0.$$

This is also called a *exterior form*.

**Example 5.2.8.** If  $\sigma$  is a 0-tensor or a 1-tensor, then it is alternating.

**Proposition 5.2.9.** *TFAE.*

1.  $T$  is alternating.
2.  $T(X_1, \dots, X_k) = 0$  whenever  $\{X_1, \dots, X_k\}$  is linearly dependent.
3.  $T(X_1, \dots, X_i, X_{i+1}, \dots, X_k) = -T(X_1, \dots, X_{i+1}, X_i, \dots, X_k)$ .

**Definition 5.2.10.** The space of alternating covariant  $k$ -tensors is a subspace of  $T^k(V)$  that is denoted by  $\bigwedge^k(V)$ . Given  $T \in T^k(V)$ , define the *alternation* of  $T$  as

$$\text{Alt}(T) : (V_1, \dots, V_k) \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(V_{\alpha(1)}, \dots, V_{\sigma(k)}).$$

**Example 5.2.11.**  $\text{Alt}(T)(X, Y, Z) = \frac{1}{6}(T(X, Y, Z) + T(Y, Z, X) + T(Z, X, Y) - T(Y, X, Z) - T(Z, Y, X) - T(X, Z, Y))$ .

**Example 5.2.12.** Let  $\{w^1, \dots, w^n\}$  be a cobasis for the real vector space  $V$ . Then

$$\begin{aligned} & \text{Alt}(w^1 \otimes \dots \otimes w^n)(e_1, \dots, e_n) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) w^1 \otimes \dots \otimes w^n(e_{\sigma(1)}, \dots, e_{\sigma(n)}) \\ &= \frac{1}{n!} \text{sgn}(\text{id}_n) w^1 \otimes \dots \otimes w^n(e_1, \dots, e_n) \\ &= \frac{1}{n!}. \end{aligned}$$

**Proposition 5.2.13.**

1.  $\text{Alt}(T) \in \bigwedge^k(V)$ .
2.  $\text{Alt}(T) = T \iff T \in \bigwedge^k(V)$ .
3.  $\text{Alt} : T^k(V) \rightarrow \bigwedge^k(V)$  is linear.

### 5.3 Lecture 18

**Lemma 5.3.1.** Let  $\{w^1, \dots, w^n\}$  be a cobasis for the real vector space  $V$ . Let  $k \leq n$ . Then

$$A := \{\text{Alt}(w^{i_1} \otimes \dots \otimes w^{i_k}) : 1 \leq i_1 < \dots < i_k \leq n\}$$

is a basis for  $\bigwedge^k(V)$ .

*Proof.* It's clear from our previous proposition, that  $A$  spans  $\bigwedge^k(V)$ . It remains to show that  $A$  is linearly independent.

**Exercise 5.3.2.** Prove the following statements.

1. If  $(i_1, \dots, i_k)$  is not pairwise distinct, then  $\text{Alt}(w^{i_1} \otimes \dots \otimes w^{i_k}) = 0$ .
2.  $\text{Alt}(w^{i_1} \otimes \dots \otimes w^{i_j} \otimes w^{i_{j+1}} \otimes \dots \otimes w^{i_k}) = -\text{Alt}(w^{i_1} \otimes \dots \otimes w^{i_{j+1}} \otimes w^{i_j} \otimes \dots \otimes w^{i_k})$ .

Therefore,  $\text{span}(A) = \text{span}\{\text{Alt}(w^{i_1} \otimes \dots \otimes w^{i_k}) : 1 \leq i_1 \leq \dots \leq i_k \leq n\}$ .

**Exercise 5.3.3.** Show that this implies that  $A$  is linearly independent. □

**Corollary 5.3.4.** If  $\dim(V) = n$ , then  $\dim \bigwedge^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ .

**Definition 5.3.5.** Define the *wedge product* as the map

$$\wedge : \bigwedge^k(V) \times \bigwedge^l(V) \rightarrow \bigwedge^{k+l}(V) \quad (w, q) \mapsto w \wedge q := \frac{(k+l)!}{k!l!} \text{Alt}(w \otimes q).$$

This is like the tensor product. The *exterior algebra*  $A^*$  is the algebra of alternating tensors under the wedge product.

**Corollary 5.3.6.** The set  $\{w^{i_1} \wedge \dots \wedge w^{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$  is a basis for  $\bigwedge^k(V)$ .

*Proof.* For each  $(i_1, \dots, i_k)$ , one can show that  $w^{i_1} \wedge \dots \wedge w^{i_k}$  and  $\text{Alt}(w^{i_1} \otimes \dots \otimes w^{i_k})$  differ by a real factor. □

**Remark 5.3.7.** Let  $B := \{e_1, \dots, e_n\}$  be the standard basis for  $V$ . Note that  $\det \in \bigwedge^n(V)$  by Proposition 5.2.13. But  $\bigwedge^n(V) = 1$ , so that  $\det = c(w^1 \wedge \dots \wedge w^n)$ . But evaluating both sides at  $(e_1, \dots, e_n)$  gives  $1 = c(1) = c$ . Thus,

$$\det = w^1 \wedge \dots \wedge w^n.$$

**Proposition 5.3.8.** Suppose that  $\omega$ ,  $\omega$ ,  $\eta$ , and  $\eta'$  are exterior forms. The following are properties of the wedge product.

1. (Bilinearity) If  $a, a' \in \mathbb{R}$ , then

$$\begin{aligned}(a\omega + a'\omega') \wedge \eta &= a(\omega \wedge \eta) + a'(\omega' \wedge \eta) \\ \eta \wedge (a\omega + a'\omega') &= a(\eta \wedge \omega) + a'(\eta \wedge \omega').\end{aligned}$$

2. (Associativity)

$$(\eta \wedge \omega) \wedge \omega' = \eta \wedge (\omega \wedge \omega').$$

3. (Anticommutativity) If  $\omega \in \bigwedge^k(V)$  and  $\eta \in \bigwedge^l(V)$ , then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

**Corollary 5.3.9.** If  $\omega$  is a 1-form, then  $\omega \wedge \omega = 0$ .

4. If  $\omega^1, \dots, \omega^k \in \bigwedge^1(V)$ , then

$$\omega^1 \wedge \dots \wedge \omega^k(v_1, \dots, v_k) = \det(\omega^j(v_i)).$$

**Definition 5.3.10.** Let  $M^n$  be a smooth manifold. Define the *alternating bundle of rank  $k$*  as

$$\bigwedge^k(M) = \coprod_{p \in M} \bigwedge^k(T_p M).$$

A smooth section of  $\bigwedge^k(M)$  is called a *differential  $k$ -form*.

In local coordinates we have a basis  $\{\frac{\partial}{\partial x^i}|_p\}_{1 \leq i \leq n}$  for  $T_p M$  and a corresponding dual basis  $\{dx^i\}$ . Then for any  $w \in \bigwedge^k(M)$ , we can write

$$w = \sum_{1 \leq i_1 < \dots < i_k \leq n} w_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

locally at  $p$ . Let  $I := \{i_1 < \dots < i_k\}$ . Since

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} \left( \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \delta_J^I$$

where  $\delta_J^I = 1$  if and only if  $I = J$  as sets, it follows that  $w_{i_1, \dots, i_k} = w(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}})$ . We abbreviate this by writing

$$w = w_I dx^I,$$

where we tacitly sum over the  $I$ .

