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Abstract

These notes are based on Alexander Voronov's lectures for the course "Rational Homotopy Theory" at UMN. Any mistake in what follows is my own.

Contents

1 Introduction

1.1 Lecture 1

In algebraic topology, we want to associate to any topological space X an algebraic object $A(X)$ such that the mapping A is homotopy invariant, i.e.,

$$X \sim Y \implies A(X) \sim A(Y).$$

In an ideal world, we have a converse so that we can learn everything about X by computing $A(X)$. Indeed, topology is hard, and algebra is easy. In reality, however, we can just learn *something* about X from $A(X)$.

Example 1.1.1. If $\widetilde{H}_0(X) = 0$, then X is path connected. (Of course, the converse also holds.)

In rational homotopy theory (RHT), the mapping A is invariant under *rational equivalence*, i.e.,

$$X \overset{\mathbb{Q}}{\sim} Y \implies A(X) \cong A(Y).$$

The reverse implication holds under certain assumptions, e.g., that X and Y are simply connected. The object $A(X)$ is pretty simple, a differential graded-commutative algebra (DGCA) over $\mathbb{Q}$.

Example 1.1.2. The so-called Sullivan picture of A has the following properties.

1. $A(S^3) = \mathbb{Q}[g_3],\ |g_3| = 3,\ dg_3 \equiv 0.$

   Hence $A(S^3)$ is the exterior algebra $\mathbb{Q} \oplus \mathbb{Q}g_3$ with $d = 0$.

2. $A(S^4) = \mathbb{Q}[g_4, g_7],\ dg_4 \equiv 0,\ dg_7 \equiv g_4^2.$

Moreover, there exists a "Koszul-dual" picture of Quillen models used to define A. This deals with differential graded-Lie algebras (DGLA's) along with L_∞-algebras. (This picture appears to be new for a course on RHT.)

We plan to cover the following topics, time permitting.

1. DGCA's
2. model categories
3. simplicial sets
4. polynomial de Rham algebras
5. spectral sequences
6. Quillen models (L_∞-versions)
7. Mysterious Duality between physics and math via RHT

Here are a few good references for these topics.

- Bousfield, A. K.; Gugenheim, V. K. A. M. *On PL de Rham theory and rational homotopy type.*
- Félix, Y.; Halperin, S.; Thomas, J.-C. *Rational homotopy theory.*

- Julian Holstein's online notes *Rational homotopy theory.*

First, let's review some concepts and facts from homotopy theory. Any based space (X,x) gives rise to the following algebraic objects.

- the graded-abelian group $H_\bullet(X;\mathbb{Z})$
- the graded-commutative algebra $H^\bullet(X;\mathbb{Z})$
- for any integer $n > 0$, the group $\pi_n(X,x)$ of homotopy components
- the pointed set $\pi_0(X,x)$ of path components

We call $\pi_1(X,x)$ the fundamental group and $\pi_n(X,x)$ a higher homotopy group when $n \geq 2$.

For any subspaces $A \subset X$ and $B \subset Y$, we have the set

$$[(X,A),(Y,B)]$$

of homotopy classes of maps $(X,A) \to (Y,B)$ of pairs, i.e., maps $f : X \to Y$ such that $f(A) \subset B$. A homotopy between two such maps is an ordinary homotopy $H : X \times I \to Y$ between them such that $H(a,t) \in B$ for any $(a,t) \in A \times I$.

Definition 1.1.3. For any $(X,x) \in \mathsf{Top}_*$ and integer $n \geq 1$, define the *n-th homotopy group* $\pi_n(X,x)$ as the set

$$\underbrace{[(I^n,\partial I^n),(X,x)]}_{[(S^n,N),(X,x)]}$$

equipped with the binary operation

$$(f * g)(t_1,\dots,t_n) \equiv \begin{cases} g(t_1,\dots,t_{n-1},2t_n) & t_n \leq \frac{1}{2} \\ f(t_1,\dots,t_{n-1},2t_n - 1) & t_n \geq \frac{1}{2} \end{cases},$$

i.e., concatenation in the last variable.

The Eckmann-Hilton argument tells us

1. that we obtain the same group law by concatenating in any other variable and
2. that π_n is abelian when $n \geq 2$.

1.2 Lecture 2

Definition 1.2.1. A map $f : X \to Y$ is a *weak (homotopy) equivalence (w.e.)* if the induced map

$$\pi_n(f) : \pi_n(X,x) \to \pi_n(Y,f(x))$$

is an isomorphism for every $n \geq 0$ and $x \in X$.

Is it cheating to replace homotopy equivalence (h.e.) with w.e.? Not quite, in the following sense.

Theorem 1.2.2 (Whitehead). *Any w.e. between CW complexes is a h.e.*

Recall that a CW complex is a space of the form $\bigcup_{n\geq -1} X^n = \operatorname{colim}_n X^n$ where $X^{-1} \equiv \varnothing$ and X^n is obtained from X^{n-1} by adjoining n-cells. In other words, X^n is precisely the pushout

$$\begin{tikzcd} \coprod_{i\in I} S^{n-1} \arrow[r, "f"] \arrow[d, hook] & X^{n-1} \arrow[d] \\ \coprod_{i\in I} D^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X^n \end{tikzcd},$$

where f denotes the attaching map. In concrete form, $X^n = X^{n-1} \cup_f (\coprod_{i\in I} D^n)$.

Note 1.2.3. The CW complexes $S^3 \times \mathbb{RP}^2$ and $\mathbb{RP}^3 \times S^2$ have isomorphic homotopy groups but are *not* weakly equivalent.

Proof. The double cover $\mathbb{Z}_2 \to S^n \to \mathbb{RP}^n$ induces a LES

$$\begin{tikzcd} \cdots \arrow[r] & \pi_k(\mathbb{Z}_2) \arrow[r] & \pi_k(S^n) \arrow[r] & \pi_k(\mathbb{RP}^n) \arrow[dll] \\ & \pi_{k-1}(\mathbb{Z}_2) \arrow[r] & \cdots \arrow[r] & \pi_0(\mathbb{RP}^n) \end{tikzcd}.$$

If $n \geq 2$, then we deduce that the higher homotopy groups of $\mathbb{RP}^n$ are isomorphic to those of S^n, so that

$$\pi_k(\mathbb{RP}^n) = \begin{cases} 0 & k = 0 \\ \mathbb{Z}_2 & k = 1 \\ \pi_k(S^n) & k \geq 2 \end{cases}.$$

This means that $S^3 \times \mathbb{RP}^2$ and $\mathbb{RP}^3 \times S^2$ have the same homotopy groups.

At the same time, they have different cohomology algebras over $\mathbb{Z}_2$.

$$\begin{aligned} H^\bullet(S^3 \times \mathbb{RP}^2; \mathbb{Z}_2) &\cong \mathbb{Z}_2[\xi] \otimes \mathbb{Z}_2[x] \big/ (x^3), \ |\xi| = 3, \ |x| = 1 \\ &\ncong H^\bullet(\mathbb{RP}^3 \times S^2; \mathbb{Z}_2) \end{aligned}$$

Thus, they are not homotopy equivalent, hence not weakly equivalent, by Theorem 1.2.2. □

Theorem 1.2.4 (CW approximation). *For any space X, there is some CW complex Z along with a weak equivalence $Z \xrightarrow{w.e.} X$.*

Theorem 1.2.4 means that the category of spaces localized at weak equivalences is equivalent to the category of CW complexes localized at homotopy equivalences. The category of CW complexes with homotopy classes of maps is called the *homotopy category of spaces*.

Theorem 1.2.5. *Suppose that $f : X \to Y$ is a map of connected CW complexes that induces isomorphisms on $\pi_1(-)$ and $H_n(-; \mathbb{Z})$ for all $n \geq 2$. Then f is a weak equivalence (hence homotopy equivalence).*

Idea of proof. Look at the homotopy fiber of f (also called the *mapping cocone* of f):

$$\begin{tikzcd}
\mathrm{cocone}(f) \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & E_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& Y^I \arrow[r, "\mathsf{eval}_0"] \arrow[d, "\mathsf{eval}_1"] & Y \\
* \arrow[r] & Y &
\end{tikzcd}.$$

The space E_f is homotopy equivalent to X. Now apply the Hurewicz theorem to $\mathrm{cocone}(f)$. $\square$

Let's now turn to a basic concept of RHT: *rationalization.*

Definition 1.2.6. We say that a path connected space X is *rational* if $H_n(X;\mathbb{Z})$ is a $\mathbb{Q}$-vector space for all $n \geq 1$.

Example 1.2.7. We construct the space $S^1_{\mathbb{Q}}$ as the colimit of the following sequence

$$X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow X_{k+1} \hookrightarrow \cdots$$

in Top_*. Let $X_1 = \left(S^1, 1\right)$. For any $k \geq 1$, define $w_k : S^1 \to S^1$ by $z \mapsto z^k$ and let W_k denote the pointed mapping cylinder of w_k. Also, define the embeddings

$$s_k : S^1 \hookrightarrow W_k,\ z \mapsto (z,1)$$
$$t_k : S^1 \hookrightarrow W_k,\ z \mapsto [(z,0)].$$

Finally, for each $k \geq 2$, define X_k inductively as the pushout

$$\begin{tikzcd}
S^1 \arrow[r, "s_k"] \arrow[d, "t_{k-1}"'] & W_k \arrow[dd] \\
W_{k-1} \arrow[d] & \\
X_{k-1} \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X_k
\end{tikzcd}.$$

Note that $X_n \overset{\text{h.e.}}{\sim} S^1$ for all $n \geq 1$. It turns out, however, that the colimit

$$X \coloneqq \mathrm{colim}_n\, X_n = \bigcup_n X_n$$

is the Eilenberg-MacLane space $K(\mathbb{Q},1)$.

1.3 Lecture 3

We have maps out of S^1 into all finite stages of $S^1_{\mathbb{Q}}$.

$$\begin{tikzcd}
X_1 \arrow[r, hook] & \cdots \arrow[r, hook] & X_k \arrow[r, hook] & X_{k+1} \arrow[r, hook] & \cdots \\
& & S^1 \arrow[ull, "\underbrace{i_1}_{\mathrm{id}_{S^1}}"] \arrow[u, "i_k"'] \arrow[ur, "i_{k+1}"'] & &
\end{tikzcd}$$

It's easy to see that $X_k \sim S^1$ for all $k \geq 1$, so that $\pi_1(X_k) \cong \mathbb{Z}$. The inclusion $X_{k-1} \to X_k$, however, induces the mapping

$$\begin{aligned} \mathbb{Z} &\to \mathbb{Z} \\ 1 &\mapsto k \cdot 1 \end{aligned}$$

on π_1.

Claim 1.3.1. *All higher homology groups of X vanish.*

Proof. Let $n \in \mathbb{Z}_{\geq 2}$. Any map $\Delta^n \to X$ factors through X_k for some k because Δ^n is compact. Thus, any singular n-cycle σ in X will be a cycle in X_k, which is homotopy equivalent to S^1. This means that σ will be a boundary in X_k and thus in X. Thus, $H_n(X;\mathbb{Z}) = 0$. □

Consider the loop $\gamma : S^1 \xrightarrow{i_1} X_1 \hookrightarrow X$ in X. Its homotopy class is nontrivial, for otherwise there is a null-homotopy $S^1 \times I \to X$ factoring through some X_k. Since $X_k \sim S^1$, this would imply that S^1 is contractible, which is false.

Specifically, $[\gamma] = k!\,[\delta]$ where $[\delta]$ denotes the bottom generator of $\pi_1(X_k) \cong \mathbb{Z}$. In fact, all $\mathbb{Q}$-multiples of $[\gamma]$ belong to $\pi_1(X,1)$. Further, any map $S^1 \to X$ factors through some $X_k \hookrightarrow X$, so that its homotopy class is a rational multiple of $[\gamma]$. It follows that $\pi_1(X,1) \cong \mathbb{Q}\,[\gamma] \cong \mathbb{Q}$.

By the Hurewicz theorem, $H_1(X;\mathbb{Z}) \cong \pi_1(X,1)^{\mathrm{ab}} \cong \mathbb{Q}$. Hence X is rational. Also, it's path connected, and any map $S^n \to X$ factors through some $X_k \subset X$, where $X^k \sim S^1$. Thus, if $n \geq 2$, then $\pi_n(X,1) = 0$. We conclude that $S^1_{\mathbb{Q}} \cong K(\mathbb{Q},1)$.

Terminology. The space $S^1_{\mathbb{Q}}$ is called the *rationalization* of S^1.

Remark 1.3.2. The same construction can be done for higher spheres and their attaching maps and thus for all CW complexes.

Definition 1.3.3. Let X and Y be path connected spaces.

1. A map $f : X \to Y$ is a *rational equivalence* if the induced map $H_\bullet(f) : H_\bullet(X;\mathbb{Q}) \to H_\bullet(Y;\mathbb{Q})$ is an isomorphism.

2. A *rationalization* of X is a rational space $X_{\mathbb{Q}}$ together with a rational equivalence $X \to X_{\mathbb{Q}}$.

Lemma 1.3.4. *The pair $\left(S^1_{\mathbb{Q}}, i_1 : S^1 \to X\right)$ is a rational equivalence.*

Proof. If $n \geq 3$, then

$$\begin{aligned} H_n(S^1;\mathbb{Q}) &= 0 \\ &= H_n(S^1_{\mathbb{Q}};\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \\ &= H_n(S^1_{\mathbb{Q}};\mathbb{Q}). && \text{(UCT)} \end{aligned}$$

We also have that

$$\begin{aligned} H_2(S^1_{\mathbb{Q}};\mathbb{Q}) &= \mathrm{Tor}_1(H_1(S^1_{\mathbb{Q}};\mathbb{Z}),\mathbb{Q}) && \text{(UCT)} \\ &= 0 && (\mathrm{Tor}_1(-.\mathbb{Q}) = 0) \\ &= H_2(S^1;\mathbb{Q}). \end{aligned}$$

Finally, we have that

$$\begin{aligned} H_1(S^1_{\mathbb{Q}};\mathbb{Q}) &= H_1(S^1_{\mathbb{Q}};\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \qquad \text{(UCT)} \\ &= \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \\ &= \mathbb{Q} \\ &= H_1(S^1;\mathbb{Q}). \end{aligned}$$

The final isomorphism is given by

$$H_1(i_1) : H_1(S^1;\mathbb{Q}) \to H_1(S^1_{\mathbb{Q}};\mathbb{Q}),$$

which sends the generator $[\gamma] \in H_1(S^1;\mathbb{Z})$ to its image $[\gamma]$ as a $\mathbb{Q}$-vector space generator. □

Theorem 1.3.5. *Every connected CW complex has a rationalization.*

Definition 1.3.6. A *rational homotopy type* is the equivalence class of a path connected space under rational equivalence.

Example 1.3.7. Consider $\mathbb{RP}^2$. Recall that

$$H_n(\mathbb{RP}^2;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & n = 0 \\ 0 & n > 0 \end{cases}.$$

Hence $\mathbb{RP}^2 \to *$ is a rational equivalence, so that the rational homotopy type of $\mathbb{RP}^2$ is $[*]$. Further, $*$ is trivially a rational space. Therefore, it is a rationalization of $\mathbb{RP}^2$. At the same time, we have the nontrivial group

$$\pi_2^{\mathbb{Q}}(\mathbb{RP}^2) \coloneqq \underbrace{\pi_2(\mathbb{RP}^2)}_{\mathbb{Z}} \otimes \mathbb{Q} \cong \mathbb{Q}.$$

Thus, defining rational equivalence in terms of $H_\bullet(-;\mathbb{Q})$ rather than $H_\bullet^{\mathbb{Q}}(-)$ was simple, sensible enough.

We shall see, though, that $\mathbb{RP}^2$ is problematic for RHT because it's not nilpotent. (It's not useless, being simply connected.)

1.4 Lecture 4

Remark 1.4.1. Recall that $S^1_{\mathbb{Q}} = \bigcup_{n\geq 1} X_n = \operatorname{colim}_n X_n$ from Example 1.2.7. This is exactly the homotopy colimit of the sequence

$$S^1 \xrightarrow{\text{id}} S^1 \xrightarrow{w_2} S^1 \xrightarrow{w_3} S^1 \to \cdots.$$

Indeed, we have a commutative square

$$\begin{tikzcd} S^1 \arrow[r, "s_k"] \arrow[d, equal] & W_k \arrow[d, "\substack{\textit{collapse onto} \\ \textit{bottom}}"] \\ S^1 \arrow[r, "w_k"'] & S^1 \end{tikzcd},$$

so that s_k is the cofibrant replacement of w_k. As a result, we see that

$$\pi_1(S^1_{\mathbb{Q}}) = \operatorname{colim}_n \pi_1(S^1) = \operatorname{colim}\left(\mathbb{Z} \xrightarrow{-\cdot 1} \mathbb{Z} \xrightarrow{-\cdot 2} \mathbb{Z} \xrightarrow{-\cdot 3} \cdots\right) = \mathbb{Q}$$

because $\pi_1(-)$ preserves homotopy colimits of connected spaces.

We now turn to viable candidates, particularly the de Rham algebra, for the rational homotopy type of a space.

(1) For Top_* up to w.e., look at $\pi_\bullet(X,x)$.

Cons:

- Hard to compute.
- Fails to determine the weak homotopy type (see Note 1.2.3).

(2) If X is path connected, replace $\pi_\bullet(X,x)$ with the easier-to-compute $\pi_1(X)$ and $H_n(X;\mathbb{Z})$, $n \geq 2$.

Cons:

- Fails to determine the weak homotopy type.
  (If we introduce the coproduct on $H_\bullet(X;\mathbb{Z})$, then we could do better.)

(3) Look at $H^\bullet(X;\mathbb{Z})$, which is more comfortable as a graded-commutative ring.

Cons:

- Fails to determine the weak homotopy type.

(4) Look at the cochain complex $C^\bullet(X;\mathbb{Z})$ equipped with associative cup product.

Cons:

- Huge.
- Cup product is only homotopy commutative.

Pros:

- Determines the weak homotopy type once you extend homotopy commutativity to an E_∞-structure (Mandell's theorem).

(5) If X is a (smooth) manifold, then look at the de Rham algebra $(\Omega^\bullet(X), d)$.

Cons:

- Huge.
- Over $\mathbb{R}$ rather than $\mathbb{Q}$ or $\mathbb{Z}$.
- Defined for manifolds rather than generic spaces.

Pros:

- Graded commutative on the nose.
- Above problems can be resolved by RHT.
- Determines the rational homotopy type.

Note that $\Omega^\bullet(X) = C^\infty(S^\bullet(T^*_X[-1])) = C^\infty(\bigwedge^\bullet T^*_X)$. In local coordinates, the differential $d : \Omega^n(X) \to \Omega^{n+1}(X)$ is defined by

$$d(f(x_1,\ldots,x_m)dx_{i_1} \wedge \cdots \wedge dx_{i_n}) = \sum_{j=1}^{m} \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_n}, \quad m \equiv \dim X$$

for each $n \geq 0$.

2 Dg-commutative algebras

We want to focus on candidate (5) and create a homotopy theory of DGCA's.

Let k be any field. (It's cleaner to assume $\mathrm{char}(k) \neq 2$, and soon we'll have $\mathrm{char}(k) = 0$ and $k = \mathbb{Q}$.)

Definition 2.0.1. A *DGCA* (or *dg-commutative algebra* or *differential graded-commutative algebra*) is a (unital) k-algebra that is

(a) $\mathbb{Z}$-graded: $A = \bigoplus_{n\in\mathbb{Z}} A^n$, $a \in A^n \implies |a| = n$

$$A^m \otimes A^n \xrightarrow{mult.} A^{m+n}, \; |a \cdot b| = |a| + |b|$$

(b) graded-commutative: any two homogeneous elements a and b satisfy the *Koszul rule of signs*, i.e.,

$$a \cdot b = (-1)^{|a||b|}\, b \cdot a.$$

(c) equipped with a differential: a k-linear map $d : A \to A$ of degree 1, i.e., $d : A^n \to A^{n+1}$, such that

$$d^2 = 0, \; d(1) = 0, \; \underbrace{d(ab) = (da)\, b + (-1)^{|a|}\, a\, (db)}_{\textit{Leibniz rule}}.$$

Terminology. An element $a \in A^n$ is called $\begin{cases} \textit{even} & n \text{ is even} \\ \textit{odd} & n \text{ is odd} \end{cases}$.

For any DGCA (A, d), its cohomology $H^\bullet(A, d)$ is a graded-commutative algebra. Conversely, any graded-commutative algebra may be viewed as a DGCA with $d = 0$. It turns out that the de Rham algebra determines not only the de Rham cohomology but also all rational homotopy groups.

2.1 Lecture 5

Let's consider a few examples of DGCA's.

Example 2.1.1.

(1) Any commutative (associative) algebra is a DGCA A with $A = A^0$ and $d = 0$.

(2) Any graded-commutative algebra $A = \bigoplus_{n\in\mathbb{Z}} A^n$ is a DGCA with $d = 0$.

(3) The de Rham algebra $(\Omega^\bullet(X), d)$, with d the exterior derivative.

(4) Consider the topological n-simplex

$$\Delta^n \coloneqq \left\{ (t_0, \ldots, t_n) \mid \sum_{i=0}^{n} t_i = 1, \; t_i \geq 0 \right\}.$$

Let $\Omega^\bullet_{\mathsf{PL}}(\Delta^n)$ denote the quotient algebra

$$\Omega^\bullet(n) \coloneqq \frac{\mathbb{Q}\,[t_0, t_1, \ldots, t_n, dt_0, dt_1, \ldots, dt_n]}{\left(\sum_{i=0}^n t_i = 1, \; \sum_{i=0}^n dt_i = 0\right)}, \qquad |dt_i| = 1, \; |t_i| = 0$$

over $\mathbb{Q}$. Note that

$$\Omega^0_{\mathsf{PL}}(\Delta^n) = \frac{\mathbb{Q}[t_0, \ldots, t_n]}{((\sum_{i=0}^n t_i) - 1)}$$
$$\Omega^\bullet_{\mathsf{PL}}(\Delta^n) = \frac{\Omega^0_{\mathsf{PL}}(\Delta^n)[dt_0, \ldots, dt_n]}{(\sum_{i=0}^n dt_i)}$$
$$dt_i \cdot dt_j = -dt_j \cdot dt_i.$$

Define the differential d on $\Omega^\bullet(n)$ as the (graded) derivation satisfying

$$d(t_i) = dt_i$$
$$d(dt_i) = 0.$$

It's clear that $d^2 = 0$. By the definition of d, we also can verify that

$$d\big(f(t_0, \ldots, t_n)dt_{i_1}\cdots dt_{i_p}\big) = \sum_{j=0}^n \frac{\partial f}{\partial t_j} dt_j dt_{i_1}\cdots dt_{i_p}.$$

We call $\Omega^\bullet(n)$ the algebra of *rational polynomial differential forms on* Δ^n. In particular, we have that

$$H^p(\Omega^\bullet(n), d) = \begin{cases} \mathbb{Q} & p = 0 \\ 0 & p \neq 0 \end{cases}$$

as the affine plane $\{\sum_{i=0}^n t_i = 1\} \supset \Delta^n$ is contractible.

A *homomorphism* $f : A \to B$ *of DGCA's* is a mapping that respects grading, multiplication, and differentials. Denote the category of DGCA's over k by dgca_k. This has the category $\mathsf{dgca}_k^{\geq 0}$ of DGCA's over k concentrated in degree ≥ 0 as a full subcategory.

Definition 2.1.2. A morphism $f : A \to B$ of DGCA's is a *quasi-isomorphism* if it induces an isomorphism

$$H^\bullet(f) : H^\bullet(A) \xrightarrow{\sim} H^\bullet(B)$$

on cohomology.

Let V be a graded-vector space $\bigoplus_{m\in\mathbb{Z}} V^m$. The *free graded-commutative algebra on* V is

$$S^\bullet(V) \coloneqq \bigoplus_{n\geq 0} S^n(V) = \frac{T^\bullet(V)}{\Big(v \otimes w - (-1)^{|v||w|}\, w \otimes v\Big)}, \qquad v, w \in V,$$

where $T^\bullet(V)$ denotes the tensor algebra $\bigoplus_{n\geq 0} V^{\otimes n}$ on V.

