Perry Hart
Homotopy and K-theory seminar
Talk #15
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Abstract

We begin higher Waldhausen K-theory. The main sources for this talk are the following.

- *nLab*.
- Charles Weibel's *The* K*-book: an introduction to algebraic* K*-theory*. Chapter IV.8.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

For the original development, see Friedhelm Waldhausen's *Algebraic K-theory of spaces* (1985), 318-419.

Let $\mathscr{C}$ be a Waldhausen category. Our goal is to construct the K-theory $K(\mathscr{C})$ of $\mathscr{C}$ as a based loop space ΩY endowed with a loop completion map $\iota : |w\mathscr{C}| \to K(\mathscr{C})$ where $w\mathscr{C}$ denotes the subcategory of weak equivalences. This will produce a function $\operatorname{ob}\mathscr{C} \to |w\mathscr{C}| \to \Omega Y$. Further, we'll require of $K(\mathscr{C})$ certain limit and coherence properties, eventually rendering $K(\mathscr{C})$ the underlying infinite loop space of a spectrum $\mathbf{K}(\mathscr{C})$, called the algebraic K-theory spectrum of $\mathscr{C}$.

Definition 1. Let $\mathscr{C}$ be a category equipped with a subcategory $co(\mathscr{C})$ of morphisms called *cofibrations*. The pair $(\mathscr{C}, co\mathscr{C})$ is a *category with cofibrations* if the following conditions hold.

1. (W0) Every isomorphism in $\mathscr{C}$ is a cofibration.
2. (W1) There is a base point $*$ in $\mathscr{C}$ such that the unique morphism $* \rightarrowtail A$ is a cofibration for any $A \in \operatorname{ob}\mathscr{C}$.
3. (W2) We have a *cobase change*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, dotted] \\ C \arrow[r, tail, dotted] & B \cup_A C \end{tikzcd} .$$

We see that $B \coprod C$ always exists as the pushout $B \cup_* C$ and that the cokernel of any $i : A \rightarrowtail B$ exists as $B \cup_A *$ along $A \to *$. We call $A \rightarrowtail B \twoheadrightarrow B/A$ a *cofiber sequence*.

Definition 2. A *Waldhausen category* $\mathscr{C}$ is a category with cofibrations together with a subcategory $w\mathscr{C}$ of morphisms called *weak equivalences* such that every isomorphism in $\mathscr{C}$ is a w.e. and the following "Gluing axiom" holds.

1. (W3) For any diagram

$$\begin{tikzcd} C \arrow[d, "\sim"'] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"'] & B \arrow[d, "\sim"'] \\ C' & A' \arrow[l] \arrow[r, tail] & B' \end{tikzcd} ,$$

   the induced map $B \cup_A C \to B' \cup_{A'} C'$ is a w.e.

Definition 3. A Waldhausen category $(\mathscr{C}, w)$ is *saturated* if whenever fg makes sense and is a w.e., then f is a w.e. iff g is.

Definition 4. We now introduce the main concept to be generalized.

Let $\mathscr{C}$ be a category with cofibrations. Let the *extension category* $S_2\mathscr{C}$ have as objects the cofiber sequences in $(\mathscr{C}, co\mathscr{C})$ and as morphisms the triples (f', f, f'') such that

$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d, "f'"] & X \arrow[r, two heads] \arrow[d, "f"] & X'' \arrow[d, "f''"] \\ Y' \arrow[r, tail] & Y \arrow[r, two heads] & Y'' \end{tikzcd}$$

commutes. This is pointed at $* \rightarrowtail * \twoheadrightarrow *$.

Definition 5. Suppose an arbitrary triple (f', f, f'') as above has the property that whenever f' and f'' are w.e., then so is f. Then we say $\mathscr{C}$ is *extensional* or *closed under extensions*.

Say that the morphism (f', f, f'') is a cofibration if f', f'', and $Y' \cup_{X'} X \to Y$ are cofibrations in $\mathscr{C}$. Say that the same triple is a weak equivalence if f', f, and f'' are w.e. in $\mathscr{C}$. This makes $S_2\mathscr{C}$ into a Waldhausen category.