*Remark 5.3.11.* Write  $w = w_I dx^I$  and  $w = \tilde{w}_J d\tilde{x}^J$  locally where  $x, \tilde{x} : U \rightarrow \mathbb{R}^m$  are charts. A direct computation shows that

$$\tilde{w}_J = w\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}}\right) = \sum_I \det(k \times k \text{ minor of } \frac{\partial x}{\partial \tilde{x}} \text{ relative to } i_1, \dots, i_k \text{ and } j_1, \dots, j_k).$$

## 5.4 Lecture 19

**Definition 5.4.1.** Let  $F : M \rightarrow N$  be smooth and  $\omega \in \bigwedge^k(N)$ . Define the *pullback*  $F^*\omega$  of  $\omega$  by  $F$  as the differential  $k$ -form on  $M$  given by

$$F^*\omega|_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)),$$

*Remark 5.4.2.* The pullback  $F : \Omega^k(N) \rightarrow \Omega^k(M)$  is a linear map over  $\mathbb{R}$ .

**Proposition 5.4.3 (Naturality of the pullback).**  $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$ .

**Lemma 5.4.4.** In any local coordinates,  $F^*(\sum_I \omega_I dy^{i_1} \wedge \dots \wedge dy^{i_k}) = \sum_I (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F)$ .

*Proof.* ?? □

**Example 5.4.5.** Consider the transformation to polar coordinates  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $(r, \theta) \mapsto (r \cos \theta, r \sin \theta)$ . This is the identity map with respect to different atlases on  $\mathbb{R}^2$ . Our previous lemma together with computational properties of  $\wedge$  gives

$$\begin{aligned} dx \wedge dy &= d(r \cos \theta) \wedge d(r \sin \theta) \\ &= (\cos \theta dr - r \sin \theta d\theta) \wedge (\sin \theta dr + r \cos \theta d\theta) \\ &= (\cos \theta dr - r \sin \theta d\theta) \wedge \sin \theta dr + (\cos \theta dr - r \sin \theta d\theta) \wedge r \cos \theta d\theta \\ &= (\cos \theta dr \wedge \sin \theta dr) - (r \sin \theta d\theta \wedge \sin \theta dr) + (\cos \theta dr \wedge r \cos \theta d\theta) - (r \sin \theta d\theta \wedge r \cos \theta d\theta) \\ &= -(r \sin \theta d\theta \wedge \sin \theta dr) + (\cos \theta dr \wedge r \cos \theta d\theta) \\ &= r \sin^2 \theta (dr \wedge d\theta) + r \cos^2 \theta (dr \wedge d\theta) \\ &= r dr \wedge d\theta. \end{aligned}$$

## 5.5 Lecture 20

**Definition 5.5.1.** Let  $\Omega^k(M)$  and  $A^k(M)$  denote the vector space of differential  $k$ -forms on the manifold  $M$ . Let  $\omega \in A^k(M)$  and write  $\omega_I dx^I$  in local coordinates. Define the *exterior derivative* of  $\omega$  as

$$d\omega = d\omega_I \wedge dx^I.$$

Note that  $d\omega = \frac{\partial}{\partial x^j} \omega_I dx^j \wedge dx^I$ . We call the operation  $d : A^k(M) \rightarrow A^{k+1}(M)$  *exterior differentiation*.

*Remark 5.5.2.* If we view  $\Omega^k : \mathbf{Diff}^{\text{op}} \rightarrow \mathbf{Vec}_{\mathbb{R}}$  as the presheaf sending each smooth map  $f$  to the pullback  $f^*$ , then the exterior derivative becomes a natural transformation  $\Omega^k \Rightarrow \Omega^{k+1}$ .

**Definition 5.5.3.** Let  $\omega \in A^k(M)$ .

1. We say that  $\omega$  is *closed* if  $d\omega = 0$ .
2. We say that  $\omega$  is *exact* if  $\omega = d\eta$  for some  $\eta \in A^{k-1}(M)$ .

**Lemma 5.5.4.** Suppose  $M = \mathbb{R}^n$ .

1.  $d$  is linear over  $\mathbb{R}$ .
2.  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ .
3.  $d \circ d \equiv 0$ .
4.  $d(F^*\omega) = F^*(d\omega)$ .

*Proof.* The first statement is obvious, and the last is an easy computation. Now, write  $\omega = udx^I$  and  $\eta = vdx^J$ . By linearity, it suffices to compute  $d(udx^I \wedge vdx^J)$  in order to verify the second statement.

$$\begin{aligned}
d(\omega \wedge \eta) &= d(udx^I \wedge vdx^J) \\
&= d(uvdx^I \wedge dx^J) \\
&= (vdu + udv) \wedge dx^I \wedge dx^J \\
&= (du \wedge dx^I) \wedge (vdx^J) \wedge (dv \wedge udx^I) \wedge dx^J \\
&= (du \wedge dx^I) \wedge (vdx^J) \wedge (-1)^k(udx^I) \wedge (dv \wedge dx^J) \\
&= d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.
\end{aligned}$$

To prove the third statement, first observe that if  $k = 1$  and we write  $\omega = \omega_j dx^j$ , then

$$d\omega = \sum_{i,j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j = \sum_{i < j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i > j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j = \sum_{i < j} \left( \frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j.$$

This together with Clairaut implies that

$$d(du) = \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i \wedge dx^j = \sum_{i < j} \left( \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j = 0.$$

Drop the assumption that  $k = 1$ . Then expanding  $d(d\omega)$  gives a sum of two summations of wedge products. One of which contains the term  $d(d\omega_j)$ , and the other contains the term  $d(dx^{j_i})$ . These both equal zero, hence the entire expression  $d(d\omega)$  vanishes.  $\square$

**Corollary 5.5.5.** *The exterior derivative is well-defined.*

*Proof.* Let  $(U, \phi)$  be a chart for  $M$ . Notice that

$$d\omega = \phi^* d(\phi^{-1*} \omega).$$

Let  $(V, \psi)$  be another chart. Then

$$(\phi \circ \psi^{-1})^* d(\phi^{-1*} \omega) = d((\phi \circ \psi^{-1})^* \phi^{-1*} \omega).$$

Since  $(\phi \circ \psi^{-1})^* = \psi^{-1*} \circ \phi^*$  and  $F^* \circ F^{-1*} = \text{id}$  for any diffeomorphism  $F$ , it follows that

$$\psi^{-1*} \circ \phi^* d(\phi^{-1*} \omega) = d(\psi^{-1*} \omega).$$

$\implies$

$$\phi^* d(\phi^{-1*} \omega) = \psi^* d(\psi^{-1*} \omega).$$

$\square$

**Corollary 5.5.6.** *Any exact form is closed.*

*Remark 5.5.7.* It is not the case that any closed form is exact. Let  $M := \mathbb{R}^2 \setminus \{0\}$ . Define the 1-form  $\omega : M \rightarrow TM$  by  $(x, y) \mapsto \frac{x dy - y dx}{x^2 + y^2}$ . On the one hand, a direct computation shows that  $d\omega = 0$ . On the other hand, recall from calculus that  $\omega$  is exact on a connected open  $\omega \subset M$  if and only if  $\int_c \omega = 0$  for any closed curve  $c \subset \omega$ . But if  $\gamma : [0, 2\pi] \rightarrow M$  is given by  $(\cos \theta, \sin \theta)$ , then

$$\int_{\gamma} \omega = \int_0^{2\pi} d\theta = 2\pi \neq 0,$$

hence  $\omega$  is not exact.

