

Abstract

These notes are based on Ron Donagi's lectures for the course "Complex Algebraic Geometry" at UPenn along with Daniel Huybrechts's *Complex Geometry*. Any mistake in what follows is my own.

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1 A cursory overview of algebraic geometry

1.1 Lectures 1-4

These lectures consisted of informal surveys of certain fundamental concepts of algebraic geometry. They were meant as previews of various topics that we will cover rigorously. The following problems give a taste of the initial concepts presented.

Projective space.

Recall that projective space \mathbb{P}^n of dimension n is by definition the quotient of $\mathbb{A}^{n+1} \setminus \{0\}$ by the rescaling action of $\text{GL}(1)$. This is a compactification of n -dimensional affine space \mathbb{A}^n . Show that \mathbb{P}^n has an open covering by $n+1$ many open subsets U_0, \dots, U_n each isomorphic to \mathbb{A}^n . Work out all of the transition functions.

For simplicity, assume that $\mathbb{A}^n = \mathbb{R}^n$. For each $i \in \{1, \dots, n+1\}$, let $\widetilde{U}_i = \{\vec{x} \in \mathbb{R}^{n+1} : x_i \neq 0\}$. Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$ denote the natural projection and let $U_i = \pi(\widetilde{U}_i)$. Since \widetilde{U}_i is saturated and open, we know that $\pi|_{\widetilde{U}_i}$ is a quotient map. Define $f_i : U_i \rightarrow \mathbb{R}^n$ by $[x_1, \dots, x_{n+1}] \mapsto \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right)$, whose inverse is given by $(x_1, \dots, x_n) \mapsto [x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n]$. Since $f_i \circ \pi$ is continuous, so is f_i . Hence we see that f_i is a homeomorphism.

Now, consider any transition map $f_i \circ f_j^{-1}$. Let $1 \leq i, j \leq n+1$. If $i = j$, then the transition map is the identity, which is smooth. So assume, wlog, that $j < i$. Then

$$f_j \circ f_i^{-1}(x_1, \dots, x_n) = \left(\frac{x_1}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, \dots, \frac{x_n}{x_j}\right)$$

when $(x_1, \dots, x_n) \in f_i(\widetilde{U}_i \cap \widetilde{U}_j)$. This is clearly smooth. ■

Classification of conics.

Classify, up to linear automorphism, all conics (i.e., 1-dimensional quadratic hypersurfaces) in \mathbb{RP}^2 .

First of all, we may assume that any conic $C(x, y, z) = 0$ in \mathbb{RP}^2 is a homogeneous polynomial of degree two. As a result, we can write $C(x, y, z) = v^t A v$ for some symmetric 3×3 matrix A over \mathbb{R} where

$$v := \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Since A is orthogonally diagonalizable, we have that $C(v) = (Tv)^t D(Tv)$ for some orthogonal matrix T and some diagonal matrix D . Thus, C is isomorphic to a curve in \mathbb{RP}^2 defined by $0 = a_1 x^2 + a_2 y^2 + a_3 z^3$. We can apply another invertible change of variables to ensure that $a_i \in \{-1, 0, 1\}$ for each $i = 1, 2, 3$.

Likewise, any quadratic hypersurface in \mathbb{CP}^n is isomorphic to a curve in \mathbb{CP}^n defined by $\sum_{i=1}^{n+1} a_i x_i^2 = 0$ where $a_i \in \{-1, 0, 1\}$ for each $i = 1, \dots, n+1$.

We now see that there are exactly five distinct automorphism classes of conics in \mathbb{RP}^2 :

- (a) $[x^2 = 0]$ (a double line),

- (b) $[x^2 + y^2 = 0]$ (a single point),
- (c) $[x^2 - y^2 = (x - y)(x + y) = 0]$ (the union of two distinct lines that intersect at a single point),
- (d) $[x^2 + y^2 + z^2 = 0]$ (the empty set), and
- (e) $[x^2 + y^2 - z^2 = 0]$ (a curve C whose affine part is either an ellipse, a parabola, or a hyperbola, depending on whether C intersects the line at infinity at zero, one, or two points, respectively).

■

Plane cubics.

Recall that a plane curve C is *smooth* if it has no *singular* points, i.e., points at which all first-order partial derivatives of C vanish.

1. Consider a smooth plane cubic X with an inflection point, i.e., a non-singular point at which the tangent line to X has intersection multiplicity at least 3 with X . Show that X is isomorphic, up to linear automorphism of the ambient space, to a smooth plane cubic in *Weierstrass form* $y^2 = x^3 + ax + b$.
2. Find an expression of the form $J := \frac{a^m}{b^n}$ for some integers m and n that is invariant under linear changes of coordinates sending our equation of X to another Weierstrass form.
3. A slight problem with J is that it goes to infinity when $b = 0$ even though the plane cubic X remains smooth. Can you suggest another function $j(a, b)$ which has the same invariance property as J but remains finite (i.e. is holomorphic) whenever X is smooth? Relate j to J .

1. Any linear automorphism of \mathbb{C}^3 induces a projective transformation of \mathbb{CP}^2 (i.e., an element of $\text{PGL}(3, \mathbb{C})$), and the group of all linear automorphisms of \mathbb{C}^3 acts transitively on the set of all 2-dimensional subspaces of \mathbb{C}^3 . It follows that for any two lines $L_1, L_2 \subset \mathbb{CP}^2$, there is some projective transformation $\psi : \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ such that $\psi(L_1) = L_2$. In particular, we can assume that the tangent line L to our given inflection point P is precisely the line at infinity $\{z = 0\}$ and that any other line touching P is precisely the line $\{x = 0\}$. This means that P corresponds to the point $[0, 1, 0]$ on our new, isomorphic copy X' of X . Note that X' is smooth since any projective transformation preserves smoothness by the familiar multivariable chain rule.

Let X' be the zero locus of the homogeneous polynomial $f(x, y, z)$ of degree three. This has the form

$$Ax^3 + Bxy^2 + Cyx^2 + Dy^3 + zg(x, yz)$$

where $g(x, y, z)$ denotes a homogenous polynomial of degree two. We have that $d = 0$ because $[0, 1, 0]$ lies on X' . Further, we have that $b = c = 0$ because $\{z = 0\}$ is tangent to an inflection point. Therefore, the affine part of X' has the form

$$f(x, y, 1) = Ax^3 + Bx^2 + Cxy + Dy^2 + Ex + Fy + G$$

where both A and D are nonzero. By scaling this appropriately, we may write

$$f(x, y, 1) = x^3 + Bx^2 + Cxy - y^2 + Ex + Fy + G$$

for some $B, C, E, F, G \in \mathbb{C}$.

It remains to show that there exists an invertible affine map (or change of variables) $\mathbb{C}^2 \rightarrow \mathbb{C}^2$ that sends the zero locus of $f(x, y, 1)$ to the zero locus of a cubic polynomial in Weierstrass form. To this end, first define the mapping $f : (x, y) \mapsto \left(x, \frac{y+Cx+F}{2}\right)$. A tedious computation shows that φ transforms the affine part of X' to the curve X'' given by $\frac{y^2-F^2-C^2x^2}{4} - \frac{CFx}{2} = x^3 + Bx^2 + Ex + G$, i.e.,

$$y^2 = 4x^3 + (C^2 + 4B)x^2 + (4E + 2CF)x + (4G + F^2).$$

Substituting $2y$ for y , we can write X'' as

$$y^2 = x^3 + \underbrace{\left(\frac{C^2}{4} + B\right)}_{k_1} x^2 + \underbrace{\left(E + \frac{CF}{2}\right)}_{k_2} x + \underbrace{\left(G + \frac{F^2}{4}\right)}_{k_3}.$$

Next, define the mapping $g : (x, y) \mapsto \left(x - \frac{k_1}{3}, y\right)$. By another tedious computation, we see that g sends X'' to the curve given by

$$y^2 = x^3 + \left(k_2 - \frac{(k_1)^2}{3}\right)x + \left(k_3 + \frac{2(k_1)^3}{27} - \frac{k_1 k_2}{3}\right).$$

Since f and g are invertible affine maps, we are done.

2. Note that any linear change of variables preserving Weierstrass form must look like $(x, y) \mapsto (r^2x, r^3y)$ with $r \neq 0$. Let $J = \frac{A^3}{B^2}$. It's straightforward to check that J is invariant under any affine map of the form $(x, y) \mapsto (r^2x, r^3y)$.
3. Let $j = (27)(4^4) \left(\frac{A^3}{4A^3+27B^2}\right)$. This quantity is well-known as the *j-invariant*. Though tedious, checking that j is invariant under any suitable linear change of variables is still straightforward. In addition, elementary algebra shows that $J = \frac{j}{(27)(4^4)} + \frac{A^3(4A^3+27B^2)}{B^2(4A^3+27B^2)}$. ■

2 Complex analysis

2.1 Lecture 5

First, let's review some basic concepts about functions of a single complex variable.

Definition 2.1.1. Let $z_0 \in \mathbb{C}$. A function $f = u + iv : U \subset \mathbb{C} \rightarrow \mathbb{C}$ is *holomorphic* or *analytic* if at least one of the following equivalent conditions holds.

- Both u and v are C^1 , and f satisfies the Cauchy-Riemann equations, i.e.,

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

- $\frac{\partial f}{\partial \bar{z}} = 0$, where $\frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)$.

- The Cauchy integral formula holds, i.e.,

$$f(w) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\eta)}{\eta - w} d\eta$$

for any closed circular path γ centered at w in U .