Definition 6. Let $q \geq 0$. Let the *arrow category* $\mathrm{Ar}[q]$ *on* $[q]$ have as objects ordered pairs (i, j) with $i \leq j \leq q$ and as morphisms commutative diagrams of the form

$$\begin{tikzcd} i \arrow[r, "\leq"] \arrow[d, "\leq"'] & j \arrow[d, "\leq"] \\ i' \arrow[r, "\leq"'] & j' \end{tikzcd} .$$

We view $[q]$ a full subcategory of $\mathrm{Ar}[q]$ via the embedding $[q] \xrightarrow{k \mapsto (0,k)} \mathrm{Ar}[q]$.

Remark 1.

1. Any triple $i \leq j \leq k$ determines the morphisms $(i, j) \to (i, k)$ and $(i, k) \to (j, k)$. Conversely, any morphism in the arrow category is a composition of such triples.

2. $\mathrm{Ar}[q] \cong \mathbf{Fun}([1], [q])$ by identifying each pair (i, j) with the functor satisfying $0 \mapsto i$ and $1 \mapsto j$.

Example 7. The category $\mathrm{Ar}[2]$ is generated by the commutative diagram

$$\begin{tikzcd} (0,0) \arrow[r] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\ & (1,1) \arrow[r] & (1,2) \arrow[d] \\ & & (2,2) \end{tikzcd} .$$

Definition 8. Let $\mathscr{C}$ be a category with cofibrations and $q \geq 0$. Define $S_q\mathscr{C}$ as the full subcategory of $\mathbf{Fun}(\mathrm{Ar}[q], \mathscr{C})$ generated by $X : \mathrm{Ar}[q] \to \mathscr{C}$ such that

1. $X_{j,j} = *$ for each $j \in [q]$.

2. $X_{i,j} \rightarrowtail X_{i,k} \twoheadrightarrow X_{j,k}$ is a cofiber sequence for any $i < j < k$ in $[q]$. Equivalently, if $i \leq j \leq k$ in $[q]$, then the square

$$\begin{tikzcd} X_{i,j} \arrow[r, tail] \arrow[d, two heads] & X_{i,k} \arrow[d, two heads] \\ X_{j,j} = * \arrow[r, tail] & X_{j,k} \end{tikzcd}$$

   is a pushout.

This is pointed at the constant diagram at $*$.

Note 9. A generic object in $S_q\mathscr{C}$ looks like

$$\begin{tikzcd}
* \arrow[r, tail] & X_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & X_{q-1} \arrow[r, tail] \arrow[d, two heads] & X_q \arrow[d, two heads] \\
 & * \arrow[r, tail] & \cdots \arrow[r, tail] & X_{q-1}/X_1 \arrow[r, tail] \arrow[d, two heads] & X_q/X_1 \arrow[d, two heads] \\
 & & \ddots & \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\
 & & & * \arrow[r, tail] & X_q/X_{q-1} \arrow[d, two heads] \\
 & & & & *
\end{tikzcd} . \tag{$*$}$$

where X_q corresponds to $X_{0,q}$ and X_j/X_i to $X_{i,j}$ for any $1 \le i \le j \le q$.

Definition 10. Let $(\mathscr{C}, co\mathscr{C})$ be a category with cofibrations. Let $coS_q\mathscr{C} \subset S_q\mathscr{C}$ consist of the morphisms $f : X \rightarrowtail Y$ of $\mathrm{Ar}[q]$-shaped diagrams such that for each $1 \le j \le q$ we have

$$\begin{tikzcd}
X_{0,j-1} \arrow[r, tail] \arrow[d, "f_{0,j-1}"'] & X_{0,j} \arrow[d] \arrow[ddr, bend left, "f_{0,j}"] & \\
Y_{0,j-1} \arrow[r, tail] \arrow[drr, tail, bend right] & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} \arrow[dr, tail, dashed] & \\
 & & Y_{0,j}
\end{tikzcd} .$$

