Perry Hart K-theory reading seminar UPenn October 3, 2018

#### Abstract

This is an introduction to universal properties in category theory. The main sources for this talk are the following.

- nLab.
- John Rognes's Lecture Notes on Algebraic K-Theory, Ch. 4.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4.

# 1 Universal arrows

**Definition 1.1.** An object X of  $\mathscr{C}$  is *initial* if for each  $Y \in \text{ob}\,\mathscr{C}$ , there is a unique morphism  $f: X \to Y$ . Moreover, we say that X is *terminal* if for each  $Z \in \text{ob}\,\mathscr{C}$ , there is a unique morphism  $g: Z \to X$ . Either condition is called a *universal property* of X.

Any property P of  $\mathscr{C}$  has a dual property  $P^{\text{op}}$  of  $\mathscr{C}^{\text{op}}$  obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P. Then P is true of  $\mathscr{C}^{\text{op}}$  is true of  $\mathscr{C}^{\text{op}}$ .

**Example 1.2.** Being initial and being terminal are dual properties.

**Lemma 1.3.** Any two initial objects of  $\mathscr{C}$  are canonically isomorphic. The same holds for any two terminal objects of  $\mathscr{C}$ .

*Proof.* Let X and X' be two initial objects. Compose the two unique morphisms  $X \to X'$  and  $X' \to X$  to get an isomorphism between X and X'. Apply duality to this argument for the case of terminal objects.  $\square$ 

We can think of a universal property as follows. Let  $F: \mathscr{D} \to \mathscr{C}$  be a functor and  $X \in \text{ob}\,\mathscr{C}$ . A universal arrow from X to F is an ordered pair (Y,f) with  $Y \in \text{ob}\,\mathscr{D}$  and  $f: X \to F(Y)$  a morphism of  $\mathscr{C}$  with the property that for any  $X' \in \text{ob}\,\mathscr{D}$  and morphism  $f': X \to F(X')$  of  $\mathscr{C}$ , there exists a unique morphism  $\hat{f}: Y \to X'$  of  $\mathscr{D}$  such that  $F(\hat{f}) \circ f = f'$ .

$$X \xrightarrow{f} F(Y)$$

$$\downarrow^{F(\hat{f})}$$

$$F(X')$$

Dually, a universal arrow from F to X is an ordered pair (Y, f) with  $Y \in \text{ob } \mathscr{D}$  and  $f : F(Y) \to X$  of  $\mathscr{C}$  with the property that for any  $X' \in \text{ob } \mathscr{D}$  and morphism  $f' : F(X') \to X$ , there exists a unique morphism

 $\hat{f}: X' \to Y$  such that  $f' = f \circ F(\hat{f})$ .

$$F(X') \xrightarrow{F(\hat{f})} F(Y)$$

$$\downarrow^{f}$$

$$X$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

### Definition 1.4.

- 1. Let  $F: \mathscr{C} \to \mathscr{D}$  be a functor and  $Y \in \text{ob } \mathscr{D}$ . The *slice* or *left fiber category*, denoted by (F/Y) or  $(F \downarrow Y)$ , has as objects pairs (X, f) where  $f: F(X) \to Y$  and as morphisms from  $f: F(X) \to Y$  to  $f': F(X') \to Y$  morphisms  $g: X \to X'$  such that  $f = f' \circ F(g)$ .
- 2. The coslice or right fiber category, denoted by (Y/F) or  $(Y \downarrow F)$ , has as objects pairs (X, f) where  $f: Y \to F(X)$  and as morphisms from  $f: Y \to F(X)$  to  $f': Y \to F(X')$  morphisms  $g: X \to X'$  such that  $f' = F(g) \circ f$ .

If  $F^{\text{op}}: C^{\text{op}} \to D^{\text{op}}$  is opposite to the functor  $F: \mathscr{C} \to \mathscr{D}$  and  $Y \in \text{ob } \mathscr{D}$ , then  $(Y/F)^{\text{op}} = F^{\text{op}}/Y$ . Thus, the left and right fiber categories are dual in the sense that P(Y, F) is true of any right fiber category Y/F iff  $P^{\text{op}}(Y, F)$  is true of any left fiber category F/Y.