Example 2.1.3. Let $\langle x_1, \ldots, x_n\rangle_k$ denote the graded-vector space over k spanned by elements $x_1, \ldots, x_n$ of chosen degrees, i.e., the direct sum $kx_1 \oplus \cdots \oplus kx_n$. The polynomial algebra $k[x_1, \ldots, x_n]$ is precisely the free algebra $S(\langle x_1, \ldots, x_n\rangle_k)$.

Consider a cochain complex $(V = \bigoplus_{m\in\mathbb{Z}} V^m, d)$. The *free DGCA* $(S^\bullet(V), d)$ *on the complex* V has differential

$$d(v_1\cdots v_n) \equiv \sum_{i=1}^n (-1)^{\epsilon_i}\, v_1\cdots v_{i-1} dv_i \cdot v_{i+1}\cdots v_n, \qquad \epsilon_i \equiv \sum_{k=1}^{i-1} |v_k|$$

(which arises directly from the Leibniz rule).

Lemma 2.1.4. *Let V be the cohain complex*

$$0 \to \underbrace{\mathbb{Q}^n}_{\deg 0} \xrightarrow{\mathrm{id}} \underbrace{\mathbb{Q}^n}_{\deg 1} \to 0.$$

Then $\Omega^\bullet(n) = (S^\bullet(V), d)$.

Proof. Notice that

$$\begin{aligned}\Omega(n) &= \frac{\mathbb{Q}[t_0, t_1, \ldots, t_n, dt_0, dt_1, \ldots, dt_n]}{(\sum_{i=0}^n t_i = 1,\ \sum_{i=0}^n dt_i = 0)} \\ &\cong \mathbb{Q}[t_1, \ldots, t_n, dt_1, \ldots, dt_n],\ d(t_1) \equiv dt_i,\ d(dt_i) \equiv 0\end{aligned}$$

as $t_0 = 1 - t_1 - \cdots - t_n$ and $dt_0 = -dt_0 - \cdots - dt_n$. This algebra is clearly isomorphic to $S^\bullet(V)$ with V written as

$$0 \to \mathbb{Q}t_1 \oplus \cdots \oplus \mathbb{Q}t_n \xrightarrow{d} \mathbb{Q}dt_1 \oplus \cdots \oplus \mathbb{Q}dt_n \to 0.$$

□

The forgetful functor $\mathsf{dgca}_k \xrightarrow{U} \mathsf{dgVect}_k$ into cochain complexes (forgetting multiplication) is right adjoint to the free DGCA functor $(V, d) \mapsto (S(V), d)$. This generalizes the familiar adjunction

$$\begin{tikzcd} \mathsf{gca}_k \arrow[r, bend left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{gVect}_k \arrow[l, bend left, "S"] \end{tikzcd}.$$

Definition 2.1.5. A DGCA (A, d) is *semifree* if $(A, 0)$ is free, i.e., there exists a graded-vector space V such that $S(V) \cong A$.

Definition 2.1.6 (Augmented DGCA). An *augmentation* on a DGCA A over k is a homomorphism $\epsilon : A \to k$.

This is similar to a choice of basepoint of a space. We call the maximal ideal $m \coloneqq \ker \epsilon$ the *augmentation ideal* of A and the quotient ${m}/{m^2}$ the *space of indecomposables*.

Note 2.1.7. There is a split SES

$$\begin{tikzcd} 0 \arrow[r] & m \arrow[r] & A \arrow[r] & k \arrow[r] \arrow[l, bend left, "\substack{unit\\ 1 \mapsfrom 1}"] & 0 \end{tikzcd}$$

of graded-vector spaces, which induces a sequence

$$\begin{tikzcd} 0 \arrow[r] & k \arrow[r] & A \arrow[r] & m \arrow[r] \arrow[dr, two heads] & 0 \\ & & & & {m}/{m^2} \end{tikzcd}.$$

We can form the category dgca/k with DGCA morphisms that respect augmentation. This is the slice category of dgca_k over k.

2.2 Lecture 6

We now turn to minimal models. The de Rham algebra $(\Omega^\bullet(X), d)$ for a manifold X is too large, whereas the cohomology $H^\bullet_{\mathrm{dR}}(X)$ is too small from the RHT perspective. Minimal models lie between these two pictures in a certain sense.

Definition 2.2.1. Let A be a DGCA.

1. We say that A is *connected* if
   (a) $A = A^{\geq 0} \coloneqq \bigoplus_{n\geq 0} A^n$ and
   (b) the map $k \to A^0$ given by $1 \mapsto 1$ is an isomorphism.
2. We say that A is *simply connected* if A is connected and $A^1 = 0$.

Any connected DGCA A is augmented. Indeed, the subalgebra $A^+ \coloneqq \bigoplus_{n>0} A^n$ is a maximal ideal, and we have an augmentation $A \to {}^{A}\!/_{A^+} = A^0 = k$. Here the augmentation ideal m is precisely A^+.

Example 2.2.2. Recall that $\Omega(n) = \mathbb{Q}[t_1, \ldots, t_n, dt_1, \ldots, dt_n]$. This is augmented by the mapping

$$\begin{aligned} \epsilon : \quad t_i &\mapsto 0 \\ dt_i &\mapsto 0, \end{aligned}$$

which is the same as the augmentation

$$f(t_1, \ldots, t_n) dt_{i_1} \cdots dt_{i_p} \overset{\epsilon}{\longmapsto} \begin{cases} f(0) & p = 0 \\ 0 & p \geq 1 \end{cases}.$$

The algebra, however, is not connected, for $\Omega(n)^0 = \mathbb{Q}[t_1, \ldots, t_n] \ncong \mathbb{Q}$. Even so, it's not simply connected, as

$$\Omega(n)^1 = \mathbb{Q}[t_1, \ldots, t_n] \otimes_{\mathbb{Q}} \underbrace{\langle dt_1, \ldots, dt_n \rangle_{\mathbb{Q}}}_{\mathbb{Q}\text{-span}} \neq 0.$$

Now, the augmentation ideal m is the homogeneous ideal

$$\ker \epsilon = (t_1, \ldots, t_n, dt_1, \ldots, dt_n).$$

This means that the space of indecomposables is the $2n$-dimensional vector space

$$\begin{aligned} {}^{m}\!/_{m^2} &= \langle t_1, \ldots, t_n, dt_1, \ldots, dt_n \rangle_{\mathbb{Q}} \\ &= \underbrace{T_0^*(T[-1]\Delta^n)}_{\textit{tangent bundle over } \Delta^n} \end{aligned}$$

over $\mathbb{Q}$. Here, $T[-1]\Delta^n$ denotes the graded-manifold defined by the graded-commutative algebra $\Omega(n)$. (In this sense, we have shifted Δ^n by -1.)

Definition 2.2.3. A *Sullivan algebra* is a DGCA $S(V)$ for a graded-vector space $V = \bigoplus_{n\geq 1} V^n$ that satisfies the *Sullivan (nilpotent) condition*, i.e., there is a filtration

$$V(0) \subset V(1) \subset \cdots \subset V(m) \subset V(m+1) \subset \cdots, \qquad V = \bigcup_{n\geq 0} V(n)$$

of V by graded-vector subspaces (in the sense that $V(m) = \bigoplus_{n\geq 1} (V(m) \cap V^n)$) such that

- $dV(0) = 0$ and
- $dV(n) \subset S(V(n-1))$.

Note that a Sullivan algebra is automatically augmented and connected, with

$$\begin{aligned} m = S(V)^+ &\coloneqq \bigoplus_{n>0} S(V)^n \\ &= S^{\geq 1}(V) \coloneqq \bigoplus_{p\geq 1} S^p(V) \end{aligned}$$

$$m/m^2 = V.$$

Definition 2.2.4. A connected DGCA A is *minimal* if $\operatorname{im} d \subset \underbrace{(A^+)^2}_{m^2}$ (i.e., d is decomposable).

Note 2.2.5. Let $A = S(V)$. Then A is connected if and only if $V = \bigoplus_{n\geq 1} V^n$, and A is simply connected if and only if $V = \bigoplus_{n\geq 2} V^n$.

Lemma 2.2.6. *Suppose that a simply connected semifree DGCA $S(V)$ is minimal. Then it's Sullivan.*

Proof. As $S(V)$ is simply connected, Note 2.2.5 implies that $V = \bigoplus_{n\geq 2} V^n$. For each $m \geq 0$, let

$$V(m) = \bigoplus_{n=2}^{m} V^n,$$

where $V(0) = V(1) = 0$. The family $\{V(m)\}_{m\geq 0}$ of graded-subspaces of V is an exhaustive filtration of V. As $S(V)$ is minimal, we have that $dV^n \subset S^{\geq 1}(V)\cdot S^{\geq 1}(V)$ for every $n \geq 0$. We also have that $dV^n \subset S(V)^{n+1}$. It follows that

$$dV^n \subset \left(S^{\geq 1}(V)\cdot S^{\geq 1}(V)\right) \cap S(V)^{n+1} \subset S\left(\bigoplus_{i=2}^{n-1} V^i\right) = S(V(n-1))$$

because $V^n \cdot V^i \subset S(V)^{\geq n+2}$ whenever $i \geq 2$. Since $S(V(m-1)) \subset S(V(m))$ for all $m \geq 1$, we conclude that $dV(n) \subset S(V(n-1))$. Thus, $\{V(m)\}_{m\geq 0}$ satsifies the Sullivan condition. □

2.3 Lecture 7

Definition 2.3.1 (Sullivan model).

1. A *Sullivan model of a DGCA A* is a Sullivan algebra $S(V)$ together with a quasi-isomorphism $S(V) \xrightarrow{q\text{-}iso.} A$.
2. A *Sullivan minimal model $M(X)$ of a manifold X* is a minimal Sullivan model of $\Omega^\bullet(X)$, the de Rham algebra of X. (This is defined for ground field $k = \mathbb{R}$.)

Example 2.3.2. Consider the n-sphere S^n where $n \geq 1$. Recall that the cohomology of $\Omega^\bullet(S^n)$ is exactly

$$H^\bullet(S^n;\mathbb{R}) = \mathbb{R}[\omega]/(\omega^2) \cong \mathbb{R}\oplus\mathbb{R}\,\omega, \qquad |\omega| \equiv n$$

We want to find a DGCA $M(S^n)$ together with a quasi-isomorphism $M(S^n) \to \Omega^\bullet(S^n)$. To begin, define the map

$$\begin{aligned} \mathbb{R}[x] &\xrightarrow{\varphi} \Omega^\bullet(S^n), \qquad |x| \equiv n,\ dx \equiv 0 \\ x &\mapsto \underbrace{\omega \in \Omega^n(S^n)}_{\text{volume form on } S^n}, \qquad d\omega = 0. \end{aligned}$$

By construction, this map induces a surjection on cohomology. We must consider two cases.

- Suppose that n is odd. Then the degree of x is odd, so that $x^2 = 0$. This means that $\mathbb{R}[x] = \mathbb{R} \oplus \mathbb{R}\, x$, so that φ induces an isomorphism on cohomology. Thus, $(\mathbb{R}[x], dx \equiv 0)$ is a Sullivan minimal model of S^n.

- Suppose that n is even. We want to kill x^2 in the cohomology of $\mathbb{R}[x]$. To this end, adjoin y to $\mathbb{R}[x]$ such that $|y| = 2n-1$ and $dy = x^2$. Clearly, x^2 is killed in cohomology. The map

$$\begin{aligned} \mathbb{R}[x,y] &\xrightarrow{\varphi} \Omega^\bullet(S^n) \\ x &\mapsto \omega \\ y &\mapsto 0. \end{aligned}$$

is a morphism of DGCA's. Since y has odd degree, all higher powers of it are killed in $\mathbb{R}[x,y]$. Also, y itself is not a cocycle and thus does not affect cohomology. Thus, φ induces an isomorphism on cohomology. This means that $\left(\mathbb{R}[x,y], dx \equiv 0, dy \equiv x^2\right)$ is a Sullivan model of S^n.

Notation.

1. We may write the DGCA $(k[x_1, \ldots, x_n], dx_1 = 0, dx_2 = \cdots, \ldots, dx_n = \cdots)$ as

$$k[x_1, \ldots, x_n \mid dx_2 = \cdots, \ldots, dx_n = \cdots].$$

For example, $M(S^n) = \begin{cases} \mathbb{R}[x] & n \text{ odd} \\ \mathbb{R}\left[x, y \mid dy = x^2\right] & n \text{ even} \end{cases}.$

2. We may write the DGCA $k[x_1, y_1, x_2, y_2, x_3, x_4, \ldots \mid dx_1 = y_1, dx_2 = y_2, dx_3 = \cdots, dx_4 = \cdots, \ldots]$ as

$$k[x_1, dx_1, x_2, dx_2, x_3, x_4, \ldots \mid dx_3 = \cdots, dx_4 = \cdots, \ldots].$$

For example, $\Omega(n) = \mathbb{Q}[t_1, \ldots, t_n, dt_1, \ldots, dt_n]$.

Note 2.3.3 (Serre). Recall that $\pi_3(S^2) \cong \mathbb{Z}$. In general, $\pi_{2n-1}(S^n) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$ when n is even. Further, $\pi_n(S^n) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}$ for all $n \geq 1$. In all other cases, however, $\pi_\bullet(S^n)$ is a torsion group.

This will be a common pattern for us. The generators of $\pi_n(X) \otimes \mathbb{Q}$ $(\mathcal{N} \geq 1)$ correspond to free generators of $M(X)$.

Example 2.3.4. We want to find $M(S^3 \vee S^3)$. The algebra $\Omega^\bullet(S^3 \vee S^3)$ consists of pairs (ω_1, ω_2) of differential forms on S^3 that agree at the base point. Choose two volume forms ω_1 and ω_2 for the two copies of S^3. Consider the map

$$\begin{aligned} \mathbb{R}[x,y] &\xrightarrow{\psi} \Omega^\bullet(S^3 \vee S^3), \qquad |x| \equiv 3,\ |y| \equiv 3 \\ x &\mapsto \omega_1 \\ y &\mapsto \omega_2. \end{aligned}$$

The element xy represents a nontrivial cocycle of degree 6. To kill this in cohomology, adjoin a degree-5 variable z to $\mathbb{R}[x,y]$ such that $dz = xy$. We still have nontrivial cocycles xz and yz of degree 8. Adjoin two degree-7 variables a and b such that $da = xz$ and $db = yz$. We can continues this process to infinity to obtain a Sullivan minimal model of $S^3 \vee S^3$.

Remark 2.3.5. In both Example 2.3.2 and Example 2.3.4, the minimality of $M(X)$ holds by construction of d. Also, $M(X)$ is simply connected unless $X = S^1$, in which case $M(S^1) = \mathbb{R}[x]$ where $|x| = 1$. In this case, let $V(0) = \mathbb{R}x$ and $V = V(0)$. Since $dV(0) = 0$, we see that $M(S^1)$ satisfies the Sullivan condition. Thanks to Lemma 2.2.6, it follows that $M(X)$ is a Sullivan algebra in all cases.

2.4 Lecture 8

Let us establish a sufficient condition for a DGCA to have a minimal Sullivan model.

Definition 2.4.1. A k-DGCA A is *homologically connected* if $A = \bigoplus_{n\geq 0} A^n$ and $H^0(A) \cong k$.

Theorem 2.4.2. *Every homologically connected DGCA has a minimal Sullivan model.*

Proof. For simplicity, assume that A is *homologically simply connected*, i.e., it is homologically connected *and* has $H^1(A) = 0$. For the general case, see either of the following sources.

- *Volume I* of FHT. This uses relative Sullivan models for the proof.
- *Volume II* of FHT. This uses structure theory of Sullivan algebras.

In our case, thanks to Lemma 2.2.6, it suffices to construct a simply connected minimal model of A, i.e.,

- a DGCA $(S(V), d)$ such that $V = \bigoplus_{n\geq 2} V^n$ and $\operatorname{im} d \subset (S(V)^+)^2$ along with
- a quasi-isomorphism $(S(V), d) \xrightarrow{q\text{-}iso.} (A, d)$.

Plan. Define V^n and $d \restriction_{S(V^{\leq n})}$ inductively so that $V^1 = 0$ and $dV^n \subset S(V^{\leq n-1})$. This will imply the minimality condition

$$dV^n \subset S(V^{\leq n-1})^{n+1} \subset S^{\geq 2}(V^{\leq n-1}) = \left(S(V^{\leq n-1})^+\right)^2$$

because $S^{\geq 2}(V^{\leq n-1}) = V^{\leq n-1} \cdot S(V^{\leq n-1})^+$. At the same time, define a homomorphism

$$f_n : \left(S(V^{\leq n}), d\right) \to (A, d), \qquad f_n \restriction_{V^{\leq n-1}} = f_{n-1} \restriction_{V^{\leq n-1}}$$

such that $H^{\leq n}(f_n)$ is an isomorphism and $H^{n+1}(f_n)$ is an injection. Finally, let $V = \bigoplus_{n\geq 2} V^n$ and $f \restriction_{V^n} = f_n \restriction_{V^n}$.

For our base case, let $V^1 = 0$ and define $d = 0$ on $S(\overbrace{V^{\leq 1}}^{0}) = k$. Define $f_1 : k \to A$ as the unit map, i.e., the map induced by the zero map $V^1 \xrightarrow{0} A$. The map $H^0(f_1)$ is an isomorphism because $H^0(A) \cong k$. Further, the map $H^1(f_1)$ is an isomorphism as the zero map from 0 to $H^1(A) = 0$. Finally, $H^2(f_1)$ is trivially an injective map.

For our inductive step, suppose that $n \geq 1$ and that we've constructed V_n and d along with a map $f_n : (S(V^{\leq n}), d) \to (A, d)$ inducing an isomorphism on $H^{\leq n}$ and an injection on H^{n+1}. Pick cocycles $a_i \in A^{n+1}$ representing a basis of $\operatorname{coker} H^{n+1}(f_n)$, so that

$$\operatorname{coker} H^{n+1}(f_n) = \bigoplus_{i\in I} k\,[a_i]\,.$$

Likewise, pick cocycles $b_j \in S(V^{\leq n})^{n+2}$ so that

$$\ker H^{n+2}(f_n) = \bigoplus_{j\in J} k\,[b_j]\,.$$

For each $j \in J$, we may pick $c_j \in A^{n+1}$ such that $f_n(b_j) = dc_j$. Now let

$$V^{n+1} = \left(\bigoplus_{i\in I} kx_i\right) \oplus \left(\bigoplus_{j\in J} ky_j\right), \qquad |x_i| = |y_j| = n+1.$$

Extend d to $S(V^{\leq n+1})$ by $dx_i \equiv 0$ and $dy_j \equiv b_j$, so that $d \restriction_{V^{n+1}}$ takes values in $S(V^{\leq n})^{n+2}$. Since b_j is a cocycle, this extension of d is a differential map.

Define $f_{n+1} : V^{\leq n+1} \to A$ by

$$\begin{aligned} f_{n+1} \restriction_{V^{\leq n}} &= f_n \restriction_{V^{\leq n}} \\ f_{n+1}(x_i) &= a_i \\ f_{n+1}(y_j) &= c_j. \end{aligned}$$

It remains to verify the following two properties.

Claim 2.4.3.

(1) $H^{\leq n+1}(f_{n+1})$ is epic.

(2) $H^{\leq n+2}(f_{n+1})$ is monic.

2.5 Lecture 9

Proof.

1. We see that $H^{\leq n+1}(f_{n+1})$ is onto by our construction of f_{n+1}. All elements of $H^{n+1}(A)$ missed by $H^{n+1}(f_n)$ are now covered by the x_i.

2. The map $H^{\leq n}(f_{n+1})$ is monic as it equals $H^{\leq n}(f_n)$. Further, $H^{n+2}(f_{n+1})$ is monic by construction. No new cocycles are in $S(V^{\leq n+1})^{n+2}$ as compared to $S(V^{\leq n})^{n+2}$. Indeed,

$$S(V^{\leq n+1})^{n+2} = S(V^{\leq n})^{n+2} \tag{$*$}$$

because $V^1 = 0$. Finally, consider the map $H^{n+1}(f_{n+1})$. This may be problematic as we have added cocycles of degree $n+1$, namely the x_i.

Let $[z] \in \ker H^{n+1}(f_{n+1})$. We must show that $[z] = 0$. The representative z belongs to $S(V^{\leq n+1})^{n+1}$ and has the form

$$w + \sum_{i\in I} \lambda_i x_i + \sum_{j\in J} \lambda_j y_j, \qquad w \in S(V^{\leq n})^{n+1}.$$

As z is a cocycle, we have that

$$\begin{aligned} dz = 0 &\implies \sum_j \lambda_j b_j = -dw \\ &\implies \sum_j \lambda_j\,[b_j] = 0 \\ &\implies \lambda_j = 0 && \text{(the } [b_j] \text{ linearly independent)} \\ &\implies dw = 0. \end{aligned}$$

Applying $H^{n+1}(f_{n+1})$ to the equation

$$[z] = [w] + \sum_{i\in I} \lambda_i [x_i]$$

in cohomology produces

$$\sum_i \lambda_i [a_i] = -H^{n+1}(f_n)([w]).$$

This belongs to $\operatorname{im} H^{n+1}(f_n)$. As the $[a_i]$ are linearly independent, it follows that $\lambda_i = 0$ for all i. This means that $z = w$. Hence $z \in S(V^{\le n})$, and $[z] \in \ker H^{n+1}(f_n)$. By our induction hypothesis, we conclude that $[z] = 0$.

□

This completes our main proof. Note that equation ($*$) relies on the fact that $H^1(A) = 0$. □

Let us preview the topic of homotopy theory in $\mathsf{dgca}_{\mathbb{Q}}^{\ge 0}$. Recall the algebra $\Omega^\bullet(n)$ from Example 2.1.1(4). We have a tensor product-preserving functor $\Omega(-) : \mathsf{Top}^{\mathrm{op}} \to \mathsf{dgca}_{\mathbb{Q}}^{\ge 0}$ such that $\Omega(\Delta^n) = \Omega^\bullet(n)$. Let $H : X \times I \to Y$ be an ordinary homotopy. We may take its image $\Omega(H) : A \to \Omega(1) \otimes B$ under $\Omega(-)$.

Consider the inclusions of endpoints

$$\Delta^0 \underset{d_1}{\overset{d_0}{\rightrightarrows}} \Delta^1$$

in Δ^1. Let $\partial_i = \Omega(d_i)$ for each $i = 0, 1$.

We have that $\Omega(0) = \mathbb{Q}$ and

$$\Omega(1) = \frac{\mathbb{Q}[t_0, t_1, dt_0, dt_1]}{(t_0 + t_1 = 1,\ dt_0 + dt_1 = 0)}.$$

Each map $\partial_i : \Omega(1) \to \mathbb{Q}$ is given by

$$\partial_i(dt_i) = 0$$
$$\partial_i(t_i) = 0.$$

Definition 2.5.1. Two maps $f, g : A \to B$ in $\mathsf{dgca}_{\mathbb{Q}}^{\ge 0}$ are *homotopic* if there exists a map $H : A \to \Omega(1) \otimes B$ of DGCA's such that

$$(\partial_1 \otimes \mathrm{id}) \circ H = f$$
$$(\partial_0 \otimes \mathrm{id}) \circ H = g.$$

Exercise 2.5.2 (Homotopy invariance). *If the maps f and g are homotopic, then they induce equal maps on cohomology: $H^\bullet(f) = H^\bullet(g)$.*

3 Model categories

3.1 Lecture 10

Definition 3.1.1. Let $\mathcal{C}$ be a category. Suppose that the square

$$\begin{array}{ccc} U & \longrightarrow & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ V & \longrightarrow & B \end{array}$$

commutes in $\mathcal{C}$. If there exists a lift $q : V \to E$ for this diagram, then we say that

- i has the *left lifting property* (*LLP*) with respect to p and that
- p has the *right lifting property* (*RLP*) with respect to i.

Example 3.1.2. Let $\mathcal{C}$ be the category R-mod of modules over R.

