

Abstract

These notes are based on Julius Shaneson's lectures for the course "Algebraic Topology, Part I" at UPenn. Any mistake in what follows is my own.

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1 Background material

1.1 Lecture 1

Here are the topics for the course:

- fiber bundles over cell complexes,
- spectral sequences,
- characteristic classes, and
- cobordism theory.

To start, let's review some basic concepts from homology theory.

Definition 1.1.1. A (finite) cell complex is a (topological) space X that can be written as $\bigcup_{n=0}^K X^n$ for some $K \in \mathbb{N}$ (called the *dimension of X*) where

- X^0 is chosen to be finite,
- $X^n = \frac{X^{n-1} \amalg D_1^n \amalg \cdots \amalg D_{k_n}^n}{x \sim \varphi_i(x)}$,
- $D_i^n \cong D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ for each $i \in \{1, \dots, k_n\}$, and
- $\varphi_i : \partial D_i^n = S^{n-1} \rightarrow X^{n-1}$, called an *attaching map*.

Terminology. Each D_i^n is called an *n-cell of X* .

Every attaching map $\varphi_i : \partial D_i^n \rightarrow X^{n-1}$ can be extended to a *characteristic map* given by the composite

$$D_i^n \hookrightarrow X^{n-1} \amalg D_1^n \amalg \cdots \amalg D_{k_n}^n \rightarrow X^n \hookrightarrow X.$$

Example 1.1.2. There are at least two ways of endowing S^2 with a cell structure.

1. $X^0 \equiv \{N, S\}$, $X^1 \equiv X^0 \cup_{\varphi_1} D_1^1 \cup_{\varphi_2} D_2^1$ where each φ_i is an embedding, and $X^2 \equiv X^1 \cup_{\varphi'_1} D_1^2 \cup_{\varphi'_2} D_2^2$ where each φ'_i is an embedding.
2. $\text{pt} \cup_{\varphi} D^2$ where φ identifies the equator of the upper half-sphere with pt .

Definition 1.1.3. A cell complex X is *regular* if every characteristic map $D_i^n \rightarrow X$ is an embedding.

Definition 1.1.4. Given a family of functors $\{H_n : \mathbf{Top}^2 \rightarrow \mathbf{Ab}\}_{n \in \mathbb{N}}$ where \mathbf{Top}^2 denotes the category of (topological) pairs, we say that H_i is a *homology functor* if each of the following properties holds.

1. (LES) For any pair (X, A) of space, there is a natural long exact sequence

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \longrightarrow \cdots,$$

where $H_i(Z) := H_i(Z, \emptyset)$ for any space Z .

2. (Excision) If $\text{cl}(A) \subset \bigcup_{\text{open}} U \subset X$, then $H_i(X \setminus A, U \setminus A) \cong H_i(X, U)$.

$$3. \text{ (Dimension) } H_i(\mathbf{pt}) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases}.$$

4. (Homotopy) If f and g are homotopic, then $f_* = g_*$, where $h_* := H_i(h)$ for any map $h : (X, A) \rightarrow (Y, B)$.

Theorem 1.1.5. *There exists a family of homology functors.*

Example 1.1.6. In singular homology theory, we have that $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}.$

Let X be a cell complex. Let $C_n(X)$ denote the free abelian group on the set of all n -cells of X . Define $\partial : C_n(X) \rightarrow C_{n-1}(X)$ by $\partial[D_i^n] = \sum_{j=1}^{k_{n-1}} \lambda_{ij}[D_j^{n-1}]$ where λ_{ij} is defined, up to sign, as follows. Consider the map

$$S^{n-1} \xlongequal{\quad} \partial D_i^n \xrightarrow{\varphi_i} X^{n-1} \twoheadrightarrow \frac{X^{n-1}}{X^{n-2} \cup (\text{all cells of dim. } n-1 \text{ except } D_j^{n-1})} \xlongequal{\quad} D^{n-1} / \partial D_j^{n-1} \xlongequal{\quad} S^{n-1}.$$

ω

Then let λ_{ij} satisfy $\omega_*(x) = \lambda_{ij}x$ with x a chosen generator (i.e., orientation) of $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Terminology. The integer λ_{ij} is called the *degree* of ω , denoted by $\deg(\omega)$.

Theorem 1.1.7. $\partial_n \partial_{n+1} = 0$, and $H_n(X) \cong \ker \partial_n / \text{im } \partial_{n+1}$, which is independent of our choice of generator x .

Example 1.1.8. Suppose that $f : S^n \rightarrow S^n$ is smooth. By Sard's theorem, we can find a regular value $x \in S^n$. There is some neighborhood U of x such that $f^{-1}(U) = U_1 \cup \dots \cup U_n$ for some n . Using the inverse function theorem and the compactness of S^n , it follows that f^{-1} is of the form $\{x_1, \dots, x_n\}$. Note that the differential $(df)_{x_i} : S_{x_i}^n \rightarrow S_x^n$ satisfies $\det(df)_{x_i} = \pm 1$. In fact,

$$\deg(f) = \sum_{i=1}^n \det(df)_{x_i}.$$

Exercise 1.1.9. *Prove that any finite cell complex $X = X^K$ is homotopy equivalent to a regular cell complex.*

(Hint: Consider the map $S^{n-1} \rightarrow X^{n-1} \times D^n$ given by $x \mapsto (\varphi(x), x)$ where φ denotes an attaching map of X .)

Proof. Let us construct recursively a finite sequence A^0, A^1, \dots, A^K of spaces such that each A^i carries the structure of a regular cell complex and is homotopy equivalent to X^i . For each $n \in \{1, \dots, K\}$, let k_n denote the necessarily finite number of attaching maps $\varphi_{\alpha_1}, \dots, \varphi_{\alpha_{k_n}} : S^{n-1} \rightarrow X^{n-1}$ for the n -skeleton of X . Let

$$A^0 = X^0 \times D_{\alpha_1}^1 \times \dots \times D_{\alpha_{k_1}}^1,$$

viewed as a product of finite cell-complexes. Note that the topology of A^0 is precisely the product topology. Thus, A^0 is homotopy equivalent to X^0 as D^1 is contractible. Now, suppose that $0 \leq n \leq K-1$ and that we

have constructed our desired space A^n . This means that there is some homotopy equivalence $\gamma_n : X^n \rightarrow A^n$. Form A^{n+1} by attaching finitely many $(n+1)$ -cells $e_{\alpha_1}^{n+1}, \dots, e_{\alpha_{k_{n+1}}}^{n+1}$ to $Z_n \equiv A^n \times D_{\alpha_1}^{n+1} \times \dots \times D_{\alpha_{k_{n+1}}}^{n+1}$ via the maps

$$\psi_{\alpha_i} : S^n \rightarrow A^n \times D_{\alpha_1}^{n+1} \times \dots \times D_{\alpha_{k_{n+1}}}^{n+1}$$

$$x \mapsto \left(\gamma_n \circ \varphi_i(x), 0, \dots, 0, \underbrace{x}_{i\text{-th spot}}, 0, \dots, 0 \right)$$

where Z_n is viewed as a product of finite cell complexes (whose topology is precisely the product topology). It is easy to see that A^{n+1} is homotopy equivalent to X^{n+1} . Moreover, since each map ψ_{α_i} is an embedding and any n -disk has the structure of a regular cell complex, we see from our construction of (A^i) that A^K has the structure of a regular cell complex. By design, this space is homotopy equivalent to X^K , thereby completing our proof. \square

1.2 Lecture 2

Example 1.2.1 (Real projective space). Recall that $\mathbb{RP}^n = S^n / x \sim -x$. Then $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\pi_{n-1}} D^n$ where $\pi_{n-1} : S^{n-1} \rightarrow \mathbb{RP}^{n-1}$ denotes the canonical projection. Thus, \mathbb{RP}^n is an n -dimension cell complex with $(\mathbb{RP}^n)^m = \mathbb{RP}^m$ for each integer $0 \leq m \leq n$.

Now, for each $0 \leq m \leq n$, we have that $C_m(\mathbb{RP}^n) \cong \mathbb{Z}$ with generator $[D^m]$. To determine $\partial[D^m] \in C_{m-1}(\mathbb{RP}^n)$, we must find the degree of the map

$$\begin{array}{ccccccc} S^{m-1} & \longrightarrow & \mathbb{RP}^{m-1} & \longrightarrow & \mathbb{RP}^{m-1} / \mathbb{RP}^{m-1} & \equiv & D^{m-1} / \partial D^{m-1} \equiv S^{m-1} \\ & & & & \searrow & & \nearrow \\ & & & & \varphi & & \end{array}$$

Assume, for simplicity, that $m = 2$. Choose a regular value $p \in S^1$ so that $\varphi^{-1}(p) = \{N, S\}$. Let φ_T and φ_B denote the restrictions of φ to the top and bottom components of $S^1 \setminus \{(-1, 0), (1, 0)\}$, respectively. Note that both of these are homeomorphisms and thus have degrees equal to ± 1 . If $a : S^{m-1} \rightarrow S^{m-1}$ denotes the antipodal map, we have that $\varphi_B \circ a = \varphi_T$. Hence $(d\varphi)_S \circ (da)_N = (d\varphi)_N$. Since $\deg(a) = \det(da) = (-1)^m$, it follows that

$$\deg(\varphi) = \begin{cases} \pm 2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}.$$

Thus, we get a chain complex

$$0 \longrightarrow C_n(\mathbb{RP}^n) \xrightarrow{\kappa_1} C_{n-1}(\mathbb{RP}^n) \xrightarrow{\kappa_2} \dots \xrightarrow{0} C_2(\mathbb{RP}^n) \xrightarrow{\pm 2} C_1(\mathbb{RP}^n) \xrightarrow{0} C_0(\mathbb{RP}^n) \longrightarrow 0$$

$$\text{where } \kappa_1 = \begin{cases} 0 & n \text{ odd} \\ \pm 2 & n \text{ even} \end{cases} \text{ and } \kappa_2 = \begin{cases} \pm 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}.$$

This proves that

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}_2 & i = 1 \\ 0 & i = 2 \\ \mathbb{Z}_2 & i < n \text{ odd} \\ 0 & i < n \text{ even} \\ 0 & i > n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & i = n \text{ even} \end{cases}.$$

Example 1.2.2. $H_{2i}(\mathbb{CP}^n) \cong \mathbb{Z}$.

Next, let's introduce some fundamental concepts from homotopy theory.

Definition 1.2.3. Let $M(X, Y)$ denote the set of maps $X \rightarrow Y$.

1. For any compact $C \subset X$ and open $U \subset Y$, let

$$N(C, U) = \{f : X \rightarrow Y \mid f(C) \subset U\}.$$

The *compact-open topology* on $M(X, Y)$ consists of all unions of finite intersections of subsets of the form $N(C, U)$. Under this topology, $M(X, Y)$ is called a *mapping space*.

2. The *n-th loop space* of a pointed space (X, x) is

$$\Omega^{n-1}(X, x) := M((D^{n-1}, \partial D^{n-1}), (X, x)),$$

which is a subset of $M(D^{n-1}, X)$.

Definition 1.2.4 (Higher homotopy groups). If $n \geq 2$, then the *n-th homotopy group* of (X, x) is

$$\pi_n(X, x) := \pi_1(\Omega^{n-1}X, e_x).$$

Recall that $\pi_1(-)$ is a functor $\mathbf{Top}_* \rightarrow \mathbf{Grp}$. Also, $\Omega^{n-1}(-)$ is a functor $\mathbf{Top}_* \rightarrow \mathbf{Top}$ defined on morphisms $f : (X, x) \rightarrow (Y, y)$ by post-composition with f . Therefore, it's easy to see that $\pi_n(-)$ is a functor $\mathbf{Top}_* \rightarrow \mathbf{Grp}$ as well.

Notation. Let $f_* = \pi_n(f)$ for any $f : (X, x) \rightarrow (Y, y)$.

Proposition 1.2.5. *There is a homeomorphism $M(X \times Y, Z) \cong M(X, M(Y, Z))$ so long as Y is locally compact and Hausdorff.*

In particular, we have a composite

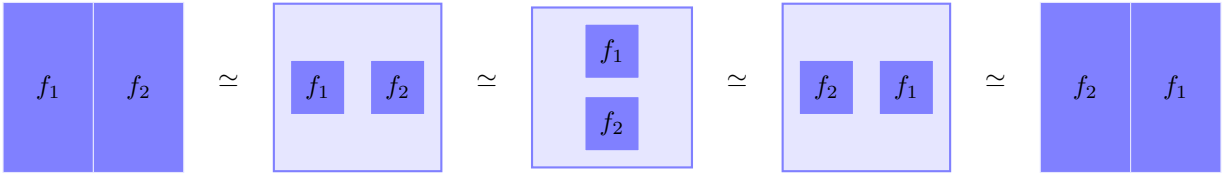
$$M([0, 1], \{0, 1\}), (M((D^{n-1}, \partial), (X, x)), e_x) \hookrightarrow M([0, 1], M(D^{n-1}, X)) \xrightarrow{\cong} M([0, 1] \times D^{n-1}, X),$$

whose image is precisely $M((D^n, \partial), (X, x)) \cong M((S^n, \text{pt}), (X, x))$. This proves that $\pi_n(X, x)$ consists of all homotopy classes of maps $(I^n, \partial) \rightarrow (X, x)$ under the operation $[f] * [g] = [f * g]$ where

$$f * g(t_1, \dots, t_n) = \begin{cases} f(2t_1, t_2, \dots, t_n) & 0 \leq t_1 \leq \frac{1}{2} \\ f(2t_1 - 1, t_2, \dots, t_n) & \frac{1}{2} \leq t_1 \leq 1 \end{cases}.$$

Lemma 1.2.6. *If $n \geq 2$, then $\pi_n(X, x)$ is abelian.*

Proof. Let f_1 and f_2 be maps $(I^n, \partial) \rightarrow (X, x)$. We must find a homotopy $F : I^n \times I \rightarrow X$ between $f_1 * f_2$ and $f_2 * f_1$ such that $F(t_1, \dots, t_n, s) = x$ for any $(t_1, \dots, t_n) \in \partial I^n$ and $s \in I$. To this end, first shrink the domains of f_1 and f_2 to small n -cubes in I^n (thereby thickening the set of points mapped to x under F), then slide these small cubes past each other, and finally enlarge them to their original sizes as follows.



□

Remark 1.2.7. A map $f : S^{n-1} \rightarrow X$ is homotopic to the constant map if and only if there is some g such that

$$\begin{array}{ccc} & D^n & \\ \uparrow & \searrow g & \\ S^{n-1} & \xrightarrow{f} & X \end{array}$$

commutes.

Theorem 1.2.8 (Whitehead). *If $\psi : X \rightarrow Y$ is a map of connected cell complexes, then f is a homotopy equivalence if and only if $\psi_* : \pi_n(X, x) \rightarrow \pi_n(Y, y)$ is an isomorphism for each $n \in \mathbb{N}$.*

A map $f : X \rightarrow Y$ of path connected spaces is a *weak homotopy equivalence* if it induces an isomorphism $\pi_n(X) \rightarrow \pi_n(Y)$ for each $n \in \mathbb{N}$. Since any cell complex is locally path connected, it is connected if and only if it is path connected. Hence Theorem 1.2.8 says that ψ is a homotopy equivalence if and only if it is a weak homotopy equivalence.

1.3 Lecture 3

Definition 1.3.1. If $x \in A \subset X$, then the n -th relative homotopy group $\pi_n(X, A, x)$ consists of all homotopy classes of maps $(D^n, S^{n-1}, x_0) \rightarrow (X, A, x)$.

We see that

$$M((D^n, S^{n-1}, x), (X, A, x_0)) \cong M((I^n, I^{n-1} \times \{1\}, \underbrace{\partial I^n \setminus \text{Int}(I^{n-1} \times \{1\})}_{\partial_0 I^n}), (X, A, x_0))$$

by considering the homeomorphism $(I^n / \partial_0 I^n, \partial I^n / \partial_0 I^n) \cong (D^n, S^{n-1})$. Therefore, $\pi_n(X, A, x)$ can be viewed as consisting of all homotopy classes of maps $(I^n, \partial I^n, \partial_0 I^n) \rightarrow (X, A, x)$.

Definition 1.3.2. In order to interpret an exact sequence involving objects in the category of pointed sets, we define the *kernel of a function* $f : (X, x) \rightarrow (Y, y)$ of pointed sets as $\ker f \equiv f^{-1}(y)$.

Proposition 1.3.3.

