

### Abstract

We begin higher Waldhausen  $K$ -theory. The main sources for this talk are the following.

- $n\text{Lab}$ .
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Ch. IV.8.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

For the original development, see Friedhelm Waldhausen's *Algebraic K-theory of spaces* (1985).

Let  $\mathcal{C}$  be a Waldhausen category. Our goal is to construct the  $K$ -theory  $K(\mathcal{C})$  of  $\mathcal{C}$  as a based loop space  $\Omega Y$  endowed with a loop completion map  $\iota : |w\mathcal{C}| \rightarrow K(\mathcal{C})$  where  $w\mathcal{C}$  denotes the subcategory of weak equivalences. This will produce a function  $\text{ob } \mathcal{C} \rightarrow |w\mathcal{C}| \rightarrow \Omega Y$ . Further, we'll require of  $K(\mathcal{C})$  certain limit and coherence properties, thereby making  $K(\mathcal{C})$  the underlying infinite loop space of a spectrum  $\mathbf{K}(\mathcal{C})$ , called the *algebraic K-theory spectrum* of  $\mathcal{C}$ .

**Definition 1.** Let  $\mathcal{C}$  be a category equipped with a subcategory  $\text{co } \mathcal{C}$  of morphisms called *cofibrations*. The pair  $(\mathcal{C}, \text{co } \mathcal{C})$  is a *category with cofibrations* if the following conditions hold.

1. (W0) Every isomorphism in  $\mathcal{C}$  is a cofibration.
2. (W1) There is a base point  $*$  in  $\mathcal{C}$  such that the unique morphism  $* \rightarrow A$  is a cofibration for any  $A \in \text{ob } \mathcal{C}$ .
3. (W2) We have a *cobase change*

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ C & \rightarrow & B \cup_A C \end{array} .$$

We see that  $B \amalg C$  always exists as the pushout  $B \cup_* C$  and that the cokernel of any  $i : A \rightarrow B$  exists as  $B \cup_A *$  along  $A \rightarrow *$ . We call  $A \rightarrow B \rightarrow B/A$  a *cofiber sequence*.

**Definition 2.** A *Waldhausen category*  $\mathcal{C}$  is a category with cofibrations together with a subcategory  $w\mathcal{C}$  of morphisms called *weak equivalences* such that every isomorphism in  $\mathcal{C}$  is a w.e. and the following “Gluing axiom” holds.

1. (W3) For any diagram

$$\begin{array}{ccccc} C & \longleftarrow & A & \longrightarrow & B \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ C' & \longleftarrow & A' & \longrightarrow & B' \end{array} ,$$

the induced map  $B \cup_A C \rightarrow B' \cup_{A'} C'$  is a w.e.

**Definition 3.** A Waldhausen category  $(\mathcal{C}, w)$  is *saturated* if whenever  $fg$  makes sense and is a w.e., then  $f$  is a w.e. iff  $g$  is.

Now, let us introduce the main concept to be generalized.

**Definition 4.** Let  $\mathcal{C}$  be a category with cofibrations. Let the *extension category*  $S_2\mathcal{C}$  have as objects the cofiber sequences in  $(\mathcal{C}, \text{co}\mathcal{C})$  and as morphisms the triples  $(f', f, f'')$  such that

$$\begin{array}{ccccc} X' & \twoheadrightarrow & X & \twoheadrightarrow & X'' \\ \downarrow f' & & \downarrow f & & \downarrow f'' \\ Y' & \twoheadrightarrow & Y & \twoheadrightarrow & Y'' \end{array}$$

commutes. This is pointed at  $* \twoheadrightarrow * \twoheadrightarrow *$ .

**Definition 5.** Suppose an arbitrary triple  $(f', f, f'')$  as above has the property that whenever  $f'$  and  $f''$  are w.e., then so is  $f$ . In this case, we say  $\mathcal{C}$  is *extensional* or *closed under extensions*.

Say that the morphism  $(f', f, f'')$  is a cofibration if  $f'$ ,  $f''$ , and  $Y' \cup_{X'} X \rightarrow Y$  are cofibrations in  $\mathcal{C}$ . Say that the same triple is a weak equivalence if  $f'$ ,  $f$ , and  $f''$  are w.e. in  $\mathcal{C}$ . This makes  $S_2\mathcal{C}$  into a Waldhausen category.

