

## Abstract

These notes are based on Anindya De’s “Theory of Computation” lectures at UPenn along with Michael Sipser’s *Introduction to the Theory of Computation, 3rd ed.* and Arora and Barak’s *Computational Complexity: A Modern Approach*. Any mistake in what follows is my own.

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# 1 Automata theory

## 1.1 Lecture 1

### Definition 1.1.1.

1. An *alphabet* is a nonempty finite set of characters, e.g.,  $\Sigma := \{0, 1\}$ .
2. A *string* is a finite ordered sequence of elements from a given alphabet  $\Sigma$ . The empty sequence  $\epsilon$  is allowed.
3. Let  $\Sigma^*$  denote the set of all finite-length strings over  $\Sigma$ . Any subset of  $\Sigma^*$  is called a (*formal*) *language*.

**Example 1.1.2.** Both the set of binary strings representing prime numbers and the set of binary strings with an even number of 1's are languages.

**Remark 1.1.3.** Consider a function  $f : \{0, 1\}^* \rightarrow \{0, 1\}$ . The computation of  $f$  is equivalent to determining whether  $x \in \underbrace{f^{-1}}_{\text{a language}} \subset \{0, 1\}^*$ . Thus, computing any boolean function is the same as determining membership in some language.

**Note 1.1.4.** Finite automata are characterized by  $O(1)$  memory and passing over their inputs exactly once.

**Definition 1.1.5.** Formally, an *m-state deterministic finite automaton* (DFA) is an ordered 5-tuple

$$M := (Q, \Sigma, q_0, \delta, Q_F)$$

where  $|Q| = m$ ,  $\Sigma$  is an alphabet,  $q_0 \in Q$ ,  $\delta : Q \times \Sigma \rightarrow Q$ , and  $Q_F \subset Q$ . We call  $\delta$  the *transition function* of  $M$ .

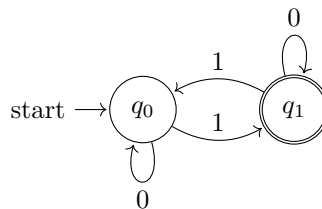
Intuitively, an automaton  $M$  is a DFA if

- (a)  $M$  has a finite number of states  $Q$ ,
- (b)  $M$  has a unique *starting state*  $q_0$ ,
- (c) for every state  $q$  and every symbol  $\sigma \in \Sigma$ , there is a unique next state  $\delta(q, \sigma)$ ,
- (d) computation begins at the starting state and applies  $\delta$  in order, and
- (e) certain states  $Q_F$  are designated as *final states*.

**Definition 1.1.6.** Let  $x := x_1x_2 \cdots x_n \in \Sigma^*$ . Set  $q_0(x) = q_0$ . For each  $1 \leq i \leq n$ , define  $q_i(x) = \delta(q_{i-1}(x), x_i)$ . If  $x = \epsilon$ , then  $n = 0$ , so that  $q_0(x) = q_0$  as well. We say that  $x$  is *accepted by*  $M$  if  $q_n(x) \in Q_F$ . We define  $L(M) = \{x \in \Sigma^* : M \text{ accepts } x\}$ .

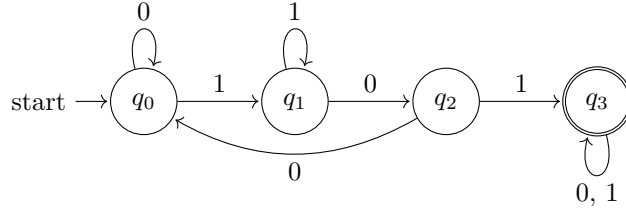
**Example 1.1.7.** Set  $\Sigma = \{0, 1\}$ .

1. Let  $M$  denote



Then  $L(M)$  consists of all binary strings with an even number of 0's.

2. Let  $M$  denote



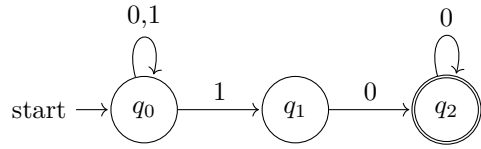
Then  $L(M) = \{x \in \Sigma^* : x = y101z \text{ for some strings } y \text{ and } z\}$ .

**Definition 1.1.8.** A language  $L$  is *regular* if there is some DFA  $M$  such that  $L(M) = L$ .

*Remark 1.1.9.* Every regular expression induces a DFA, and vice versa. Thus, they have equal expressive power. The former gives rules for generating legitimate strings whereas the latter recognizes membership of a language.

**Definition 1.1.10.** A *nondeterministic finite automaton* (NFA) is an ordered quintuple  $(Q, \Sigma, q_0, \delta, Q_F)$  of the sort above except that  $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$ .

**Example 1.1.11.** Set  $\Sigma = \{0, 1\}$  and  $M =$



If  $x = 0100$ , then

$$\begin{aligned}
 q_0(x) &= \{q_0\} \\
 q_1(x) &= \{q_0\} \\
 q_2(x) &= \{q_0, q_1\} \\
 q_3(x) &= \{q_0, q_2\} \\
 q_4(x) &= \{q_0, q_2\}.
 \end{aligned}$$

## 1.2 Lecture 2

**Definition 1.2.1.** Let  $\delta$  denote a transition function for the NFA  $N$ . Define the *multi-step transition function*

$$\hat{\delta} : Q \times \Sigma^* \rightarrow \mathcal{P}(Q)$$

inductively as follows.

$$\begin{aligned}
 \hat{\delta}(q, \epsilon) &= \{q\} \\
 \hat{\delta}(q, x) &= \bigcup_{\gamma \in \hat{\delta}(q, y)} \delta(\gamma, \sigma) \quad x = y\sigma, \ y \in \Sigma^*, \ \sigma \in \Sigma.
 \end{aligned}$$

**Note 1.2.2.** If  $p \in \hat{\delta}(q, y)$  and  $r \in \hat{\delta}(p, \sigma)$ , then  $r \in \hat{\delta}(q, y\sigma) = \hat{\delta}(q, x)$ .

**Definition 1.2.3.** A string  $x$  is *accepted by*  $N$  if  $\hat{\delta}(q_0, x) \cap Q_F \neq \emptyset$ . Let  $L(N) := \{x \in \Sigma^* : N \text{ accepts } x\}$ .

*Remark 1.2.4.* Every DFA is an NFA, in which case we have that  $\hat{\delta}(q, x) = \left\{ \delta \left( \hat{\delta}(q, y), \sigma \right) \right\}$ . It's not the case, however, that any NFA is a DFA.

**Theorem 1.2.5.** For any NFA  $N$ , there is some DFA  $M$  such that  $L(N) = L(M)$ .

*Proof.* Write  $N = (Q, \Sigma, q_0, \delta_N, Q_F)$ . Define the DFA

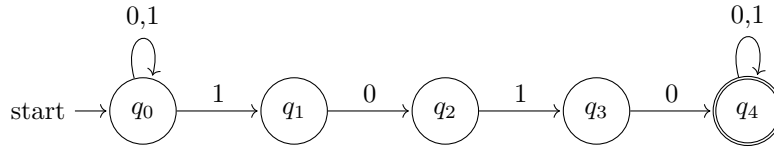
$$M = (\mathcal{P}(Q), \Sigma, q_0^{(1)}, \delta_M, Q_F^{(1)})$$

where  $q_0^{(1)} = \{q_0\}$ ,  $\delta_M(Q', \sigma) = \bigcup_{\gamma \in Q'} \delta_N(\gamma, \sigma)$  and  $Q_F^{(1)} := \{R \subset Q : R \cap Q_F \neq \emptyset\}$ . For any string  $x$ , one can use induction on  $|x|$  to show that if  $R \subset Q$ , then

$$\tilde{\delta}_M(R, x) = \bigcup_{p \in R} \widehat{\delta}_N(p, x)$$

where  $\tilde{\delta}_M : \mathcal{P}(Q) \times \Sigma^* \rightarrow \mathcal{P}(Q)$  denotes the obvious extension of  $\delta_M$  to strings. By setting  $R = \{q_0\}$ , we are done.  $\square$

**Example 1.2.6.** Let  $L \subset \{0, 1\}^*$  consist of those strings  $x$  such that “1010” appears in  $x$ . We can easily capture  $L$  with the following NFA, but writing a DFA that captures  $L$  is much harder.



**Definition 1.2.7.** An NFA is called an  $\epsilon$ -NFA if its transition function is of the form  $\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow \mathcal{P}(Q)$ . In this case, we call  $\delta$  an  $\epsilon$ -transition.

**Definition 1.2.8.** Let  $q$  be a state. The  $\epsilon$ -closure of  $q$  is the set of states that can be reached from  $q$  by taking finitely many  $\epsilon$ -transitions. This determines a function  $\epsilon\text{-cl}(-) : \mathcal{P}(Q) \rightarrow \mathcal{P}(Q)$ .

**Note 1.2.9.** We have that  $\hat{\delta}(q, x) = \begin{cases} \epsilon\text{-cl}(q) & x = \epsilon \\ \bigcup_{r \in \hat{\delta}(q, y)} \epsilon\text{-cl}(\delta(r, \sigma)) & x = y\sigma \end{cases}$ .

**Theorem 1.2.10.** For any  $\epsilon$ -NFA  $N$ , there is some DFA  $M$  such that  $L(N) = L(M)$ .

*Proof.* Use a similar argument to the proof for an NFA. In particular, set

$$q_0^{(1)} = \epsilon\text{-cl}(q_0)$$

and

$$\begin{aligned} \delta_M(R, \sigma) &= \bigcup_{r \in R} \bigcup_{p \in \epsilon\text{-cl}(r)} \bigcup_{s \in \delta_N(p, \sigma)} \epsilon\text{-cl}(s) \\ &= \bigcup_{r \in R} \bigcup_{s \in \delta_N(r, \sigma)} \epsilon\text{-cl}(s). \end{aligned}$$

$\square$

### 1.3 Lecture 3

**Proposition 1.3.1.** Let  $L_1, L_2 \subset \Sigma^*$  be regular.

(a)  $\overline{L_1} := \Sigma^* \setminus L_1 = (L_1)^c$  is regular.

**Corollary 1.3.2.** If  $L$  is finite or cofinite, then it is regular.

(b)  $L_1 \cup L_2$  is regular.

(c)  $L_1 \cap L_2$  is regular.

(d) Define  $L_1 \cdot L_2 = \{xy \mid x \in L_1 \wedge y \in L_2\}$ . Then  $L_1 \cdot L_2$  is regular.

*Proof.* By assumption, there exist DFA's  $M_1$  and  $M_2$  such that  $L_1 = L(M_1)$  and  $L_2 = L(M_2)$ .

- (a) Construct a new DFA  $M_3$  by making every final state of  $M_1$  non-final and vice versa. Then  $L(M_3) = \overline{L_1}$ .
- (b) Construct an  $\epsilon$ -NFA  $M_3$  as follows. Take a starting state  $q_0$ . Attach an  $\epsilon$ -transition from  $q_0$  to the starting state of  $M_1$  and an  $\epsilon$ -transition from  $q_0$  to the starting state of  $M_2$ . Then  $L(M_3) = L_1 \cup L_2$ .
- (c) Note that  $L_1 \cap L_2 = (\overline{L_1} \cup \overline{L_2})^c$ . Now apply (a) and (b).
- (d) Construct an  $\epsilon$ -NFA  $M_3$  as follows. Given any final state  $q$  of  $M_1$ , add an  $\epsilon$ -transition from  $q$  to the start state of  $M_2$ . Then  $L(M_3) = L_1 \cdot L_2$ .

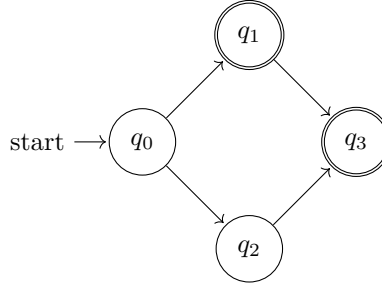
□

**Definition 1.3.3.** Let  $L$  be any language. Let  $L^k := \underbrace{L \cdot L \cdots L}_{k \text{ times}}$  for each integer  $k \geq 1$ . Moreover, define  $L^0 = \{\epsilon\}$  (which is regular). Then define

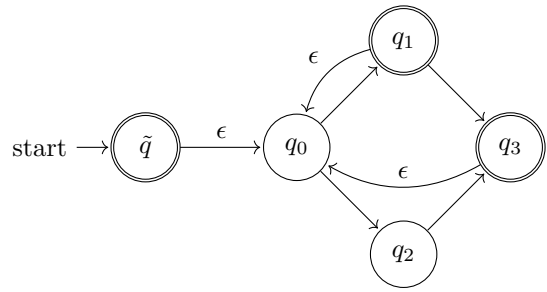
$$L^* = \bigcup_{k \geq 0} L^k.$$

**Proposition 1.3.4.** If  $L$  is regular, then  $L^*$  is regular as well.

*Proof.* There is some DFA  $M$  such that  $L(M) = L$ . Without loss of generality, write  $M$  as



Now, let  $\widetilde{M}$  denote the automaton



Then  $L(\widetilde{M}) = L^*$ . □

*Remark 1.3.5.* There exists a canonical set isomorphism  $\{F : F \text{ is a finite automaton.}\} \cong \mathbb{N}$ . Also, we have that  $\{0,1\}^* \cong \mathbb{N}$ . But then  $\mathbb{R} \cong \mathcal{P}(\mathbb{N}) \cong \mathcal{P}(\{0,1\}^*) = \{L : L \text{ is a language over } \{0,1\}\}$ . Since there is a surjection

$$\{F : F \text{ is a finite automaton.}\} \rightarrow \{L : L \text{ is a regular language over } \{0,1\}\},$$

it follows that there are uncountably many non-regular languages over  $\{0,1\}$ .