**Proposition 1.5.** Let  $F: \mathcal{D} \to \mathscr{C}$  be a functor and  $x \in \text{ob } C$ . Then  $u: x \to Fr$  is a universal arrow from x to F iff it is initial object of the coslice  $(x \downarrow F)$ . Dually,  $u': Fr' \to x$  is a universal arrow from F to x iff it is a terminal object of the same category.

*Proof.* Suppose that u is universal and  $f: x \to Fy$  is another object of  $(x \downarrow F)$ . Then there is some unique  $\hat{f}: r \to y$  such that  $F\left(\hat{f}\right) \circ u = f$ . Thus  $F\left(\hat{f}\right)$  is a unique morphism of the coslice.

Conversely, suppose that u is initial. Then for any object  $f: x \to Fy$  of  $(x \downarrow F)$ , there is some unique arrow  $Sg: Fr \to Fy$  such that  $Sg \circ u = f$ . Hence setting  $\hat{f} = g$  makes u a universal arrow.

Corollary 1.6. Any two universal arrows from x to F can be canonically identified by Lemma 1.3.

# 2 (Co)limits

**Definition 2.1.** A zero object of  $\mathscr{C}$  is an object that is both initial and terminal.

**Example 2.2.** The unique initial object of **Set** is  $\emptyset$ , and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in iso(**Set**).

Given  $X \in \text{ob}\,\mathscr{C}$ , the undercategory  $X/\mathscr{C}$  has as objects morphisms in  $\mathscr{C}$  of the form  $i: X \to Y$  where X is fixed. Given  $i: X \to Y$  and  $i': X \to Y'$  in ob  $X/\mathscr{C}$ , define the set of morphisms from i to i' as the morphisms  $f: Y \to Y'$  where



commutes. (We call i the structure morphism.) Composition and identity carry over exactly from  $\mathscr{C}$ .

Likewise, given  $x \in \text{ob}\,\mathscr{C}$ , the overcategory  $\mathscr{C}/X$  has as objects morphisms in  $\mathscr{C}$  of the form  $i: Y \to X$  where X is fixed. Given  $i: Y \to X$  and  $i': Y' \to X$  in  $\text{ob}\,\mathscr{C}/X$ , define the set of morphisms from i to i' as the morphisms  $f: Y \to Y'$  where

$$Y \xrightarrow{f} Y' \downarrow_{i'} X$$

commutes. Composition and identity carry over exactly from  $\mathscr{C}$ .

Remark 2.3. If  $X \in \text{ob}\,\mathcal{C}$ , then  $(X/\mathcal{C})^{\text{op}} = \mathcal{C}^{\text{op}}/X$ . Thus, the under- and overcategories are dual in the sense that  $P(X,\mathcal{C})$  is true of any undercategory  $X/\mathcal{C}$  iff  $P^{\text{op}}(X,\mathcal{C})$  is true of any overcategory  $\mathcal{C}/X$ .

**Lemma 2.4.** For any  $X \in \mathcal{C}$ , the identity morphism on X is an initial object  $X/\mathcal{C}$ . Dually, it is a terminal object in  $\mathcal{C}/X$ .

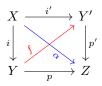
*Proof.* Any  $i: X \to Y$  is itself the unique morphism from  $\mathrm{Id}_X$  to i.

**Lemma 2.5.** Let X be an initial object of  $\mathscr{C}$ . The identity morphism on X is a zero object  $\mathscr{C}/X$ . Dually, if  $Y \in \text{ob}\,\mathscr{C}$  is terminal, then  $\text{Id}_Y$  is a zero object in  $Y/\mathscr{C}$ .

*Proof.* We already know that  $\mathrm{Id}_X$  is terminal. If  $p:Y\to X$  is an object in  $\mathscr{C}/X$ , then there is a unique morphism  $f:X\to Y$ . Then  $f\circ p$  must equal  $\mathrm{Id}_X$ .

**Example 2.6.** Let (X, x) be a pointed set with  $X = \{x\}$ . Let  $\mathbf{Set}_*$  denotes the category of pointed sets with base point preserving functions. Since  $\mathbf{Set}_* \cong X/\mathbf{Set}$ , it follows that X is a zero object in  $\mathbf{Set}_*$ .