- f has a power series representation on U .

Definition 2.1.2. A bijective function $f : U \subset \mathbb{C} \rightarrow V \subset \mathbb{C}$ is *biholomorphic* if it is holomorphic and its inverse is holomorphic. In this case, we say that U is *biholomorphic to* V , written as $U \approx V$.

Fact 2.1.3.

- (a) (**Maximum modulus principle**) If $U \subset \mathbb{C}$ is a domain, $f : U \rightarrow \mathbb{C}$ is holomorphic, and $|f|$ has a local maximum, then f is constant.
- (b) (**Liouville's theorem**) Any bounded entire function is constant.
- (c) (**Riemann extension theorem**) If $\epsilon > 0$ and $f : B_{\epsilon}(z) \setminus \{z\} \subset \mathbb{C} \rightarrow \mathbb{C}$ is bounded and holomorphic, then f can be extended to a holomorphic function on $B_{\epsilon}(z)$.
- (d) (**Riemann mapping theorem**) If $U \subsetneq \mathbb{C}$ is simply connected and open, then $U \approx B_1(0)$.
- (e) (**Residue theorem**) If $f : B_{\epsilon}(0) \setminus \{0\}$ is holomorphic, then f can be expanded in a Laurent series $\sum_{n=-\infty}^{\infty} a_n z^n$ such that $a_{-1} = \frac{1}{2\pi i} \oint f(z) dz$.

Next, let's look at some basic concepts about functions of several complex variables.

Definition 2.1.4. A function $f = u + iv : U \subset \mathbb{C}^n \rightarrow \mathbb{C}$ is *holomorphic* if at least one of the following equivalent conditions holds.

- f is holomorphic in each variable individually.
- Both u and v are C^1 , and f satisfies the Cauchy-Riemann equations,

$$u_{x_i} = v_{y_i}$$

$$u_{y_i} = -v_{x_i}$$

for each $i = 1, \dots, n$.

- $\sum_{i=1}^n \frac{\partial f}{\partial \bar{z}_i} = 0$.
- f has a power series representation on U ,

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1, \dots, k_n} z_1^{k_1} \cdots z_n^{k_n}.$$

Note 2.1.5. Statements (a), (b), and (c) of Fact 2.1.3 generalize to functions of several variables, as does the Cauchy integral formula:

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\eta_i - z_i| = \epsilon_i} \frac{f(\eta_1, \dots, \eta_n)}{(\eta_1 - z_1) \cdots (\eta_n - z_n)} d\eta_1 \cdots d\eta_n$$

where $\eta_i > 0$ for each $i = 1, \dots, n$.

Theorem 2.1.6 (Hartog). *If $n > 1$, then any holomorphic function $f : B_\epsilon(0) \setminus \{0\} \subset \mathbb{C}^n \rightarrow \mathbb{C}$ extends to a holomorphic function on $B_\epsilon(0)$.*

Definition 2.1.7. Let X be a (topological) space. A sheaf F on X is a presheaf on X such that for any open $U \subset X$ and any open cover $\{U_i\}_{i \in J}$ of U , there is an exact sequence

$$0 \longrightarrow F(U) \longrightarrow F(U_i) \longrightarrow F(U_{ij})$$

where $U_{ij} := U_i \cap U_j$.

Example 2.1.8. Let S be a suitable object (such as a set or R -module) and $x \in X$. The *skyscraper sheaf* of X supported at x with value S is given by

$$\left(U \subset X \atop \text{open} \right) \mapsto \begin{cases} S & x \in U \\ \text{pt} & x \notin U \end{cases}.$$

Definition 2.1.9. A *ringed space* is a pair (X, \mathcal{J}) where X is a space and \mathcal{J} is a sheaf of rings on X .

Remark 2.1.10. Given any standard object (X, \mathcal{J}) , we can define a *geometric object* as a ringed space locally isomorphic to (X, \mathcal{J}) .

Definition 2.1.11 (Vector bundle). Let X and V be complex manifolds. Let $\pi : V \rightarrow X$ be holomorphic. We say that π is a (*holomorphic*) *vector bundle of rank n* if for any $x \in X$, there exist an open set $U \ni x$ in X and an isomorphism $\pi^{-1}(U) \xrightarrow{\cong} U \times \mathbb{C}^n$ such that the *transition maps* $U_{ij} \times \mathbb{C}^n \rightarrow U_{ij} \times \mathbb{C}^n$ are holomorphic and fiber linear.

Any vector bundle $\pi : V \rightarrow X$ induces a sheaf on X given by

$$F(U) = \Gamma(U, \pi^{-1}(U)).$$

Example 2.1.12.

1. The sheaf induced by the trivial bundle $\mathbf{1} := X \times \mathbb{C}$ is denoted by \mathcal{O}_X .
2. The tangent bundle TX of a smooth manifold X induces the sheaf of vector fields on X .
3. The cotangent bundle T^*X induces the sheaf $\Omega^1(X)$ of one-forms on X .
4. The alternating bundle $\bigwedge^p X$ of rank p induces the sheaf $\Omega^p(X)$ of p -forms on X .

3 Line bundles

3.1 Lecture 6

Definition 3.1.1. A *line bundle* is a vector bundle of rank 1.

Definition 3.1.2. Let X be a complex manifold. A *sheaf F of \mathcal{O}_X -modules* is a sheaf on X such that for any open set U in X ,

- $F(U)$ is a module over $\mathcal{O}_X(U)$ and
- if $U \subset V \subset X$, then $(f \cdot a) \upharpoonright_U = f \upharpoonright_U \cdot a \upharpoonright_U$.

Example 3.1.3 (Sheaf of sections). Let X be a complex manifold and J be a vector bundle over X . For any open $U \subset X$, let

$$\mathcal{L}_J(U) = \Gamma(U, L).$$

This inherits a vector space structure from the family of fibers of V . Also, any relation of the form $U_1 \subset U_2 \subset U$ induces a linear map $\Gamma(U_2, L) \rightarrow \Gamma(U_1, L)$ given by $\sigma \mapsto \sigma \upharpoonright_{U_1}$. Thus, $\mathcal{L}_J(-)$ is a sheaf of vector spaces. Moreover, it is easily seen to be a sheaf of \mathcal{O}_X -modules.

Since any vector bundle is locally trivial, we see that \mathcal{L}_J is *locally free*, i.e., for any $x \in X$, there exist an (open) neighborhood U of x in X and an isomorphism $\varphi : \mathcal{L}_J(U) \rightarrow \bigoplus_{i=1}^{\text{rank}(J)} \mathcal{O}_X(U)$ such that for any open set $V \subset U$, the square

$$\begin{array}{ccc} \mathcal{L}_J(U) & \xrightarrow{\cong} & \bigoplus_{i=1}^{\text{rank}(J)} \mathcal{O}_X(U) \\ \downarrow & & \downarrow \\ \mathcal{L}_J(V) & \xrightarrow{\cong} & \bigoplus_{i=1}^{\text{rank}(J)} \mathcal{O}_X(V) \end{array}$$

commutes. In other words, \mathcal{L}_J is locally isomorphic to $(\mathcal{O}_X)^{\oplus \text{rank}(J)}$.

Definition 3.1.4. A sheaf F on a complex manifold X is *invertible* if there exist an open cover $\{U_i\}$ of X and a family of holomorphic isomorphisms $\varphi_i : \mathcal{O}_{U_i} \rightarrow \mathcal{L}_J \upharpoonright_{U_i}$.

Example 3.1.5. If J is a line bundle, then \mathcal{L}_J is invertible.

Consider the composite

$$\mathcal{O}_{U_i \cap U_j} \xrightarrow{\varphi_i} \mathcal{L}_J \upharpoonright_{U_i \cap U_j} \xrightarrow{\varphi_j^{-1}} \mathcal{O}_{U_i \cap U_j}, \quad 1 \mapsto g_{ij}.$$

From this, we can construct a line bundle L over X by defining the total space as

$$\coprod_i (U_i \times \mathbb{C}) / \sim$$

where $(x, \lambda)_i \sim (y, \mu)$ if $x = y$ and $\mu = g_{ij}\lambda$.

Definition 3.1.6 (Divisor). A *divisor on a complex manifold X* is a locally finite \mathbb{Z} -combination of irreducible holomorphic hypersurfaces of X . Equivalently, it is a subset of X locally defined by the vanishing of a holomorphic function.

Example 3.1.7. If $X = \mathcal{A}^1$, then any divisor D on X is of the form

$$D = \sum m_i p_i, \quad p_i \in \mathcal{A}^1, \quad m_i \in \mathbb{Z}.$$

Terminology. Each m_i is known as the *multiplicity of p_i* .

Any divisor D defines a line bundle $\mathcal{O}_X(D)$ on X and a holomorphic map $X \dashrightarrow \mathbb{P}(V^\vee)$ where $V \equiv \Gamma(X, \mathcal{O}_X(D))$. It is also true that any line bundle defines a divisor. It follows that

$$(\text{line bundles}) \xleftarrow{\sim} (\text{invertible sheaves}) \xleftarrow{\sim} (\text{divisors module linear equiv.}) . \quad (\dagger)$$

Consider the case where $D = \text{pt.}$. Let $f \in \Gamma(U, \mathcal{O}_U)$ and let $U_i = X \setminus D$, which is a tubular neighborhood of D . Note that $U_i = f^{-1}(\mathbb{C} \setminus \text{hyperplane})$. Define $\mathcal{O}_X(D)$ as the line bundle with transition functions of the form $f|_{U_i \cap U_j}$.

