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Homotopy and K-theory seminar
Talk #16
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Abstract

We continue doing higher Waldhausen K-theory. The main sources for this talk are the following.

- *nLab.*
- Charles Weibel's *The* K-*book: an introduction to algebraic* K-*theory.* Chapter V.2.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

Remark 1. Recall that $|wS_\bullet\mathscr{C}|$ is an H-space via the map

$$\coprod : |wS_\bullet\mathscr{C}| \times |wS_\bullet\mathscr{C}| \cong |wS_\bullet\mathscr{C} \times wS_\bullet\mathscr{C}| \to |wS_\bullet\mathscr{C}|.$$

This produces an H-space structure $(K(\mathscr{C}), +)$.

Definition 1. Let $\mathscr{B}$ and $\mathscr{C}$ be Waldhausen categories. We say that $F' \rightarrowtail F \twoheadrightarrow F''$ is a *short exact sequence* or *cofiber sequence of exact functors* if every $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$ is a cofiber sequence and $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration in $\mathscr{C}$ for every $A \rightarrowtail B$ in $\mathscr{B}$.

Remark 2. Let $\mathscr{C}$ be a Waldhausen category. Let $(\eta) : A \rightarrowtail B \twoheadrightarrow C$ be an object in $S_2\mathscr{C}$. Define the source s, target t, and quotient q functors $S_2\mathscr{C} \to \mathscr{C}$ by $s(\eta) = A$, $t(\eta) = B$, and $q(\eta) = C$. Then $s \rightarrowtail t \twoheadrightarrow q$ is a cofiber sequence of functors. Since defining a cofiber sequence of exact functors $\mathscr{B} \to \mathscr{C}$ is equivalent to defining an exact functor $\mathscr{B} \to S_2\mathscr{C}$, we may restrict our attention to $s \rightarrowtail t \twoheadrightarrow q$ when proving things about a given cofiber sequence of exact functors $\mathscr{B} \to \mathscr{C}$. We say that $S_2\mathscr{C}$ is universal in this sense.

Theorem 1. *(Extension theorem) Let $\mathscr{C}$ be Waldhausen. The exact functor $(s, q) : S_2\mathscr{C} \to \mathscr{C} \times \mathscr{C}$ induces a homotopy $K(S_2\mathscr{C}) \simeq K(\mathscr{C}) \times K(\mathscr{C})$. The functor $\coprod : (A, B) \to (A \rightarrowtail A \coprod B \twoheadrightarrow B)$ is a homotopy inverse.*

Proof. Let $\mathscr{C}^w_m$ denote the category of m-length sequences of weak equivalences. For each n, define $s_n\mathscr{C}^w_m$ as the commutative diagram

$$\begin{tikzcd}
X^0_1 \arrow[r, tail] \arrow[d, "\sim"'] & X^0_2 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X^0_n \arrow[d, "\sim"'] \\
X^1_1 \arrow[r, tail] \arrow[d, "\sim"'] & X^1_2 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X^1_n \arrow[d, "\sim"'] \\
\vdots \arrow[d, "\sim"'] & \vdots \arrow[d, "\sim"'] & & \vdots \arrow[d, "\sim"'] \\
X^m_1 \arrow[r, tail] & X^m_2 \arrow[r, tail] & \cdots \arrow[r, tail] & X^m_n
\end{tikzcd}.$$

This is naturally isomorphic to an (m, n)-bisimplex in $N_\bullet wS_\bullet\mathscr{C}$, which is thus isomorphic to the bisimplicial set $s_\bullet\mathscr{C}^w_{(-)}$. One can show that the source s and quotient q functors $S_2\mathscr{C} \to \mathscr{C}$ give a homotopy equivalence $s \times q : s_\bullet S_2(\mathscr{C}^w_m) \to s_\bullet\mathscr{C}^w_m \times s_\bullet\mathscr{C}^w_m$ for each m. Thus, we get a homotopy equivalence

$$s_\bullet S_2(\mathscr{C}^w_{(-)}) \simeq s_\bullet\mathscr{C}^w_{(-)} \times s_\bullet\mathscr{C}^w_{(-)}$$

between bisimplicial sets. But we already have that $s_\bullet\mathscr{C}^w_{(-)} \cong N_\bullet wS_\bullet\mathscr{C}$, completing the proof. ☐

