

## Abstract

We introduce the concept of a natural transformation in category theory, leading to equivalences and adjunctions. The main sources for this talk are the following.

- nLab.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 3.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 1.

## 1 Natural transformations

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F$  and  $G$  be functors  $\mathcal{C} \rightarrow \mathcal{D}$ . A *natural transformation*  $\phi : F \Rightarrow G$  is a function  $A \mapsto f_A$  from  $\text{ob } \mathcal{C}$  to  $\text{mor } \mathcal{D}$  such that  $f_A$  is a map  $F(A) \rightarrow G(A)$  and the following diagram commutes for any morphism  $h : A \rightarrow B$  in  $\mathcal{C}$ .

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ f_A \downarrow & & \downarrow f_B \\ GA & \xrightarrow{Gh} & GB \end{array}$$

In symbols, this may be written as  $f_B h_* = h_* f_A$ , where  $f_A$  is called a *component* of  $\phi$ .

**Note 1.1.** If every  $f_A$  is an isomorphism, then the maps  $(f_A)^{-1}$  define a natural transformation  $G \Rightarrow F$ .

If each  $f_A$  is an isomorphism, then we say that  $\phi$  is a *natural isomorphism*. Note that if  $\mathcal{D}$  is a groupoid (i.e., a category in which every morphism is an isomorphism), then  $\phi$  must be a natural isomorphism.

Let  $F$ ,  $G$ , and  $H$  be functors  $\mathcal{C} \rightarrow \mathcal{D}$ . The *identity natural transformation*  $\text{Id}_F : F \Rightarrow F$  is given by  $A \mapsto \text{Id}_{F(A)}$ . Moreover, given natural transformations  $\phi : F \rightarrow G$  and  $\psi : G \rightarrow H$ , define the *composite natural transformation*  $\psi \circ \phi$  by  $A \mapsto (\psi \circ \phi)_A := \psi_A \circ \phi_A$ .

**Lemma 1.2.** A natural transformation  $\phi : F \Rightarrow G$  is a natural isomorphism iff it has an inverse  $\phi^{-1} : G \Rightarrow F$ .

*Proof.* This follows from Note 1.1 along with our definition of a composite natural transformation.  $\square$

**Example 1.3.**

1. Let  $R$  and  $S$  be commutative rings. Any ring homomorphism  $f : R \rightarrow S$  induces a ring homomorphism  $\text{GL}_n(f) : \text{GL}_n(R) \rightarrow \text{GL}_n(S)$  satisfying

$$f(\det(A)) = \det \left( \text{GL}_n(f)(A) \right).$$

By viewing  $\mathrm{GL}_n$  and  $R \mapsto R^*$  as functors from **Ring** to **Grp** and  $\det_R : \mathrm{GL}_n(R) \rightarrow R^*$  as a morphism in **Grp**, we see that  $\det_R$  defines a natural transformation  $\phi : \mathrm{GL}_n \Rightarrow f^*$  where  $f^*$  denotes  $f \restriction_{R^*} : R^* \rightarrow S^*$ .

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{\mathrm{GL}_n(f)} & \mathrm{GL}_n(S) \\ \det_R \downarrow & & \downarrow \det_S \\ R^* & \xrightarrow{f^*} & S^* \end{array}$$

2. Consider the power set functor  $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$ , given on objects by  $A \mapsto \mathcal{P}(A)$  and on morphisms  $g$  by  $\mathcal{P}g(S) = g(S)$ . Then the function  $f_A : A \rightarrow \mathcal{P}(A)$  given by  $a \mapsto \{a\}$  defines a natural transformation  $\phi : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{P}$ .
3. Set  $\mathcal{C} = \mathcal{D} = \mathbf{Grp}$ ,  $F = \mathrm{Id}_{\mathcal{C}}$ , and  $G = (-)^{\mathrm{ab}}$ . Then given a group  $H$ , the natural projection  $f : H \twoheadrightarrow H^{\mathrm{ab}}$  induces a natural transformation  $\phi : F \Rightarrow G$ .
4. We can view preorders  $(P, \leq)$  and  $(Q, \leq)$  as small categories and functors  $F, G : P \rightarrow Q$  as order-preserving functions. Then there is a unique natural transformation  $\phi : F \Rightarrow G$  iff  $F(x) \leq G(x)$  for every  $x \in P$ .
5. The inversion isomorphism from a group  $G$  to its opposite group  $G^{\mathrm{op}}$  defines a natural transformation  $\phi : \mathrm{Id}_{\mathbf{Grp}} \Rightarrow ((-)^{\mathrm{op}} : \mathbf{Grp} \rightarrow \mathbf{Grp})$ . In this sense,  $G$  is naturally isomorphic to  $G^{\mathrm{op}}$ .