**Theorem 5.5.8 (Unique differentiation theorem).** *The exterior derivative is the unique operation  $\bar{d} : A^k(M) \rightarrow A^{k+1}$  satisfying the three above properties along with the property that  $\bar{d}f$  equals the differential of  $f$  for any  $f \in C^\infty(M)$ .*



**Proposition 5.5.9 (Naturality of the exterior derivative).** *If  $F$  is a smooth map, then  $d(F^*\omega) = F^*(d\omega)$ .*

*Proof.* This follows from the case  $M = \mathbb{R}^n$ , which is stated in Lemma 5.5.4.  $\square$

**Definition 5.5.10.** Let  $V$  be a finite-dimensional vector space. For each  $v \in V$ , define *interior multiplication* as the linear map  $i_v : \bigwedge^k(V) \rightarrow \bigwedge^{k-1}(V)$  given by  $i_v\omega(w_1, \dots, w_{k-1}) = \omega(v, w_1, \dots, w_{k-1})$ . Let  $v \lrcorner \omega := i_v\omega$ . Then we may extend interior multiplication as follows. For each  $X \in \mathcal{X}(M)$  and  $\omega \in A^k(M)$ , define the  $(k-1)$ -form  $X \lrcorner \omega$  by  $p \mapsto X_p \lrcorner \omega_p$ .

## 5.6 Lecture 21

**Definition 5.6.1.** Let  $V$  be a finite-dimensional vector space. Suppose that  $\{e_1, \dots, e_n\}$  and  $\{f_1, \dots, f_n\}$  are two bases for  $V$ . We say that they are *co-oriented* if the change-of-basis matrix has positive determinant.

This gives us two equivalence classes of bases for  $V$ , which we call the *orientations* for  $V$ . If  $[E_1, \dots, E_n]$  is a given orientation for  $V$ , then we call any basis in it (*positively*) *oriented* and any basis not in it *negatively oriented*.

**Definition 5.6.2 (Orientation).** An *orientation* on a manifold  $M$  is a continuous choice of orientation on each  $T_p M$ .

Equivalently, if  $\{(U_\alpha, \phi_\alpha)\}$  denotes the smooth structure on  $M$ , we say that  $M$  is *orientable* if the Jacobian  $D[\phi_\beta \circ \phi_\alpha^{-1}]$  has positive determinant on  $\phi_\alpha(U_\alpha \cap U_\beta)$  for any  $\alpha, \beta$ .

**Example 5.6.3.**  $\mathbb{S}^n$  is orientable for any  $n \geq 1$ . For each  $p \in \mathbb{S}^n$ , say that  $(v_1, \dots, v_n)$  is positively oriented on  $T_p \mathbb{S}^n$  if  $(p, v_1, \dots, v_n)$  is positively oriented on  $\mathbb{R}^{n+1}$ , i.e. is co-oriented with the standard basis for  $\mathbb{R}^{n+1}$ .

**Lemma 5.6.4.** *Let  $\pi : E \rightarrow M$  be a smooth vector bundle and  $V \subset E$  open. If  $V_p$  is a convex subspace of  $E_p$  for every  $p \in M$ , then there is some  $\sigma \in \Gamma(E)$  such that  $\sigma_p \in V_p$  for every  $p$ .*

*Proof.* Find a cover of  $E$  by local trivializations  $U_\alpha$  over  $M$  along with smooth sections  $\sigma_\alpha$  of them. We get some partition of unity  $\psi_\alpha$  subordinate to  $(U_\alpha)$ . Define  $\sigma : M \rightarrow E$  as  $\sum_\alpha \psi_\alpha \sigma_\alpha$ , so that  $\sigma \in \Gamma(E)$ . Then  $\sigma_p$  belongs to  $V_p$  by convexity.  $\square$

**Proposition 5.6.5.** *Suppose that  $M$  is an  $n$ -manifold. Any nowhere vanishing  $n$ -form on  $M$  gives rise to a unique orientation on  $M$ . Conversely, any orientation on  $M$  gives rise to a nowhere vanishing  $n$ -form on  $M$ .*

*Proof.* First, let  $\omega \in A^n(M)$  be nowhere vanishing. For each  $p \in M$ , we see that  $\omega_p$  defines an orientation  $O_M^p$  on  $M$  by saying that  $[e_1, \dots, e_n] \in O_M$  if and only if  $\omega_p(e_1, \dots, e_n) > 0$ . It remains to show that if  $p \in M$ , then we can find some chart  $U_p$  around  $p$  and some local frame  $(E_1, \dots, E_n)_p$  on  $U_p$  such that  $\omega_q(E_1|_q, \dots, E_n|_q) > 0$  for every  $q \in U_p$ . To see this, pick any  $U_p$  and local frame  $(E_1, \dots, E_n)_p$  on  $U_p$ . Write  $\omega = f dE^1 \wedge \dots \wedge dE^n$  locally for some smooth  $f : U_p \rightarrow \mathbb{R}$ . Since  $\omega$  is nowhere vanishing, it follows that

$$\omega(E_1, \dots, E_n) = f \neq 0.$$

Since  $f$  is continuous and  $M$  connected, we see that  $f > 0$  or  $f < 0$ . We may assume that  $f > 0$  as otherwise we choose  $(-E_1, \dots, -E_n)_p$  instead.

Conversely, given  $p \in M$  and an orientation  $O_M^p$  on  $T_p M$ , say that  $w \in \bigwedge^n(T_p M)$  is positively oriented if  $w(e_1, \dots, e_n) > 0$  for any  $[e_1, \dots, e_n] \in O_M^p$ . Then the subspace  $\bigwedge_+^n(T_p M)$  is open and convex. By our previous lemma, we are done.  $\square$

**Definition 5.6.6.** A diffeomorphism  $F : M \rightarrow N$  between two oriented manifolds is *orientation-preserving* if the isomorphism  $dF_p$  maps positively oriented bases for  $T_p M$  to positively oriented bases for  $T_{F(p)} N$  for each  $p \in M$ . It is called *orientation-reversing* if it maps positively oriented bases to negatively oriented ones.

**Note 5.6.7.**  $F$  is orientation-preserving  $\iff \det(dF_p) > 0$  for each  $p \in M \iff F^*\omega$  is positively oriented for any positively oriented form  $\omega$ .

**Lemma 5.6.8.** *The antipodal map  $\alpha : \mathbb{S}^n \rightarrow \mathbb{S}^n$  is orientation-preserving if and only if  $n$  is odd.*

*Proof.* We have the commutative diagram

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{\alpha} & \mathbb{S}^n \\ \downarrow & & \downarrow \\ \mathbb{R}^{n+1} & \xrightarrow{\hat{\alpha}} & \mathbb{R}^{n+1} \end{array}$$

where  $\hat{\alpha} : \vec{x} \mapsto -\vec{x}$ . By inspecting  $\det(I_{n+1})$ , we see that  $\hat{\alpha}$  is orientation-preserving if and only if  $n$  is odd. Thus, the restriction  $\alpha$  has the same property.  $\square$

**Corollary 5.6.9.**  $\mathbb{RP}^n$  is not orientable when  $n$  is even.

*Proof.* Suppose, for contradiction, that  $\mathbb{RP}^n$  admits some orientation. Apply Proposition 5.5.9 to obtain a nowhere vanishing  $n$ -form  $\omega$  on  $\mathbb{RP}^n$ . If  $\pi : \mathbb{S}^n \rightarrow \mathbb{RP}^n$  denotes the natural projection, then we also obtain the nowhere vanishing  $n$ -form  $\pi^*\omega$  on  $\mathbb{S}^n$ . Applying the same proposition shows that this determines the usual orientation on  $\mathbb{S}^n$ . Note that  $\pi \circ \alpha = \pi$ , so that  $\alpha^*\pi^*\omega = \pi^*\omega$ . But this implies that  $\alpha$  preserves the orientation of  $\mathbb{S}^n$ , contrary of our previous lemma.  $\square$

**Proposition 5.6.10.**  $\mathbb{RP}^n$  is orientable only if  $n$  is even.

**Definition 5.6.11 (Manifold with boundary).** Let  $\mathbb{H}^n := \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0\}$ .