1. If $n \geq 2$, then $\pi_n(X, A, x)$ is, in fact, a group.
2. If $n \geq 3$, then $\pi_n(X, A, x)$ is abelian.
3. We have a long exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n(A, x) & \longrightarrow & \pi_n(X, x) & \longrightarrow & \pi_n(X, A, x) \xrightarrow{\partial} \pi_{n-1}(A, x) \\ & & & & & & \swarrow \\ & & \pi_{n-1}(X, x) & \longleftarrow & \cdots & \longrightarrow & \pi_0(A, x) \longrightarrow \pi_0(X, x) \longrightarrow 0 \end{array}$$

with $\partial[f] = [f \downarrow_{I^{n-1}}]$.

Theorem 1.3.4 (Hurewicz). Let $n \in \mathbb{Z}_{\geq 2}$. If $\pi_i(X) = 0$ for each $i < n$, then $\pi_n(X) \cong H_n(X)$.

Note 1.3.5. This result can't be improved in general. For example, $\pi_3(S^2) \cong \mathbb{Z}$, whereas $H_3(S^2) = 0$.

Let $A \subset X$ be a subcomplex. Recall that $H_i(X, A) \cong H_i(X/A, *)$ for each $i \geq 1$. But it is *not* the case that $\pi_i(X, A) \cong \pi_i(X/A, *)$, for otherwise $\pi_i(S^n) \cong \pi_i(D^n, S^{n-1}) \cong \pi_i(S^{n-1})$, which is known to be false exactly when $i > 2n - 2$.

Example 1.3.6. $\pi_4(S^3) \cong \mathbb{Z}_2 \not\cong \pi_4(S^4)$.

Finally, let's review the notion of a fibration of spaces.

Consider any map $f : Z \rightarrow B$. Recall that if $p : E \rightarrow B$ is a covering projection, then TFAE.

1. There exists a unique $\hat{f} : Z \rightarrow E$ such that $p \circ \hat{f} = f$.
2. $f_*(\pi_1(Z)) \subset p_*(\pi_1(E))$.

The existence of \hat{f} follows from the fact that any covering space satisfies the homotopy lifting property.

Definition 1.3.7 (Fibration). Suppose that $p : E \rightarrow B$ is any map. We say that p is a (*Serre*) *fibration* if it satisfies the homotopy lifting property, i.e., given a commutative square

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}_0} & E \\ \downarrow & & \downarrow p \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

where X is a cell complex, there is some G such that

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}_0} & E \\ \downarrow & \nearrow G & \downarrow p \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

commutes.

Theorem 1.3.8. *If $p : E \rightarrow B$ is a fibration with $e \in F := p^{-1}(b)$, then*

$$p_* : \pi_i(E, F, e) \xrightarrow{\cong} \pi_i(B, b, b) = \pi_i(B, b).$$

Proof. Let $f : (I^n, \partial I^n) \rightarrow (B, b)$. To prove that p_* is surjective, it suffices to find some $G : (I^n, \partial I^n) \rightarrow (E, F)$ such that

$$\begin{array}{ccccc} \partial_0 I^n & \longrightarrow & \{e\} & \hookrightarrow & F \hookrightarrow E \\ \downarrow & & \nearrow G & & \downarrow p \\ I^{n-1} \times [0, 1] & \xrightarrow{\quad f \quad} & & & B \end{array}$$

commutes, for in this case $[p \circ G'] = [f]$. Since p is a fibration, there is some G such that

$$\begin{array}{ccccc} I^{n-1} \times \{0\} & \longrightarrow & \{e\} & \hookrightarrow & F \hookrightarrow E \\ \downarrow & & \nearrow G' & & \downarrow p \\ I^{n-1} \times [0, 1] & \xrightarrow{\quad f \quad} & & & B \end{array}$$

commutes. But $(I^n, \partial_0 I^n) \cong (I^n, I^{n-1} \times \{0\})$, and thus such a G' is enough. \square

Corollary 1.3.9. *We have a long exact sequence*

$$\cdots \longrightarrow \pi_i(F, e) \longrightarrow \pi_i(E, e) \longrightarrow \pi_i(B, b) \xrightarrow{\partial} \pi_{i-1}(F, e) \longrightarrow \cdots$$

Example 1.3.10.

1. Suppose that

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}} & B \times F \\ \downarrow & & \downarrow \pi_B \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

commutes. Then $\hat{f}(x, 0) = (\hat{f}_1(x, 0), \hat{f}_2(x, 0))$ where $\hat{f}_1(x, 0) = f(x, 0)$. Let $G(X, t) = (f(x, t), \hat{f}_2(x, 0))$. Then

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}_0} & B \times F \\ \downarrow & \nearrow G & \downarrow \pi_B \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

commutes, so that π_B is a fibration. (Moreover, $\pi_n(B \times F) \cong \pi_n(B) \times \pi_n(F)$.)

2. Let $A \subset X$ be a subcomplex. The map $\varphi : M(X, Y) \rightarrow M(A, Y)$ defined by $f \mapsto f|_A$ is a fibration.

3. Define the *Hopf fibration* as the quotient map

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1\} \twoheadrightarrow S^3 / \sim_x = \mathbb{CP}^1 = S^2.$$

Corollary 1.3.11. $\pi_3(S^3) \cong \pi_3(S^2)$.

Proof. Since we have an exact sequence

$$\pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) ,$$

it suffices to show that both $\pi_3(S^1)$ and $\pi_2(S^1)$ are trivial. To this end, note that since $\pi_1(S^k) = 0$ for every $k > 1$, we can always find, for any $f : S^k \rightarrow S^1$, a map \hat{f} such that

$$\begin{array}{ccc} & & \mathbb{R} \\ & \nearrow \hat{f} & \downarrow e^{2\pi i x} \\ S^k & \xrightarrow{f} & S^1 \end{array}$$

commutes. Thus, f is homotopic to the constant map. Since f was arbitrary, our proof is complete. \square

Definition 1.3.12. A map $p : E \rightarrow B$ is *locally trivial* if for any $b \in B$, there exist a neighborhood $U \ni b$ in B , a space F , and a homeomorphism $\varphi : p^{-1}(U) \xrightarrow{\cong} U \times F$ such that $\pi_U \circ \varphi = p \upharpoonright_{p^{-1}(U)}$.

Theorem 1.3.13. *Any locally trivial map $p : E \rightarrow B$ is a fibration whenever B is a cell complex.*

Exercise 1.3.14. *Prove that the Hopf fibration is locally trivial.*

Proof. For each $k \in \{0, 1\}$, let $U_k = \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_k \neq 0\}$. Then U_0 and U_1 form an open cover of \mathbb{CP}^1 . Note that the preimage of U_k under the Hopf fibration q is precisely $\{(z_0, z_1) \in S^3 \mid z_k \neq 0\}$. Define $f : q^{-1}(U_k) \rightarrow U_k \times S^1$ by

$$(z_0, z_1) \mapsto \left([z_0, z_1], \frac{z_k}{|z_k|} \right).$$

This is clearly continuous. Further, define the map $g : U_k \times S^1 \rightarrow q^{-1}(U_k)$ by

$$([z_0, z_1], e^{i\theta}) \mapsto \frac{e^{i\theta} |z_k|}{z_k |z_0, z_1|} (z_0, z_1).$$

Since U_k is a saturated open set, we have that the restriction of q to $q^{-1}(U_k)$ is a quotient map. But $g \circ q \upharpoonright_{q^{-1}(U_k)}$ is continuous, so that g is also continuous by the characteristic property of quotient maps. Finally, it is easy to verify that g and f are inverses of each other and that $\pi_{U_k} \circ f = p \upharpoonright_{q^{-1}(U_k)}$. \square

1.4 Lecture 4

Theorem 1.4.1. *Let $A \subset X$ be a subcomplex. Define $r : M(X, Y) \rightarrow M(A, Y)$ by $r(f) = f \upharpoonright_A$. Then r is a fibration.*

Proof. We must fill any diagram of the form

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{\hat{f}} & M(X, Y) \\ \downarrow & \nearrow F & \downarrow r \\ Z \times [0, 1] & \xrightarrow{f} & M(A, Y) \end{array} .$$

It suffices to find a map \bar{F} such that

$$\begin{array}{ccc}
 Z \times \{0\} \times X & \xrightarrow{\hat{f}} & Y \\
 \downarrow & \nearrow \bar{F} & \parallel \\
 Z \times [0, 1] \times X & & Y \\
 \uparrow & \nearrow \bar{f} & \\
 Z \times [0, 1] \times A & &
 \end{array}$$

commutes for, in this case, we can set $F(z, t)(x) = \bar{F}(z, t, x)$.

Note 1.4.2. Suppose that such an \bar{F} exists. Define $g : Z \times X \rightarrow Y$ by $g(z, x) = \hat{f}(z, 0, x)$. Define $h : Z \times X \times [0, 1] \rightarrow Y$ by $H(z, x, t) = \bar{F}(z, t, x)$. Then

$$\begin{array}{ccc}
 Z \times X \times \{0\} & & \\
 \downarrow & \searrow g & \\
 Z \times X \times [0, 1] & \xrightarrow{H} & Y \\
 \uparrow & \nearrow K & \\
 Z \times A \times [0, 1] & &
 \end{array}$$

commutes where $K(z, a, t) = \bar{f}(z, t, a)$. In the case where $Z = \mathbf{pt}$, this means that if $K : A \times [0, 1] \rightarrow Y$ is a homotopy from a map $f : A \rightarrow Y$ and g extends f to X , then there exists a homotopy $H : X \times [0, 1] \rightarrow Y$ such that $H \upharpoonright_{A \times [0, 1]} = K$. In other words, the extension problem for cell complexes is a homotopy problem.

Let's return to proving our theorem. By induction, it suffices to consider just the case where $X = A \cup_{\varphi} D^n$, with characteristic map $\chi : D^n \rightarrow X$. Thus, it suffices to find a map w such that

$$\begin{array}{ccc}
 Z \times D^n \times \{0\} & & \\
 \downarrow & \searrow \text{id}_Z \times (g \circ \chi) & \\
 Z \times D^n \times [0, 1] & \xrightarrow{w} & Y \\
 \uparrow & \nearrow K & \\
 Z \times S^{n-1} \times [0, 1] & \xrightarrow{\text{id}_Z \times \varphi \times \text{id}_{[0, 1]}} & Z \times A \times [0, 1]
 \end{array}$$

commutes for, in this case, we can set $H(z, x, t) = g \cup_{\varphi} w$, thereby making

$$\begin{array}{ccc}
 Z \times D^n \times \{0\} & & \\
 \downarrow & \searrow \text{id}_Z \times (g \circ \chi) & \\
 Z \times D^n \times [0, 1] & \xrightarrow{\text{id}_Z \times \chi \times \text{id}_{[0, 1]}} & Z \times X \times [0, 1] \xrightarrow{H} Y \\
 \uparrow & \nearrow w & \nearrow K \\
 Z \times S^{n-1} \times [0, 1] & \xrightarrow{\text{id}_Z \times \varphi \times \text{id}_{[0, 1]}} & Z \times A \times [0, 1]
 \end{array}$$

commute. To this end, define the retraction $u : D^n \times [0, 1] \rightarrow D^n \times \{0\} \cup S^{n-1} \times [0, 1]$ by picking a point $*$ directly above the cylinder $D^n \times [0, 1]$ and then sending any point x in the cylinder to the unique point

where $D^n \times \{0\} \cup S^{n-1} \times [0, 1]$ intersects the line containing $*$ and x . Now, define w so that

$$\begin{array}{ccc} Z \times (D^n \times [0, 1]) & \xrightarrow{w} & Y \\ \text{id}_Z \times u \downarrow & \nearrow \text{id}_Z \times (g \circ \chi \cup K \circ (\varphi \times \text{id}_{[0, 1]})) & \\ Z \times (D^n \times \{0\} \cup S^{n-1} \times [0, 1]) & & \end{array}$$

commutes. □

Exercise 1.4.3. Let $x \in X$. Consider the loop space $\Omega(X, x) \equiv M((S^1, \text{pt}), (X, x))$. Prove that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$.

Proof. Consider the path space $PX \equiv \{\gamma : [0, 1] \rightarrow X \mid \gamma(0) = x\}$ of (X, x) , equipped with the compact-open topology. We claim that PX is contractible. Indeed, define $K : PX \times [0, 1] \rightarrow PX$ by

$$(\gamma, t) \mapsto (s \mapsto \gamma(s(1-t))).$$

Then K is a homotopy from id_{PX} to the constant map at the constant path at x .

Define the map $p : PX \rightarrow X$ by $\gamma \mapsto \gamma(1)$. Then $p^{-1}(x) = \Omega(X)$. By Corollary 1.3.9, it suffices to show that p is a fibration. To this end, suppose that the square

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\hat{f}} & PX \\ \downarrow & & \downarrow p \\ Y \times [0, 1] & \xrightarrow{f} & X \end{array}$$

commutes. Define $H : Y \times [0, 1] \rightarrow PX$ by $(y, t) \mapsto H(y, t)$ where

$$H(y, t)(s) = \begin{cases} \hat{f}(y)((1+t)s) & 0 \leq s \leq \frac{1}{1+t} \\ f(y, (1+t)s - 1) & \frac{1}{1+t} \leq s \leq 1 \end{cases}.$$

We see that H is continuous when viewed as a function of (y, t, s) and thus is continuous. It is easy to check that

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{\hat{f}} & PX \\ \downarrow & \nearrow H & \downarrow p \\ Y \times [0, 1] & \xrightarrow{f} & X \end{array}$$

commutes, as desired. □

Let $p : E \rightarrow B$ be a map. Recall that the pullback of p along $f : X \rightarrow B$ is given explicitly as

$$f^*E \equiv \{(x, e) \in X \times E \mid f(x) = p(e)\}.$$

Let f^*p denote the map $\pi_X \upharpoonright_{f^*E}$.

Proposition 1.4.4. If p is a fibration, then so is f^*p .

Lemma 1.4.5. If p is locally trivial, then so is f^*p .

Proof. Let $a \in X$. Since p is locally trivial by assumption, we can find a neighborhood U of $f(a)$ in B and a homeomorphism $\varphi : p^{-1}(U) \rightarrow U \times F$. Observe that

$$(f^*p)^{-1}(f^{-1}(U)) = \{(x, e) \mid f(x) = p(e), f(x) \in U\} \subset f^{-1}(U) \times p^{-1}(U).$$

Further, we have a map $\psi : f^{-1}(U) \rightarrow p^{-1}(U) \rightarrow f^{-1}(U) \times F$ given by $(x, e) \mapsto (x, \pi_F(\varphi(e)))$. Define $\lambda : f^{-1}(U) \times F \rightarrow (f^*p)^{-1}(f^{-1}(U))$ by $(x, y) \mapsto (x, \varphi^{-1}(f(x), y))$. Using the fact that

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ & \searrow p & \downarrow \pi_U \\ & & U \end{array}$$

commutes, it is easy to check that ψ and λ are inverses of each other. \square

1.5 Lecture 5

Theorem 1.5.1. *Let B be a cell complex and let $p : E \rightarrow B$ be locally trivial. Then p is a fibration.*

Proof. It suffices to prove the following claim:

If $h : Z \rightarrow X \times [0, 1]$ is locally trivial, $X = \bigcup_{i=0}^n X^i$ is a cell complex, and $\sigma_0 : X \times \{0\} \rightarrow Z$ satisfies $h \circ \sigma_0 = \text{id}_{X \times \{0\}}$, then there is some map $\sigma : X \times [0, 1] \rightarrow Z$ such that $\sigma_{X \times \{0\}} = \sigma_0$ and $h \circ \sigma = \text{id}_{X \times [0, 1]}$.

For, in this case, Lemma 1.4.5 implies that given any commutative square

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{\hat{f}} & E \\ \downarrow & & \downarrow p \\ X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

we can find some σ such that

$$\begin{array}{ccccc} & & f^*E & \longrightarrow & E \\ & \nearrow \sigma_0 & \downarrow \sigma & \nearrow & \downarrow p \\ X \times \{0\} & \hookrightarrow & X \times [0, 1] & \xrightarrow{f} & B \end{array}$$

commutes where $\sigma_0(x, 0) = (x, 0, \hat{f}(x, 0))$.