**Definition 6.** Let  $q \geq 0$ . Let the *arrow category*  $\text{Ar}[q]$  on  $[q]$  have as objects ordered pairs  $(i, j)$  with  $i \leq j \leq q$  and as morphisms commutative diagrams of the form

$$\begin{array}{ccc} i & \xrightarrow{\leq} & j \\ \leq \downarrow & & \downarrow \leq \\ i' & \xrightarrow{\leq} & j' \end{array}$$

We view  $[q]$  as a full subcategory of  $\text{Ar}[q]$  via the embedding  $[q] \xrightarrow{k \mapsto (0, k)} \text{Ar}[q]$ .

**Note 7.**

1. Any triple  $i \leq j \leq k$  determines the morphisms  $(i, j) \rightarrow (i, k)$  and  $(i, k) \rightarrow (j, k)$ . Conversely, any morphism in the arrow category is a composite of such triples.
2.  $\text{Ar}[q] \cong \mathbf{Fun}([1], [q])$  by identifying each pair  $(i, j)$  with the functor satisfying  $0 \mapsto i$  and  $1 \mapsto j$ .

**Example 8.** The category  $\text{Ar}[2]$  is generated by the commutative diagram

$$\begin{array}{ccccc} (0, 0) & \longrightarrow & (0, 1) & \longrightarrow & (0, 2) \\ & & \downarrow & & \downarrow \\ & & (1, 1) & \longrightarrow & (1, 2) \\ & & & & \downarrow \\ & & & & (2, 2) \end{array}$$

**Definition 9.** Let  $\mathcal{C}$  be a category with cofibrations and  $q \geq 0$ . Define  $S_q\mathcal{C}$  as the full subcategory of  $\mathbf{Fun}(\text{Ar}[q], \mathcal{C})$  generated by  $X : \text{Ar}[q] \rightarrow \mathcal{C}$  such that

1.  $X_{j,j} = *$  for each  $j \in [q]$ .
2.  $X_{i,j} \twoheadrightarrow X_{i,k} \twoheadrightarrow X_{j,k}$  is a cofiber sequence for any  $i < j < k$  in  $[q]$ . Equivalently, if  $i \leq j \leq k$  in  $[q]$ , then the square

$$\begin{array}{ccc} X_{i,j} & \twoheadrightarrow & X_{i,k} \\ \downarrow & & \downarrow \\ X_{j,j} = * & \twoheadrightarrow & X_{j,k} \end{array}$$

is a pushout.

This is pointed at the constant diagram at  $*$ .

**Note 10.** A generic object in  $S_q\mathcal{C}$  looks like

$$\begin{array}{ccccccc} * & \twoheadrightarrow & X_1 & \twoheadrightarrow & \cdots & \twoheadrightarrow & X_{q-1} & \twoheadrightarrow & X_q \\ & & \downarrow & & & & \downarrow & & \downarrow \\ & & * & \twoheadrightarrow & \cdots & \twoheadrightarrow & X_{q-1}/X_1 & \twoheadrightarrow & X_q/X_1 \\ & & & & & & \downarrow & & \downarrow \\ & & & & \ddots & & \vdots & & \vdots \\ & & & & & & * & \twoheadrightarrow & X_q/X_{q-1} \\ & & & & & & & & \downarrow \\ & & & & & & & & * \end{array} \quad (\dagger)$$

where  $X_q$  corresponds to  $X_{0,q}$  and  $X_j/X_i$  to  $X_{i,j}$  for any  $1 \leq i \leq j \leq q$ .

**Definition 11.** Let  $(\mathcal{C}, \text{co}\mathcal{C})$  be a category with cofibrations. Let  $\text{co}S_q\mathcal{C} \subset S_q\mathcal{C}$  consist of the morphisms  $f : X \twoheadrightarrow Y$  of  $\text{Ar}[q]$ -shaped diagrams such that for each  $1 \leq j \leq q$  we have

$$\begin{array}{ccccc} X_{0,j-1} & \twoheadrightarrow & X_{0,j} & & \\ f_{0,j-1} \downarrow & & \downarrow & \searrow f_{0,j} & \\ Y_{0,j-1} & \twoheadrightarrow & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} & & \\ & \searrow & & \nearrow & \\ & & & & Y_{0,j} \end{array}$$

**Proposition 12.** If  $f : X \rightarrow Y$  is a cofibration of  $S_q\mathcal{C}$ , then

$$\begin{array}{ccc} X_{i,j} & \twoheadrightarrow & X_{i,k} \\ f_{i,j} \downarrow & & \downarrow f_{i,k} \\ Y_{i,j} & \twoheadrightarrow & Y_{i,k} \end{array}$$

for any  $i \leq j \leq k$  in  $[q]$ .<sup>1</sup>

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<sup>1</sup>Lemma 8.3.12 (Rognes).