Theorem 2. *(The additivity theorem) Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors $\mathscr{B} \to \mathscr{C}$. Then $F_* \simeq F'_* + F''_*$ as maps $K(\mathscr{B}) \to K(\mathscr{C})$. Hence $F_* = F'_* + F''_*$ as maps $K_i(\mathscr{B}) \to K_i(\mathscr{C})$.*

Proof. As $S_2\mathscr{C}$ is universal, it suffices to prove that $t_* \simeq s_* + q_*$. Notice that the two compositions $\mathscr{C} \times \mathscr{C} \xrightarrow{\coprod} S_2\mathscr{C} \underset{s\coprod q}{\overset{t}{\rightrightarrows}} \mathscr{C}$ are the same. The extension theorem implies that $K(\coprod) : K(\mathscr{C}) \times K(\mathscr{C}) \to K(S_2\mathscr{C})$ is a homotopy equivalence. Since the H-space structure on $K(\mathscr{C})$ is induced by $\coprod$, we get $t_* \simeq s_* + q_*$. ☐

Definition 2. Let $\mathscr{C}$ be Waldhausen. We say that a sequence $* \to A_n \to \cdots \to A_0 \to *$ is *admissibly exact* if each morphism in the sequences can be written as a cofiber sequence $A_{i+1} \twoheadrightarrow B_i \rightarrowtail A_i$.

Corollary 3. *Suppose that* $* \to F^0 \to F^1 \to \cdots \to F^n \to *$ *is an admissibly exact sequence of exact functors* $\mathscr{B} \to \mathscr{C}$*. Then* $\sum_i (-1)^i F^i_* = 0$ *as maps* $K_i(\mathscr{B}) \to K_i(\mathscr{C})$*.*

Proof. Induct on n. ∎

Corollary 4. *Let* $F' \rightarrowtail F \twoheadrightarrow F''$ *be a short exact sequence of exact functors* $\mathscr{B} \to \mathscr{C}$*. Then*

$$F''_* \simeq F_* - F_* \simeq 0.$$

This implies that the homotopy fiber of $F''_* : K(\mathscr{B}) \to K(\mathscr{C})$ *is homotopy equivalent to* $K(\mathscr{B}) \vee \Omega K(\mathscr{C})$*.*

Definition 3. Let $\mathscr{C}$ be a Waldhausen category. Recall the arrow category $\mathrm{Ar}(\mathscr{C})$ of $\mathscr{C}$ consisting of morphisms in $\mathscr{C}$ as objects and commutative squares as morphisms. Let s and t denote the source and target functors $\mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$, respectively.

A functor $T : \mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$ is a *(mapping) cylinder functor* on $\mathscr{C}$ if it comes equipped with natrual transformations $j_1 : s \Rightarrow T$, $j_2 : t \Rightarrow T$, and $p : T \Rightarrow t$ such that for any $f : A \to B$, we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "j_2"'] \arrow[dl, "="] \\ & B & \end{tikzcd}.$$

Moreover, T must satisfy the following axioms.

1. T sends every initial morphism $* \to A$ to A for any $A \in \mathrm{ob}\,\mathscr{C}$.
2. $j_1 \coprod j_2 : A \coprod B \rightarrowtail T(f)$ is a cofibration for any $f : A \to B$.
3. Given a morphism $(a, b) : f \to f'$ in $\mathrm{Ar}(\mathscr{C})$, if both a and b are w.e. in $\mathscr{C}$, then so is $T(f) \to T(f')$.
4. Given a morphism $(a, b) : f \to f'$ in $\mathrm{Ar}(\mathscr{C})$, if both a and b are cofibrations in $\mathscr{C}$, then so is $T(f) \to T(f')$. Also, the map $A' \coprod_A T(f) \coprod_B B' \to T(f')$ induced by axiom 2 is a cofibration in $\mathscr{C}$.
5. (Cylinder Axiom) The map $p : T(f) \to B$ is a w.e. in $\mathscr{C}$.

Definition 4. Let T be a cylinder functor on $\mathscr{C}$.

1. We call $T(A \to *)$ the *cone* of A, denoted by $\mathrm{cone}(A)$.
2. We call $\mathrm{cone}(A)/A$ the *suspension* of A, denoted by ΣA.