For induction, let us assume that our claim is true for each X^0, X^1, \dots, X^{n-1} . We may assume, wlog, that $X = D^n$. It suffices to find a map $\tau : S^{n-1} \times [0, 1] \rightarrow Z$ such that $h \circ \tau = \text{id}_{S^{n-1} \times [0, 1]}$ and

$$\begin{array}{ccccc} & & Z & & \\ & \nearrow \sigma_0 & \downarrow h & \nwarrow \tau & \\ D^n \times \{0\} & \hookrightarrow & D^n \times [0, 1] & \hookleftarrow & S^{n-1} \times [0, 1] \\ & \nwarrow & & \nearrow & \\ & & S^{n-1} \times \{0\} & & \end{array}$$

commutes as there is a retraction

$$r : D^n \times [0, 1] \rightarrow D^n \times \{0\} \cup S^{n-1} \times [0, 1].$$

To this end, fix a positive integer m . For each $j \in \{0, 1, \dots, m\}$, let $a_j = \frac{j}{m}$ and let $I_j = [a_j, a_{j+1}]$. Since $D^n \times [0, 1]$ is compact, by making m large enough, we can ensure that $h \upharpoonright_{h^{-1}(I_{j_1} \times \dots \times I_{j_{n+1}})}$ is trivial.

Claim. $h \upharpoonright_{h^{-1}(I_{j_1} \times \dots \times I_{j_n} \times [0, 1])}$ is also trivial.

Proof. We may assume, wlog, that h is trivial on $h^{-1}(I_{j_1} \times \dots \times I_{j_n} \times I_j)$ for any j . Let $k \in \{0, 1, \dots, m-1\}$ and assume, for induction, that h is trivial on $h^{-1}(I_{j_1} \times \dots \times I_{j_n} \times [0, a_k])$. Let $K = I_{j_1} \times \dots \times I_{j_n}$ and $J = K \times [0, a_k]$ and $\tilde{J} = K \times [a_k, a_{k+1}]$. By assumption, there exists a homeomorphism $\varphi : h^{-1}(J) \xrightarrow{\cong} J \times F$ such that $\pi_J \circ \varphi = h$. Likewise, there exists a homeomorphism $\psi : h^{-1}(\tilde{J}) \xrightarrow{\cong} \tilde{J} \times F$ such that $\pi_{\tilde{J}} \circ \psi = h$. If $\psi = \varphi$ on $K \times \{a_k\}$, then $h^{-1}(J \cup \tilde{J}) \xrightarrow{\varphi \cup \psi} (J \cup \tilde{J}) \times F$ is a well-defined trivialization, in which case we're done. With this in mind, let

$$w = \left(\varphi \upharpoonright_{K \times \{a_k\}} \circ (\psi \upharpoonright_{K \times \{a_k\}})^{-1} \right) \times \text{id}_{[a_k, a_{k+1}]}.$$

Note that

$$\begin{array}{ccc} h^{-1}(\tilde{J}) & \xrightarrow{\psi} & \tilde{J} \times F \\ h \downarrow & & \downarrow w \times \text{id}_F \\ \tilde{J} & \xleftarrow{\pi_{\tilde{J}}} & \tilde{J} \times F \end{array}$$

commutes and that $\gamma := ((w \times \text{id}_F) \circ \psi)$ agrees with φ on $K \times \{a_k\}$. Hence we may take $\varphi \cup \gamma$ as our desired trivialization. \square

As a result, we may assume that h is trivial on its entire domain, i.e., that h is the projection

$$(D^n \times [0, 1]) \times F \twoheadrightarrow D^n \times [0, 1].$$

Moreover, by induction, we can find a right inverse $\sigma : X^{n-1} \times [0, 1] \rightarrow D^n \times [0, 1]$ of $h : Z \rightarrow X^{n-1} \times [0, 1]$ that extends $\sigma_0 \upharpoonright_{X^{n-1} \times \{0\}}$. But X^{n-1} consists of all $(n-1)$ -dimensional faces of the n -cube, and thus we have a map

$$\tau \equiv \sigma \upharpoonright_{\partial I^n \times [0, 1]} : S^{n-1} \times [0, 1] \rightarrow (D^n \times [0, 1]) \times F,$$

which has the form $(x, t) \mapsto (x, t, \tilde{\tau}(x, t))$. Further, σ_0 has the form $(x, 0) \mapsto (x, 0, \tilde{\sigma}_0(x, 0))$. Therefore, our desired map $\sigma : D^n \times [0, 1] \rightarrow (D^n \times [0, 1]) \times F$ is given by

$$\sigma(x, t) = (x, t, (\tilde{\sigma}_0 \cup \tilde{\tau})(r(x, t))).$$

\square

2 Fiber bundles

Definition 2.0.1. A *topological group* G is a group such that both multiplication $G \times G \xrightarrow{\mu} G$ and inversion $G \xrightarrow{\iota} G$ are continuous.

Definition 2.0.2 (Fiber bundle). Let G be a topological group.

1. A *fiber* F of G is a space equipped with a faithful (i.e., injective) group action $\rho : G \rightarrow \text{Homeo}(F) \subset M(F, F)$.

2. An atlas for the structure of a (fiber) bundle with group G and fiber F on a map $p : E \rightarrow B$ consists of

- (a) a family $(U_\alpha, h_\alpha)_{\alpha \in A}$ where each U_α is open and each h_α is a homeomorphism $p^{-1}(U_\alpha) \rightarrow U_\alpha \times F$ and
- (b) a family of continuous *transition functions* $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow G\}_{\alpha, \beta \in A}$

such that

- i $B = \bigcup_{\alpha \in A} U_\alpha$,
- ii $\pi_{U_\alpha} \circ h_\alpha = p|_{p^{-1}(U_\alpha)}$, and
- iii $x \in U_\alpha \cap U_\beta \implies h_\beta \circ h_\alpha^{-1}(x, f) = (x, h_{\beta\alpha}(x) \cdot f)$

3. Two atlases are *compatible* if their union is an atlas.

4. A *bundle structure on B* is a maximal atlas on p .

Terminology. If B is equipped with a bundle structure, then we say that p is a (fiber) bundle.

Example 2.0.3.

1. The tangent bundle $\pi : TM \rightarrow M$ of a smooth n -manifold M is a bundle with group $\text{GL}(n, \mathbb{R})$.

Proof. Let (U, φ) be any coordinate chart for M with coordinate functions (x^i) . Define $h : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ by

$$v^i \frac{\partial}{\partial x^i} (p) \mapsto (p, (v^1, \dots, v^n)).$$

It is clear that $\pi_U(h(p)) = \pi(c)$ for any $c \in \pi^{-1}(U)$. To see that h is a homeomorphism, note that the composite $(\varphi \times \text{id}_{\mathbb{R}^n}) \circ h : \pi^{-1}(U) \rightarrow \varphi(U) \times \mathbb{R}^n$ is given by

$$v^i \frac{\partial}{\partial x^i} (p) \mapsto (x^1(p), \dots, x^n(p), v^1, \dots, v^n),$$

the inverse of which is given by $(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto v^i \frac{\partial}{\partial x^i} (\varphi^{-1}(x))$. Therefore, $(\varphi \times \text{id}_{\mathbb{R}^n}) \circ h$ is given locally by

$$(x^1, \dots, x^n, v^1, \dots, v^n) \mapsto \left(\tilde{x}^1(x), \dots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^j}(x) v^j, \dots, \frac{\partial \tilde{x}^n}{\partial x^j}(x) v^j \right),$$

which is smooth. Thus, h is a diffeomorphism as the composite of two diffeomorphisms. In particular, h is a homeomorphism.

It remains to describe the transition functions $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow \text{GL}(n, \mathbb{R})\}$ for TM . Note that

$$\begin{array}{ccccc} U_{\alpha\beta} \times \mathbb{R}^n & \xleftarrow{h_\alpha} & \pi^{-1}(U_{\alpha\beta}) & \xrightarrow{h_\beta} & U_{\beta\alpha} \times \mathbb{R}^n \\ & \searrow \pi_1 & \downarrow \pi & \swarrow \pi_1 & \\ & & U_{\alpha\beta} & & \end{array}$$

commutes. In particular, $\pi_1 \circ h_\beta \circ h_\alpha^{-1} = \pi_1$, which implies that $h_\beta \circ h_\alpha^{-1}(u, v) = (u, f(u, v))$ for some smooth map $f : U_{\alpha\beta} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. This must be a linear isomorphism when restricted to $\{u\} \times \mathbb{R}^n$ for any $u \in U_{\alpha\beta}$, which is uniquely determined by an element $h_{\beta\alpha}(u)$ of $\text{GL}(n, \mathbb{R})$ (provided that we have fixed a basis of \mathbb{R}^n). Hence

$$h_\beta \circ h_\alpha^{-1}(u, v) = (u, h_{\beta\alpha}(u)v).$$

Since the map $h_{\beta\alpha} : U_{\alpha\beta} \rightarrow \text{GL}(n, \mathbb{R})$ is continuous, our proof is complete. \square

2. Let $p : E \rightarrow B$ be any bundle with group $\{e\}$. Then p is the trivial bundle, i.e., is isomorphic to the projection map.

Proof. We have that $h_\beta = h_\alpha$ on $p^{-1}(U_\alpha \cap U_\beta) = p^{-1}(U_\alpha) \cap p^{-1}(U_\beta)$, so that $h \equiv \bigcup_{\alpha \in A} h_\alpha$ is a well-defined homeomorphism $E \cong B \times F$. \square

2.1 Lecture 6

Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure with group G and fiber F on $p : E \rightarrow B$. Let $U = U_\alpha \cap U_\beta \cap U_\gamma$. Consider the commutative diagram

$$\begin{array}{ccccccc} & & & p^{-1}(U) & & & \\ & \nearrow h_\alpha^{-1} & & & \nwarrow h_\gamma & & \\ U \times F & \xrightarrow{h_\alpha^{-1}} & p^{-1}(U) & \xrightarrow{h_\beta} & U \times F & \xrightarrow{h_\beta^{-1}} & p^{-1}(U) & \xrightarrow{h_\gamma} & U \times F \end{array}.$$

The bottom row is given by $(u, f) \mapsto (u, h_{\beta\alpha}(u) \cdot f) \mapsto (u, h_{\gamma\beta}(u) \cdot (h_{\beta\alpha}(u) \cdot f)) = (u, (h_{\gamma\beta}(u)h_{\beta\alpha}(u)) \cdot f)$, and the top composite is given by $(u, f) \mapsto (u, h_{\gamma\alpha}(u) \cdot f)$. It follows that

$$h_{\gamma\beta}(u)h_{\beta\alpha}(u) = h_{\gamma\alpha}(u)$$

for each $u \in U$. This property is known as the *cocycle condition*.

Theorem 2.1.1. *Let G be a topological group acting on a space F . Suppose that $\{U_\alpha\}$ is an open cover of B and $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow G\}$ is a family of continuous functions satisfying the cocycle condition. Then there exists a bundle $p : E \rightarrow B$ with group G , fiber F , and transition functions $h_{\beta\alpha}$.*

Proof sketch. Let $E = \coprod_\alpha U_\alpha \times F / \sim$ where $(u, f)_\alpha \sim (u, h_{\beta\alpha}(u) \cdot f)_\beta$. Define $p : E \rightarrow B$ by $(u, f) \mapsto u$. \square

Definition 2.1.2 (Bundle map). A morphism of bundles p_1 and p_2 with group G and fiber F is a commutative square of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{g}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{g} & B_2 \end{array}.$$

Suppose that (\hat{g}, g) is a bundle map $p_1 \rightarrow p_2$. Let $\{(U_\alpha, h_\alpha)\}$ and $\{(V_\beta, k_\beta)\}$ be bundle structures on B_2 and B_1 , respectively. We have a commutative diagram

$$\begin{array}{ccccc} & & d_{\alpha\beta} & & \\ & \nearrow & & \searrow & \\ (g^{-1}(U_\alpha) \cap V_\beta) \times F & \xrightarrow{k_\beta^{-1}} & p_1^{-1}(g^{-1}(U_\alpha) \cap V_\beta) & \xrightarrow{\hat{g}} & p_2^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times F \\ & \searrow \pi_1 & \downarrow & & \downarrow & \swarrow \pi_1 & \\ & & g^{-1}(U_\alpha) \cap V_\beta & \xrightarrow{g} & U_\alpha & & \end{array},$$

so that $d_{\alpha\beta}(x, f) = (g(x), \lambda_{\alpha\beta}(x) \cdot f)$ for some continuous map $\lambda_{\alpha\beta} : g^{-1}(U_\alpha) \cap V_\beta \rightarrow G$. Letting $W = g^{-1}(U_\alpha \cap U_{\alpha'}) \cap (V_\beta \cap V_{\beta'})$, we have that

$$h_{\alpha'\alpha}(w)\lambda_{\alpha\beta}(w)k_{\beta\beta'}(w) = \lambda_{\alpha'\beta'}(w) \quad (\dagger)$$

for every $w \in W$.

Exercise 2.1.3 (Pullback bundle). Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure on $p : E \rightarrow B$ with group G and consider the pullback diagram

$$\begin{array}{ccc} g^*E & \longrightarrow & E \\ g^*p \downarrow & & \downarrow p \\ X & \xrightarrow{g} & B \end{array}$$

Define $h'_{\beta\alpha} : g^{-1}(U_\alpha) \cap g^{-1}(U_\beta) \rightarrow G$ as the composite $h_{\beta\alpha} \circ g$ restricted to $g^{-1}(U_\alpha \cap U_\beta)$. Show that the family $\{h'_{\beta\alpha}\}$ induces a bundle structure on g^*p .

Theorem 2.1.4. Every bundle map

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{g}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

factors as

$$\begin{array}{ccccc} E_1 & \xrightarrow{\tau} & g^*E_2 & \xrightarrow{\bar{g}} & E_2 \\ p_1 \downarrow & & \downarrow g^*p_2 & & \downarrow p_2 \\ B_1 & \xrightarrow{\text{id}_{B_1}} & B_1 & \xrightarrow{g} & B_2 \end{array}$$

where $\tau(e) = (p_1(e), \hat{g}(e))$ for any $e \in E_1$.

2.2 Lecture 7

Note 2.2.1. If $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow G\}$ is a family of transition functions, then

$$h_{\alpha\beta}(x) = (h_{\beta\alpha}(x))^{-1}$$

for any $x \in U_\alpha \cap U_\beta$. In particular, $h_{\alpha\alpha}(x) = (h_{\alpha\alpha}(x))^{-1}$.

Theorem 2.2.2. Any bundle map of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{g}} & E_2 \\ & \searrow p_1 & \downarrow p_2 \\ & & B \end{array}$$

is an isomorphism.

Proof. Note that

$$\begin{array}{ccccc} & & p_2^{-1}(U_\alpha \cap U_\beta) & & \\ & \swarrow h_\beta & \uparrow \hat{g} & \searrow h_\alpha & \\ (U_\alpha \cap U_\beta) \times F & \xleftarrow{k_\beta} & p_1^{-1}(U_\alpha \cap U_\beta) & \xleftarrow{k_\alpha^{-1}} & (U_\alpha \cap U_\beta) \times F \\ & \searrow & \downarrow & \swarrow & \\ & & U_\alpha \cap U_\beta & & \end{array}$$

commutes. We have that $h_\beta \circ \hat{g} \circ k_\alpha^{-1}(x, f) = (x, \lambda_{\beta\alpha}(x) \cdot f)$. Thus, if $h_\alpha(e) = (x, f)$, then $h_\alpha(\hat{g}(e)) = (x, \lambda_{\alpha\alpha}(x) \cdot d)$. Let

$$(\hat{g})^{-1}(e) = k_\alpha^{-1}(x, \lambda_{\alpha\alpha}(x)^{-1} \cdot f)$$

where $(x, f) = h_\alpha(e)$. If this is well-defined on E_2 (??), then it indeed equals the inverse of \hat{g} . Moreover, by Note 2.2.1, it is easy to check that $d_{\alpha'\beta'}(x)^{-1}$ satisfies (†), and thus it can be shown that $(\hat{g})^{-1}$ is a bundle map. □

Corollary 2.2.3. *Every bundle $E \rightarrow X$ is isomorphic to the pullback of E by id_X .*

Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure with group G and fiber G on $p : E \rightarrow X$. In particular,

$$\begin{array}{ccc} U_\alpha \times G & \xleftarrow{h_\alpha} & p^{-1}(U_\alpha) \\ \pi_1 \downarrow & \swarrow p & \\ U_\alpha & & \end{array}$$

commutes. Define the free action $E \times G \rightarrow E$ by

$$e \cdot g = h_\alpha^{-1}(h_\alpha(e) \cdot g).$$

where $p(e) \in U_\alpha$ and $(u, h) \cdot g \equiv (u, hg)$. This is well-defined because it does not depend on our choice of α . Indeed, suppose that $p(e)$ also belongs to U_β . We have that $h_\alpha(e) = (p(e), h)$ and $h_\beta(e) = (p(e), h')$ for some $h, h' \in G$. Then $e \cdot g = h_\alpha^{-1}(p(e), hg)$, and we must show that this equals $h_\beta^{-1}(p(e), h'g)$. Note that $h_\beta(e \cdot g) = (p(e), h_{\beta\alpha}(p(e))hg)$. But

$$(p(e), h_{\beta\alpha}(p(e))h) = h_\beta(h_\alpha^{-1}(p(e), h)) = (p(e), h'),$$

so that $h_{\beta\alpha}(p(e))h = h'$, and thus $h_\beta(e \cdot g) = (p(e), h'g)$, as desired.