1. An  $n$ -dimensional manifold with boundary  $M$  is a second-countable Hausdorff space that is locally homeomorphic to either an open Euclidean ball or an open subset in  $\mathbb{H}^n$ .
2. Any point  $p \in M$  is called an *interior point* if it belongs to a chart homeomorphic to an open ball.
3. The point  $p$  is called a *boundary point* if it belongs to a boundary chart that maps  $p$  into  $\partial\mathbb{H}^n$ .

**Note 5.6.12.** Every point in  $M$  is either an interior or a boundary point, but not both.

**Proposition 5.6.13.** *The set of boundary points  $\partial M$  is an  $(n-1)$ -dimensional embedded submanifold of  $M$ .*

*Remark 5.6.14.* Moreover,  $\partial M$  inherits an orientation from  $M$  when  $M$  is oriented. This is called the *induced* or *Stokes orientation*. Indeed, we may construct a smooth outward-pointing vector field  $N$  along  $\partial M$ , which is nowhere tangent to  $\partial M$ . Therefore, if  $\omega$  denotes the orientation form for  $M$ , then the form  $i_{\partial M}^*(N \lrcorner \omega)$  is an orientation form for  $\partial M$ .

**Example 5.6.15.**  $\mathbb{S}^n$  is orientable as the boundary of the closed unit ball.

## 6 Integration

### 6.1 Lecture 22

**Definition 6.1.1.** A *singular  $k$ -cell* on  $M^n$  is a smooth map  $\sigma : [0, 1]^k \rightarrow M$ .

*Remark 6.1.2.* Note that 0-cells are precisely points in  $M$  and 1-cells are precisely smooth curves in  $M$ .

**Definition 6.1.3.** Let  $A_0^k(\mathbb{R}^k)$  denote the space of  $k$ -forms with compact support. Let  $\omega \in A_0^k(\mathbb{R}^k)$  and write  $\omega = f dx^1 \wedge \dots \wedge dx^k$ . Define

$$\int_{\mathbb{R}^k} \omega = \int_{\mathbb{R}^k} f(x^1, \dots, x^k) dx^1 \dots dx^k.$$

**Exercise 6.1.4.** *Given another coordinate representation  $\omega = g y^1 \wedge \dots \wedge y^k$  with  $\det(\frac{\partial x}{\partial y}) > 0$ , show that  $\int_{\mathbb{R}^k} f(x^1, \dots, x^k) dx^1 \dots dx^k = \int_{\mathbb{R}^k} g(y^1, \dots, y^k) dy^1 \dots dy^k$ . Thus, our last definition makes sense.*

**Definition 6.1.5.** Let  $\omega \in A^k(M)$  and  $\sigma$  be a singular  $k$ -cell on  $M$ . Define

$$\int_{\sigma} \omega = \int_{[0,1]^k} \sigma^* \omega.$$

**Proposition 6.1.6.** Let  $p : [0,1]^k \rightarrow [0,1]^k$  be a diffeomorphism. If  $p$  is orientation-preserving, then  $\int_{\sigma} \omega = \int_{\sigma \circ p} \omega$ . If  $p$  is orientation-reversing, then  $\int_{\sigma} \omega = -\int_{\sigma \circ p} \omega$ .

**Definition 6.1.7.** A singular  $k$ -chain on  $M$  is a formal finite  $\mathbb{R}$ -combination  $\sigma = \sum_{i=1}^N a_i \sigma_i$  of singular  $k$ -cells on  $M$ . Define

$$\int_{\sigma} \omega = \sum_{i=1}^N a_i \int_{\sigma_i} \omega.$$

**Definition 6.1.8.** Let  $\sigma$  be a singular  $k$ -cell on  $M$ . Let  $i = 1, \dots, 2k$  and  $\alpha = 0, 1$ . Define the  $(i, \alpha)$ -face of  $\sigma$  as the smooth map  $\sigma_{(i, \alpha)}$  given by

$$\sigma_{(i, \alpha)}(x^1, \dots, x^k) = \sigma(x^1, \dots, x^{i-1}, \alpha, x^i, \dots, x^k).$$

Moreover, define the *boundary* of  $\sigma$  as the  $(k-1)$ -chain

$$\partial \sigma = \sum_{i=1}^k (-1)^{i+1} (\sigma_{(i, 1)} - \sigma_{(i, 0)}).$$

**Definition 6.1.9.** If  $\sigma := \sum_{i=1}^N a_i \sigma_i$  is a singular  $k$ -chain, then define the *boundary* of  $\sigma$  as the  $(k-1)$ -chain

$$\partial \sigma = \sum_{i=1}^N a_i \partial \sigma_i.$$

Note that  $\int_{\partial \sigma} \omega = \sum_{i=1}^N a_i \int_{\partial \sigma_i} \omega$ .

**Definition 6.1.10.** A singular  $k$ -chain  $\sigma$  is a *closed chain* if  $\partial \sigma = 0$ .

**Exercise 6.1.11.** Show that if  $\sigma$  is any singular  $k$ -chain, then  $\partial \sigma$  is closed.

**Theorem 6.1.12 (Stokes' theorem for chains).** Let  $\sigma$  be a  $k$ -chain and  $\omega \in A^{k-1}(M)$ . Then  $\int_{\sigma} d\omega = \int_{\partial \sigma} \omega$ .

*Proof.* For now, assume that  $M = \mathbb{R}^k$  and  $\sigma = I^k$ . As the smooth structure on  $\mathbb{R}^k$  is global, we may write  $\omega = f dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$  for some distinguished  $1 \leq i \leq k$  and some smooth  $f : \mathbb{R}^k \rightarrow \mathbb{R}$ . We compute

$$\begin{aligned} d\omega &= df \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k \\ &= \left( \sum_{j=1}^k \frac{\partial f}{\partial x^j} dx^j \right) \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k \\ &= (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^k. \end{aligned}$$

Now, apply Fubini and the fundamental theorem of calculus (FTC) to obtain

$$\begin{aligned}
\int_{\sigma} d\omega &= (-1)^{i-1} \int_{[0,1]^k} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge dx^k \\
&= (-1)^{i-1} \int_0^1 \cdots \int_0^1 \left( \int_0^1 \frac{\partial f}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^k \\
&= (-1)^{i-1} \int_0^1 \cdots \int_0^1 (f(x^1, \dots, \underbrace{1}_{i\text{-th position}}, \dots, x^k) - f(x^1, \dots, \underbrace{0}_{i\text{-th position}}, \dots, x^k)) dx^1 \cdots \widehat{dx^i} \cdots dx^k \\
&= (-1)^{i-1} \left( \int_{[0,1]^{k-1}} f(x^1, \dots, 1, \dots, x^k) dx^1 \cdots \widehat{dx^i} \cdots dx^k - \int_{[0,1]^{k-1}} f(x^1, \dots, 0, \dots, x^k) dx^1 \cdots \widehat{dx^i} \cdots dx^k \right) \\
&= (-1)^{i-1} \left( \int_{\sigma_{(i,1)}} \omega - \int_{\sigma_{(i,0)}} \omega \right).
\end{aligned}$$

Moreover, we compute

$$\int_{\partial\sigma} \omega = \sum_{j=1}^k (-1)^{j-1} \left( \int_{\sigma_{(j,1)}} \omega - \int_{\sigma_{(j,0)}} \omega \right).$$

Since  $x^j$  is constant along the  $(j, \alpha)$ -face for each  $\alpha = 0, 1$ , it follows that  $dx^j = 0$ . Therefore,

$$\int_{\partial\sigma} \omega = (-1)^{i-1} \left( \int_{\sigma_{(i,1)}} \omega - \int_{\sigma_{(i,0)}} \omega \right) = \int_{\sigma} d\omega.$$