## 1.4 Lecture 4

**Lemma 1.4.1 (Pumping).** *Let  $L$  be a regular language. Then there exists  $n_0 \in \mathbb{N}$  such that for any  $x \in L$  with  $|x| \geq n_0$ , we may write  $x = wyz$  such that*

- (a)  $|y| > 0$ ,
- (b)  $|wy| \leq n_0$ , and
- (c) if  $i \geq 0$ , then  $wy^iz \in L$  where  $y^i := \underbrace{y \cdots y}_{i \text{ times}}$ .

In this case, we call the minimal such  $n_0$  the pumping length of  $L$ .

*Proof.* By assumption, there is some DFA  $M = (Q, \Sigma, \delta, q_0, Q_F)$  such that  $L = L(M)$ . Let  $n_0 = |Q|$ . Let  $x \in L$  such that  $|x| \geq n_0$ . Set  $q_i = \hat{\delta}(q_0, x_0 \cdots x_i)$  for each  $i \geq 0$ . There exist  $0 \leq i < j \leq n_0$  such that  $q_i = q_j$ . Define the three strings

$$w = x_1 \cdots x_i \quad y = x_{i+1} \cdots x_j \quad z = x_{j+1} \cdots x_m$$

where  $|x| = m$ . It is straightforward to verify that these satisfy conditions (a), (b), and (c).  $\square$

**Corollary 1.4.2.**

1. Our last proof shows that any regular language  $L$  has pumping length  $\leq |Q|$ .
2. If  $L(M) \neq \emptyset$ , then there exists  $x \in L(M)$  such that  $|x| \leq |Q|$ .

**Example 1.4.3.** Let  $n_0 \geq 0$  be an integer. Suppose that  $x = wyz$  with  $|y| > 0$  and  $|wy| \leq n_0$ .

1. Define  $L = \{1^{2^n} : n \geq 0\}$ . Let  $x := 1^{2^{n_0+1}}$ . But note that  $|wy^iz| = |wyz| + (i-1)|y| = 2^{n_0+1} + (i-1)|y|$ . Hence if  $i = 2$ , then  $|wy^iz| = 2^{n_0+1} + |y| \leq 2^{n_0+1} + n_0 < 2^{n_0+2}$  since  $n_0 < 2^{n_0+1}$ , in which case  $2^{n_0+1} < |wy^iz|$  as well. Therefore,  $L$  is not a regular language.
2. Define  $L = \{ww : w \in \{0,1\}^*\}$ . Let  $x := 0^{n_0}10^{n_0}1$ . If  $|y| = m$  with  $0 < m \leq n_0$ , then  $wz = 0^{n_0-m}10^{n_0}1$ , which does not belong to  $L$ . Hence  $L$  is not a regular language.

*Aside.* Let  $D$  be a DFA with  $|Q_D| = n$ . Then  $D$  recognizes an infinite language if and only if it accepts some string  $s$  such that  $n \leq |s| \leq 2n$ .

*Proof.* The ( $\Leftarrow$ ) direction follows from the pumping lemma. Conversely, suppose that  $L(D)$  is infinite. Then  $D$  contains some path  $p$  of states from the start state to a final state as well as some cycle of states  $c$  such that  $c \cap p \neq \emptyset$ . Note that  $|c| \leq n$  and  $|p| \leq n$ . Hence we can apply  $c$  sufficiently many times to get our desired string.  $\square$

## 2 Computability theory

**Definition 2.0.1.** A *Turing machine* is a 7-tuple

$$(Q, \Sigma, \Gamma, \delta, q_0, Q_F, Q_R)$$

where  $\Gamma \supset \Sigma$  is finite such that there is some *null character*  $\perp \in \Gamma \setminus \Sigma$ ,  $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ , and  $Q_F, Q_R \subset Q$  such that  $Q_F \cap Q_R = \emptyset$ . We call  $\Sigma$  the *input alphabet*,  $\Gamma$  the *tape alphabet*,  $Q_F$  the set of *accepting states*, and  $Q_R$  the set of *rejecting states*.

**Note 2.0.2.** On any given input  $x$ , a TM can either *accept* or *reject* or *loop* (i.e., fail to halt on  $x$ ).

*Remark 2.0.3.* A Turing machine is supposed to act as a minimal model of computation. It should be able to write, be able to move left and right, and have unconstrained memory.

## 2.1 Lecture 5

*Remark 2.1.1.* Every finite automaton may be viewed as a Turing machine  $M$  with the following properties.

- (a)  $M$  never writes on the tape.
- (b)  $M$ 's read-write head moves to the right only.
- (c)  $M$  writes on just a finite portion of the tape.
- (d)  $M$  either accepts or rejects immediately after reading the input string.

*Remark 2.1.2.* Adding a stay option  $S$  to the set  $\{L, R\}$  would not increase a Turing machine's computational power.

**Definition 2.1.3.** Let  $M$  be a Turing machine. Let  $q \in Q$  and  $u, v \in \Gamma^*$ . We say that the *configuration* of  $M$  is  $uqv$  if

- (a) the current state of  $M$  is  $q$ ,
- (b) the current tape contents is precisely  $uv$ , and
- (c) the current head location is the first symbol of  $v$ .

We call this an *accepting configuration* if  $q \in Q_F$  and a *rejecting configuration* if  $q \in Q_R$ .

**Definition 2.1.4.** Let  $a, b, c \in \Gamma$  and  $u, v \in \Gamma^*$ . Let  $p, q \in Q$ . We say that the configuration  $C_1$  *yields* the configuration  $C_2$  in the following cases.

- (a)  $uapbv$  yields  $uqacv$  when  $\delta(p, b) = (q, c, L)$ .
- (b)  $uapbv$  yields  $uacqv$  when  $\delta(p, b) = (q, c, R)$ .

We write  $C_1 \vdash C_2$ .

**Definition 2.1.5.** We say that the TM  $M$  *accepts* the input  $w$  if there is some sequence  $C_1, \dots, C_k$  of configurations such that  $C_1$  is  $\perp q_0 w$ , each  $C_i$  yields  $C_{i+1}$  (in which case we write  $C_1 \vdash^* C_k$ ), and  $C_k$  is an accepting configuration. We define the *language of the TM  $M$*  as

$$L(M) = \{x \in \Sigma^* : M \text{ accepts } x\}.$$

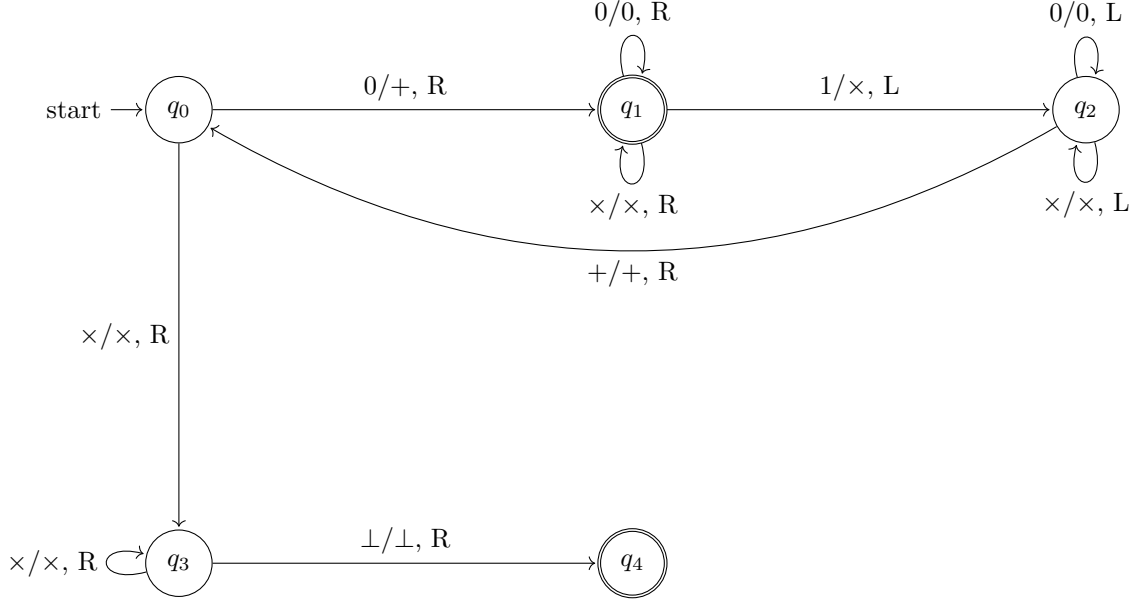
**Definition 2.1.6.** Let  $L$  be a language.

1. We say that  $L$  is *Turing-recognizable* or *recursively enumerable* if there is some TM  $M$  such that  $L = L(M)$ .
2. We say that  $L$  is *decidable* or *recursive* if there is some TM  $M$  such that  $L = L(M)$  and  $M$  halts on every input. In this case, we say that  $M$  is an *algorithm*.

**Note 2.1.7.** Every decidable language is recursively enumerable.

**Example 2.1.8.** By the pumping lemma, one may show that the language  $L := \{0^n 1^n : n \geq 1\}$  is not regular. But  $L$  is decidable.

*Proof.* Set  $\Sigma = \{0, 1\}$  and  $\Gamma = \Sigma \cup \{\perp, +, \times\}$ . Define the TM  $M$  as follows.



Then  $L(M) = L$ . □

*Remark 2.1.9.* The *Church-Turing thesis* states that our pre-theoretic notion of algorithm is entirely captured by decidability (equivalently,  $\lambda$ -computability).

## 2.2 Lecture 6

**Definition 2.2.1.** A *multi-tape Turing machine* is exactly like an ordinary Turing machine except that the former's transition function is of the form

$$\delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k$$

where  $k \in \mathbb{N}$ .

**Theorem 2.2.2.** For any  $k \in \mathbb{N}$  and any language  $L$ , if there is some  $k$ -tape TM  $M$  such that  $L(M) = L$ , then there is some single-tape TM  $M'$  such that  $L = L(M')$ . Moreover,  $T$  steps of a  $k$ -tape TM can be simulated using  $O_k(T^2)$  steps of a single-tape TM.

*Proof.* Write  $M = (Q, \Sigma, \Gamma, \delta, q_0, Q_F, Q_R)$ . Let

$$\{\#\} \bigcup_{\gamma \in \Gamma} \{\gamma, \dot{\gamma}\}$$

be the tape alphabet of  $M'$ . Construct  $M'$  so that its tape always has  $\#$  in a cell separating the current contents of  $M$ 's different tapes and  $\dot{\gamma}$  whenever the head of the tape of  $M$  containing  $\gamma$  is currently at  $\gamma$ . We make  $M'$  scan its tape once to determine the positions of the heads of  $M$ 's tapes, then scan it again to update its contents according to  $\delta$ . □

**Example 2.2.3.** Let  $L = \{w \in \{0, 1\}^* \mid w \text{ is a palindrome}\}$ . This cannot be recognized by a TM running in better than quadratic time but can be recognized by a 2-tape TM running in linear time. Thus, computational models may differ in complexity even when they don't in decidability.

**Definition 2.2.4.** A *nondeterministic Turing machine*  $M$  is exactly like an ordinary Turing machine except that  $M$ 's transition function is of the form

$$\delta : Q \times \Gamma \rightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

such that  $M$  accepts on an input as long as at least one branch of computation accepts.



**Theorem 2.2.5.** *Let  $M$  be a NDTM and let  $L = L(M)$ . Then there exists a TM  $M'$  such that  $L = L(M')$ .*

*Proof.* By Theorem 2.2.2, it suffices to construct a multi-tape TM that simulates  $M$ . Construct a tree with branches corresponding to threads of computation given by  $M$  and nodes corresponding to configurations of  $M$ . We construct a 3-tape TM  $D$  as follows.

Tape 1 always contains the input string  $w$  and nothing else. Tape 2 contains just the string on the tape of the current node. Setting  $m = \max\{|A| : A \in \text{im } \delta\}$ , tape 3 contains a string over  $\{1, \dots, m\}$  that corresponds to the “address” of the current node.

We make  $D$  simulate a breadth-first search of the tree as follows.

1. Initialize tape 1 with  $w$  and tape 3 with  $\epsilon$ .
2. Copy tape 1 to tape 2.
3. Move to the node given by the next symbol on tape 3.
4. Read the configuration of  $M$  on  $w$  that is determined by this node.
5. Accept or reject if this is an accepting or rejecting configuration, respectively.
6. Replace the current string on tape 3 with the next string under the string order.
7. Do step 1.

□

**Note 2.2.6.**

1. This simulation has time complexity  $O(m^T)$  where  $T$  denotes the steps taken by  $M$ .
2. We can modify the proof of our last theorem to show that if  $M$  always halts on each branch of computation, then  $M'$  always halts. Thus, a language is decidable if and only if a NDTM decides it.

*Remark 2.2.7.* There is an injective function  $\iota$  from the set of Turing machines into  $\{0, 1\}^*$  because any TM's transition function admits a finite description. In particular, the set of TM is countable. Let  $\langle M \rangle$  denote the binary encoding of the TM  $M$ . Let any  $x \notin \text{im } \iota$  correspond to the TM that immediately halts and outputs zero on every input. As a result, every binary string corresponds to some Turing machine.