Corollary 5. *The induced suspension map* $\Sigma : K(\mathscr{C}) \to K(\mathscr{C})$ *is a homotopy inverse for the* H*-space* $K(\mathscr{C})$*.*

Proof. Note that axiom 3 gives us a cofiber sequence $A \rightarrowtail \mathrm{cone}(A) \twoheadrightarrow \Sigma A$. Therefore, $1 \rightarrowtail \mathrm{cone} \twoheadrightarrow \Sigma$ is an exact sequence of functors. By the cylinder axiom, we know that cone is null-homotopic. It follows by the additivity theorem that $\Sigma_* + 1 = \mathrm{cone}_* = *$. ∎

Theorem 6. *(Waldhausen localization theorem) Let* $\mathscr{C}$ *be a category with cofibrations. Equip it with two Waldhausen subcategories* $v(\mathscr{C})$ *and* $w(\mathscr{C})$ *of weak equivalences such that* $v(\mathscr{C}) \subset w(\mathscr{C})$*. Assume that* $(\mathscr{C}, w)$ *admits a cylinder functor. Suppose that* $w(\mathscr{C})$ *is saturated and closed under extensions. Let* $\mathscr{C}^w$ *denote the Waldhausen subcategory of* $(\mathscr{C}, v)$ *consisting of any* A *where* $* \to A$ *is in* $w(\mathscr{C})$ *[[Are the initial morphisms the only w.e.?]]. Then*

$$K(A^w) \to K(\mathscr{C}, v) \to K(\mathscr{C}, w)$$

is a homotopy fibration sequence.

Proof. Recall that a small bicategory is a bisimplicial set such that each row/column is the nerve of a category. Note that $v_{(-)}w_{(-)}\mathscr{C}$ is a bicategory whose bimorphisms are commutative squares of the form

$$\begin{tikzcd}
(-) \arrow[r, "w'"] \arrow[d, "v"'] & (-) \arrow[d, "v'"] \\
(-) \arrow[r, "w"] & (-)
\end{tikzcd}.$$

It turns out that treating $w\mathscr{C}$ as a bicategory with a single vertical morphism proves that $w\mathscr{C} \simeq v_{(-)}w_{(-)}\mathscr{C}$. This gives $wS_n\mathscr{C} \simeq v_{(-)}w_{(-)}S_n\mathscr{C}$ for each n.

Now, let $v_{(-)}\mathrm{co}w_{(-)}\mathscr{C}$ denote the subcategory of the above squares where the horizontal maps are also cofibrations. One can show that the inclusion $v_{(-)}\mathrm{co}w_{(-)}\mathscr{C} \subset v_{(-)}w_{(-)}\mathscr{C}$ is a homotopy equivalence. Since each $S_n\mathscr{C}$ inherits a cylinder functor from $\mathscr{C}$, we simplicial bi-subcategory $v_{(-)}\mathrm{co}w_{(-)}S_\bullet\mathscr{C}$ such that the inclusion intro $v_{(-)}w_{(-)}S_\bullet\mathscr{C}$ is a homotopy equivalence. We have now obtained the following diagram.

$$\begin{tikzcd}
vS_\bullet C^w \arrow[r] & vS_\bullet C \arrow[r] \arrow[d] & v_{(-)}\mathrm{co}w_{(-)}S_\bullet C \arrow[d, "\simeq"] \\
& wS_\bullet C \arrow[r, "\simeq"] & v_{(-)}w_{(-)}S_\bullet C
\end{tikzcd}$$

It therefore suffices to show that the top row is a fibration. [[What about the left vertical morphism?]] You do this by using the relative K-theory space construction. See Weibel IV.8.5.3 and V.2.1 for the details. □

Definition 5. Let $\mathscr{A}$ be an exact category embedded in an abelian category $\mathscr{B}$ and let $\mathbf{Ch}^b(\mathscr{A})$ denote the category of bounded chain complexes in $\mathscr{A}$. One can verify that $\mathbf{Ch}^b(\mathscr{A})$ is Waldhausen where the cofibrations $A_\bullet \rightarrowtail B_\bullet$ are precisely the degree-wise admissible monomorphisms (i.e., those giving a short exact sequence $A_n \to B_n \to {}^{B_n}\!/_{A_n}$ in $\mathscr{A}$ for each n) and the w.e. are precisely the chain maps which are quasi-isomorphisms of complexes in $\mathbf{Ch}(\mathscr{B})$.

