## Abstract

This project briefly describes the isometries of  $\mathbb{C}^2$ . In particular, it classifies five important groups of such maps in the category **Top** of topological spaces. Thanks to Steven Rosenberg for his guidance on this topic.

## 1 Isometries of $\mathbb{C}^2$ over $\mathbb{R}$

If M is a metric space, then let  $\mathrm{Isom}(M)$  denote the set of all isometries of M. For now, let  $(\mathbb{C}^2, \|\cdot\|)$  denote the normed vector space  $\mathbb{C}^2$  over  $\mathbb{R}$  where  $\|\cdot\|: \mathbb{C}^2 \to [0, \infty)$  is given by

$$||(z,w)|| = \sqrt{z\bar{z} + w\bar{w}}.$$

That is,  $\|\cdot\|$  is exactly the norm induced by the (Euclidean) inner product  $\langle (z, w), (z, w) \rangle$ . Then  $\mathbb{C}^2 \cong \mathbb{R}^4$  as normed vector spaces via the map  $T: \mathbb{C}^2 \to \mathbb{R}^4$  given by

$$(a+bi, a'+b'i) \mapsto (a, a', b, b').$$
 (\*)

Endow  $\mathbb{C}^2$  and  $\mathbb{R}^4$  with the standard Euclidean metrics d and d', respectively. Since  $||T(\vec{v})|| = ||\vec{v}||$  and T is linear, we see that

$$d(\vec{v}, \vec{x}) = \|\vec{v} - \vec{x}\| = \|T(\vec{v}) - T(\vec{x})\| = d'(T(\vec{v}), T(\vec{x}))$$

for any  $\vec{v}, \vec{x} \in \mathbb{C}^2$ . Likewise, we see that

$$d(T^{-1}(\vec{y}), T^{-1}(\vec{z})) = \|T^{-1}(\vec{y}) - T^{-1}(\vec{z})\| = \|\vec{y} - \vec{z}\| = d'(\vec{y}, \vec{z})$$

for any  $\vec{y}, \vec{z} \in \mathbb{R}^4$ . Thus, the map  $f \mapsto T \circ f \circ T^{-1}$  defines a group isomorphism  $\mathrm{Isom}(\mathbb{C}^2) \stackrel{\cong}{\longrightarrow} \mathrm{Isom}(\mathbb{R}^4)$ , provided that both  $\mathrm{Isom}(\mathbb{C}^2)$  and  $\mathrm{Isom}(\mathbb{R}^4)$  are, in fact, groups under composition. Certainly they are closed under composition and contain the identity map. Also, every isometry f of a given metric space  $(X, \rho)$  must be injective. Indeed, if  $x \neq y$  but f(x) = f(y), then  $\rho(x, y) \neq 0 = \rho(f(x), f(y))$ , which is impossible. Since the inverse of f must also be an isometry, it just remains to show that f is surjective in order to prove that the two are groups. This is the content of Corollary 1.12 below.

Consider the group  $O(4) := \{ f \in Isom(\mathbb{R}^4) : f \text{ fixes } \vec{0} \}$ . For each  $\vec{v} \in \mathbb{R}^4$ , define  $T_{\vec{v}} : \mathbb{R}^4 \to \mathbb{R}^4$  by  $\vec{x} \mapsto \vec{x} + \vec{v}$ .

**Lemma 1.1.** Any  $A \in \text{Isom}(\mathbb{R}^4)$  can be written uniquely as  $T_{A(\vec{0})} \circ g$  for some  $g \in O(4)$ .

Proof. Define  $g: \mathbb{R}^4 \to \mathbb{R}^4$  by  $A(\vec{v}) - A(\vec{0})$ . Then  $g \in \mathrm{O}(4)$ , and  $A(\vec{v}) = T_{A(\vec{0})} \circ g(\vec{v})$  for any  $\vec{v}$ . Further, if  $A = T_{A(\vec{0})} \circ k$  for some  $k \in \mathrm{O}(4)$ , then  $g(\vec{v}) = A(\vec{v}) - A(\vec{0}) = k(\vec{v})$ , thereby proving uniqueness.  $\square$ 

**Definition 1.2.** A matrix  $X \in \mathbb{M}^4(\mathbb{R})$  is *orthogonal* if its column vectors are orthonormal.

**Proposition 1.3.** The following are equivalent.

- (a) X is orthogonal.
- (b)  $X \in GL(4, \mathbb{R})$  with  $X^T = X^{-1}$ .