Given a morphism  $\alpha: X \to Z$  in  $\mathscr{C}$ , define the under-and-overcategory  $(X/\mathscr{C}/Z)_{\alpha}$  as having triples (Y, i, p) as objects where  $i: X \to Y$  and  $p: Y \to Z$  are morphisms in  $\mathscr{C}$  such that  $p \circ i = \alpha$ . Define the set of morphisms from (Y, i, p) to (Y', u', p') as the set of morphisms  $f: Y \to Y'$  such that



commutes. If  $\alpha = \operatorname{Id}_X$ , then we call  $(X/\mathscr{C}/X)_{\operatorname{Id}_X}$  the category of *retractive* objects over X, with each triple (Y, i, p) being a retraction of Y onto X.

**Example 2.7.** If  $F: \mathscr{C} \to \mathscr{C}$  is the identity functor, then the undercategory  $Y/\mathscr{C}$  equals the right fiber category Y/F, and the overcategory  $\mathscr{C}/Y$  equals the left fiber category F/Y.

**Definition 2.8.** Let  $\mathscr{J}$  be a category. A diagram of shape  $\mathscr{J}$  in  $\mathscr{C}$  is a functor  $F:\mathscr{J}\to\mathscr{C}$ .

**Definition 2.9.** Given a functor  $F: \mathscr{J} \to \mathscr{C}$  and  $X \in \text{ob}\,\mathscr{C}$ , a cone over F consists of an apex  $X \in \text{ob}\,\mathscr{C}$  and  $legs\ f_j: X \to F(j)$  for each  $j \in \text{ob}\,\mathscr{J}$  such that for any  $\alpha: j \to j'$ , the triangle

$$X \xrightarrow{f_j} F(j)$$

$$\downarrow_{F\alpha}$$

$$F(j')$$

commutes.

This is simply a natural transformation  $\Delta_{\mathscr{J}}X\Rightarrow F$  where  $\Delta_{\mathscr{J}}X$  denotes the constant functor on  $\mathscr{J}$  at X. If  $\mathscr{J}$  is small, then  $\Delta_{\mathscr{J}}$  is just a functor from  $\mathscr{C}$  to  $\mathbf{Fun}(\mathscr{J},\mathscr{C})$ .

**Definition 2.10.** The category of cones over F is the right fiber category X/F. The category of cones under F is the left fiber category F/X.

**Definition 2.11.** Let  $\mathscr{C}$  and  $\mathscr{D}$  be categories and  $g: Y \to Z$  a morphism in  $\mathscr{D}$ . Let  $\Delta_{\mathscr{C}}g: \Delta_{\mathscr{C}}Y \Rightarrow \Delta_{\mathscr{C}}Z$  be the natural transformation with components  $X \mapsto g$ .

- 1. A colimit  $colim_{\mathscr{C}} F$  of the functor  $F : \mathscr{C} \to \mathscr{D}$  is an object Y of  $\mathscr{D}$  together with a natural transformation  $i : F \Rightarrow \Delta_{\mathscr{C}} Y$  such that for any  $Z \in ob \mathscr{D}$  and natural transformation  $j : F \Rightarrow \Delta_{\mathscr{C}} Z$ , there is a unique morphism  $g : Y \to Z$  such that  $j = \Delta_{\mathscr{C}} g \circ i$ .
- 2. We say that  $\mathscr{D}$  admits/has  $\mathscr{C}$ -shaped colimits if each functor  $G:\mathscr{C}\to\mathscr{D}$  has a colimit.
- 3. We sat that  $\mathscr{D}$  is *cocomplete* if each functor  $G:\mathscr{C}\to\mathscr{D}$  with  $\mathscr{C}$  small has a colimit.

If  $\mathscr C$  is small, then a colimit of  $F:\mathscr C\to\mathscr D$  is just an initial object in the right fiber category  $F/\Delta_{\mathscr C}$ , which has as objects pairs  $(Z,j:F\to\Delta Z)$  and as morphisms from (Y,i) to (Z,j) the morphisms  $g:Y\to Z$  in  $\mathscr D$  such that  $\Delta g\circ i=j$ .