- We say that an object P is *projective* if the unique map $0 \to P$ has the LLP w.r.t. all surjections.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[d, two heads] \\ P \arrow[r] \arrow[ur, dashed] & N \end{tikzcd}$$

- We say that an object I is *injective* if the unique map $I \to 0$ has the RLP w.r.t. all injections.

$$\begin{tikzcd} N \arrow[r] \arrow[d, hook] & I \arrow[d] \\ M \arrow[r] \arrow[ur, dashed] & 0 \end{tikzcd}$$

Definition 3.1.3. Let $f : A \to B$ and $f' : A' \to B'$ be maps in $\mathcal{C}$. We say that f is a *retract* of f' if we can factor the id morphism

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "\mathrm{id}"'] & B \end{tikzcd}$$

through f' as follows.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & A' \arrow[r] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r] & B \end{tikzcd}$$

Definition 3.1.4. A *(closed) model category* is a category $\mathcal{M}$ equipped with three classes of morphisms closed under composition.

- the class $\mathscr{W}$ of *weak equivalences*
- the class $\mathscr{F}$ of *fibrations*
- the class $\mathscr{C}$ of *cofibrations*

A map in $\mathscr{F} \cap \mathscr{W}$ is called an *acyclic* or *trivial* fibration. Likewise, a map in $\mathscr{C} \cap \mathscr{W}$ is called an *acyclic* or *trivial* cofibration.

Moreover, $\mathcal{M}$ must satisfy the following five axioms.

(MC1) It has small limits and colimits, i.e., is complete and cocomplete.

(MC2) The class $\mathscr{W}$ has the *2-out-of-3* property. For any two morphisms f and g with fg defined, if two of f, g, and fg belong to $\mathscr{W}$, then so does the third.

(MC3) The classes $\mathscr{F}$, $\mathscr{C}$, and $\mathscr{W}$ are all closed under retracts.

(MC4) Any cofibration has the LLP w.r.t. all acyclic fibrations.

Any acyclic cofibration has the LLP w.r.t. all fibrations.

(MC5) Any morphism f in $\mathcal{M}$ can be functorially factored in two ways:

$$f = pi, \quad p \text{ acyclic fibration, } i \text{ cofibration}$$
$$f = qj, \quad q \text{ fibration, } j \text{ acyclic cofibration.}$$

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d] & \bullet \arrow[d] & \mapsto & \bullet \arrow[r, "i"] \arrow[d] & \bullet \arrow[r, "p"] \arrow[d, dashed] & \bullet \arrow[d] \\ \bullet \arrow[r, "g"'] & \bullet & & \bullet \arrow[r, "i'"'] & \bullet \arrow[r, "p'"'] & \bullet \end{tikzcd}.$$

Terminology. Let $X \in \operatorname{ob}\mathcal{M}$. We say that

- X is *fibrant* if the unique map $X \to 1$ is a fibration and that
- X is *cofibrant* if the unique map $0 \to X$ is a cofibration.

Example 3.1.5. The category Top has the structure of a model category. The class $\mathscr{W}$ consists of all weak homotopy equivalences. The class $\mathscr{F}$ consists of all Serre fibrations, maps with the RLP w.r.t. all inclusions of the form

$$D \times \{0\} \hookrightarrow D \times I, \qquad D \text{ an } n\text{-dimensional ball.}$$

(We could replace balls with CW complexes.) Thus, a map p is a fibration if and only if it has the HLP w.r.t. all balls:

$$\begin{tikzcd} D \times \{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ D \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}.$$

The class $\mathscr{C}$ must consist of all maps with the LLP w.r.t. Serre fibrations that are weak equivalences. One can prove that $\mathscr{C}$ equals the class of all retracts of relative CW complexes.

3.2 Lecture 11

Let R be a (unital) commutative ring. Consider the category $\mathsf{dg}\text{-}R\text{-}\mathsf{mod}^{\leq 0}$ of complexes of R-modules bounded by zero on the right. This has a model category structure, with

$\mathscr{W}$ = the class of quasi-isos.

$\mathscr{F}$ = the class of cochain maps f such that f_n is surjective for all $n < 0$

$\mathscr{C}$ = the class of cochain maps f such that f_n is injective and $\operatorname{coker} f_n$ is a projective module for all $n \leq 0$.

This is known as the *projective* model category structure on $\mathsf{dg}\text{-}R\text{-}\mathsf{mod}^{\leq 0}$. All complexes are fibrant, and a complex is cofibrant if and only if it is term-wise projective.

Likewise, the category $\mathsf{dg}\text{-}R\text{-}\mathsf{mod}^{\geq 0}$ has an *injective* model category structure, with

$\mathscr{W}$ = the class of quasi-isos.

$\mathscr{F}$ = the class of cochain maps f such that f_n is surjective and $\ker f_n$ is an injective module for all $n \geq 0$

$\mathscr{C}$ = the class of cochain maps f such that f_n is injective $n > 0$.

Here a complex is fibrant if and only if it is term-wise injective.

Now consider the category $\mathsf{dg}\text{-}R\text{-}\mathsf{mod}$ of possibly unbounded complexes. This has a projective model category structure:

$\mathscr{W}$ = the class of quasi-isos.

$\mathscr{F}$ = the class of cochain maps f such that f_n is surjective for all $n \in \mathbb{Z}$

$\mathscr{C}$ = the class of cochain maps that have the LLP w.r.t. all acyclic fibrations.

A useful description of $\mathscr{C}$ is much harder in this case. The following sources derive one in detail:

- Hovey
- Dwyer and Spalinski
- Goerss and Schemmerhorn.

We can equip $k\text{-}\mathsf{dgca}^{\geq 0}$ with a similar model category structure:

$\mathscr{W}$ = the class of quasi-isos.

$\mathscr{F}$ = the class of all maps f such that f_n is surjective for all $n \geq 0$

$\mathscr{C}$ = the class of all maps that have the LLP w.r.t. all acyclic fibrations.

Let's turn to a few basic properties of model categories.

Lemma 3.2.1. *Suppose that $\mathcal{M}$ is a model category. The cofibrations in $\mathcal{M}$ are precisely the maps with the LLP w.r.t. all acyclic fibrations.*

Proof. Suppose that $f : K \to L$ has the LLP w.r.t. all acyclic fibrations. We must show that f is a cofibrations. By MC5, we may factor f as

$$K \xrightarrow[\text{}]{\substack{i \\ \text{cof.}}} L' \xrightarrow{\substack{p \\ \text{acyclic fib.}}} L.$$

Apply the LLP of f to get a lift $h : L \to L'$ for the commutative sqaure

$$\begin{tikzcd} K \arrow[r, "i"] \arrow[d, "f"'] & L' \arrow[d, "p"] \\ L \arrow[r, "\mathrm{id}"'] & L \end{tikzcd}.$$

It's easy to check that

$$\begin{tikzcd} K \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & K \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & K \arrow[d, "f"] \\ L \arrow[r, "h"'] \arrow[rr, bend right, "\mathrm{id}"'] & L' \arrow[r, "p"'] & L \end{tikzcd}$$

commutes. Hence f is a retract of i. By MC3, it follows that f is a cofibration. $\square$

Remark 3.2.2. The other three versions of Lemma 3.2.1 are true and are proven similarly.

Corollary 3.2.3.

(1) Both cofibrations and acyclic cofibrations are stable under pushouts and coproducts.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r] \arrow[ur, phantom, "\ulcorner"] & D \end{tikzcd} \qquad i \text{ cofibration} \implies j \text{ cofibration}$$

(2) Both fibrations and acyclic fibrations are stable under pullbacks and pushouts.

(3) If $(\mathcal{M}, \mathscr{W}, \mathscr{F}, \mathscr{C})$ is a model category, then so is $(\mathcal{M}^{\mathrm{op}}, \mathscr{W}^{\mathrm{op}}, \mathscr{C}^{\mathrm{op}}, \mathscr{F}^{\mathrm{op}})$.

Lemma 3.2.4. *Cofibrations are stable under transfinite composition, i.e., for any sequence*

$$X_0 \xrightarrow{\textit{cofib.}} X_1 \xrightarrow{\textit{cofib.}} X_2 \xrightarrow{\textit{cofib.}} \cdots$$

of cofibrations, the map $X_0 \to \operatorname{colim}_{i \geq 0} X_i$ is a cofibration.

3.3 Lecture 12

Consider any model category $(\mathcal{M}, \mathscr{W}, \mathscr{F}, \mathscr{C})$.

Definition 3.3.1 (Homotopy category). The *homotopy category* $\mathrm{Ho}(\mathcal{M})$ of $\mathcal{M}$ is the localization $\mathcal{M}\left[\mathscr{W}^{-1}\right]$ w.r.t. the class $\mathscr{W}$ of weak equivalences.

In general, for any category $\mathcal{M}$ and any class $\mathscr{W}$ of morphisms in $\mathcal{C}$, the localization w.r.t. $\mathscr{W}$ is a category $\mathcal{M}\left[\mathscr{W}^{-1}\right]$ together with a functor $\mathcal{M} \to \mathcal{M}\left[\mathscr{W}^{-1}\right]$ that sends any $w \in \mathscr{W}$ to an isomorphism and satsifies the following universal property. For any functor $\mathcal{M} \to \mathcal{C}$ sending $\mathscr{W}$ to isomorphism in $\mathcal{C}$, there is a unique factorization of the form

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[rr, bend left] & \mathcal{M}\left[\mathscr{W}^{-1}\right] \arrow[r, dashed, "\exists!"'] & \mathcal{C} \end{tikzcd} .$$

A direct construction is possible (modulo set-theoretic subtleties) as long as $\mathscr{W}$ satisfies properties similar to those for localization of associative rings:

- $\mathscr{W}$ is a subcategory (like a multiplicative subset of a ring);
- the so-called Ore condition.

We'll use the specific properties of a model category instead.

We now begin developing homotopy theory in model categories. We must generalize the notion of a homotopy $H : X \times I \to Y$ for spaces, where $I = \Delta^1$ or $[0,1]$. Let $\mathcal{M}$ be a model category,

Notation.

- The symbol $\hookrightarrow$ denotes a cofibration.
- The symbol $\twoheadrightarrow$ denotes a fibration.

- The symbol $\xrightarrow{\sim}$ denotes a weak equivalence.

Definition 3.3.2. Let $A \in \operatorname{ob}\mathcal{M}$. A *cylinder object* for A is an object $\mathrm{Cyl}(A)$ in $\mathcal{M}$ along with a commutative triangle

$$\begin{tikzcd} A \coprod A \arrow[r, hook, "i"] \arrow[dr, "\nabla"'] & \mathrm{Cyl}(A) \arrow[d, "q"', "\wr"] \\ & A \end{tikzcd}.$$

Note that cylinders always exist in a model category with q moreover being an acyclic fibration.

Definition 3.3.3. A *left homotopy* between maps $f, g : A \to B$ in $\mathcal{M}$ via $\mathrm{Cyl}(A)$ is a map $H : \mathrm{Cyl}(A) \to B$ such that

$$\begin{tikzcd} A \coprod A \arrow[r, hook, "i"] \arrow[dr, "f+g"'] & \mathrm{Cyl}(A) \arrow[d, "H"] \\ & B \end{tikzcd}$$

commutes. In this case, write $f \overset{\ell}{\sim} g$.

Definition 3.3.4.

1. Let $A \in \operatorname{ob}\mathcal{M}$. A *path space object* for A is an object A^I in $\mathcal{M}$ along with a commutative triangle

$$\begin{tikzcd} A \times A & A^I \arrow[l, two heads, "p"'] \\ A \arrow[u, "\Delta"] \arrow[ur, "r", "\sim"'] & \end{tikzcd}.$$

2. A *right homotopy* between maps $f, g : A \to B$ in $\mathcal{M}$ via B^I is a map $H : A \to B^I$ such that

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[dr, "(f,g)"'] & B^I \arrow[d, two heads, "p"] \\ & B \times B \end{tikzcd}$$

   commutes. In this case, write $f \overset{r}{\sim} g$.

Notation. The expression $f \sim g$ means that $f \overset{\ell}{\sim} g$ and $f \overset{r}{\sim} g$.

Example 3.3.5. Let $\mathcal{M} = \mathsf{Top}$. We have that $f \overset{\ell}{\sim} g \iff f \overset{r}{\sim} g$ thanks to the natural isomorphism

$$\mathrm{Hom}(A \times I, B) \cong \mathrm{Hom}(A, B^I), \qquad B^I = \mathrm{Map}(I, B)$$

in A and B.

For a notion of homotopy classes of maps in model categories, let

$$\pi^\ell(A, B) = \mathrm{Hom}(A, B) \big/ \overset{\ell}{\sim}$$
$$\pi^r(A, B) = \mathrm{Hom}(A, B) \big/ \overset{r}{\sim}.$$

These are sets of equivalence classes for the equivalence relations generated by left and right homotopy.

Lemma 3.3.6.

(1) If $f \overset{\ell}{\sim} g : A \to B$, then $hf \overset{\ell}{\sim} hg$ for any map $h : B \to C$.

(2) If C is fibrant and $f \overset{\ell}{\sim} g : B \to C$, then $fk \overset{\ell}{\sim} gk$ for any map $k : A \to B$.

(3) If C is fibrant, then we have a well-defined composition map

$$\begin{aligned} \pi^\ell(A,B) \times \pi^\ell(B,C) &\to \pi^\ell(A,C) \\ (f,g) &\mapsto g \circ f. \end{aligned}$$

(4) If A is cofibrant, then left homotopy between maps $A \to B$ is an equivalence relation on $\mathrm{Hom}(A,B)$.

(5) If B is cofibrant and $h : X \to Y$ is an acyclic fibration or a weak equivalence between fibrant objects, then the mapping $f \overset{h_*}{\longmapsto} h \circ f$ induces a bijection

$$\pi^\ell(B,X) \xrightarrow{\cong} \pi^\ell(B,Y).$$

For a proof of this, see Chapter 1 of Hovey.

3.4 Lecture 13

Suppose that $\mathcal{M}$ is a model category. We want to study the relationship between left and right homotopy.

Lemma 3.4.1. *Let $A, B \in \mathrm{ob}\,\mathcal{M}$ and $f, g : A \to B$. Suppose that A is cofibrant and that $f \overset{\ell}{\sim} g$. For any path space object B^I, we have that $f \overset{r}{\sim} g$ via B^I.*

Proof. By assumption, we have a commutative triangle

$$\begin{tikzcd} A \coprod A \arrow[r, hook, "i"] \arrow[dr, "f+g"'] & \mathrm{Cyl}(A) \arrow[d, "H"] \\ & B \end{tikzcd} .$$

We want a commutative triangle of the form

$$\begin{tikzcd} A \arrow[r, dotted] \arrow[dr, "{(f,g)}"'] & B^I \arrow[d, "p"] \\ & B \times B \end{tikzcd} .$$

To this end, construct the commutative square

$$\begin{tikzcd} A \arrow[r, "r \circ f"] \arrow[d, hook, "i_0"'] & B^I \arrow[d, two heads, "p"] \\ \mathrm{Cyl}(A) \arrow[r, "{(f\circ q, H)}"'] & B \times B \end{tikzcd} , \qquad (*)$$

where r and q are as in the definition of path space object and cylinder object, respectively. We want to find a lift $K : \mathrm{Cyl}(A) \to B^I$ for this square.

First, note that i_0 is a cofibration as the composition of two cofibrations. Indeed, the inclusion map $A \xrightarrow{j_0} A \coprod A$ is a cofibration as a pushout

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, hook, "j_0"] \\ A \arrow[r] & A \coprod A \end{tikzcd}$$

of the cofibration $0 \to A$. Since $i_0 = i \circ j_0$, this is also a cofibration. Second, note that i_0 is also a weak equivalence. Indeed, the composite

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Cyl}(A) \arrow[r, "q"] & A \end{tikzcd}$$

is a weak equivalence, and q is a weak equivalence. Hence i_0 is one by the two-out-of-three property. It follows that i_0 is an acyclic cofibration. By the LLP, there exists a lift $K : \mathrm{Cyl}(A) \to B^I$ for $(*)$.

It's easy to check that $K \circ i_1$ is a right homotopy between f and g via B^I, i.e., the triangle

$$\begin{tikzcd} A \arrow[r, "K \circ i_1"] \arrow[dr, "{(f,g)}"'] & B^I \arrow[d, "p"] \\ & B \times B \end{tikzcd}$$

commutes.

□

Corollary 3.4.2. *Let A be cofibrant and B be fibrant.*

(1) Suppose that $f \overset{\ell}{\sim} g$ via a cylinder object $\mathrm{Cyl}(A)$. Then $f \overset{r}{\sim} g$ via any path space object B^I, and $f \overset{\ell}{\sim} g$ via any cylinder object $\mathrm{Cyl}(A)$.

(2) $f \overset{\ell}{\sim} g \iff f \overset{r}{\sim} g$.

Under these assumptions, the expression $[A, B]$ will denote $\pi^\ell(A, B)$ or $\pi^r(A, B)$ (which are equal).

Theorem 3.4.3 ("Whitehead"). *Suppose that both A and B are fibrant and cofibrant. Then a map $f : A \to B$ is a weak equivalence if and only if it is a homotopy equivalence.*

For a proof of this, see chapter 1 of Hovey. Note that Whitehead's theorem for CW complexes follows from Theorem 3.4.3 for the standard model structure on Top. Here, every object is fibrant, and the cofibrant objects are precisely retracts of CW complexes.

3.5 Lecture 14

Let $\mathcal{M}$ be a model category. Recall the economic description of the homotopy category of $\mathcal{M}$:

$$\mathrm{Ho}(\mathcal{M}) = \mathcal{M}\left[\mathscr{W}^{-1}\right], \qquad \mathscr{W} \text{ class of w.e.'s.}$$

We now turn to an alternative description.

Let $X \in \mathrm{ob}\,\mathcal{M}$. By MC5, we can factor the map $X \to 1$ as

$$X \overset{i_X}{\underset{\sim}{\hookrightarrow}} RX \twoheadrightarrow 1$$

and the map $0 \to X$ as

$$0 \hookrightarrow QX \overset{p_X}{\underset{\sim}{\twoheadrightarrow}} X\ .$$

We call RX and QX the *fibrant* and *cofibrant replacement* of X, respectively. Note that both R and Q are functors $\mathcal{M} \to \mathcal{M}$. Also, they induce natural transformations

$$\begin{aligned} Q &\Rightarrow \mathrm{id}_{\mathcal{M}}, \quad & p_X &: QX \xrightarrow{\sim} X \\ \mathrm{id}_{\mathcal{M}} &\Rightarrow R, \quad & i_X &: X \xrightarrow{\sim} RX. \end{aligned}$$

Both QRX and RQX are fibrant and cofibrant. Indeed, the map $QRX \xrightarrow{p_{RX}} RX \to 1$ is a fibration as the composition of fibrations. Likewise, the map $0 \to QX \xrightarrow{i_{QX}} RQX$ is a cofibration as the composition of cofibrations.

Theorem 3.5.1. *The functor $RQ : \mathcal{M} \to \mathcal{M}$ induces an isomorphism*

$$\mathrm{Ho}(\mathcal{M}) \xrightarrow{\cong} (\mathrm{ob}\,\mathcal{M}, \mathrm{Hom}(X,Y) \equiv [RQX, RQY])$$

of categories.

By Theorem 3.4.3, we also have an equivalence

$$\begin{tikzcd} (\text{fibrant cofibrant objects of } \mathcal{M}, [X,Y]) \arrow[r, bend left, "i"] & (\mathrm{ob}\,\mathcal{M}, [RQX, RQY]) \arrow[l, bend left, "RQ"] \end{tikzcd}$$

of categories where i denotes the inclusion functor.

Corollary 3.5.2.

(1) The category $\mathrm{Ho}(\mathsf{Top})$ is equivalent to the category of retracts of CW complexes with homotopy classes of maps between CW complexes. (The latter is equivalent to the category of CW complexes with homotopy classes of maps.)

(2) The derived category of R-modules

$$D^{\le 0}(R\text{-}\mathsf{mod}) := \mathrm{Ho}(\underbrace{\mathsf{dg}\text{-}R\text{-}\mathsf{mod}^{\le 0}}_{\substack{\textit{proj. model}\\ \textit{structure}}}) = \mathsf{dg}\text{-}R\text{-}\mathsf{mod}^{\le 0}\left[q\text{-}iso.^{-1}\right]$$

is equivalent to the category of projective complexes with cochain homotopy classes of cochain maps.

Proof sketch of Theorem 3.5.1. Let $\mathcal{M}_{RQ} = (\mathrm{ob}\,\mathcal{M}, [RQX, RQY])$. We want to show that $\mathcal{M}_{RQ}$ has the universal property of $\mathcal{M}\left[\mathscr{W}^{-1}\right]$. First, let's show that

$$\left(\mathrm{id}_{\mathrm{ob}\,\mathcal{M}}, RQ_{\mathrm{Mor}(\mathcal{M})}\right) : \mathcal{M} \longrightarrow \mathcal{M}_{RQ}$$

takes weak equivalences to isomorphisms. Let $f : X \to Y$ be a w.e. in $\mathcal{M}$. We have a commutative diagram

$$\begin{tikzcd} RQX \arrow[r, "RQf"] & RQY \\ QX \arrow[u, "i_{QX}", "\wr"'] \arrow[r, "Qf"] \arrow[d, "p_X"', "\wr"] & QY \arrow[u, "i_{QY}"', "\wr"] \arrow[d, "p_Y", "\wr"'] \\ X \arrow[r, "\sim", "f"'] & Y \end{tikzcd} . \qquad (*)$$

By applying the 2-out-of-3 property twice, we see that RQf is a weak equivalence. Hence it is an isomorphism in $\mathcal{M}_{RW}$ by Theorem 3.4.3.

Next, let's show that $\mathcal{M}_{RQ}$ has the universal property. Let $\mathcal{C}$ be any category and suppose that $\mathcal{M} \xrightarrow{G} \mathcal{C}$ is any functor taking $\mathscr{W}$ to isomorphims. We must find a unique functor $\widetilde{G} : \mathcal{M}_{RQ} \to \mathcal{C}$ such that

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "RQ"] \arrow[dr, "G"'] & \mathcal{M}_{RQ} \arrow[d, "\widetilde{G}"] \\
& \mathcal{C}
\end{tikzcd}$$

commutes. Define $\widetilde{G}$ on objects by $X \mapsto G(X)$. Consider any homotopy class of the form $[f] : RQX \to RQY$. Define the map $\widetilde{G}([f])$ as follows.

$$\begin{tikzcd}
X & Y & & X \arrow[r, dashed, "\widetilde{G}([f])"] & Y \\
QX \arrow[u, "\wr"] \arrow[d, "\wr"'] & QY \arrow[u, "\wr"'] \arrow[d, "\wr"] & \overset{G}{\longmapsto} & GQX \arrow[u, "\cong"] \arrow[d, "\cong"'] & GQY \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
RQX \arrow[r, "f"'] & RQY & & GRQX \arrow[r, "G(f)"'] & GRQY
\end{tikzcd} .$$

Now, by applying G to a diagram like $(*)$, we see that $G(h) = \widetilde{G}(RQh)$ for any map $h : X \to Y$ in $\mathcal{M}$.

It remains to show that $\widetilde{G}([f])$ (equivalently, $G(f)$) is independent of the choice of representative f.

3.6 Lecture 15

Claim 3.6.1. *Let $f, g : A \to B$ be maps in $\mathcal{M}$. If $f \overset{\ell}{\sim} g$ (or $f \overset{r}{\sim} g$) in $\mathcal{M}$, then $G(f) = G(g)$.*

Proof. Suppose that $f \overset{\ell}{\sim} g$. This gives us a homotopy

$$\begin{tikzcd}
A \coprod A \arrow[r] \arrow[dr, "f+g"'] & \mathrm{Cyl}(A) \arrow[d, "H"] \\
& B
\end{tikzcd} .$$

We also have commutative diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, "f"] \arrow[dr, "i_0"'] & A \coprod A \arrow[r, "f+g"] \arrow[d] & B \\
& \mathrm{Cyl}(A) \arrow[ur, "H"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, "g"] \arrow[dr, "i_1"'] & A \coprod A \arrow[r, "f+g"] \arrow[d] & B \\
& \mathrm{Cyl}(A) \arrow[ur, "H"'] &
\end{tikzcd} .$$

Note that $q \circ i_0 = q \circ i_1 = \mathrm{id}_A$, where q is a weak equivalence. Hence

$$G(w)G(i_0) = G(w)G(i_1) = \mathrm{id}_{G(A)},$$

and $G(w)$ has an inverse. This implies that $G(i_0) = G(i_1)$, so that

$$G(f) = G(H)G(i_0) = G(H)G(i_1) = G(g).$$

□

This completes our main proof. □

In summary, the homotopy category $\mathrm{Ho}(\mathcal{M})$ is characterized as either $\mathcal{M}\left[\mathscr{W}^{-1}\right]$ or $(\mathrm{ob}\,\mathcal{M}, [RQ-, RQ-])$. We'll mostly use the latter as it's easier to work with.