Corollary 1.4. Any orthogonal matrix  $X \in \mathbb{M}^4(\mathbb{R})$  preserves the inner product, i.e.,  $\langle X\vec{v}, X\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle$  for any  $\vec{v}, \vec{w} \in \mathbb{R}^4$ .

*Proof.* We have that 
$$X\vec{v} \bullet X\vec{w} = \vec{v} \bullet X^T X\vec{w} = \vec{v} \bullet I\vec{w} = \vec{v} \bullet \vec{w}$$
.

Notation. The symbol • will denote the Euclidean inner product.

Corollary 1.5. If  $X \in \mathbb{M}^4(\mathbb{R})$  is orthogonal, then  $|\det(X)| = 1$ .

*Proof.* We have that 
$$1 = \det(I) = \det(XX^T) = \det(X)^2$$
.

**Lemma 1.6.** If  $X \in \mathbb{M}^4(\mathbb{R})$  is orthogonal, then  $X \in O(4)$ .

*Proof.* By Corollary 1.4, X preserves the inner product, which implies that

$$||X\vec{v} - X\vec{w}||^2 = ||X\vec{v}||^2 - 2X\vec{v} \cdot X\vec{w} + ||X\vec{w}||^2$$
$$= ||\vec{v}||^2 - 2\vec{v} \cdot \vec{w} + ||\vec{w}||^2$$
$$= ||\vec{v} - \vec{w}||^2$$

for any  $\vec{v}, \vec{w} \in \mathbb{R}^4$ . Thus,  $d'(X\vec{v}, X\vec{w}) = d'(\vec{v}, \vec{w})$ , and  $X \in O(4)$ .

**Definition 1.7.** An invertible linear operator T on a finite-dimensional vector space is orientation-preserving if det  $M_T > 0$  and orientation-reversing if det  $M_T < 0$  where  $M_T$  denotes the matrix of T.

Soon we shall prove that  $O(4) \subset GL(4,\mathbb{R})$ . Therefore, it makes sense to introduce the group

$$SO(4) := \left\{ f \in Isom(\mathbb{R}^4) : f \text{ fixes } \vec{0} \text{ and is orientation-preserving} \right\}.$$

Let  $\{\vec{e}_1, \dots, \vec{e}_4\}$  denote the standard basis of  $\mathbb{R}^4$ . We are now ready to establish a so-called TRF-decomposition of  $Isom(\mathbb{R}^4)$ .

**Theorem 1.8.** Let  $\mathcal{F}: \mathbb{R}^4 \to \mathbb{R}^4$  be given either by the identity map or the reflection  $(a, b, c, d) \mapsto (a, b, c, -d)$ . Let  $A \in \text{Isom}(\mathbb{R}^4)$ . Then we have

$$A = T_{A(\vec{0})} \circ R' \circ \mathcal{F}$$

for some  $R' \in SO(4)$ .

Proof. By Lemma 1.1, we have that  $A = T_{A(\vec{0})} \circ g$  for some  $g \in O(4)$ . Since g is an isometry, we know that  $\|\vec{x} - \vec{y}\|^2 = \|g(\vec{x}) - g(\vec{y})\|^2$  for any  $\vec{x}, \vec{y} \in \mathbb{R}^4$ . As g fixes  $\vec{0}$ , it follows that  $\|g(\vec{v})\| = \vec{v}$  for any  $\vec{v} \in \mathbb{R}^4$ . We can apply the additivity of the inner product to get

$$\begin{split} \left\|g(\vec{v})\right\|^2 + \left\|g(\vec{w})\right\|^2 - 2\langle g(\vec{v}), g(\vec{w})\rangle &= \langle g(\vec{v}) - g(\vec{w}), g(\vec{v}) - g(\vec{w})\rangle \\ &= \langle \vec{v} - \vec{w}, \vec{v} - \vec{w}\rangle \\ &= \left\|\vec{v}\right\|^2 + \left\|\vec{w}\right\|^2 - 2\langle \vec{v}, \vec{w}\rangle. \end{split}$$

We can cancel terms to find that g preserves the inner product. Note that our proof of this fact actually applies to any element of O(4).