Remark 2.12. There is a natural bijection  $\mathscr{D}(Y,Z) \cong \mathbf{Fun}(\mathscr{C},\mathscr{D})(F,\Delta Z)$  iff  $Y = \mathrm{colim}_{\mathscr{C}} F$ .

**Proposition 2.13.** Any two colimits are canonically isomorphic.

*Proof.* When  $\mathscr{C}$  is small, this is immediate from Lemma 1.3. But note that the proof of Lemma 1.3 does not require that  $\mathscr{C}$  be locally small (a property which Rognes stipulates of any category).

Assume that  $\mathscr{D}$  has  $\mathscr{C}$ -shaped colimits and that  $\mathscr{C}$  is small. Then a (possibly global) choice function  $\operatorname{colim}_{\mathscr{C}}: \operatorname{\mathbf{Fun}}(\mathscr{C},\mathscr{D}) \to \mathscr{D}$  given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor  $\Delta_{\mathscr{C}}: \mathscr{D} \to \operatorname{\mathbf{Fun}}(\mathscr{C},\mathscr{D})$ . Indeed, for any functor  $F: \mathscr{C} \to \mathscr{D}$ , there is a bijection  $\mathscr{D}(\operatorname{colim}_{\mathscr{C}} F, Z) \cong \operatorname{\mathbf{Fun}}(\mathscr{C}, \mathscr{D})(F, \Delta_{\mathscr{C}} Z)$ .

**Definition 2.14.** A *limit* of the functor  $F: \mathscr{C} \to \mathscr{D}$  is the colimit of  $F^{\mathrm{op}}: \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$ .

Explicitly, a limit for  $F: \mathscr{C} \to \mathscr{D}$  is an object Z of  $\mathscr{D}$  along with a natural transformation  $p: \Delta_{\mathscr{C}}Z \Rightarrow F$  such that for any  $Y \in \text{ob} \mathscr{D}$  and natural transformation  $q: \Delta_{\mathscr{C}}Y \Rightarrow F$ , there is a unique morphism  $g: Y \to Z$  such that  $q = p \circ \Delta_{\mathscr{C}}g$ .

Remark 2.15. The colimit of a functor F is the limit of  $F^{op}$ . Hence limit and colimit are dual properties, and our results so far for colimits can be dualized for limits.

**Example 2.16.** If  $\mathscr{C}$  is the empty category, then the empty functor  $F:\mathscr{C}\to\mathscr{D}$  satisfies  $F/\Delta_{\mathscr{C}}\cong\mathscr{D}$ , so that the colimit is an initial object of  $\mathscr{D}$ .

**Definition 2.17.** Let  $\mathscr{J}$  be a discrete small category. Consider a diagram  $\{A_i\}_{i\in \mathrm{ob}}\mathscr{J}$  of shape  $\mathscr{J}$ .

1. The limit of this diagram is called the *product*  $\prod_i A_i$ , equipped with projections  $\pi_i : \prod_i A_i \to A_i$  such that for every  $f_i : U \to A_i$  there is some unique map  $f := (f_i) : U \to \prod_i A_i$  satisfying  $\pi_i \circ f = f_i$ .

2. Dually, the colimit of the diagram is called the *coproduct*  $\coprod_i A_i$ , equipped with inclusions  $u_i: A_i \to \coprod_i A_i$  such that for any  $f_i: A_i \to Y$ , there is some unique map  $f:=(f_i): \coprod_i A_i \to Y$  satisfying  $f_i=f\circ u_i$ .

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Let  $\mathscr{J}$  be the category  $\bullet \rightrightarrows \bullet$ . Then a diagram of shape  $\mathscr{J}$  looks like  $A \stackrel{f}{\underset{g}{\Longrightarrow}} B$ . A cone over this with apex C and legs  $f_1: C \to A$  and  $f_2: C \to B$  satisfies  $f f_1 = f_2 = g f_1$ .

**Definition 2.18.** If such an object C together with  $f_1$  is the limit of the diagram, then we say it is the equalizer of f and g. Dually, the colimit is called the coequalizer of f and g.