**Definition 1.4.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories with  $\mathcal{C}$  small. The *functor category*  $\mathbf{Fun}(\mathcal{C}, \mathcal{D}) := \mathcal{D}^{\mathcal{C}}$  has functors  $F : \mathcal{C} \rightarrow \mathcal{D}$  as objects and natural transformations as morphisms.

*Remark 1.5.* Any Grothendieck universe models ZFC, in particular **Replacement**. This ensure that for any two functors  $F, G : \mathcal{C} \rightarrow \mathcal{D}$ , the class of natural transformation  $\phi : F \Rightarrow G$  is a set. This means that  $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$  is locally small, a condition of our definition of a category.

**Definition 1.6.** Given a category  $\mathcal{C}$ , the *arrow category*  $\mathrm{Ar}(\mathcal{C})$  of  $\mathcal{C}$  has as objects morphisms  $f : X_0 \rightarrow X_1$  in  $\mathcal{C}$  and as morphisms  $M : (f : X_0 \rightarrow X_1) \rightarrow (g : Y_0 \rightarrow Y_1)$  the pairs  $(M_0, M_1)$  of morphisms  $M_0 : X_0 \rightarrow Y_0$  and  $M_1 : X_1 \rightarrow Y_1$  such that

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ M_0 \downarrow & & \downarrow M_1 \\ Y_0 & \xrightarrow{g} & Y_1 \end{array}$$

commutes.

**Note 1.7.**

1.  $\mathrm{Ar}(\mathcal{C}) \cong \mathbf{Fun}([1], \mathcal{C})$ .
2.  $\mathbf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathbf{Fun}(\mathcal{C}, \mathbf{Fun}(\mathcal{D}, \mathcal{E}))$ .

## 2 Equivalences

Usually, it is useful to make our notion of *sameness* between categories weaker than *isomorphism*.

**Definition 2.1.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an *equivalence* if there is a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$ , called the *quasi-inverse* of  $F$ , such that  $F \circ G \cong \text{Id}_{\mathcal{D}}$  and  $G \circ F \cong \text{Id}_{\mathcal{C}}$ . In this case, we say that  $F$  and  $G$  are *equivalent categories*. Moreover, we say that a property of  $\mathcal{C}$  is *categorical* if it is invariant under categorical equivalence.

**Example 2.2.** Let  $k$  be a field. Let the category  $\mathbf{Mat}_k$  have natural numbers as objects and morphisms  $n \rightarrow p$  given by  $p \times n$  matrices over  $k$ . Let  $\mathbf{fdMod}$  denote the category of finite-dimensional vector spaces with linear maps as morphisms. These two categories are equivalent. Indeed, send the natural number  $n$  to  $k^n$  in one direction and the space  $V$  to  $\dim V$  in the other direction.

**Definition 2.3.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *essentially surjective* if for each object  $Z$  of  $\mathcal{D}$ , there is some object  $Y$  of  $\mathcal{C}$  such that  $F(Y) \cong Z$ .

**Theorem 2.4.** A functor is an equivalence iff it is full, faithful, and essentially surjective.<sup>1</sup>

**Definition 2.5.** A *skeleton* of  $\mathcal{C}$  is a full subcategory  $\mathcal{C}' \subset \mathcal{C}$  such that each element of  $\text{ob } \mathcal{C}$  is isomorphic to exactly one element of  $\text{ob } \mathcal{C}'$ .

An application of Theorem 2.4 yields the following result.

**Lemma 2.6.** Let  $\mathcal{C}'$  be a skeleton of  $\mathcal{C}$ . Then the inclusion functor  $\mathcal{C}' \hookrightarrow \mathcal{C}$  is an equivalence.

**Lemma 2.7.** Any two skeleta  $\mathcal{C}', \mathcal{C}'' \subset \mathcal{C}$  are isomorphic.

*Proof.* Define  $F : \mathcal{C}' \rightarrow \mathcal{C}''$  on objects by  $F(X) = Y$  where  $X \cong Y$  via a chosen isomorphism  $h_X$  and on morphisms  $f \in \mathcal{C}(X, Y)$  by  $F(f) = h_Y \circ f \circ (h_X)^{-1}$ . To get  $F^{-1}$ , define  $G : \mathcal{C}'' \rightarrow \mathcal{C}'$  by similarly choosing an isomorphism  $(h_X)^{-1}$  for each  $X \in \text{ob } \mathcal{C}''$ .  $\square$

*Remark 2.8.* Both Lemma 2.6 and Lemma 2.7 are logically equivalent to the axiom of choice, as is the statement that every category admits a skeleton.