Theorem 7. *(Gillet-Waldhausen) Let $\mathscr{A}$ be an exact category closed under kernels of surjections. Then the exact inclusion $\mathscr{A} \to \mathbf{Ch}^b(\mathscr{A})$ induces a homotopy equivalence $K(\mathscr{A}) \simeq K\,\mathbf{Ch}^b(\mathscr{A})$. Hence*

$$K_i(\mathscr{A}) = K_i\,\mathbf{Ch}^b(\mathscr{A})$$

for every i.

Proof. Apply the localization theorem. See Weibel, V.2.2. □

Definition 6. Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between Waldhausen categories. We say that F satisfies the *approximate lifting property* if for any map $b : F(A) \to B$ in $\mathscr{B}$, there is some map $a : A \to A'$ in $\mathscr{A}$ and some w.e. $b' : F(A') \simeq B$ in $\mathscr{B}$ so that

$$\begin{tikzcd}
F(A') \arrow[r, dashed, "\simeq"] & B \\
F(A) \arrow[u, dashed, "F(a)"] \arrow[ur, "b"] &
\end{tikzcd}.$$

commutes. In this way, we can lift to w.e.

Proposition 8. *Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between Waldhausen categories such that the following hold.*

1. *F satisfies the approximate lifting property.*
2. *$\mathscr{A}$ admits a cylinder functor.*

3. *A morphism f in $\mathscr{A}$ is a w.e. iff $F(f)$ is a w.e. in $\mathscr{B}$.*

Then $wF : w\mathscr{A} \to w\mathscr{B}$ is a homotopy equivalence.

Corollary 9. *(Waldhausen approximation theorem) With the same conditions as before, we have*

$$K(\mathscr{A}) \simeq K(\mathscr{B}).$$

Proof. One can show that each functor $S_n\mathscr{A} \to S_n\mathscr{B}$ is exact and also has the approximate lifting property. The previous proposition thus gives degree-wise homotopy equivalence between the bisimplicial map $wS_\bullet\mathscr{A} \to wS_\bullet\mathscr{B}$, which is enough. $\square$

Definition 7. Let $\mathscr{A}$ be an abelian category $\mathbf{Ch}(\mathscr{A})$ denote the category of chain complexes over $\mathscr{A}$. We say that a complex $C_\bullet$ is *homologically bounded* if only finitely many $H_i(C_j)$ are nonzero. Let $\mathbf{Ch}^{hb}_{\pm}$ denote the subcategory of bounded below (respectively, bounded above) complexes.

Example 8. Let $\mathscr{A}$ be an abelian category. By homology theory, we have that $\mathbf{Ch}^b(\mathscr{A}) \subset \mathbf{Ch}^{hb}_{-}(\mathscr{A})$ and $\mathbf{Ch}^{hb}_{+}(\mathscr{A}) \subset \mathbf{Ch}^{hb}(\mathscr{A})$ have the approximate lifting property. We also have that $\mathbf{Ch}^b(\mathscr{A}) \subset \mathbf{Ch}^{hb}_{+}(\mathscr{A})$ and $\mathbf{Ch}^{hb}_{+}(\mathscr{A}) \subset \mathbf{Ch}^{hb}(\mathscr{A})$ satisfy the dual of the approximate lifting property. Thus, we can apply the approximation theorem and Gillet-Waldhausen to see that

$$K(\mathscr{A}) \simeq K\,\mathbf{Ch}^b(\mathscr{A}) \simeq K\,\mathbf{Ch}^{hb}_{-} \simeq K\,\mathbf{Ch}^{hb}_{+}(\mathscr{A}) \simeq K\,\mathbf{Ch}^{hb}(\mathscr{A}).$$

Definition 9. (The following notion is due to Hovey-Shipley-Smith.) A *symmetric spectrum* $\mathbf{X}$ in topological spaces in a sequence of based Σ_n-spaces (X_n) endowed with structure maps $\sigma : X_n \wedge S^1 \to X_{n+1}$ such that $\sigma^k : X_n \wedge S^k \to X_{n+k}$ is $(\Sigma_n \times \Sigma_k)$-equivariant for any $n, k \geq 0$, where $S^k := \underbrace{S^1 \wedge \cdots \wedge S^1}_{k\text{-times}}$. A map $\mathbf{f} : \vec{x} \to \mathbf{Y}$ of symmetric spectra is a sequence $(f_n : X_n \to Y_n)$ of based Σ_n-equivariant maps such that for each $n \geq 0$, the square

$$\begin{tikzcd}
X_n \wedge S^1 \arrow[r, "f_n \wedge \mathrm{Id}"] \arrow[d, "\sigma"'] & Y_n \wedge S^1 \arrow[d, "\sigma"] \\
X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1}
\end{tikzcd}$$

commutes. Let Sp^Σ denote the category of symmetric spectra in topological spaces.