**Lemma 13.**  $(S_q\mathcal{C}, \text{co } S_1\mathcal{C})$  is a category with cofibrations.

*Proof.* First notice that the composite of two cofibrations  $g \circ f : X \rightarrow Y \rightarrow Z$  is a cofibration because we have

$$\begin{array}{ccccc}
 X_{0,j-1} & \xrightarrow{\quad} & X_{0,j} & \xrightarrow{\quad f_{0,j} \quad} & Y_{0,j} \\
 f_{0,j-1} \downarrow & & \downarrow & & \downarrow \\
 Y_{0,j-1} & \xrightarrow{\quad} & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} & \xrightarrow{\quad} & Y_{0,j} \\
 g_{0,j-1} \downarrow & & \downarrow & & \downarrow \\
 Z_{0,j-1} & \xrightarrow{\quad} & X_{0,j} \cup_{X_{0,j-1}} Z_{0,j-1} & \xrightarrow{\quad} & Y_{0,j} \cup_{Y_{0,j-1}} Z_{0,j-1} \xrightarrow{\quad} Z_{0,j}
 \end{array}$$

It's clear that any isomorphism or initial morphism in  $S_q\mathcal{C}$  is a cofibration.

To see that (W2) is satisfied, let  $f : X \rightarrow Y$  and  $g : X \rightarrow Z$  be morphisms in  $S_q\mathcal{C}$ . It's easy to verify that each component  $f_{i,j} : X_{i,j} \rightarrow Y_{i,j}$  is a cofibration. Thus, each pushout  $W_{i,j} := Y_{i,j} \cup_{X_{i,j}} Z_{i,j}$  exists. These form a functor  $W : \text{Ar}[q] \rightarrow \mathcal{C}$ . If  $i < j < k$ , then we have  $W_{i,j} \rightarrow W_{i,k} \rightarrow W_{j,k}$  because the left morphism factors as the composite of two cofibrations

$$\begin{array}{ccccc}
 Z_{i,j} & \xrightarrow{\quad} & Z_{i,k} & & \\
 f_{i,j} \cup \text{Id} \downarrow & & \downarrow f_{i,j} \cup \text{Id} & & \\
 Y_{i,j} \cup_{X_{i,j}} Z_{i,j} & \xrightarrow{\quad} & Y_{i,j} \cup_{X_{i,j}} Z_{i,k} & \xrightarrow{\quad} & Y_{i,k} \cup_{X_{i,k}} Z_{i,k} \\
 & & \uparrow \text{Id} \cup g_{i,k} & & \uparrow \text{Id} \cup g_{i,k} \\
 & & Y_{i,j} \cup_{X_{i,j}} X_{i,k} & \xrightarrow{\quad} & Y_{i,k}
 \end{array}$$

The fact that colimits commute with each other ensures that  $W_{j,k} \cong W_{i,k} / W_{i,j}$ . Hence  $W$  is the pushout of  $f$  and  $g$ . To verify that this is a cofibration, we must check that the pushout map  $W_{0,j-1} \cup_{Z_{0,j-1}} Z_{0,j} \rightarrow W_{0,j}$  is a cofibration. But this follows from the pushout square

$$\begin{array}{ccc}
 Y_{0,j-1} \cup_{X_{0,j-1}} X_{0,j} & \xrightarrow{\quad} & Y_{0,j} \\
 \downarrow & & \downarrow \\
 Y_{0,j-1} \cup_{X_{0,j-1}} Z_{0,j} & \xrightarrow{\quad} & Y_{0,j} \cup_{X_{0,j}} Z_{0,j}
 \end{array}$$

□

**Definition 14.** Let  $(\mathcal{C}, w\mathcal{C})$  be a Waldhausen category. Let  $wS_q\mathcal{C} \subset S_q\mathcal{C}$  consist of the morphisms  $f : X \xrightarrow{\sim} Y$  of  $\text{Ar}[q]$ -shaped diagrams such that the component  $f_{0,j} : X_{0,j} \rightarrow Y_{0,j}$  is a w.e. in  $\mathcal{C}$  for each  $1 \leq j \leq q$ .