## 3 Adjunctions

**Definition 3.1.**

1. Let  $Z \in \text{ob } \mathcal{C}$ . Define the contravariant functor  $\mathcal{Y}_Z : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  on objects by  $Y \mapsto \mathcal{C}(Y, Z)$  and on morphisms by sending  $f : X \rightarrow Y$  in  $\mathcal{C}$  to the map  $f^* : \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$  given by  $g \mapsto gf$ .

We call  $\mathcal{C}(-, Z) := \mathcal{Y}^Z$  the set-valued functor *represented by*  $Z$  in  $\mathcal{C}$ .

2. Let  $X \in \text{ob } \mathcal{C}$ . Define the functor  $\mathcal{Y}^X : \mathcal{C} \rightarrow \mathbf{Set}$  on objects by  $Y \mapsto \mathcal{C}(X, Y)$  and on morphisms by sending  $g : Y \rightarrow Z$  to the map  $g_* : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$  given by  $f \mapsto gf$ .

We call  $\mathcal{C}(X, -) := \mathcal{Y}^X$  the set-valued functor *corepresented by*  $X$  in  $\mathcal{C}$ .

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<sup>1</sup>Theorem 3.2.10 (Rognes).

A functor of the form  $\mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{D}$  is called a *bifunctor*. In particular, define  $\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$  on objects by  $(X, X') \rightarrow \mathcal{C}(X, X')$  and on morphisms by sending  $(f, f') : (X, X') \rightarrow (Y, Y')$  to the map  $\mathcal{C}(f, f') : \mathcal{C}(X, X') \rightarrow \mathcal{C}(Y, Y')$  given by  $g \mapsto f'gf$ .

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be functors.

**Definition 3.2 (Kan).** Consider the set-valued bifunctors  $\mathcal{D}(F(-), -), \mathcal{C}(-, G(-)) : \mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$ . An *adjunction from  $F$  to  $G$*  is a natural isomorphism

$$\phi : \mathcal{D}(F(-), -) \Rightarrow \mathcal{C}(-, G(-)).$$

If such a  $\phi$  exists, then we say that  $(F, G)$  is an *adjoint pair (of functors)*.

Note that  $\phi$  is natural in the sense that for any map  $c : X' \rightarrow X$  in  $\mathcal{C}$  and  $d : Y \rightarrow Y'$  in  $\mathcal{D}$ , the square

$$\begin{array}{ccc} \mathcal{D}(FX, Y) & \xrightarrow{\phi_{X,Y}} & \mathcal{C}(X, GY) \\ c^* d_* \downarrow & & \downarrow c^* d_* \\ \mathcal{D}(FX', Y') & \xrightarrow{\phi_{X',Y'}} & \mathcal{C}(X', GY') \end{array}$$

commutes in  $\mathbf{Set}$ .

**Proposition 3.3.** *Left and right adjoints are both unique up to unique isomorphism.*

*Terminology.* We call  $F$  the *left adjoint* to  $G$  and  $G$  the *right adjoint* to  $F$ . In symbols,  $F \dashv G$ .

**Note 3.4.** It is straightforward to check that any adjoint triple  $F \dashv G \dashv H$  yields two new adjunctions:

$$\begin{aligned} GF &\dashv GH \\ FG &\dashv HG \end{aligned}$$

**Definition 3.5.** Given an adjunction  $\phi : \mathcal{D}(F(-), -) \Rightarrow \mathcal{C}(-, G(-))$ , define the *unit morphism*

$$\eta_X = \phi_{X, FX} (\text{Id}_{FX}) \in \mathcal{C}(X, GF(X))$$

and the *counit morphism*

$$\epsilon_Y = \phi_{GY, Y}^{-1} (\text{Id}_{GY}) \in \mathcal{D}(FG(Y), Y).$$

**Lemma 3.6.** *The unit morphisms  $(\eta_X)_{X \in \text{ob } \mathcal{C}}$  define a natural transformation  $\eta : \text{Id}_{\mathcal{C}} \Rightarrow GF$ , and the counit morphisms  $(\epsilon_Y)_{Y \in \text{ob } \mathcal{D}}$  define a natural transformation  $\epsilon : FG \Rightarrow \text{Id}_{\mathcal{D}}$ .*