Proposition 1. *If $f : X \to Y$ is a cofibration of $S_q\mathscr{C}$, then*

$$\begin{tikzcd}
X_{i,j} \arrow[r, tail] \arrow[d, tail, "f_{i,j}"'] & X_{i,k} \arrow[d, tail, "f_{i,k}"] \\
Y_{i,j} \arrow[r, tail] & Y_{i,k}
\end{tikzcd}$$

for any $i \le j \le k$ in $[q]$.

Proof. See Rognes, Lemma 8.3.12. ☐

Lemma 2. *$(S_q\mathscr{C}, coS_1\mathscr{C})$ is a category with cofibrations.*

Proof. First notice that the composite of two cofibrations $g \circ f : X \to Y \to Z$ is a cofibration because we have

$$\begin{tikzcd}
X_{0,j-1} \arrow[r, tail] \arrow[d, "f_{0,j-1}"'] & X_{0,j} \arrow[d] \arrow[dr, bend left, "f_{0,j}"] & & \\
Y_{0,j-1} \arrow[r, tail] \arrow[d, "g_{0,j-1}"'] & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} \arrow[r, tail] \arrow[d] & Y_{0,j} \arrow[d] \arrow[dr, bend left, "f_{0,j}"] & \\
Z_{0,j-1} \arrow[r, tail] & X_{0,j} \cup_{X_{0,j-1}} Z_{0,j-1} \arrow[r, tail] & Y_{0,j} \cup_{Y_{0,j-1}} Z_{0,j-1} \arrow[r, tail] & Z_{0,j}
\end{tikzcd} .$$

It's clear that any isomorphism or initial morphism in $S_q\mathscr{C}$ is a cofibration.

To see that (W2) is satisfied, let $f : X \to Y$ and $g : X \to Z$ be morphisms in $S_q\mathscr{C}$. It's easy to verify that each component $f_{i,j} : X_{i,j} \to Y_{i,j}$ is a cofibration. Thus, each pushout $W_{i,j} \coloneqq Y_{i,j} \cup_{X_{i,j}} Z_{i,j}$ exists. These form a functor $W : \mathrm{Ar}[q] \to \mathscr{C}$. If $i < j < k$, then we have $W_{i,j} \rightarrowtail W_{i,k} \twoheadrightarrow W_{j,k}$ because the left morphism factors as the composite of two cofibrations

$$\begin{tikzcd}
Z_{i,j} \arrow[r, tail] \arrow[d, "f_{i,j}\cup \mathrm{Id}"'] & Z_{i,k} \arrow[d, "f_{i,j}\cup \mathrm{Id}"] & \\
Y_{i,j} \cup_{X_{i,j}} Z_{i,j} \arrow[r, tail] & Y_{i,j} \cup_{X_{i,j}} Z_{i,k} \arrow[r, tail] & Y_{i,k} \cup_{X_{i,k}} Z_{i,k} \\
& Y_{i,j} \cup_{X_{i,j}} X_{i,k} \arrow[u, "\mathrm{Id}\cup g_{i,k}"] \arrow[r, tail] & Y_{i,k} \arrow[u, "\mathrm{Id}\cup g_{i,k}"']
\end{tikzcd}.$$

The fact that colimits commute confirms that $W_{j,k} \cong {}^{W_{i,k}}\!/\!_{W_{i,j}}$ Hence W is the pushout of f and g. To verify that this is a cofibration, we must check that the pushout map $W_{0,j-1} \cup_{Z_{0,j-1}} Z_{0,j} \to W_{0,j}$ is a cofibration. But this follows from the pushout square

$$\begin{tikzcd}
Y_{0,j-1} \cup_{X_{0,j-1}} X_{0,j} \arrow[r, tail] \arrow[d] & Y_{0,j} \arrow[d] \\
Y_{0,j-1} \cup_{X_{0,j-1}} Z_{0,j} \arrow[r, tail] & Y_{0,j} \cup_{X_{0,j}} Z_{0,j}
\end{tikzcd}.$$

$\square$

Definition 11. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Let $wS_q\mathscr{C} \subset S_q\mathscr{C}$ consist of the morphisms $f : X \xrightarrow{\sim} Y$ of $\mathrm{Ar}[q]$-shaped diagrams such that the component $f_{0,j} : X_{0,j} \to Y_{0,j}$ is a w.e. in $\mathscr{C}$ for each $1 \le j \le q$.