Now, it follows that  $\|g(\vec{e}_i)\|^2 = \|\vec{e}_i\|^2 = 1$  for each i = 1, 2, 3, 4, so that  $\|g(\vec{e}_i)\| = 1$ . Similarly, we can deduce that  $\langle g(\vec{e}_i), g(\vec{e}_j) \rangle = 0$  if  $i \neq j$ . Thus,  $\{g(\vec{e}_i)\}_{i=1,2,3,4}$  is an orthonormal (hence linearly independent) set. Let

$$M \coloneqq \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ g(\vec{e}_1) & g(\vec{e}_2) & g(\vec{e}_3) & g(\vec{e}_4) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Then  $M^TM = MM^T = I$ , so that M is invertible with  $M^T = M^{-1}$ . Lemma 1.6 implies that  $M \in O(4)$ . The isometry  $f := M^{-1} \circ g : \mathbb{R}^4 \to \mathbb{R}^4$  satisfies  $f(\vec{0}) = \vec{0}$  and  $f(\vec{e}_i) = \vec{e}_i$  for each i.

Since  $f \in O(4)$ , it follow that

$$f(\vec{x}) \bullet f(\vec{e}_i) = \vec{x} \bullet \vec{e}_i = f(\vec{x}) \bullet \vec{e}_i = \vec{x} \bullet \vec{e}_i$$

for each i. Writing  $\vec{x} = \sum_{i=1}^4 c_i \vec{e}_i$  for some  $c_i \in \mathbb{R}$ , we have that  $f(\vec{x}) \bullet \vec{e}_i = \left(\sum_{i=1}^4 c_i \vec{e}_i\right) \bullet \vec{e}_i = c_i$ , and thus  $f(\vec{x}) = \vec{x}$ . Hence f = Id, so that M = g. We deduce that any isometry of  $\mathbb{R}^4$  that fixes  $\vec{0}$  is given by an orthogonal matrix.

By Corollary 1.5,  $det(g) = \pm 1$ . If det(g) = 1, then  $g \in SO(4)$ , and we're done. Assume that det(g) = -1. Note that the reflection

$$\phi(a, b, c, d) \equiv (a, b, c, -d)$$

is given by the matrix

$$S \coloneqq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since it's clear that  $\phi \in O(4)$ , we see that  $g \circ \phi \in O(4)$ . Also,  $\det(gS) = \det(g) \det(S) = (-1)(-1) = 1$ . Therefore,  $g \circ \phi \in SO(4)$ . As  $\phi = \phi^{-1}$ , it follows that  $(g \circ \phi) \circ \phi = g \circ (\phi^2) = g$ . Now, set  $R' = g \circ \phi$  and  $\mathcal{F} = \phi$ , thereby completing out proof.

By inspecting our last proof, we obtain several quick results.

Corollary 1.9. If  $X \in \mathbb{M}^4(\mathbb{R})$  preserves the inner product, then X is orthogonal.

Corollary 1.10. We have that

$$\begin{aligned} \mathrm{O}(4) &= \left\{ X \in \mathrm{GL}(4,\mathbb{R}) : X \text{ is orthogonal} \right\} \\ \mathrm{SO}(4) &= \left\{ X \in \mathrm{GL}(4,\mathbb{R}) : X \text{ is orthogonal and } \det(X) = 1 \right\}. \end{aligned}$$

Corollary 1.11. A function f is an element of  $\text{Isom}(\mathbb{R}^4)$  if and only if there exist  $M \in O(4)$  and  $\vec{b} \in \mathbb{R}^4$  such that for any  $\vec{x} \in \mathbb{R}^4$ ,  $f(\vec{x}) = M\vec{x} + \vec{b}$ . In this case,  $M = R' \circ \mathcal{F}$  with notation as in Theorem 1.8.