Alternatively, let

$$(\mathcal{O}_X(D))(U) = \{g : U \rightarrow \mathbb{C} \mid g \text{ is meromorphic, } \overbrace{fg}^{\text{product}} \text{ is holomorphic}\}.$$

For example, let $X = \mathbb{P}^1$ and D be a point p . Let (x_0, x_1) denote local coordinates on X near p . Let g be meromorphic in these coordinates and let $f(x_0, x_1) = \frac{x_1}{x_0}$. Then fg is holomorphic, i.e., g has a pole of order at most one at p .

Question.

1. What is $\Gamma(\mathbb{P}^1, \mathcal{O}_X)$?
2. What is $\Gamma(\mathbb{P}^1, \mathcal{O}_X(D))$?

In fact, it can be shown that

$$\Gamma(\mathbb{P}^1, \mathcal{O}_X(m, p)) = \begin{cases} \mathbb{C}\langle 1, x, \dots, x^m \rangle & m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In general, D is defined locally, and thus so is $\mathcal{O}_U(D)$. Specifically, $\Gamma(U, \mathcal{O}_U(D))$ consists of all holomorphic functions $f : U \setminus \text{supp}(D) \rightarrow \mathbb{C}$ such that if $D = \sum m_i Y_i$ and $Y_i \cap U = \{f_i = 0\}$, then $g \prod_i f_i^{m_i}$ is holomorphic in U .

Example 3.1.8 (Veronese embedding). Let $X = \mathbb{P}^1$ and p be as before.

1. Let $D = \mathcal{O}(2p)$. Consider the space $V := \Gamma(\mathbb{P}^1, \mathcal{O}(2p)) = \mathbb{C}\langle 1, x, x^2 \rangle$. Define the map $\varphi_{\mathcal{O}(2p)} : \mathbb{P}^1 \rightarrow \mathbb{P}^2$ by

$$\varphi_{\mathcal{O}(2p)}(x) = \underbrace{(1, x, x^2)}_{(x, y, z)}.$$

This is an embedding. Its image is precisely the smooth curve given by $y^2 = xz$.

2. Let $D = \mathcal{O}(3p)$. Then the image of the map $\varphi_{\mathcal{O}(3p)} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ given by $x \mapsto (1, x, x^2, x^3)$ is a so-called twisted cubic.

The line bundle L on X determines the map $X \dashrightarrow \mathbb{P}(\Gamma(X, L)^\vee)$ directly, as follows.

$$x \mapsto \ker \left(\Gamma(X, L) \xrightarrow{\text{eval}_x} L_p \right)$$

Definition 3.1.9. The *base locus* of L is $\mathcal{BL}(L) \equiv \{x \in X \mid s(x) = 0 \text{ for each } s \in \Gamma(X, L)\}$.

Note that we get a map $X \setminus \mathcal{BL}(L) \rightarrow \mathbb{P}(\Gamma(X, L)^\vee)$.

Now, let's consider a slight generalization of our preceding discussion. Let $V \subset \Gamma(X, L)$. This induces a map

$$\begin{array}{ccc} X & \dashrightarrow & \mathbb{P}(V^\vee) \\ \uparrow & \nearrow & \\ X \setminus \mathcal{BL}(V) & & \end{array}.$$

Let $X = \mathbb{P}^1$ and $p = \{x = 0\}$. Then $V \subset \Gamma(\mathbb{P}^1, \mathcal{O}(2)) = \mathbb{C}\langle 1, x, x^2 \rangle$, and

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{\varphi_{\mathcal{O}(2)}} & \mathbb{P}^2 \\ & \searrow \varphi_V & \downarrow \rho \\ & & \mathbb{P}^1 \end{array}$$

commutes where ρ denotes the linear projection. Note that φ_V is a morphism so long as the center of ρ is not in the image of $\varphi_{\mathcal{O}(2)}$. In this case, we have that

$$\begin{aligned} \varphi_{\mathcal{O}(2)}(x) &= \frac{a + by + cx^2}{d + ex + fx^2} \\ \rho(x) &= \frac{a + bx}{c + dx}. \end{aligned}$$

3.2 Lecture 7

Let L_1 and L_2 be line bundles over X with transition functions $\{g_1^{kl} : U_{kl} \rightarrow \mathbb{C}^*\}$ and $\{g_2^{ij} : U_{ij} \rightarrow \mathbb{C}^*\}$, respectively. We can take a refinement $\{U_i \cap U_k\}$ where both L_1 and L_2 are trivial. Define $L^1 \otimes L^2$ as the line bundle with transition functions $\{g_1^{kl} g_2^{ij} : U_{ij} \cap U_{kl} \rightarrow \mathbb{C}^*\}$. Further, define $(L^1)^{-1}$ as the line bundle with transition functions $\{(g_1^{kl})^{-1} : U_{kl} \rightarrow \mathbb{C}^*\}$. Note that, locally, $L^1 \otimes (L^1)^{-1} \cong \mathcal{O}_X$.

Definition 3.2.1. We say that a divisor $D = \sum_i m_i Y_i$ is effective if $m_i \geq 0$ for each i .

Let $V = \Gamma(X, \mathcal{O}_X(D))$ and let D be effective. Note that $\mathbb{C}\langle D \rangle \subset V$. We have that $\text{supp}(D) = \varphi^{-1}(\text{hyperplane})$ where $(\mathbb{C}\langle 0 \rangle)^\perp$ is precisely the hyperplane in $\mathbb{P}(V^\vee)$.

Example 3.2.2. Let $X = \mathbb{P}^1$.

1. Let $x = \frac{x_1}{x_0}$ and $D = p := \{x = 0\}$. Then $V = \mathbb{C}\langle 1, x \rangle$, and the map $\varphi_V : \mathbb{P}^1 \rightarrow \mathbb{P}(V^\vee)$ is given by $c \mapsto y := \frac{x}{1}$.
2. Let $D = m(\infty)$ with $m > 0$. Then $V = \mathbb{C}\langle x_1, \dots, x_m^m \rangle$, and the map $\varphi_{m\infty} : \mathbb{P}^1 \rightarrow \mathbb{P}^m$ is given by

$$\begin{aligned} (x_0, x_1) &\mapsto (x_0^m, x_0^{m-1}x_1, \dots, x_0x_1^{m-1}, x_1^m) \\ x &\mapsto (1, x, \dots, x^m). \end{aligned}$$

3. Let $D = p_1 + \dots + p_m$ where $p_i = [1 : t_i]$. Let $x = \frac{x_1}{x_0}$, so that ∞ is given by $x_0 = 0$. Then $V = \mathbb{C}\langle 1, \underbrace{\frac{1}{x - t_1}, \dots, \frac{1}{x - t_m}}_{a_0, \frac{a_1}{x - t_1}, \dots, \frac{a_m}{x - t_m}} \rangle$. This can be viewed as the space of all regular meromorphic functions on open subsets of \mathbb{P}^1 having poles of order at most m . The image of $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^m$ is precisely the hyperplane $\{a_0 = 0\}$.

Example 3.2.3. Let X be an elliptic curve, i.e., a space of the form \mathbb{C}/Λ . Let p be the image of 0 and let $D = mp$.

1. Let $m = 1$. Then $V = \Gamma(X, \mathcal{O}_X(D))$, which consists of all maps $f : X \rightarrow \mathbb{P}^1$ such that $f^{-1}(\infty) = \{0\}$. These are precisely the constant maps, so that $V \cong \mathbb{C}\langle s \rangle$ where s is a holomorphic section of $\mathcal{O}_X(D)$.

vanishing at p and is meromorphic on \mathcal{O}_X .

$$\begin{array}{ccc} X & \dashrightarrow & \mathbb{P}^0 \\ \uparrow & \nearrow & \\ X \setminus p & & \end{array}$$

It follows that $\mathcal{BL}(\mathcal{O}_X(D)) = p$.

2. Let $m = 2$. Then $V = \mathbb{C}\langle 1, p \rangle$, and $\varphi_{2p} : X \rightarrow \mathbb{P}^1$ is precisely the D -th Weierstrass function. Moreover, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(p) \longrightarrow \mathcal{O}_X(2p) \longrightarrow \mathcal{O}_p \longrightarrow \cdots$$

3. Let $m = 3$. Then $V = \langle 1, p, p' \rangle$, and the image of $\varphi_{3p} : X \rightarrow \mathbb{P}^2$ is given by $y^2 = x^3 + ax + b$.

Example 3.2.4. Let $X = \mathbb{P}^2$. Let $D = m \underbrace{(\text{line at } \infty)}_{\{z=0\}}$.

1. Let $m = 0$. Then $V = \mathbb{C}\langle 1 \rangle$, and $\mathcal{BL} = \emptyset$.
2. Let $m = 1$. Then $C = \mathbb{C}\langle \frac{x}{z}, \frac{y}{z}, \frac{z}{z} \rangle \cong \mathbb{C}\langle 1, X, Y \rangle$, and $\mathcal{BL} = \emptyset$. The map $\varphi_D : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is precisely the identity.
3. Let $m = 2$. Then $V = \langle \frac{x^2}{z^2}, \frac{x^4}{z^2}, \frac{y^2}{z^2}, \frac{x}{z}, \frac{y}{z}, \frac{z}{z} \rangle$, and the map $\varphi_D : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ is an embedding given by $(x, y, z) \mapsto \langle x^2, xy, y^2, xz, yz, z^2 \rangle$.