Proposition 3. *Let f be a w.e. in $S_q\mathscr{C}$. Each component $f_{i,j} : X_{i,j} \to Y_{i,j}$ is a w.e. in $\mathscr{C}$.*

Proof. Apply the Gluing axiom to the diagram

$$\begin{tikzcd}
X_{0,j} \arrow[d, "\cong"'] & X_{0,i} \arrow[l, tail] \arrow[r] \arrow[d, "\cong"'] & * \arrow[d, "="'] \\
Y_{0,j} & Y_{0,i} \arrow[l, tail] \arrow[r] & *
\end{tikzcd}.$$

Then $X_{i,j} \cong X_{0,j} \cup_{X_{0,i}} * \xrightarrow{\sim} Y_{0,j} \cup_{Y_{0,i}} * \cong Y_{i,j}$, as desired. $\square$

Lemma 4. *$(S_q\mathscr{C}, wS_q\mathscr{C})$ is a Waldhausen category.*

Definition 12. Let $\mathscr{C}$ be a category with cofibrations. If $\alpha : [p] \to [q]$, then define $\alpha^* : S_q\mathscr{C} \to S_p\mathscr{C}$ by

$$\alpha^*(X : \mathrm{Ar}[q] \to \mathscr{C}) = X \circ \mathrm{Ar}(\alpha) : \mathrm{Ar}[p] \to \mathrm{Ar}[q] \to \mathscr{C}.$$

It's easy to check that this satisfies the two conditions of a diagram in $S_p\mathscr{C}$. Moreover, the face maps d_i are given by deleting the row $X_{i,-}$ and the column containing X_i in $(*)$ of Note 9 and then reindexing as necessary. The degeneracy maps s_i are given by duplicating X_i and then reindexing such that $X_{i+1,i} = 0$. [[Not sure the s_i work.]]

Proposition 5. *Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Each functor $\alpha^* : S_q\mathscr{C} \to S_p\mathscr{C}$ is exact, so that $(S_\bullet\mathscr{C}, wS_\bullet\mathscr{C})$ is a simplicial Waldhausen category.*

The nerve $N_\bullet wS_\bullet \mathscr{C}$ is a bisimplicial set with (p,q)-bisimplices the diagrams of the form

$$\begin{tikzcd}
* \arrow[r, tail] & X_1^0 \arrow[r, tail] \arrow[d, "\sim"'] & X_2^0 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X_q^0 \arrow[d, "\sim"'] \\
* \arrow[r, tail] & X_1^1 \arrow[r, tail] \arrow[d, "\sim"'] & X_2^1 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X_q^1 \arrow[d, "\sim"'] \\
 & \vdots \arrow[d, "\sim"'] & \vdots \arrow[d, "\sim"'] & & \vdots \arrow[d, "\sim"'] \\
* \arrow[r, tail] & X_1^p \arrow[r, tail] & X_2^p \arrow[r, tail] & \cdots \arrow[r, tail] & X_q^p
\end{tikzcd}$$

such that $X_{i,j}^k \cong X_j^k / X_i^k$ for every $i \le j \le q$ and $k \in [p]$.

Lemma 6. *There is a natural map $N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 \to N_\bullet wS_\bullet \mathscr{C}$, which automatically induces a based map $\sigma : \Sigma|w\mathscr{C}| \to |wS_\bullet \mathscr{C}|$ of classifying spaces.*

Proof. We can treat $N_\bullet wS_\bullet \mathscr{C}$ as the simplicial set $[q] \mapsto N_\bullet wS_q \mathscr{C}$. This defines a right skeletal structure on $N_\bullet wS_\bullet \mathscr{C}$.