**Theorem 2.2.8.** *There is a TM (denoted by  $U_{\text{TM}}$ ) taking two strings as inputs,  $\langle M \rangle$  and  $x$ , (i.e., one string over  $\{0, 1\} \times \Sigma$ ) such that*

1. *if  $M$  accepts  $x$ , then  $U_{\text{TM}}$  accepts,*
2. *if  $M$  rejects  $x$ , then  $U_{\text{TM}}$  rejects, and*
3. *if  $M$  does not halt on  $x$ , then neither does  $U_{\text{TM}}$ .*

*We call  $U_{\text{TM}}$  a universal Turing machine. Moreover, if  $M$  takes  $T$  steps on  $w$ , then  $U_{\text{TM}}$  takes  $O(T \log T)$  steps on  $\langle M, w \rangle$ .*

*Proof.* This is like constructing an interpreter for a programming language within the language itself. See Arora and Barak, Theorem 1.13 for a high-level proof. □

## 2.3 Lecture 7

**Note 2.3.1.** Alternatively, we can view  $U_{\text{TM}}$  as a ternary function with input  $(\langle M \rangle, w, 1^k)$  such that

1. if  $M$  accepts (resp. rejects) on  $w$  in  $\leq k$  steps, then  $U_{\text{TM}}$  accepts (resp. rejects), and
2. if  $M$  does not halt on  $w$  in  $\leq k$  steps, then  $U_{\text{TM}}$  will reach a special state.

**Lemma 2.3.2.** Let  $A_{\text{TM}}$  denote the language  $\{\langle M, w \rangle \mid M \text{ accepts } w\}$ . Then  $A_{\text{TM}}$  is recursively enumerable.

*Proof.* Observe that  $L(U_{\text{TM}}) = A_{\text{TM}}$ . □

**Theorem 2.3.3.**  $A_{\text{TM}}$  is undecidable.

*Proof.* Suppose, for contradiction, that there is some  $M$  that decides  $A_{\text{TM}}$ . Design a new TM  $N$  as follows.

- (a) Given a binary string  $x$ , run  $M$  on  $(\langle M_x \rangle, \langle M_x \rangle)$  where  $M_x$  denotes the TM corresponding to  $x$ .
- (b) Let  $N$  reject when  $M$  accepts and  $N$  accept when  $M$  rejects.

Then

$$N(\langle N \rangle) = \begin{cases} \text{accept} & N \text{ does not accept } \langle N \rangle \\ \text{reject} & N \text{ accepts } \langle N \rangle \end{cases},$$

which is impossible. □

**Lemma 2.3.4.** If  $L$  is decidable, then clearly  $\bar{L}$  is decidable.

**Lemma 2.3.5.** If both  $L$  and  $\bar{L}$  are recursively enumerable, then  $L$  is decidable.

*Proof.* Find some  $M_1$  and  $M_2$  such that  $L = L(M_1)$  and  $\bar{L} = L(M_2)$ . Construct some TM  $M$  as follows.

---

**Algorithm 1:** pseudocode describing  $M$

---

```

Input: the string  $w$ 
1  $T = 1$ ;
2 while the current state is a non-halting state do
3   run  $U_{\text{TM}}$  on  $(\langle M_1 \rangle, w)$  for  $T$  steps;
4   if  $U_{\text{TM}}$  accepts then
5     | accept
6   else
7     run  $U_{\text{TM}}$  on  $(\langle M_2 \rangle, w)$  for  $T$  steps ;
8     if  $U_{\text{TM}}$  accepts then
9       | reject
10    else
11      |  $T += 1$ 
12    end
13  end
14 end

```

---

□

**Corollary 2.3.6.**  $\overline{A_{\text{TM}}}$  is not recursively enumerable.

**Definition 2.3.7.** A function  $f : \Sigma^* \rightarrow \Sigma^*$  is *computable* if there is some TM  $M$  such that for any string  $w$ ,  $M$  halts on  $w$  with its tape containing just  $f(w)$ .

**Definition 2.3.8.** Let  $L \subset \Sigma^*$  and  $L' \subset \Sigma^*$  be languages. We say that  $L$  *many-one reduces* to  $L'$ , written as  $L \leq_m L'$ , if there is some computable function  $f : \Sigma^* \rightarrow \Sigma^*$  such that  $x \in L \iff f(x) \in L'$ . In this case, we call  $f$  the *reduction* from  $L$  to  $L'$ .

**Lemma 2.3.9.** Suppose that  $L \leq_m L'$  and that  $L'$  is decidable (resp. recursively enumerable), then  $L$  is decidable (resp. recursively enumerable). In this way,  $L'$  is at least as “hard” as  $L$ .

*Proof.* Find some  $M$  that decides (resp. recognizes)  $L'$  and some reduction  $f$  from  $L$  to  $L'$ . Construct the TM  $N$  so that on input  $w \in \Sigma^*$ , we let  $N$  compute  $f(w)$  and then output whatever  $M$  outputs on  $f(w)$ . Then  $N$  decides (resp. recognizes)  $L$ . □

## 2.4 Lecture 8

**Example 2.4.1 (Halting problem).** Let  $A_{\text{HALT}} := \{\langle M, w \rangle \mid M \text{ halts on } w\}$ . Then  $A_{\text{HALT}}$  is undecidable.

*Proof.* Recall that  $A_{\text{TM}}$  is undecidable. Thus, it suffices to show that  $A_{\text{TM}} \leq_m A_{\text{HALT}}$ . To do this, we want to design a computable function that maps any  $\langle M, w \rangle$  to another  $\langle M', w' \rangle$  such that  $M'$  halts on  $w'$  precisely when  $M$  accepts  $w$ . Construct such an  $M'$  as follows.

---

**Algorithm 2:** pseudocode describing  $M'$

---

**Input:** the string  $x$

```

1 run  $U_{\text{TM}}$  on  $(\langle M \rangle, x)$ ;
2 if  $U_{\text{TM}}$  accepts then
3   | accept
4 else
5   | while true do
6     |   pass
7   | end
8 end
```

---

Then we get a suitable function given by  $(\langle M \rangle, w) \mapsto (\langle M' \rangle, w)$ . □

**Theorem 2.4.2 (Rice).** Every nontrivial semantic property of Turing machines is undecidable. Formally, let  $C$  be any subset of the universe of all languages over a fixed alphabet. Define  $L_C = \{\langle M \rangle : L(M) \in C\}$ . Suppose that both  $L_C$  and  $\overline{L_C}$  are nonempty. Then  $L_C$  is undecidable.

*Proof.* We may assume that  $\emptyset \notin C$  for otherwise we could show that  $\overline{L_C}$  is undecidable. We know that  $L(M_y) \in C$  for some TM  $M_y$ . We show that  $A_{\text{TM}} \leq_m L_C$ . Consider any  $\langle M, w \rangle \in A_{\text{TM}}$ . Define  $M'$  as follows.

---

**Algorithm 3:** pseudocode describing  $M'$

---

**Input:** the string  $x$

```

1 run  $U_{\text{TM}}$  on  $\langle M, w \rangle$ ;
2 if  $U_{\text{TM}}$  accepts then
3   | run  $U_{\text{TM}}$  on  $\langle M_y, x \rangle$ 
4 else
5   | reject
6 end
```

---

If  $M$  accepts  $w$ , then  $L(M') = L(M_y)$ . If  $M$  rejects  $w$ , then  $L(M') = \emptyset$ . If  $M$  does not halt on  $w$ , then  $L(M') = \emptyset$ . □

**Example 2.4.3.** Let  $A_{\text{fin}} := \{\langle M \rangle : L(M) \text{ is finite}\}$ . Then  $A_{\text{fin}}$  is undecidable.

**Example 2.4.4.** Moreover,  $A_{\text{fin}}$  is not recursively enumerable.

*Proof.* Recall that  $\overline{A_{\text{TM}}}$  is not recursively enumerable. We show that  $\overline{A_{\text{TM}}} \leq_m A_{\text{fin}}$ . Given any  $\langle M, w \rangle$ , define  $M'$  as follows.

---

**Algorithm 4:** pseudocode describing  $M'$

---

**Input:** the string  $x$

```

1 run  $U_{\text{TM}}$  on  $\langle M, w \rangle$ ;
2 if  $U_{\text{TM}}$  accepts then
3   | accept
4 else
5   | reject
6 end
```

---

If  $M$  accepts  $w$ , then  $L(M') = \{0, 1\}^*$ . Otherwise,  $L(M') = \emptyset$ . □

**Note 2.4.5.** It's possible that both a language and its complement are not recursively enumerable.

## 3 Complexity theory

### 3.1 Lecture 9

*Remark 3.1.1.* Once we know that a question is decidable, we want to determine the amount of computational resources required to decide it. Such resources include

- (a) time
- (b) space
- (c) parallelism
- (d) communication
- (e) rounds
- (f) randomness .

We also want to study how these trade off with each other.

**Definition 3.1.2.** Given any TM  $M$ , its *time complexity* is the function  $f : \mathbb{N} \rightarrow \mathbb{N}$  where  $f(n) = \max\{\text{steps used by } M \text{ on input } w \mid |w| = n\}$ . We say that  $M$  runs in time  $f(n)$ .

**Definition 3.1.3.** Given any *time-constructible* function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , define  $\text{DTIME}(f(n)) = \{L : \exists \text{ TM } M \text{ such that } L = L(M) \text{ and } M \text{ halts on all inputs of length } n \text{ in } O(f(n)) \text{ steps}\}$ . Let  $\mathbf{P} := \bigcup_{k \geq 0} \text{DTIME}(n^k)$ .

**Proposition 3.1.4.**  $\mathbf{P}$  is independent of the variant of deterministic Turing machine used.

*Remark 3.1.5.* Given any convex body, we want to compute its volume. In 1989, Dyer and Frieze proved that this is solvable in  $O(n^{23})$  steps. It is now known that it's solvable in  $O(n^2)$  steps.

**Example 3.1.6.** For any  $k \geq 0$ ,  $\text{DTIME}(n^k) \subset \text{DTIME}(n^{k+1})$ . Consider the case where  $k = 2$ . Then we can show that this containment is proper by using diagonalization. Indeed, define the language  $L$  as follows.

1. If  $x$  is of the form  $w10^i$  for some  $w$  and some  $i$ , then let  $x \notin L$ .
2. Otherwise, let  $M_w$  be the TM corresponding to  $w$ .
3. In this case, run  $M_w$  on  $x$  for  $n^2$  steps where  $n$  denotes  $|x|$ .
4. If  $M_w$  does not halt in so many steps, then let  $x \notin L$ .
5. Else, let  $x \in L$  when  $M_w$  rejects and let  $x \notin L$  when  $M_w$  accepts.

Steps 2 and 3 together take  $O(n^2 \log n)$  steps. It follows that  $L \in \text{DTIME}(n^2 \log n) \subset \text{DTIME}(n^3)$ . But  $L \notin \text{DTIME}(n^2)$ .

**Theorem 3.1.7 (Time hierarchy).** Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be any function. Suppose that  $g(n) = \omega(f(n) \log f(n))$ . Then  $\text{DTIME}(f(n)) \subsetneq \text{DTIME}(g(n))$ .

**Definition 3.1.8.** We say that a NDTM  $M$  has *time complexity*  $t(n)$  if every branch of  $M$  runs in time  $t(n)$ . Define  $\text{NTIME}(t(n)) = \{L : \exists \text{ NDTM } M \text{ such that } L(M) = L \text{ and every branch of } M \text{ halts on any input of length } n \text{ in } O(t(n)) \text{ steps}\}$ . Let  $\mathbf{NP} := \bigcup_{k \geq 0} \text{NTIME}(n^k)$ .

**Proposition 3.1.9.**  $\text{NTIME}(t(n)) \subset \text{DTIME}(2^{O(t(n))})$ .

**Example 3.1.10.** Let  $\varphi$  be a Boolean formula in conjunctive normal form (CNF). Suppose that  $\varphi$  contains  $n$  clauses and  $m$  literals. Then the size of the representation in bits of  $\varphi$  is of order  $O(2m \cdot n) = O(mn)$ . Also, deciding whether  $\varphi$  evaluates to 1 takes  $O(2m \cdot n) = O(mn)$  steps.

Define  $\text{CNF-SAT} = \{\langle \varphi \rangle : \varphi \text{ is satisfiable}\}$ . Notice that this can be decided by a nondeterministic Turing machine in linear time. There is, however, no known deterministic Turing machine running faster than brute force.

### 3.2 Lecture 10

**Lemma 3.2.1.** A language  $L$  belongs to **NP** if and only if there exist a deterministic TM  $V(\cdot, \cdot)$  and constants  $c_1, c_2 > 0$  such that  $L = \{x \mid \exists y. |y| \leq |x|^{c_1} \wedge V(x, y) = 1 \text{ where } V \text{ runs in time } |x \cdot y|^{c_2}\}$ . We call  $V$  a verifier and  $y$  a witness.

*Proof.*

( $\implies$ ) There is some NDTM  $M$  running in polynomial time such that  $L(M) = L$ . Say that  $M$  runs in time  $n^c$ . The sequence of choices along any branch of  $M$  can be represented by a binary string of length  $n^c$ . Define  $V$  as the algorithm taking inputs  $x$  and  $y$  with  $y \in \{0, 1\}^{n^c}$  and executing  $M$  on  $x$  with choice of branch given by  $y$ . Then  $V$  runs in polynomial time, and  $x \in L \iff V(x, y) = 1$  for some  $y$ .

( $\impliedby$ ) Given an input  $x$ , define a NDTM  $M$  that first guesses a witness  $y$  in a separate tape and then runs  $V$  on  $(x, y)$  in polynomial time.  $\square$

**Note 3.2.2.** Equivalently, we could have made  $V$  run in time  $|x|^{c_2}$  while dropping the requirement that  $|y|$  be polynomial in  $|x|$ .

**Definition 3.2.3.** An *independent set* of a graph  $G = (V, E)$  is a set  $I \subset V$  such that no two points in  $I$  are connected by an edge.

**Example 3.2.4.** The following languages are in **NP**.

1.  $\text{IND-SET} := \{\langle G, k \rangle : G \text{ is an (undirected) graph with an independent set of size at least } k\}$
2.  $\text{3-COLOR} := \{\langle G \rangle : G \text{ has a 3-coloring}\}$
3.  $\text{Composite} := \{x \mid x \text{ is a composite number}\}$
4.  $\text{PRIMES} := \{n \mid n \text{ is prime}\}$

**Definition 3.2.5.** We say that  $L_1$  *polynomially many-one reduces to*  $L_2$  (written as  $L_1 \leq_m^p L_2$ ) if there is some TM  $M$  running in polynomial time such that  $x \in L_1 \iff M(x) \in L_2$ .