We now want to develop homological algebra in model categories. Suppose that $F : \mathcal{M} \to \mathcal{N}$ is a functor between model categories. This induces a functor $\mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$ whenever F preserves weak equivalences. Usually, however, this is too strong a condition. To find weaker conditions, we should look at Quillen adjunctions.

Definition 3.6.2. An adjoint pair $(F : \mathcal{M} \to \mathcal{N}, G : \mathcal{N} \to \mathcal{M})$ of functors is a *Quillen adjunction* if F preserves cofibrations and G preserves fibrations. In this case, we call F a *left Quillen functor* and G a *right Quillen functor*.

Note 3.6.3. Consider an adjunction $F \dashv G$. TFAE.

1. The pair (F, G) is a Quillen adjunction.
2. F preserves both cofibrations and acyclic cofibrations.
3. G preserves both fibrations and acyclic fibrations.

Example 3.6.4. Recall the free-forgetful adjunction

$$\begin{tikzcd} \mathsf{dg\text{-}Vect}^{\leq 0}_{\mathbb{Q}} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathsf{dgca}^{\geq 0}_{\mathbb{Q}} \arrow[l, bend right, "U"'] \end{tikzcd}, \qquad (V, d) \mapsto (S(V), d).$$

For both of these model categories, $\mathscr{W}$ consists of all quasi-isomorphisms and $\mathscr{F}$ consists of all degree-wise surjective maps. Thus, (F, U) is a Quillen adjunction because U preserves both fibrations and acyclic fibrations. (In fact, it preserves weak equivalences.)

Lemma 3.6.5.

1. *Any left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ induces a functor $LF : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{N})$, called the left derived functor of F.*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "Q"] \arrow[dr] & \mathcal{M} \arrow[r, "F"] & \mathcal{N} \arrow[r] & \mathrm{Ho}(\mathcal{N}) \\ & \mathrm{Ho}(\mathcal{M}) \arrow[urr, dashed, "LF"'] & & \end{tikzcd}$$

2. *Dually, any right Quillen functor $G : \mathcal{N} \to \mathcal{M}$ induces a functor $RG : \mathrm{Ho}(\mathcal{N}) \to \mathrm{Ho}(\mathcal{M})$, called the right derived functor of G.*

$$\begin{tikzcd} \mathcal{N} \arrow[r, "R"] \arrow[dr] & \mathcal{N} \arrow[r, "G"] & \mathcal{M} \arrow[r] & \mathrm{Ho}(\mathcal{M}) \\ & \mathrm{Ho}(\mathcal{N}) \arrow[urr, dashed, "RG"'] & & \end{tikzcd}$$

3.7 Lecture 16

By the universal property of localization, it suffices to show that the composite

$$\mathcal{M} \xrightarrow{Q} \mathcal{M} \xrightarrow{F} \mathcal{N} \to \mathrm{Ho}(\mathcal{N})$$

sends weak equivalences to isomorphisms. Recall that F preserves acyclic cofibrations. Hence the composite $\mathcal{M} \xrightarrow{F} \mathcal{N} \to \mathrm{Ho}(\mathcal{N})$ sends acyclic cofibrations to isomorphisms.

Lemma 3.7.1 (Ken Brown). *Consider any functor $F : \mathcal{M} \to \mathcal{C}$. Suppose that F sends acyclic cofibrations between cofibrant objects to isomorphisms. Then F sends every weak equivalence between cofibrant objects to an isomorphism.*

Proof. Suppose that $f : A \to B$ is a weak equivalence with both A and B cofibrant. We must show that $F(f)$ is an isomorphism. Form a factorization

$$A \coprod B \overset{j}{\hookrightarrow} C \underset{\sim}{\overset{p}{\twoheadrightarrow}} B$$

of $f + \mathrm{id}_B$. We have that $f = p \circ j \circ i_A$ and $\mathrm{id}_B = p \circ j \circ i_B$, so that

$$F(f) = F(p) \circ F(j \circ i_A)$$
$$F(\mathrm{id}_B) = F(p) \circ F(j \circ i_B).$$

With a "bootstrap" method, we can show that each of $F(j \circ i_B)$, $F(j \circ i_A)$, and $F(p)$ is an isomorphism.

We claim that $j \circ i_B : B \to C$ is an acyclic cofibration between cofibrant objects. We've assumed that B is cofibrant. Note that i_B is a cofibration as the pushout of $0 \hookrightarrow A$, where A is cofibrant by assumption. As j is also a cofibration, so is $j \circ i_B$ as the composition of two cofibrations. Thus, the composite $0 \hookrightarrow B \xrightarrow{j \circ i_B} C$ is also a cofibration, i.e., C is cofibrant. Further, we have that

$$\begin{tikzcd} B \arrow[r, hook, "j\circ i_B"] \arrow[rr, bend left, "\mathrm{id}_B"] & C \arrow[r, "p", "\sim"'] & B \end{tikzcd}$$

commutes, so that $j \circ i_B$ is a w.e. by the 2-out-of-3 property. This implies that $F(j \circ i_B)$ is an isomorphism.

Similarly, $j \circ i_A$ is a cofibration between cofibrant objects. Also, $f = p \circ (j \circ i_A)$, where f is a weak equivalence. Hence $j \circ i_A$ is acyclic. It follows that $F(j \circ i_A)$ is also an isomorphism.

Finally, since $\mathrm{id}_{F(B)} = F(p) \circ F(j \circ i_B)$, we see that $F(p)$ is an isomorphism. This means that $F(f)$ is an isomorphism as the composition $F(p) \circ F(j \circ i_A)$ of isomorphisms. □

It's clear that Lemma 3.6.5 follows from Lemma 3.7.1 because Q preserves weak equivalences.

Note 3.7.2. If a functor $F : \mathcal{M} \to \mathcal{C}$ sends weak equivalences between cofibrant objects to isomorphisms, then we may define the *left derived functor* $LF : \mathrm{Ho}(\mathcal{M}) \to \mathcal{C}$ of F via $F \circ Q$.

Example 3.7.3. Let $\mathcal{M} = \mathsf{dg}\text{-}R\text{-}\mathsf{mod}^{\leq 0}$ with the projective model structure. Let N be an R-module. We have a natural bijection

$$\mathrm{Hom}_R(M^\bullet \otimes_R N, K^\bullet) \cong \mathrm{Hom}_R(M^\bullet, \underline{\mathrm{Hom}}_R(N, K^\bullet))$$

in $M^\bullet, K^\bullet \in \operatorname{ob}\mathcal{M}$. Here, $\underline{\operatorname{Hom}}_R(N, K^\bullet)$ consists of all homogeneous chain maps of any degree. The right adjoint $\operatorname{Hom}(N, -)$, however, fails to preserve fibrations for it's only left exact. This means that the adjunction $- \otimes N \dashv \operatorname{Hom}(N, -)$ is not a Quillen adjunction.

If N is projective, then it is a Quillen adjunction. In this case, the right derived functor carries no new information, in the sense that

$$R\underline{\operatorname{Hom}}(N, K^\bullet) = \underline{\operatorname{Hom}}(N, RK^\bullet) = \underline{\operatorname{Hom}}(N, K^\bullet).$$

Suppose that N is arbitrary. We can still define $L(- \otimes N)$ by

$$M^\bullet \mapsto M^\bullet \overset{\mathbb{L}}{\otimes}_R N \coloneqq QM^\bullet \otimes_R N, where$$

$QM^\bullet$ is a projective resolution. In particular, by viewing the R-module M as the complex $0 \to M \to 0$, we can define

$$\operatorname{Tor}_{-n}(M, N) = H^n(QM \otimes N).$$

Likewise, we can define Ext in terms of either $\underline{\operatorname{Hom}}(QM, N)$ or $\underline{\operatorname{Hom}}(N, RM)$ with the dual model structure.

3.8 Lecture 17

Our goal is to prove the main theorem on Quillen adjunctions.

Theorem 3.8.1. *For any Quillen adjunction*

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend right, "F"'] \arrow[r, phantom, "\top"] & \mathcal{N} \arrow[l, bend right, "G"'] \end{tikzcd},$$

the pair $(\mathbb{L}F, \mathbb{R}G)$ is an adjoint pair.

Proof. We must exhibit a natural bijection

$$\operatorname{Hom}_{\operatorname{Ho}(\mathcal{N})}(\mathbb{L}FA, X) \cong \operatorname{Hom}_{\operatorname{Ho}(\mathcal{M})}(A, \mathbb{R}GX)$$

in A and X. First, note that the natural transformation $\left\{p_A : QA \xrightarrow{\sim} A\right\}_{A \in \operatorname{ob}\mathcal{M}}$ induces a bijection

$$\operatorname{Hom}_{\operatorname{Ho}(\mathcal{M})}(A, \mathbb{R}GX) \xrightarrow{p_A^*} \operatorname{Hom}_{\operatorname{Ho}(\mathcal{M})}(QA, GRX)$$

because $\operatorname{Ho}(\mathcal{M}) = \mathcal{M}\left[\mathscr{W}^{-1}\right]$. Likewise, the natural transformation $\left\{i_X : X \xrightarrow{\sim} RX\right\}_{X \in \operatorname{ob}\mathcal{N}}$ induces a bijection

$$\operatorname{Hom}_{\operatorname{Ho}(\mathcal{N})}(\mathbb{L}FA, X) \xrightarrow{(i_X)_*} \operatorname{Hom}_{\operatorname{Ho}(\mathcal{N})}(FQA, RX).$$

Next, note that

$$\operatorname{Hom}_{\operatorname{Ho}(\mathcal{M})}(QA, GRX) = [RQQA, RQGRX] = [QA, GRX].$$

Indeed, the map $\pi^\ell(QA, QGRX) \xrightarrow{p_*} \pi^\ell(QA, GRX)$ is a bijection by Lemma 3.3.6(5), and the map $\pi^r(QA, QGRX) \xrightarrow{p^*} \pi^r(QQA, QGRX)$ is a bijection by the dual of Lemma 3.3.6(5). These bijections arise from the commutative square

$$\begin{tikzcd} QQA \arrow[r, dashed] \arrow[d, two heads, "\wr"] & QGRX \arrow[d, two heads, "\wr"] \\ QA \arrow[r, dashed] \arrow[ur] & GRX \end{tikzcd}.$$

The two $R's$ are attached in a similar way. Moreover, we have a similar identification

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{N})}(FQA, RX) = [FQA, RX].$$

It suffices to show that $F \dashv G$ induces a natural bijection $[FQA, RX] \cong [QA, GRX]$.

Lemma 3.8.2. *If B is cofibrant and Y fibrant, then the bijection $\mathrm{Hom}_{\mathcal{M}}(B, GY) \xrightarrow{(-)^{\#}} \mathrm{Hom}_{\mathcal{N}}(FB, Y)$ induces a bijection $[B, GY] \xrightarrow{\cong} [FB, Y]$.*

Proof. It suffices to show that

(a) $f \overset{\ell}{\sim} g \implies f^{\#} \overset{\ell}{\sim} g^{\#}$ and

(b) $f^{\#} \overset{r}{\sim} g^{\#} \implies f \overset{\ell}{\sim} g$.

For (a), suppose that we have a homotopy of the form

$$\begin{tikzcd} B \coprod B \arrow[r, hook] \arrow[dr, "f+g"'] & \mathrm{Cyl}(B) \arrow[d, "H"] \\ & GY \end{tikzcd}.$$

This gives rise to a commutative triangle

$$\begin{tikzcd} F(B \coprod B) \arrow[r, hook] \arrow[dr, "(f+g)^{\#}"'] & F\mathrm{Cyl}(B) \arrow[d, "H^{\#}"] \\ & Y \end{tikzcd}.$$

We claim that this is a homotopy from $f^{\#}$ to $g^{\#}$. Since F preserves coproducts as a left adjoint, it suffices to show that $F\mathrm{Cyl}(B)$ is a cylinder object for FB. Note that $\mathrm{Cyl}(B)$ is cofibrant and that F preserves weak equivalences between cofibrant objects by our proof of Lemma 3.7.1. Thus, we have that

$$\begin{tikzcd} B \coprod B \arrow[r, hook] \arrow[dr, "\nabla"'] & \mathrm{Cyl}(B) \arrow[d, "\wr"] \\ & B \end{tikzcd} \overset{F}{\longmapsto} \begin{tikzcd} FB \coprod FB \arrow[r, hook] \arrow[dr, "\nabla"'] & F\mathrm{Cyl}(B) \arrow[d, "\wr"] \\ & FB \end{tikzcd}.$$

This means that $F\mathrm{Cyl}(B)$ is a cylinder object for FB. ☐

Taking QA for B and RX for Y completes our main proof. ☐

3.9 Lecture 18

Definition 3.9.1. A Quillen adjunction $F \dashv G$ is a *Quillen equivalence* if the pair $(\mathbb{L}F, \mathbb{R}F)$ is an equivalence $\mathrm{Ho}(\mathcal{M}) \simeq \mathrm{Ho}(\mathcal{N})$ of categories.

Our big goal is to establish a Quillen equivalence like $\mathsf{Top}^{\mathbb{Q}} \xrightarrow{\sim} \mathsf{dgca}^{\geq 0}_{\mathbb{Q}}$.

Let's look at homotopy (co)limits in a model category $\mathcal{M}$. We have all (co)limits in $\mathcal{M}$ by MC1. How do they descend to $\mathrm{Ho}(\mathcal{M})$? In Top, pushouts fails to respect weak equivalences. For example, the disk D^2 is contractible, but the pushouts

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} S^1 \arrow[r] \arrow[d] & D^2 \arrow[d] \\ * \arrow[r] & \frac{D^2 \amalg *}{S^1 \sim *} \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

are not homotopy equivalent, as the latter is equivalent to S^2. In general, we can fix this problem by "deriving" (co)limits just as we derived Quillen adjunctions by $(F, G) \mapsto (\mathbb{L}F, \mathbb{R}G)$. To this end, recall the following facts of category theory.

Proposition 3.9.2. *Let $\mathcal{C}$ be a (co)complete category and $\mathcal{J}$ be a small category. Consider the category* $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ *of $\mathcal{J}$-shaped diagrams in $\mathcal{C}$. Let* $\mathrm{const} : \mathcal{C} \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$ *denote the constant-diagram functor. We have adjunctions*

$$\mathrm{colim}_{\mathcal{J}} \dashv \mathrm{const}$$
$$\mathrm{const} \dashv \lim_{\mathcal{J}}.$$

The natural bijections

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}_{\mathcal{J}} F, C) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(F, \mathrm{const}_C)$$
$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(\mathrm{const}_C, F) \cong \mathrm{Hom}_{\mathcal{C}}(C, \lim_{\mathcal{J}} F)$$

are precisely the universal properties of $\mathrm{colim}_{\mathcal{J}} F$ and $\lim_{\mathcal{J}} F$, respectively.

We want to take $\mathbb{L}\,\mathrm{colim}_{\mathcal{J}}$ and $\mathbb{R}\lim_{\mathcal{J}}$ for the homotopy colimit and limit, respectively, provided that

- $\mathcal{C}$ has a model structure,
- $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ has a suitable model structure, and
- both $(\mathrm{colim}_{\mathcal{J}}, \mathrm{const})$ and $(\mathrm{const}, \lim)$ are Quillen adjunctions under these two model structures.

In this case, let $\mathrm{hocolim}_{\mathcal{J}} = \mathbb{L}\,\mathrm{colim}_{\mathcal{J}}$ and $\mathrm{holim}_{\mathcal{J}} = \mathbb{R}\lim_{\mathcal{J}}$. For example, $\mathrm{Fun}(\mathcal{J}, \mathcal{C})$ has a suitable model structure when $\mathcal{C}$ is a *combinatorial* model category.

Definition 3.9.3 (Homotopy pushout). Consider any span $C \xleftarrow{g} A \xrightarrow{f} B$ in a model category. Factor the composite $QA \to A \xrightarrow{f} B$ as $QA \hookrightarrow \widetilde{B} \xrightarrow{\sim} B$. Likewise, factor $QA \to A \xrightarrow{g} C$ as $QA \hookrightarrow \widetilde{C} \xrightarrow{\sim} C$. Now take the ordinary pushout

$$\begin{tikzcd} QA \arrow[r, hook] \arrow[d, hook] & \widetilde{B} \arrow[d] \\ \widetilde{C} \arrow[r] & H \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

as the homotopy pushout of our span.

Example 3.9.4. Let $\mathcal{M} = \mathsf{Top}$. Let $f : A \to B$ be a map of spaces. We can factor f as

$$A \xhookrightarrow{i_1} Mf \xrightarrow{\sim} B$$

where Mf denotes the mapping cylinder of f. Here, the inclusion i_1 is a cofibration because A is neighborhood deformation retract (NDR) of Mf. The map $Mf \xrightarrow{\sim} B$ need not be a fibration. But applying this factorization to the map $S^1 \to *$ produces

$$S^1 \hookrightarrow \underbrace{CS^1}_{D^2} \xrightarrow{\sim} * ..$$

Thus, the homotopy pushout of $* \leftarrow S^1 \to *$ is precisely

$$\begin{tikzcd} S^1 \arrow[r, hook] \arrow[d, hook] & CS^1 \arrow[d] \\ CS^1 \arrow[r] & S^2 \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} .$$

This is the same as the ordinary pushout of $* \leftarrow S^1 \hookrightarrow D^2$.

It's enough to replace just one $*$ with CS^1 because Top is *left proper*. In particular, the homotopy pushout of $B \xleftarrow{f} A \to *$ is precisely the pushout of $B \xleftarrow{f} A \hookrightarrow CA$, i.e., the mapping cone of f. This homotopy pushout is known as the *homotopy cofiber* of f.

3.10 Lecture 19

Let us return to our main example of a model category: $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$. Our big goals are

1. to do homotopy theory in this category and compare its homotopy category to that of $\mathsf{Top}^{\mathbb{Q}}$ and
2. to prove that minimal Sullivan models are unique.

Theorem 3.10.1. *The following subcategories of $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ define the structure of a model category:*

$$\begin{aligned} \vec{q} &= \{f : A \to B \mid f \text{ q-iso.}\} \\ \mathscr{F} &= \{f : A \to B \mid f^n : A^n \to B^n \text{ onto for all } n \geq 0\} \\ \mathscr{C} &= \{f : A \to B \mid f \text{ has LLP w.r.t. acyclic fibrations}\} . \end{aligned}$$

Corollary 3.10.2. *The category $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}/\mathbb{Q}$ of augmented $\mathbb{Q}$-DGCA's inherits a model category structure from $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$.*

Proof. For any model category $\mathcal{M}$ and any $A \in \operatorname{ob}\mathcal{M}$, the slice category $\mathcal{M}/A$ over A has a model category structure where a morphism

$$\begin{tikzcd} N \arrow[rr, "f"] \arrow[dr] & & N' \arrow[dl] \\ & A & \end{tikzcd}$$

is a fibration, cofibration, or weak equivalence if and only if f is such a map in $\mathcal{M}$. ∎

Before proving Theorem 3.10.1, let's better understand cofibrations in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$.

Lemma 3.10.3. *The following morphisms are cofibrations in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$.*

(i) The unit map $u : \mathbb{Q} \to \mathbb{Q}[x]$, defined by $1 \mapsto 1$, where $|x| \geq 0$.

(ii) The map $\theta_n : \mathbb{Q}[x] \to \mathbb{Q}[y, dy]$ defined by $x \mapsto dy$ where $n \equiv |x| = |y| + 1 \geq 1$.

(iii) The map $\theta_0 : \mathbb{Q}[x] \to \mathbb{Q}$ defined by $x \mapsto 0$ where $|x| \equiv 0$.

Proof.

(i) This holds if and only if $\mathbb{Q}[x]$ is cofibrant. We want to find a lift $\mathbb{Q}[x] \to A$ for the commutative square

$$\begin{tikzcd} \mathbb{Q} \arrow[r] \arrow[d, "u"'] & A \arrow[d, two heads, "\wr"', "f"] \\ \mathbb{Q}[x] \arrow[r, "x\mapsto b"'] \arrow[ur, dashed] & B \end{tikzcd}.$$

Note that $db = 0$ because $dx = 0$. Let $f^* = H^*(f)$. Since this map is onto, we can find a class $[a] \in H^\bullet(A)$ such that $[b] = f^*(a)$. Then $f(a) = b + db'$ for some $b' \in B$ where $da = 0$. As f is onto, there is some $a' \in A$ such that $f(a') = b'$. We have that $f(a - da') = b + db' - db' = b$. Thus, we may send x to $a - da'$ to make the lower triangle commute. The upper one commutes automatically for $\mathbb{Q}$ is the initial object of $\mathsf{dgca}^{\geq 0}_{\mathbb{Q}}$.

(ii) We want to find a lift $\mathbb{Q}[y, dy] \to A$ for the commutative square

$$\begin{tikzcd} x \arrow[r, mapsto] & a \\ \mathbb{Q}[x] \arrow[r] \arrow[d, "\theta"'] & A \arrow[d, two heads, "\wr"', "f"] \\ \mathbb{Q}[y, dy] \arrow[r] & B \\ dy \arrow[r, mapsto] & b \end{tikzcd}.$$

Note that y is mapped to an element $b' \in B$ such that $db' = b$. Hence $db = 0$. Also, $da = 0$ because $dx = 0$. Now we may use an argument similar to part (i) to define our lift.

(iii) We want to find a lift for the commutative square

$$\begin{tikzcd} \mathbb{Q}[x] \arrow[r, "x\mapsto a"] \arrow[d, "\theta_0"'] & A \arrow[d, two heads, "\wr"', "f"] \\ \mathbb{Q} \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}.$$

The only possibility for this lift is the unique map $\mathbb{Q} \to A$. Again, we have that $da = 0$. We also have that $f(a) = 0$, so that $f^*([a]) = 0$ in cohomology. As f^* is an isomorphism, it follows that $[a] = 0$. Hence $a = da'$ for some $a' \in A^{-1}$. Since $A^{-1} = 0$, we see that $a = 0$. This means that the upper triangle commutes, thereby completing our proof.

□

Note 3.10.4. The composite $\theta_n \circ u : \mathbb{Q} \to \mathbb{Q}[y, dy]$ is a cofibration. It's easy to check that it's also a weak equivalence. Therefore,

$$H^*(\mathbb{Q}[y, dy]) = \mathbb{Q}.$$

This is not true, however, when the base field has characteristic $p > 0$. In this case, y^p is a nontrivial cocycle:

$$d(y^p) = py^{p-1}dy = 0.$$

Proof of Theorem 3.10.1.

MC1: Recall that a category has (small) limits if and only if it has products and equalizers. Dually, a category has colimits if and only if it has coproducts and coequalizers. In $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$, coproducts have the form $\bigotimes_{i\in I} A_i$ (where I is a set), and products have the form $\prod_{i\in I} A_i$ where

$$(\pi_{i\in I} A_i)^p \equiv \pi_{i\in I}\left(A_i^p\right)$$

and all operations are defined componentwise. Moreover, coequalizers are quotients of the form $B \diagup \mathrm{im}(f-g)$ where f and g are parallel maps, and equalizers are subalgebras of the form $\ker(f-g)$.

3.11 Lecture 20

MC2: We must show that $\mathscr{W}$ satisfies two-out-of-three. Note that the class of isomorphisms satisfies two-out-of-three. Simply apply this fact to the equation $H(fg) = H(f)H(g)$ for any two composable maps $f, g \in \mathrm{Mor}(\mathsf{dgca}_{\mathbb{Q}}^{\geq 0})$.

MC3: Suppose that $f : A \to B$ is a retract of $f : A' \to B'$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_A"] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & B' \arrow[r] & B
\end{tikzcd}.$$

If f' is a (co)fibration, then so is f thanks to a diagram chase.