Note 2.2.4. $E/G \cong \{p^{-1}(x) \mid x \in X\} \cong X$.

Definition 2.2.5 (Balanced product). Let F be a space. The *balanced product* $E \times_G F$ of E and F is the quotient space $E \times F / \sim$ where

$$(e, f) \sim (eg, g^{-1}f)$$

for any $e \in E$ and $f \in F$.

By the universal property of the quotient space, there is a unique map \bar{p} such that

$$\begin{array}{ccc} E \times F & \twoheadrightarrow & E \times_G F \\ p \circ \pi_E \downarrow & \swarrow \bar{p} & \\ X & & \end{array} \quad (\star)$$

Notation. Let $\mathcal{B}(X, G, \rho, F)$ denote the set of all isomorphism classes of bundles over X with group G and fiber F .

Lemma 2.2.6. \bar{p} is a bundle with group G and fiber F .

Proof. As $(g, f) \sim (e_G, gf)$, we see that $(U \times G) \times_G F \cong U \times F$. Thus, we can endow \bar{p} with local trivializations and transition functions that are exactly similar to those for p . □

Proposition 2.2.7. *The function $p \mapsto \bar{p}$ defines a set isomorphism $\mathcal{B}(X, G, \rho, G) \xrightarrow{\cong} \mathcal{B}(X, G, \rho, F)$.*

Let $p_1 : E \rightarrow B_1$ and $p_2 : E \rightarrow B_2$ be bundles. Let $e_1 \in E_1$, $e_2 \in E_2$, and $b_1 \in B_1$.

Question. Can we find a bundle map

$$\begin{array}{ccc} E_1 & \dashrightarrow & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \dashrightarrow & B_2 \end{array}$$

such that $e_1 \mapsto e_2$ and $e_1 \mapsto b_1$?

Define the action $G \times E_2 \rightarrow E_2$ by $g * e_2 = e_2 \cdot g^{-1}$. From this, we obtain a bundle

$$\psi : \underbrace{E_1 \times_G E_2}_{(E_1 \times E_2)/G} \rightarrow E_1 \times_G \text{pt} \cong B_1$$

with fiber E_2 .

Lemma 2.2.8. *There is a one-to-one correspondence between bundle maps $p_1 \rightarrow p_2$ and sections of ψ .*

Proof. Suppose that σ is a section of ψ . As G acts freely on $E_1 \times E_2$, we see that for any $e \in E_1$, there exists a unique \tilde{e} such that $\sigma(p(e)) = [(e, \tilde{e})]$. Define $\hat{g} : E_1 \rightarrow E_2$ by $e \mapsto \tilde{e}$. This respects the action of G and thus must be a bundle map. \square

Now, let $A \subset B_1$ and suppose that

$$\begin{array}{ccc} p_1^{-1}(A) & \longrightarrow & E_2 \\ \downarrow & \alpha & \downarrow p_2 \\ A & \longrightarrow & B_2 \end{array}$$

is a bundle map. Then α extends when ???. Also, the corresponding section

$$\sigma : A \rightarrow p^{-1}(A) \times_G E_2 \subset E_1 \times_G E_2$$

extends.

Definition 2.2.9 (Principal bundle). Let G be a topological group. A *principal G -bundle* is a fiber bundle with group G and fiber G with G acting on itself by left translation.

Theorem 2.2.10. *Let f and g be homotopic maps $X \rightarrow Y$. Let $p : E \rightarrow Y$ be any bundle with group G and fiber F . Then $f^*p \cong g^*p$.*

2.3 Lecture 8

Before proving this, we wish to determine when, given any two bundles $p_1 : E_1 \rightarrow B_1$ and $p_2 : E_2 \rightarrow B_2$ and any map $g : B_1 \rightarrow B_2$, we can find a map \hat{g} such that

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{g}} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ B_1 & \xrightarrow{g} & B_2 \end{array}$$

commutes.

Define the *diagonal action* ΔG of G on $E_1 \times E_2$ by

$$(e_1, e_2) \cdot h = (e_1 \cdot h, e_2 \cdot h),$$

so that $E_1 \times_G E_2 = E_1 \times E_2 / \Delta G$. By (\star) , we can find a unique map τ such that

$$\begin{array}{ccc} E_1 \times_G E_2 & & \\ \downarrow & \searrow \tau & \\ B_1 & \xleftarrow{\pi_1} & B_1 \times B_2 \end{array}$$

commutes.

Exercise 2.3.1. Show that \hat{g} exists if and only if there is some $\lambda : B_1 \rightarrow E_1 \times_G E_2$ such that $\tau(\lambda(b_1)) = (b_1, g(b_1))$.

Proof.

(\Leftarrow) As G acts freely on $E_1 \times E_2$, we see that $(e, e') \sim (e, e'') \implies e' = e''$ for any $e', e'' \in E_2$. Hence for any $e \in E_1$, there exists a unique $\hat{e} \in E_2$ such that $\lambda(p_1(e)) = [(e, \hat{e})]$. Let $\hat{g}(e) = \hat{e}$. Then \hat{g} is clearly continuous and G -equivariant, and thus (\hat{g}, g) is a bundle map.

(\Rightarrow) Consider the homeomorphism $\varphi : B_1 \xrightarrow{\cong} E_1 / G$ with $\varphi(b) = p_1^{-1}(b)$. Let $b \in B_1$. Let $\varphi(b) = [e]$. Define $\lambda : B_1 \rightarrow E_1 \times_G E_2$ by $\lambda(b) = [(e, \hat{g}(e))]$. Since \hat{g} is G -equivariant, we see that λ is well-defined. Further, λ is continuous as the quotient of the map

$$f : E_1 \rightarrow E_1 \times E_2, \quad f(x) = (x, \hat{g}(x))$$

by G . Finally, it is easy to check that $\tau(\lambda(b_1)) = (b_1, g(b_1))$ for any $b_1 \in B_1$. \square

Lemma 2.3.2. τ is locally trivial, hence a fibration.

Proof. Locally, we have that $E_1 \cong U \times G$ and $E_2 \cong V \times G$, so that $E_1 \times E_2 \cong U \times V \times G \times G$. It follows that, locally, $E_1 \times_G E_2 \cong U_1 \times U_2 \times G \times G / \Delta G$ where $\Delta G \equiv \{(g, g) \mid g \in G\}$. \square

Remark 2.3.3. In fact, τ is a bundle with fiber $G \times G / \Delta G \cong G$.

Proof of Theorem 2.2.10. Due to Proposition 2.2.7, we may assume that p is a principal G -bundle. By assumption, there is some homotopy $H : X \times I \rightarrow Y$ from f to g . Let $\omega = H^*p$. Then

$$\begin{aligned} f^*p &= \omega \upharpoonright_{\omega^{-1}(X \times \{0\})} : \omega^{-1}(X \times \{0\}) \rightarrow X \times \{0\} \cong X \\ g^*p &= \omega \upharpoonright_{\omega^{-1}(X \times \{1\})} : \omega^{-1}(X \times \{1\}) \rightarrow X \times \{1\} \cong X. \end{aligned}$$

Therefore, it suffices to show that $f^*p \times \text{id}_I \cong \omega$ such that the diagram

$$\begin{array}{ccccc} f^*E \times I & \xrightarrow{\cong} & H^*E & \longrightarrow & E \\ f^*p \times \text{id}_I \downarrow & & \downarrow \omega & & \downarrow p \\ X \times I & \xlongequal{\quad} & X \times I & \xrightarrow{H} & Y \end{array}$$

commutes. For, in this case, our isomorphism restricts over $X \times \{1\}$, i.e., $g^*p = \omega \upharpoonright_{X \times \{1\}} \cong f^*p$. It thus suffices to exhibit a bundle map $f^*p \times I \rightarrow \omega$ over $\text{id}_{X \times I}$ that equals the identity over $\omega \upharpoonright_{X \times \{0\}} = f^*p$.

Remark 2.3.4. It is easy to show that there is some bundle map $f^*p \times \text{id}_I \rightarrow \omega$. Indeed, by the homotopy lifting property, we obtain a section σ fitting into the commutative diagram

$$\begin{array}{ccc} & (f^*E \times I) \times_G H^*E & \\ \lambda_0 \nearrow & \downarrow \tilde{\gamma} \sigma & \\ X \times \{0\} & \longrightarrow & X \times I \end{array},$$

in which case we obtain our desired map by Lemma 2.2.8. As mentioned, however, we want a bundle map that equals the identity over f^*p .

To get such a map, we must find a section λ such that

$$\begin{array}{ccccc} & (f^*E \times I) \times_G H^*E & & & \\ \lambda_0 \nearrow & \downarrow \tilde{\gamma} \lambda & \searrow \tau & & \\ X \times \{0\} & \longrightarrow & X \times I & \xrightarrow{\Delta} & (X \times I) \times (X \times I) \end{array}$$

commutes. But λ must exist since τ is a fibration by virtue of Lemma 2.3.2.

□

Corollary 2.3.5. *Any bundle over a contractible space B is trivial.*

Proof. Let $i : \text{pt} \rightarrow B$ and $\pi : B \rightarrow \text{pt}$ denote inclusion and projection, respectively. Then

$$\begin{aligned} p &\cong (\text{id})^* p \\ &\cong (i\pi)^* p \\ &\cong \pi^* \underbrace{i^* p}_{\text{trivial}}, \end{aligned}$$

which is trivial since the pullback of a trivial bundle is trivial.

□

Corollary 2.3.6. *Every bundle p over $X \times I$ is isomorphic to $(p \upharpoonright_{p^{-1}(X \times \{0\})}) \times \text{id}_I$.*

Example 2.3.7. Consider $S^1 \subset \mathbb{R}^2$ with center the origin. Let $p : E \rightarrow S^1$ be a bundle with group G and fiber F . Cover S^1 with the open intervals $I_1 := S^1 \setminus \{-1\}$ and $I_2 := S^1 \setminus \{1\}$. We may assume that $F = p^{-1}(-1)$. Then $E = E_1 \cup E_2$ where $E_i \cong I_i \times F$ via, say, φ_i for each $i = 1, 2$. By Corollary 2.3.6, we see that

$$\varphi_1 \upharpoonright_{\varphi_1^{-1}(\{1\} \times F)} = \varphi_2 \upharpoonright_{\varphi_2^{-1}(\{-1\} \times F)} = \text{id}_F.$$

Moreover, the transition function $\varphi_2^{-1} \circ \varphi_1 \upharpoonright_{p^{-1}(1)} : F \rightarrow F$ is given by multiplication by some $g \in G$. Hence the map $G \rightarrow \mathcal{B}(S^1, G, F)$ is surjective. In fact, it can be shown that this maps descends to an isomorphism

$$\pi_0(G) \cong G/G_0 \xrightarrow{\cong} \mathcal{B}(S^1, G, F)$$

where G_0 denotes the connected component of e_G .

For example, if $G = F = \text{GL}(n, \mathbb{R})$, then $\pi_0(G)$ consists of the set of matrices with positive determinant and the set of matrices with negative determinant, so that $\mathcal{B}(S^1, G, F) \cong \mathbb{Z}_2$.

Example 2.3.8. The set $\mathcal{B}(S^2, G, F)$ is isomorphic to the set of homotopy classes of maps $S^1 \rightarrow G$. As it turns out, we can ignore base points, so that $\mathcal{B}(S^2, G, F) \cong \pi_1(G)$.

For example, if $G = F = \text{SO}(2)$, then $G \cong S^1$, so that $\mathcal{B}(S^2, G, F) \cong \mathbb{Z}$.

2.4 Lecture 9

Theorem 2.4.1. *Let X be a cell complex with $\dim X \leq n$. Let $A \subset X$ be a subcomplex. Let $p : E \rightarrow X$ be a bundle with fiber F such that $\pi_i(F, f) = 0$ for each $i \leq n-1$. Suppose that $\sigma_0 : A \rightarrow E$ satisfies $p \circ \sigma_0(a) = a$ for each $a \in A$. Then σ_0 extends to a section $\sigma : X \rightarrow E$ of p .*

$$\begin{array}{ccc} & E & \\ \sigma_0 \nearrow & \downarrow p & \nwarrow \sigma \\ A & \hookrightarrow & X \end{array}$$

Proof. First, assume that X is a regular complex. Since X is finite, we may assume that $X = A \cup_{S^{k-1}} D^k$ where $k \leq n$. Further, we may assume, wlog, that $X = D^k$. Thus, we must find a section σ such that

$$\begin{array}{ccc} & E & \\ \sigma_0|_{S^{k-1}} \nearrow & \downarrow p & \nwarrow \sigma \\ S^{k-1} & \hookrightarrow & D^k \end{array}$$

commutes. Since D^k is contractible, we have that $E \cong D^k \times F$. Then $\sigma_0(x) = (x, \tilde{\sigma}_0(x))$ for each $x \in S^{k-1}$. But $\tilde{\sigma}_0(x) : S^{k-1} \rightarrow F$ extends to a map $\tilde{\sigma} : D^k \rightarrow F$ because $\pi_{k-1}(F) = 0$. Hence we can take σ to be the map defined by $x \mapsto (x, \tilde{\sigma}(x))$.

Next, drop the assumption that X is regular. Using Exercise 1.1.9, we get a homotopy equivalence

$$\begin{array}{ccc} & h \nearrow & \\ (X, A) & & (\overline{X}, \overline{A}) \\ & g \nwarrow & \\ & \text{regular} & \end{array}$$

of pairs. Define $\overline{A} \rightarrow g^*E$ by $\bar{\sigma}_0(a) = (a, \sigma_0(g(a)))$. By our preceding discussion, this extends to a section $\bar{\sigma}$ on \overline{X} .¹ We wish to find σ such that

$$\begin{array}{ccc} g^*E & \longrightarrow & E \\ \bar{\sigma} \uparrow \left(\begin{array}{c} \downarrow p \\ \downarrow \end{array} \right) \bar{\sigma} & & \uparrow \left(\begin{array}{c} \downarrow p \\ \downarrow \end{array} \right) \sigma \\ \overline{X} & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ \overline{A} & \xrightarrow{g} & A \end{array}$$

commutes. But since $p \cong h^*g^*p$, we have a commutative diagram

$$\begin{array}{ccccc} g^*E & \longleftarrow & h^*g^*E & \xrightarrow{\cong} & E \\ \bar{\sigma} \uparrow \left(\begin{array}{c} \downarrow g^*p \\ \downarrow \end{array} \right) \bar{\sigma} & & h^*g^*p \downarrow & & \downarrow p \\ \overline{X} & \xleftarrow{h} & X & & \end{array},$$

from which we obtain our desired section σ . □

Notation. $[X, Y] := (\text{homotopy classes of maps } X \rightarrow Y)$.

Corollary 2.4.2. *Let $p : E \rightarrow B$ be a principal G -bundle and suppose that $\pi_i(E) = 0$ for any $i \leq n-1$. The function $\chi_X : [X, B] \rightarrow \mathcal{B}(X, G, G)$ given by $f \mapsto f^*p$ is bijective.*

¹As $\dim \overline{X} > \dim X$, we tacitly rely on the fact that $\pi_i(F)$ is trivial for large enough i .

Proof.

Surjective: Let $p_1 : E_1 \rightarrow X$ be a bundle. Due to Theorem 2.1.4, it suffices to find a bundle map (\hat{f}, f) such that

$$\begin{array}{ccc} E_1 & \xrightarrow{\hat{f}} & E \\ p_1 \downarrow & & \downarrow \\ X & \xrightarrow{f} & B \end{array}$$

commutes. Such a map can be found precisely when there exists a section of the bundle $E_1 \times_G E \rightarrow X$, which holds by applying Theorem 2.4.1 to the case where $A = \emptyset$.