Finally, assume that  $M$  is arbitrary and  $\sigma$  is an arbitrary  $k$ -cell on  $M$ . By the special case just proved, we have that

$$\int_{\sigma} d\omega = \int_{[0,1]^k} \sigma^*(d\omega) = \int_{[0,1]^k} d(\sigma^*\omega) = \int_{\partial[0,1]^k} \sigma^*\omega = \int_{\partial\sigma} \omega.$$

This clearly remains true if  $\sigma$  is a  $k$ -chain on  $M$ . □

**Corollary 6.1.13.** *The FTC occurs precisely when  $\sigma = I^1$  and  $\omega = f$ . This shows that Stokes' theorem for chains is equivalent to the FTC.*

## 6.2 Lecture 23

**Lemma 6.2.1.** *Let  $M$  be an oriented manifold. Let  $\omega \in A^n(M)$ . Let  $\sigma_1$  and  $\sigma_2$  be singular  $n$ -cells on  $M$  that can be extended to diffeomorphisms on (open) neighborhoods of  $[0, 1]^n$ . Suppose that both are orientation-preserving. If  $\text{supp } \omega \subset \sigma_1([0, 1]^n) \cap \sigma_2([0, 1]^n)$ , then  $\int_{\sigma_1} \omega = \int_{\sigma_2} \omega$ .*

*Proof.* Since  $\text{supp } \omega \subset \sigma_1([0, 1]^n) \cap \sigma_2([0, 1]^n)$ , Proposition 6.1.6 implies that

$$\int_{\sigma_1} \omega = \int_{\sigma_2 \circ (\sigma_2^{-1} \circ \sigma_1)} \omega = \int_{\sigma_2} \omega.$$

□

**Definition 6.2.2.** Let  $\omega \in A^n(M)$ . Let  $\sigma$  be an orientation-preserving singular  $n$ -cell on  $M$ . If  $\text{supp } \omega \subset \sigma([0, 1]^n)$ , then, by our previous lemma, we may define

$$\int_M \omega = \int_{\sigma} \omega.$$

In general, there exists an open cover  $(U_{\alpha})$  of  $M$  such that each  $U_{\alpha} \subset \sigma_{\alpha}([0, 1]^n)$  where  $\sigma_{\alpha}$  is some orientation-preserving singular  $n$ -cell on  $M$ . Find a partition of unity  $(\phi_{\alpha})$  subordinate to this cover. Note

that each  $\phi_\alpha \omega$  belongs to  $A^n(M)$  and is supported in  $U_\alpha$ . If  $\omega$  is compactly supported, then  $\text{supp } \omega$  intersects at most finitely many  $\text{supp } \phi_\alpha$ . In this case, we define

$$\int_M \omega = \sum_\alpha \int_M \phi_\alpha \omega$$

as this sum is finite.

**Lemma 6.2.3.** *If  $(V_\beta, \psi_\beta)$  is another such partition of unity, then  $\sum_\beta \int_M \psi_\beta \omega = \sum_\alpha \int_M \phi_\alpha \omega$ . Hence our previous definition makes sense.*

*Proof.*

$$\begin{aligned} \sum_\alpha \int_M \phi_\alpha \omega &= \sum_\alpha \int_M \phi_\alpha \sum_\beta \psi_\beta \omega \\ &= \sum_\alpha \sum_\beta \int_M \phi_\alpha \psi_\beta \omega = \sum_\beta \sum_\alpha \int_M \psi_\beta \phi_\alpha \omega \\ &= \sum_\beta \int_M \psi_\beta \sum_\alpha \phi_\alpha \omega = \sum_\beta \int_M \psi_\beta \omega. \end{aligned}$$

□

**Note 6.2.4.** If  $\omega$  is not assumed to be compact, then  $\int_M \omega$  may be infinite but is still well-defined.

**Theorem 6.2.5 (Stokes).** *Let  $M$  be an oriented compact  $n$ -manifold with boundary. If  $\omega \in A^{n-1}(M)$ , then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

*Proof.* There are three cases to consider.

Case 1: Suppose that there is some orientation-preserving  $n$ -cell  $\sigma$  on  $M$  such that  $\text{supp } \omega \subset \text{Int}(\text{im } \sigma)$  and  $\text{im } \sigma \cap \partial M = \emptyset$ . By Stokes' theorem for chains, it follows that

$$\int_M d\omega = \int_\sigma d\omega = \int_{\partial\sigma} \omega = 0 = \int_{\partial M} \omega.$$

Case 2: Suppose that there is some orientation-preserving  $n$ -cell  $\sigma$  on  $M$  such that  $\text{supp } \omega \subset \text{im } \sigma$ ,  $\text{im } \sigma \cap \partial M = \sigma_{(n,0)}([0, 1]^{n-1})$ , and  $\text{supp } \omega \cap \text{im } \partial\sigma \subset \sigma_{(n,0)}$ . By Stokes' theorem for chains, it follows that

$$\int_M d\omega = \int_\sigma d\omega = \int_{\partial\sigma} \omega = (-1)^n \int_{\sigma_{(n,0)}} \omega.$$

Note that if  $\mu$  is the usual orientation on  $\mathbb{H}^n$ , then the induced orientation on the boundary  $\partial\mathbb{H}^n$  is equal to  $(-1)^n \mu$ . Therefore,  $\sigma_{(n,0)} : [0, 1]^{n-1} \rightarrow \partial M$  is orientation-preserving if and only if  $n$  is even. In either case, we have that

$$(-1)^n \int_{\sigma_{(n,0)}} \omega = \int_{\partial M} \omega,$$

which completes this case.

Case 3: In general, there exist an open cover  $(U_\alpha)$  of  $M$  and a partition of unity  $(\phi_\alpha)$  subordinate to it

such that each  $\phi_\alpha \omega$  is an  $(n-1)$ -form of the kind in Case 1 or Case 2. Since  $\sum_\alpha \phi_\alpha$  is constant, we see that  $0 = d(\sum_\alpha \phi_\alpha) = \sum_\alpha d\phi_\alpha$ . Hence  $\sum_\alpha d\phi_\alpha \wedge \omega = 0$ , so that  $\sum_\alpha \int_M d\phi_\alpha \wedge \omega = 0$ . From this we compute

$$\begin{aligned} \int_M d\omega &= \int_M \sum_\alpha \phi_\alpha d\omega \\ &= \sum_\alpha \int_M \phi_\alpha d\omega \\ &= \sum_\alpha \int_M d\phi_\alpha \wedge \omega + \phi_\alpha d\omega \\ &= \sum_\alpha \int_M d(\phi_\alpha \omega) \\ &= \sum_\alpha \int_{\partial M} \phi_\alpha \omega \\ &= \int_{\partial M} \omega. \end{aligned}$$

□

## 7 De Rham cohomology

### 7.1 Lecture 24

**Definition 7.1.1.** Given a manifold  $M^n$  and integer  $k \geq 1$ , define the real vector spaces

$$\begin{aligned} Z^k(M) &= \{\omega \in A^k(M) : d\omega = 0\} \\ B^k(M) &= \{d\eta : \eta \in A^{k-1}(M)\}. \end{aligned}$$

Since  $B^k(M) \subset Z^k(M)$ , we may form the quotient space

$$H_{\text{dR}}^k(M) := Z^k(M) / B^k(M),$$

called the  $k$ -th de Rham cohomology group of  $M$ .

*Remark 7.1.2.* This is the same as the singular cohomology group over  $\mathbb{R}$ .