**Lemma 3.2.6.** If  $L_1 \leq_m^p L_2$  and  $L_2 \in \mathbf{P}$ , then  $L_1 \in \mathbf{P}$ .

**Definition 3.2.7.** We say that  $L$  is **NP**-complete if  $L \in \mathbf{NP}$  and for any  $L' \in \mathbf{NP}$ ,  $L' \leq_m^p L$ .

### 3.3 Lecture 11

**Definition 3.3.1.** A (*Boolean*) *circuit* is a directed acyclic graph with a unique sink node such that

1. each node has indegree at most 2
2. each internal node is labelled by  $\wedge$ ,  $\vee$ , or  $\neg$ ,
3. each leaf node is labeled by a Boolean variable, and
4. each edge is labeled by the Boolean value given as the output of the prior node.

The *size of a circuit* is the number of its internal nodes.

*Remark 3.3.2.* This is an example of a non-uniform model of computation as we must specify a new circuit for each input size.

**Lemma 3.3.3.** Every function  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  can be computed by a circuit of size  $O(2^n)$ .

*Proof.* We use induction to show that any  $g : \{0, 1\}^n \rightarrow \{0, 1\}$  can be computed by a circuit of size at most  $3 \cdot 2^n - 4$ , which is enough. When  $n = 1$ , there are four cases to consider.

- (a) If  $g = \text{id}_{\{0,1\}}$ , then  $g$  can be computed by a circuit of size 0.
- (b) If  $g(0) = 1$  and  $g(1) = 0$ , then  $g(x) = \neg x$ .

(c) If  $g(0) = g(1) = 0$ , then  $g(x) = x \wedge \neg x$ .

(d) If  $g(0) = g(1) = 1$ , then  $g(x) = x \vee \neg x$ .

Hence the base case holds. Now, define  $g_0, g_1 : \{0, 1\}^{n-1} \rightarrow \{0, 1\}$  by  $g_0(y) = g(0, y)$  and  $g_1(y) = g(1, y)$ . Then  $g$  satisfies

$$g(y) = (\neg y_1 \wedge g_0(y_2, \dots, y_n)) \vee (y_1 \wedge g_1(y_2, \dots, y_n))$$

for each  $y$ . By induction,  $g$  can be computed by a circuit of size at most  $4 + 2(3 \cdot 2^{n-1} - 4) = 3 \cdot 2^n - 4$ .  $\square$

**Lemma 3.3.4.** *Let  $M = (Q, \Sigma, \Gamma, q_0, \delta, Q_F, Q_R)$  be a TM. Suppose that on any input of size  $n$ ,  $M$  halts in at most  $t$  steps with  $t \geq n$ . Then there is a circuit of size  $O(t^2 \cdot (|\Gamma| \cdot |Q|)^3) = O(t^2)$  that outputs 1 on a string  $x$  of length  $n$  if and only if  $M$  accepts  $x$ .*

*Proof.* We want to encode a given configuration of  $M$ , which we may assume uses at most  $t$  cells of tape. To do this, we take  $\log |\Gamma|$  bits, 1 bit, and  $\log |Q|$  bits to encode the content of the current cell, whether or not the head is located at this cell, and, if so, the current state, respectively. For each  $i, j \geq 0$ , the bit  $b_{j,l+1}$  representing the  $j$ -th cell at time  $l + 1$  depends precisely on the three bits  $b_{j-1,l}$ ,  $b_{j,l}$ , and  $b_{j+1,l}$ . If  $B := 1 + \log |\Gamma| + \log |Q|$ , then every bit of the encoding of the configuration of  $M$  at time  $l + 1$  depends on  $3B$  bits. This determines a Boolean function  $f : \{0, 1\}^{3B} \rightarrow \{0, 1\}$  that computes the next configuration. Our previous lemma implies that  $f$  can be computed by a circuit of size  $O(2^{3B})$  and hence by one of size  $O((|\Gamma| \cdot |Q|)^3)$ . Thus, there is a circuit of size  $O(t \cdot t \cdot (|\Gamma| \cdot |Q|)^3)$  that simulates  $M$  on inputs of size  $n$ .  $\square$

**Example 3.3.5 (Cook-Levin theorem).** Define  $\text{CIRCUIT-SAT} = \{\langle C \rangle : \exists x. C(x) = 1\}$ . This is certainly in **NP**. We claim that it is **NP**-complete.

*Proof.* If  $L \in \text{NP}$ , then there is an efficient (i.e., polynomial-time) algorithm  $V$  such that

$$\forall x \in L. \exists y \in \Sigma^*. |y| \leq |x|^{O(1)} \wedge V(x, y) = 1 \wedge (x \notin L \implies \forall y. V(x, y) = 0).$$

Thus, for each  $n \in \mathbb{N}$ , we can use our previous lemma to construct a circuit of size  $n^{O(1)}$  such that  $C(x, y) = V(x, y)$  for each string  $x$  of size  $n$  and each string  $y$  with  $|y| \leq n^{O(1)}$ . This means that for each string  $x$ , we can construct a circuit  $C_x(\cdot)$  of size  $|x|^{O(1)}$  such that  $C_x(y) = V(x, y)$  for any  $y$  with  $|y| \leq |x|^{O(1)}$ . The mapping  $M : x \mapsto \langle C_x(\cdot) \rangle$  satisfies  $x \in L \iff M(x) \in \text{CIRCUIT-SAT}$ , as desired.  $\square$

### 3.4 Lecture 12

**Corollary 3.4.1.** *Showing that  $\text{CIRCUIT-SAT}$  is not in **P** is equivalent to showing that **P**  $\neq$  **NP**.*

**Corollary 3.4.2.** *Suppose that  $\text{CIRCUIT-SAT} \leq_m^p L$  and  $L \in \text{NP}$ . Then any  $L' \in \text{NP}$  satisfies  $L' \leq_m^p L$ , i.e.,  $L$  is **NP**-complete.*

**Note 3.4.3.** Our last two corollaries hold with  $\text{CIRCUIT-SAT}$  replaced by any **NP**-complete language.

**Definition 3.4.4.** A Boolean formula in CNF is a *3cnf formula* if each clause contains exactly 3 literals.

**Example 3.4.5.**

1. Define  $3\text{-SAT} = \{\langle \varphi \rangle \mid \varphi \text{ is a 3cnf formula that is satisfiable}\}$ . This is certainly in **NP**. We claim that it is **NP**-complete.

*Proof.* It suffices to show that  $\text{CIRCUIT-SAT} \leq_m^p 3\text{-SAT}$ . We must construct an efficient algorithm  $M(-)$  such that the circuit  $C$  is satisfiable if and only if  $\varphi := M(\langle C \rangle)$  is satisfiable. If  $C$  has size  $n$ , then, wlog, we can use the associativity of our Boolean operations to add at most  $n^k$  internal nodes to  $C$  such that each gate labeled by  $\wedge$  or  $\vee$  takes exactly two inputs.

Let  $g_1, \dots, g_n$  and  $x_1, \dots, x_m$  denote the Boolean values given by the edges and inputs of  $C$ , respectively. Relabel  $g_1, \dots, g_n, x_1, \dots, x_m$  as  $w_1, \dots, w_{n+m}$ . Let  $\varphi$  be the 3cnf formula in the variables  $w_1, \dots, w_{n+m}$  where each clause of  $\varphi$  corresponds either to  $C$ 's output value  $w_s \vee w_s \vee w_s$  or to one of  $C$ 's internal edges. In the latter case, we can give the following descriptions.

- If  $w_j = \neg w_i$ , then  $\varphi$  contains exactly one clause of the form

$$(w_i \vee w_j) \wedge (\neg w_i \vee \neg w_j).$$

- If  $w_h = w_i \wedge w_j$  in  $C$ , then  $\varphi$  contains exactly one clause of the form

$$(w_i \vee w_j \vee \neg w_h) \wedge (w_i \vee \neg w_j \vee \neg w_h) \wedge (\neg w_i \vee w_j \vee \neg w_h) \wedge (\neg w_i \vee \neg w_j \vee w_h).$$

- If  $w_h = w_i \vee w_j$  in  $C$ , then  $\varphi$  contains exactly one clause of the form

$$(w_i \vee w_j \vee \neg w_h) \wedge (w_i \vee \neg w_j \vee w_h) \wedge (\neg w_i \vee w_j \vee w_h) \wedge (\neg w_i \vee \neg w_j \vee w_h).$$

By construction,  $\varphi$  is satisfiable if and only if  $C$  is. The algorithm  $M : \langle C \rangle \mapsto \varphi$  is linear in  $n^k$ , hence efficient. Hence it is a suitable reduction.  $\square$

2. It's clear that IND-SET is in **NP**. We claim that this is **NP**-complete.

*Proof.* We show that  $3\text{-SAT} \leq_m^p \text{IND-SET}$ . Let  $\varphi$  be a 3cnf-formula and write  $\varphi = c_1 \wedge c_2 \wedge c_3 \wedge \dots \wedge c_m$ . For each clause  $c_i$ , create a triangle  $t_i$  with vertices corresponding to the three literals in  $c_i$ . Let  $G_\varphi$  denote the graph obtained from the graph  $\coprod_{i=1}^m t_i$  by adding an edge between any two conflicting vertices  $v$  and  $\neg v$  in  $\coprod_{i=1}^m t_i$ . Then the algorithm  $\varphi \mapsto \langle G_\varphi, m \rangle$  defines a suitable reduction.  $\square$

3. We say that  $K \subset V$  is a *vertex cover* of a graph  $G = (V, E)$  if any  $(x, y) \in E$  has  $x \in K$  or  $y \in K$ . Define  $\text{VC} = \{ \langle G, k \rangle \mid G \text{ has a vertex cover of size at most } k \}$ . Then  $\text{IND-SET} \leq_m^p \text{VC}$ , so that VC is **NP**-complete.

*Proof.* Let  $G$  be a graph with an independent set  $S$  with  $|S| \geq k$ . Then  $V \setminus S$  is a vertex cover of  $G$ . Conversely, if  $G$  has a vertex cover  $K$  of size at most  $|V| - k$ , then  $V \setminus K$  is an independent set of size at least  $k$  in  $G$ . Thus, the algorithm  $\langle G, k \rangle \mapsto \langle G, |V| - k \rangle$  defines a suitable reduction.  $\square$

### 3.5 Lecture 13

**Definition 3.5.1.** We say that a language  $L$  is **NP-hard** if  $L' \leq_m^p L$  for any  $L' \in \text{NP}$ .

**Definition 3.5.2.** Let  $G = (V, E)$  be a graph. A subset  $C \subset V$  is a *clique* in  $G$  if any two distinct points in  $C$  are adjacent. Let  $\text{CLIQUE} := \{ \langle G, k \rangle \mid G \text{ has a clique of size at least } k \}$ .

**Definition 3.5.3.** If  $G = (V, E)$  is a graph, then the graph  $\overline{G} = (V, E')$  where  $E' = \{ (x, y) \mid (x, y) \notin E \}$  is the *complement graph* of  $G$ .

**Proposition 3.5.4.** A set  $S$  of vertices in a graph  $G$  is an independent set in  $G$  if and only if it is a clique in  $\overline{G}$ .

**Example 3.5.5.**  $\text{IND-SET} \leq_m^p \text{CLIQUE}$  via  $\langle G, k \rangle \mapsto \langle \overline{G}, k \rangle$ . Hence CLIQUE is **NP-hard**.

**Definition 3.5.6.** Let  $G = (V, E)$  be an (undirected) weighted graph with weight function  $w : E \rightarrow \mathbb{R}_{\geq 0}$ . Suppose  $C \subset V$ . A *Steiner tree* in  $G$  is a connected subgraph of  $G$  with no cycles that contains each vertex in  $C$ .

*Remark 3.5.7.* Any connected subgraph of minimum weight must be a tree.

*Problem.* Given a weighted graph  $G$  and set of vertices  $C$  in  $G$ , find the Steiner tree in  $G$  of minimum weight that contains  $C$ .

**Definition 3.5.8.** Let  $G = (V, E)$ . A subset  $C \subset V$  has a *Steiner tree of total weight  $W$*  if there exists a connected subgraph of  $G$  that contains every vertex in  $C$  and has weight at most  $W$ .

**Example 3.5.9.** Let  $\text{Steiner-tree} := \{ \langle G, C, W \rangle \mid C \text{ has a Steiner tree of total weight at most } W \}$ . This is **NP-complete**.

*Proof.* It's clear that **Steiner-tree** is in **NP**. To show that it is also **NP**-hard, we prove that  $\text{VC} \leq_m^p \text{Steiner-tree}$ . Let  $G = (V, E)$  be a graph with a vertex cover  $S$  of size  $k$ .

Let

$$V' = \bigcup_{v \in V} [v] \bigcup_{(u,v) \in E} [u, v].$$

Build  $E'$  as follows.

- (a) Let  $([u], [v]) \in E'$  for any  $u, v \in V$  and set  $w'([u], [v]) = 1$ .
- (b) If  $(u, v) \in E$ , then let  $([u], [u, v]), ([u, v], [v]) \in E'$  and set  $w'([u], [u, v]) = w'([u, v], [v]) = 1$ .
- (c) If  $(v, w) \in E$  and  $u, v, w$  are pairwise distinct, then let  $([u], [v, w]) \in E'$  and set  $w'([u], [v, w]) = 2$ .
- (d) Finally, for any  $(u, v), (w, z) \in E$ , let  $([u, v], [w, z]) \in E'$  with weight

$$w'([u, v], [w, z]) = \begin{cases} 2 & (u, v) \text{ and } (w, z) \text{ share a vertex} \\ 3 & \text{otherwise} \end{cases}.$$

Set  $C = \{[u, v] : (u, v) \in E\}$ . Also, set  $W = |E| + k - 1$ . Let  $G' = (V', E', w')$ . Note that we have constructed  $G'$  in polynomial time.