Injective: Suppose that $\chi_X(f) = \chi_X(g)$. We must show that $f \simeq g$, i.e., that there is some bundle map (\hat{H}, H) such that

$$\begin{array}{ccccc} & & & & \text{curved arrow} \\ & & & & \nearrow \\ f^*p \times \{0, 1\} & \hookrightarrow & f^*p \times I & \xrightarrow{\hat{H}} & E \\ \nwarrow \cong & & \downarrow & & \downarrow p \\ f^*p \cup g^*p & & X \times I & \xrightarrow{H} & B \\ & \nearrow & & & \\ & X \times \{0, 1\} & & & \end{array}$$

commutes. This is equivalent to finding a section λ such that

$$\begin{array}{ccc} (X \times \{0, 1\}) \times B & \xleftarrow{\tau} & (f^*p \times I) \times_G E \\ \gamma \uparrow & \nearrow \lambda_0 & \downarrow \lambda \\ X \times \{0, 1\} & \hookrightarrow & X \times I \end{array}$$

commutes where

$$\gamma(x, t) = \begin{cases} (x, t, f(x)) & t = 0 \\ (x, t, g(x)) & t = 1 \end{cases}.$$

But this exists by Theorem 2.4.1 because $\pi_i(E) = 0$ by assumption. \square

Definition 2.4.3 (Classifying space). A *classifying space for principal G -bundles* is a space B such that χ_X is bijective for every cell complex X .

Example 2.4.4. Let $G = \{\pm 1\}$. Then any principal G -bundle over X is a two-fold covering space of X , i.e., a subgroup of index two in $\pi_1(X)$, i.e., a nontrivial homomorphism $\pi_1 X \rightarrow G$.

For example, let $\{U_i\}$ denote the usual open covering of $\mathbb{R}P^n = S^n/G$. Let $\pi : S^n \rightarrow \mathbb{R}P^n$ denote the projection map. We have that $\pi^{-1}(U_i) \cong U_i \times G$. Indeed, define $h_i : \pi^{-1}(U_i) \rightarrow U_i \times G$ by

$$(x_0, \dots, x_n) \mapsto \left([x_0, \dots, x_n], \frac{x_i}{|x_i|} \right),$$

the inverse of which is given by

$$\begin{aligned} (y_0, \dots, y_n) &\mapsto ([x_0, \dots, x_n], \epsilon) \\ y_k &\equiv \epsilon x_k \cdot \frac{|x_i|}{x_i}. \end{aligned}$$

Note that any transition function $h_{ji} : U_i \cap U_j \rightarrow G$ is given by $h_{ji}(x) = -1$.

Using the fact that π_1 is the abelianization of H_1 along with the universal coefficient theorem for cohomology, one can prove the following.

Proposition 2.4.5. $\mathcal{B}(X, \mathbb{Z}_2, F) \cong [X, \mathbb{RP}^n] \cong \text{Hom}(\pi_1(X), \mathbb{Z}_2) \cong H^1(X; \mathbb{Z}_2)$.

Let $w_1 \in H^1(\mathbb{RP}^n; \mathbb{Z}_2) \cong \mathbb{Z}_2$ be nonzero. Let $p_1 : E \rightarrow X$ be a \mathbb{Z}_2 -bundle. We call $w_1(p_1) := f^*w_1 \in H^1(X; \mathbb{Z}_2)$ the *first Stiefel-Whitney class* of p .

2.5 Lecture 10

Example 2.5.1. Let $n \in \mathbb{N}$. Recall that \mathbb{CP}^n , by definition, consists of all the complex lines in \mathbb{C}^{n+1} . Let $G = S^1$. Then G acts on \mathbb{C}^{n+1} by $g \cdot (z_0, \dots, z_n) = (gz_0, \dots, gz_n)$. We have that $\mathbb{CP}^n \cong S^{2n+1}/\sim$ where $z \sim \zeta \cdot z$ for any $\zeta \in S^1$. Consider the projection map $\pi : S^{2n+1} \rightarrow \mathbb{CP}^n$. For each $i \in \{0, \dots, n\}$, let $H_i = \{z \in \mathbb{CP}^n \mid z_i = 0\} \cong \mathbb{CP}^{n-1}$ and let $U_i = \mathbb{CP}^n \setminus H_i$. Then the U_i form an open cover of \mathbb{CP}^n . Define $h_i : \pi^{-1}(U_i) \rightarrow U_i \times S^1$ by $(z_0, \dots, z_n) \mapsto ([z_0, \dots, z_n], \frac{z_i}{|z_i|})$.

Exercise 2.5.2.

1. Prove that h_i is a homeomorphism.
2. Find the transition functions $h_{ij} : U_j \cap U_i \rightarrow S^1$.

Proof.

1. It is obvious that h_i is continuous. Define $g_i : U_i \times S^1 \rightarrow \pi^{-1}(U_i)$ by

$$([z_0, \dots, z_n], \epsilon) \mapsto (y_0, \dots, y_n)$$

$$y_k \equiv \epsilon z_k \cdot \frac{|z_i|}{z_i}, \quad k = 0, \dots, n.$$

It is easy to check that this is well-defined and that g_i is the inverse of h_i . It remains to show that g_i is continuous. Consider the quotient map $q := \pi \times \text{id}_{S^1} : S^{2n+1} \times S^1 \rightarrow \mathbb{CP}^n \times S^1$. Let $\tilde{U}_i = \{z \in S^{2n+1} \mid z_i \neq 0\}$. Note that $g_i \circ q|_{\tilde{U}_i \times S^1}$ is clearly continuous. But $\tilde{U}_i \times S^1$ is both open in $S^{2n+1} \times S^1$ and saturated with respect to q . Hence $q|_{\tilde{U}_i \times S^1}$ is a quotient map, so that g_i is continuous.

2. Note that

$$h_i \circ h_j^{-1}([z_0, \dots, z_n], \epsilon) = \left([z_0, \dots, z_n], \epsilon \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}\right)$$

for any $[z_0, \dots, z_n] \in U_i \cap U_j$. This implies that

$$h_{ij}([z_0, \dots, z_n]) = \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}.$$

□

It follows that π is a principal S^1 -bundle. Since each homotopy group $\pi_i(S^{2n+1})$ is trivial, Corollary 2.4.2 implies that

$$\mathcal{B}(X, S^1, F) \cong [X, \mathbb{CP}^n],$$

which for large enough n , is isomorphic to $[X, \mathbb{CP}^\infty]$ where X denotes any cell complex and

$$\mathbb{CP}^\infty \equiv \bigcup_{k \in \mathbb{N}} \mathbb{CP}^k$$

equipped with the weak topology.

Definition 2.5.3. An *Eilenberg-MacLane space of type $K(G, n)$* is a space satisfying

$$\begin{cases} \pi_i K = 0 & i \neq n \\ \pi_i K \cong G & i = n \end{cases}.$$

Theorem 2.5.4. If X is a cell complex, then $[X, K(G, n)] \cong H^n(X; G)$.

Example 2.5.5. By inspecting the long exact sequence

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_2(S^{2n+1}) & \longrightarrow & \pi_2(\mathbb{CP}^n) & & \\ & & \searrow & & \swarrow & & \\ & & \pi_1(S^1) & \longrightarrow & \pi_1(S^{2n+1}) & \longrightarrow & \cdots \end{array},$$

$\underbrace{\pi_1(S^1)}_{\mathbb{Z}}$

we see that \mathbb{CP}^n is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2)$. Moreover, there is a commutative triangle

$$\begin{array}{ccc} \mathbb{CP}^\infty & \longleftrightarrow & \mathbb{CP}^n \\ \uparrow & \nearrow & \\ S^i & & \end{array}$$

for any $i \in \mathbb{N}$. Thus, $\pi_i(\mathbb{CP}^\infty) = \pi_i(\mathbb{CP}^n)$ when n is large enough. This means that \mathbb{CP}^∞ is also an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2)$. By Theorem 2.5.4, we have that

$$\mathcal{B}(X, S^1, F) \cong H^2(X; \mathbb{Z})$$

whenever X is a cell complex.

For us, a CW complex refers to a cell complex X for which there may be infinitely many attaching maps of any dimension. In this name, “C” stands for the property *closure-finite*, i.e., every open cell e^i is contained in a finite subcomplex of X . Further, “W” stands for the weak topology, with which X is equipped.

Remark 2.5.6. Each of our results holds even if we assume that a certain space is merely a CW complex rather than a cell complex.

We want to ensure that a classifying space for any topological group G exists. There are at least two famous ways of finding a classifying space for G .

Theorem 2.5.7 (Milnor construction). *There is some functor $\mathbf{TopGrp} \rightarrow \mathbf{PrinBund}$ that maps each topological group G to a principal G -bundle*

$$E_G \xrightarrow{p_G} B_G$$

such that B_G is a CW complex and $\pi_i(E_G) = 0$.

This means that B_G is a classifying space for principal G -bundles. Moreover, by applying our LES on homotopy groups to p_G , we see that $\pi_i(B_G) \cong \pi_{i-1}(G)$.

Our next method is slightly less powerful than Theorem 2.5.7 in that it produces a classifying space for principal G -bundles only over pointed connected (equivalently, path connected) CW complexes. At the same time, it produces “classifying” objects in settings other than that of principal G -bundles.

Theorem 2.5.8 (Brown representability). *Consider the homotopy category $\mathrm{Ho}(\mathbf{CW}_*^{\mathrm{conn}})$ of pointed connected CW complexes. Let F be a functor $\mathrm{Ho}(\mathbf{CW}_*^{\mathrm{conn}})^{\mathrm{op}} \rightarrow \mathbf{Set}_*$ with the following properties.*

(i) (Wedge axiom) F takes coproducts in $\mathrm{Ho}(\mathbf{CW}_*^{\mathrm{conn}})$ to products in \mathbf{Set}_* , i.e.,

$$F\left(\bigvee_{\alpha} X_{\alpha}\right) \cong \prod_{\alpha} F(X_{\alpha}).$$

(ii) (Mayer-Vietoris axiom) F takes weak pushouts to weak pullbacks, i.e., the universal morphism

$$F(B \cup_A C) \rightarrow F(B) \times_{F(A)} F(C)$$

is a surjection (or split epimorphism) for any two cofibrations $A \rightarrow B$ and $A \rightarrow C$ in $\mathbf{CW}_*^{\mathrm{conn}}$.

Then F is representable.²

By the Yoneda lemma, this means that there exists a pointed connected CW complex B along with an element $b \in F(B)$ such that the set map $[X, B] \rightarrow F(X)$ given by $g \mapsto F(g)(b)$ is a natural bijection in X . In particular, Theorem 2.5.8 applied to the functor taking any pointed connected CW complex to the set of all pointed principal G -bundles over X makes the object (B'_G, b) representing F a classifying space for G .

Remark 2.5.9. The converse of Theorem 2.5.8 is true because any representable contravariant functor takes colimits to limits.

Proof sketch of Theorem 2.5.8.

We say that a pointed connected CW complex B along with an element $b \in F(B)$ *spherically represents* F if the set map $\nu_X : [X, B] \rightarrow F(X)$ given by $g \mapsto F(g)(b)$ is a natural bijection in $X \in \{S^n \mid n \in \mathbb{Z}_{\geq 1}\}$.

Suppose that (B, b) and (B', b') are two objects spherically representing F . Let $f : B \rightarrow B'$ be a map such that $F(f)(b') = b$. Then f induces a weak homotopy equivalence. By Theorem 1.2.8, f must be a homotopy equivalence.

Now, let X be a pointed CW complex and $x \in F(X)$. It is known that there exist an object (B, b) spherically representing F and a map

$$\varphi : X \rightarrow B, \quad F(\varphi)(b) = x. \quad (\bullet)$$

Therefore, it suffices to prove the following assertion.

Claim. *Any object (B, b) spherically representing F represents F .*

Proof. We must show that ν_X is a bijection. In the interest of space, let us prove just that it is surjective. Let $x \in F(X)$ and consider the coproduct

$$X \xrightarrow{i_1} X \vee B \xleftarrow{i_2} B.$$

By the wedge axiom, we have that $F(X \vee B) \cong F(X) \times F(B)$ with $F(i_1) \cong \pi_1$ and $F(i_2) \cong \pi_2$. Thus, we have an element $(x, b) \in F(X \vee B)$ such that $F(i_1)(x, b) = x$ and $F(i_2)(x, b) = b$. Thanks to (\bullet) , we can find

²According to a certain [MathOverflow answer](#), Theorem 2.5.8 holds when F is instead a functor $\mathbf{CW} \rightarrow \mathbf{Ab}$.

an object (\widehat{B}, \hat{b}) spherically representing F along with a map $\varphi : X \vee B \rightarrow \widehat{B}$ such that $F(\varphi)(\hat{b}) = (x, b)$. It follows that

$$\begin{aligned} F(\varphi \circ i_1)(\hat{b}) &= x \\ F(\varphi \circ i_2)(\hat{b}) &= b. \end{aligned}$$

Hence $\varphi \circ i_2$ is a homotopy equivalence $B \rightarrow \widehat{B}$, with homotopy inverse, say, η . This implies that

$$F(\eta \circ \varphi \circ i_2)(b) = x,$$

so that ν_X is surjective. □

□

Let us turn to the question of uniqueness of a classifying space, having just considered the question of existence.

Lemma 2.5.10. *Let $p_1 : E_1 \rightarrow B_1$ and $p_2 : E_2 \rightarrow B_2$ be classifying spaces for principal G -bundles. Then $B_1 \simeq B_2$.*

Proof. By Corollary 2.4.2, there is some map $f : B_1 \rightarrow B_2$ such that $f^*p_2 \cong p_1$. Likewise, there is some map $g : B_2 \rightarrow B_1$ such that $g^*p_1 \cong p_2$. Therefore,

$$\begin{aligned} (f \circ g)^*p_2 &\cong g^*f^*p_2 \\ &\cong g^*p_1 \\ &\cong p_2 \\ &\cong \text{id}_{B_2}^*p_2. \end{aligned}$$

Therefore, $f \circ g \simeq \text{id}_{B_2}$. Similarly, $g \circ f \simeq \text{id}_{B_1}$. □

In particular, $B_G \simeq B'_G$.

Example 2.5.11. $B_{S^1} = \mathbb{CP}^\infty$.

Let $H \leq G$. Consider the commutative square

$$\begin{array}{ccc} E_G & \xrightarrow{q} & E_{G/H} \\ p_G \downarrow & & \downarrow r \\ B_G & \xlongequal{\quad} & E_{G/H} \end{array} \quad .$$

Note that, locally, r looks like the trivial map with fiber G/H . Thus, q locally looks like the map

$$U \times G \rightarrow U \times G/H.$$

This shows that if the natural projection $G \rightarrow G/H$ is a principal H -bundle, then so is q . In this case, we have that $B_H \simeq E_{G/H}$ by Corollary 2.4.2 together with Lemma 2.5.10.

Theorem 2.5.12. *If G is a Lie group and H is a closed subgroup of G , then the natural projection $G \rightarrow G/H$ is a principal H -bundle.*

Definition 2.5.13. The *orthogonal group* $O(n, \mathbb{R})$ is the group of $n \times n$ real matrices A such that $AA^t = A^t A = I_n$, equivalently, $Av \bullet Aw = v \bullet w$ for any $v, w \in \mathbb{R}^n$. We call such an A *orthogonal*.

In particular, if A is orthogonal, then $\|Av\| = \|v\|$ for any $v \in \mathbb{R}^n$.

Example 2.5.14. The orthogonal group $O(n, \mathbb{R})$ is a closed subgroup of $GL(n, \mathbb{R})$ because $O(n, \mathbb{R}) = f^{-1}(I_n)$ where $f : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ is given by $X \mapsto XX^t$. Let $\gamma : GL(n, \mathbb{R}) \rightarrow O(n, \mathbb{R})$ denote the map given by the Gram-Schmidt procedure. Let $i : O(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$ denote the inclusion map. Then γ and i are homotopy inverses of each other, so that

$$GL(n, \mathbb{R}) \simeq O(n, \mathbb{R}).$$

Since $\pi : GL(n, \mathbb{R}) \rightarrow \underbrace{GL(n, \mathbb{R})/O(n, \mathbb{R})}_M$ is an $O(n, \mathbb{R})$ -bundle by Theorem 2.5.12, our LES on homotopy groups applied to π shows that $\pi_i(M) = 0$ for each $i \in \mathbb{N}$. Further, our LES applied to the M -bundle $r : B_{O(n, \mathbb{R})} \rightarrow B_{GL(n, \mathbb{R})}$ shows that

$$\pi_i(B_{O(n, \mathbb{R})}) \cong \pi_i(B_{GL(n, \mathbb{R})})$$

for each i . By Theorem 1.2.8, it follows that

$$B_{O(n, \mathbb{R})} \simeq B_{GL(n, \mathbb{R})}.$$

An exactly similar argument proves that $B_{U(n, \mathbb{C})} \simeq B_{GL(n, \mathbb{C})}$.

Eventually, we want to describe $H^*(B_G)$. This will lead us to the notion of a spectral sequence.