$H_{\text{dR}}^k(M)$  can be thought of as a quantitative measure of the number of holes in  $M$ .

**Theorem 7.1.3.** If  $M$  and  $N$  are continuously homotopy equivalent, then  $H_{\text{dR}}^k(M) \cong H_{\text{dR}}^k(N)$  for each  $k \geq 1$ .

**Lemma 7.1.4 (Poincaré).** If  $M$  is (smoothly) contractible, then  $H_{\text{dR}}^k(M) = 0$  for each  $k \geq 1$ .

*Proof.* Assume that  $k = 1$ . For each  $t \in [0, 1]$ , define  $\iota_t : M \rightarrow M \times [0, 1]$  by  $p \mapsto (p, t)$ .

**Claim.** If  $\omega$  is any closed 1-form on  $M \times [0, 1]$ , then  $\iota_1^* \omega - \iota_0^* \omega$  is exact.

*Proof.* If  $\pi_M : M \times [0, 1] \rightarrow M$  denotes the projection and  $(U, x^i)$  denotes local coordinates on  $M$ , then  $(\pi_M^{-1}(U), (\bar{x}^i, t))$  is a coordinate chart on  $M \times [0, 1]$  where  $\bar{x}^i := x^i \circ \pi_M$ . We thus have that  $\omega = w_i d\bar{x}^i + f dt$ . For each  $\alpha = 0, 1$ , we see that

$$\iota_\alpha^* \omega = \iota_\alpha^* (w_i d\bar{x}^i + f dt) = w_i(-, \alpha) dx^i + 0.$$

Moreover,

$$\begin{aligned} 0 &= d\omega \\ &= dw_i \wedge d\bar{x}^i + df \wedge dt \\ &= (\text{terms not involving } dt) + \frac{\partial w_i}{\partial t} dt \wedge d\bar{x}^i \\ &\quad + \frac{\partial f}{\partial \bar{x}^i} d\bar{x}^i \wedge dt. \end{aligned}$$

This implies that  $\frac{\partial w_i}{\partial t} = \frac{\partial f}{\partial x^i}$  for each  $i$ . For each  $p \in U$ , we compute the sum

$$w_i(p, 1) - w_i(p, 0) = \int_0^1 \frac{\partial w_i}{\partial t}(p, t) dt = \int_0^1 \frac{\partial f}{\partial x^i}(p, t) dt.$$

As a result,

$$\iota_1^* \omega - \iota_0^* \omega = \left( \int_0^1 \frac{\partial f}{\partial x^i}(p, t) dt \right) dx^i.$$

Define  $g : U \rightarrow \mathbb{R}$  by  $\int_0^1 f(p, t) dt$ , so that  $\frac{\partial g}{\partial x^i} = \int_0^1 \frac{\partial f}{\partial x^i}(p, t) dt$ . It follows that  $\iota_1^* \omega - \iota_0^* \omega = \frac{\partial g}{\partial x^i} dx^i = dg$ .

Since the pullback is coordinate-independent,  $g$  is as well.  $\square$

By assumption, there is some smooth map  $H : M \times [0, 1] \rightarrow M$  such that  $H \circ \iota_1 = \text{id}_M$  and  $H \circ \iota_0 = e_{p_0}$  where  $p_0 \in M$ . Let  $\omega$  be a closed 1-form on  $M$ . Then  $H^* \omega$  is closed since pullbacks commute with exterior derivatives. Recall that the pullback is a contravariant functor. By our claim, it follows that  $\iota_1^* H^* \omega - \iota_0^* H^* \omega = \omega - 0 = \omega$  is closed.

The generalization of this result to any positive integer  $k$  proceeds as follows.

We have the decomposition  $T_{(p,t)} M \times [0, 1] = \ker d\pi|_{(p,t)} \oplus \ker d\pi_M|_{(p,t)}$  where  $\pi : M \times [0, 1] \rightarrow [0, 1]$  denotes projection. Then any 1-form  $\omega$  on  $M \times [0, 1]$  may be written uniquely as  $\omega = \omega_1 + \omega_2$  such that  $\omega_i(v_1 + v_2) = \omega(v_i)$  for each  $i = 1, 2$ . Hence there is some unique  $f : M \times [0, 1] \rightarrow \mathbb{R}$  such that  $\omega_2 = f dt$ . In general, one can show that if  $\omega$  is a  $k$ -form on  $M \times [0, 1]$ , then we can write  $\omega$  uniquely as

$$\omega = \omega_1 + (dt \wedge \eta)$$

where  $\omega_1(v_1, \dots, v_k) = 0$  if some  $v_i \in \ker d\pi_M|_{(p,t)}$  and  $\eta$  is a  $(k-1)$ -form with the analogous property.

**Lemma 7.1.5.** *Define the  $(k-1)$ -form  $I\omega$  on  $M$  by*

$$I\omega|_p(v_1, \dots, v_{k-1}) = \int_0^1 \eta(p, t)(d\iota_t|_{(p,t)}(v_1), \dots, d\iota_t|_{(p,t)}(v_{k-1})) dt.$$

*Then  $\iota_1^* \omega - \iota_0^* \omega = d(I\omega) + I(d\omega)$ . In particular,  $\iota_1^* \omega - \iota_0^* \omega$  is exact whenever  $d\omega = 0$ .*

*Proof.* For an argument similar to our  $k = 1$  case, see *Spivak*, Theorem 7.17. In particular,  $I\omega$  and  $\eta$  correspond to our  $g$  and  $f$ , respectively.  $\square$

$\square$

**Corollary 7.1.6.** *By Remark 5.5.7,  $\mathbb{R}^2 \setminus \{0\}$  is not contractible.*

## 7.2 Lecture 25

**Corollary 7.2.1.** *If  $M$  is closed (i.e., compact without boundary) and orientable, then  $M$  is not contractible.*

*Proof.* There is some positively oriented orientation form  $\omega$  on  $M$ . Then  $d\omega = 0$ , and  $\int_M \omega > 0$ . But if  $\omega = d\eta$  for some form  $\eta$ , then  $\underbrace{\int_M \omega = \int_{\partial M} \eta}_{\text{Stokes}} = 0$ , a contradiction. Hence  $H^n(M) \neq 0$ .  $\square$

**Example 7.2.2.**  $\mathbb{S}^n$  is not contractible.

**Theorem 7.2.3.** *If  $M$  is a (connected) orientable  $n$ -manifold, then there is an isomorphism*

$$\underbrace{H_c^n(M)}_{\text{compactly supported}} \xrightarrow{\cong} \mathbb{R}, \quad [\omega] \mapsto \int_M \omega.$$

*Proof.* Take for granted that the statement holds when  $M = \mathbb{R}^n$ . There is some compactly supported orientation form  $\omega$  on  $M$  such that  $\int_M \omega \neq 0$  and  $\text{supp } \omega \subset \bigcup_{\text{open}} U \subset M$ . Let  $\omega'$  be a compactly supported

$n$ -form on  $M$ . Find any partition of unity  $(\phi_\alpha)$  on  $M$ . Then  $\omega' = \phi_1 \omega' + \dots + \phi_k \omega'$ . Thus, we may assume that  $\text{supp } \omega' \subset V$  where  $V \approx \mathbb{R}^n$ . We want to show that  $\omega' = c\omega + d\eta$  for some  $c \in \mathbb{R}$  and some  $\eta \in A^{n-1}(M)$ . Since  $M$  is connected, there is some sequence  $U = V_1, V_2, \dots, V_r = V$  of open sets such that  $V_i \approx \mathbb{R}^n$  and  $V_i \cap V_{i+1} \neq \emptyset$  for each  $i = 1, \dots, r-1$ . For each  $i = 1, \dots, r-1$ , find forms  $\omega_i$  on  $M$  such  $\int_M \omega_i \neq 0$  and  $\text{supp } \omega_i \subset V_i \cap V_{i+1}$ . It follows that

$$\begin{aligned}\omega_1 &= c_1 \omega + d\eta_1 \\ \omega_2 &= c_2 \omega_1 + d\eta_2 \\ &\vdots \\ \omega' &= c_r \omega_{r-1} + d\eta_r,\end{aligned}$$

as desired. □

*Remark 7.2.4.* If  $M$  and  $N$  are closed orientable  $n$ -manifolds and  $f : M \rightarrow N$  is smooth, then the pullback  $f^*$  induces a linear map  $f^* : H_{\text{dR}}^n(N) \rightarrow H_{\text{dR}}^n(M)$ . We thus get a linear map  $f^* : \mathbb{R} \rightarrow \mathbb{R}$ , which shows that there is some real number  $a$  such that

$$\int_M f^* \omega = a \int_N \omega$$

for every  $\omega \in H_{\text{dR}}^n(N)$ . Such a scalar  $a$  is called the *degree* of  $f$ .