**Claim.**  $G'$  has a Steiner tree containing  $C$  with weight at most  $|E| + k - 1$ .

*Proof.* Write  $S = \{v_1, \dots, v_k\}$ . Let  $S' := \{[v_1], \dots, [v_k], \bigcup_{(u,v) \in E} [u, v]\}$  and  $E_{S'} := \{(x, y) \in E' \mid w'(x, y) = 1 \text{ and } x, y \in S'\}$ . Since  $S$  is a vertex cover, we see that  $(S', E_{S'})$  is a connected subgraph of  $G'$  that contains  $C$  and has weight  $|E| + k - 1$ .  $\square$

**Claim.** If  $C$  has a Steiner tree  $T$  of total weight  $W \leq |E| + k - 1$ , then  $G$  has a vertex cover of size  $k$ .

*Proof.* Alter  $T$  as follows.

- Replace any edge of weight 2 between  $[w]$  and  $[u, v]$  with the edges  $([w], [u])$  and  $([u], [u, v])$ .
- Replace any edge of weight 2 between  $[u, v]$  and  $[v, w]$  with the edges  $([u, v], [v])$  and  $([v], [v, w])$ .
- Replace any edge of weight 3 between  $[u, v]$  and  $[w, z]$  with the edges  $([u, v], [v]), ([v], [w]),$  and  $([w], [w, z])$ .

The resultant graph  $T'$  is connected and contains  $C$ . Note that  $T'$  has weight at most  $|E| + k - 1$  where each edge of  $T'$  has weight 1. This implies that  $T'$  spans at most  $|E| + k$  vertices. Since  $T'$  contains  $C$ , it follows that  $T'$  contains a set  $R$  of vertices of the form  $[v]$  such that  $|R| \leq k$ . If  $(x, y) \in E$ , then  $[x, y]$  is connected to some  $[v_0]$  by edges of weight 1. This means that  $[x]$  or  $[y]$  is in  $T'$ , so that  $[x]$  or  $[y]$  is in  $R$ . This shows that  $R$  is a vertex cover for  $G$ .  $\square$

$\square$

*Remark 3.5.10.* Any undirected weighted (connected) graph can be endowed with a metric by taking the shortest path between any two vertices. Our choice of weights in part (d) of our construction of  $E'$  made  $G'$  a metric space.

**Example 3.5.11.** Let  $\text{SUBSET-SUM} := \{\langle a_1, \dots, a_k, t \rangle \mid a_i, t \geq 0, \exists S \subset [k]. \sum_{i \in S} a_i = t\}$ . **SUBSET-SUM** is **NP**-complete.



*Proof.* It's clear that this is in **NP**. We show that  $\text{VC} \leq_m^p \text{SUBSET-SUM}$ . Let  $G = (V, E)$  such that  $|V| = n$  and  $|E| = m$ . We can make  $E$  totally ordered. Suppose that  $G$  has a vertex cover  $C$  of size  $k$ . For each  $v \in V$ , we define an integer  $a_v \geq 0$  in base-4 (written from left to right) consisting of  $m+1$  digits. Further, for each  $e \in E$ , we define an integer  $b_e \geq 0$  in base-4 consisting of  $m+1$  digits. Specifically, if  $0 \leq i \leq |E| - 1$  and  $(u, v) \in E$  is the  $i$ -th edge, then define both  $a_u$  and  $a_v$  as the integer

$$0 \cdots 0 \underbrace{1}_{i\text{-th digit}} 0 \cdots 01$$

and define  $b_{(u,v)}$  as the integer

$$0 \cdots 0 \underbrace{1}_{i\text{-th digit}} 0 \cdots 00.$$

Now, set  $t = k \cdot 4^m + \sum_{i=0}^{m-1} 2 \cdot 4^i$ .

Let  $S = \{a_v \mid v \in C\} \cup \{b_{(u,v)} \mid \text{exactly one of } u \text{ and } v \text{ belongs to } C\}$ . Note that we can construct  $S$  in polynomial time. It's straightforward to check that the terms of  $S$  sum to  $t$ .

**Claim.** Suppose that there are  $U \subset V$  and  $T \subset E$  such that  $t = \sum_{u \in U} a_u + \sum_{(u,v) \in T} b_{(u,v)}$ . Then  $U$  is a vertex cover for  $G$  of size at most  $k$ .

*Proof.* Since  $t < (k+1)4^m$  and each  $a_u > 4^m$ , it follows that  $|U| \leq k$ . Note that, in base-4, each of the first  $m$  digits of  $t$  equals 2. Thus, for each  $(u, v) \in E$ , at least two of  $a_u$ ,  $a_v$ , and  $b_{(u,v)}$  contribute to the summation  $\sum_{u \in U} a_u + \sum_{(u,v) \in T} b_{(u,v)}$ . This implies that at least one of  $u$  and  $v$  belongs to  $U$ . Hence  $U$  is a vertex cover for  $G$ .  $\square$

$\square$

*Remark 3.5.12.* Using dynamic programming, one can show that there is a  $\text{poly}(k, t)$  algorithm deciding SUBSET-SUM. This result, however, does not imply that SUBSET-SUM  $\in \mathbf{P}$ , because the size of the whole input  $\langle a_1, \dots, a_k, t \rangle$  is on the order of  $k \log t$ .

## 3.6 Lecture 14

**Definition 3.6.1.** Let  $S : \mathbb{N} \rightarrow \mathbb{N}$ .

1. Define the *space complexity class*  $\text{DSpace}(S(n)) = \{L \mid \exists \text{TM } M \text{ such that } L = L(M) \text{ and on any input of length } n, M \text{ touches at most } S(n) \text{ cells (on its work tape)}\}$ .
2. Define  $\text{NSpace}(S(n)) = \{L \mid \exists \text{NDTM } M \text{ such that } L = L(M) \text{ and on any input of length } n, M \text{ halts on every branch of computation and touches at most } S(n) \text{ cells on any branch}\}$ .

Unless we state otherwise, we assume that  $S(n) \geq \log n$ .

**Definition 3.6.2.** Let  $M$  be a Turing machine that always halts. Define the *configuration graph*  $G_{M,x}$  of  $M$  on input  $x$  as the directed graph  $(V, E)$  where  $V$  consists of the possible configurations of  $M$  on  $x$  and  $E = \{(C, C') : C \vdash C'\}$ . By making  $M$  erase the contents of its work tapes right before halting, we may assume that  $M$  has exactly one accepting configuration  $C_{\text{accept}}$  on  $x$ .

**Note 3.6.3.** Since  $M$  always halts, it can never reach the same configuration more than once. Thus,  $G_{M,x}$  is a directed acyclic graph.

**Lemma 3.6.4.**  $\text{NSpace}(S(n)) \subset \text{DTIME}(2^{O(S(n))})$ .

*Proof.* Let  $L \in \text{NSpace}(S(n))$  with  $L(M) = L$ . Given any input  $x$  of length  $n$ , we can use  $O(S(n))$  bits to describe the current contents of the tape,  $O(\log S(n))$  bits to describe the current state, and  $O(1)$  bits to describe the current location of the head. Thus, we need  $O(S(n)) + O(1) + O(\log S(n)) = O(S(n))$  bits to describe any vertex of  $G_{M,x}$ .

Note that the number of configurations of  $M$  is at most  $2^{O(S(n))}$  (provided that any configuration of a NDTM yields at most two distinct configurations). Therefore, we can construct  $G_{M,x}$  in  $2^{O(S(n))}$  steps. Now apply the standard linear-time BFS for connectivity to  $G_{M,x}$  to decide if there is a path from  $C_{\text{start}}$  to  $C_{\text{accept}}$ .  $\square$

**Corollary 3.6.5.**

1.  $\text{DTIME}(S(n)) \subset \text{DSpace}(S(n)) \subset \text{NSpace}(S(n)) \subset \text{DTIME}(2^{O(S(n))})$ .
2.  $\text{DTIME}(S(n)) \subset \text{NTIME}(S(n)) \subset \text{NSpace}(S(n))$ .

*Remark 3.6.6.* It is not known whether these chains of containment can be improved.

**Definition 3.6.7.**

1. Define  $\text{PSPACE} = \bigcup_{k \geq 0} \text{DSpace}(n^k)$ .
2. Define  $\text{NPSPACE} = \bigcup_{k \geq 0} \text{NSpace}(n^k)$ .

**Note 3.6.8.**

1.  $\mathbf{P} \subset \text{PSPACE}$ .
2.  $\mathbf{NP} \subset \text{NPSPACE}$ .

**Proposition 3.6.9.** Let  $\mathbf{L} := \text{DSpace}(\log n)$  and  $\mathbf{NL} := \text{NSpace}(\log n)$ .

1. Let  $L_1 = \{\langle x, y, z \rangle \mid x \cdot y = z\}$  and  $L_2 = \{\langle x, y, z \rangle \mid x + y = z\}$ . Then  $L_1, L_2 \in \mathbf{L}$ .
2. Let  $\text{DIR-REACH} := \{\langle G, s, t \rangle \mid G \text{ is directed and the vertex } t \text{ is reachable from } s\}$ . Then  $\text{DIR-REACH} \in \mathbf{NL}$ .

*Remark 3.6.10.* Omer Reingold has shown that  $\text{REACH} := \{\langle G, s, t \rangle \mid G \text{ is undirected and the vertex } t \text{ is reachable from } s\}$  belongs to  $\mathbf{L}$ .

**Theorem 3.6.11 (Savitch).** Recall that  $\text{NTIME}(S(n)) \subset \text{DTIME}(2^{O(S(n))})$ . But  $\text{NSpace}(S(n)) \subset \text{DSpace}(S^2(n))$ .

**Corollary 3.6.12.**  $\text{PSPACE} = \text{NPSPACE}$ .

### 3.7 Lecture 15

**Theorem 3.7.1 (Savitch).** Recall that  $\text{NTIME}(S(n)) \subset \text{DTIME}(2^{O(S(n))})$ . But we have that

$$\text{NSpace}(S(n)) \subset \text{DSpace}(S^2(n)).$$

*Proof.* Let  $L \in \text{NSpace}(S(n))$  with  $L(M) = L$ . Recall that the configuration graph  $G_{M,x}$  has at most  $T_0 := 2^{O(S(n))}$  nodes. Consider the following recursive algorithm.

---

```

Input: the string  $x$ 
1 for  $j \in \{1, \dots, T_0\}$  do
2   if  $\text{REACH}(C_{\text{start}}, j, \frac{T_0}{2})$  and  $\text{REACH}(j, C_{\text{accept}}, \frac{T_0}{2})$  then
3     output "yes"
4   else
5     output "no"
6   end
7 end

```

---

Denote the space complexity of the preceding algorithm by  $\mathcal{L}(T_0)$ . Note that we can reuse the space used by the first recursive call for the second recursive call. Since we need  $\log T_0$  cells to encode the counter, it follows that

$$\mathcal{L}(T_0) = \log T_0 + \mathcal{L}(T_0/2) + O(1).$$

Note that the recursion depth is precisely  $\log T_0$ . Using this, we compute

$$\begin{aligned}
 \mathcal{L}(T_0) &= \log T_0 + \mathcal{L}(T_0/2) \\
 &= \log^2 T_0 + \mathcal{L}(1) = \log^2 2^{O(S(n))} + \mathcal{L}(1) \\
 &= O(S^2(n)) + O(S(n)) = O(S^2(n)).
 \end{aligned}$$

□

**Corollary 3.7.2.**  $\text{NL} \subset \text{P}$  because  $\text{DIR-REACH}$  is  $\text{NL}$ -complete and, by our last proof, has a  $\log^2 n$ -space deterministic algorithm.

**Note 3.7.3.**

1. We have that  $\text{NTIME}(\text{poly}(n)) \subset \text{NPSACE}(\text{poly}(n)) \subset \text{DPSACE}(\text{poly}(n)) = \text{PSPACE}$ .
2.  $\text{PSPACE}$  is closed under complementation.

**Example 3.7.4.**

1. Let  $\Sigma_2\text{-SAT} := \{\varphi \text{ Boolean} \mid \forall \bar{x} \exists \bar{y} (\varphi(\bar{x}, \bar{y}) = 1)\}$ . It is unclear that this (or its complement) belongs to  $\text{NP}$ .
2. Let  $\text{TQBF-SAT} := \{\varphi(\bar{x}_1, \dots, \bar{x}_n) \mid Q_1 \bar{x}_1 Q_2 \bar{x}_2 Q_3 \bar{x}_3 \cdots Q_n \bar{x}_n (\varphi(\bar{x}_1, \dots, \bar{x}_n) = 1), Q_i \in \{\forall, \exists\}\}$ . This stands for the set of *totally quantified Boolean formulas*. It belongs to  $\text{PSPACE}$ .

*Proof.* Construct an algorithm  $T(\varphi)$  as follows.

- If  $\varphi$  is quantifier-free, then evaluate it directly. Accept if it evaluates to 1 and reject otherwise.
- If  $\varphi = \exists x \psi$ , then recursively call  $T$  on  $\psi$  once with  $x = 0$  and once with  $x = 1$ . Accept if either of these recursive calls accepts and reject otherwise.
- If  $\varphi = \forall x \psi$ , then recursively call  $T$  on  $\psi$  once with  $x = 0$  and once with  $x = 1$ . Accept if both of these recursive calls accept and reject otherwise.