2.6 Lecture 11

Before moving to spectral sequences, let us look at a couple more examples of fiber bundles.

Example 2.6.1. Let $\{e_i\}_{1 \leq i \leq n}$ denote the standard basis of \mathbb{R}^n . Consider the map $\rho : GL(n, \mathbb{R}) \rightarrow \mathbb{R}^n \setminus \{0\}$ given by $A \mapsto Ae_n$ and its restriction $\tau : O(n, \mathbb{R}) \rightarrow S^{n-1}$. Note that $\rho^{-1}(e_n)$ consists of all $n \times n$ matrices of the form

$$\left(\begin{array}{c|c} B & 0 \\ \hline * & 1 \end{array} \right)$$

where B denotes an invertible $(n-1) \times (n-1)$ matrix. This means that $\rho^{-1}(e_n) \simeq GL(n-1, \mathbb{R})$. Similarly, we see that $\tau^{-1}(e_n) \simeq O(n-1, \mathbb{R})$. Moreover, both ρ and τ are locally trivial. In particular, this yields a LES

$$\begin{array}{ccccccc} & & & \cdots & \longrightarrow & \pi_{i+1}(S^{n-1}) & \\ & & & & \swarrow & & \\ \pi_i(O(n-1)) & \longrightarrow & \pi_i(O(n)) & \longrightarrow & \pi_i(S^{n-1}) & \cdot & \\ & & & \nwarrow & & & \\ \pi_{i-1}(O(n-1)) & \longrightarrow & \cdots & & & & \end{array}$$

Since $\pi_i(S^{n-1})$ is trivial for any $0 \leq i \leq n-2$, we see that the map $\pi_i(O(n-1)) \rightarrow \pi_i(O(n))$ is an isomorphism for any $i \leq n-3$ and an epimorphism when $i = n-2$. The same result holds with $O(n)$ replaced by $GL(n, \mathbb{R})$.

Example 2.6.2. Consider the *Stiefel manifold* $V_{n+k,k}$ consisting of orthonormal k -frames (i.e., k -tuples) in \mathbb{R}^{n+k} . If we view the standard basis of \mathbb{R}^k as the “zero element” of $V_{n+k,k}$, then we have a “short exact sequence”

$$0 \longrightarrow O(n) \xhookrightarrow{i} O(n+k) \xrightarrow{p_1} V_{n+k,k} \longrightarrow 0$$

where i is given by $A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array} \right)$ and p_1 is given by $A \mapsto (Ae_{n+1}, \dots, Ae_{n+k})$. In this case,

$$V_{n+k,k} \cong \frac{O(n+k)}{O(n)},$$

a coset space. Note that i induces an isomorphism $\pi_i(O(n)) \xrightarrow{\cong} O(n+k)$ for each $i \leq n-2$ and an epimorphism when $i = n-1$.

Claim. *The map p_1 is a fiber bundle.*

Proof. Let $F \in V_{n+k,k}$ and choose any orthonormal basis B of the n -plane orthogonal to F . For any n -plane near B , take the orthogonal projection of B onto B' and then apply the Gram-Schmidt process to the new basis to obtain an orthonormal basis \underline{B}' of B' . The assignment $B \mapsto \underline{B}'$ is continuous, and the space of all n -planes orthogonal to any $(n+k)$ -plane near F is identifiable with $V_n(\mathbb{R}^n) \cong O(n)$. Therefore, we get a trivialization around F , which was arbitrary. \square

Using the LES obtained from Corollary 1.3.9, we see that $\pi_i(V_{n+k,k}) = 0$ for each $i \leq n-1$. Consider now the *Grassmann manifold*

$$G_{n+k,k} \equiv \frac{O(n+k)}{O(n) \times O(k)}$$

where each pair $(A, B) \in O(n) \times O(k)$ is identified with the orthogonal $(n+k) \times (n+k)$ matrix $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array} \right)$.

Note that $G_{n+k,k}$ may be viewed as the space of all k -dimensional planes in \mathbb{R}^{n+k} .

Proposition 2.6.3.

1. *The space $E_{O(k)}$ consists of all orthonormal k -frames in \mathbb{R}^∞ .*
2. *The Grassmannian $G_{\infty,k} \equiv B_{O(k)} = B_{GL(k)}$ consists of all k -planes in \mathbb{R}^∞ .*
3. *Similarly, the space $B_{U(k)}$ consists of all k -planes in \mathbb{C}^∞ .*

Define $p_2 : V_{n+k,k} \rightarrow G_{n+k,k}$ by sending each $v \in V_{n+k,k}$ to the subspace of \mathbb{R}^{n+k} spanned by v .

Claim. *The map p_2 is a principal $O(k)$ -bundle.*

Proof. This follows from the fact that $O(n+k) \rightarrow G_{n+k,k}$ is a principal $O(n) \times O(k)$ -bundle. \square

As a result, $\pi_i(G_{n+k,k}) = 0$ for each $i \leq n-2$.

3 Spectral sequences

We are given a fibration:

$$\begin{array}{c} F \\ \downarrow \\ E \\ \downarrow \pi \\ X \end{array}$$

where X is a connected cell complex and $F = \pi^{-1}(x)$ for some distinguished point x .

Question. What is $H_n(E)$ if we know $H_n(F)$ and $H_n(X)$?

Recall that $H_n(X) = \frac{\ker \partial_n}{\text{im } \partial_{n+1}}$ where ∂_n is defined as the composite

$$\begin{array}{ccccc} \overbrace{H_n(X^n, X^{n-1})}^{C_n(X)} & \longrightarrow & H_{n-1}(X^{n-1}) & \longrightarrow & \overbrace{H_{n-1}(X^{n-1}, X^{n-2})}^{C_{n-1}}(X) \\ & & & \searrow \partial_n & \nearrow \end{array}$$

where $H_i(X^n, X^{n-1}) = 0$ for any $i \neq n$. Furthermore, letting $E_n = \pi^{-1}(X_n)$, we have that $H_*(E_n, E_{n-1}) = C_*(X) \otimes H_*(F)$.

At this point, it is useful to generalize our situation by developing the theory of spectral sequences. For each $r \in \mathbb{Z}_{\geq 0}$, let $\{E_{p,q}^r\}_{p,q \in \mathbb{Z}}$ be a family of abelian groups and let $\{d_r^{p,q} : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r\}_{p,q \in \mathbb{Z}}$ be a family of maps (called *differentials*) such that

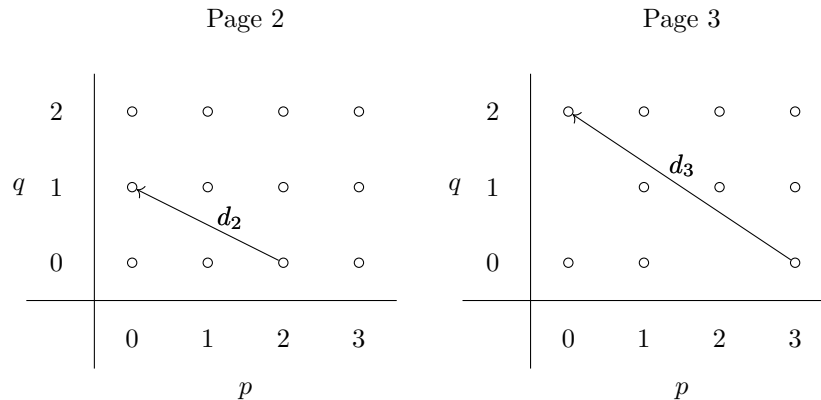
(a) $d_r^{p,q} \circ d_r^{p+r,q-r+1} = 0$ and

(b) $E_{p,q}^{r+1} = \frac{\ker d_r^{p,q}}{\text{im } d_r^{p+r,q-r+1}}$.

Such a sequence $(E^r, d_r)_{r \in \mathbb{Z}_{\geq 0}}$ of pairs is called a *homological spectral sequence*, and each double complex (E^r, d_r) is called the *r-th page* of the sequence.

Note 3.0.1. $E^{r+1} = H_*(E^r, d_r)$.

We shall consider only *first-quadrant* spectral sequences, i.e., those for which $E_{p,q}^r = 0$ unless $p, q \geq 0$.



As a result, there is some $k \in \mathbb{N}$ such that $E^r = E^{r+1}$ for any $r \geq k$.

Notation. $E^\infty := E^k$.

Definition 3.0.2 (Convergence). We say that a spectral sequence $E^* := (E^r, d_r)$ *converges* to a sequence of abelian groups $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$, written as

$$E^* \Rightarrow \{A_n\},$$

if for each n , there exists a filtration

$$\cdots \subset A_{-1,n+1} = \{0\} \subset A_{0,n} \subset \cdots \subset A_{n-1,1} \subset A_{n,0} = A_n$$

of A_n such that $\frac{A_{p,q}}{A_{p-1,q+1}} \cong E_{p,q}^\infty$.

Theorem 3.0.3. *Let B be a simply connected, path connected cell complex with n -skeleton B^n and suppose that $\pi : E \rightarrow B$ is a fibration with fiber F . There exists a (first-quadrant) spectral sequence (E^r, d_r) that*

(a) *converges to $\{H_n(E)\}_{n \in \mathbb{Z}_{\geq 0}}$ and*

(b) *satisfies $E_{p,q}^2 \cong H_p(B; H_q(F))$.*

The filtration $D_{p,q} := (H_n(E))_{p+q=n}$ witnessing this convergence is given by $\text{im}(H_n(\pi^{-1}(B^p)) \rightarrow H_n(E))$.

Remark 3.0.4. This holds without the hypothesis that B is a cell complex.

Example 3.0.5. Consider the path space fibration

$$\begin{array}{c} \Omega X \\ \downarrow \\ PX \\ \downarrow \\ X \end{array}.$$

Recall that PX is contractible. Let $n \geq 2$ and $X = S^n$. Then

$$E_{p,q}^2 \cong H_p(S^n; H_q(\Omega S^n)) = \begin{cases} H_q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise} \end{cases},$$

and $(E^r, d_r) \Rightarrow \{\mathbb{Z}, 0, 0, \dots\}$. This means that $d_k = 0$ for any $k \neq n$, so that

$$\begin{aligned} E^2 &= E^3 = \cdots = E^n \\ E^{n+1} &= E^{n+2} = \cdots = E^\infty. \end{aligned}$$

As a result, each differential $d_n^{p,q}$ is an isomorphism provided that $(p, q) \neq (n, 1 - n)$ for, otherwise, $E_{p,q}^{n+1}$ is

nontrivial, which is impossible. Hence the n -th page looks like

$$\begin{array}{c|ccc}
 3(n-1) & H_{3n-3}(\Omega S^n) & & H_{3n-3}(\Omega S^n) \\
 & & \swarrow \text{iso.} & \\
 2(n-1) & H_{2n-2}(\Omega S^n) & & H_{2n-2}(\Omega S^n) \\
 & & \swarrow \text{iso.} & \\
 n-1 & H_{n-1}(\Omega S^n) & & H_{n-1}(\Omega S^n) \\
 & & \swarrow \text{iso.} & \\
 0 & \mathbb{Z} & & \mathbb{Z} \\
 \hline
 & 0 & & n
 \end{array}$$

This implies that $H_q(\Omega S^n) \cong H_{q+n-1}(\Omega X)$ for any $q \in \mathbb{Z}_{\geq 0}$. But ΩS^n is path connected since S^n is simply connected. By induction, it follows that

$$H_q(\Omega X) \cong \begin{cases} \mathbb{Z} & q \equiv 0 \pmod{n-1} \\ 0 & \text{otherwise} \end{cases}.$$

3.1 Lecture 12

Suppose that

$$\begin{array}{c}
 F \\
 \downarrow \\
 E \\
 \downarrow \pi \\
 B
 \end{array}$$

is a fibration with B simply connected and F path connected. Thanks to Theorem 3.2.5, we have the inclusion

$$E_{0,n}^\infty \cong \frac{D_{0,n}}{D_{-1,n+1}} = D_{0,n} \subset H_n(E)$$

as well as a commutative diagram

$$\begin{array}{c}
 \overbrace{H_p(B; H_0(F))}^{H_p(B)} = E_{p,0}^2 \longleftrightarrow E_{p,0}^3 \\
 \quad \quad \quad \nearrow \\
 E_{p,0}^4 \longleftrightarrow \cdots \longleftrightarrow E_{p,0}^p \\
 \quad \quad \quad \nearrow \\
 E_{p,0}^\infty \longleftarrow H_p(E) = D_{p,0} \\
 \quad \quad \quad \nwarrow \pi_* \\
 \quad \quad \quad \nearrow
 \end{array}$$

$$\begin{array}{c}
 \xrightarrow{i_*} \\
 \underbrace{H_q(F)}_{H_0(B, H_q(F))} \longrightarrow E_{0,q}^3 \longrightarrow \cdots \longrightarrow E_{0,q}^\infty \hookrightarrow H_q(E) .
 \end{array}$$
$$\begin{array}{ccc}
\pi_n(B) & \xrightarrow{\partial} & \pi_{n-1}(F) \\
h_n \downarrow & & \downarrow h_{n-1} \\
H_n(B) & \xrightarrow{\partial} & H_{n-1}(F) \\
\cong \uparrow & & \Downarrow \\
E_{n,0}^2 & & E_{0,n-1}^3 \\
\uparrow & & \downarrow \\
\vdots & & \vdots \\
\uparrow & & \Downarrow \\
E_{n,0}^n & \xrightarrow{d_n} & E_{0,n-1}^n
\end{array}$$
$$h(\gamma) = f_*(\tau) \in H_n(X).$$
$$[f : (D^n, S^{n-1}, x_0) \rightarrow (X, A, \text{pt})] \mapsto f_*(\sigma)$$

Proof of Theorem 3.1.1. Suppose that $\pi_i(X) = 0$ for each $1 \leq i \leq n-1$. For induction, assume that h_{n-1} is an isomorphism for any path connected space. From Example 3.0.5, we gather that the n -th page of the spectral sequence induced by the path space fibration $\Omega X \rightarrow PX \rightarrow X$ looks like

$$\begin{array}{ccc}
n-1 & H_{n-1}(\Omega X) & \bullet & H_{n-1}(\Omega X) \\
& & \nwarrow d_n & \\
0 & \mathbb{Z} & & H_n(X) \\
\hline
& 0 & & n
\end{array}$$

where d_n is an isomorphism. Thanks to our inductive hypothesis together with Exercise 1.4.3, we have now a commutative square of the form

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\cong} & \pi_{n-1}(\Omega X) \\ h_n \downarrow & & \downarrow h_{n-1} \cong \\ H_n(X) & \xrightarrow[\cong]{d_n} & H_{n-1}(\Omega X) \end{array} \quad (*)$$

This implies that h_n is an isomorphism. It remains to verify our base case. Note that $\pi_1(\Omega X)$ is isomorphic to $\pi_2(X)$ and thus abelian. It can be shown directly that h_1 factors as a composite

$$\begin{array}{ccccc} \pi_1(\Omega X) & \xrightarrow{\cong} & \pi_1(\Omega X)^{\text{ab}} & \xrightarrow{\cong} & H_1(\Omega X) \\ & \searrow & & \nearrow & \\ & & h_1 & & \end{array}$$

of isomorphisms. Hence h_2 must be an isomorphism in light of (*). \square

Question. Does a similar argument work for Theorem 3.1.2?

Answer. Yes, in the sense that there is a spectral sequence proof of it. Specifically, there is a relative version of the spectral sequence. Suppose that B' is a simply connected subspace of B and let $E' = \pi^{-1}(B')$, yielding a fibration $F \rightarrow E' \xrightarrow{\pi|_{E'}} B'$. Then there exists a spectral sequence E^* converging to $H_*(E, E')$ such that

$$E_{p,q}^2 \cong H_p(B, B'; H_q(F)).$$

One can use this to deduce Theorem 3.1.2 from Theorem 3.1.1. \square

Corollary 3.1.3. *Let X be path connected.*

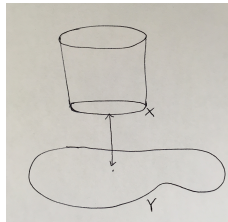
1. $H_1(X) \cong \pi_1^{\text{ab}}(X)$.
2. If X is simply connected and $H_i(X) = 0$ for every $1 \leq i \leq n-1$, then $\pi_i(X) = 0$ for every $1 \leq i \leq n-1$.
3. If $\pi_i(X) = 0$ for each $0 \leq i \leq n-1$, then $\tilde{H}_i(X) = 0$ for each $0 \leq i \leq n-1$.