**Example 2.19.** The equalizer in **Set** of  $f, g: X \to Y$  is the subset  $X' := \{x \in X : f(x) = g(x)\}$  together with the inclusion function  $X' \hookrightarrow X$ . The coequalizer of (f, g) is  $Y/\sim$  together with the quotient map on B where  $\sim$  is the smallest equivalence relation under which  $f(x) \sim g(x)$  for every x.

**Example 2.20.** The equalizer in **Grp** is defined as in Example 2.19 except that the relation  $\sim$  becomes a certain minimal normal subgroup.

Now, let  $\mathscr{J}$  be the category  $\bullet \to \bullet \leftarrow \bullet$ . Then a diagram of this shape looks like  $B \xrightarrow{f} D \xleftarrow{g} A$ , and a cone over this diagram looks like

$$\begin{array}{c|c}
C & \xrightarrow{j} & A \\
\downarrow \downarrow & & \downarrow g \\
B & \xrightarrow{f} & D
\end{array}$$

**Definition 2.21.** If such an object C together with i and j is the limit of this diagram, then we call it the pullback of f and g, denoted by  $B \times_D A$ .

We can perform an analogous construction for  $\mathscr{J}^{\text{op}}$ . Then the colimit of the resulting diagram is called the *pushout*, denoted by  $B \cup_D A$ .

**Example 2.22.** The pullback in **Set** of  $f: X \to Z$  and  $g: Y \to Z$  is the subset  $\{(x, y) \in X \times Y : f(x) = g(y)\}$ , called the *fiber product* of X and Y over Z.

All coequalizers  $A \stackrel{f}{\underset{g}{\Longrightarrow}} B \stackrel{h}{\longrightarrow} C$  can be obtained from taking binary coproducts and pushouts as follows.

$$\begin{array}{ccc}
A \coprod A \xrightarrow{(f,g)} B \\
(\operatorname{Id}_A,\operatorname{Id}_A) \downarrow & & \downarrow h \\
A & \longrightarrow & C
\end{array}$$

Therefore, any category with binary coproducts and pushouts has coequalizers.

Moreover, any colimit of a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$
 (\*)

is precisely the coequalizer of

$$\coprod_{n} X_{n} \xrightarrow{\operatorname{Id}} \coprod_{(u_{n+1} \circ f_{n})} \coprod_{n} X_{n}.$$

Therefore, any category with coequalizers and small coproducts has colimits of diagrams like (\*). This fact can be generalized as follows.

## Theorem 2.23 (Freyd).

- (i) If & has equalizers and small (resp. finite) products, then it has small (resp. finite) limits.
- (ii) If  $\mathscr C$  has pullbacks and a terminal object, then it has finite limits.

Proof.

1. Let  $F: \mathscr{J} \to \mathscr{C}$  be any diagram with  $\mathscr{J}$  small. Consider the following two morphisms in  $\mathscr{C}$ :

$$f, g: \prod_{j \in \text{ob } \mathscr{J}} F_j \to \prod_{\alpha: i \to j} F_j$$
$$\pi_{\alpha: i \to j} \circ f \equiv \pi_j$$
$$\pi_{\alpha: i \to j} \circ g \equiv F(\alpha) \circ \pi_i.$$

Then  $\lim_{\mathscr{I}} F$  is precisely the equalizer of f and g.

2. Thanks to part (i), it suffices to show that ℒ has equalizers and finite products. By assumption, there is some terminal object 1. Then any product A<sub>1</sub> × A<sub>2</sub> can be realized as the pullback of A<sub>1</sub> → 1 ← A<sub>2</sub>. By induction, ℒ has finite products. Moreover, for morphisms f, g: A → B, note that any cone over the diagram

$$A \xrightarrow{(\mathrm{Id}_A,g)} A \times B \xleftarrow{(\mathrm{Id}_A,f)} A$$

yields morphisms  $h: A \to C$  and  $k: C \to A$  such that h = k and fk = gh. As a result, the pullback for this diagram is an equalizer of f and g, and thus our proof is complete.

Corollary 2.24. Both Set and Grp are complete and cocomplete (or bicomplete).

It turns out that adjoints interact nicely with (co)limits under mild conditions.