*Proof.* For simplicity, let us just prove that  $\epsilon$  is a natural transformation. We must check that

$$\begin{array}{ccc} FG(Y) & \xrightarrow{FG(y)} & FG(Y') \\ \epsilon_Y \downarrow & & \downarrow \epsilon_{Y'} \\ Y & \xrightarrow{y} & Y' \end{array}$$

commutes for any map  $y : Y \rightarrow Y'$  in  $\mathcal{D}$ . By the naturality of  $\phi$ , we have that

$$\begin{aligned}
y \circ \epsilon_Y &= y \circ \phi^{-1}(\text{Id}_{GY}) \\
&= \phi^{-1}(Gy \circ \text{Id}_{GY}) \\
&= \phi^{-1}(\text{Id}_{GY'} \circ Gy) \\
&= \phi^{-1}(\text{Id}_{GY'}) \circ FG(y) \\
&= \epsilon_{Y'} \circ FG(y),
\end{aligned}$$

as required.  $\square$

Moreover, one can verify that the unit and counit of  $\phi$  satisfy the *triangle identities*,

$$\epsilon_{FX} \circ F\eta_X = 1_{FX} \quad (\Delta_1)$$

$$G\epsilon_Y \circ \eta_{GY} = 1_{GY}, \quad (\Delta_2)$$

for any  $X \in \text{ob } \mathcal{C}$  and  $Y \in \text{ob } \mathcal{D}$ .

Conversely, suppose that  $F$  and  $G$  come equipped with two natural transformations

$$\begin{aligned}
\eta : \text{Id}_{\mathcal{C}} &\Rightarrow GF \\
\epsilon : FG &\Rightarrow \text{Id}_{\mathcal{D}}
\end{aligned}$$

satisfying the triangle identities. Then we get an adjunction  $\phi$  from  $F$  to  $G$  with component

$$\phi_{X,Y} : \mathcal{D}(FX, Y) \rightarrow \mathcal{C}(X, GY), \quad f \mapsto Gf \circ \eta_X.$$

Indeed, define  $\psi_{X,Y} : \mathcal{C}(X, GY) \rightarrow \mathcal{D}(FX, Y)$  by  $g \mapsto \epsilon_Y \circ Fg$ . We have that

$$\begin{aligned}
\psi_{X,Y}(\phi_{X,Y}(f)) &= \psi_{X,Y}(Gf \circ \eta_X) \\
&= \epsilon_Y \circ F(Gf \circ \eta_X) \\
&= \epsilon_Y \circ F(Gf) \circ F\eta_X \\
&= f \circ \epsilon_{FX} \circ F\eta_X && \text{(naturality of } \epsilon) \\
&= f. && ((\Delta_1))
\end{aligned}$$

Likewise, we have that  $\phi_{X,Y}(\psi_{X,Y}(g)) = g$ . Hence  $\phi_{X,Y}$  is a natural isomorphism in both  $X$  and  $Y$  with inverse  $\psi_{X,Y}$ .

Even so,  $\mathcal{C} \xrightleftharpoons[F]{G} \mathcal{D}$  need *not* be an equivalence of categories, as  $\eta$  and  $\epsilon$  may not be isomorphisms. Further, a given equivalence  $\mathcal{C} \xrightleftharpoons[R]{L} \mathcal{D}$  of categories need *not* be an adjunction, as its associated natural transformations

$$\begin{aligned}
\eta' : \text{Id}_{\mathcal{C}} &\Rightarrow RL \\
\epsilon' : LR &\Rightarrow \text{Id}_{\mathcal{D}}
\end{aligned}$$

may not satisfy the triangle inequalities. Nevertheless,  $(L, R)$  is an adjoint pair with unit  $\eta'$  and counit another natural transformation defined in terms of  $\eta'$  and  $\epsilon'$ . By symmetry,  $(R, L)$  is also an adjoint pair.

**Example 3.7 (Monad).** Let  $(\mathcal{C}, \otimes, 1)$  be a monoidal category. A *monoid* in  $\mathcal{C}$  is an object  $M$  equipped with a *multiplication* map  $\mu : M \otimes M \rightarrow M$  and a *unit* map  $\eta : 1 \rightarrow M$  that satisfy certain coherence properties expressing that  $\mu$  is associative and that  $\eta$  is a two-sided identity. Given two monoids  $(M, \mu, \eta)$  and  $(M', \mu', \eta')$  in  $\mathcal{C}$ , a map  $f : M \rightarrow M'$  in  $\mathcal{C}$  is a *morphism of monoids* if it satisfies

$$f \circ \mu = \mu' \circ (f \otimes f) \quad f \circ \eta = \eta'.$$

A *comonoid*  $N$  in  $\mathcal{C}$  is a monoid in  $\mathcal{C}^{\text{op}}$ , equipped with a *comultiplication* map  $\delta : N \rightarrow N^2$  and a *counit* map  $\epsilon : N \rightarrow 1$ .