If  $m$  denotes the size of  $\varphi$ , then  $\mathcal{L}(m) = m^{O(1)} + \mathcal{L}(m-1) + O(1) = m^{O(1)} + \mathcal{L}(m-1) = O(m^k)$  for some  $k$ .  $\square$

### 3.8 Lecture 16

**Definition 3.8.1.** A language  $L$  is  $\text{PSPACE}$ -complete if it belongs to  $\text{PSPACE}$  and for any  $L' \in \text{PSPACE}$ ,  $L' \leq_m^p L$ .

**Example 3.8.2.**  $\text{TQBF-SAT}$  is  $\text{PSPACE}$ -complete.

*Proof.* Let  $L \in \text{PSPACE}$  with  $L(M) = L$ . Given any input  $x$  with  $|x| = n$ , we want to construct a  $\text{TQBF } \varphi_{c,c',i}$  of size  $O(S(n)^2)$  that is satisfiable if and only if there is a path of length at most  $2^i$  from  $c$  to  $c'$  in the configuration graph  $G_{M,x}$ . This will imply that

$$\hat{\varphi} := \varphi_{C_{\text{start}}, C_{\text{accept}}, O(S(n))}$$

is true if and only if  $M$  accepts  $x$  if and only if  $x \in L$ .

We have previously constructed such a  $\varphi_{c_1, c_2, 0}$ . Moreover, if  $i \geq 1$ , then we see that

$$\begin{aligned} \varphi_{c_1, c_2, i} &\equiv \exists c (\varphi_{c_1, c, i-1} \wedge \varphi_{c, c_2, i-1}) \\ &\equiv \exists c \forall D^1 \forall D^2 ((D^1 = c_1 \wedge D^2 = c) \vee (D^1 = c) \wedge (D^2 = c_2)) \implies \varphi_{D^1, D^2, i-1}. \end{aligned}$$

It follows that  $|\varphi_{c_1, c_2, i}| \leq |\varphi_{c_1, c_2, i-1}| + O(S(n))$ , so that  $|\hat{\varphi}| \leq O(S(n)^2)$ , as desired.  $\square$

**Proposition 3.8.3.** A language  $L$  belongs to  $\text{NL}$  if and only if there exist  $c \in \mathbb{N}$  and a deterministic TM  $V(\cdot, \cdot)$  consisting of one read-only input tape, one work tape, and one read-only, single-axis proof tape such that  $V$ 's work tape uses  $O(\log |(\text{first input})|)$  space and

$$L = \{x \mid \exists y. |y| \leq |x|^c \wedge V(x, y) = 1\}.$$

### 3.9 Lecture 17

#### Definition 3.9.1.

1. Let  $M$  be a TM consisting of one read-only input tape, one work tape, and one write-only, single-axis output tape such that  $M$ 's work tape uses  $O(\log n)$  space. We call  $M$  a *log space transducer*.
2. Let  $A$  and  $B$  be languages. We say that  $A$  is *log space reducible to  $B$* , written as  $A \leq_l B$ , if there is some log space transducer  $M : \Sigma^* \rightarrow \Sigma^*$  such that  $x \in A \iff M(x) \in B$ .
3. A language  $L$  is **NL-complete** if it is in **NL** and  $L' \leq_l L$  for any  $L' \in \mathbf{NL}$ .

**Proposition 3.9.2.** *If  $L \in \mathbf{NL}$  and  $L' \leq_l L$ , then  $L' \in \mathbf{NL}$ .*

**Example 3.9.3.** DIR-REACH is **NL-complete**.

*Proof.* See Arora and Barak, Theorem 4.16. □

**Theorem 3.9.4 (Immerman-Szelepcsényi).**

$$\mathbf{NL} = \mathbf{co-NL}.$$

*Proof.* Since DIR-REACH is **NL-complete**, it suffices to show that  $\text{DIR-REACH}^c \in \mathbf{NL}$ . Let  $G = (V, E)$  be a graph of size  $n$  and  $s, t \in V$ . Let  $C_i$  denote the set of vertices  $v \in V$  reachable from  $s$  in at most  $i$  steps. We can assume that  $V$  is ordered  $(v_1, \dots, v_n)$  since each index can be described in  $\log n$  bits.

First, let  $v \in V$ . Given that  $|C_i| = k$ , we can verify that  $v \notin C_i$  as follows.

1. Propose a list  $v_{s_1}, \dots, v_{s_m}$  of vertices and a path  $s \rightsquigarrow v_{s_i}$  for each  $1 \leq i \leq m$ .
2. Write the next  $v_{s_i}$  on the work tape and write the proposed path  $s \rightsquigarrow v_{s_i}$  on the proof tape.
3. Verify that the proposed path is valid. If not, then *reject*.
4. Otherwise, verify that  $s_i > s_{i-1}$  and  $v_{s_i} \neq v$ . If not, then *reject*.
5. Repeat steps 2-4 until there is no  $v_{s_i}$  left.
6. Verify that  $m = k$  by using a counter on the work tape. If not, then *reject*. Otherwise, *accept*.

Second, given that  $|C_{i-1}| = k$ , we can verify that  $v \notin C_i$  as follows.

1. Propose a list  $v_{s_1}, \dots, v_{s_m}$  of vertices and a path  $s \rightsquigarrow v_{s_i}$  for each  $1 \leq i \leq m$ .
2. Write the next  $v_{s_i}$  on the work tape and write the proposed path  $s \rightsquigarrow v_{s_i}$  on the proof tape.
3. Verify that the proposed path is valid. If not, then *reject*.
4. Otherwise, verify that  $s_i > s_{i-1}$ ,  $v_{s_i} \neq v$ , and  $v_{s_i}$  is not a neighbor of  $v$ . If not, then *reject*.
5. Repeat steps 2-4 until there is no  $v_{s_i}$  left.
6. Verify that  $m = k$  by using a counter on the work tape. If not, then *reject*. Otherwise, *accept*.

Finally, let  $c \in \mathbb{N}$ . Given that  $|C_{i-1}| = k$ , we can verify that  $|C_i| = c$  as follows.

1. Write the next  $v_i$  on the work tape (where  $1 \leq i \leq n$ ).
2. Decide if  $v_i \in C_i$  using our two previous algorithms.
3. Repeat steps 1-2 until there is no  $v_i$  left.
4. Determine the number  $r$  of vertices in  $C_i$  by using a counter on the work tape. If  $r = c$ , then *accept*. Otherwise, *reject*.

Note that each of our three verifiers uses  $O(\log n)$  space on its work tape and is polynomial in  $n$  on its proof tape. Apply our final algorithm iteratively  $n$ -times to verify the size of  $C_n$ . Since we can reuse space on the work tape, our space complexity on it remains  $O(\log n)$ . Next, apply our first algorithm to verify that  $t \notin C_n$ , in which case  $t$  is not reachable from  $s$ .  $\square$

**Corollary 3.9.5.** *If  $s(n) \geq \log n$ , then  $\text{NSPACE}(s(n)) = \text{co-NSPACE}(s(n))$ .*

*Remark 3.9.6.* Bertrand's postulate implies that some prime number between  $N$  and  $2N$  always exists. Suppose that we want to find the least such prime  $\tilde{p}$ . Consider the probability  $\mathbb{P}$  that a randomly chosen number between  $N$  and  $2N$  is prime. From the prime number theorem, it is known that  $\mathbb{P} \approx \frac{N}{\log N}$ . As a result, we can apply the AKS primality test  $O(\log N)$  times to find  $\tilde{p}$  with high probability.

### 3.10 Lecture 18

#### Definition 3.10.1.

1. We call a TM a *probabilistic/randomized Turing machine* if it consists of an input tape, a work tape, and a “random bits” tape.
2. Let  $p : \mathbb{N} \rightarrow \mathbb{N}$  be a polynomial. A probabilistic TM  $M(\cdot, \cdot)$  *decides  $L$  with respect to  $p$*  if
  - for any  $x \in L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{p(|x|)}}[M(x, r) = 1] \geq \frac{2}{3}$  and
  - for any  $x \notin L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{p(|x|)}}[M(x, r) = 0] \geq \frac{2}{3}$ .

**Definition 3.10.2.** A language  $L$  is in  $\underbrace{\text{BPTIME}(t(n))}_{\text{bounded error probabilistic time}}$  if there exists a randomized TM  $M$  running in time  $t(|x|)$  (with probability 1) such that  $M$  decides  $L$  with respect to  $t(n)$ . Let  $\mathbf{BPP} := \bigcup_{c \geq 0} \text{BPTIME}(n^c)$ .

**Note 3.10.3.** It's clear that  $\mathbf{P} \subset \mathbf{BPP}$ .

**Proposition 3.10.4.**  $\mathbf{P} \neq \mathbf{NP} \implies \mathbf{P} = \mathbf{BPP}$ .

*Remark 3.10.5.*

1. Computing the value of the determinant of an  $n \times n$  matrix of integers via cofactor expansion takes  $\omega(n!)$  steps. Computing it via Gaussian elimination, however, takes  $O(n^3)$  steps.
2. Computing the value of the determinant of an  $n \times n$  matrix of linear forms over  $\mathbb{Z}$  via cofactor expansion is exponential in  $n$ . There is no known deterministic polynomial time algorithm for such a computation.

#### Proposition 3.10.6.

- (a) Let  $L$  be an  $n \times n$  matrix of linear forms in  $\mathbb{Z}[x_1, \dots, x_n]$  whose coefficients are in  $[-2^n, 2^n]$ . Then  $\det L$  is a polynomial in  $x_1, \dots, x_n$  with (total) degree  $n$  and each coefficient an integer  $\leq 2^{O(n^2)}$ .
- (b) Let  $p(x)$  be a univariate polynomial of degree  $d \geq 0$ . Let  $S \subset \mathbb{Z}$  be finite. Then  $\mathbb{P}[p(x) = 0] \leq \frac{d}{|S|}$  for any  $x \in_R S$ .

**Lemma 3.10.7 (DeMillo-Lipton-Schwartz-Zippel).** Let  $p(x_1, \dots, x_n)$  be a multivariate polynomial of degree at most  $d \geq 0$ . Let  $S \subset \mathbb{Z}$  be finite. Then for any  $a_1, \dots, a_n$  randomly chosen with replacement from  $S$ ,

$$\mathbb{P}[p(a_1, \dots, a_n) = 0] \leq \frac{d}{|S|}.$$

*Proof.* We use induction on  $n$ . If  $n = 1$ , then this is exactly Proposition 3.10.6(b). Now, we can write

$$p(x_1, \dots, x_n) = \sum_{i=0}^d x_1^i q_i(x_2, \dots, x_n)$$

where  $\deg q_i \leq d - i$  for each  $i$ . Let  $k$  be maximal such that  $q_k(x_2, \dots, x_n) \neq 0$ . Let  $E$  denote the event that  $q_k(x_2, \dots, x_n) = 0$ . By induction together with Proposition 3.10.6(b), it follows that

$$\begin{aligned} \mathbb{P}[p(a_1, \dots, a_n) = 0] &\leq \mathbb{P}[E] + \mathbb{P}[p(a_1, \dots, a_n) = 0 \mid \neg E] \\ &\leq \frac{d-k}{|S|} + \frac{k}{|S|} = \frac{d}{|S|}. \end{aligned}$$

□

**Example 3.10.8.** Let  $L$  be an  $n \times n$  matrix of linear forms in  $\mathbb{Z}[x_1, \dots, x_n]$  whose coefficients are in  $[-2^n, 2^n]$ . Define the probabilistic TM  $A$  on input  $\langle L \rangle$  as follows.

1. Set  $S = \{1, \dots, 100n\}$ .
2. Choose  $a_1, \dots, a_n$  randomly from  $S$  with replacement
3. Evaluate  $\det(a_1, \dots, a_n)$ . If this equals 0, then *accept*. Otherwise, *reject*.

Then  $A$  accepts  $\langle L \rangle$  with probability 1 when  $\det L = 0$ . Also, it rejects with probability  $\geq \frac{99}{100}$  when  $\det L \neq 0$  because

$$\mathbb{P}[\det(a_1, \dots, a_n) = 0] \leq \frac{1}{100}.$$

Since evaluating a polynomial is polynomial in its degree, we see that  $A$  is polynomial in  $n$ .

### 3.11 Lecture 19

**Example 3.11.1.** Let  $G = (V_1, V_2, E)$  be a bipartite graph with  $|V_1| = |V_2| = n$ . A *perfect matching* is a permutation  $\sigma$  of  $\{1, 2, \dots, n\}$  such that  $(i, \sigma(i)) \in E$  for each  $i = 1, \dots, n$ .

Let  $M = (m_{i,j})$  be the  $n \times n$  matrix with

$$m_{i,j} = \begin{cases} X_{ij} & (i, j) \in E \\ 0 & (i, j) \notin E \end{cases}.$$

Since

$$\det M = \sum_{\sigma \in S_n} (-1)^{\text{sgn } \sigma} \prod_{i=1}^n m_{i, \sigma(i)},$$

we see that  $\det M \neq 0$  if and only if  $G$  has some perfect matching. By Example 3.10.8, it follows that deciding whether or not a finite graph has a perfect matching is in **BPP**.

**Lemma 3.11.2 (Chernoff bound).** Let  $X_1, \dots, X_n$  be independent boolean-valued random variables. Let  $\mathbb{E}[X_i] = \mu$ . Then  $Z = \frac{X_1 + \dots + X_n}{n}$ , so that  $\mathbb{E}[Z] = \mu$ . Then

$$\mathbb{P}[|Z - \mu| \geq t] \leq e^{-\frac{t^2 n \mu}{4}}$$

for any  $t \in (0, 1)$ .