In general, if $H \subset \mathbb{P}^n$ is a hyperplane, then $\varphi_{\mathcal{O}(dH)} : \mathbb{P}^n \rightarrow \mathbb{P}^{\binom{d+n}{n}-1}$ is given by

$$(x_0, \dots, x_n) \mapsto (d\text{-th order homogenous polynomials}),$$

known as the d -th order Veronese embedding on \mathbb{P}^n .

Example 3.2.5. Let $X = \mathbb{P}^2$ with coordinates (x, y, z) . Let H denote the hyperplane given by $z = 0$ and let $D = 2H$. Then $V = \{s \in \Gamma(\mathcal{O}(2H)) \mid s(0, 0, 1) = 0\}$, and

$$\begin{array}{ccc} V & \hookrightarrow & \Gamma(\mathcal{O}(2H)) \\ \cong \uparrow & & \uparrow \cong \\ \mathbb{C}\langle x^2, xy, y^2, xz, yz \rangle & \hookrightarrow & \mathbb{C}\langle x^2, xy, y^2, xz, yz, z^2 \rangle \end{array}$$

commutes. Further, $\mathcal{BL}(V) = \{0\} = [0, 0, 1]$, and φ_V is a map $\mathbb{P}^2 \setminus \{0\} \rightarrow \mathbb{P}^4$ but does not extend to \mathbb{P}^2 . Indeed, we have that

$$\begin{aligned} \lim_{\substack{(0,y,1) \\ y \rightarrow 0}} \varphi_V &= \lim_{y \rightarrow 0} (0, 0, y^2, 0, y) = (0, 0, 0, 0, 1) \\ \lim_{\substack{(x,0,1) \\ x \rightarrow 0}} \varphi_V &= \lim_{x \rightarrow 0} (x^2, 0, 0, x, 0) = (0, 0, 0, 1, 0). \end{aligned}$$

Note that for any $p \in X$, there exist \tilde{X} and $\pi : \tilde{X} \rightarrow X$ such that π restricted to $\pi^{-1}(X \setminus p)$ is an isomorphism and $\pi^{-1}(p)$ is a divisor on \tilde{X} that is isomorphic to \mathbb{P}^1 .

Proposition 3.2.6. *Let $Y \subset X$ be a submanifold of codimension $k \geq 2$. Let $\varphi : X \setminus Y \rightarrow Z$. Then there exist \tilde{X} and $\pi : \tilde{X} \rightarrow X$ such that π restricted to $\pi^{-1}(X \setminus Y)$ is an isomorphism and restricted to $\underbrace{\pi^{-1}(Y)}_{\text{divisor on } \tilde{X}}$ is a bundle with each fiber isomorphic to \mathbb{P}^{k-1} .*

Notation. In this case, the space \tilde{X} is denoted by $\text{Bl}_Y(X)$.

3.3 Lecture 8

Recall our correspondence (†). We can add to it the class of all maps

$$X \setminus \mathcal{BL}(L) \rightarrow \mathbb{P}(\Gamma(X, L)^\vee).$$

Let's turn now to some higher-dimensional examples.

Example 3.3.1. Let $X = \mathbb{P}^2$, $L = \mathcal{O}(2)$, and $V = \{s \in \Gamma(X, \mathcal{O}(2)) \mid \text{linearity condition}\}$. Then $\varphi_V : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. Consider any homogenous polynomial $\sum a_{ijk}x^i y^j z^k$. Then our linearity condition may take any of the following forms.

- $\sum a_{ijk}x^i y^j z^k = 0$ where a_{ijk} ranges over

$$\{a_{000}, a_{120}, a_{020}, a_{101}, a_{011}, a_{002}\}.$$

- $a_{002} = 0$
- $a_{002} + a_{001} = 0$.

In the case of either of these last two, we get a map

$$\mathbb{P}^2 \xrightarrow{\varphi_V} \mathbb{P}^5 \xrightarrow{\psi} \mathbb{P}^4$$

for any $p \in \mathbb{P}^5$. There are two scenarios to consider.

- Suppose that $p \notin \text{im } \varphi_V$. Then $\psi \circ \varphi_V$ is a morphism.
- Suppose that $p = \varphi_V(001)$. Then ψ blows up at p . Consider the map $\varphi_V : \mathbb{P}^2 \setminus p \hookrightarrow \mathbb{P}^4$ given by $(x, y, z) \mapsto \underbrace{(x^2, xy, y^2, xz, y^2)}_{(x, y, z, u, v)}$. The image of this map is precisely $\text{im } \varphi_V \amalg \underbrace{\mathbb{P}^1}_{\{x=y=z=0\}} \subset \mathbb{P}^4$.

Terminology. In this setting, \mathbb{P}^1 is called an *exceptional divisor*.

Note that the equations

$$xz = y^2$$

$$zu = yv$$

$$xv = yu$$

together generate the relevant ideal.

Remark 3.3.2. If we took L to be $\mathcal{O}(n)$ with $n \neq 2$, then our generators would still be quadratic.

Now, fix a and b and let $x = \epsilon a$, $y = \epsilon b$, and $z = 1$ where $\epsilon \rightarrow 0$. Then

$$\begin{aligned}\varphi_V(x, y, z) &= (\epsilon^2 a^2, \epsilon^2 ab, \epsilon^2 b^2, \epsilon a, \epsilon b) \\ &\sim (\epsilon a^2, \epsilon ab, \epsilon b^2, a, b) \\ &\rightarrow (0, 0, 0, a, b).\end{aligned}$$

Question. Is $\text{im } \varphi_V$ a manifold at $00010 = \varphi_V(1, b, a)$?

We have that

$$\begin{aligned}zu - yv &\rightarrow \frac{z}{u} = \frac{y}{u} \frac{v}{u} \\ xv = yu &\rightarrow \frac{x}{u} \frac{v}{u} = \frac{y}{u}.\end{aligned}$$

More generally, let X be a complex n -manifold and let $p \in X$. Then $\text{Bl}_p X = (X \setminus p) \coprod \underbrace{\mathbb{P}^{n-1}}_{\mathbb{P}(T_p X)}$. There at least two ways of extending the map

$$X \setminus p \xrightarrow{\varphi_V} \mathbb{P}^n \dashrightarrow^{\psi} \mathbb{P}^{n-1}$$

so that its image is a manifold at every point.

- (a) Provided that $\psi \circ \varphi_V$ is an embedding, then we can take $\text{Bl}_p(X)$ to be the closure of $X \setminus p$ in \mathbb{P}^{n-1} .
- (b) Let U is any polydisk containing the origin. We can replace $(X \setminus p) \cup U$ with $(X \setminus p) \cup \tilde{U}$ where \tilde{U} denotes the blow-up of U at 0.

More generally still, let $Y^m \subset X^n$ be a closed submanifold. Then $\tilde{X} := \text{Bl}_Y(X) = (X \setminus Y) \coprod \underbrace{\mathbb{P}(N_Y X)}_{\text{normal bundle}}$.

$$\begin{array}{ccc} \mathbb{P}^{n-m-1} & \longrightarrow & \mathbb{P}(N_Y X) \\ & & \downarrow \\ & & Y \end{array}$$

We wish to find a line bundle L over Y and a subspace $V \subset \Gamma(X, L)$ such that $\mathcal{BL}_V = Y$. In this case, the closure of the image of $\varphi_V : X \setminus Y \rightarrow \mathbb{P}(V^\vee)$ determines $(X \setminus Y) \cup \tilde{U}$ on U where U denotes any tubular neighborhood of Y in X .

Alternatively, if we are given an embedding

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{P}^{n-c} & \hookrightarrow & \mathbb{P}^n \end{array}$$

where c denotes the codimension of Y , then we can take $\text{Bl}_Y(X)$ to be the closure of $\text{Bl}_{\mathbb{P}^{n-c}}(\mathbb{P}^n \cap (X \setminus Y))$.

Example 3.3.3. Consider \mathbb{P}^3 with coordinates (x, y, z, w) . We wish to resolve the cone $\{x^2 = y^2\} \subset \mathbb{P}^3$. Let $p = \{x = z = 0\}$. We have a commutative diagram

$$\begin{array}{ccccccc}
 & & Y & \hookrightarrow & \text{cone} & & \\
 & & \downarrow & & \downarrow & & \\
 \{x = z = 0\} & \longrightarrow & \mathbb{P}^0 & \longrightarrow & \mathbb{P}^3 & \longrightarrow & \mathbb{P}^{\binom{5}{2}-1} \\
 & \nearrow & & & \uparrow & & \downarrow \\
 & X & & & \text{Bl}_{\mathbb{P}^0}(\mathbb{P}^3) & \longrightarrow & \mathbb{P}^8 \\
 & & & & \downarrow & & \\
 & & & & X \setminus p & &
 \end{array}$$

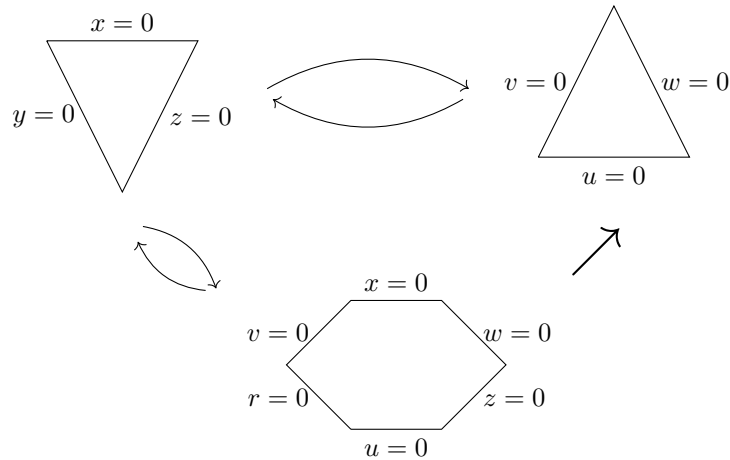
Then the exceptional divisor in $\text{Bl}_p(\mathbb{P}^2)$ is isomorphic to $\mathbb{P}^2 \cong \mathbb{P}(T_p\mathbb{P}^2)$, and the exceptional divisor in $\text{Bl}_p(X)$ is isomorphic to the cone.