### 7.3 Lecture 26

**Theorem 7.3.1.** *Let  $M$  and  $N$  be closed orientable  $n$ -manifolds and  $f : M \rightarrow N$  be smooth. By Sard's theorem, find some regular value  $q$  of  $f$ . For each  $p \in f^{-1}(q)$ , define  $\text{sgn}_p f = \begin{cases} 1 & df_p \text{ orientation-preserving} \\ -1 & df_p \text{ orientation-reversing} \end{cases}$ .*

*Then*

$$\deg f = \sum_{p \in f^{-1}(q)} \text{sgn}_p f$$

*where we set  $\deg f = 0$  if  $f^{-1}(q) = \emptyset$ . In particular,  $\deg f$  is always an integer.*

*Proof.* Since  $f$  has constant rank  $n$  and  $\{q\}$  is closed, we see that  $f^{-1}(q)$  is a compact 0-dimensional submanifold of  $M$  and thus must be finite. Write  $f^{-1}(q) = \{p_1, \dots, p_k\}$ . Find respective charts  $U_1, \dots, U_k$  which are pairwise disjoint so that each  $u_i \in U_i$  is a regular point of  $f$ . Find a chart  $(V, y^i)$  around  $q$  such that the components of  $f^{-1}(V)$  are precisely the  $U_i$ . Set  $\omega = g dy^1 \wedge \dots \wedge dy^n$  where  $g$  is nonnegative and compactly supported in  $V$ . This implies that  $f^* \omega \subset f^{-1}(V) = U_1 \sqcup \dots \sqcup U_k$ . Therefore,

$$\int_M f^* \omega = \sum_{i=1}^k \int_{U_i} f^* \omega.$$

Since each  $f|_{U_i} : U_i \rightarrow V$  is a diffeomorphism, we have that

$$\int_{U_i} f^* \omega = \begin{cases} \int_V \omega & f|_{U_i} \text{ orientation-preserving} \\ -\int_V \omega & f|_{U_i} \text{ orientation-reversing} \end{cases}.$$

As a result,

$$\int_M f^* \omega = \left( \sum_{p \in f^{-1}(q)} \text{sgn}_p f \right) \int_V \omega = \left( \sum_{p \in f^{-1}(q)} \text{sgn}_p f \right) \int_M \omega.$$

□



**Example 7.3.2.** Let  $A_n : \mathbb{S}^n \rightarrow \mathbb{S}^n$  denote the antipodal map. Choose  $p_0 \in \mathbb{S}^n$ , which is a regular value of  $A_n$ . Hence  $\deg A_n = (-1)^{n-1}$ .

**Theorem 7.3.3.** Suppose that  $f, g : M \rightarrow N$  are (smoothly) homotopic maps. Then  $f^* = g^*$  as linear maps. If  $M$  and  $N$  are compact orientable  $n$ -manifolds, it follows that  $\deg f = \deg g$ .

*Proof.* By assumption, there exists a smooth map  $H : M \times [0, 1] \rightarrow M$  such that  $H \circ \iota_0 = f$  and  $H \circ \iota_1 = g$ . Let  $\omega \in Z^k(N)$ . We apply Lemma 7.1.5 (including its notation) to compute

$$\begin{aligned} g^*\omega - f^*\omega &= (H \circ \iota_1)^*\omega - (H \circ \iota_0)^*\omega \\ &= \iota_1^*(H^*\omega) - \iota_0^*(H^*\omega) \\ &= d(IH^*\omega) + I(dH^*\omega) = d(IH^*\omega). \end{aligned}$$

This implies that  $f^*([\omega]) = g^*([\omega])$ , as required.  $\square$

**Corollary 7.3.4.** (Hairy ball theorem) If  $n$  is even, then there is no non-vanishing vector field on  $\mathbb{S}^n$ .

*Proof.* The identity  $\text{id}_{\mathbb{S}^n}$  has degree 1 and thus is not homotopic to the antipodal map  $A_n$ . Suppose, for contradiction, that there is some non-vanishing  $X \in \mathcal{X}(\mathbb{S}^n)$ . For each  $p \in \mathbb{S}^n$ , there is a unique great semicircle  $\gamma_p$  traveling from  $p$  to  $A(p)$  whose tangent vector at  $p$  equals  $cX_p$  for some  $c \in \mathbb{R}$ . The smooth map  $H(p, t) = \gamma_p(t)$  defines a homotopy between  $\text{id}_{\mathbb{S}^n}$  and  $A_n$ , a contradiction.  $\square$

## 8 Integral curves and flows

### 8.1 Lecture 27

**Definition 8.1.1.** Let  $M$  be a manifold and  $X \in \mathcal{X}(M)$ . We say that  $\gamma : J \rightarrow M$  is an *integral curve* for  $X$  if  $\gamma'(t) = X_{\gamma(t)}$  for any  $t \in J$ .

**Example 8.1.2.** Set  $M = \mathbb{R}^2$ ,  $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ , and  $\gamma(t) = (x(t), y(t))$ . Then  $\gamma'(t) = x'(t) \frac{\partial}{\partial x} + y'(t) \frac{\partial}{\partial y}$ . The system  $\begin{cases} x(t) = x'(t) \\ y(t) = y'(t) \end{cases}$  determines that  $\gamma(t) = e^t(x(0), y(0))$ .

*Remark 8.1.3.* In general, define the vector field  $x^i \frac{\partial}{\partial x^i}$  on a chart  $(U, x^i)$  for the  $n$ -manifold  $M$ . Then given an integral curve  $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$  for  $X$  where  $\gamma^i = \gamma \circ x^i$ , we obtain the system

$$\gamma'^i(t) = X^i(\gamma^1(t), \dots, \gamma^n(t)).$$

Given that  $\gamma(0) = p$ , we have an initial value problem, to which we can always find a local solution.

**Theorem 8.1.4 (Fundamental theorem for autonomous ODEs).** Let  $U \subset \mathbb{R}^n$  be open and  $X : U \rightarrow \mathbb{R}^n$  is a smooth vector field. Consider the initial value problem.

$$(*) \quad \begin{cases} \gamma'^i(t) = X^i(\gamma^1(t), \dots, \gamma^n(t)) \\ \gamma(t_0) = (c^1, \dots, c^n) \end{cases}.$$

1. (Existence) Let  $t_0 \in \mathbb{R}$  and  $x_0 \in U$ . There exist some interval  $J_0 \ni t_0$  and open subset  $U_0 \subset U$  such that for each  $c \in U_0$ , there is some  $C^1$  curve  $\gamma : J_0 \rightarrow U_0$  that solves  $(*)$ .
2. (Uniqueness) Any two differentiable solutions to  $(*)$  agree on the common domain.
3. (Smoothness) Let  $J_0$  and  $U_0$  be as before. Define  $\theta : J_0 \times U_0 \rightarrow U$  by  $(t, x) \mapsto \gamma_x(t)$  where  $\gamma_x : J_0 \rightarrow U$  uniquely solves  $(*)$  with initial condition  $\gamma(t_0) = x$ . Then  $\theta$  is smooth.