Let $n \geq 2$ and pick any generator $[f]$ of $\pi_{n-1}(\Omega S^n) \cong \pi_n(S^n) \cong \mathbb{Z}$. By Theorem 3.1.1, the induced map $f_* : H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(\Omega S^n)$ is an isomorphism.

Remark 3.1.4. Let $g : X \rightarrow Y$ be any map of spaces. Recall the mapping cylinder

$$\text{Cyl}(g) \equiv \frac{(X \times I) \amalg Y}{(x, 0) \sim g(x)}$$

of g .



This is precisely the pushout of the span $X \times I \xleftarrow{\sigma_0} X \xrightarrow{g} Y$. As it turns out, g factors as

$$\begin{array}{ccccc} & & g & & \\ & \searrow & & \nearrow & \\ X & \xleftarrow{\iota} & \text{Cyl}(g) & \xrightarrow{h} & Y \end{array}$$

for some deformation retraction h . Further, ι is a cofibration, the dual notion to a fibration.

Consider the subspace of ΩS^n consisting of all great circles passing through, say, the north pole. This is clearly homeomorphic to S^{n-1} . Thus, we get a LES in homology

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \underbrace{H_{2n-2}(S^{n-1})}_0 & \longrightarrow & \underbrace{H_{2n-2}(\Omega S^n)}_{\mathbb{Z}} & \xrightarrow{\cong} & H_{2n-2}(\Omega S^n, S^{n-1}) \\ & & \swarrow & & \searrow & & \\ & & \underbrace{H_{2n-3}(S^{n-1})}_0 & \longrightarrow & H_{2n-3}(\Omega S^n) & \xrightarrow{\cong} & H_{2n-3}(\Omega S^n, S^{n-1}) \longrightarrow \cdots \\ & & & & \swarrow & & \\ & & & & \underbrace{H_{n+1}(S^{n-1})}_0 & \longrightarrow & H_{n+1}(\Omega S^n) \xrightarrow{\cong} H_{n+1}(\Omega S^n, S^{n-1}) \\ & & & & \swarrow & & \\ & & & & \underbrace{H_n(S^{n-1})}_0 & \longrightarrow & H_n(\Omega S^n) \xrightarrow{\cong} H_n(\Omega S^n, S^{n-1}) \\ & & & & \swarrow & & \\ & & & & H_{n-1}(S^{n-1}) & \xrightarrow[\cong]{f_*} H_{n-1}(\Omega S^n) \xrightarrow{0} H_{n-1}(\Omega S^n, S^{n-1}) \\ & & & & & \swarrow & \\ & & & & & \underbrace{H_{n-2}(S^{n-1})}_0 & \longrightarrow \cdots \end{array}$$

From this, we deduce that

$$H_i(\Omega S^n, S^{n-1}) = \begin{cases} 0 & i \leq 2n-3 \\ \mathbb{Z} & i = 2n-2 \end{cases}.$$

By Corollary 3.1.3(2), this means that

$$\pi_i(\Omega S^n, S^{n-1}) = \begin{cases} 0 & i \leq 2n-3 \\ \mathbb{Z} & i = 2n-2 \end{cases}.$$

This yields a LES in homotopy

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{2n-2}(\Omega S^n) & \longrightarrow & \pi_{2n-2}(\Omega S^n, S^{n-1}) & & \\ & & \swarrow & & \searrow & & \\ & & \pi_{2n-3}(S^{n-1}) & \longrightarrow & \pi_{2n-3}(\Omega S^n) & \longrightarrow & \underbrace{\pi_{2n-3}(\Omega S^n, S^{n-1})}_0 \\ & & & & \swarrow & & \\ & & & & \pi_{2n-4}(S^{n-1}) & \xrightarrow{\cong} & \underbrace{\pi_{2n-4}(\Omega S^n)}_{\pi_{2n-3}(S^n)} \longrightarrow \underbrace{\pi_{2n-4}(\Omega S^n, S^{n-1})}_0 \longrightarrow \cdots \end{array},$$

which proves the following statement.

Theorem 3.1.5 (Suspension). *If $0 \leq i \leq 2n - 4$, then $\pi_i(S^{n-1}) \cong \pi_{i+1}(S^n)$.*

This can be generalized as follows.

Exercise 3.1.6 (Freudenthal suspension). *Let $i \in \mathbb{Z}_{\geq 1}$ and suppose that the space X satisfies $\pi_n(X) = 0$ for each $0 \leq n \leq i - 1$. Show that*

$$\pi_i(X) \cong \pi_{i+1}(SX)$$

through around dimension $2i - 3$ (figure this out exactly), where $S(-)$ denotes the suspension functor.

Proof. We could use a spectral sequence argument together with the relative Hurewicz theorem. Rather, let us show that it follows from the following famous theorem in homotopy theory:

Theorem 3.1.7 (Blakers-Massey). *Suppose that*

$$\begin{array}{ccc} X & \hookrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & C \end{array}$$

is a pushout diagram. Also, suppose that $\pi_n(B, X) = 0$ and $\pi_m(A, X) = 0$ for each $0 \leq n \leq k_1$ and each $0 \leq m \leq k_2$. The map $(A, X) \rightarrow (C, B)$ of pairs induces an isomorphism $\pi_n(A, X) \xrightarrow{\cong} \pi_n(C, B)$ for each $0 \leq n \leq k_1 + k_2 - 1$.

Now, let us prove the Freudenthal suspension theorem with an upper bound of $2i - 2$ rather than $2i - 3$. To start, decompose SX into the union of two cones C_+X and C_-X that meet at a copy of X . Note that SX is precisely the pushout of the diagram

$$C_-X \longleftarrow X \longrightarrow C_+X.$$

As C_+X is contractible, the LES of homotopy groups for the pair (C_+X, X) shows that the map $\partial : \pi_{n+1}(C_+X, X) \rightarrow \pi_n(X)$ is an isomorphism. Similarly, we see that the map $\iota : \pi_{n+1}(SX) \rightarrow \pi_{n+1}(SX, C_-X)$ induced by inclusion is an isomorphism. Consider the sequence of homomorphisms

$$\pi_n(X) \xrightarrow{\partial^{-1}} \pi_{n+1}(C_+X, X) \xrightarrow{\psi} \pi_{n+1}(SX, C_-X) \xrightarrow{\iota^{-1}} \pi_{n+1}(SX)$$

where ψ is induced by pullback. Since $\pi_n(X) = 0$ for each $0 \leq n \leq i - 1$, the LES for the pair $(C_{\pm}X, X)$ also shows that $\pi_n(C_{\pm}X) = 0$ for any $0 \leq n \leq i$. By Theorem 3.1.7, it follows that ψ is an isomorphism so long as $n + 1 \leq 2i - 1$. Hence $\pi_n(X) \cong \pi_{n+1}(SX)$ for any $n \leq 2i - 2$, as desired.

Furthermore, the upper bound of $2i - 2$ is sharp. Indeed, we have that

- $\pi_0(S^2) = \pi_1(S^2) = 0$,
- $\pi_3(S^2) \cong \mathbb{Z}$, and
- $\pi_4(S^3) \cong \mathbb{Z}/2$.

If we could increase our upper bound to $2i - 1$, then we would have an isomorphism $\pi_3(S^2) \cong \pi_4(S^3)$, which is impossible. \square

3.2 Lecture 13

As expected, spectral sequences have exact analogues in cohomology. Before introducing them, let us review a bit of singular cohomology theory. Let X be a cell complex and let $n \in \mathbb{Z}_{\geq 0}$. Recall that $C_n(X)$ the free abelian group on the set of all n -cells of X and the boundary map $\partial_n : C_n(X) \rightarrow C_{n-1}(X)$. Let

$$C^n(X) = \text{Hom}(C_n(X), \mathbb{Z})$$

and define the homomorphism $\delta^n : C^n(X) \rightarrow C^{n+1}(X)$ by

$$\delta^n(\varphi) = \varphi \circ \partial_n.$$

Theorem 3.2.1. $H^n(X; \mathbb{Z}) \cong \frac{\ker \delta^{n+1}}{\text{im } \delta^n}.$

Example 3.2.2. $H^*(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}[x] / (x^{n+1})$ with $|x| = 2$.
dim.

Theorem 3.2.3 (Poincaré duality). *If M is a connected orientable n -manifold, then $H_i(M) \cong H^{n-i}(M)$.*

Now, a cohomological spectral sequence consists of the following data:

- for each $r \in \mathbb{Z}_{\geq 0}$, a family of abelian groups $\{E_r^{p,q}\}_{p,q \in \mathbb{Z}}$ and
- a family of maps $\{d_r^{p,q} : E_r^{p,q} \rightarrow E_r^{p+r, q-r+1}\}_{p,q \in \mathbb{Z}}$ (called *differentials*) such that
- $d_r^{p,q} \circ d_r^{p-r, q+r-1} = 0$ and
- $E_{r+1}^{p,q} = \frac{\ker d_r^{p,q}}{\text{im } d_r^{p-r, q+r-1}}.$

Again, we shall consider only *first-quadrant* spectral sequences, i.e., those for which $E_r^{p,q} = 0$ unless $p, q \geq 0$.

As a result, there is some $k \in \mathbb{N}$ such that $E_r = E_{r+1}$ for any $r \geq k$.

Notation. $E_\infty := E_k$.

Definition 3.2.4 (Convergence). We say that a spectral sequence $E_* := (E_r, d_r)$ *converges* to a sequence of abelian groups $\{D^n\}_{n \in \mathbb{Z}_{\geq 0}}$, written as

$$E_* \Rrightarrow \{D^n\},$$

if for each n , there exists a filtration

$$\dots \subset D^{n+1, -1} = \{0\} \subset D^{n, 0} \subset \dots \subset D^{1, n-1} \subset D^{0, n} = D^n$$

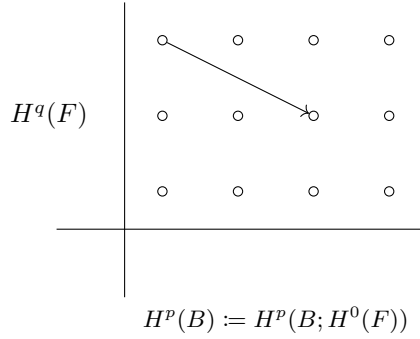
of D^n such that $\frac{D^{p,q}}{D^{p+1, q-1}} \cong E_\infty^{p,q}.$

Theorem 3.2.5. *Let B be simply connected and path connected and suppose that $\pi : E \rightarrow B$ is a fibration with fiber F . There exists a (first-quadrant) spectral sequence (E^r, d_r) that*

- (a) *converges to $\{H^n(E)\}_{n \in \mathbb{Z}_{\geq 0}}$ and*
- (b) *satisfies $E_2^{p,q} \cong H^p(B; H^q(F))$.*

In pictures, we have

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$$\begin{array}{c}
 H^q(E) \cong D^{0,q} \twoheadrightarrow E_{\infty}^{0,q} \hookrightarrow \cdots \hookrightarrow E_2^{0,q} \cong H^q(F) \\
 \searrow \quad \quad \quad \nearrow \\
 \quad \quad \quad i^* \quad \quad \quad \\
 \nearrow \quad \quad \quad \searrow \\
 H^p(B) \cong E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{\infty}^{p,0} \hookrightarrow H^p(E) \\
 \searrow \quad \quad \quad \nearrow \\
 \quad \quad \quad \pi^* \quad \quad \quad
 \end{array}$$

Let X be a cell complex. Recall the *cup product* operation $H^i(X) \times H^j(X) \xrightarrow{\smile} H^{i+j}(X)$ on cohomology, which is both bilinear and *anti-commutative* in the sense that

$$x \smile y = (-1)^{ij} y \smile x.$$

Consider the constant map $C_0(X) \rightarrow \mathbb{Z}$ given by $D^0 \mapsto 1$, which corresponds to an element $\mathbf{1}$ of $H^0(X)$ via Theorem 3.2.1. We have that

$$-\mathbf{1} \smile x = x \smile \mathbf{1} = \mathbf{1}.$$

Suppose that Y is another cell complex. Let $x \in H^i(X)$ and $y \in H^j(X)$ and let f denote a map $Y \rightarrow X$. Then

$$f^*(x \smile y) = f^*(x) \smile f^*(y),$$

i.e., f^* is a graded ring homomorphism. Now, $X \times Y$ carries a cell complex structure with n -cells of the form

$$D^i \times D^j, \quad i + j = n$$

and n -skeleton

$$(X \times Y)^n \equiv \bigcup_{i+j=n} X^i \times Y^j.$$

We have that

$$C_n(X \times Y) \cong C_n(X) \otimes_{\mathbb{Z}} C_n(Y)$$

and, in light of the fact that $\partial(D^i \times D^j) = (\partial D^i \times D^j) \cup (D^i \times \partial D^j)$, that

$$\partial[D^i \times D^j] = \partial[D^i] \otimes D^j + (-1)^i [D^i] \otimes \partial[D^j].$$

Consider any two maps $f : C_i(X) \rightarrow \mathbb{Z}$ and $g : C_j(X) \rightarrow \mathbb{Z}$, extending them both by 0 to the entire graded abelian group $C_*(X)$. Define $f \otimes g : C_m(X \times Y) \cong C_m(X) \otimes C_m(Y) \rightarrow \mathbb{Z}$ by

$$(f \otimes g)(u \otimes v) = f(u) \cdot g(v).$$

Proposition 3.2.6. $\delta(f \otimes g) = \delta f \otimes g + (-1)^i f \otimes \delta g$.

As it turns out, this means that the map $(f, g) \mapsto (f \otimes g)$ induces an operation $H^i(X) \times H^j(Y) \xrightarrow{\times} H^{i+j}(X \times Y)$ on cohomology known as the *cross product*. The relation between the cup and cross product has the form $\Delta^*(x \times y) = x \smile y$, where $\Delta : X \rightarrow X \times X$ denotes the diagonal map.

In general, let R_1, R_2 , and R_3 be commutative rings and let $\mu : R_1 \times R_2 \rightarrow R_3$ denote “multiplication.” This induces the cup product on cohomology

$$\begin{aligned} H^i(X; R_1) \times H^j(X; R_2) &\xrightarrow{\smile} H^{i+j}(X; R_3) \\ \Downarrow \\ H^p(B, H^q(F)) \times H^{p'}(B, H^{q'}(F)) &\xrightarrow{\smile} H^{p+p'}(B, H^{q+q'}(F)) \\ E_2^{p,q} \times E_2^{p',q'} &\xrightarrow{\smile} E_2^{p+p',q+q'}. \end{aligned}$$

Proposition 3.2.7. For any $r \in \mathbb{Z}_{\geq 2}$, there is a certain operation $\smile_r : E_r^{p,q} \times E_r^{p',q'} \rightarrow E_r^{p+p',q+q'}$ such that

$$d_r(x \smile y) = d_r(x) \smile y + (-1)^{p+q} x \smile d_r(y).$$

Construction. Let $r \in \mathbb{Z}_{\geq 2}$ and suppose, for induction, that we have already constructed \smile_r . Let $x \in E_r^{p,q}$ and $y \in E_r^{p',q'}$. Suppose that $d_r x = d_r y = 0$, so that $d_r(x \smile y) = 0$. If $y = d_r(z)$, then

$$x \smile y = x \smile d_r(z) = d(x \smile z) \pm \underbrace{d_r(x)}_0 \smile z.$$

by induction. This means that \smile_r induces a pairing \smile_{r+1} on E_{r+1} . To complete our induction on r , simply take the ordinary cup product on cohomology to be \smile_2 . \square

Now, given the filtration

$$\{0\} \subset D^{n,0} \subset \dots \subset D^{0,n} \subset H^n(E),$$

the operation \smile_r on E_r carries $D^{p,q} \times D^{p',q'}$ to $D^{p+p',q+q'}$ where $p+q = p'+q' = n$, thereby inducing a pairing

$$\smile_\infty : E_\infty^{p,q} \times E_\infty^{p',q'} \rightarrow E_\infty^{p+p',q+q'}$$

on E_∞ .