Example 3.3.4. Consider the quadratic map $\varphi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ given by $(x, y, z) \mapsto \left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) = \underbrace{(yz, xz, xy)}_{(u,v,w)}$. Let

$$V = \{s \in \Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \mid s(001) = 0, s(010) = 0, s(100) = 0\},$$

which is isomorphic to $\Gamma(\underbrace{\mathcal{I}_{3 \text{ points}}}_{\text{ideal sheaf on 3 points}} \otimes \mathcal{O}(2))$. The fact that $\varphi^{-1} = \varphi$ yields the following properties.

- The line $z = 0$ collapses to the point $u = v = 0$.
- The line $y = 0$ collapses to the point $u = v = 0$.
- The point $y = z = 0$ blows up to the line $u = 0$.



This hexagon is called the *del Pazzo surface of degree three*, denoted by dP_3 . Each of its lines is isomorphic to \mathbb{P}^1 .

Note 3.3.5. Suppose that C is a smooth curve and that $\dim X < 2$. Then $\varphi : C \setminus \text{pt} \rightarrow X$ automatically extends. But if C were singular or $\dim X \geq 2$, then this would be false.

3.4 Lecture 9

Definition 3.4.1 (Picard group). Let X be a complex manifold. The *Picard group* $\text{Pic}(X)$ of X is the group of all isomorphism classes of line bundles over X under \otimes .

Let $n \in \mathbb{N}$ and consider the family of line bundles $\{\mathcal{O}(k) \mid k \in \mathbb{Z}\}$ over \mathbb{P}^n .

Proposition 3.4.2. $\text{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$ with generator $\mathcal{O}(1)$.

Let $\mathbb{P}^n = \mathbb{P}(V)$. We have an exact sequence

$$0 \longrightarrow L \longrightarrow \mathcal{O}_{\mathbb{P}^n} \otimes_{\mathbb{C}} V \longrightarrow \dots$$

We have that

1. $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = \mathbb{C}\langle z_0, \dots, z_n \rangle = V^\vee$,
2. $\Gamma(\mathbb{P}^n, \mathcal{O}(-1)) = 0$, and
3. $\Gamma(\mathbb{P}^n, \mathcal{O}(k)) = \begin{cases} \text{Sym}^k(V^\vee) & k \geq 0 \\ 0 & k < 0 \end{cases}$.

Let $U_i = \{z \in \mathbb{P}^n \mid z_i \neq 0\}$ for each $i \in \{0, 1, \dots, n\}$, so that $\mathbb{P}^n = \bigcup_{i=0}^n U_i$. Let $Z_{ij} = \frac{z_j}{z_i}$, thereby endowing each U_i with local coordinates. Let s be a section of \mathcal{O} , so that

$$s = (s_i \in \Gamma(U_i, \mathcal{O}))_{i=0}^n.$$

Note that Z_i defines a section on U_j with $s_j = \frac{z_i}{z_j} = Z_{ji}$ for each $j = 0, \dots, n$.

$$\begin{array}{ccc} s_j & \longmapsto & Z_{jk} \cdot s_k \\ \parallel & & \parallel \\ \frac{z_i}{z_j} & \longmapsto & Z_{jk} \cdot \frac{z_i}{z_k} \end{array}$$

We can establish the following properties.

1. If $\mathcal{O} = \mathcal{O}(1)$, then $s_i = Z_{ij}s_j$.
2. If $\mathcal{O} = \mathcal{O}(-1)$, then $s_i = Z_{ji}s_j$.
3. If $\mathcal{O} = \mathcal{O}(k)$, then $s_i = (Z_{ij})^k s_j$.

In summary,

	\mathcal{O} (trivial)	$\mathcal{O}(-1)$	$\mathcal{O}(1)$	$\mathcal{O}(k)$
LB	$\mathbb{P}^n \times \mathbb{C}$	tautological	dual	
Sheaf	1	Z_{ji}	Z_{ij}	$(Z_{ij})^k$
Divisor	0	$-H_{\text{h.p.}}$	$+H$	kH
Map	pt	undefined	id	$\left\{ \begin{array}{ll} \text{Veronese} & k > 0 \\ \text{undefined} & k < 0 \\ \text{pt} & k = 0 \end{array} \right.$

Let X be a complex n -manifold. Then T_X consists of all local sections on an open set U with coordinates, say, z_1, \dots, z_n . The set $\{\frac{\partial}{\partial z_i}\}$ is a basis for this, with each section of the form $\sum_{i=1}^n f_i \frac{\partial}{\partial z_i}$ where each f_i belongs to $\Gamma(U, \mathcal{O})$. For any other basis $\{\frac{\partial}{\partial w_i}\}$, we have that

$$\frac{\partial}{\partial w_i} = \sum \frac{\partial z_j}{\partial w_i} \frac{\partial}{\partial z_j}.$$

Note that $T_V \cong \mathcal{O}_V \otimes_{\mathbb{C}} V$. In general, $\Omega_V^i \cong \mathcal{O}_V \otimes \wedge^i V^\vee$.

Question. What is $T_{\mathbb{P}(V)}$?

Note 3.4.3 (Bundle associated to an n -manifold).

1. $T_X^\vee = \Omega \equiv \Omega^1$, whose transition functions are precisely the inverses of the transposes of those for T_X .
2. Let $\Omega^i = \wedge^i \Omega^1$. If $i = n$, then we call this space the *canonical sheaf* K_X or the *dualized sheaf* ω_X .
3. Recall the map $\wedge^i : \mathrm{GL}(n) \rightarrow \mathrm{GL}\left(\binom{n}{i}\right)$. If $i = n$, then this is precisely the determinant map.

Consider the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0$$

$$1 \longmapsto (z_i) \quad .$$

$$(a_i) \longmapsto \sum a_i \frac{\partial}{\partial z_i}$$

Terminology. The vector field given by $\sum z_i \frac{\partial}{\partial z_i}$ is known as the *Euler vector field*.

Moreover, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underbrace{\mathcal{O}_{\mathbb{P}(V)}}_{\mathbb{C}} & \longrightarrow & \underbrace{\mathcal{O}_{\mathbb{P}(V)}(1) \otimes V}_{V^\vee} & \longrightarrow & T_{\mathbb{P}(V)} \longrightarrow , \\ & & & & \downarrow & & \downarrow \\ & & & & \mathcal{O}_V(1) \otimes V & \xrightarrow{\cong} & T_V \longrightarrow 0 \end{array} \quad .$$

Terminology. The top row of this diagram is known as the *Euler sequence*.

Therefore, the *weight* of V equals -1 , whereas the weight of V^\vee equals $+1$.

Informally, any holomorphic function f on V is the same as a direct sum of homogenous functions of degree k , i.e., has the form

$$\bigoplus_{k=0}^{\infty} \Gamma(\mathbb{P}(V), \mathcal{O}(k)),$$

called the *Taylor expansion* of f .

Note 3.4.4. In general, we have an exact sequence

$$0 \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}^{(n+1)} \longrightarrow T_{\mathbb{P}}(-1) \longrightarrow 0 ,$$

which becomes the Euler sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \oplus \mathcal{O}(1) \longrightarrow T_{\mathbb{P}^1} \longrightarrow 0$$

in the case where $n = 1$. It follows that

$$K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Lemma 3.4.5. *If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of vector spaces, then*

$$\det(B) = \det(A) \otimes \det(C).$$

Corollary 3.4.6. $\mathcal{O}(2) \cong \det(\mathcal{O}(1) \oplus \mathcal{O}(1)) = \det(\mathcal{O}) \otimes \det(T) = \det(T).$

Remark 3.4.7. Similarly, we can show that $\det(T_{\mathbb{P}^n}) = \mathcal{O}_{\mathbb{P}^n}(n+1).$

Suppose that $X \subset Y$ is a submanifold of codimension 1. Then we have a short exact sequence

$$0 \longrightarrow T_X \longrightarrow (T_Y)|_X \longrightarrow N_{X/Y} \longrightarrow 0.$$

Lemma 3.4.8. $N_{X/Y} \cong \mathcal{O}_Y(X)|_X.$

In other words, if $L \in \text{Pic}(Y)$, $s \in \Gamma(Y, L)$, and $X = \{s = 0\}$, then $N_{X/Y} \cong L|_X.$

Theorem 3.4.9 (Adjunction formula). $K_X \cong (K_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(X))|_X.$

Proof. Note that $(K_Y^{-1})|_X = K_X^{-1} \otimes N_{X/Y}.$ Thus,

$$\begin{aligned} K_X &\cong K_Y|_X \otimes N_{X/Y} \\ &\cong K_Y|_X \otimes \mathcal{O}_Y(X)|_X \\ &\cong (K_Y \otimes \mathcal{O}_Y(X))|_X. \end{aligned}$$

□

3.5 Lecture 10

Proof of Lemma 3.4.8. Let $s \in \Gamma(Y, L)$. We can write $s = fs_0$, so that $ds = s_0df + fds_0$. Consider the short exact sequence

$$0 \longrightarrow T_X \longrightarrow T_Y|_X \xrightarrow{ds} L \longrightarrow 0.$$

Thus, ds transforms just as s_0 does. □

Example 3.5.1.