Suppose that f' is a quasi-isomorphism. Consider the commutative diagram

$$\begin{tikzcd}
HA \arrow[r] \arrow[d, "Hf"'] \arrow[rr, bend left, "\mathrm{id}_{HA}"] & HA' \arrow[r] \arrow[d, "Hf'"] & HA \arrow[d, "Hf"] \\
HB \arrow[r] \arrow[rr, bend right, "\mathrm{id}_{HB}"'] & HB' \arrow[r] & HB
\end{tikzcd}.$$

It's easy to see that $H(f)$ must be both injective and surjective, hence an isomorphism. Thus, f is a quasi-isomorphism as well.

MC4(i): By definition of $\mathscr{C}$, every cofibration has the LLP w.r.t. acyclic fibrations.

MC5: We mimic the "small object" argument from model category theory. Let $f : A \to B$ be a map.

First, we want to factor f as

$$A \overset{\sim}{\underset{\beta}{\hookrightarrow}} A_f \underset{\psi}{\twoheadrightarrow} B\ .$$

Recall our basic cofibrations

$$\mathbb{Q} \xrightarrow{u} \mathbb{Q}[x]$$

$$\underbrace{\mathbb{Q}[x]}_{S(n)} \xrightarrow{\theta_n} \underbrace{\mathbb{Q}[y, dy]}_{D(n)}, \quad n \geq 1$$

$$\underbrace{\mathbb{Q}[x]}_{S(0)} \xrightarrow{\theta_0} \underbrace{\mathbb{Q}}_{D(0)}$$

from Lemma 3.10.3.

Remark 3.11.1. Think of $S(n)$ and $D(n)$ as the n-sphere and n-disk, respectively. Indeed, the de Rham functor sends the quotient map $D^n \to \underbrace{D^n \diagup \partial D^n}_{S^n}$ to θ_n.

Let

$$A_f = A \otimes \underbrace{\left(\bigotimes_{b \in B} D(|b| + 1) \right)}_{(\otimes_{b\in B} \mathbb{Q}[y_b, dy_b])}, \qquad |y_b| = |b|\,.$$

Note that the inclusion $A \to A_f$ is a cofibration as a coproduct of cofibrations:

$$A \otimes \left(\bigotimes_{b \in B} \left(\mathbb{Q} \overset{\sim}{\hookrightarrow} D(|b| + 1) \right) \right).$$

Moreover, it induces an isomorphism $H(A) \xrightarrow{\cong} H(A_f)$ on cohomology thanks to the Künneth formula and the weak equivalence $\mathbb{Q} \xrightarrow{\sim} D(|b| + 1)$. Hence it is an acyclic cofibration, and we may use it for β. Now, define $\psi : A_f \to B$ by

$$\begin{aligned} a \otimes 1 &\mapsto f(a) \\ 1 \otimes y_b &\mapsto b \\ a \otimes y_b &\mapsto f(a) \cdot b. \end{aligned}$$

This is surjective in each degree as $1 \otimes y_b \mapsto b$ for all $b \in \coprod_{n \geq 0} B^n$. Thus, ψ is a fibration.

Next, we want to factor f as

$$A \hookrightarrow B_f \overset{\sim}{\twoheadrightarrow} B\ .$$

We shall construct B_f as the sequential colimit $\operatorname{colim}_{n \geq 1} B_f(n)$ of a diagram of the form

$$\begin{tikzcd} A \arrow[r, hook, "\beta_1"] \arrow[d, "f"'] & B_f(1) \arrow[r, hook, "\beta_2"] \arrow[dl, bend left, "\psi_1"] & B_f(2) \arrow[r, hook, "\beta_3"] \arrow[dll, bend left, "\psi_2"] & \cdots \arrow[r] & B_f \arrow[dllll, dotted, bend left, "\psi"] \\ B \end{tikzcd}.$$

Here the map $A \to B_f$ is a cofibration as the transfinite composition of cofibrations. Let

$$B_f(1) = A \otimes \left(\bigotimes_{b \in B} D(|b| + 1) \right) \otimes \left(\bigotimes_{z \in Z(B)} S(|z|) \right).$$

Define β_1 and ψ_1 as for A_f. Then β_1 is a cofibration, and ψ_1 is a fibration. Also, $H(\psi_1)$ is surjective. To define β_2, form the pushout square

$$\begin{tikzcd}
\bigotimes_{(w,b)\in K} S(|w|) \arrow[r, "x_w \mapsto w"] \arrow[d, hook, "\bigotimes_{(w,b)\in R} \theta_{|w|}"'] & B_f(1) \arrow[d, "\beta_2"] \arrow[ddr, bend left, "\psi_1"] & \\
\bigotimes_{(w,b)\in R} D(|b|+1) \arrow[r] \arrow[drr, bend right, "y_b \mapsto b"'] & B_f(2) \arrow[dr, dashed, "\psi_2"] \arrow[ul, phantom, "\ulcorner", very near start] & \\
 & & B
\end{tikzcd} \quad , \qquad R \equiv \{(w,b) \in B_f(1) \times B \mid \ dw = 0,\ \psi_1(w) = db\}\,.$$

Notice that β_2 is a cofibration as well as an inclusion. The map $B_f(2) \xrightarrow{\psi_2} B$ is an isomorphism on $\operatorname{im}\beta_2$ in homology. Indeed, it's onto because $H(\psi_1)$ is onto, and it's injective because we've killed $\ker H(\psi_1)$. Indeed, for any $[w] \in \ker H(\psi_1)$, there is some y_b such that $dy_b = w$ by the commutativity of the pushout square.

Iterate this construction to get $B_f(3)$, $B_f(4)$, $\ldots$, B_f, where $B_f = \bigcup_{n\geq 0} B_f(n)$.

3.12 Lecture 21

Claim 3.12.1. *The map $\psi : B_f \to B$ is an acyclic fibration.*

Proof. It is a fibration because it is onto. To see that it's a weak equivalence, first note that $H(\psi)$ is surjective because $H(\psi_1)$ is onto. Next, suppose that $w \in B_f$ satisfies $\psi(w) = db$ for some $b \in B$. Since $w \in B_f(n)$ for some $n \geq 0$, we have that $db = \psi(w) = \psi_n(w)$. This means that w is a coboundary by our construction of $B_f(n) \xrightarrow{\psi_n} B_f(n+1)$ as a pushout. Therefore, $H(\psi)$ is also injective. □

MC4(ii): Let $f : A \to B$ be an acyclic cofibration. We must show that this has the LLP w.r.t. all fibrations. By MC5, we can factor f as

$$A \xhookrightarrow[\alpha]{\sim} A_f \xrightarrow[\pi]{} \!\!\!\!\twoheadrightarrow B\ ,$$

where $A_f = A \otimes (\bigotimes_{b\in B} D(|b|+1))$. One can check that α has the LLP w.r.t. all fibrations because $\mathbb{Q} \to \mathbb{Q}[y, dy]$ does. Thus, it suffices to show that f is a retract of α. We have that π is a weak equivalence by MC2. Thus, we can apply MC4(ii) to find a lift $B \to A$ for the commutative square

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, hook, "f"'] & A_f \arrow[d, two heads, "\pi", "\wr"'] \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dashed] & B
\end{tikzcd} \ .$$

This gives us a retract diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"'] \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[d, "f"'] & A \arrow[r, "\mathrm{id}_A"'] \arrow[d, "\alpha"] & A \arrow[d, "f"] \\
B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & A_f \arrow[r, "\pi"] & B
\end{tikzcd} \ ,$$

as desired.

□

3.13 Lecture 22

Let's return to Sullivan algebras.

Theorem 3.13.1. *Every Sullivan algebra $S(V)$ (Definition 2.2.3) is cofibrant in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$.*

Proof. By definition of a Sullivan algebra, we have a filtration

$$\mathbb{Q} \subset V(0) \subset V(1) \subset V(2) \subset \cdots, \qquad V = \bigcup_{n \geq 0} V(n)$$

and thus a filtration

$$\mathbb{Q} \subset S(V(0)) \subset S(V(1)) \subset S(V(2)) \subset \cdots, \qquad S(V) = \bigcup_{n \geq 0} S(V(n))$$

Since cofibrations are closed under transfinite composition, it suffices to show that $S(V(0))$ is cofibrant and that $S(V(n-1)) \subset S(V(n))$ is a cofibration for all $n \geq 1$.

- Recall that $dV(0) = 0$ by definition of a Sullivan algebra. Thus, $S(V(0))$ is cofibrant because it's free with $d = 0$. (This is the same reason that $\mathbb{Q}[x]$ is cofirbant.)
- Recall that $dV(n) \subset S(V(n-1))$ by definition of a Sullivan algebra. We have a SES

$$0 \to V(n-1) \to V(n) \to {}^{V(n)}\!\diagup_{V(n-1)} \to 0,$$

which is split exact as it consists of vector spaces. Hence we can find a subspace $W(n) \subset V(n)$ such that $V(n) = V(n-1) \oplus W(n)$. Let $\{e_i \mid i \in I\}$ denote a basis of $W(n)$. We now have a pushout square

$$\begin{tikzcd} \bigotimes_{i \in I} \mathbb{Q}[x_i] \arrow[r, "x_i \mapsto de_i"] \arrow[d, hook, "\bigotimes_{i\in I} \theta_{|x_i|}"'] & S(V(n-1)) \arrow[d, "i"] \arrow[ddr, bend left] & \\ \bigotimes_{i \in I} \mathbb{Q}[y_i, dy_i] \arrow[r, "y_i \mapsto e_i"'] \arrow[drr, bend right] & S(V(n)) \arrow[dr, dashed, "\exists!"] & \\ & & A \end{tikzcd} \quad , \quad |x_i| \equiv |e_i| + 1, \ |y_i| \equiv |e_i|,$$

where i denote the inclusion map. Therefore, i is a cofibration as the pushout of a cofibration.

□

Homotopy groups in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$

Consider an augmented DGCA $A \xrightarrow{\epsilon} \mathbb{Q} \in \mathrm{ob}(\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}/\mathbb{Q})$. Let $m = \ker \epsilon$, so that $dm \subset m$. Note that $A = \mathbb{Q} \oplus m$ as a complex. Recall the space of indecomposables $IA \coloneqq {}^{m}\!\diagup_{m^2}$. This inherits a differential from A, with grading

$$IA = \bigoplus_{n \geq 0} \frac{m_n}{m_n \cap m^2} \cong \bigoplus_{n \geq 0} \frac{m_n + m^2}{m^2}.$$

Definition 3.13.2. For each $k \in \mathbb{Z}_{\geq 0}$, the *$k$-th homotopy group of A* is the $\mathbb{Q}$-vector space

$$\pi^k(A) \coloneqq H^k(IA, d).$$

We have that

$$D(1) = \mathbb{Q}[t, dt] = \frac{\mathbb{Q}[t_0, t_1, dt_0, dt_1]}{\binom{t_0+t_1-1}{dt_0+dt_1}} = \Omega(1), \qquad |t| \equiv 0.$$

Let $B \in \mathrm{ob}(\mathsf{dgca}_{\mathbb{Q}}^{\geq 0})$. Define $(\partial_0, \partial_1) : \underbrace{B \otimes \Omega(1)}_{\Omega_1 \oplus m_B \otimes \Omega(1)} \to B \times B$ by

$$(\partial_0, \partial_1)(b_0 t_0 + b_1 t_1) = (b_0, b_1)$$
$$(\partial_0, \partial_1)(b_0' dt_0 + b_1' dt_1) = (0, 0)$$

with $b_i, b_i' \in m_B$. By the Künneth formula, we have a commutative triangle

$$\begin{tikzcd} B \arrow[rr, hook, "\sim"] \arrow[dr, "\Delta"'] & & B \otimes \Omega(1) \arrow[dl, two heads, "(\partial_0,\partial_1)"] \\ & B \times B & \end{tikzcd}.$$

Hence $B \otimes \Omega(1)$ is a path space object for B in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$. Moreover, the algebra

$$B \otimes \Omega(1) \coloneqq \mathbb{Q} \otimes m_B \otimes \Omega(1)$$

is a path space object for B in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}/\mathbb{Q}$:

$$\begin{tikzcd} B \arrow[rr, "\sim"] \arrow[dr, "\Delta"'] & & B \otimes \Omega(1) \arrow[dl, two heads, "(\partial_0,\partial_1)"] \\ & B \times_{\mathbb{Q}} B & \end{tikzcd}.$$

3.14 Lecture 23

Our current goal is to prove that any minimal Sullivan model is unique (up to isomorphism).

Lemma 3.14.1. *Suppose that $f, g : A \to B$ are maps of augmented DGCA's such that $f \overset{r}{\sim} g$ via $B \otimes \Omega(1)$. Then $\pi(f) = \pi(g)$.*

Remark 3.14.2. If A is cofibrant and $f \overset{\ell}{\sim} g$, then $f \overset{r}{\sim} g$ via $B \otimes \Omega(1)$, and thus $\pi(f) = \pi(g)$.

Proof. By assumption, we have a homotopy

$$\begin{tikzcd} A \arrow[r, "H"] \arrow[dr, "(f,g)"'] & B \otimes \Omega(1) \arrow[d, "(\partial_0,\partial_1)"] \\ & B \times_{\mathbb{Q}} B \end{tikzcd}.$$

Applying $\pi(-)$ to this produces the commuting triangle

$$\begin{tikzcd} \pi(A) \arrow[r, "\pi(H)"] \arrow[dr, "\pi(f,g)"'] & \pi(B \otimes \Omega(1)) \arrow[d, "\pi(\partial_0,\partial_1)"] \\ & \pi(B \times_{\mathbb{Q}} B) \end{tikzcd}.$$

Note that $B \times B = B \oplus B$ as a chain complex and that

$$B \times_{\mathbb{Q}} B = (\mathbb{Q} \times_{\mathbb{Q}} \mathbb{Q}) \times (m_B \times_{\mathbb{Q}} m_B) = \mathbb{Q} \oplus (m_B \oplus m_B)$$

because limits commute with limits. Thus, $I(B\times_{\mathbb{Q}} B) = IB\oplus IB$, and the diagram

$$\begin{tikzcd} H(IA) \arrow[r, "\pi(H)"] \arrow[dr, bend right, "\pi(f,g)"'] & H(IB\otimes\Omega(1)) \arrow[r, "\cong", "K\ddot{u}nneth"'] \arrow[d, "\pi(\partial_0,\partial_1)"] & H(IB) \arrow[dl, bend left, "\pi(\Delta)"] \\ & H(IB\oplus IB) & \end{tikzcd}$$

commutes. Since $\pi(\Delta)(b) = (b,b)$, it follows that $\pi(\partial_0) = \pi(\partial_1)$. This means that $\pi(f) = \pi(g)$. □

Theorem 3.14.3. *Suppose that $f : S(V_1)\to S(V_2)$ is a weak equivalence of minimal Sullivan algebras. Then f is an isomorphism.*

Proof. We always have a commutative triangle

$$\begin{tikzcd} S(V_1) \arrow[rr, "f"] \arrow[dr, "\epsilon_1"'] & & S(V_2) \arrow[dl, "\epsilon_2"] \\ & \mathbb{Q} & \end{tikzcd}$$

as any augmentation of the form $S(V)\to\mathbb{Q}$ maps V to 0 when V is positively graded. Hence it suffices to prove that f is an isomorphism as a map in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$. It's clear that all objects of $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ are fibrant (where the zero algebra is the terminal object). Further, both $S(V_1)$ and $S(V_2)$ are cofibrant by Theorem 3.13.1. By Theorem 3.4.3, we can find a map $g : S(V_2)\to S(V_1)$ such that $fg\sim \mathrm{id}_{S(V_2)}$ and $gf\sim\mathrm{id}_{S(V_1)}$. Lemma 3.14.1 implies that

$$\begin{gathered} \pi(fg) = \mathrm{id}_{\pi(S(V_2))} \quad \pi(gf) = \mathrm{id}_{\pi(S(V_1))} \\ \Downarrow \\ \pi(f)\pi(g) = \mathrm{id}_{\pi(S(V_2))} \quad \pi(g)\pi(f) = \mathrm{id}_{\pi(S(V_1))}, \end{gathered}$$

i.e., $\pi(f)$ is an isomorphism with inverse $\pi(g)$. By the minimality of d on $S(V_1)$ and $S(V_2)$, we see that $d\restriction_{IS(V_1)} = 0 = d\restriction_{IS(V_2)}$. Hence $\pi(f)$ defines the map $If : IS(V_1)\xrightarrow{\cong} IS(V_2)$.

Remark 3.14.4. Holstein claims that $\pi(g)\pi(f) = \mathrm{id}_{IS(V_1)} \implies gf = \mathrm{id}_{S(V_1)}$. This is false. We just have that $V_1 \xrightarrow{gf\restriction_{V_1}} V_1$ is the identity map modulo V_1^2.

Lemma 3.14.5 (Cf. Jacobian conjecture). *Let $f : S(V_1)\to S(V_2)$ be a morphism of DGCA's with V_1 and V_2 positively graded. Suppose that $If : IS(V_1)\to IS(V_2)$ is an isomorphism. Then f is an isomorphism.*

Proof. Consider the maps $f\restriction_{V_1} : V_1\to S^{\geq 1}(V_2)$ and $f\restriction_{S^k(V_1)} : S^k(V_1)\to S^{\geq k}(V_2)$. We have that

$$f = \sum_{r=0}^{\infty} f_r, \qquad f_r : S^k(V_i)\to S^{k+r}(V_2).$$

Note that f is determined by $f\restriction_{V_1}$ because $S(V_1)$ is semifree. ...?? □

As If is an isomorphism, we see from Lemma 3.14.5 that f is also an isomorphism.

□

3.15 Lecture 24

Let us finish proving that any minimal Sullivan model is unique.

Theorem 3.15.1. *A minimal Sullivan model $M \xrightarrow{\sim} A$ of a DGCA A is unique up to isomorphism.*

Proof. We shall find an isomorphism $M \xrightarrow{g} M'$ such that the triangle

$$\begin{tikzcd} M \arrow[r, "f", "\sim"'] \arrow[d, dashed] & A \\ M' \arrow[ur, "f'"', "\sim"] & \end{tikzcd}$$

commutes up to homotopy.

Since M is cofibrant and any $X \in \operatorname{ob} \mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ is fibrant, both left and right homotopy are equivalence relations on $\operatorname{Hom}(M, X)$, with

$$\pi^{\ell}(M, X) = \pi^{r}(M, X) = [M, X].$$

Also, since f' is a weak equivalence between fibrant objects, the function

$$[M, M'] \xrightarrow{f' \circ -} [M, A]$$

is a bijection. In particular, there is some $g : M \to M'$ such that $f' \circ g = f$ up to homotopy. This implies that $H(f) = H(f' \circ g)$ as the cohomology functor is homotopy invariant. Thus, $f' \circ g$ is a weak equivalence. By two-out-of-three, g is also a weak equivalence. By Theorem 3.14.3, g is an isomorphism. $\square$

4 Simplicial sets

Recall the *simplicial category* Δ:

$$\operatorname{ob} \Delta \equiv \left\{ \overbrace{\{0, 1, \ldots, n\}}^{[n]} \mid n \geq 0 \right\}$$
$$\operatorname{Hom}([m], [n]) \equiv \left\{ [m] \xrightarrow{f} [n] \mid f \text{ nondecreasing} \right\}$$

For any category $\mathcal{C}$, a *simplicial object* in $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$, and a *cosimplicial object* is a functor $\Delta \to \mathcal{C}$. We denote the category of simplicial objects in $\mathcal{C}$ by $\mathsf{s}\mathcal{C}$. We call sSet the category of simplicial sets.

Let K be a simplicial set. For each $n \geq 0$, let K_n denote the set $K([n])$. A simplicial set is then a family of set $K_\bullet = \{K_n \mid n \geq 0\}$ together with a map $K_n \xrightarrow{K(\alpha)} K_m$ for each $[m] \xrightarrow{\alpha} [n]$. An element of K_n is called an *n-simplex*. There exists a unique factorization of α in Δ:

$$\alpha = \delta^{i_1} \cdots \delta^{i_p} \sigma_{j_1} \cdots \sigma_{j_e} \qquad i_1 \leq i_2 \leq \cdots \leq i_p$$
$$j_1 \leq j_2 \leq \cdots \leq j_e.$$

Here, the i-th *face map* $\delta^i : [n-1] \to [n]$ skips i, and the j-th *degeneracy map* $\sigma_j : [n+1] \to [n]$ sends j and $j+1$ to j. In Δ, these maps satisfy relations such as

$$\delta^j \delta^i = \delta^i \delta^{j-1}, \qquad i < j.$$

Therefore, any simplicial set is determined by a family $K_\bullet$ of sets together with face $K_n \xrightarrow{d_i \coloneqq K(\delta^i)} K_{n-1}$ and degeneracy $K_n \xrightarrow{s^i \coloneqq K(\sigma_i)} K_{n+1}$ maps satisfying relations such as $d_i d_j = d_{j-1} d_i, \quad i < j$. A morphism $K_\bullet \to L_\bullet$ of simplicial sets is a family of maps $\{K_n \to L_n \mid n \geq 0\}$ commuting with each d_i and s^i.

Example 4.0.1.

1. Let X be a space. Let $\delta^n \subset \mathbb{R}^{n+1}$ denote the (standard) geometric n-simplex. Recall the set $\mathrm{Sing}_n(X) = \{\gamma : \Delta^n \to X \mid \gamma \text{ continuous}\}$ of singular n-simplices. Then $S_n(X) \coloneqq \mathbb{Z}\mathrm{Sing}_n(X)$ consists of all singular n-chains of X, with differential

   $$\partial j = \sum_i (-1)^i d_i \gamma$$

   where $d_i\gamma$ restricts γ to the i-th face of Δ^n.

2. The *standard n-simplex* is the simplicial set

   $$\Delta[n] \coloneqq \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathsf{Set}.$$

   The set $\Delta[n]_i$ of i-simplices is precisely $\mathrm{Hom}_\Delta([i], [n])$. For example, $\Delta[0]_i$ consists of a single simplex.

   For any simplicial set $K_\bullet$, the Yoneda lemma gives us a natural bijection

   $$\begin{gathered} \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], K_\bullet) \cong K_n \\ \Downarrow \\ \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], \Delta[m]) \cong \mathrm{Hom}_\Delta([n], [m]). \end{gathered}$$

4.1 Lecture 25

We want to define the geometric realization of a simplicial set.

Example 4.1.1. Recall the geometric n-simplex:

$$\Delta^n = \left\{ (t_0, t_1, \ldots, t_n) \mid \sum_{i=0}^n t_i = 1,\ t_i \geq 0 \right\}.$$

The mapping $[n] \mapsto \Delta^n$ defines a cosimplicial space $\Delta(-)$ sending each $[m] \xrightarrow{\alpha} [n]$ in Δ to

$$\begin{gathered} \Delta^m \to \Delta^n \\ i\text{-th vertex} \mapsto \alpha(i)\text{-th vertex}, \\ \text{extended by linearity.} \end{gathered}$$

For any space X, the composite $[n] \mapsto \Delta^n \mapsto \mathrm{Hom}_{\mathsf{Top}}(\Delta^n, X)$ gives rise to a simplciial set $\mathrm{Sing}_\bullet(X)$. The *geometric realization* $|-| : \mathsf{sSet} \to \mathsf{Top}$ functor will be left adjoint to $\mathrm{Sing}_\bullet : \mathsf{Top} \to \mathsf{sSet}$.

Let $A_\bullet$ be a simplicial set. Define $|A_\bullet|$ as the coequalizer (or colimit) of the diagram

$$\coprod_{[m]\xrightarrow{\alpha}[n]} A_n \times \Delta^m \underset{\coprod \mathrm{id}_{A_n} \times \Delta(\alpha)}{\overset{\coprod A(\alpha)\times \mathrm{id}_{\Delta^m}}{\rightrightarrows}} \coprod_{k\geq 0} A_k \times \Delta^k$$

of spaces. This means that

$$|A_\bullet| = \coprod_{k\geq 0} A_k \times \Delta^k \diagup \sim.$$

Here, $A_n \times \Delta^n \ni (x,p) \sim (y,q) \in A_{n-1} \times \Delta^{n-1}$ if $\left(s^i \times \mathrm{id}\right)(y,p) = (x,p)$ and $\left(\mathrm{id} \times S^i\right)(y,p) = (y,q)$, i.e.,

$$s^i y = x \text{ and } S^i p = q.$$

Likewise, $A_{n-1} \times \Delta^{n-1} \ni (y,q) \sim (x,p) \in A_n \times \Delta^n$ when

$$d_i x = y \text{ and } D_i q = p.$$

Exercise 4.1.2. $|\Delta[n]| \cong \Delta^n$.