**Proposition 15.** Let  $f$  be a w.e. in  $S_q\mathcal{C}$ . Each component  $f_{i,j} : X_{i,j} \rightarrow Y_{i,j}$  is a w.e. in  $\mathcal{C}$ .

*Proof.* Apply the Gluing axiom to the diagram

$$\begin{array}{ccccc}
 X_{0,j} & \longleftarrow & X_{0,i} & \longrightarrow & * \\
 \cong \downarrow & & \cong \downarrow & & = \downarrow \\
 Y_{0,j} & \longleftarrow & Y_{0,i} & \longrightarrow & *
 \end{array}$$

Then  $X_{i,j} \cong X_{0,j} \cup_{X_{0,i}} * \xrightarrow{\sim} Y_{0,j} \cup_{Y_{0,i}} * \cong Y_{i,j}$ , as desired. □

**Lemma 16.**  $(S_q\mathcal{C}, wS_q\mathcal{C})$  is a Waldhausen category.

**Definition 17.** Let  $\mathcal{C}$  be a category with cofibrations. If  $\alpha : [p] \rightarrow [q]$ , then define  $\alpha^* : S_q\mathcal{C} \rightarrow S_p\mathcal{C}$  by

$$\alpha^*(X : \text{Ar}[q] \rightarrow \mathcal{C}) = X \circ \text{Ar}(\alpha) : \text{Ar}[p] \rightarrow \text{Ar}[q] \rightarrow \mathcal{C}.$$

It's easy to check that this satisfies the two conditions of a diagram in  $S_p\mathcal{C}$ . Moreover, the face maps  $d_i$  are obtained by deleting the row  $X_{i,-}$  and the column containing  $X_i$  in  $(\dagger)$  and then reindexing as necessary. The degeneracy maps  $s_i$  are given by duplicating  $X_i$  and then reindexing such that  $X_{i+1,i} = 0$ .

**Proposition 18.** Let  $(\mathcal{C}, w\mathcal{C})$  be a Waldhausen category. Each functor  $\alpha^* : S_q\mathcal{C} \rightarrow S_p\mathcal{C}$  is exact, so that  $(S_\bullet\mathcal{C}, wS_\bullet\mathcal{C})$  is a simplicial Waldhausen category.

The nerve  $N_\bullet wS_\bullet\mathcal{C}$  is a bisimplicial set with  $(p, q)$ -bisimplices the diagrams of the form

$$\begin{array}{ccccccc} * & \longrightarrow & X_1^0 & \longrightarrow & X_2^0 & \longrightarrow & \cdots \longrightarrow X_q^0 \\ & & \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ * & \longrightarrow & X_1^1 & \longrightarrow & X_2^1 & \longrightarrow & \cdots \longrightarrow X_q^1 \\ & & \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ & & \vdots & & \vdots & & \vdots \\ & & \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ * & \longrightarrow & X_1^p & \longrightarrow & X_2^p & \longrightarrow & \cdots \longrightarrow X_q^p \end{array}$$

such that  $X_{i,j}^k \cong X_j^k / X_i^k$  for every  $i \leq j \leq q$  and  $k \in [p]$ .

**Lemma 19.** There is a natural map  $N_\bullet w\mathcal{C} \wedge \Delta_\bullet^1 \rightarrow N_\bullet wS_\bullet\mathcal{C}$ , which automatically induces a based map  $\sigma : \Sigma|w\mathcal{C}| \rightarrow |wS_\bullet\mathcal{C}|$  of classifying spaces.

*Proof.* We can treat  $N_\bullet wS_\bullet\mathcal{C}$  as the simplicial set  $[q] \mapsto N_\bullet wS_q\mathcal{C}$ . This defines a right skeletal structure on  $N_\bullet wS_\bullet\mathcal{C}$ .

