

**Abstract**

These notes are based on Scott Weinstein’s “Model Theory” lectures at UPenn along with David Marker’s *Model Theory: An Introduction*. Any mistake in what follows is my own.

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# 1 Introduction

## 1.1 Lecture 1

Recall the structure  $\mathbb{N} := \langle \omega, S, 0 \rangle$  where

- $\omega$  denotes the set of natural numbers  $\{0, 1, 2, \dots\}$ ,
- $S$  is interpreted as the successor function  $\omega \rightarrow \omega$ , and
- the constant symbol  $0$  is interpreted as the natural number  $0$ .

The formal language  $\mathcal{L}$  for which  $\mathbb{N}$  is a structure consists of the first-order (FO) logical symbols

$$\forall, \exists, \wedge, \neg, \vee, \rightarrow, =$$

along with non-logical symbols such as  $0$ ,  $S^n 0 := \underbrace{S \cdots S}_n 0$ , and  $S^n x$ . Let FO denote the set of all (first-order)  $\mathcal{L}$ -sentences.

The *theory of  $\mathbb{N}$*  is

$$\text{Th}(\mathbb{N}) := \{\varphi \in \text{FO} \mid \mathbb{N} \models \varphi\},$$

which consists of all sentences satisfied by  $\mathbb{N}$ . Further, for any subset  $\Sigma \subset \text{FO}$ , consider the set

$$\text{Cn}(\Sigma) := \{\varphi \in \text{FO} \mid \Sigma \models \varphi\}$$

of consequences of  $\Sigma$ .

*Question.* Can we find a theory  $\Sigma$  (other than  $\text{Th}(\mathbb{N})$ ) such that  $\text{Cn}(\Sigma) = \text{Th}(\mathbb{N})$ ?

Let  $\Delta = \{\forall x (Sx \neq 0), \forall x \forall y (Sx = Sy \rightarrow x = y), \forall x (x \neq 0 \rightarrow \exists y (Sy = x))\}$ . Each element of  $\Delta$  is clearly true in  $\mathbb{N}$ , i.e.,  $\mathbb{N} \models \Delta$ . But is it the case that  $\text{Cn}(\Delta) = \text{Th}(\mathbb{N})$ ? No, provided that we allow ourselves access to monadic second-order sentences. Specifically, consider the *induction axiom* **IA**:

$$\forall P ((P(0) \wedge \forall x (P(x) \rightarrow P(Sx)) \rightarrow \forall x (P(x))). \quad (*)$$

This is clearly true in  $\mathbb{N}$ . Consider, however, a new structure  $\mathbb{A} := \langle \omega \cup \mathbb{Z}, S, 0 \rangle$ . Then  $\Delta \subset \text{Th}(\mathbb{A})$ , and we have a  $\mathbb{Z}$ -chain in  $\mathbb{A}$  (pretending, for the moment, that the universe  $|\mathbb{A}|$  has the usual order  $<$ ):

$$\dots \xrightarrow{\quad} -(n+1)^{\mathbb{A}} \xrightarrow{\quad} -n^{\mathbb{A}} \xrightarrow{\quad} \dots \xrightarrow{\quad} -1^{\mathbb{A}} \xrightarrow{\quad} 0^{\mathbb{A}} \xrightarrow{\quad} 1^{\mathbb{A}} \xrightarrow{\quad} \dots$$

The second-order sentence  $(*)$  with  $P$  instantiated by the “initial segment”  $\mathbb{Z}_{\geq -1}$  is not true in  $\mathbb{A}$ , so that  $\mathbb{A} \not\models \text{IA}$ . In this case,  $\text{IA} \in \text{Th}(\mathbb{N}) \setminus \text{Cn}(\Delta)$ .

Nevertheless, we want to restrict ourselves to FO. Recall that two structures  $\mathbb{B}$  and  $\mathbb{C}$  are *elementarily equivalent* if  $\text{Th}(\mathbb{B}) = \text{Th}(\mathbb{C})$ .

*Question.* Are  $\mathbb{A}$  and  $\mathbb{N}$  elementarily equivalent?

If we can find some sentence belonging to  $\text{Th}(\mathbb{N}) \setminus \text{Th}(\mathbb{A})$ , then  $\text{Cn}(\Delta) \neq \text{Th}(\mathbb{N})$ .

**Definition 1.1.1.** A theory  $\Sigma$  is *categorical* if for any structures  $\mathbb{B}$  and  $\mathbb{C}$ , if  $\mathbb{B} \models \Sigma$  and  $\mathbb{C} \models \Sigma$ , then  $\mathbb{B} \cong \mathbb{C}$ .

**Example 1.1.2.**  $\Delta' := \Delta + \text{IA}$  is categorical.

Perhaps exhibiting that the usual order  $<$  on  $\omega$  is definable in  $\mathbb{N}$  would reveal that  $\mathbb{A} \neq \mathbb{N}$ . For this, we must find a (well-formed) formula  $\theta(x, y)$  such that for every  $n, m \in \omega$ ,

$$m < n \iff \mathbb{N} \models \theta[n, m].$$

Thanks to Lagrange's four square theorem, we could define  $<$  on the positive integers. But it's unclear how to proceed further.

**Theorem 1.1.3.** *If  $\mathbb{B}$  is infinite, then for every infinite cardinal  $\kappa$ , there is some  $\mathbb{C}$  such that  $\mathbb{C} \equiv \mathbb{B}$  and  $\text{card}(\mathbb{C}) = \kappa$ .*

**Corollary 1.1.4.** *If  $\mathbb{B}$  is infinite, then there exists a  $\mathbb{C}$  such that  $\mathbb{C} \equiv \mathbb{B}$  and  $\mathbb{C} \not\equiv \mathbb{B}$ .*

Therefore,  $\Delta$  does *not* categorically describe  $\mathbb{N}$ . Now, consider the structure  $\tilde{\mathbb{N}}$  obtained from  $\mathbb{N}$  by adding a single point  $\bullet$  fixed by  $\mathbf{S}$ . Then the sentence  $\forall x (Sx \neq x)$  is true in  $\mathbb{N}$  but not in  $\tilde{\mathbb{N}}$ . Moreover,  $\Delta \subset \text{Th}(\tilde{\mathbb{N}})$ , which proves that

$$\text{Cn}(\Delta) \neq \text{Th}(\mathbb{N}).$$

With this in mind, let  $\Sigma = \Delta \cup \{\forall x (S^n x \neq x) \mid n \in \omega\}$ . To show that  $\text{Cn}(\Sigma) = \text{Th}(\mathbb{N})$ , it suffices to show that for any  $\mathbb{B}$ , if  $\mathbb{B} \models \Sigma$ , then  $\mathbb{B} \equiv \mathbb{N}$ . To this end, for any cardinal  $\kappa$ , let  $\mathbb{A}_\kappa = \mathbb{N} \cup (\kappa \times \mathbb{Z})$ , which is precisely the structure obtained from  $\mathbb{N}$  by adding  $\kappa$  many disjoint  $\mathbb{Z}$ -chains.

*Question.* How many structures that

- (a) satisfy  $\Sigma$  and
- (b) are of cardinality  $\kappa$

are there up to isomorphism?

If  $\kappa < \omega$ , then  $\text{card}(\mathbb{A}_\kappa) = \aleph_0$ . Also, if  $\kappa > \omega$ , then  $\text{card}(\mathbb{A}_\kappa) = \kappa$ , so that  $\Sigma$  is  $\kappa$ -categorical, i.e., every structure satisfying both (a) and (b) is isomorphic to  $\mathbb{A}_\kappa$ .