1. Let $Y = \mathbb{P}^3$. Suppose that \tilde{X} is a smooth curve of degree d . Then $K_Y = \mathcal{O}(-3)$, and $K_X = \mathcal{O}(d-3)|_X$. Further, if g denotes the genus of a surface, then Bézout's theorem implies that

$$\begin{aligned} 2g - 2 &= \deg(K_X) = d(d-3) \\ &\Downarrow \\ g &= 1 + \frac{d(d-3)}{2} = \frac{(d-1)(d-2)}{2}. \end{aligned}$$

In particular,

d	g
1	0
2	0
3	1
4	3
5	6

2. Let $Y = \mathbb{P}^n$ and let $X \subset Y$ be of dimension d . Note that $K_X = \mathcal{O}_X$ precisely when $d = n + 1$. In particular,

n	X
2	cubic / elliptic curve
3	quartic (a K_3 surface)
4	quintic

Let $p_1, \dots, p_n \in \mathbb{P}^N$, let $m_1, \dots, m_n \in \mathbb{Z}_{\geq 1}$, and let $d \in \mathbb{Z}$. We wish to describe

$$\Gamma(\mathcal{J}_{\Sigma_{m_i p_i}}(d)) := (\mathcal{J}_{\Sigma_{m_i p_i}} \otimes \mathcal{O}(d)).$$

For simplicity, let $N = 2$.

Definition 3.5.2. If $n = 1$, then *imposition* is $\text{Imp}_m \equiv \text{codim}(\Gamma(\mathcal{J}_{mp}(d), \Gamma(\mathcal{O}(d))))$.

Proposition 3.5.3. $\text{Imp}_m = \binom{m+1}{2}$.

Definition 3.5.4. Consider the space Γ .

1. The *actual dimension* of Γ is the dimension of Γ as a vector space.
2. The *virtual dimension* $\text{vd}(\Gamma)$ of Γ is the quantity $\binom{d+2}{2} - 1 - \sum_i \binom{m_i+1}{2}$.
3. The *expected dimension* of Γ is the quantity $\max(\text{vd}(\Gamma), 0)$.

Conjecture 3.5.5. *The actual dimension always equals the expected dimension.*

Answer. This is **false**. For example, let $N = 2$, $d = 1$, $m_i = 1$, and $n = 3$. Then $\Gamma = 0$, so that $\mathbb{P}(\Gamma) = \emptyset$. Hence the expected dimension is zero, but the actual dimension is positive whenever the p_i are co-linear. \square

This leads us to the following modification of Conjecture 3.5.5.

Conjecture 3.5.6. *If the p_i are in general position, then the actual dimension equals the expected dimension.*

Answer. This is **false**. To see this, let $d = 2$ and $N = n = m_i = 2$. Consider a conic C through five points. Here, our conjecture holds. But if instead $N = 2$, $d = 4$, $n = 5$, and $m_i = 2$, then the virtual dimension is precisely $\binom{4+2}{2} - 5 \cdot 3 = 0$. Since the square of C exists, it follows that our conjecture fails. \square

We can improve Conjecture 3.5.6 as follows.

Conjecture 3.5.7. *If the actual dimension is different from the expected dimension, then $\Gamma(\mathcal{J}_{\sum m_i p_i}(d))$ has a base curve.*

Answer. This is **unknown**. See the article “[Linear Systems of Plane Curves](#)” by Rick Miranda. \square

Consider the map $|\mathcal{O}(d)| : \mathbb{P}^2 \hookrightarrow \mathbb{P}^{\binom{d+2}{2}-1}$. We also have a map

$$\begin{array}{ccc} & \mathbb{P}^2 & \xrightarrow{|\mathcal{J}_{\sum p_i}(d)|} \mathbb{P}^{\dim-1} \\ & \uparrow & \nearrow \\ \mathrm{Bl}_{p_1, \dots, p_n}(\mathbb{P}^2) & \xlongequal{\quad} & \widetilde{\mathbb{P}^2} \end{array}$$

Proposition 3.5.8. *Consider the blow-up $\pi : \underbrace{\widetilde{\mathbb{P}^2}}_X \rightarrow \mathbb{P}^2$. We have that*

$$\mathrm{Pic}(X) \cong \mathbb{Z}\langle \pi^*(\mathcal{O}(1)), E_1, \dots, E_n \rangle$$

where E_i denotes the divisor collapsing to p_i .

Remark 3.5.9.

Good: $\pi^*\mathcal{O}(d) - \sum m_i E_i \longleftrightarrow \mathcal{J}_{\sum m_i p_i}(d)$.

Better: $\Gamma(X, ") = \Gamma(\mathbb{P}^2, ")$.

Best: $\pi_*(") = "$.

Conjecture 3.5.10. *Any line bundle $L := (\pi^*\mathcal{O}(d) - \sum m_i E_i)$ has the expected dimension of the space of sections unless $\mathcal{BL}(L)$ contains a (-1) -curve, i.e., a smooth curve C of genus zero such that $C^2 = -1$.*

Example 3.5.11 ((-1) -curve). Let $d = 1$, $n = 2$, and $m_1 = m_2 = 1$. If $C \in \mathcal{O}(1)(-p - q)$, then $C^2 = 1^2 - 1 - 1 = -1$. In general,

$$\mathcal{O}(d) \left(\left(-\sum m_i E_i \right) \left(\mathcal{O}(d) - \sum m_i p_i \right) \right) = dd' - \sum m_i m'_i.$$

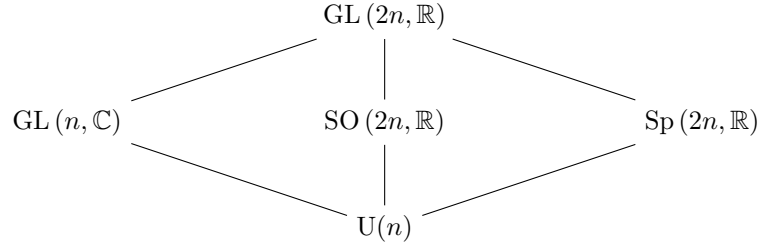
In \mathbb{P}^2 , this means the number of intersections other than the p_i .

Space	C^2
$\mathcal{O}(1)$	1
$\mathcal{O}(1)(-p)$	0
$\mathcal{O}(1)(-p - q)$	-1
\vdots	
$\mathcal{O}(2)$	4
$\mathcal{O}(2)(-p_1)$	3
$\mathcal{O}(2)(-p_1 - p_2)$	2
\vdots	
$\mathcal{O}(2)(-p_1 - \dots - p_4)$	0
$\mathcal{O}(2)\left(-\sum_{i=1}^5 p_i\right)$	-1

4 Kähler manifolds

4.1 Lecture 11

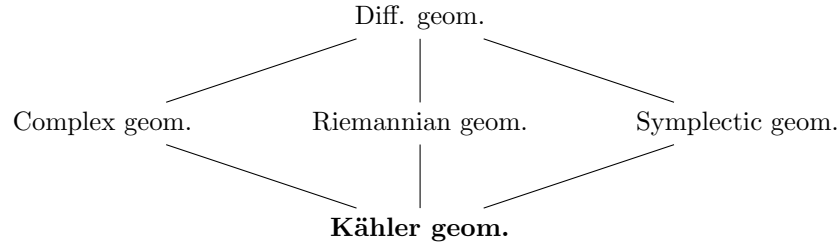
Consider the following Hasse diagram of subgroups:



where $\text{Sp}(2n, \mathbb{R})$ denotes the group of real $2n \times 2n$ *symplectic matrices*, i.e., matrices M satisfying

$$M^t \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} M = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Similarly, we can view various areas of geometry as refinements of certain others. Specifically,



Before investigating Kähler geometry, we establish some basic geometric concepts.

Definition 4.1.1. Let X be a real manifold. An *almost complex structure on X* is a bundle map $I : TX \rightarrow TX$ such that $I^2 = -1$.

Note that the eigenvalues of I are precisely i and $-i$.

Notation.

1. Let $T^{1,0}$ denote the eigenspace of i .
2. Let $T^{0,1}$ denote the eigenspace of $-i$.

Any complex manifold X has a natural almost complex structure. Indeed, given local coordinates x_i, y_i on X , define I by $\frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$ and $\frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}$. It follows that any manifold with an almost complex structure has even dimension.

Now, consider the complexification of our tangent bundle, $T^{\mathbb{C}}X \equiv TX \otimes_{\mathbb{R}} \mathbb{C}$.

Proposition 4.1.2.

1. $T^{\mathbb{C}}X \cong T^{1,0} \oplus T^{0,1}$.

$$2. T^{\mathbb{C}}X \cong T^{*1,0} \oplus T^{*0,1}$$

Define, formally, the complex coordinates $z_j = x_j + iy_j$. Note that $T^{\mathbb{C}}X$ has as basis $\{\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}\}$ and that $T^{*\mathbb{C}}X$ has as basis $\{dz_j, d\bar{z}_j\}$ where $dz_j \equiv dx_j + idy_j$.