Definition 10. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. The *external n-fold $S_\bullet$-construction* on $\mathscr{C}$ is the n-multisimplicial Waldhausen category

$$(S_\bullet \cdots S_\bullet\mathscr{C}, wS_\bullet \cdots S_\bullet\mathscr{C}).$$

It multidegree $(q_1, \ldots, q_n)$, it has as objects the diagrams $X : \mathrm{Ar}[q_1] \times \cdots \times \mathrm{Ar}[q_n] \to \mathscr{C}$ such that

1. $X((i_1, j_1), \ldots, (i_n, j_n)) = *$ if $i_k = j_k$ for some $1 \leq k \leq n$.
2. $X(\ldots, (i_t, j_t), \ldots) \rightarrowtail X(\ldots, (i_t, k_t), \ldots) \twoheadrightarrow X(\ldots, (j_t, k_t), \ldots)$ is a cofiber sequence in the $(n-1)$-fold iterated $S_\bullet$-construction for any $i_t \leq j_t \leq k_t$ in $[q_t]$.

Definition 11. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. The *internal n-fold $S_\bullet$-construction* on $\mathscr{C}$ is the simplicial Waldhausen category

$$(S^{(n)}_\bullet\mathscr{C}, wS^{(n)}_\bullet\mathscr{C}).$$

It has as q-simplices the functor categories $(S_q \cdots S_q\mathscr{C}, wS_q \cdots S_q\mathscr{C})$ whose objects are the $(\mathrm{Ar}[q])^n$-shaped diagrams $X : (\mathrm{Ar}[q])^n \to \mathscr{C}$ such that

1. $X((i_1, j_1), \ldots, (i_n, j_n)) = *$ if $i_k = j_k$ for some $1 \leq k \leq n$.
2. $X(\ldots, (i_t, j_t), \ldots) \rightarrowtail X(\ldots, (i_t, k_t), \ldots) \twoheadrightarrow X(\ldots, (j_t, k_t), \ldots)$ is a cofiber sequence in the $(n-1)$-fold iterated $S_\bullet$-construction for any $i_t \leq j_t \leq k_t$ in $[q]$.

Note that Σ_n acts on $S_\bullet^{(n)}\mathscr{C}$ by $(\pi \cdot X)(\dots,(i_t,j_t),\dots) = X(\dots,(i_{\pi^{-1}(t)},j_{\pi^{-1}(t)}),\dots)$.

Definition 12. The *(symmetric) algebraic K-theory spectrum* $\mathbf{K}(\mathscr{C},w)$ of a small Waldhausen category $(\mathscr{C},w\mathscr{C})$ has n-th space $K(\mathscr{C},w)_n = |wS_\bullet^{(n)}\mathscr{C}|$ based at $*$. There is a Σ_n-action on $K(\mathscr{C},w)_n$ induced by permuting the order of the internal $S_\bullet$-constructions. Moreover, we have

$$|wS_\bullet^{(n)}\mathscr{C}| \wedge S^1 \cong |wS_\bullet^{(n)}S_\bullet\mathscr{C}|^{(1)} \subset |wS_\bullet^{(n)}S_\bullet\mathscr{C}| \cong |wS_\bullet^{(n+1)}\mathscr{C}|$$

, where $^{(1)}$ denotes the 1-skeleton with respect to the last simplicial direction. This determines the structure map σ. Then σ^k is $(\Sigma_n \times \Sigma_k)$-invariant.

Theorem 10. *For any $i \geq 0$, we have that $K_i(\mathscr{C},w) = \pi_{i+1}K(\mathscr{C},w)_1 \cong \pi_i\mathbf{K}(\mathscr{C},w)$.*

Proof. See Rognes, Lemma 8.7.4. □

Remark 3. In this way, we encode our algebraic K-theory in an infinite loop space.