Corollary 1.12. Every  $f \in \text{Isom}(\mathbb{R}^4)$  and every  $g \in \text{Isom}(\mathbb{C}^2)$  are invertible, so that both  $\text{Isom}(\mathbb{C}^2)$  and  $\text{Isom}(\mathbb{R}^4)$  are groups under composition.

*Proof.* Thanks to Corollary 1.11, we can write  $f(\vec{x}) = M\vec{x} + \vec{b}$ . Then it's easy to verify that  $f^{-1}(\vec{x}) = M^{-1}\vec{x} - M^{-1}\vec{b}$ .

Moreover, with T given by (\*), we find that  $g = T \circ h \circ T^{-1}$  for some  $h \in \text{Isom}(\mathbb{R}^4)$ . Hence g is the composite of three invertible functions and thus is invertible.

**Note 1.13.** The decomposition of A given in Theorem 1.8 is unique.

Proof. Suppose  $A(\vec{x}) = M\vec{x} + \vec{b} = M'\vec{x} + \vec{b}'$  for every  $\vec{x} \in \mathbb{R}^4$ . Then  $\vec{b} = \vec{b}'$ , so that M = M'. Moreover, if  $M = T \circ \mathcal{F}$  for some  $T \in SO(4)$ , then  $T = M \circ \mathcal{F}$ . This shows that the decomposition  $A = T_{A(\vec{0})} \circ g \circ \mathcal{F}$  given in Theorem 1.8 is, indeed, unique.

## 2 Isometries of $\mathbb{C}^2$ over $\mathbb{C}$

Now, view  $\mathbb{C}^2$  as a two-dimensional vector space over  $\mathbb{C}$ . Recall that the Hermitian inner product  $H: \mathbb{C}^2 \times \mathbb{C}^2 \to [0, \infty)$  is defined by  $H(x, y) = x_1 \bar{y}_1 + x_2 \bar{y}_2$ .

**Definition 2.1.** For any  $n \in \mathbb{N}$ , a matrix  $X \in \mathbb{M}^n(\mathbb{C})$  is *unitary* if its column vectors are orthonormal with respect to H.

Let U(n) denote the set of all unitary matrices. Lemma 2.5 below indicates that these are isometries of  $\mathbb{C}^2$ .

**Proposition 2.2.** The following are equivalent.

- (a)  $X \in U(2)$ .
- (b)  $X \in GL(2,\mathbb{C})$  with  $X^* = X^{-1}$ , where  $X^*$  denotes the conjugate transpose of X.

Corollary 2.3. U(n) is a group under composition for each n = 1, 2.

*Proof.* First, note that  $U(1) = \{z \in \mathbb{C} : |z| = 1\} = S^1$ , which is a group because the complex modulus is multiplicative and  $|z| = 1 \implies |z^{-1}| = \frac{|\bar{z}|}{|z|^2} = 1$ . Next, consider U(2). It suffices to verify closure. If  $A, B \in U(2)$ , then

$$(AB)^*(AB) = B^*A^*AB = B^*B = I,$$

and thus  $AB \in U(2)$ .

Note that U(2) is nonabelian. Indeed, let  $A := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $B := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . These are unitary, but  $0 \neq AB = -BA$ .

Corollary 2.4. Every  $2 \times 2$  unitary matrix X has  $|\det(X)| = 1$ , where  $|\cdot|$  denote the complex modulus.

*Proof.* We have that 
$$1 = \det(I) = \det(XX^*) = \det(X)\det(X^*) = \det(X)\overline{\det(X)} = |\det(X)|$$
.