Notation.

$$1. \bigwedge^k X := \bigwedge^k T^*X.$$

$$2. \bigwedge^{p,q} X := \bigwedge^p T^{*1,0}X \otimes_{\mathbb{C}} \bigwedge^q T^{*0,1}X.$$

Note 4.1.3. Let X be an n -dimensional complex manifold.

$$1. \left(\bigwedge^k T^*X \right) \otimes \mathbb{C} = \bigwedge_{\mathbb{C}}^k (T^*X \otimes \mathbb{C}).$$

$$2. \left(\bigwedge^k X \right) \otimes \mathbb{C} = \bigoplus_{p+q=k} \bigwedge^{p,q} X.$$

Therefore, $\left(\bigwedge^k X \right) \otimes \mathbb{C}$ can be decomposed according to the counting equation $\binom{2n}{k} = \sum \binom{n}{p} \binom{n}{q}$.

Let U and V be open in \mathbb{C}^n . Let $f : U \rightarrow V$ be holomorphic. Then the map $df : TU \rightarrow TV$ extends to a map $df^{\mathbb{C}} : T^{\mathbb{C}}U \rightarrow T^{\mathbb{C}}V$ that preserves both $T^{1,0}$ and $T^{0,1}$.

Let $\mathcal{A}^{p,q} = \Gamma(\bigwedge^{p,q})$, i.e., $\mathcal{A}^{p,q}(U) = \Gamma(U, \bigwedge^{p,q})$. Consider the exterior derivative $d : \mathcal{A}^k \rightarrow \mathcal{A}^{k+1}$. With π denoting the projection map, define the operators

$$\partial = \pi^{p+1,q} \circ d$$

$$\bar{\partial} = \pi^{p,q+1} \circ d$$

on $\mathcal{A}^{p,q}$. Locally, we have that

$$df = \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i = \frac{\partial f}{\partial z_i} dz_i + \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$$

for any $f \in \mathcal{A}^{0,0}$. By the Cauchy-Riemann equations, it follows that f is holomorphic if and only if $\bar{\partial}f = 0$.

Remark 4.1.4. Any (p, q) -form locally looks like $f_{IJ} dz_I \wedge \bar{z}_J$.

Proposition 4.1.5.

$$1. d = \partial + \bar{\partial}.$$

$$2. \partial^2 = 0 = \bar{\partial}^2.$$

$$3. \partial \bar{\partial} = -\bar{\partial} \partial.$$

$$4. \partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \partial\beta \text{ for any } \alpha \in \mathcal{A}^{p,q} \text{ and } \beta \in \mathcal{A}^{r,s}.$$

Lemma 4.1.6 (Single-variable Poincaré). Consider the disk $B_\epsilon \subset \overline{B_\epsilon} \subset U \subset \mathbb{C}$ where U is open. Let $\alpha = f d\bar{z} \in \mathcal{A}^{0,1}(U)$ and

$$g(z) = \frac{1}{2\pi i} \int_{\overline{B_\epsilon}} \frac{f(w)}{w - z} dw \wedge d\bar{w}.$$

Then $\bar{\partial}g = \alpha$.

Lemma 4.1.7 (Multi-variable Poincaré). *Consider the polydisk $B_\epsilon \subset \overline{B_\epsilon} \subset U \subset \mathbb{C}^n$ where U is open. Let $\alpha \in \mathcal{A}^{p,q}$ with $q > 0$ and $\bar{\partial}\alpha = 0$. Then there is some $\beta \in \mathcal{A}^{p,q-1}(B_\epsilon)$ such that $\bar{\partial}\beta = \alpha$.*

Remark 4.1.8. If U is contractible, then any differential form on U is closed if and only if it is exact.

Let $U \subset \mathbb{C}^n$ be open and let I denote the natural almost complex structure on U . Let g be a Riemannian metric on U .

Definition 4.1.9 (Hermitian metric).

1. We say that g is *compatible with I* or *(almost) Hermitian* if $g(u, v) = g(Iu, Iv)$.
2. If g is Hermitian, then the real $(1, 1)$ -form $\omega \in \mathcal{A}^{1,1}(U) \cap \mathcal{A}^2(U)$ defined by

$$\omega(u, v) = g(Iu, v)$$

is called the *fundamental form of g* .

Notation. $h := g - i\omega$.

Definition 4.1.10. A Hermitian matrix M is *positive-definite* if $z^* M z > 0$ for every nonzero complex column vector z .

Note that h is a positive-definite form in the sense that, locally, its component functions define a positive-definite matrix at any given point.

Example 4.1.11. Let $g = \underbrace{dx^2}_{dx \otimes dx} + dy^2 = \sum_{i=1}^n dx_i^2 + dy_i^2 \in T^* \otimes T^* \subset (T^* \otimes T^*) \otimes_{\mathbb{R}} \mathbb{C}$. Since

$$dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = -2idx \wedge dy,$$

it follows that

$$\omega = dx \otimes dy - dy \otimes dx = \frac{i}{2} dz \wedge d\bar{z}.$$

Moreover, we see that

$$\begin{aligned} h &= z - i\omega \\ &= dx^2 - idxdy + idydx + dy^2 \\ &= dx(dx - idy) + idy(dx + idy) \\ &= (dx + idy)(dx - idy) \\ &= dz \otimes d\bar{z}. \end{aligned}$$

For each $z \in \mathbb{C}^n$, define the matrix $(h_{ij})(z)$ by

$$h_{ij}(z_1, \dots, z_n) = h\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Proposition 4.1.12. *Let I be an almost complex structure on $U \subset X$ and let g be compatible with I . Then $d\omega = 0$ if and only if for each $x \in X$, there exist a neighborhood U' of x and a holomorphic map $f : U' \rightarrow U$ such that f^*g oscialates the standard metric to the second order, i.e., $(h_{ij}) = \text{id} + O(|z|^2)$.*

Notation. In this case, we write $h \approx \text{id}$.

Definition 4.1.13 (Kähler manifold). Consider the four-tuple (X, I, g, ω) . We say that X is a *Kähler manifold* if $d\omega = 0$. In this case, we call g a *Kähler metric on X* and ω a *Kähler form*.

Definition 4.1.14. Let (X, I, g, ω) be a Kähler structure with $\dim X = n$.

1. The *Lefschetz operator* $L : \bigwedge^k X \rightarrow \bigwedge^{k+2} X$ is defined by $\alpha \mapsto \alpha \wedge \omega$.
2. The *Hodge *-operator* $*$: $\bigwedge^k X \rightarrow \bigwedge^{2n-k} X$ is defined by the property

$$\alpha \wedge * \beta = \hat{g}(\alpha, \beta) \omega^n$$

where \hat{g} is induced by g and ω^n denotes the (positively oriented) volume form on X .

3. The *dual Lefschetz operator* $\Lambda : \bigwedge^k X \rightarrow \bigwedge^{k-2} X$ is defined as the composite $*^{-1} \circ L \circ *$.

Note 4.1.15.

1. In coordinates in which $h \approx \text{id}$, we have that $*dx^I = dx^{\partial}$ where $\partial := I^{\mathbb{C}}$??.
2. Λ is \mathcal{O} -linear.

4.2 Lecture 12

Proposition 4.2.1. Let X be a complex manifold. Let ω be a closed real positive-definite form of type $(1, 1)$, i.e., locally, $\omega = \frac{i}{2} \sum h_{ij} dz_i \wedge d\bar{z}_j$ such that the matrix $(h_{ij}(p))$ is positive-definite for each p . Then there exists a Kähler metric g on X such that ω equals the fundamental form of g .

Since every Kähler form is positive-definite, it follows that the set \mathbb{K}_X of all Kähler forms on X is precisely the set of all closed real positive-definite forms of type $(1, 1)$.

Definition 4.2.2. Let V be a vector space over \mathbb{R} . A subset $C \subset V$ is a *convex cone* if $av_1 + bv_2 \in C$ for any $v_1, v_2 \in C$ and any $a, b \in \mathbb{R}_{>0}$.

Corollary 4.2.3. Suppose that X is compact. Then \mathbb{K}_X is an open convex cone in the infinite-dimensional real vector space $S := \{\omega \in \mathcal{A}^{(1,1)}(X) \cap \mathcal{A}^2(X) \mid d\omega = 0\}$.

Idea. The fact that \mathbb{K}_X is a convex cone follows from the fact that the set of all positive-definite matrices is a convex cone. It remains to show that \mathbb{K}_X is open. Since X is compact, it has a finite open cover $\{U_i\}$. The set $P_{U_i} \subset S$ of all forms that are positive-definite on U_i is open. Thus, $\bigcap_i P_{U_i} = \mathbb{K}_X$ is also open. \square

Remark 4.2.4. It turns out that $S \cong H^2(X, \mathbb{R})$.

Example 4.2.5.

1. The form $\omega \equiv \frac{i}{2} dz \wedge d\bar{z}$ is Kähler on \mathbb{C} and is exact.
2. The same form descends to a Kähler form on the torus \mathbb{C}/Λ , which is not exact.
3. Consider the inclusion $i : X \rightarrow Y$ of a closed submanifold. If ω is Kähler on Y , then $i^*\omega$ is Kähler on X .