Example 3.2.8. Consider the fiber bundle $S^1 \rightarrow S^{2n+1} \twoheadrightarrow \mathbb{CP}^n$, so that

$$E_2^{p,q} \cong H^p(\mathbb{CP}^n; H^q(S^1)).$$

Pick a generator x of the group $H^1(S^1) \cong \mathbb{Z}$. Then the cohomology ring $H^*(S^1)$ is isomorphic to $\mathbb{Z}[x]/(x^2)$, and

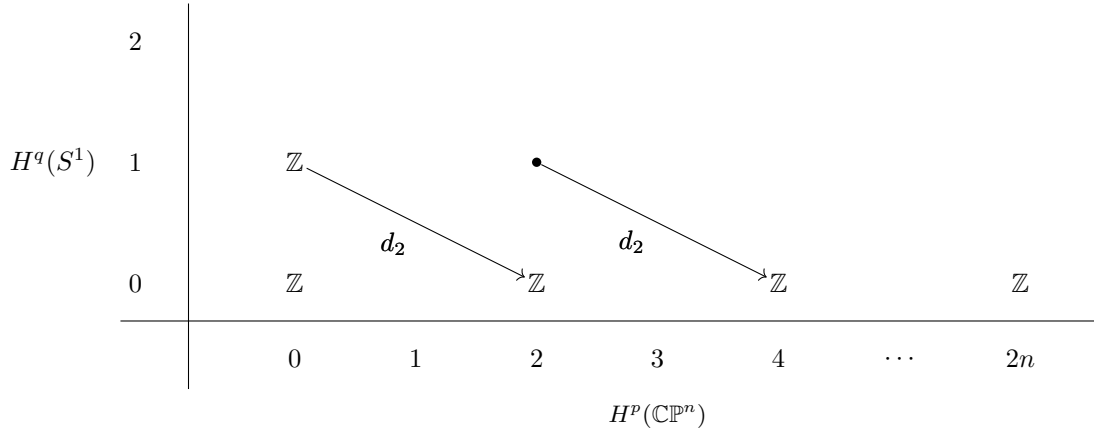
$$H^i(S^1) \cong \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases}.$$

Moreover, recall that

$$H^i(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & 0 \leq i \leq 2n, i \equiv 0 \pmod{2} \\ 0 & \text{otherwise} \end{cases},$$

which yields

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where each d_2 is an isomorphism. Suppose that x is a generator of $H^1(S^1)$ and let $c = d_2(x)$. Then

$$d_2(c \smile x) = c \smile d_2(x) = c^2,$$

which is a generator of $H^4(\mathbb{CP}^n)$. Similarly, c^i is a generator of $H^{2i}(\mathbb{CP}^n)$ for each $i \in \mathbb{Z}_{\geq 0}$.

By letting $c^0 = 1$ and making n large enough, we have determined the ring structure of $H^*(\mathbb{CP}^\infty)$.

Theorem 3.2.9. *If c_1 is a generator of $H^2(\mathbb{CP}^\infty) \cong \mathbb{Z}$, then $H^*(B_{S^1}) = H^*(\mathbb{CP}^\infty) \cong \mathbb{Z}[c_1]$.*

Example 2.5.11

4 Characteristic classes

4.1 Lecture 14

Recall the space $B_{U(n)} = B_{GL(n, \mathbb{C})}$ of n -planes in \mathbb{C}^∞ as well as the space $B_{O(n)} = B_{GL(n, \mathbb{R})}$ of n -planes in \mathbb{R}^∞ . We want to classify the graded rings $H^*(B_{U(n)}; \mathbb{Z})$ and $H^*(B_{O(n)}; \mathbb{Z}_2)$. Let's begin with the former.

Consider the embedding $U(n-1) \hookrightarrow U(n)$ given by $A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}$. Consider also the mapping $U(n) \rightarrow S^{2n-1}$ given by $A \mapsto Ae_n$. These fit into a “short exact sequence”

$$0 \rightarrow U(n-1) \rightarrow U(n) \rightarrow S^{2n-1} \rightarrow 0 \quad (1)$$

of spaces, and thus the bundle

$$\frac{U(n)}{U(n-1)} \rightarrow BU(n-1) \xrightarrow{\pi} BU(n) \quad (2)$$

has fiber S^{2n-1} .

Remark 4.1.1. Let us make (1) precise. Note that $U(n)$ acts transitively on S^{2n-1} because any point in S^{2n-1} belongs to at least one orthonormal bases of \mathbb{C}^n . This means that S^{2n-1} is a *homogenous* $U(n)$ -space.

Proposition 4.1.2. Any homogenous G -space X is homeomorphic to the coset space $G/\text{Stab}_G(\vec{o})$ where \vec{o} is any choice of “identity.”

In our case, let $\vec{o} = e_n$, so that $\text{Stab}_{U(n)}(\vec{o}) = U(n-1)$.

Now, (2) induces a spectral sequence E_* converging to $H^*(BU(n-1))$ such that

$$E_2^{p,q} = H^p(BU(n); H^q(S^{2n-1}))$$

and

$$\begin{array}{ccccccc} H^p(BU(n)) & \longrightarrow & E_2^{p,0} & \longrightarrow & E_3^{p,0} & \longrightarrow & \cdots \longrightarrow E_\infty^{p,0} \hookrightarrow H^p(BU(n-1)) \\ & & & & & \searrow & \nearrow \\ & & & & & \pi^* & \end{array}$$

commutes. If $p < 2n$, then $H^p(BU(n)) \cong E_\infty^{p,0}$, in which case π^* is injective. Also, we have that $E_2^{p-k,k} = E_\infty^{p-k,k}$ whenever $0 < p-k < 2n+1$, so that π^* is surjective whenever $p < 2n$. It follows that π^* is an isomorphism

$$H^p(BU(n)) \cong H^p(BU(n-1)) \quad (3)$$

when $p \leq 2n-1$.

Consider the differential $d_2 : \underbrace{H^{2n-1}(S^{2n-1})}_{\mathbb{Z}} \rightarrow H^{2n}(BU(n))$ and pick a generator g_n of $H^{2n-1}(S^{2n-1})$, i.e., an orientation of S^{2n-1} . Let

$$c_n = d_2(g_n).$$

In light of (3), we see that

$$c_i \in H^{2i}(BU(n)) \cong \cdots \cong H^{2i}(BU(i+1)) \cong H^{2i}(BU(i))$$

for each $i \leq n$. Abusing notation, we shall write $c_i \in H^{2i}(BU(i))$.

Theorem 4.1.3. $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$

Proof. Proceed by induction on $n \in \mathbb{N}$. Our base case holds by virtue of Theorem 3.2.9. Assume that

$$H^*(BU(n-1)) \cong \mathbb{Z}[c_1, \dots, c_{n-1}].$$

Note that $E_{2n} = E_\infty$ and $E_{2n-1} = E_2$, yielding

Page $2n - 1$

$$\begin{array}{c}
 \begin{array}{c} 2n-1 \\ \vdots \\ 0 \end{array} \quad \begin{array}{c} g_n \\ \searrow d_{2n-1} \\ \rightarrow c_n \end{array} \\
 \begin{array}{c} H^q(S^{2n-1}) \\ \vdots \\ 0 \end{array} \quad \begin{array}{c} \mathbb{Z} \quad 0 \quad c_1 \quad 0 \quad c_2 \quad \cdots \end{array} \\
 \begin{array}{c} 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots \quad 2n \end{array} \\
 \begin{array}{c} H^p(BU(n)) \end{array}
 \end{array}$$

??

□

We call c_i the i -th *Chern class* of $BU(n)$. The following may be seen as a real analogue to Theorem 4.1.3.

Theorem 4.1.4. $H^*(BO(n), \mathbb{Z}_2) \cong \mathbb{Z}[w_1, \dots, w_n]$ for certain cohomology classes $w_i \in H^i(BO(n), \mathbb{Z}_2)$.

We call w_i the i -th *Stiefel-Whitney class* of $BO(n)$.

Example 4.1.5. Let M be a smooth n -manifold. Consider the tangent bundle TM with group $O(n)$ and fiber \mathbb{R}^n . Let $f : M \rightarrow BO(n)$ denote the classifying map for $TM \xrightarrow{\pi} M$, so that $f^*\gamma_n = \pi$ where γ_n denotes the universal $O(n)$ -bundle. For each $i \in \{1, 2, \dots, n\}$, the i -th Stiefel-Whitney class for M is exactly the element

$$w_i(M) \equiv f^*w_i(E_{O(n)})$$

of $H^i(M, \mathbb{Z}_2)$. Note that w_i is natural in the sense that $w_i(f^*E_{O(n)}) = f^*w_i(E_{O(n)})$. Now, the complexification of π is the vector bundle $T_{\mathbb{C}}M$, with classifying map $f : M \rightarrow BU(n)$. In this case, the i -th Chern class for M is exactly the element

$$c_i(M) \equiv f^*c_i(E_{U(n)})$$

of $H^{2i}(M, \mathbb{Z})$. In general, any vector bundle ξ with group $O(n) \subset U(n)$ and fiber \mathbb{R}^n has a complexification $\xi \otimes \mathbb{C}$ with group $U(n)$ and fiber \mathbb{C}^n . Every fiber E_x of ξ is converted to a fiber $E_x \oplus E_x$ of $\xi \otimes \mathbb{C}$ with scalar multiplication given by $i \cdot (x, y) \equiv (-y, x)$. The i -th *Pontryagin class* for M is exactly

$$p_i(M) \equiv (-1)^i c_{2i}(TM \otimes \mathbb{C}).$$

For any two complex vector bundles ξ and η over X with groups $U(n)$ and $U(m)$, respectively, we can form a new vector bundle $\xi \oplus \eta$ with group $U(n+m)$ and fiber $\mathbb{C}^n \times \mathbb{C}^m$ such that

$$h_{\alpha\beta}(\xi \oplus \eta) \equiv \begin{bmatrix} h_{\alpha\beta}(\xi) & 0 \\ 0 & h_{\alpha\beta}(\eta) \end{bmatrix}.$$

Terminology. The sum $c(\xi) := 1 + c_1 + c_2 + \cdots + c_n \in H^*(X, \mathbb{Z})$ is called the *total Chern class* for η . If ξ is instead a real vector bundle, then we have the *total Stiefel-Whitney class* $w(\xi) := 1 + w_1 + w_2 + \cdots + w_n \in H^*(X, \mathbb{Z}_2)$.

Theorem 4.1.6 (Whitney sum formulas).

1. $c(\xi \oplus \eta) = c(\xi) \smile c(\eta)$, i.e., $c_k(\xi \oplus \eta) = \sum_{i+j=k} c_i(\xi) \smile c_j(\eta)$, $c_0 \equiv 1$.
2. $w(\xi \oplus \eta) = w(\xi) \smile w(\eta)$, i.e., $w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) \smile w_j(\eta)$, $w_0 \equiv 1$.

Exercise 4.1.7. Let ξ and η be fiber bundles over a CW complex X with group $U(1) \cong S^1$ and fiber \mathbb{C} , i.e., (complex) line bundles over X . Consider the tensor bundle $\xi \otimes \eta$.

- (a) Show that the set $\mathcal{LB}(X)$ of isomorphism classes of line bundles over X is an abelian group under \otimes .
- (b) Compute $c_1(\xi \otimes \eta)$ where c_1 denotes the first Chern class.

Proof.

- (a) It is easy to see that \otimes is commutative and associative. It is also easy to see that the natural isomorphism $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ extends to an isomorphism between $\xi \otimes \eta$ and the trivial line bundle $X \times \mathbb{C}$, which is thus the identity element of $\mathcal{LB}(X)$.

It remains to show that $\mathcal{LB}(X)$ has all inverses. To this end, consider the conjugate line bundle $\bar{\xi}$. If $h_{\beta\alpha} : U_\alpha \cap U_\beta \rightarrow S^1$ is any transition function for ξ , then the corresponding transition function for $\xi \otimes \bar{\xi}$ is given by

$$\tilde{h}_{\beta\alpha} := h_{\beta\alpha} \otimes \overline{h_{\beta\alpha}} : U_\alpha \cap U_\beta \rightarrow S^1, \quad x \mapsto h_{\beta\alpha}(x) \cdot \overline{h_{\beta\alpha}(x)}.$$

But this means that every transition function for $\xi \otimes \bar{\xi}$ is the constant map at 1. As a result,

$$\tilde{h}_\beta \circ \tilde{h}_\alpha^{-1}(x, f) = (x, f), \quad (\star)$$

i.e., the local trivializations \tilde{h}_α and \tilde{h}_β for $\xi \otimes \bar{\xi}$ agree on $U_\alpha \cap U_\beta$. By gluing together these local trivializations, we get a well-defined bundle isomorphism

$$\tilde{h} \equiv \bigcup_{\alpha} h_{\alpha} : \xi \otimes \bar{\xi} \xrightarrow{\cong} X \times \mathbb{C}.$$

This proves that $\bar{\xi}$ is the inverse of ξ .

- (b) Since $U(1)$ is a topological group, we can apply the Milnor construction to it to obtain the universal bundle

$$p_{U(1)} : L \rightarrow BU(1) = \mathbb{CP}^\infty.$$

For each $i \in \{1, 2\}$, let $\pi_i : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \rightarrow \mathbb{CP}^\infty$ denote the i -th projection. Consider now the tensor bundle

$$\tilde{L} := \underbrace{\pi_1^*(L)}_{L_1} \otimes \underbrace{\pi_2^*(L)}_{L_2}$$

over $\mathbb{CP}^\infty \times \mathbb{CP}^\infty$. Then $c_1(\tilde{L})$ belongs to $H := H^2(\mathbb{CP}^\infty \times \mathbb{CP}^\infty; \mathbb{Z})$. From lecture, we know that $H^*(\mathbb{CP}^\infty; \mathbb{Z})$ is precisely the infinite cyclic group generated by $c_1(L)$. In particular, it is a finitely-generated abelian group. Further, \mathbb{CP}^∞ is a CW complex as the colimit of CW complexes. Since $H^0(\mathbb{CP}^\infty; \mathbb{Z}) = \mathbb{Z}$ and $H^1(\mathbb{CP}^\infty; \mathbb{Z}) = 0$, it follows from the Künneth theorem that

$$H \cong (H^2(\mathbb{CP}^\infty; \mathbb{Z}) \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes H^2(\mathbb{CP}^\infty; \mathbb{Z})) \cong H^2(\mathbb{CP}^\infty; \mathbb{Z}) \oplus H^2(\mathbb{CP}^\infty; \mathbb{Z}). \quad (1)$$

Under this sequence of isomorphisms, $c_1(\tilde{L}) \in H$ corresponds uniquely to $(nc_1(L), mc_1(L))$ for some $n, m \in \mathbb{Z}$. Choose any point $P \in \mathbb{CP}^\infty$ and let $i : \{P\} \times \mathbb{CP}^\infty \hookrightarrow \mathbb{CP}^\infty \times \mathbb{CP}^\infty$ denote the inclusion map. Taking the pullback of L along i yields a line bundle i^*L over $\{P\} \times \mathbb{CP}^\infty \cong \mathbb{CP}^\infty$ that is isomorphic to L_2 viewed over \mathbb{CP}^∞ . Hence $c_1(i^*L) = c_1(L)$. But $i^*(c_1(L)) = mc_1(L)$, so that $c_1(L) = mc_1(L)$ by naturality of c_1 . It follows that $m = 1$. Similarly, we see that $n = 1$.

By inspection, $c_1(L_1) + c_1(L_2)$ also corresponds uniquely to $(c_1(L), c_1(L))$ under (1). This means that

$$c_1(L_1) + c_1(L_2) = c_1(\tilde{L}). \quad (2)$$

From (2), we can derive that $c_1(\xi \otimes \eta) = c_1(\xi) + c_1(\eta)$. Indeed, there must be maps $g_1, g_2 : X \rightarrow BU(1)$ such that $\xi \cong f_1^*L$ and $\eta \cong f_2^*L$. Note that the map $F := (f_1, f_2) : X \rightarrow BU(1) \times BU(1)$ satisfies

$$F^*(L_1) = F^*(\pi_1^*(L)) \cong (\pi_1 \circ F)^*(L) = f_1^*L \cong \xi \quad (3)$$

$$F^*(L_2) = F^*(\pi_2^*(L)) \cong (\pi_2 \circ F)^*(L) = f_2^*L \cong \eta.$$

Now, recalling that for any line bundle E over X , the local trivializations and transition functions of $F^*(E)$ are given by precomposition with F , we easily see that

$$c_1(F^*(L_1) \otimes F^*(L_2)) = c_1(F^*(L_1 \otimes L_2)). \quad (4)$$

Finally, by combining (2), (3), (4), and the naturality of c_1 , we get

$$\begin{aligned} c_1(\xi \otimes \eta) &= c_1(F^*(L_1) \otimes F^*(L_2)) \\ &= c_1(F^*(L_1 \otimes L_2)) \\ &= F^*(c_1(L_1 \otimes L_2)) \\ &= F^*(c_1(L_1) + c_1(L_2)) \\ &= F^*(c_1(L_1)) + F^*(c_1(L_2)) \\ &= c_1(F^*(L_1)) + c_1(F^*(L_2)) \\ &= c_1(\xi) + c_1(\eta). \end{aligned}$$

□