From a linear-algebraic perspective, we see that U(2) is the complex analogue of O(4). Group-theoretically, however, we can construct an embedding  $F: U(2) \hookrightarrow SO(4)$  as follows. For each  $M \in U(2)$ , write

$$M = \begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + i \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = A + iB$$

<sup>&</sup>lt;sup>1</sup>As a result, SO(4) is nonabelian and hence not isomorphic to SO(2).

and set  $F(M) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ . It's easy to verify that F(M) is orthogonal. Also, note that

$$\det(F(M)) = 1 \cdot \det \begin{pmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \end{pmatrix} \cdot 1$$

$$= \det \begin{pmatrix} \begin{bmatrix} I & 0 \\ iI & I \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} I & 0 \\ -iI & I \end{bmatrix} \end{pmatrix}$$

$$= \det \begin{pmatrix} \begin{bmatrix} A+iB & -B \\ 0 & A-iB \end{bmatrix} \end{pmatrix}$$

$$= \det(A+iB) \det(A-iB)$$

$$= \det(A)^2 + \det(B)^2$$

$$= |\det(M)|^2 = 1.$$

Therefore, F is well-defined. To verify that F is a homomorphism, note that if N = C + Di, then MN = (AC - BD) + (AD + BC)i. In this case

$$F(MN) = \begin{bmatrix} AC - BD & -AD - BC \\ AD + BC & AC - BD \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} C & -D \\ D & C \end{bmatrix} = F(M)F(N).$$

Furthermore, if  $F(M) \in \ker(F)$ , then  $A = I_2$  and  $B = 0_2$ , i.e.,  $M = I_2$ . Hence  $\ker(F)$  is trivial, and thus F is an injective homomorphism, as desired.

In fact, the  $2 \times 2$  unitary matrices are precisely those elements of SO(4) which preserve the Hermitian inner product H. This provides us with a geometric distinction between U(2) and SO(4).

**Lemma 2.5.** A map  $R \in \mathbb{M}^2(\mathbb{C})$  satisfies H(R(x), R(y)) = H(x, y) for any  $x, y \in \mathbb{C}^2$  if and only if  $R \in U(2)$ .

*Proof.* Note that  $H(x,y) = \bar{x}^T y$ . Then

$$H(Rx, Ry) = H(x, y) \iff \overline{Rx}^T Ry = \overline{x}^T y$$

$$\iff \overline{x}^T (\overline{R}^T R) y = \overline{x}^T y$$

$$\iff \overline{R}^T R = I.$$

Let us look now at the complex analogue of SO(4). The map  $D: \mathrm{U}(2) \to \mathrm{U}(1)$  given by  $D(X) = \det(X)$  is well-defined by Corollary 2.4. As det is multiplicative, it is also a homomorphism. For any  $e^{i\theta} \in \mathbb{C}$ , we see that  $M:=\begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \in \mathrm{U}(2)$  and  $D(M)=e^{i\theta}$ , which means that D is surjective. Now note that

$$\ker D = K := \{X \in \mathrm{U}(2) : \det(X) = 1\}.$$

This yields an isomorphism  $U(2)/K \cong U(1)$  in the category **Grp** of groups.

Let  $\mathrm{SU}(2) \coloneqq \ker(D)$ . Then  $\mathrm{SU}(2)$  consists precisely of those  $2 \times 2$  unitary matrices which are orientation-preserving. Let  $W \in \mathrm{SU}(2)$  and write  $W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Since  $\det(W) = 1$ , we find that  $W^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ .

Since  $W^* = W^{-1}$ , it follows that  $d = \bar{a}$  and  $-\bar{b} = c$ . Therefore,  $\det(W) = \|(a,c)\|^2 = a\bar{a} + c\bar{c} = 1$ , and  $W = \begin{bmatrix} a & c \\ -\bar{c} & \bar{a} \end{bmatrix}$ . Conversely, the column vectors of such a matrix are orthonormal. Hence

$$\mathrm{SU}(2) = \left\{ X \in \mathbb{M}^2(\mathbb{C}) : X = \begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix} \text{ with } x\bar{x} + y\bar{y} = 1 \right\}.$$

Theorem 2.6.  $U(2) \cong (SU(2) \times U(1)) / \mathbb{Z}_2$  in Grp.