Note 4.2.6. Let $f : X \rightarrow Y$ be holomorphic and let ω be a Kähler form on Y . It is *not* necessarily true that $f^*\omega$ is Kähler on X . For example, if $f(x) = \mathbf{pt}$ for all $x \in X$, then $f^*\omega$ is the zero form and thus not positive. In general, f must be injective. For example, if $f : C \rightarrow \mathbb{C}$ is a double cover where C is a Riemann surface, then C inherits a Kähler form only outside the *ramification of f* , i.e., the set

$$\{c \in C \mid \text{there is no neighborhood } U \text{ of } c \text{ such that } f|_U \text{ is injective}\}.$$

This is precisely the set of points at which df is nonzero.

Example 4.2.7.

1. Consider the open cover $\{U_i\}_{1 \leq i \leq n}$ of \mathbb{P}^n where $U_i \equiv \{z \in \mathbb{P}^n \mid z_i \neq 0\}$. Define $\varphi_i : U_i \xrightarrow{\cong} \mathbb{C}^n$ by

$$(z_0, \dots, z_n) \mapsto \underbrace{\left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i} \right)}_{(w_1, \dots, w_n)}.$$

Then $\{(U_i, \varphi_i)\}$ is a holomorphic atlas on \mathbb{P}^n . For each i , let

$$\omega_i = \frac{i}{2\pi} \partial \bar{\partial} \log \left(\sum_{l=0}^n \left| \frac{z_l}{z_i} \right|^2 \right).$$

By way of φ_i , this becomes

$$\frac{i}{2\pi} \partial \bar{\partial} \log \left(1 + \sum_{k=1}^n |w_k|^2 \right).$$

Exercise 4.2.8. Show that $\omega_i|_{U_i \cap U_j} = \omega_j|_{U_i \cap U_j}$.

Therefore, the ω_i patch together to form a metric ω on \mathbb{P}^n , known as the *Fubini-Study metric*.

Exercise 4.2.9. Show that ω is closed, real, positive, and of type $(1,1)$.

It follows that ω is a Kähler metric.

2. Any branched cover of \mathbb{P}^n admits a Kähler metric (which must be different from the pullback of a Kähler metric on \mathbb{P}^n). For example, consider an elliptic curve $E \rightarrow \mathbb{P}^1$, which fits into a commutative square

$$\begin{array}{ccc} E & \longrightarrow & \mathbb{P}^1 \\ \parallel & & \uparrow \text{---} \\ E & \hookrightarrow & \mathbb{P}^2 \end{array}.$$

Definition 4.2.10. A complex manifold is *projective* if it is isomorphic to a closed submanifold of projective space.

Proposition 4.2.11. Any projective complex manifold is Kähler.

Proof. This follows from Example 4.2.5(3) together with Example 4.2.7(1). □

Definition 4.2.12. Let X be a complex manifold. Let D be a first-order operator on $\mathcal{A}^*(X)$.

1. The *adjoint* of D is

$$D^* \equiv -* \circ D \circ *$$

2. The *Laplacian* associated to D is

$$\Delta_D \equiv DD^* + D^*D.$$

Definition 4.2.13. The *Laplace operator* is $\Delta \equiv dd^* + d^*d$.

Example 4.2.14.

1. Let $D = \partial$. Then $\partial^* (f_{IJ} dz^I \wedge dz^J) = \sum_{i \in I} f_{IJ} dz^{I-i} \wedge d\bar{z}^J$.
2. Let $D = d$. Let (x_1, \dots, x_n) be local coordinates on X . Then

$$\begin{aligned} d(f dx^I) &= \sum_{i \notin I} \frac{\partial f}{\partial x_i} dx^i \wedge dx^I \\ d^*(f dx^I) &= \sum_{i \in I} \frac{\partial f}{\partial x_i} dx^{I-i}. \end{aligned}$$

Therefore,

$$\begin{aligned} d \circ d^* (f dx^I) &= \frac{\partial^2}{\partial x_i \partial x_j} dx^{I-i \cup j} \\ &= \sum_{\substack{i \in I \\ j \notin I}} \dots + \sum_{i=j \in I} \dots \\ d^* \circ d (f dx^I) &= 0 + \sum_{i=j \notin I} \dots, \end{aligned}$$

$$\text{so that } \Delta_D = \sum \frac{\partial^2 f}{\partial x_i^2}.$$

Theorem 4.2.15 (Kähler identities). Let (X, I, g, ω) be a Kähler manifold.

1. $[\bar{\partial}, L] = 0 = [\partial, L]$.
2. $[\partial^*, \Lambda] = 0 = [\bar{\partial}^*, \Lambda]$.
3. $[\bar{\partial}^*, L] = i\partial$ and $[\partial^*, L] = -i\bar{\partial}$.
4. $\Delta_\partial = \Delta_{\bar{\partial}} = \frac{1}{2}\Delta$, and Δ commutes with $*$, ∂ , $\bar{\partial}$, ∂^* , $\bar{\partial}^*$, L , and Λ .

5 Lie algebras

Let G be any Lie group. For any $g \in G$, $\ell_g : G \rightarrow G$ is an isomorphism of \mathbb{C} -manifolds. Thus, if V is a vector field on G , then so is $(\ell_g)_* V$.

Definition 5.0.1. We say that V is *left-invariant* if $(\ell_g)_* V = V$ for every $g \in G$.

Definition 5.0.2 (Lie algebra). The *Lie algebra* \mathfrak{G}_G of G is the space of left-invariant vector fields on G under the Lie bracket.

Consider the commutative diagram

$$\begin{array}{ccc} \mathfrak{G}_G & \hookrightarrow & (\mathcal{X}(G), [-, -]) \\ & \searrow \alpha & \downarrow \text{eval}_1 \\ & & T_1(G) \end{array} .$$

Proposition 5.0.3. α is an isomorphism of vector spaces.

Example 5.0.4. Let $G = \text{GL}(n, \mathbb{C})$, which is a complex Lie group. We have that $\text{GL}(n, \mathbb{C})$ is an open submanifold of the vector space $M_n(\mathbb{C})$. Hence \mathfrak{G}_G is isomorphic to $M_n(\mathbb{C})$ under the *commutator bracket*, which is given by $[A, B] = AB - BA$.

Definition 5.0.5 (Matrix exponential). Define the map $e^{(\cdot)} : M_n(\mathbb{C}) \rightarrow \text{GL}_n(\mathbb{C})$ by

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

This is well-defined. Indeed, letting $\|\cdot\|$ denote the operator norm, we see that $\frac{\|A^n\|}{n!} \leq \frac{\|A\|^n}{n!}$ on any bounded subset $S \subset \mathbb{C}^n$. But $\sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} = e^{\|A\|}$ on S , and thus e^A converges uniformly on S . Moreover, one can show that its limit must be invertible.

Exercise 5.0.6. Let $G = \text{SL}_2(\mathbb{C})$, which is complex Lie group. Show that

$$\mathfrak{G}_G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C}) \mid a + d = 0 \right\}.$$

Proof. Any element X of \mathfrak{G}_G generates a local flow $\theta : D \subset \mathbb{R} \times G \rightarrow G$. Since X is left-invariant, it is complete. In particular, the maximal integral curve θ^1 is defined on \mathbb{R} . Left-invariance also implies that for any $s \in \mathbb{R}$, $L_{\theta^1(s)} \circ \theta^1$ is an integral curve starting at $\theta^1(s)$. But the curve given by $t \mapsto \theta^1(s+t)$ is also an integral curve starting at $\theta^1(s)$. Hence $\theta^1(s+t) = \theta^1(s)\theta^1(t)$. By the uniqueness of maximal integral curves, this proves that $\theta^1(s)$ is a smooth group homomorphism $\mathbb{R} \rightarrow G$, known as a *one-parameter subgroup* of G . Moreover, any one-parameter subgroup γ of G has the form $\gamma(t) = e^{tA}$ where $A = \gamma'(0) \in T_1(G) \subset T_1(\text{GL}_2(\mathbb{C})) \cong M_2(\mathbb{C})$. It follows that

$$\begin{aligned} X \in T_1(G) &\iff \forall t \in \mathbb{R}, e^{tX} \in G \\ &\iff \forall t \in \mathbb{R}, \det(e^{tX}) = 1 \\ &\iff \forall t \in \mathbb{R}, e^{t \text{tr}(X)} = 1 \\ &\iff \forall t \in \mathbb{R}, t \text{tr}(X) = 0 \\ &\iff \text{tr}(X) = 0. \end{aligned}$$

□

Intuitively, Theorem 4.2.15 means that the space $\mathcal{A}^{p,q}(X)$ has a symmetry encoded in the $\text{SL}_2(\mathbb{C})$ -action.

5.1 Lecture 13

Definition 5.1.1. Let V be a vector space endowed with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. The *orthogonal group* $O(V, \langle \cdot, \cdot \rangle)$ is the group of all linear maps $f : V \rightarrow V$ such that $\langle fx, fy \rangle = \langle x, y \rangle$ for any $x, y \in V$.

Example 5.1.2. Consider the Lie group $G := O(\mathbb{R}^n)$. Define the smooth map $\varphi : GL_n(\mathbb{R}) \rightarrow GL_n(\mathbb{R})$ by $A \mapsto AA^t$, which has constant rank. Then $G = \varphi^{-1}(I_n)$, so that $T_{I_n}G = \ker d\varphi_{I_n}$. Since $d\varphi_{I_n}(A) = A^t + A$ for any $A \in M_n(\mathbb{R})$, it follows that \mathfrak{G}_G consists of all $n \times n$ skew-symmetric matrices.

\vdots