For example, a monoid in the monoidal category  $(\text{End}(\mathcal{C}), \circ, \text{Id}_{\mathcal{C}})$  of endofunctors of  $\mathcal{C}$  is called a *monad on  $\mathcal{C}$* . A comonoid in  $\text{End}(\mathcal{C})$  is called a *comonad on  $\mathcal{C}$* .

Explicitly, a monad on  $\mathcal{C}$  consists of an endofunctor  $T : \mathcal{C} \rightarrow \mathcal{C}$  together with two natural transformations  $\eta : \text{Id}_{\mathcal{C}} \rightarrow T$  and  $\mu : T^2 \rightarrow T$  such that the following diagrams commute:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ \mu_T \downarrow & & \downarrow \mu \\ T^2 & \xrightarrow{\mu} & T \end{array} \quad \begin{array}{ccc} T & \xrightarrow{\eta_T} & T^2 & \xleftarrow{T\eta} & T \\ & \searrow & \downarrow \mu & \swarrow & \\ & & T & & \end{array}.$$

These are precisely the *associativity* and *unit* laws, respectively. Now, let  $(F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C})$  be an adjoint pair with unit  $\eta : \text{Id}_{\mathcal{C}} \rightarrow G \circ F$  and counit  $\epsilon : F \circ G \rightarrow \text{Id}_{\mathcal{D}}$ . We then have a natural transformation  $(G \circ F)^2 \rightarrow G \circ F$  given componentwise by

$$G(\epsilon_{FX}) : GF GF X \rightarrow GF X$$

One can check that  $(G \circ F, \eta, G\epsilon_F)$  is a monad on  $\mathcal{C}$ .

Dually, a comonad  $R : \mathcal{C} \rightarrow \mathcal{C}$  on  $\mathcal{C}$  satisfies the relations

$$\begin{aligned} \delta_R \circ \delta &= R\delta \circ \delta \\ \epsilon_R \circ \delta &= \text{Id}_R = R\epsilon \circ \delta. \end{aligned}$$

Moreover, any adjoint pair  $(F : \mathcal{C} \rightarrow \mathcal{D}, G : \mathcal{D} \rightarrow \mathcal{C})$  with unit  $\eta$  and counit  $\epsilon$  induces a comonad  $(G, \epsilon, \delta)$  on  $\mathcal{D}$  where

$$\begin{aligned} G &\equiv F \circ G : \mathcal{D} \rightarrow \mathcal{D} \\ \delta &\equiv F\eta_G : G \rightarrow G^2. \end{aligned}$$

**Theorem 3.8.** *The category of monoids in  $\mathcal{C}$  is equivalent to the category of  $\mathcal{C}$ -enriched categories with one object.*

**Example 3.9.**

- (1) The forgetful functor  $U : \mathbf{Grp} \rightarrow \mathbf{Set}$  has a left adjoint  $F : \mathbf{Set} \rightarrow \mathbf{Grp}$  sending a set to the free group generated by  $A$ .
- (2) Let  $R$  be a ring. The forgetful functor  $U : R\text{-}\mathbf{Mod} \rightarrow \mathbf{Set}$  has a left adjoint  $R(-)$  sending a set  $S$  to  $\bigoplus_{s \in S} R$ , the free  $R$ -module generated by  $S$ .

The forgetful functor has no right adjoint in either Example 3.9(1) or Example 3.9(2). It does, however, have one in the following setting.

**Example 3.10.** The forgetful functor  $U : \mathbf{Top} \rightarrow \mathbf{Set}$  has a left adjoint that sends a set to the same set equipped with the discrete topology. It also has a right adjoint that sends a set to the same set equipped with the indiscrete topology.

**Definition 3.11.** A subcategory  $\mathcal{C} \subset \mathcal{D}$  is *reflective* if the inclusion functor has a left adjoint and is *coreflective* if the inclusion functor has a right adjoint.

**Example 3.12.**

1. The full subcategory  $\mathbf{Ab} \subset \mathbf{Grp}$  is reflexive as the inclusion functor is right adjoint to  $(-)^{\text{ab}}$ .
2. Let  $\mathbf{Ab}_T \subset \mathbf{Ab}$  denote the subcategory of torsion groups. This is coreflective as the inclusion functor is right adjoint to the functor sending an abelian group to its torsion subgroup.