*Proof.* Define  $\psi : \mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{U}(2)$  by  $(A, k) \mapsto kA$ . This map is certainly a well-defined homomorphism. Moreover, for any  $X \in \mathrm{U}(2)$ , note that  $\sqrt{\det(X)} \in \mathrm{U}(1)$  and  $\frac{1}{\sqrt{\det(X)}} X \in \mathrm{SU}(2)$ , so that

$$\psi\left(\frac{1}{\sqrt{\det(X)}}X, \sqrt{\det(X)}\right) = X.$$

Thus,  $\psi$  is surjective. Finally, notice that  $\ker \psi = \{\pm(I,1)\} \cong \mathbb{Z}_2$ . By the first isomorphism theorem, we get an isomorphism  $\tilde{\psi} : \mathrm{U}(2) \xrightarrow{\cong} (\mathrm{SU}(2) \times \mathrm{U}(1)) / \mathbb{Z}_2$ , as desired.

It turns out that SU(2) is the same as the group of unit quaternions.

Theorem 2.7.  $SU(2) \cong S^3$  in Grp.

*Proof.* For any  $x := (x_1, x_2, x_3, x_4) \in S^3$ , let  $z = x_1 + x_2 i \in \mathbb{C}$  and  $w = x_3 + x_4 i \in \mathbb{C}$ . Then x = z + w j. Define the map  $f: S^3 \to \mathrm{SU}(2)$  by

$$f(x) = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}.$$

We see that  $|x|^2 = |z|^2 + |w|^2 = \det \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$ . Hence  $x \in S^3$  if and only if  $\det \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} = 1$ , which establishes a clear bijection. It remains to check that f is a homomorphism. Let  $y \in S^3$  so that y = p + qj. Then since  $jw = \bar{w}j$  and  $jz = \bar{z}j$ , we obtain

$$xy = pz + pwj + q(jz) + p(jw)j = (pz - p\bar{w}) + pw + q\bar{z}j$$
.

Finally, we compute

$$f(yx) = \begin{bmatrix} pz - q\bar{w} & pw + q\bar{z} \\ -pw + q\bar{z} & pz - q\bar{w} \end{bmatrix}$$

$$= \begin{bmatrix} pz - q\bar{w} & pw + q\bar{z} \\ -\bar{p}\bar{w} - \bar{q}z & \bar{p}\bar{z} - \bar{q}w \end{bmatrix}$$

$$= \begin{bmatrix} p & q \\ -\bar{q} & \bar{p} \end{bmatrix} \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$$

$$= f(y)f(x).$$

## 3 Topology of Isom( $\mathbb{C}^2$ )

Let us turn our attention to providing the groups

- SU(2)
- U(2)
- SO(4)
- O(4)
- $\operatorname{Isom}(\mathbb{R}^4)$

with topological characterizations, having treated them only as algebraic objects thus far. The first four of these groups are topological spaces as subsets of normed vector spaces. The last group, Isom( $\mathbb{R}^4$ ), has the metric topology induced by

$$d(f,g) \equiv \max\{|f(x) - g(x)| : x \in \mathbb{R}^4, |x| \le 1\},\$$

which is a modest generalization of the metric induced by the familiar operator norm in the theory of finite-dimensional vector spaces.

Remark 3.1. All five groups are actually Lie groups.

Theorem 3.2.  $SU(2) \cong S^3$  in Top.

*Proof.* We claim that the map f from Theorem 2.7 is a homeomorphism. Indeed, note that as  $S^3$  is a closed and bounded subset of Euclidean space, it is compact. Also, SU(2) is Hausdorff as a topological group. Thus, it suffices to show that f is continuous. By identifying each matrix in f's codomain with a vector in  $\mathbb{C}^4$ , we find that continuity follows from the fact that complex conjugation is continuous along with the fact that continuity is preserved by addition and multiplication.

Corollary 3.3. SU(2) is simply connected.