If  $q = 0$ , then  $wS_0\mathcal{C} = S_0\mathcal{C} = *$ , so that  $N_\bullet wS_0\mathcal{C} = *$  as well. If  $q = 1$ , then  $wS_1\mathcal{C} \cong w\mathcal{C}$ . Thus, the right 1-skeleton is equal to  $N_\bullet w\mathcal{C} \wedge \Delta_\bullet^1$ , which in turn must be equal to the image  $I$  of the canonical map

$$\coprod_{q \leq 1} N_\bullet wS_q\mathcal{C} \times \Delta_\bullet^q \rightarrow N_\bullet wS_\bullet\mathcal{C}.$$

Now, the degeneracy map  $s_0$  collapses  $\{*\} \times \Delta_\bullet^1$ , and the face maps  $d_0$  and  $d_1$  collapse  $N_\bullet w\mathcal{C} \times \partial\Delta_\bullet^1$ . Therefore,  $I$  must equal

$$N_\bullet w\mathcal{C} \wedge \Delta_\bullet^1 = \frac{N_\bullet w\mathcal{C} \times \Delta_\bullet^1}{\{*\} \times \Delta_\bullet^1 \cup N_\bullet w\mathcal{C} \times \partial\Delta_\bullet^1}.$$

We have defined a natural inclusion map  $\lambda : N_\bullet w\mathcal{C} \wedge \Delta_\bullet^1 \rightarrow N_\bullet wS_\bullet\mathcal{C}$ .

Since  $\Delta_{\bullet}^1$  is isomorphic to the unit interval and the map  $\lambda$  agrees on the endpoints, we can pass to  $S^1$  during the suspension. Hence  $\lambda$  induces the desired map  $\sigma$ .<sup>2</sup>  $\square$

**Note 20.** The axiom (W3) implies that  $w\mathcal{C}$  is closed under coproducts, making  $|wS_{\bullet}\mathcal{C}|$  into an  $H$ -space via the map

$$\coprod : |wS_{\bullet}\mathcal{C}| \times |wS_{\bullet}\mathcal{C}| \cong |wS_{\bullet}\mathcal{C} \times wS_{\bullet}\mathcal{C}| \rightarrow |wS_{\bullet}\mathcal{C}|.$$

**Definition 21.** Let  $(\mathcal{C}, w\mathcal{C})$  be a Waldhausen category. Define the *algebraic  $K$ -theory space*

$$K(\mathcal{C}, w) = \Omega|N_{\bullet}wS_{\bullet}\mathcal{C}|.$$

Then we have a right adjoint  $\iota : |w\mathcal{C}| \rightarrow K(\mathcal{C}, w)$  to the based map  $\sigma$ .

Moreover, let  $F : (\mathcal{C}, w\mathcal{C}) \rightarrow (\mathcal{D}, w\mathcal{D})$  be an exact functor. Let

$$K(F) = \Omega|wS_{\bullet}F| : K(\mathcal{C}, w) \rightarrow K(\mathcal{D}, w).$$

This is the *algebraic  $K$ -theory functor*  $K : \mathbf{Wald} \rightarrow \mathbf{Top}_{\ast}$ .

Recall that any exact category  $\mathcal{A}$  is a Waldhausen category with cofibrations the admissible exact sequences and w.e. the isomorphisms. Waldhausen showed that  $|iS_{\bullet}\mathcal{A}|$  (where  $i$  denotes the iso. category) and  $BQ\mathcal{A}$  are homotopy equivalent. Therefore, our current definition of higher algebraic  $K$ -theory agrees with Quillen's.

**Example 22.** Let  $R$  be a ring. Define the *algebraic  $K$ -theory space of  $R$*  as

$$K(R) = K(\mathbf{P}(R), i)$$

where the w.e.  $i$  are precisely the injective  $R$ -linear maps with projective cokernel and the cofibrations are precisely the  $R$ -linear maps.

**Example 23.** Assume that  $\mathcal{C}$  is a small Waldhausen category where  $w\mathcal{C}$  consists of the isomorphisms in  $\mathcal{C}$ . If  $s_n\mathcal{C}$  denotes the set of objects of  $S_n\mathcal{C}$ , then we get a simplicial set  $s_{\bullet}\mathcal{C}$ . Waldhausen showed that the inclusion  $|s_{\bullet}\mathcal{C}| \hookrightarrow |iS_{\bullet}\mathcal{C}|$  is a homotopy equivalence. This makes  $\Omega|s_{\bullet}\mathcal{C}|$  into a so-called simplicial model for  $K(\mathcal{C}, w)$ .