Lemma 4.1.3. *The pair* $(|-|, \mathrm{Sing}_\bullet)$ *is an adjoint pair.*

Proof. We must exhibit a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Top}}(|A_\bullet|, X) \cong \mathrm{Hom}_{\mathsf{sSet}}(A_\bullet, \mathrm{Sing}_\bullet(X))$$

of sets. We have that

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Top}}(\mathrm{coeq}(\textstyle\coprod \rightrightarrows \coprod), X) &\cong \mathrm{eq}(\prod_{k\geq 0} \mathrm{Hom}_{\mathsf{Top}}(A_k \times \Delta^k, X) \rightrightarrows \prod_{[m]\to[n]} \mathrm{Hom}_{\mathsf{Top}}(A_n \times \Delta^m, X)) \\
&\cong \mathrm{eq}(\prod_{k\geq 0} \mathrm{Hom}_{\mathsf{Set}}(A_k, \mathrm{Sing}_k(X)) \rightrightarrows \prod_{[m]\to[n]} \mathrm{Hom}_{\mathsf{Set}}(A_n, \mathrm{Sing}_m(X))) \\
&\cong \mathrm{Hom}_{\mathsf{sSet}}(A_\bullet, \mathrm{Sing}_\bullet(X)),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Top}}(A_k \times \Delta^k, X) &\cong \mathrm{Hom}_{\mathsf{Top}}(A_k, \mathrm{Hom}_{\mathsf{Top}}(\Delta^k, X)) && \text{(tesnor-hom adjunction)} \\
&\cong \mathrm{Hom}_{\mathsf{Set}}(A_k, \underbrace{\mathrm{Hom}_{\mathsf{Top}}(\Delta^k, X)}_{\mathrm{Sing}_k(X)}). && (A_k \text{ discrete})
\end{aligned}$$

□

Theorem 4.1.4. *The following data is a model category structure on* sSet*:*

$$\begin{aligned}
\mathscr{W} &\equiv \{f : K_\bullet \to L_\bullet \mid |f| : |K_\bullet| \to |L_\bullet| \textit{ is a weak equivalence}\} \\
\mathscr{C} &\equiv \{f : K_\bullet \to L_\bullet \mid f \textit{ monic, i.e., } f_n \textit{ is injective for all } n \geq 0\} \\
\mathscr{F} &= \{f : K_\bullet \to L_\bullet \mid f \textit{ has the RLP w.r.t all acyclic cofibrations}\}.
\end{aligned}$$

For a proof of this theorem, see *More Concise Algebraic Topology* by May and Ponto.

4.2 Lecture 26

Recall the maps

$$\begin{aligned} \mathbb{Q}[x] &\xrightarrow{x\mapsto dy} \mathbb{Q}[y, dy] \text{ cofibration} \\ \mathbb{Q} &\longrightarrow \mathbb{Q}[y, dy] \text{ acyclic cofibration} \end{aligned}$$

in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$. What are analogous of these in sSet?

Definition 4.2.1. For each integer $0 \leq k \leq n$, the *k-horn* of $\Delta[n]$ is the sub-simplicial set $\Lambda_k[n] \subset \Delta[n]$ obtained by leaving out $[n] \xrightarrow{\mathrm{id}} [n]$ from $\Delta[n]_n$ and the k-th face d_k from $\Delta[n]_{n-1}$ along with all of their degeneracies.

In other words, all of its n-simplices are degenerate, and

$$\Lambda_k[n]_i = \left\{[i] \xrightarrow{f} [n] \in \Delta[n]_i \mid \operatorname{im} f \notin \{0, 1, \ldots, k-1, k+1, \ldots, n\}\right\}, \quad 0 \leq i < n.$$

Example 4.2.2.

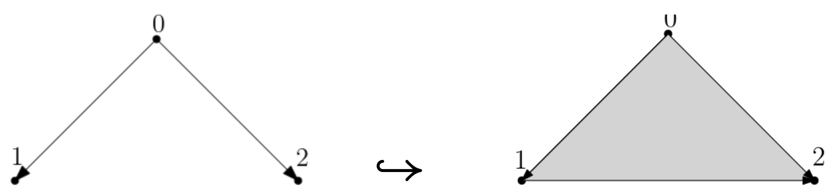

Figure 1: $\Lambda_0[2]$ (left) and $\Delta[2]$ (right)

Definition 4.2.3 (Kan complex). A simplical set $K_\bullet$ is a *Kan complex* if every map $\Lambda_k[n] \to K$ can be extended to a map $\Delta[n] \to K$:

$$\begin{tikzcd} \Lambda_k[n] \arrow[r] \arrow[d, hook] & K \\ \Delta[n] \arrow[ur, dashed] & \end{tikzcd}.$$

Proposition 4.2.4.

1. *The set of maps $\{\Lambda_k[n] \subset \Delta[n]\}$ is a set of generating acyclic cofibrations, i.e., $K_\bullet \to L_\bullet$ is a fibration if and only if it has the RLP w.r.t. every inclusion of the form $\Lambda_k[n] \subset \Delta[n]$.*

2. *The set of maps $\{\partial[n] \subset \Delta[n]\}$ is a set of generating cofibrations.*

Corollary 4.2.5. *A simplicial set is fibrant if and only if it's a Kan complex.*

Example 4.2.6. Any simplicial group $G : \Delta^{\mathrm{op}} \to \mathsf{Grp}$ is a Kan complex when viewed as a simplicial set. (This is Prop. 8.2.8 of Weibel's *Hom. Alg.*) For example, recall the inclusion $\Lambda_0[2] \subset \Delta[2]$ from Example 4.2.2. Apply G to this to get

$$\begin{tikzcd} & g_0 \arrow[dl, "g_{01}"'] \arrow[dr, "g_{02}"] \arrow[d, phantom, "?"] & \\ g_1 \arrow[rr, dashed] & & g_2 \end{tikzcd}.$$

We want to find a 2-simplex $g \coloneqq g_{012} \in G_2$ along with a 1-simplex $g_{12} \in G_1$ such that

$$d_2 g = g_{01}$$
$$d_1 g = g_{02}$$
$$d_0 g = g_{12}.$$

It seems that taking

$$g = \underbrace{(s_0 g_{02})\left(s_0^2 g_0^{-1}\right)(s_0 g_{01})}_{s_0(g_{02}(s_0 g_0^{-1})g_{01})}, \qquad G_0 \xrightarrow{s_0} G_1 \xrightarrow{s_0} G_2$$

doesn't work. But note that $g_{12} = g_{02} g_{01}^{-1}$. Taking $g = s_0 g_{12}$ may work.

Definition 4.2.7. Let X be a space.

1. We say that X is *weak Hausdorff* if the image in X of a compact Hausdorff space is closed.
2. We say that X is *compactly generated* if a subspace $A \subset X$ is closed if and only if $A \cap K$ is closed in K for every compact $K \subset X$.

Let WHCG denote the full subcategory of Top on all weak Hausdorff compactly generated spaces. This is a convenient category of spaces in the sense that it's cartesian closed:

$$\mathrm{Hom}_{\mathsf{WHCG}}(X \times Y, Z) \cong \mathrm{Hom}_{\mathsf{WHCG}}(X, Z^Y).$$

Moreover, it contains all CW-complexes.

Theorem 4.2.8.

1. *The adjunction*
$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathsf{WHCG} \arrow[l, bend left, "\mathrm{Sing}"] \end{tikzcd}$$
is a Quillen equivalence.
2. *There exists a Quillen equivalence* $\mathsf{WHCG} \rightleftarrows \mathsf{Top}$.

5 Polynomial de Rham algebras

5.1 Lecture 27

We turn to defining the polynomial deRham functor. This should be both an extension and a "rationalization" of the de Rham functor.

$$\mathsf{Man}^{\mathrm{op}} \to \mathsf{dgca}_{\mathbb{R}}^{\geq 0}, \; M \mapsto (\Omega^\bullet(M), d)$$
$$\Downarrow$$
$$\mathsf{Top}^{\mathrm{op}} \to \mathsf{dgca}_{\mathbb{Q}}^{\geq 0}, \text{ or}$$
$$\mathsf{sSet}^{\mathrm{op}} \to \mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$$

Recall the cosimplicial space $\Delta^\bullet$ of all geometric n-simplices $\Delta^n \subset \mathbb{R}^{n+1}$. Moreover, $\Omega^\bullet$ is a contravariant functor to $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$, with

$$\Omega(n) = \frac{\mathbb{Q}[t_0, \dots, t_n, dt_0, \dots, dt_n]}{(\sum_{i=0}^n t_i = 1, \sum_{i=0}^n dt_i)}.$$

This is called the *polynomial de Rham (dR) algebra* on Δ^n. Indeed, we can make $\Omega(\Delta^\bullet)$ a simplicial object $\Delta^{\mathrm{op}} \to \mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ as follows.

Let $\Omega_n = \Omega(n)$. Let $u : [m] \to [n]$ be a morphism in Δ. Define $\Omega(f) : \Omega(n) \to \Omega(m)$ by

$$t_i \mapsto \sum_{j \in u^{-1}(i)} t_j.$$

This means that $\sum_{i=0}^n t_i \mapsto \sum_{j=0}^m t_j$. The degeneracy s_i and face ∂_i maps are given, respectively, by

$$s_i t_k = \begin{cases} t_{k+1} & i < k \\ t_k + t_{k+1} & i = k \\ t_k & i > k \end{cases}$$

$$\partial_i t_k = \begin{cases} t_{k-1} & i < k \\ 0 & i = k \\ t_k & i > k \end{cases}.$$

By construction, we have, for example, that

$$\partial_i(t_k t_e) = \partial_i(t_k)\partial_i(t_e)$$
$$\partial_i(dt_k) = d(\partial_i t_k).$$

We conclude that $\Omega_\bullet \in \operatorname{ob} s(\mathsf{dgca}_{\mathbb{Q}}^{\geq 0})$.

The *polynomial (PL) de Rham functor* $\mathcal{A} : \mathsf{sSet}^{\mathrm{op}} \to \mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ is defined by

$$K \mapsto \mathrm{Hom}_{\mathsf{sSet}}(K, \Omega_\bullet)$$
$$(\mathrm{Hom}_{\mathsf{sSet}}(K, \Omega_\bullet))^p \equiv \mathrm{Hom}_{\mathsf{sSet}}(K, \Omega_\bullet^p)$$
$$df \equiv d_\Omega \circ f, \quad f : K \to \Omega_\bullet^p$$
$$(f \cdot g)(x) \equiv f(x) \cdot g(x), \quad x \in K_n.$$

For any space X, the *PL de Rham algebra* of X is

$$\Omega_{\mathsf{PL}}(X) \equiv \underbrace{\mathrm{Hom}_{\mathsf{sSet}}(\mathrm{Sing}_\bullet(X), \Omega_\bullet)}_{\mathcal{A}(\mathrm{Sing}_\bullet(X))},$$

i.e., the image of X under the composite

$$\mathsf{Top}^{\mathrm{op}} \xrightarrow{\mathrm{Sing}_\bullet} \mathsf{sSet}^{\mathrm{op}} \xrightarrow{\mathcal{A}} \mathsf{dgca}_{\mathbb{Q}}^{\geq 0}.$$

Note that

$$
\begin{aligned}
\mathcal{A}(0) &= \mathcal{A}(\Delta[0]) \\
&= \mathrm{Hom}_{\mathsf{sSet}}(\Delta[0], \Omega) \\
&\cong \Omega([0]) && \text{(Yoneda)} \\
&= \Omega(0) \\
&= \mathbb{Q}.
\end{aligned}
$$

Now, define $\mathcal{F} : \mathsf{dgca}_{\mathbb{Q}}^{\geq 0} \to \mathsf{sSet}^{\mathrm{op}}$ by

$$
\begin{aligned}
&B \mapsto \mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \Omega) \\
&\left(\mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \Omega)\right)_n \equiv \mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \Omega(n)).
\end{aligned}
$$

We see that $\mathcal{F}(\mathbb{Q}) = 0 = \Delta[0]$ because for all $n \geq 0$, there is a unique map $\mathbb{Q} \to \Omega(n)$.

5.2 Lecture 28

Our big goal is to show that the pair $(\mathcal{F}, \mathcal{A})$ of functors induces a Quillen equivalence on certain subcategories (which are related to nilpotence). To begin, we want to prove that it is a Quillen adjunction. This serves as our main adjunction for the course.

Lemma 5.2.1. *The pair $(\mathcal{F}, \mathcal{A})$ is an adjoint pair.*

Proof. We must find a natural bijection

$$
\begin{aligned}
\mathrm{Hom}_{\mathsf{sSet}^{\mathrm{op}}}(\mathcal{F}(B), K) &\cong \mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \mathcal{A}(K)), \text{ i.e.,} \\
\mathrm{Hom}_{\mathsf{sSet}}(K, \mathcal{F}(B)) &\cong \mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \mathcal{A}(K)), \text{ i.e.,} \\
\mathrm{Hom}_{\mathsf{sSet}}(K, \mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \Omega)) &\cong \mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \mathrm{Hom}_{\mathsf{sSet}}(K, \Omega)).
\end{aligned}
$$

For each $f : K \to \mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \Omega)$, define $f^\sharp : B \to \mathrm{Hom}_{\mathsf{sSet}}(K, \Omega)$ by

$$b \mapsto (x \mapsto f(x)(b)).$$

Conversely, for each $g : B \to \mathrm{Hom}_{\mathsf{sSet}}(K, \Omega)$, define $g^\flat : K \to \mathrm{Hom}_{\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}}(B, \Omega)$ by

$$x \mapsto (b \mapsto g(b)(x)).$$

Then $-^\sharp$ and $-^\flat$ are inverses of each other. □

Corollary 5.2.2. *The adjunction $\mathcal{F} \dashv \mathcal{A}$ restricts to an adjunction* $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}/\mathbb{Q} \rightleftarrows \mathsf{sSet}_*^{\mathrm{op}}$.

For a sense of why Corollary 5.2.2 is true, notice that $\mathcal{F}$ sends any map $B \to \mathbb{Q}$ to the pointed simplicial set $\underbrace{\mathcal{F}(\mathbb{Q})}_{*} \to \mathcal{F}(B)$ and that $\mathcal{A}$ sends any pointed simplicial set $* \to K$ to the map $\mathcal{A}(K) \to \underbrace{\mathcal{A}(*)}_{\mathbb{Q}}$ over $\mathbb{Q}$.

We now want to show that $(\mathcal{F}, \mathcal{A})$ forms a Quillen adjunction. We must show that

- $\mathcal{F}$ sends cofibrations in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ to cofibrations in $\mathsf{sSet}^{\mathrm{op}}$ (i.e., fibrations in sSet) and

- $\mathcal{A}$ sends fibrations in $\mathsf{sSet}^{\mathrm{op}}$ (i.e., cofibrations in sSet) to fibrations in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$.

Lemma 5.2.3. *The functor $\mathcal{A}$ sends cofibrations in sSet to fibrations in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$.*

Proof. We must show that $\mathcal{A}$ sends any inclusion $K \hookrightarrow L$ to a degree-wise surjection $\mathcal{A}(L) \to \mathcal{A}(K)$. This map is surjective if and only if we can find lifts of the form

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & \Omega^p_\bullet \arrow[d] \\ L \arrow[r] \arrow[ur, dashed] & * \end{tikzcd}$$

where $p \geq 0$. In this case, $K \hookrightarrow L$ has the LLP w.r.t each map of the form $\Omega^p \to *$. Thus, it suffices to show that $\Omega^p \to *$ is an acyclic fibration.

- By Example 4.2.6, Ω^p is a Kan complex as a simplicial (abelian) group.
- Note that $\Omega^p \to *$ is a map of fibrant, cofibrant objects. By Theorem 3.4.3, this map is a weak equivalence if and only if it's a homotopy equivalence. Thus, it suffices to show that $\Omega^p \to *$ is a homotopy equivalence with homotopy inverse $0 \hookrightarrow \Omega^p$. We want a left homotopy

$$\Omega^p \times \Delta[1] \to \Omega^p$$

between $\Omega^p \xrightarrow{\mathrm{id}} \Omega^p$ and $\Omega^p \to 0 \to \Omega^p$ (the constant map at 0). Note that

$$(\Omega^p \times \Delta[1])_n = \Omega(n)^p \times \Delta[1]_n = \bigoplus_{\alpha \in \Delta[1]_n} \Omega(n)^p.$$

We shall define the left homotopy after discussing homotopies in sSet.

□

5.3 Lecture 29

Our current plan has three parts.

1. Finishing the main theorem: $(\mathcal{F}, \mathcal{A})$ induces a Quillen equivalence on suitable subcategories
2. Quillen (minimal) models of sSet and $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ in the L_∞-world
3. Application to Mysterious Duality in string theory

To finish our main theorem, we must look at the notion of homotopy in sSet.

Theorem 5.3.1 (Fundamental adjunction). *The pair $(\mathcal{F}, \mathcal{A})$ is a Quillen adjunction*

$$\mathsf{dgca}_{\mathbb{Q}}^{\geq 0} \rightleftarrows \mathsf{sSet}^{\mathrm{op}} .$$

Proof. First, we know that this is an adjunction by Lemma 5.2.1. Second, up to our discussion of homotopy in sSet, we know that $\mathcal{A}$ sends fibrations in $\mathsf{sSet}^{\mathrm{op}}$ to fibrations in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ by Lemma 5.2.3. Finally, we must show that $\mathcal{F}$ sends cofibrations in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ to cofibrations in $\mathsf{sSet}^{\mathrm{op}}$, i.e., fibrations in sSet.

With this in mind, we want to show that the maps

$$\Omega^p \underset{0}{\overset{\mathrm{id}}{\rightrightarrows}} \Omega^p$$

are left homotopic. A *left homotopy* for simplicial sets or for simplicial $\mathbb{Q}$-vector spaces is a left homotopy with respect to the cylinder object

$$\begin{tikzcd} K \coprod K \arrow[r, hook, "\mathrm{id}\times\Delta[\delta^1] + \mathrm{id}\times\Delta[\delta^0]"] \arrow[dr, "\nabla"'] & K \times \Delta[1] \arrow[d, "\wr"] & {|K| \times |\Delta[1]|} \arrow[d, "\wr"] \\ & K \times * & {|K|} \end{tikzcd} \Longleftarrow \quad , \qquad |\Delta[1]| = \Delta^1 \sim *,$$

in the sense of Definition 3.3.3. In general, a left homotopy $K \times \Delta[1] \to L$ is equivalent to a map $K \to \underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Delta[1], L)$ thanks to the natural isomorphism

$$\mathrm{Hom}_{\mathsf{sSet}}(K \times X, L) \cong \mathrm{Hom}_{\mathsf{sSet}}(K, \underline{\mathrm{Hom}}_{\mathsf{sSet}}(X, L)), \qquad \underline{\mathrm{Hom}}_{\mathsf{sSet}}(X, L)_n \equiv \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n] \times X, L).$$

This definition of the internal hom as a simplicial set is motivated by the equations

$$\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n] \times X, L) = \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n], \underline{\mathrm{Hom}}_{\mathsf{sSet}}(X, L)) = \underline{\mathrm{Hom}}_{\mathsf{sSet}}(X, L)_n.$$

In particular, note that

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathsf{sSet}}(\Delta[1], L)_n &= \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n] \times \Delta[1], L) \\ &\quad\updownarrow \\ &\quad \mathrm{Hom}_{\mathsf{sSet}}(\Delta[n+1], L) \\ &\quad \| \\ &\quad L_{n+1}. \end{aligned} \tag{EZ-AW}$$

This means that a *simplicial* homotopy $K \times \Delta[1] \to L$ is the same as a collection of maps $K_n \to L_{n+1}$. In particular, a homotopy $\mathrm{id}_{\Omega^p(n)} \overset{\ell}{\sim} 0$ amounts to a formula of the form $\Omega^p(n) \to \Omega^p(n+1)$ for each $p \geq 0$:

$$\begin{aligned} 1 &\mapsto T^2 \\ t_i &\mapsto T \cdot t_{i+1} \\ dt_i &\mapsto T \cdot dt_{i+1} - dT \cdot t_{i+1}, \qquad T \equiv t_1 + \cdots + t_n. \end{aligned}$$

Next, we want to show that $\mathcal{F}$ sends cofibrations in $\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}$ to fibrations in sSet. Suppose that $i : B \to C$ is such a cofibration. To see that $\mathcal{F}_i$ is a fibration, it suffices to check the RLP of $\mathcal{F}_i$ with respect to all generating acyclic cofibrations,

$$\Lambda[n]_k \overset{\sim}{\hookrightarrow} \Delta[n] \qquad n, k \geq 0 \ .$$

Our adjunction $\mathcal{F} \dashv \mathcal{A}$ gives us an equivalence

$$\begin{tikzcd} \Lambda[n]_k \arrow[r] \arrow[d, hook, "\wr"'] & \mathcal{F}C \arrow[d, "\mathcal{F}i"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathcal{F}B \end{tikzcd} \iff \begin{tikzcd} B \arrow[r] \arrow[d, hook, "i"'] & \mathcal{A}(\Delta[n]) \arrow[d, two heads] \\ C \arrow[r] \arrow[ur, dashed] & \mathcal{A}(\Lambda[n]_k) \end{tikzcd}.$$

Thus, it suffices to show that the map $\mathcal{A}(\Delta[n]) \to \mathcal{A}(\Lambda[n]_k)$ is an acyclic fibration, i.e., that the induced map

$$\underbrace{H^\bullet(\mathcal{A}(\Delta[n]))}_{\mathbb{Q}} \overset{?}{\xrightarrow{\sim}} H^\bullet(\mathcal{A}(\Lambda[n]_k))$$

is an isomorphism.

For any simplicial set K, define the *simplicial cochain complex of K $C^\bullet(K)$* by

$$C^n(K) = \operatorname{Hom}(K_n, \mathbb{Q})$$
$$df(x) = \sum_{i=0}^{n+1} (-1)^i f(d_i x), \qquad x \in K_{n+1}.$$

Proposition 5.3.2. $H^\bullet(C^\bullet(K)) = H^\bullet(|K|)$.

Next, define $\rho : \mathcal{A}(K) \to C^\bullet(K)$ by integration of PL differential forms over geometric simplices:

$$\mathcal{A}(K)^p \ni \omega \mapsto \left(K_p \ni x \mapsto \int_{\Delta^p} \omega(x)\right),$$

where $\omega(x) \in \Omega(p)^p$.

5.4 Lecture 30

Theorem 5.4.1 (Simplicial de Rham). *The cochain map ρ is a quasi-isomorphism of $\mathbb{Q}$-complexes, i.e., induces an isomorphism*

$$H^\bullet(\mathcal{A}(K)) \xrightarrow{\sim} H^\bullet(C^\bullet(K))$$

of graded-$\mathbb{Q}$-vector spaces.

Proof sketch.

1. **Poincaré lemma:**

$$\begin{aligned} H^\bullet(\mathcal{A}(\Delta[n])) &\cong \mathbb{Q} \\ &\cong H^\bullet(\Delta^n, \mathbb{Q}) \\ &\cong H^\bullet(C^\bullet(\Delta[n])). && \text{(Proposition 5.3.2)} \end{aligned}$$

2. **Mayer-Vietoris sequence:**

   For any pushout square

$$\begin{tikzcd} K \arrow[r, "j"] \arrow[d, hook, "i"'] \arrow[dr, phantom, "\ulcorner"] & M \arrow[d, "g"] \\ L \arrow[r, "h"'] & N \end{tikzcd}$$

in sSet, we have a short exact sequence

$$0 \longrightarrow \mathcal{A}(N) \xrightarrow{(\mathcal{A}(h),-\mathcal{A}(g))} \mathcal{A}(L) \oplus \mathcal{A}(M) \xrightarrow{\mathcal{A}(i)+\mathcal{A}(j)} \mathcal{A}(K) \longrightarrow 0$$

of $\mathbb{Q}$-vector spaces. This has a LES in cohomology, known as the M-V sequence for $H^\bullet(\mathcal{A}(N))$.

Likewise, we have a short exact sequence

$$0 \longrightarrow C^\bullet(N) \longrightarrow C^\bullet(L) \oplus C^\bullet(M) \longrightarrow C^\bullet(K) \longrightarrow 0$$

along with the M-V sequence for $H^\bullet(C^\bullet(K))$.