**Corollary 3.11.3.** Let  $M$  be a randomized polynomial-time TM and  $L \subset \Sigma^*$  be a language. Suppose that there exists  $c > 0$  such that

- for any  $x \in L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{\text{poly}(|x|)}}[M(x, r) = 1] \geq \frac{1}{2} + n^{-c}$  and
- for any  $x \notin L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{\text{poly}(|x|)}}[M(x, r) = 0] \geq \frac{1}{2} + n^{-c}$

where  $n$  denotes  $|x|$ . Then for any  $c' > 0$ , there exist a randomized polynomial-time TM  $M'$  such that

- for any  $x \in L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{\text{poly}(|x|)}}[M'(x, r) = 1] \geq 1 - 2^{-n^{c'}}$  and
- for any  $x \notin L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{\text{poly}(|x|)}}[M'(x, r) = 0] \geq 1 - 2^{-n^{c'}}$ .

*Proof.* Set  $m = n^{2c+2c'+100}$ . Define  $M'$  as follows. On any input  $x$ , run  $M(x)$   $m$  times, with outputs  $y_1, \dots, y_m$ . Accept if  $M$  accepts  $x$  more than  $\frac{m}{2}$  times and reject otherwise.

For each  $i \in \{1, \dots, m\}$ , define the random variable

$$X_i = \begin{cases} 1 & y_i = \chi_L(x) \\ 0 & \text{otherwise} \end{cases}.$$

Then  $\mathbb{E}[X_i] = \mathbb{P}[X_i = 1] \geq \mu := \frac{1}{2} + n^{-c}$ . We can apply the Chernoff bound to get

$$\begin{aligned} \mathbb{P}\left[\sum_{i=1}^m X_i \leq \frac{m}{2}\right] &\leq \mathbb{P}\left[\left|\frac{\sum_{i=1}^m X_i}{m} - \mu\right| \geq n^{-c}\right] \\ &\leq e^{\frac{-n^{-2c}(n^{2c+2c'+100})(\frac{1}{2}+n^{-c})}{4}} \\ &= \frac{1}{e^{\frac{n^{2c'+100}(\frac{1}{2}+n^{-c})}{4}}} \\ &= \frac{1}{e^{\frac{n^{2c'+100}}{8} + \frac{n^{2c'-c+100}}{4}}} \\ &= \frac{1}{e^{\frac{n^{2c'+100}+2n^{2c'-c+100}}{8}}} \\ &\leq \frac{1}{e^{\frac{1}{8}n^{2c'+100}}} \leq \frac{1}{2^{n^{c'}}}. \end{aligned}$$

□

*Remark 3.11.4.* Call a random bit  $r$  *bad for  $x$*  if  $M(x, r) \neq \chi_L(x)$  and *good for  $x$*  otherwise. For any  $x$  of length  $n$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{\text{poly}(|x|)}}[r \text{ is bad for } x] \leq 2^{-n^c}$ . Thus,  $\mathbb{P}_{r \in_R \{0,1\}^{\text{poly}(|x|)}}[r \text{ is bad for some } x \text{ of length } n] \leq 2^{-n^c} \cdot 2^n \ll 2^{-n}$  when  $c$  is large, and  $\mathbb{P}_{r \in_R \{0,1\}^{\text{poly}(|x|)}}[r \text{ is good for every } x \text{ of length } n] \geq 1 - 2^{-n}$ .

**Definition 3.11.5.**

1. Let **RP** consist of those languages  $L$  for which there is some efficient randomized TM  $M$  such that

- for any  $x \in L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{t(|x|)}}[M(x, r) = 1] \geq \frac{2}{3}$  and
- for any  $x \notin L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{t(|x|)}}[M(x, r) = 0] = 1$

where  $t(n)$  denotes the time complexity of  $M$ .

2. Let **co-RP** consist of those languages  $L$  for which there is some efficient randomized TM  $M$  such that

- for any  $x \in L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{t(|x|)}}[M(x, r) = 1] = 1$  and
- for any  $x \notin L$ ,  $\mathbb{P}_{r \in_R \{0,1\}^{t(|x|)}}[M(x, r) = 0] \geq \frac{2}{3}$

where  $t(n)$  denotes the time complexity of  $M$ .

Let **ZPP** := **RP**  $\cap$  **co-RP**.

**Note 3.11.6.**  $L \in \mathbf{ZPP}$  if there exists a randomized algorithm that runs in expected polynomial time and never errs.

## 3.12 Lecture 20

**Theorem 3.12.1.** If there exist  $L \in \text{DTIME}(2^{O(n)})$  and  $\gamma > 0$  such that  $L$  requires circuits of size at least  $2^{\gamma n}$ , then **P** = **BPP**.

**Note 3.12.2.** Recall that any algorithm running in time  $t(n)$  can be simulated by circuits of size  $t(n)^2$ . Let  $L := \{1^n \mid n \in \mathbb{N}, \text{ the TM represented by the number } n \text{ in binary halts when its input equals } n\}$ . Then  $L$  is undecidable **but has polynomial size circuits**.

**Theorem 3.12.3 (Adleman).** *Every  $L \in \mathbf{BPP}$  has polynomial size circuits. In other words,  $\mathbf{BPP} \subset \mathbf{P}/\text{poly}$ .*

*Proof.* Let  $L(M) = L$  where  $M$  is a randomized TM that runs in polynomial time. Let  $n \in \mathbb{N}$ . Remark 3.11.4 implies that there is some random bit  $r_n$  such that for any string  $x$  of size  $n$ ,  $M(x, r_n) = \chi_L(x)$ . From this, we can obtain a circuit  $C_{r_n}$  of size quadratic in the running time of  $M$  such that

$$C_{r_n}(x) = M(x, r_n) = \chi_L(x)$$

for each  $x$  of size  $n$ . □

**Corollary 3.12.4.**  $\mathbf{BPP} \subset \mathbf{EXP}$ .

**Definition 3.12.5 (Polynomial hierarchy).** 1. For any  $i \in \mathbb{N}$ ,  $\Sigma_p^i$  is the class of languages  $L$  for which there exist a polynomial-time computable predicate  $P$  and polynomials  $p_1(\cdot), \dots, p_i(\cdot)$  such that

$$x \in L \iff \exists \bar{x}_1 \in \{0, 1\}^{p_1(|x|)} \forall \bar{x}_2 \in \{0, 1\}^{p_2(|x|)} \dots \exists \bar{x}_i \in \{0, 1\}^{p_i(|x|)} (P(x, \bar{x}_1, \dots, \bar{x}_i) = 1).$$

2. For any  $i \in \mathbb{N}$ ,  $\Pi_p^i$  is the class of languages  $L$  for which there exists a polynomial-time computable predicate  $P$  and polynomials  $p_1(\cdot), \dots, p_i(\cdot)$  such that

$$x \in L \iff \forall \bar{x}_1 \in \{0, 1\}^{p_1(|x|)} \exists \bar{x}_2 \in \{0, 1\}^{p_2(|x|)} \dots \forall \bar{x}_i \in \{0, 1\}^{p_i(|x|)} (P(x, \bar{x}_1, \dots, \bar{x}_i) = 1).$$

The *polynomial hierarchy* is the set of languages  $\mathbf{PH} \equiv \bigcup_{i \in \mathbb{N}} \Sigma_p^i$ .

*Aside.* For a given (finite) *vocabulary*  $\sigma$ , let  $\mathbf{STRUC}[\sigma]$  denote the set of all finite structures of  $\sigma$  equipped with a total ordering. Consider the set

$$\mathcal{B} := \{I_b \mid I_b : \mathbf{STRUC}[\sigma] \rightarrow \{0, 1\}, \sigma \text{ is a vocabulary}\}.$$

For each function  $I_b : \mathbf{STRUC}[\sigma] \rightarrow \{0, 1\}$ , let

$$\mathcal{L}_{I_b} = \{A \in \mathbf{STRUC}[\sigma] \mid I_b(A) = 1\},$$

called a *Boolean query*.

Consider the set  $\mathcal{B} := \{\mathcal{L}_{I_b} \mid I_b \in \mathcal{B}\}$  of all Boolean queries. Let  $\mathbf{SO}$  denote the set of all Boolean queries expressible in second-order logic. That is,  $\mathcal{L}_{I_b} \in \mathbf{SO}$  iff there is some second-order sentence  $\varphi$  such that

$$I_b(A) = 1 \iff A \models \varphi.$$

It is known that  $\mathbf{SO} = \mathbf{PH}$ . This is a well-known result of so-called *descriptive complexity theory*, a branch of finite model theory.

**Example 3.12.6.** We see that  $\mathbf{MAX-CLIQUE} \in \Sigma_p^2$  because it is defined by the formula “there exists a choice of vertices  $V_1$  such that for any choice of vertices  $V_2$ ,  $V_1$  is a clique of size  $k$  and  $V_2$  is either not a clique or of size smaller than  $k$ .”

### 3.13 Lecture 21

**Note 3.13.1.**

1.  $\Sigma_p^0 = \mathbf{P}$ .
2.  $\Sigma_p^1 = \mathbf{NP}$ .
3.  $\Sigma_p^k \subset \Sigma_p^{k+1} \cap \Pi_p^{k+1}$ .



$$4. \Pi_p^k \subset \Sigma_p^{k+1} \cap \Pi_p^{k+1}.$$

$$5. \text{co-}\Sigma_p^k = \Pi_p^k.$$

**Lemma 3.13.2.** *If  $k > 0$  and  $\Sigma_p^k = \Pi_p^k$ , then  $\Sigma_p^{k+1} = \Sigma_p^k$ .*

*Proof.* For convenience, let  $k = 1$ . Note that  $L \in \Sigma_p^2$  if and only if some formula

$$\varphi(x) := \exists \bar{y}_1 \forall \bar{y}_2 (P(x, \bar{y}_1, \bar{y}_2))$$

defines  $L$ . But, by assumption,  $\varphi(x)$  is equivalent to some formula  $\exists \bar{y}_1 \exists \bar{y}_2 (P'(x, \bar{y}_1, \bar{y}_2))$ .  $\square$

**Theorem 3.13.3 (Sipser-Gács).**  $\mathbf{BPP} \subset \Sigma_p^2 \cap \Pi_p^2$ .

*Proof.* Since  $\mathbf{BPP} = \text{co-BPP}$ , it suffices to show that  $\mathbf{BPP} \subset \Sigma_p^2$ . If  $L \in \mathbf{BPP}$ , then there exists an efficient algorithm  $A$  such that  $\mathbb{P}_{r \in_R \{0,1\}^{\text{poly}(n)}}[A(x, r) = \chi_L(x)] \geq \frac{2}{3}$  where  $n$  denotes  $|x|$ . Define  $A'$  to run  $A(x, r_1), \dots, A(x, r_s)$  and take the majority. Then  $A'$  uses  $st$  random bits, and

$$\mathbb{P}_{r_1, \dots, r_s}[A'(x, r_1, \dots, r_s) = \chi_L(x)] \geq 1 - 2^{-s(n)}.$$

By choosing  $s \gg 10t^2$ , we see that  $\mathbb{P}[A'(x, \bar{r}) = \chi_L(x)] \geq 1 - \frac{1}{100m^2}$  where  $m$  denotes the number of random bits used.

**Claim.**  $x \in L \iff \exists \bar{y}_1, \dots, \bar{y}_m \in \{0,1\}^m \forall \bar{z} \in \{0,1\}^m \bigvee_{j=1}^m A'(x, \bar{y}_j \oplus \bar{z}) = 1$ .

*Proof.*

( $\implies$ ) Suppose that  $x \in L$ . It suffices to show that

$$\mathbb{P}_{\bar{y}_1, \dots, \bar{y}_m}[\exists \bar{z} \in \{0,1\}^m \bigwedge_{j=1}^m A'(\bar{x}, \bar{y}_j \oplus \bar{z}) \neq 1] < 1.$$

Note that  $\mathbb{P}_{\bar{y}_1, \dots, \bar{y}_m}[\bigwedge_{j=1}^m A'(\bar{x}, \bar{y}_j \oplus \bar{z}) \neq 1] \leq \frac{1}{(100m^2)^m}$  for any  $\bar{z} \in \{0,1\}^m$ . Therefore,

$$\mathbb{P}_{\bar{y}_1, \dots, \bar{y}_m}[\exists \bar{z} \in \{0,1\}^m \bigwedge_{j=1}^m A'(\bar{x}, \bar{y}_j \oplus \bar{z}) \neq 1] \leq \frac{2^m}{(100m^2)^m} < 1.$$

( $\impliedby$ ) Suppose that  $x \notin L$ . Fix  $y_1, \dots, y_m$ . Note that  $\mathbb{P}_{z \in_R \{0,1\}^*}[A(x, y_j \oplus z) = 1] \leq \frac{1}{100m^2}$  for each  $j = 1, \dots, m$ . This implies that

$$\mathbb{P}_{z \in_R \{0,1\}^*}[\bigvee_{j=1}^m A(x, y_j \oplus z) = 1] \leq \frac{1}{100m^2} \leq \frac{m}{100m^2} = \frac{1}{100m} < 1.$$

$\square$

$\square$

## 3.14 Lecture 22

**Definition 3.14.1.**

1. Let  $V, P : \{0,1\}^* \rightarrow \{0,1\}^*$  be mappings and  $x$  a binary string. Let  $r \in_R \{0,1\}^*$ . A  $k$ -round (randomized) interaction of  $V$  and  $P$  on  $x$  and  $r$  is the sequence of length  $k$  consisting of the following strings.

$$\begin{aligned} a_1 &= V(x, r) \\ a_2 &= P(x, a_1) \\ &\vdots \\ a_{2i+1} &= V(x, r, a_1, \dots, a_{2i}) \\ a_{2i+2} &= P(x, a_1, \dots, a_{2i+1}) \end{aligned}$$

Let  $\text{out}(V, P)(x, r)$  denote the final string of this sequence. We call  $V$  a *verifier* and  $P$  a *prover*.