**Proposition 2.25.** Let  $F: \mathscr{C} \to \mathscr{D}$  and  $G: \mathscr{D} \to \mathscr{C}$  be functors such that (F, G) is an adjoint pair. Let  $\mathscr{E}$  be small category. If  $X: \mathscr{E} \to \mathscr{C}$  is a functor whose colimit exists, then

$$\operatorname{colim}_{\mathscr{E}}(F \circ X) = F(\operatorname{colim}_{\mathscr{E}} X).$$

Dually, if  $Y : \mathcal{E} \to \mathcal{D}$  is a functor whose limit exists, then

$$\lim_{\mathscr{E}}(G\circ Y)=G(\lim_{\mathscr{E}}Y).$$

*Proof.* We have the following chain of bijections natural in  $Y \in \mathcal{D}$ :

$$\begin{split} \mathscr{D}(F(\operatorname{colim}_{\mathscr{E}}X),Y) &\cong \mathscr{C}(\operatorname{colim}_{\mathscr{E}}X,G(Y)) \\ &\cong \lim_{\mathscr{E}}\mathscr{C}(X(-),G(Y)) \\ &\cong \lim_{\mathscr{E}}\mathscr{D}(F(X(-)),Y) \\ &\cong \mathbf{Fun}(\mathscr{E},\mathscr{D})(F\circ X,\Delta Y). \end{split}$$

The second bijection exists because both sets can be identified with the components of all natural transformations  $X \Rightarrow \Delta G(Y)$ .

## 3 Fibrations

**Definition 3.1.** Let  $F: \mathscr{C} \to \mathscr{D}$  be a functor. The *fiber category*  $F^{-1}(Y)$  is the full subcategory of  $\mathscr{C}$  generated by the objects X with F(X) = Y.

**Definition 3.2.** Suppose  $\mathscr{C}$  has a terminal object 1.

- 1. Given a morphism  $p: 1 \to Y$ , the fiber of f at p is the pullback  $f^{-1}(p)$  of  $1 \to Y \leftarrow X$ .
- 2. The *cofiber* of a morphism  $f: X \to Y$  is the pushout Y/X of  $1 \leftarrow X \to Y$ .

Let  $F: \mathscr{C} \to \mathscr{D}$  be a functor. For each  $Y \in \text{ob } \mathscr{D}$ , there is a full and faithful functor  $F^{-1}(Y) \to F/Y$  given by  $X \mapsto (X, \operatorname{Id}_Y)$ .

**Definition 3.3.** We say that  $\mathscr{C}$  is a *precofibered category* over  $\mathscr{D}$  if F admits a left adjoint given by  $(Z, g: F(Z) \to Y) \mapsto g_*(Z)$ .

Likewise, there is a full and faithful functor  $F^{-1}(Y) \to Y/F$ . We say that  $\mathscr C$  is a *prefibered category* over  $\mathscr D$  if this functor admits a right adjoint given by  $(Z,g:Y\to F(Z))\mapsto g_*(Z)$ .

**Definition 3.4.** Let  $F: \mathscr{C} \to \mathscr{D}$  be a functor.

1. Let  $f:c'\to c$  be a morphism in  $\mathscr C$ . We say f is *cartesian* if for any morphism  $f':c''\to c$  in  $\mathscr C$  and any morphism  $g:F(c'')\to F(c')$  in  $\mathscr D$  such that  $Ff\circ g=Ff'$ , there exists a unique  $\phi:c''\to c$  such that  $f'=f\circ\phi$  and  $F\phi=g$ .

In other words, any filler of

$$c'' \xrightarrow{- \stackrel{\exists !}{---}} c' \downarrow_f$$

can be lifted to a filler in  $\mathcal{D}$ .

2. We say that F is a fibration if for any  $c \in \mathscr{C}$  and morphism  $f: d \to Fc$ , there is a cartesian  $\phi: c' \to c$  such that  $F\phi = f$ . Such an  $\phi$  is called a cartesian lifting of f to c.

**Example 3.5.** Let the category **Mod** consist of pairs (R, M) as objects where R is a ring and M is a left R-module and pairs  $(f, \tilde{f})$  as morphisms where  $f: R \to R'$  is a ring homomorphism and  $\tilde{f}: M \to M'$  is an R-linear map with M' viewed as an R-module via f. Then the forgetful functor  $U: \mathbf{Mod} \to \mathbf{Ring}$  is a fibration.