**Example 8.1.5.** For any compact manifold  $M$ , we may stipulate that the  $U_0$  form a finite cover  $\{U_1, \dots, U_k\}$  of  $M$ . Then make  $J_0$  smaller than any of the corresponding intervals  $J_1, \dots, J_k$ . This gives the smooth map  $\theta : J \times \mathbb{S}^n \rightarrow \mathbb{S}^n$  defined by  $(t, p) \mapsto \gamma_p^i(t)$ .

**Corollary 8.1.6.** Let  $X$  be a smooth vector field on  $M$  and  $p \in M$ . There is some  $\epsilon > 0$  and a smooth curve  $\gamma : (-\epsilon, \epsilon) \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma$  is an integral curve for  $X$ .

**Definition 8.1.7.** Let  $\theta : \mathbb{R} \times M \rightarrow M$  be a group action on  $M$ . We call  $\theta$  a *global flow* on  $M$  if it is smooth, i.e.,  $\theta^p(t) := \theta(t, p) : \mathbb{R} \rightarrow M$  is smooth for every  $p \in M$ . The vector field  $p \mapsto (\theta^p)'(0)$  is called the *infinitesimal generator* of  $\theta$ .

*Question.* When is a vector field an infinitesimal generator of a global flow?

**Example 8.1.8.** Define  $X = x^3 \frac{\partial}{\partial x}$  on  $\mathbb{R}^2$ . Then any integral curve  $\gamma(t) = (x(t), y(t))$  for  $X$  must satisfy

$$\begin{aligned} \frac{dx}{dt} &= x^3. \implies dx = x^3 dt. \\ \implies -\frac{1}{2x^2} &= t + c. \\ \implies x(t) &= \frac{1}{\sqrt{c - 2t}}, \end{aligned}$$

which is not smooth on  $\mathbb{R}$ . Hence  $X$  does not generate global flow.

**Lemma 8.1.9 (Escape lemma).** Let  $X \in \mathcal{X}(M)$  and  $\gamma$  be an integral curve for  $X$ . If the domain of  $\gamma$  does not equal  $\mathbb{R}$ , then  $\text{im } \gamma$  is not contained in any compact set.

*Remark 8.1.10.* If  $M$  is compact, then every smooth vector field on  $M$  generates a global flow.

**Theorem 8.1.11 (Fundamental theorem on flows).** Let  $M$  be a manifold and  $X \in \mathcal{X}(M)$ . There exist some unique maximal flow domain  $D \subset \mathbb{R} \times M$  and unique flow  $\phi : D \rightarrow M$  such that  $X$  generates  $\phi$ . We call  $\phi$  the *flow* of  $X$ .

**Corollary 8.1.12.** If  $M$  is a closed manifold, then  $D = \mathbb{R} \times M$ .

## 8.2 Lecture 28

**Definition 8.2.1.** Let  $M$  be a manifold without boundary. Let  $V \in \mathcal{X}(M)$ . Let  $\theta$  denote the flow of  $V$ . For any  $W \in \mathcal{X}(M)$ . Define the *Lie derivative* of  $W$  with respect to  $V$

$$(\mathcal{L}_V W)_p = \lim_{t \rightarrow 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t}.$$

This is called the *Lie derivative* of  $W$  with respect to  $V$ .

**Note 8.2.2.** If  $p \in M$ , then  $(\mathcal{L}_V W)_p$  exists and  $\mathcal{L}_V W \in \mathcal{X}(M)$ .

**Theorem 8.2.3.** If  $V, W \in \mathcal{X}(M)$ , then  $\mathcal{L}_V W = [V, W]$ .

*Proof.* Define  $\mathcal{R}(M)$  as the set of points  $p \in M$  such that  $V_p \neq 0$ . Note that  $\text{cl}(\mathcal{R}(M)) = \text{supp } V$ . Let  $p \in M$ . We consider three cases.

1. Suppose  $p \in \mathcal{R}(M)$ . Then it's a fact that we can find smooth coordinates  $(U, u^i)$  near  $p$  such that  $V = \frac{\partial}{\partial u^1}$ . In these coordinates we thus have that  $\theta_t(u) = (u^1 + t, u^2, \dots, u^n)$ . The Jacobian of  $\theta_{-t}$  at each  $t$  equals the identity. If  $u \in U$ , it follows that

$$\begin{aligned} & d(\theta_{-t})_{\theta_t(u)}(W_{\theta_t(u)}) \\ &= d(\theta_{-t})_{\theta_t(u)} \left( W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_{\theta_t(u)} \right) \\ &= W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u. \end{aligned}$$

From this we compute

$$\begin{aligned}
(\mathcal{L}_V W)_p &= \frac{d}{dt} \Big|_{t=0} W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u \\
&= \frac{\partial}{\partial u^1} W^j(u^1, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u \\
&= [V, W]_u.
\end{aligned}$$

2. Suppose that  $p \in \text{supp } V \setminus \mathcal{R}(M)$ . Since  $\text{supp } V$  is dense in  $M$  and  $TM$  is Hausdorff, it follows that  $(\mathcal{L}_V W)_p = [V, W]_p$ .
3. If  $p \in M \setminus \text{supp } V$ , then  $V$  vanishes on some neighborhood  $H$  of  $p$ . This implies that  $\theta_t = \text{id}_H$ , so that  $d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = W_p$ . Hence  $(\mathcal{L}_V W)_p = 0 = [V, W]_p$ .

□

**Definition 8.2.4.** Let  $M$  be an  $n$ -manifold. A smooth local frame  $(X_1, \dots, X_n)$  is called a *commuting* or *holonomic frame* if  $[X_i, X_j] = 0$  for any  $1 \leq i, j \leq n$ .

**Theorem 8.2.5.** Let  $(X_1, \dots, X_k)$  be a linearly independent  $k$ -tuple of smooth commuting vector fields defined on an open set  $W \subset M$ . For any  $p \in W$ , there is some chart  $(U, x^i)$  around  $p$  such that

$$X_i = \frac{\partial}{\partial x^i}$$

holds locally for each  $i = 1, \dots, k$ .

*Proof.* See Lee, Theorem 9.46.

□

## 9 Distributions

**Definition 9.0.1.** Let  $M$  be a manifold. A  $k$ -distribution on  $M$  is a rank- $k$  smooth subbundle of  $TM$ .

This means that a 1-distribution is precisely a vector field.

**Definition 9.0.2.** Let  $N \subset M$  be a nonempty submanifold and  $D$  be a distribution on  $M$ . Write  $D = \bigcup_{p \in M} D_p$  as a family of subspaces. Then  $N$  is called an *integral manifold of  $D$*  if  $D_p = T_p N$  for each  $p \in N$ . Moreover, we say that  $D$  is *integrable* if each  $p \in M$  is contained in an integrable manifold of  $D$ .

**Definition 9.0.3.** We say that a distribution  $D$  is *involutive* if  $[X, Y] \in D$  whenever  $X, Y \in D$ .

**Proposition 9.0.4.** If  $D$  is integrable, then it is involutive.

**Theorem 9.0.5 (Frobenius).** If  $D$  is involutive, then it is integrable.