If $q = 0$, then $wS_0\mathscr{C} = S_0\mathscr{C} = *$, so that $N_\bullet wS_0\mathscr{C} = *$ as well. If $q = 1$, then $wS_1\mathscr{C} \cong w\mathscr{C}$. Thus, the right 1-skeleton is equal to $N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1$, which in turn must be equal to the image I of the canonical map

$$\coprod_{q \le 1} N_\bullet wS_q \mathscr{C} \times \Delta_\bullet^q \to N_\bullet wS_\bullet \mathscr{C}.$$

Now, the degeneracy map s_0 collapses $\{*\} \times \Delta_\bullet^1$, and the face maps d_0 and d_1 collapse $N_\bullet w\mathscr{C} \times \partial\Delta_\bullet^1$. Therefore, I must equal

$$N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 = \frac{N_\bullet w\mathscr{C} \times \Delta_\bullet^1}{\{*\} \times \Delta_\bullet^1 \cup N_\bullet w\mathscr{C} \times \partial\Delta_\bullet^1}.$$

We have defined a natural inclusion map $\lambda : N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 \to N_\bullet wS_\bullet \mathscr{C}$.

Since $\Delta_\bullet^1$ is isomorphic to the unit interval and the map λ agrees on the endpoints, we can pass to S^1 during the suspension. Hence λ immediately induces the desired map σ. [[This is a tentative explanation offered by Thomas Brazelton.]] $\square$

Remark 2. The axiom (W3) implies that $w\mathscr{C}$ is closed under coproducts, making $|wS_\bullet \mathscr{C}|$ into an H-space via the map

$$\coprod : |wS_\bullet \mathscr{C}| \times |wS_\bullet \mathscr{C}| \cong |wS_\bullet \mathscr{C} \times wS_\bullet \mathscr{C}| \to |wS_\bullet \mathscr{C}|.$$

Definition 13. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Define the *algebraic K-theory space*

$$K(\mathscr{C}, w) = \Omega|N_\bullet wS_\bullet \mathscr{C}|.$$

Then we have a right adjoint $\iota : |w\mathscr{C}| \to K(\mathscr{C}, w)$ to the based map σ.

Moreover, let $F : (\mathscr{C}, w\mathscr{C}) \to (\mathscr{D}, w\mathscr{D})$ be an exact functor. Then set $K(F) = \Omega|wS_\bullet F| : K(\mathscr{C}, w) \to K(\mathscr{D}, w)$. We have thus defined the *algebraic K-theory functor* $K : \mathbf{Wald} \to \mathbf{Top}_*$.

Recall that any exact category $\mathscr{A}$ is a Waldhausen category with cofibrations the admissible exact sequences and w.e. the isomorphisms. Waldhausen showed that $|iS_\bullet \mathscr{A}|$ (where i denotes the iso category) and $BQ\mathscr{A}$ are homotopy equivalent. Hence our current definition of higher algebraic K-theory agrees with Quillen's.

Example 14. Let R be a ring. Define the *algebraic K-theory space of R* as

$$K(R) = K(\mathbf{P}(R), i)$$

where the w.e. i are precisely the injective R-linear maps with projective cokernel and the cofibrations are precisely the R-linear maps.

Example 15. Assume that $\mathscr{C}$ is a small Waldhausen category where $w\mathscr{C}$ consists of the isomorphisms in $\mathscr{C}$. If $s_n\mathscr{C}$ denotes the set of objects of $S_n\mathscr{C}$, then we get a simplicial set $s_\bullet\mathscr{C}$. Waldhausen showed that the inclusion $|s_\bullet\mathscr{C}| \hookrightarrow |iS_\bullet\mathscr{C}|$ is a homotopy equivalence. This makes $\Omega|s_\bullet\mathscr{C}|$ into a so-called simplicial model for $K(\mathscr{C}, w)$.