**Theorem 3.4.**  $U(2) \cong (SU(2) \times U(1)) / \mathbb{Z}_2$  in **Top**.

*Proof.* We claim that the map  $\tilde{\psi}$  from Theorem 2.6 is a homeomorphism. Indeed, it is clearly continuous due to the universal property of quotient spaces. Moreover, its inverse is given by

$$X \mapsto \left[ \left( X \frac{1}{\sqrt{\det X}}, \sqrt{\det X} \right) \right],$$

which is continuous because both  $\sqrt{\cdot}$  and  $\det(\cdot)$  are continuous.

**Proposition 3.5.** For any quaternions x, y, we have  $\overline{xy} = \overline{y}\overline{x}$ .

Recall that by definition  $|x| = \sqrt{x\bar{x}}$ .

Corollary 3.6. |xy| = |x| |y|.

Theorem 3.7.  $SO(4) \cong S^3 \times SO(3)$  in Top.

*Proof.* Formally, we can identify  $\mathbb{R}^4$  with the group of quaternions. For each  $q \in S^3$ , the map  $\alpha_q : \mathbb{R}^4 \to \mathbb{R}^4$  given by  $a \mapsto aq$  satisfies |aq| = |a| |q| = |a| thanks to Corollary 3.6. Hence for any  $a, b \in \mathbb{R}^4$ , we see that

$$|a - b| = |\alpha_q(a - b)| = |aq - bq|,$$

so that  $\alpha_q \in \text{Isom}(\mathbb{R}^4)$ . Further, since  $\alpha_q(0) = 0$ , it belongs to O(4). Hence it preserves the Euclidean inner product.

We construct a continuous embedding  $E: \mathrm{O}(3) \hookrightarrow \mathrm{O}(4)$  as follows. Let  $X \in \mathrm{O}(3)$  and write  $X = \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}$  where  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ . Then set

$$E(X) = (1, x, y, z) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \vdots & \vdots & \vdots \\ 0 & \vec{x} & \vec{y} & \vec{z} \\ 0 & \vdots & \vdots & \vdots \end{bmatrix},$$

which is an element of O(4). Now, define  $f: S^3 \times \mathrm{O}(3) \to \mathrm{O}(4)$  by  $(q,(1,x,y,z)) \mapsto (q,xq,yq,zq)$ . As  $\alpha_q$  preserves the norm and the inner product, it preserves orthonormality. This means that f is well-defined. It's clear that f is continuous. Moreover, f is invertible with continuous inverse  $(v,u,r,s) \mapsto (v,(1,uv^{-1},rv^{-1},sv^{-1}))$ . Note that, in fact,  $(1,uv^{-1},rv^{-1},sv^{-1}) \in \mathrm{O}(3)$  because  $\alpha_{v^{-1}}$  preserves orthonormality, so that in particular  $vv^{-1}$  must be orthogonal to each of the other three column vectors. Hence the first row vector must be (1,0,0,0), as required.

Finally, the restriction of f to  $S^3 \times SO(3)$  yields our desired homeomorphism.

Corollary 3.8.  $SO(4) \cong S^3 \times \mathbb{RP}^3$ .

Corollary 3.9.  $O(4) \cong S^3 \times O(3)$ .

Our final result classifies the entire space  $\text{Isom}(\mathbb{R}^4)$ .

**Theorem 3.10.** Isom( $\mathbb{R}^4$ )  $\cong$  O(4)  $\times$   $\mathbb{R}^4$  in **Top**.

Proof. With notation as in Corollary 1.11, define  $F: \text{Isom}(\mathbb{R}^4) \to \text{O}(4) \times \mathbb{R}^4$  by  $f \mapsto \left(M, \vec{b}\right)$ . Note 1.13 implies that F is well-defined, and Corollary 1.9 implies that it is a bijection. Note that  $F_1(f) = M = T_{-\vec{b}} \circ f$ , which is a composite of continuous functions. Further,  $F_2(f) = \vec{b} = f(\vec{0})$ . Hence each component map of F is continuous. It's clear that the inverse  $\left(M, \vec{b}\right) \to \left(\vec{x} \mapsto M\vec{x} + \vec{b}\right)$  is also continuous. Thus, F is a homeomorphism.