We see that ρ induces a morphism between the two SES's. The five lemma thus implies that if ρ induces an isomorphism on cohomology for L, M, and K, then it also induces one for N.

3. Every simplicial set is the pushout along a map of the form $\coprod_\alpha \partial\Delta[n] \hookrightarrow \coprod_\alpha \Delta[n]$.

□

Remark 5.4.2. In fact, ρ induces an isomorphism of cohomology *rings* as it can be represented as a zig-zag of quasi-isomorphism of DGA's.

$$\begin{tikzcd} & \bullet \arrow[dl, "q\text{-}iso"'] \arrow[dr, "q\text{-}iso"] & & \bullet \arrow[dl, "q\text{-}iso"'] \arrow[dr, "q\text{-}iso"] & \\ \mathcal{A}(K) \arrow[rrrr, bend right, "\rho"'] & & \cdots & & C^\bullet(K) \end{tikzcd} \qquad \Longrightarrow \qquad H^\bullet(\rho) \text{ is an iso.}$$

Now, both $H^\bullet(C^\bullet(\Delta[n]))$ and $H^\bullet(C^\bullet(\Lambda[n]_k))$ are isomorphic to $\mathbb{Q}$ thanks to Proposition 5.3.2. Thus, by Theorem 5.4.1, we have that $H^\bullet(\mathcal{A}(\Delta[n])) \cong \mathbb{Q} \cong H^\bullet(\mathcal{A}(\Lambda[n]_k))$.

□

Let's turn to the homotopy theory of our fundamental adjunction. We have an adjunction

$$\begin{tikzcd} \mathrm{Ho}(\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}) \arrow[r, bend left, "\mathbb{L}\mathcal{F}"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathsf{sSet}^{\mathrm{op}}) \arrow[l, bend left, "\mathbb{R}\mathcal{A}"] \end{tikzcd} .$$

Every object of sSet is cofibrant, and any cofibrant replacement of a cofibrant object K (such as K itself) is unique up to homotopy in a certain sense. Therefore, $\mathbb{R}\mathcal{A} = \mathcal{A}(RK) = \mathcal{A}(K)$ because RK is a cofibrant replacement of K in sSet. Moreover, $\mathbb{L}\mathcal{F}(B) = \mathcal{F}(QB)$, which is automatically a Kan complex.

Definition 5.4.3.

1. Let K be a simplicial set. A *Sullivan minimal model* of K is a minimal Sullivan model of $\mathcal{A}(K)$, i.e., a minimal Sullivan algebra $\mathcal{M}(K)$ together with a quasi-isomorphism $\mathcal{M}(K) \to \mathcal{A}(K)$.

2. Let X be a topological space. A *Sullivan model* of X is a Sullivan minimal model of $\mathrm{Sing}_\bullet(X)$.

Theorem 3.13.1 implies that $\mathcal{M}(K)$ is a cofibrant replacement of $\mathcal{A}(K)$.

We can make $\mathcal{M}$ a functor so that we have a diagram

$$\mathrm{Ho}(\underbrace{\mathsf{SMA}}_{\substack{\text{homolog.}\\ \text{conn.}\\ \text{Sull. min.}\\ \text{algebras}}}) \underset{\mathcal{M}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathrm{Ho}(\underbrace{\mathsf{sSet}_0^{\mathrm{op}}}_{\text{conn. ssets}}) .$$

This is actually an adjunction.

Homotopy groups:

Recall that for any $B \in \mathrm{ob}(\mathsf{dgca}_{\mathbb{Q}}^{\geq 0}/\mathbb{Q})$, we have $\pi^n(B) = H^n(IB)$ by definition, where $IB = {}^{m_B}\!/\!{}_{m_B^2}$ and $m = \ker(B \to \mathbb{Q})$.

Let K be a pointed simplicial set, i.e., an object of sSet_*. (Its distinguished point is precisely a map $\Delta[0] \to K$.) Assume that K is a Kan complex. Define the homotopy groups of K as follows. Let $\pi_0(K)$ be the coequalizer

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \twoheadrightarrow {}^{X_0}\!/\!{}_{X_1} .$$

This is precisely the set $[\Delta[0], K]$ of homotopy classes of maps. Further, for any $n \geq 1$, let $\pi_n(K) = [(\Delta[n], \partial\Delta[n]), (K, *)]$. Note that

$$\begin{aligned} \pi_n(K) &= [(\partial\Delta[n+1], v_0), (K, *)] \\ &= [(S^n, N), (|K|, *)] \\ &= \pi_n(|K|, *). \end{aligned}$$

Thus, we have at least four ways of defining the homotopy group. The rational homotopy groups of K are defined as

$$\pi_n^{\mathbb{Q}}(K) = \begin{cases} \pi_n(K) \otimes_{\mathbb{Z}} \mathbb{Q} & n \geq 2 \\ \frac{\pi_1(K)}{[\pi_1(K), \pi_1(K)]} \otimes_{\mathbb{Z}} \mathbb{Q} & n = 1 \end{cases} .$$

5.5 Lecture 31

We want to study the homotopy groups of a cofibrant augmented DGCA $B \to \mathbb{Q}$ and of the simplicial set $\mathcal{F}B$. Let K be a pointed Kan complex. To endow $\pi_n(K)$ with a group structure for all $n \geq 1$, let $[f], [g] \in [(\Delta[n], \partial\Delta[n]), (K, *)]$. Define $\varphi : \Lambda_n[n+1] \to K$ by

$$\varphi(i) = \begin{cases} s_0 s_0 \cdots s_0(*) & 0 \leq i \leq n-2 \\ g & i = n-1 \\ f & i = n+1 \end{cases} .$$

As K is a Kan complex, this horn may be filled to a map $\tilde{\varphi} : \Delta[n+1] \to K$. By the simplicial identities, we have that

$$d_i d_n \tilde{\varphi} = d_{n-1} d_i \tilde{\varphi} = s_0 s_0 \cdots s_0(*).$$

Let $[f] \cdot [g] = [d_n \tilde{\varphi}]$.

$$\begin{tikzcd} & 0 \arrow[dl, "f"'] \arrow[dr, dashed, "f\cdot g"] & \\ 1 \arrow[rr, "g"'] & & 2 \end{tikzcd}$$

This is a group operation. In fact, $\pi_n(\mathcal{F}B)$ is naturally a $\mathbb{Q}$-vector space and thus isomorphic to $\pi_n^{\mathbb{Q}}(\mathcal{F}B)$.

Theorem 5.5.1. *Let B be a cofibrant augmented DGCA$^{\geq 0}$.*

(1) If B is homologically connected (i.e., $H^0(B) \cong \mathbb{Q}$), then $\pi_0(\mathcal{F}B) = *$.

(2) If $n \geq 1$, then there exists a natural bijection

$$\pi_n(\mathcal{F}B) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}}(\pi^n B, \mathbb{Q})$$

of sets.

(3) if $n \geq 2$, then this bijection is a group isomorphism.

(4) If $n = 1$, then it is a group isomorphism as long as

$$x \in B^1 \implies dx \in \mathbb{Q}\left[B^0, dB^0\right] \subset B.$$

Proof sketch. First, suppose that $H^0(B) \cong \mathbb{Q}$. By Theorem 2.4.2, we have a minimal Sullivan algebra $\mathcal{M}(B)$ and a weak equivalence $\mathcal{M}(B) \xrightarrow{\sim} B$. Since both B and $\mathcal{M}(B)$ are cofibrant, our proof of Lemma 3.7.1 shows that the induced map $\mathcal{F}\mathcal{M}(B) \to \mathcal{F}B$ is a weak equivalence. Hence $|\mathcal{F}B| \simeq |\mathcal{F}\mathcal{M}(B)|$, and we have that

$$\begin{aligned} \pi_0(\mathcal{F}B) &= \pi_0(\mathcal{F}\mathcal{M}(B)) \\ &= [\Delta[0], \mathcal{F}\mathcal{M}(B)] \\ &= [\mathcal{F}\mathcal{M}(B), \Delta[0]]^{\mathrm{op}} \\ &= [\mathcal{M}(B), \underbrace{\mathcal{A}(\Delta[0])}_{\varnothing}] && (\mathbb{L}\mathcal{F} \dashv \mathbb{R}\mathcal{A}) \\ &= [\mathcal{M}(B), \mathbb{Q}] \\ &= *. && (\mathcal{M}(B) = S(V),\ V = \textstyle\bigoplus_{\geq 1} V^n) \end{aligned}$$

This proves (1).

Next, suppose that $n > 1$. By Lemma 3.8.2, we have that

$$\pi_n(\mathcal{F}B) = [\partial\Delta[n+1], \mathcal{F}B] \cong [B, \mathcal{A}(\partial\Delta[n+1])].$$

Let

$$V(n) = {}^{\mathbb{Q}[x_n]}\!\big/_{(x_n^2)}, \qquad |x_n| \equiv n.$$

This has homology $H^\bullet(S^n;\mathbb{Q})$. Further, Theorem 5.4.1 gives us a weak equivalence $\mathcal{A}(\partial\Delta[n+1]) \xrightarrow{\sim} C^\bullet(\partial\Delta[n+1])$, and $C^\bullet(\partial\Delta[n+1])$ also has homology $H^\bullet(S^n;\mathbb{Q})$. Indeed, the map $V(n) \xrightarrow{\sim} \mathcal{A}(\partial\Delta[n+1])$ defined by

$$x_n \mapsto \text{a closed PL form representing a generator of } H^n(S^n;\mathbb{Q}) \cong \mathbb{Q}$$

is a weak equivalence. This means that $[B,\mathcal{A}(\partial\Delta[n+1])] \cong [B,V(n)]$.

Proposition 5.5.2. $[B,V(n)] \cong \operatorname{Hom}_{\mathbb{Q}}(\pi^n(B),\mathbb{Q})$.

This sequence of bijections respects the group structure of π_n. Wlog, replace B with $\mathcal{M}(B)$. We have a cofibration $S(V(n-1)) \hookrightarrow S(V(n))$. By taking the pushout of this along $S(V(n-1)) \to \mathbb{Q}$, we can show that $(S(V(n)/V(n-1)), d \equiv 0)$ has the same π^n as B. This also has a coalgebra structure: $B \to B \otimes B$, which induces a monoid structure on $\mathcal{F}B$. By the Eckmann-Hilton argument, the set

$$\underbrace{[B,\mathcal{A}(\partial\Delta[n+1])]}_{[\partial\Delta[n+1],\mathcal{F}B]}$$

is a monoid with the same structure as that on $\pi_n(\mathcal{F}B)$. Moreover, the coalgebra structure defines a monoid structure on $\pi^n(B)^\vee$ identifiable as $+$.

□

5.6 Lecture 32

Let X be a rational space. Suppose that $\mathcal{F}\mathcal{M}(\mathcal{A}X) \overset{w.e.}{\sim} X$. Then

$$\begin{aligned} \pi_n(X) &\cong \pi_n(\mathbb{L}\mathcal{F}\mathcal{A}X) && (\mathcal{M}\text{ cofibrant replacement}) \\ &\cong \pi^n(\mathcal{M}(\mathcal{A}X))^\vee. && (\text{Theorem 5.5.1}) \end{aligned}$$

In particular, for all $n \geq 1$, we have that $\pi_n(S^n) \cong \pi^n(\mathcal{M}(S^n))^\vee$. As in the real de Rham case, we can guess a minimal Sullivan model of $\mathcal{A}(\partial\Delta[n+1])$:

$$\mathcal{M}(S^n) = \begin{cases} \mathbb{Q}[x_n] & n \text{ odd} \\ \mathbb{Q}\left[x_n, y_{2n-1} \mid dy_{2n-1} = x_n^2\right] & n \text{ even} \end{cases},$$

where $|x_n| = n$. Note that $\pi^k(\mathcal{M}(S^n)) = (I\mathcal{M}(S^n))^k$. Specifically, $I\mathbb{Q}[x_n] = \mathbb{Q}x_n$ when n is odd, in which case

$$\pi_k^{\mathbb{Q}}(S^n) = \begin{cases} \mathbb{Q} & k = n \\ 0 & k \neq n \end{cases}$$

for all $k \geq 1$. And when n is even, we have that

$$\begin{gathered} I\mathbb{Q}\left[x_n, y_{2n-1} \mid dy_{2n-1} = x_n^2\right] = \mathbb{Q}x_n \oplus \mathbb{Q}y_{2n-1} \\ \Downarrow \\ \pi_k^{\mathbb{Q}}(S^n) = \begin{cases} \mathbb{Q} & k = n, 2n-1 \\ 0 & \text{otherwise} \end{cases}. \end{gathered}$$

By contrast, in the non-Sullivan case, we have that $H^\bullet(S^n;\mathbb{Q}) \cong \mathbb{Q}[x_n] \big/ (x_n^2)$.

A *rationalization* of a space X is a space $X_\mathbb{Q}$ together with a map $X_\mathbb{Q} \to X$ inducing an isomorphism

$$\pi_k(X_\mathbb{Q}) \cong \pi_k(X) \otimes_\mathbb{Z} \mathbb{Q}.$$

Thus, every space of the form $\mathcal{F}B$ is a rationalization of itself.

Main theorem of RHT:

The Quillen adjunction $\mathcal{F} \dashv \mathcal{A}$

$$\mathsf{dgca}_\mathbb{Q}^{\geq 0} \rightleftarrows \mathsf{sSet}_*^{\mathrm{op}}$$

induces an equivalence of categories

$$\mathrm{Ho}(\mathsf{dgca}_\mathbb{Q}^{\geq 0,\mathrm{ft}}) \xrightarrow{\sim} \mathrm{Ho}(\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}})^{\mathrm{op}}.$$

Here, $\mathsf{dgca}_\mathbb{Q}^{\geq 0,\mathrm{ft}}$ denotes the full subcategory of cohomologically connected DGCA's of *finite type*, i.e., $\dim_\mathbb{Q} B^n < \infty$ for all $n \geq 0$ (equivalently, $\dim_\mathbb{Q} \pi^n(B) < \infty$). Further, $\mathbb{Q}\mathsf{Nil}_*^{\mathrm{ft}}$ denotes the category of cohomologically connected spaces X that are

- rational as simplicial sets, i.e., have rational homotopy groups,
- of finite type, i.e., $\dim_\mathbb{Q} H^n(X;\mathbb{Q}) < \infty$ for all $n \geq 0$, and
- nilpotent, i.e., $\pi_1(X)$ is nilpotent and acts nilpotently on $\pi_k(X)$ (via the universal cover of X) for all $k \geq 2$.

A group G is *nilpotent* if the sequence

$$G_0 \equiv G$$
$$G_{n+1} \equiv [G, G_n]$$

of groups becomes trivial for some n. Further, for any groups G and A with G acting on A, we say that G *acts nilpotently* on A if the sequence

$$A_0 \equiv A$$
$$A_{n+1} \equiv \langle ga - a \mid g \in G, a \in A_n \rangle$$

of groups becomes trivial for some n.

Example 5.6.1.

1. S^1 is nilpotent.
2. Any simply connected space is nilpotent.
3. $\mathbb{RP}^2$ is not nilpotent.
4. $S^1 \vee S^1$ is not nilpotent.

5.7 Lecture 33

Recall the main theorem of RHT. The pair $(\mathcal{F}, \mathcal{A})$ is a Quillen adjunction, which is actually a Quillen equivalence for

$$\mathsf{dgca}_{\mathbb{Q}}^{\geq 0,\mathrm{ft},\text{h-conn.}} \rightleftarrows \left(\mathsf{sSet}_{\mathbb{Q}}^{\mathrm{ft},\mathrm{nilp}}\right)^{\mathrm{op}}$$

For spaces, we may combine this equivalence with the Quillen equivalence

$$\left(\mathsf{sSet}_{\mathbb{Q}}^{\mathrm{ft},\mathrm{nilp}}\right)^{\mathrm{op}} \underset{\mathrm{Sing}_\bullet}{\overset{|-|}{\rightleftarrows}} \left(\mathsf{WHCG}_{\mathbb{Q}}^{\mathrm{ft},\mathrm{nilp}}\right)^{\mathrm{op}}$$

induced by the equivalence between $\mathsf{sSet}^{\mathrm{op}}$ and $\mathsf{WHCG}^{\mathrm{op}}$.

The rationalization of a space (in the sense of Definition 1.3.3) always exists.

Lemma 5.7.1. *The counit $K \mapsto \mathcal{F}\mathcal{A}K$ of $\mathcal{F} \dashv \mathcal{A}$ is a rationalization.*

Recall that if $(\mathcal{F} : \mathcal{C} \to \mathcal{D},\ \mathcal{A} : \mathcal{D} \to \mathcal{C})$ is any adjoint pair, then its *unit* is the natural transformation $\left\{B \xrightarrow{u_B} \mathcal{A}\mathcal{F}(B)\right\}_{B \in \mathrm{ob}\,\mathcal{C}}$ given by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(B, \mathcal{A}\mathcal{F}(B)) &\cong \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(B), \mathcal{F}(B)) \\ u_B \quad &\leftarrow\!\shortmid \quad \mathrm{id}_{\mathcal{F}(B)} \,. \end{aligned}$$

Likewise, the *counit* is the natural transformation $\left\{\mathcal{F}\mathcal{A}(K) \xrightarrow{c_K} K\right\}_{K \in \mathrm{ob}\,\mathcal{D}}$ given by

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}\mathcal{A}(K), K) &\cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{A}(K), \mathcal{A}(K)) \\ c_K \quad &\leftarrow\!\shortmid \quad \mathrm{id}_{\mathcal{A}(K)} \,. \end{aligned}$$

In our case, the adjunction counit is an isomorphism $\mathcal{F}\mathcal{A} \to \mathrm{id}_{\mathsf{sSet}^{\mathrm{op}}}$ on $\mathsf{sSet}^{\mathrm{op}}$, i.e., an isomorphism $\mathrm{id}_{\mathsf{sSet}} \to \mathcal{F}\mathcal{A}$ on sSet.

Recall that $\pi_n(\mathcal{F}(B)) \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n(B), \mathbb{Q})$ for all $n \geq 1$. Hence $H_n(\mathcal{F}(B); \mathbb{Z})$ is a $\mathbb{Q}$-vector space. It follows that the rational homotopy type of a nilpotent, finite-type space X is determined by the (quasi-)isomorphism class of its Sullivan minimal model.

Next, we'll study Quillen models via L_∞-algebras as well as dg-Lie algebras.

6 Quillen models

6.1 Lecture 34

Definition 6.1.1 (DGLA). A *dg-Lie algebra* is a chain complex $(L_\bullet \coloneqq \bigoplus_{n \in \mathbb{Z}} L_n, d : L_n \to L_{n-1})$, $d^2 = 0$, equipped with a *graded-Lie bracket* $[-,-] : L_\bullet \otimes L_\bullet \to L_\bullet$, $L_p \otimes L_q \to L_{p+q}$, such that

(1) (*graded-skew*) $[x, y] = -(-1)^{|x||y|}[y, x]$.

(2) (*graded-Jacobi*) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$.

(3) (*graded-Leibniz (derivation)*) $d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$.

We say that L is *connected* if $L = L_{>0}$.

Example 6.1.2. The following are dg-Lie algebras.

1. Any Lie algebra L, with $L = L_0$ and $d = 0$.

2. Any chain complex $(V_\bullet, d)$, with $[-,-] \equiv 0$. This is called an *abelian* dg-Lie algebra.

3. Any dg-associative algebra $(A_\bullet = \bigoplus_{n\in\mathbb{Z}} A_n, d)$, with $[a,b] \equiv ab - (-1)^{|a||b|} ba$.

   This defines a functor $(-)^{\mathsf{Lie}} : \mathsf{dga} \to \mathsf{dgla}$, which is right adjoint to the *universal enveloping algebra* $\mathcal{U} : \mathsf{dgla} \to \mathsf{dga}$.

$$\mathcal{U}L \equiv {T(L)}\big/{I}$$
$$T(L) \equiv \bigoplus_{n\geq 0} L^{\otimes n}, \quad I \equiv \left(x\otimes y - (-1)^{|x||y|} y\otimes x - [x,y] \mid x,y \in L\right)$$
$$(x_1\otimes\cdots\otimes x_p)(y_1\otimes\cdots\otimes y_q) \equiv x_1\otimes\cdots\otimes x_p\otimes y_1\otimes\cdots\otimes y_q, \ d \text{ extented as derivation from } L \text{ to } T(L)$$

Note 6.1.3.

1. There is a natural isomorphism $\mathcal{U}(L\oplus M) \cong \mathcal{U}(L)\otimes\mathcal{U}(M)$.

2. If L is an abelian DGLA, then $\mathcal{U}L = S(L)$. In particular, $\mathcal{U}(0) = k$.

3. The algebra $\mathcal{U}L$ is a dg-Hopf algebra. This means that it has extra structure coming from

   - $L \xrightarrow{x\mapsto(x,x)} L\oplus L \quad\Longrightarrow\quad \underbrace{\mathcal{U}L \xrightarrow{\Delta} \mathcal{U}L\otimes\mathcal{U}L}_{comultiplication}$,
   - $L \to 0 \quad\Longrightarrow\quad \underbrace{\mathcal{U}L \to k}_{counit}$, and
   - $\underbrace{s : L \xrightarrow{x\mapsto -x} L}_{antihomomorphism} \quad\Longrightarrow\quad \underbrace{S : \mathcal{U}L \to \mathcal{U}L}_{antipode}$.

   Note that

$$S(a\cdot b) = (-1)^{|a||b|}(Sb\cdot Sa).$$

   Also, note that

$$\begin{aligned}[x,y] \xmapsto{s} -[x,y] &= (-1)^{|x||y|}[y,x]\\ &= (-1)^{|x||y|}[-y,-x]\\ &\neq -(-1)^{|x||y|}[y,x]\\ &= [x,y]\\ &= [-x,-y].\end{aligned}$$

   Hence $s[x,y] = (-1)^{|x||y|}[sy,sx]$.

This structure makes $\mathcal{U}L$ a dg-associative algebra with a compatible dg-coassociative counital coalgebra. It's coassociative as

$$\begin{tikzcd} \mathcal{U}L \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathcal{U}L \otimes \mathcal{U}L \arrow[d, "\Delta\otimes \mathrm{id}"] \\ \mathcal{U}L \otimes \mathcal{U}L \arrow[r, "\mathrm{id}\otimes\Delta"] & \mathcal{U}L \otimes \mathcal{U}L \otimes \mathcal{U}L \end{tikzcd}$$

commutes. For comparison, multiplication is associative when

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

commutes. Moreover, Δ is cocommutative as

$$\begin{tikzcd} \mathcal{U}L \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & \mathcal{U}L \otimes \mathcal{U}L \arrow[d, "T"] \\ & \mathcal{U}L \otimes \mathcal{U}L \end{tikzcd}$$

commutes where $T(a \otimes b) \equiv (-1)^{|a||b|}\, b \otimes a$.

6.2 Lecture 35

We continue studying $\mathcal{U}L$.

Note 6.2.1. Let $M, L \in \operatorname{ob} \mathsf{dgla}$.

1. We have a morphism $L \otimes M \xrightarrow{T} M \otimes L$ defined by $(x, y) \mapsto (y, x)$ (no sign). This fits into the commutative square

$$\begin{tikzcd} \mathcal{U}(L \oplus M) \arrow[r, dash, "\cong"] \arrow[d, "\mathcal{U}T"'] & \mathcal{U}L \otimes \mathcal{U}M \arrow[d, "T"] \\ \mathcal{U}(M \oplus L) \arrow[r, dash, "\cong"'] & \mathcal{U}M \otimes \mathcal{U}L \end{tikzcd}$$

   where $V \otimes W \xrightarrow{T} W \otimes V$ is given by

$$v \otimes w \mapsto \overbrace{(-1)^{|v||w|}}^{\textit{Koszul sign}}\, w \otimes v$$

   for any k-dg-vector spaces V and W.

2. The direct sum $L \oplus M$ is the categorical *product*, not coproduct. The isomorphism $\mathcal{U}(L \oplus M) \cong \mathcal{U}L \otimes \mathcal{U}M$ is proven directly from the definition ${T(L)}\diagup_{I}$ of $\mathcal{U}L$.