Since $wS_0\mathscr{C} = *$ and every simplex of degree $n > 0$ is attached to $*$, it follows that the classifying space $|wS_\bullet\mathscr{C}|$ is connected. Therefore, we preserve any homotopical information when passing to the loop space.

Definition 16. Define the *i-th algebraic K-group* as $K_i(\mathscr{C}, w) = \pi_i K(\mathscr{C}, w)$ for each $i \geq 0$.

Proposition 7. *$\pi_1|wS_\bullet\mathscr{C}| \cong K_0(\mathscr{C}, w)$.*

Lemma 8. *The group $K_0(\mathscr{C}, w)$ is generated by $[X]$ for every $X \in \operatorname{ob}\mathscr{C}$ such that $[X'] + [X''] = [X]$ for every cofiber sequence $X' \rightarrowtail X \twoheadrightarrow X''$ and $[X] = [Y]$ for every w.e. $X \xrightarrow{\sim} Y$.*

Proof. We compute $\pi_1|N_\bullet wS_\bullet\mathscr{C}|$ based at the $(0,0)$-bisimplex $*$. Notice that $|N_\bullet wS_\bullet\mathscr{C}|$ has a CW structure [[this is reasonable visually]] with 1-cells the $(0,1)$-bisimplices and 2-cells the $(0,2)$-bisimplices $X' \rightarrowtail X \twoheadrightarrow X''$ and the $(1,1)$-bisimplices $X \xrightarrow{\sim} Y$, which are attached to the 1-cells X and Y. Any cell of dimension $n > 2$ is irrelevant to computing π_1. □

Corollary 9. *We obtain the functors $K_i : \mathbf{Wald} \to \mathbf{Top}_* \to \mathbf{Ab}$, called the algebraic K-group functors.*

Proof. By Proposition 7, we know that $K_i(\mathscr{C}, w) = \pi_{i+1}|wS_\bullet\mathscr{C}|$, which is abelian for $i \geq 1$. Moreover, note that if $X' \rightarrowtail X' \vee X'' \twoheadrightarrow X''$ and $X'' \rightarrowtail X' \vee X'' \twoheadrightarrow X'$ are cofiber sequences, then the previous lemma implies that $[X'] + [X''] = [X' \vee X''] = [X'' + X']$. Hence $K_0(\mathscr{C}, w)$ is also abelian. □

Example 17. Let X be a CW complex and $\mathcal{R}(X)$ denote the category of CW complexes Y obtained from X by attaching at least one cell such that X is a retract of Y. Equip this with cofibrations in the form of cellular inclusions fixing X and w.e. in the form of homotopy equivalences. This makes $\mathcal{R}(X)$ into a Waldhausen category. If $\mathcal{R}_f(X)$ denotes the subcategory of those Y obtained by attaching finitely many cells, then we write $A(X) \coloneqq K(\mathcal{R}_f(X))$.

Lemma 10. *$A_0(X) \cong \mathbb{Z}$.*

Proof. Weibel leaves this proof an an exercise. □

Definition 18. If $\mathscr{B}$ is a Waldhausen subcategory of $\mathscr{C}$, then it is *cofinal in $\mathscr{C}$* is for any $X \in \operatorname{ob}\mathscr{C}$, there is some $X' \in \operatorname{ob}\mathscr{C}$ such that $X \coprod X' \in \operatorname{ob}\mathscr{B}$.

Theorem 11. *Let $(\mathscr{B}, w)$ be cofinal in $(\mathscr{C}, w)$ and closed under extensions. Assume that $K_0(\mathscr{B}) = K_0(\mathscr{C})$. Then $wS_\bullet\mathscr{B} \to wS_\bullet\mathscr{C}$ is a homotopy equivalence. Therefore, $K_i(\mathscr{B}) \cong K_i(\mathscr{C})$ for every $i \geq 0$.*