2. Let  $k : \mathbb{N} \rightarrow \mathbb{N}$  be polynomial computable. A language  $L$  has a *has a  $k$ -round (randomized) interactive protocol* or *lies in the class  $\text{IP}[k]$*  if there exists a randomized TM  $V$  that  $V$  is polynomial in its first input and

- (a) (completeness) if  $x \in L$ , then there exists a prover  $P$  such that  $\langle V, P \rangle(x)$  is  $k(|x|)$ -round interaction and

$$\mathbb{P}_{r \in R \in \{0,1\}^{\text{poly}(|x|)}}[\text{out } \langle V, P \rangle(x, r) = 1] \geq \frac{2}{3}$$

and

- (b) (soundness) if  $x \notin L$ , then for any prover  $P$  such that  $\langle V, P \rangle(x)$  is  $k(|x|)$ -round interaction,

$$\mathbb{P}_{r \in R \in \{0,1\}^{\text{poly}(|x|)}}[\text{out } \langle V, P \rangle(x, r) = 1] \leq \frac{1}{3}.$$

3. Let  $\text{IP} := \bigcup_{k \in \mathbb{N}} \text{IP}[n^k]$ .

**Example 3.14.2.** Let  $\text{NIP} := \{\langle G_1, G_2 \rangle \mid G_1 \text{ and } G_2 \text{ are non-isomorphic graphs}\}$ . The following interaction shows that  $\text{NIP} \in \text{IP}$ .

- $V$  : Pick  $i \in \{1, 2\}$  uniformly randomly. Randomly permute the vertices of  $G_i$  to get a new isomorphic graph  $H$ . Send  $H$  to  $P$ .  
 $P$  : If  $H$  is not isomorphic to one of  $G_1$  and  $G_2$ , then select the other graph.  
Otherwise, select one of  $G_1$  and  $G_2$  by flipping a coin. Let  $G_j$  denote the selected graph. Send  $j$  to  $V$ .  
 $V$  : Pick  $i' \in \{1, 2\}$  uniformly randomly. Randomly permute the vertices of  $G_{i'}$  to get a new isomorphic graph  $H'$ . Send  $H'$  to  $P$ .  
 $P$  : If  $H'$  is not isomorphic to one of  $G_1$  and  $G_2$ , then select the other graph.  
Otherwise, select one of  $G_1$  and  $G_2$  by flipping a coin. Let  $G_{j'}$  denote the selected graph. Send  $j'$  to  $V$ .  
 $V$  : Accept if both  $i = j$  and  $i' = j'$ . Reject otherwise.

### 3.15 Lecture 23

**Proposition 3.15.1.**  $\text{IP}$  is closed under complementation.

*Remark 3.15.2.* The prover  $P$  of a  $k$ -round interaction can be assumed, without loss of generality, to decide languages in and only in  $\text{PSPACE}$ . As a result,  $\text{IP} \subset \text{PSPACE}$ .

**Theorem 3.15.3.**  $\text{PSPACE} \subset \text{IP}$ .

**Corollary 3.15.4.**  $\text{PSPACE} = \text{IP}$ .

**Definition 3.15.5.** Let  $A$  and  $B$  be sets of size  $2^n$  and  $2^k$ , respectively. A set of functions  $\mathcal{H} := \{h_1, \dots, h_t\}$  from  $A$  to  $B$  is *pairwise independent* if for any distinct  $x, x' \in A$  and any  $y, y' \in B$ ,

$$\mathbb{P}_{h \in \mathcal{H}}[h(x) = y \wedge h(x') = y'] = \frac{1}{2^{2k}} = \frac{1}{|B|^2}.$$

An element of such a set is called a (*pairwise independent*) *hash function*.

**Example 3.15.6.** The set of all functions  $A \rightarrow B$  is a pairwise independent set of size  $|B|^{|A|}$ .

**Definition 3.15.7.** Let  $q = 2^n$ . For each  $(s, t) \in \mathbb{F}_q \times \mathbb{F}_q$ , define  $h_{s,t} : \mathbb{F}_q \rightarrow \mathbb{F}_q$  by  $h_{s,t}(a) = a \cdot s + t$ . If  $x, y, x', y' \in \mathbb{F}_q$  with  $x \neq x'$ , then the system of equations

$$\begin{aligned} sx + t &= y \\ sx' + t &= y' \end{aligned}$$

is satisfied  $\iff \begin{bmatrix} x & 1 \\ x' & 1 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} y \\ y' \end{bmatrix} \iff \begin{bmatrix} x & 1 \\ x' & 1 \end{bmatrix}^{-1} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$ . Therefore,

$$\mathbb{P}_{(s,t) \in \mathbb{F}_q \times \mathbb{F}_q}[h_{s,t}(x) = y \wedge h_{s,t}(x') = y'] = \frac{1}{q^2},$$

which proves that  $h_{s,t}$  is a hash function.

### 3.16 Lecture 24

**Proposition 3.16.1.** *Let  $G$  and  $H$  be graphs.*

1. *If  $G \not\cong H$ , then  $\text{Aut}(G \cup H) \cong \text{Aut}(G) \times \text{Aut}(H)$ .*
2. *There is some  $k \in \mathbb{N}$  such that  $|\text{Aut}(G \cup H)| \geq 2k$  whenever  $G \cong H$  and  $|\text{Aut}(G \cup H)| \leq k$  whenever  $G \not\cong H$ .*

**Example 3.16.2 (Goldwasser-Sipser set bound protocol).** Let  $S \subset \{0, 1\}^m$  and  $K \in \mathbb{N}$ . We want to construct a verifier  $V$  and prover  $P$  such that

- $V$  can efficiently check whether any given  $x$  belongs to  $S$ , and
- it is guaranteed that either  $|S| \leq \frac{K}{2}$  or  $|S| \geq K$ .

To do this, choose  $l \in \mathbb{N}$  such that  $2^{l-2} \leq K \leq 2^{l-1}$  and  $|S| \leq 2^{l-1}$ . Let  $\mathcal{H}$  denote a pairwise independent set of mappings  $\{0, 1\}^m \rightarrow \{0, 1\}^l$ .

Have  $V$  randomly choose  $h \in \mathcal{H}$  and  $y \in \{0, 1\}^l$  and then send  $(h, y)$  to  $P$ . Next, have  $P$  send  $x \in_R \{0, 1\}^m$  to  $V$ . Finally, have  $V$  accept if and only if  $x \in S$  and  $h(x) = y$ .

**Case 1:** Suppose that  $|S| \leq \frac{K}{2}$ .

If  $x \in S$ , then  $\mathbb{P}_{h \in_R \mathcal{H}}[h(x) = y] = \frac{1}{2^l}$ . Moreover, if  $p$  denotes the quantity  $\frac{|K|}{2^l}$ , then

$$\mathbb{P}_{h \in_R \mathcal{H}}[\exists x \in S, h(x) = y] \leq \frac{|S|}{2^l} \leq \frac{|K|}{2 \cdot 2^l} = \frac{p}{2}.$$

**Case 2:** Suppose that  $|S| \geq K$ .

We compute

$$\begin{aligned} \mathbb{P}_{h \in \mathcal{H}}[\exists x \in S, h(x) = y] &\geq \sum_{x \in S} \mathbb{P}[h(x) = y] - \sum_{\substack{x, x' \in S \\ x \neq x'}} \mathbb{P}[h(x) = y \wedge h(x') = y] \\ &\geq \frac{|S|}{2^l} - \underbrace{\frac{|S|(|S| - 1)}{2}}_{\binom{|S|}{2}} \cdot \frac{1}{2^{2l}} \\ &= \frac{|S|}{2^l} \left( 1 - \frac{|S| - 1}{2} \cdot \frac{1}{2^l} \right) \\ &\geq \frac{|S|}{2^l} \left( 1 - \frac{|S|}{2 \cdot 2^l} \right) \\ &\geq \frac{K}{2^l} \left( 1 - \frac{|S|}{2 \cdot 2^l} \right) \\ &\geq \frac{3}{4} \cdot p. \end{aligned}$$

### 3.17 Lecture 25

**Lemma 3.17.1 (Sum-check protocol).** *Let  $n \in \mathbb{N}$  and choose a prime  $2^{10n} \leq p \leq 2^{20n}$ . Let  $\varphi(x_1, \dots, x_n)$  be a 3cnf formula with  $m$  clauses and let  $\tilde{\varphi}(x_1, \dots, x_n)$  be the polynomial obtained from  $\varphi$  by the following translation rules.*

- $\bar{x} \longleftrightarrow (1 - x)$ .
- $x \wedge y \longleftrightarrow x \cdot y$ .

For any  $a_1, \dots, a_i \in \mathbb{Z}_p$ , define

$$S(a_1, \dots, a_i) = \sum_{x \in \{0,1\}^{n-i}} \tilde{\varphi}(a_1, \dots, a_i, x) \mod p.$$

For any  $K \in \mathbb{N}$ , there exists an efficient interactive protocol  $(V, P)$  such that

- if  $K = S(a_1, \dots, a_i)$ , then  $V$  accepts  $\langle \varphi, p \rangle$  with probability 1 and
- if  $K \neq S(a_1, \dots, a_i)$ , then  $V$  rejects  $\langle \varphi, p \rangle$  with probability  $\geq (1 - \frac{d}{p})^{n-i}$  where  $d := 3m \leq n^3$ .

### 3.18 Lecture 26

*Proof.*

First, we construct  $(V, P)$  as follows.

1.  $\underline{V}$  : If  $n = 1$ , then compute  $\tilde{\varphi}(0) + \tilde{\varphi}(1)$ . If this equals  $K$ , then *accept*. Otherwise, *reject*.  
If  $n > 1$ , then let  $h_1(x_1) = \sum_{x_2, \dots, x_n \in \{0,1\}} \tilde{\varphi}(x_1, x_2, \dots, x_n)$ , which is a univariate polynomial of degree at most  $n^3$ . Ask  $P$  to send  $h_1(x_1)$ .
2.  $\underline{P}$  : Return  $h'_1(x_1)$  to  $V$  where  $h'_1(x_1)$  is univariate and has degree at most  $d$ .
3.  $\underline{V}$  : Compute  $h'_1(0) + h'_1(1)$ . If this equals  $K$ , then *reject*.  
Otherwise, choose  $a_1 \in_R \mathbb{F}_p$ . Recursively apply the same protocol thus far with  $K$  replaced with  $h'_1(a_1)$  and  $\sum_{x_1 \in \{0,1\}} \dots \sum_{x_n \in \{0,1\}} \tilde{\varphi}(x_1, x_2, \dots, x_n)$  replaced with

$$\sum_{x_2 \in \{0,1\}} \dots \sum_{x_n \in \{0,1\}} \tilde{\varphi}(a_1, x_2, \dots, x_n).$$

Next, we must verify the correctness of  $(V, P)$ . If  $K = \sum_{x_1, \dots, x_n \in \{0,1\}} \tilde{\varphi}(x_1, \dots, x_n)$ , then have  $P$  return  $h_i(x_i)$  for each  $i = 1, \dots, n-1$ . In this case,  $V$  accepts with probability 1.

Now, assume that  $K \neq \sum_{x_1, \dots, x_n \in \{0,1\}} \tilde{\varphi}(x_1, \dots, x_n)$ . If  $n = 1$ , then clearly  $V$  rejects with probability 1. Assume, inductively, that  $V$  rejects with high probability for any polynomial of degree  $\leq d$  in  $n-1$  variables. If  $h'_1(x_1) = h_1(x_1)$ , then  $V$  rejects with probability 1. Assume that  $h'_1(x_1) \neq h_1(x_1)$ . Note that the polynomial  $h'_1 - h_1$  is nonzero and has degree at most  $d$ . By DeMillo-Lipton, it follows that

$$\mathbb{P}_{a \in_R \mathbb{F}_p} [h'_1(a) \neq h_1(a)] \geq 1 - \frac{d}{p}.$$

Since  $s(a) \neq h(a) = \sum_{x_2 \in \{0,1\}} \dots \sum_{x_n \in \{0,1\}} \tilde{\varphi}(a_1, x_2, \dots, x_n)$ , we see, by our induction hypothesis, that  $V$  rejects its recursive input with probability  $\geq (1 - \frac{d}{p})^{n-1}$ . Thus,  $V$  rejects  $(\varphi, K, p)$  with probability

$$\geq \left(1 - \frac{d}{p}\right)^{n-1} \cdot \left(1 - \frac{d}{p}\right) = \left(1 - \frac{d}{p}\right)^n,$$

as desired.  $\square$

**Proposition 3.18.1.** *The language of all unsatisfiable 3cnf formulas is co-NP-complete.*

**Corollary 3.18.2.**  $\text{co-NP} \subset \text{IP}$ .

*Proof.* We can modify our last interactive protocol so that its first round has  $P$  send a large enough prime  $p$  to  $V$ . As a result, we can remove  $p$  from the input of  $(V, P)$ . Notice that  $\varphi$  is unsatisfiable if and only if  $\sum_{x \in \{0,1\}^n} \tilde{\varphi}(x_1, \dots, x_n) = 0$ . This, in turn, is true if and only if

$$\sum_{x \in \{0,1\}^n} \tilde{\varphi}(x_1, \dots, x_n) = 0 \mod p$$

since  $0 \leq \sum_{x \in \{0,1\}^n} \tilde{\varphi}(x_1, \dots, x_n) \leq 2^n$ . By our last proposition, we are done.  $\square$

**Theorem 3.18.3.**  $\text{IP} = \text{PSPACE}$ .

### 3.19 Final exam review session

**Theorem 3.19.1 (Space hierarchy).** *If  $f : \mathbb{N} \rightarrow \mathbb{N}$  satisfies  $f(n) \geq \log n$ , then  $\text{DSPACE}(f(n)) \subsetneq \text{DSPACE}(f^2(n))$ .*

**Example 3.19.2.** Since  $\log^2(n) \leq p(n)$  for some polynomial  $p : \mathbb{N} \rightarrow \mathbb{N}$ , the space hierarchy theorem implies that  $\mathbf{L} \subsetneq \mathbf{PSPACE}$ .