Let C be a dg-Hopf algebra, with coproduct $C \xrightarrow{\Delta} C \otimes C$ and product $C \otimes C \to C$. Consider the set of *primitive elements*

$$\mathcal{P}(C) \equiv \{x \in C \mid \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

This is a dg-vector subspace of C. It's also a dg-Lie subalgebra of C^{Lie} when equipped with the bracket $[x, y] \equiv xy - (-1)^{|x||y|} yx$. This gives rise to a functor $\mathcal{P} : \mathsf{dgha} \to \mathsf{dgla}$.

Theorem 6.2.2 (Milnor-Moore). *The diagram*

$$\mathsf{dgla}^{\geq 0}_{comm.} \underset{\mathcal{P}}{\overset{\mathcal{U}}{\rightleftarrows}} \mathsf{dgcha}^{\geq 0}_{conn.}$$

is an equivalence of categories where dgcha *denotes the category of dg-cocommutative Hopf algebras.*

Corollary 6.2.3. *For any dg-Hopf algebra H and any dg-Lie algebra L, we have isomorphisms*

$$H \xrightarrow{\sim} \mathcal{UP}(H)$$
$$L \xrightarrow{\sim} \mathcal{P}(\mathcal{U}L).$$

Let V be an abelian DGLA, so that $\mathcal{U}V = S(V)$. This is a dg-Hopf algebra with *shuffle* comultiplication.

Theorem 6.2.4 (PBW). *Let L be a DGLA. Define $\gamma : S(L) \to \mathcal{U}L$ by*

$$x_1 x_2 \cdots x_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \overbrace{(-1)^{\epsilon}}^{\substack{Koszul\\ sign}} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.$$

This is an isomorphism of dg-coalgebras.

Let's turn to topology. Let X be a simply connected pointed space. Recall the based loop space: $\Omega X = \mathrm{Map}(S^1, X)_0$. The graded-vector space $H_\bullet(\Omega X; \mathbb{Q})$ is a graded, cocommutative, connected Hopf algebra. Indeed, concatenation $\Omega X \times \Omega X \to \Omega X$ of loops makes ΩX a homotopy monoid. This structure induces multiplication on $H_\bullet(\Omega X; \mathbb{Q})$, namely the Pontryagin product. Further, the diagonal map $\Omega X \to \Omega X \times \Omega X$ induces coassociative and cocommutative comultiplication on $H_\bullet(\Omega X; \mathbb{Q})$.

Theorem 6.2.5 (Cartan-Serre). *The Hurewicz map $\pi_\bullet^{\mathbb{Q}}(\Omega X) \to H_\bullet(\Omega X; \mathbb{Q})$ induces an isomorphism*

$$\pi_\bullet^{\mathbb{Q}}(\Omega X) \xrightarrow{\sim} \mathcal{P}(H_\bullet(\Omega X; \mathbb{Q}))$$

of graded-Lie algebras.

(The commutator bracket on $\mathcal{P}H_\bullet(\Omega X; \mathbb{Q})$ is also known as the Pontryagin product.)

6.3 Lecture 36

Consider again a simply connected pointed space X. Then $H_\bullet(\Omega X; \mathbb{Q})$ is a graded-cocommutative Hopf algebra. We have an isomorphism $H_\bullet(\Omega X; \mathbb{Q}) \cong \mathcal{UP}(H_\bullet(\Omega X; \mathbb{Q}))$ of graded-Hopf algebras. It follows from Theorem 6.2.5 that

$$H_\bullet(\Omega X; \mathbb{Q}) \underset{\substack{\text{gr.}\\ \text{Hopf}\\ \text{alg.}}}{\cong} \mathcal{U}(\pi_\bullet^{\mathbb{Q}}(\Omega X)). \tag{†}$$

We still need to specify the graded-Lie algebra structure of $\pi_\bullet^{\mathbb{Q}}(\Omega X)$. Suppose that X is any pointed space. Consider the commutator map $[-,-] : \Omega X \times \Omega X \to \Omega X$, defined by

$$(\alpha, \beta) \mapsto \alpha * \beta * \alpha^{-1} * \beta^{-1}.$$

Let $f : S^p \to \Omega X$ and $g : S^q \to \Omega X$ be pointed maps. Define $[f, g]$ as the composite

$$\begin{tikzcd} S^p \times S^q \arrow[r, "f\times g"] \arrow[dr, "{[f,g]}"'] & \Omega X \times \Omega X \arrow[d, "{[-,-]}"] \\ & \Omega X \end{tikzcd}.$$

Note that $[f,g]\restriction_{S^p\vee S^q}$ is null-homotopic. By the universal property of groups, $[f,g]$ is homotopic to a map $S^{p+q}\to\Omega X$, where $S^{p+q}\cong S^p\wedge S^q\equiv\frac{S^p\times S^q}{S^p\vee S^q}$. This map is called the *Samelson product* of f and g. This gives us a product

$$\pi_p^{\mathbb{Q}}(\Omega X)\otimes\pi_q^{\mathbb{Q}}(\Omega X)\xrightarrow{[-,-]}\pi_{p+1}^{\mathbb{Q}}(\Omega X).$$

Since $\pi_n^{\mathbb{Q}}(\Omega X)\cong\pi_{n+1}^{\mathbb{Q}}(X)$, we actually have a product

$$\pi_{p+1}^{\mathbb{Q}}(X)\otimes\pi_{q+1}^{\mathbb{Q}}(X)\xrightarrow{[-,-]}\pi_{p+q+1}^{\mathbb{Q}}(X),$$

called the *Whitehead product*. This is a graded-Lie bracket on $\pi_\bullet^{\mathbb{Q}}(X)[1]$.

Example 6.3.1. Consider the sphere S^n with $n\geq 2$. We have that

$$\pi_{\bullet+1}^{\mathbb{Q}}(S^n)=\pi_\bullet^{\mathbb{Q}}(\Omega S^n)\cong\mathbb{L}(\alpha),$$

i.e., the free graded-Lie algebra on one generator. Here, α belongs to $\pi_{n-1}(\Omega S^n)$ and satisfies

$$\pi_\bullet^{\mathbb{Q}}(\Omega S^n)=\pi_{\bullet+1}^{\mathbb{Q}}(S^n)=\begin{cases}\mathbb{Q}\alpha & n\text{ odd}\\ \mathbb{Q}\alpha\oplus\mathbb{Q}[\alpha,\alpha] & n\text{ even}\end{cases}.$$

It follows that $H_\bullet(\Omega S^n;\mathbb{Q})\cong\mathcal{U}\mathbb{L}(\alpha)\cong\mathbb{T}(\alpha)$, i.e., the free graded-associative algebra on one generator.

6.4 Lecture 37

We want to consider a chain-level version of a graded-Hopf algebra.

Let X be a pointed, simply connected space. With (†) along with the fact that $H_\bullet(\Omega X;\mathbb{Q})\cong H^\bullet(\Omega X;\mathbb{Q})$, we can study the algebra $\pi_\bullet^{\mathbb{Q}}(X)[1]\cong\pi_\bullet^{\mathbb{Q}}(\Omega X)$ by looking at cochain-level enhancements of the graded-commutative algebra $H^\bullet(\Omega X;\mathbb{Q})$. This leads us to the cochain complex $(\Omega^\bullet_{\mathsf{PL}}(Y),d)$ of a space Y, reminiscent of the de Rham complex $(\Omega^\bullet(X),d_{\mathrm{dR}})$ when Y is a manifold.

Alternatively, one may try to take $(C_\bullet(\Omega X;\mathbb{Q}),d)$ the singular chain complex of ΩX. Let

$$\Omega^M X=\{(t,f)\mid t\in\mathbb{R},\ t>0,\ f:[0,t]\to X,\ f(0)=f(t)=x_0\}.$$

Multiplication by concatenation $\Omega^M X\times\Omega^M X\to\Omega^M X$ is strictly associative. This induces graded-associative multiplication

$$\begin{tikzcd} C_\bullet(\Omega^M X)\otimes C_\bullet(\Omega^M X) \arrow[r] \arrow[d, "EZ"'] & C_\bullet(\Omega^M X) \\ C_\bullet(\Omega^M X\times\Omega^M X) \arrow[ur] & \end{tikzcd}$$

at the chain level. Further, the diagonal map $\Omega^M X\xrightarrow{\Delta}\Omega^M X\times\Omega^M X$ induces a coassociative operation

$$C_\bullet(\Omega^M X)\to C_\bullet(\Omega^M X\times\Omega^M X)\xrightarrow{AW}C_\bullet(\Omega^M X)\otimes C_\bullet(\Omega^M X),$$

so that $C_\bullet(\Omega^M X)$ is a graded coassociative algebra as well. This, however, is *not* cocommutative, because the AW map breaks symmetry. Therefore, $C_\bullet(\Omega X)$ is a Hopf algebra but not cocommutative. In fact, there is no isomorphism

$$C_\bullet(\Omega^M X;\mathbb{Q}) \ncong \mathcal{U}(\mathcal{P}C_\bullet(\Omega^M X;\mathbb{Q}))$$

in general. Also, there is no isomorphism

$$H_\bullet(\mathcal{P}C_\bullet(\Omega^M X;\mathbb{Q})) \ncong \underbrace{\mathcal{P}H_\bullet(\Omega^M X;\mathbb{Q}) \cong \pi_\bullet^{\mathbb{Q}}(\Omega^M X)}_{Theorem\ 6.2.5}$$

of dg-Lie algebras.

Still, our goal is to associate a dg-Lie algebra to a given space. Quillen's solution is to work simplicially. Let K be a pointed simplicial set. Suppose that it's 1-*reduced*, i.e., its 1-skeleton equals the basepoint $*$. The *Quillen model* of K is the dg-Lie algebra

$$\lambda(K) \equiv N_\bullet \mathcal{P}\hat{\mathbb{Q}}[GK].$$

Here, GK denotes Kan's *simplicial loop group* of K, a simplicial version of ΩX. It's the simplicial group defined by

$$\begin{aligned} G_n K &= \mathrm{Fr}(K_{n+1}),\ \text{free group on } K_{n+1} \\ s_0 x &= e_n,\quad x \in K_n,\ e_n \text{ identity of } \mathrm{Fr}(K_{n+1}) \end{aligned}$$

for all $n \geq 0$. See May's *Simplicial Objects in Algebraic Topology.*

Further, $\mathbb{Q}[GK]$ denotes the group (simplicial) algebra of GK over $\mathbb{Q}$, where $\mathbb{Q}[G_n K] = \mathbb{Q}\cdot G_n K$. This is automatically a simplicial cocommutative Hopf algebra.

$$g \mapsto g \otimes g, \qquad g \in G_n K$$

Its counit serves as its augmentation. Now, $\hat{\mathbb{Q}}[GK]$ is the completion of this algebra w.r.t. the augmentation ideal I:

$$\lim_n \frac{\mathbb{Q}[GK]}{I^n}.$$

This is also a simplicial cocommutative Hopf algebra. Then the primitives $\mathcal{P}\hat{\mathbb{Q}}[GK]$ form a simplicial Lie algebra.

Finally, $N_\bullet \mathcal{P}\hat{\mathbb{Q}}[GK]$ stands for the *normalized chain complex*, i.e., the quotient of the simplicial chain complex by all degenerate simplices. This is a dg-Lie algebra.

6.5 Lecture 38

We have defined a functor $\lambda : \mathsf{sSet}_{\text{1-reduced}} \to \mathsf{dgla}_{\text{conn.}}$. We also have the *Maurer-Cartan* functor $\mathrm{MC} : \mathsf{dgla} \to \mathsf{sSet}_{\text{1-reduced}}$ (which mods out by K_1 first). With an appropriate model structure on dgla, the pair (λ, MC) is a Quillen adjunction. (The equivalences are precisely the quasi-isomorphisms of DGLA's.) When restricted to

$$\mathsf{sSet}^{\mathbb{Q}}_{\text{1-conn.}} \underset{\mathrm{MC}}{\overset{\lambda}{\rightleftarrows}} \mathsf{dgla}_{\text{conn.}} \ ,$$

the pair is a Quillen equivalence. For any DGLA g,

$$\begin{aligned}\mathrm{MC}(g) \coloneqq \mathrm{MC}(g \otimes_{\mathbb{Q}} \Omega^{-\bullet}_{\bullet}) &\equiv \left\{x \in (g \otimes_{\mathbb{Q}} \Omega^{-\bullet}_{\bullet})_1 \mid dx + \frac{1}{2}[x,x] = 0\right\} \\ &= \mathrm{Hom}_{\mathsf{dgcc}}((\Omega^{\bullet})^*, \mathrm{CE}(g)) \\ &= \mathrm{Hom}_{L_\infty}(L_\infty(\Omega^{\bullet})^*, g), \\ &\text{(as long as we think of } (\Omega^{\bullet})^* \text{ as a coalgebra, though it's not quite one)}\end{aligned}$$

where $\Omega^{-\bullet}_{\bullet}$ denotes the finite-type, simplicial DGLA of polynomial differential forms on simplices.

Note 6.5.1.

1. The tensor product of any two DFLA's is a DGLA with

$$\begin{aligned} d(\alpha \otimes f) &= d\alpha \otimes f \pm \alpha \otimes df \\ [\alpha \otimes f, \beta \otimes g] &= \pm [\alpha, \beta] \otimes fg. \end{aligned}$$

2. An algebra structure on a space need not induce a coalgebra structure on its dual space. For any k-algebra A, the dual space $A^{\bullet} \equiv \mathrm{Hom}(A, k)$ is a coalgebra when the map

$$A^* \otimes A^* \to (A \otimes A)^*$$

is an isomorphism of algebras. (It's always an isomorphism of finite-dimensional spaces.) In this case, $A \otimes A \xrightarrow{m} A$ induces comultiplication $A^{\bullet} \xrightarrow{m^*} (A \otimes A)^*$.

By contrast, a coalgebra structure $C \xrightarrow{\Delta} C \otimes C$ on a space C always induces an algebra structure $C^* \otimes C^* \to (C \otimes C)^* \xrightarrow{\Delta^*} C^*$ on C^*.

We need to introduce L_∞-minimal models of DGLA's. Quillen's original approach was based on free graded Lie algebras or semifree DGLA's. But free Lie algebras are awkward. Nowadays, we see that it's better to work with minimal L_∞-models. See "L_∞ rational homotopy of mapping spaces" by Buijs, Félix, and Murillo.

Define a mapping $\mathsf{dgla} \to \mathsf{dgcc}$ by

$$(\mathfrak{g}, d) \mapsto \mathrm{CE}_{\bullet}(\mathfrak{g}) \equiv (S(\mathfrak{g}[-1]), D)\,,$$

where $D(1) = 0$, $|D| = -1$, and $D^2 = 0$. This stands for *Chevalley-Eilenberg*. It is a graded cocommutative coalgebra under *shuffle* comultiplication:

$$\begin{aligned} \Delta &: S(\mathfrak{g}[-1]) \to S(\mathfrak{g}[-1]) \otimes S(\mathfrak{g}[-1]) \\ \Delta(x_1 \otimes \cdots \otimes x_n) &\equiv \sum_{p=0}^{n} \sum_{\sigma \in \mathrm{Sh}(p,n-p)} \left(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(p)}\right) \otimes \left(x_{\sigma(p+1)} \otimes \cdots \otimes x_{\sigma(n)}\right), \quad x_i \in \mathfrak{g}[-1]\,. \end{aligned}$$

Moreover, D is a *coderivation* in the sense that

$$\begin{tikzcd} S(\mathfrak{g}[-1]) \arrow[rr, "D"] \arrow[d, "\Delta"'] & & S(\mathfrak{g}[-1]) \arrow[d, "\Delta"] \\ S(\mathfrak{g}[-1]) \otimes S(\mathfrak{g}[-1]) \arrow[rr, "D\otimes \mathrm{id} + \mathrm{id}\otimes D"'] & & S(\mathfrak{g}[-1]) \otimes S(\mathfrak{g}[-1]) \end{tikzcd}$$

commutes. We have defined D by

$$D(x_1 \cdots x_n) = \underbrace{\sum_i \pm x_1 \cdots dx_i \cdots x_n}_{D_1(x_1\cdots x_n)} + \underbrace{\sum_{i<j} \Big(\pm \overbrace{[x_i, x_j]}^{d} x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_n \Big)}_{D_2(x_1\cdots x_n)}.$$

Now, let $\mathfrak{g}$ be a graded vector space and $D : S(\mathfrak{g}[-1]) \to S(\mathfrak{g}[-1])$ be any differential on a cofree DGCC. We may write

$$D = D_1 + D_2 + \cdots$$

where D_n is the composite linear map $S^n(\mathfrak{g}[-1]) \xrightarrow{D} S(\mathfrak{g}[-1]) \xrightarrow{proj.} \mathfrak{g}[-1]$. Note that $D_n = 0$ for all $n \geq 3$ if and only if $\mathfrak{g}$ is a DGLA. In our case, therefore, $D = D_1 + D_2$.

Define the *higher Lie* (or L_∞-) *bracket* by

$$[x_1, \ldots, x_n] = D_n(x_1 \cdots x_n) \in \mathfrak{g}, \qquad x_i \in \mathfrak{g}$$

for each $n \geq 2$, where $x_1 \cdots x_n$ is a product in the symmetric algebra (i.e., the symmetric tensor product). This gives us an L_∞-algebra.

6.6 Lecture 39

Let $\mathfrak{g}$ be a graded vector space over a field k of characteristic 0. Then $(S(\mathfrak{g}[-1]), D)$ is a graded cocommutative coaugmented (i.e., equipped with a coalgebra map $k \to S(\mathfrak{g}[-1])$) coalgebra under the shuffle coproduct, i.e., a cofree object in the category of coaugmented graded cocommutative algebras. Recall that D is a coderivation of degree -1. In general, this amounts to a family of linear maps $\{D_n : S^n(\mathfrak{g}[-1]) \to \mathfrak{g}[-1] \mid n \geq 1\}$ of degree -1 (or linear maps $D_n : S^n\mathfrak{g} \to \mathfrak{g}[n-2]$ of degree 0) where

$$D = D_1 + D_2 + D_3 + \cdots + D_n + \cdots.$$

Definition 6.6.1 (L_∞-algebra). An L_∞*-algebra* structure on a graded vector space $\mathfrak{g}$ is a differential D on the coaugmented coalgebra $S(\mathfrak{g}[-1])$.

Note that $D^2 = 0$ iff $(D_1 + D_2 + \cdots)^2 = 0$ iff

$$D_1^2 = 0 \qquad (d \equiv D,\ d^2 = 0)$$

$$d[x_1, x_3] = [dx_1, x_2] \pm [x_1, dx_2]$$

$$d[x_1, x_2, x_3] - [dx_1, x_2, x_3] \mp [x_1, dx_2, x_3] \mp [x_1, x_2, dx_3] = [[x_1, x_2], x_3] \pm [[x_2, x_3], x_1] \pm [[x_3, x_1], x_2]$$

(graded Jacobi identity up to homotopy, i.e., graded Jacobi identity for $H_\bullet(\mathfrak{g}, d)$)

... higher Jacobi identities.

Definition 6.6.2. Let $\mathfrak{g}$ and $\mathfrak{h}$ be L_∞-algebras.

1. An L_∞*-morphism* from $\mathfrak{g}$ to $\mathfrak{h}$ is a map

$$f : (S(\mathfrak{g}[-1]), D_{\mathfrak{g}}) \to (S(\mathfrak{h}[-1]), D_{\mathfrak{h}})$$

of coaugmented coalgebras that respects the differential graded structures:

$$f = \mathrm{id}_{S^0} + f_1 + f_2 + \cdots \qquad \text{(“Taylor expansion” of } f\text{)}$$

$$\underbrace{f_n : S^n(\mathfrak{g}[-1]) \to \mathfrak{h}[-1]}_{\text{linear degree-0 map}}$$

$$f \circ D_g = D_h \circ f.$$

2. An L_∞*-quasi-isomorphism* from $\mathfrak{g}$ to $\mathfrak{h}$ is an L_∞-morphism $\mathfrak{g} \to \mathfrak{h}$ such that

$$f_1 : (\mathfrak{g}[-1], d_{\mathfrak{g}}) \to (\mathfrak{h}[-1], d_{\mathfrak{h}})$$

is a quasi-isomorphism of complexes.

By definition, L_∞-quasi-isomorphisms are precisely the weak equivalences for L_∞-algebras.

Definition 6.6.3.

1. An L_∞-algebra is *minimal* if $d = 0$.

2. An L_∞-algebra is *linearly contractible* if $D_n = 0$ for all $n \geq 2$.

Theorem 6.6.4 (Kontsevich). *Every L_∞-algebra is L_∞-isomorphic to a direct sum of the form*

$$\mathfrak{g}_{min.} \oplus \mathfrak{g}_{contr.}.$$

Note 6.6.5. The algebra $S(\mathfrak{g}[-1])$ gives rise to the DGCA

$$S(\mathfrak{g}[-1])^* \equiv \hat{S}(\mathfrak{g}[-1]^*),$$

with differential $D^* = D_1^* + D_2^* + \cdots,\ \ D_n^* : \mathfrak{g}[-1]^* \to S^n(\mathfrak{g}[-1]^*)$. We have that

$$D_1 = 0 \iff D_1^* = 0 \underset{\substack{g_n = 0\\ n \leq 0}}{\iff} S(\mathfrak{g}[-1])^* \text{ minimal as DGCA.}$$

6.7 Lecture 40

To conclude, let's discuss the L_∞-model of rational homotopy types.

On the one hand, recall Quillen's approach. For any space X, take the simplicial set $\mathrm{Sing}_\bullet(X)$ and then the dg-Lie algebra $\lambda(X) \coloneqq \lambda(K)$. We have an isomorphism

$$H_\bullet(\lambda(X)) \cong \underbrace{\pi_\bullet^{\mathbb{Q}}(\Omega X)}_{\pi_\bullet^{\mathbb{Q}}[1]}$$

of graded Lie algebras. The homotopy theory of DGLA's has quasi-isomorphisms as weak equivalences, and minimal Quillen DGLA models are based on semifree DGLA's.

On the other hand, consider the modern approach. Here the homotopy theory of DGLA's corresponds to the homotopy theory of L_∞-algebras. The L_∞-quasi-isomorphisms are the weak equivalences, and the minimal L_∞-models are those L_∞-algebras with $d = 0$. For any L_∞-algebra $\mathfrak{g}$, Theorem 6.6.4 gives us a decomposition

$$\mathfrak{g} \cong \mathfrak{g}_{\text{min.}} \oplus \mathfrak{g}_{\text{contr.}}.$$

Note that

$$\begin{aligned} H_\bullet(\mathfrak{g}_{\text{min.}}, d) &= \mathfrak{g}_{\text{min.}} \\ H_\bullet(\mathfrak{g}_{\text{contr.}}, d) &= 0. \end{aligned}$$

In particular,

$$H_\bullet(\lambda(X)) = H_\bullet(\lambda(X)_{\text{min.}}) = \lambda(X)_{\text{min.}}.$$

Terminology.

1. We call $\lambda(X)_{\text{min.}}$ the *L_∞-minimal model* of X.
2. We call D_2 the *Whitehead product*.
3. We call $D_3, D_3, D_4 \ldots$ *higher Whitehead products*.

Corollary 6.7.1. *The map $\mathfrak{g}_{min.} \hookrightarrow \mathfrak{g}$ is an L_∞-quasi-isomorphism.*

Theorem 6.7.2. *An L_∞-quasi-isomorphism between two minimal L_∞-algebras is an L_∞-isomorphism.*

Definition 6.7.3. A *minimal L_∞-model* of an L_∞-algebra $\mathfrak{g}$ is a minimal L_∞-algebra $\mathfrak{g}'$ along with an L_∞-quasi-isomorphism $\mathfrak{g}' \to \mathfrak{g}$.

It turns out that there is a one-to-one correspondence

$$\text{rational homotopy types of 1-connected spaces} \quad \leftrightarrow \quad L_\infty\text{-isomorphism classes of minimal } L_\infty\text{-algebras.}$$

Finally, as long as X has finite type, we obtain a Sullivan theory for L_∞-algebras via dualization. Specifically, consider the coaugmented DGCC

$$\mathrm{CE}_\bullet(\lambda(X)_{\text{min.}}) = (S(\lambda(X)_{\text{min.}}[-1]), D).$$

We say that its dual $\mathrm{CE}_\bullet(\lambda(X)_{\text{min.}})^*$ is the *Sullivan minimal model* of X.