*Remark 24.* Since  $wS_0\mathcal{C} = *$  and every simplex of degree  $n > 0$  is attached to  $*$ , it follows that the classifying space  $|wS_{\bullet}\mathcal{C}|$  is connected. Therefore, we preserve any homotopical information when passing to the loop space.

**Definition 25.** The  *$i$ -th algebraic  $K$ -group* is  $K_i(\mathcal{C}, w) \equiv \pi_i K(\mathcal{C}, w)$  for each  $i \geq 0$ .

**Proposition 26.**  $\pi_1|wS_{\bullet}\mathcal{C}| \cong K_0(\mathcal{C}, w)$ .

**Lemma 27.** The group  $K_0(\mathcal{C}, w)$  is generated by all elements  $[X]$  such that

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<sup>2</sup>This is a tentative explanation due to Thomas Brazelton.

- $[X'] + [X''] = [X]$  for every cofiber sequence  $X' \rightarrow X \rightarrow X''$  and
- $[X] = [Y]$  for every w.e.  $X \xrightarrow{\sim} Y$ .

*Proof.* In light of Proposition 26, it suffices to compute  $\pi_1|N_\bullet wS_\bullet \mathcal{C}|$  based at the  $(0, 0)$ -bisimplex  $*$ . For this, just notice the CW structure of  $|N_\bullet wS_\bullet \mathcal{C}|$ , with 1-cells the  $(0, 1)$ -bisimplices and 2-cells the  $(0, 2)$ -bisimplices  $X' \rightarrow X \rightarrow X''$  and the  $(1, 1)$ -bisimplices  $X \xrightarrow{\sim} Y$ , which are attached to the 1-cells  $X$  and  $Y$ . Any cell of dimension  $n > 2$  is irrelevant to computing  $\pi_1$ .  $\square$

As a result, we obtain functors

$$K_i : \mathbf{Wald} \rightarrow \mathbf{Top}_* \rightarrow \mathbf{Ab}$$

known as the *algebraic K-group functors*. Indeed, thanks to Proposition 26, we know that

$$K_i(\mathcal{C}, w) = \pi_{i+1}|wS_\bullet \mathcal{C}|,$$

which is abelian for  $i \geq 1$ . Moreover, note that if  $X' \rightarrow X' \vee X'' \rightarrow X''$  and  $X'' \rightarrow X' \vee X'' \rightarrow X'$  are cofiber sequences, then Lemma 27 implies that

$$[X'] + [X''] = [X' \vee X''] = [X'' + X'].$$

Hence  $K_0(\mathcal{C}, w)$  is also abelian.

**Example 28.** Let  $X$  be a CW complex and  $\mathcal{R}(X)$  denote the category of CW complexes  $Y$  obtained from  $X$  by attaching at least one cell such that  $X$  is a retract of  $Y$ . Equip this with cofibrations in the form of cellular inclusions fixing  $X$  and w.e. in the form of homotopy equivalences. This makes  $\mathcal{R}(X)$  into a Waldhausen category.

If  $\mathcal{R}_f(X)$  denotes the subcategory of those  $Y$  obtained by attaching finitely many cells, then we denote  $K(\mathcal{R}_f(X))$  by  $A(X)$ .

**Proposition 29.**  $A_0(X) \cong \mathbb{Z}$ .

**Definition 30.** If  $\mathcal{B}$  is a Waldhausen subcategory of  $\mathcal{C}$ , then it is *cofinal in  $\mathcal{C}$*  if for any  $X \in \text{ob } \mathcal{C}$ , there is some  $X' \in \text{ob } \mathcal{B}$  such that  $X \coprod X' \in \text{ob } \mathcal{B}$ .

**Theorem 31.** Let  $(\mathcal{B}, w)$  be cofinal in  $(\mathcal{C}, w)$  and closed under extensions. Assume that  $K_0(\mathcal{B}) = K_0(\mathcal{C})$ . Then  $wS_\bullet \mathcal{B} \rightarrow wS_\bullet \mathcal{C}$  is a homotopy equivalence.

It follows that  $K_i(\mathcal{B}) \cong K_i(\mathcal{C})$  for every  $i \geq 0$ .