

Abstract

We present solutions to all exercises from Scott Weinstein’s “Model Theory” course lectures at UPenn. These are relatively self-contained and are meant to complement Weinstein’s written memoirs of our class meetings. The official reference for the course is David Marker’s *Model Theory: An Introduction*.

1. Let I be a countably infinite set. Let $\mathbb{D} := \langle I, E \rangle$ be a structure where E is an equivalence relation for which there is exactly one equivalence class of size k for each $k \in \mathbb{Z}_{\geq 1}$.

- (1) Show that the set Λ of (first-order) sentences expressing that E is an equivalence relation with exactly one equivalence class of size k for each $k \in \mathbb{Z}_{\geq 1}$ axiomatizes \mathbb{D} , i.e., $\text{Th}(\mathbb{D}) = \text{Cn}(\Lambda)$ where

$$\text{Cn}(\Lambda) := \{\varphi \in \text{FO}_{\mathbb{D}} \mid \Lambda \models \varphi\}.$$

- (2) Show that for every (first-order) formula $\theta(y, \bar{w})$ and every $\bar{a} \in I$, the set

$$\theta[\mathbb{D}, \bar{a}] := \{x \in \text{dom}(\mathbb{D}) \mid \mathbb{D} \models \theta[x, \bar{a}]\}$$

is either finite or cofinite.

- (1) It suffices to prove that Λ is complete. For, in this case, any two models of Λ must be elementarily equivalent.

Claim 1. *Let \mathbb{E} be any model of Λ of size $\kappa \geq \omega$. There exists an elementary extension $\mathbb{E}_{\kappa} \succeq \mathbb{E}$ of size κ such that \mathbb{E}_{κ} has exactly κ equivalence classes each of size κ .*

Proof. Let λ denote the cardinality of the set of all equivalence classes in $\text{dom}(\mathbb{E})$. Note that $\lambda \leq \kappa$. For every $\alpha, \beta \in \kappa$, adjoin to the language of \mathbb{E} a new constant symbol $c(\alpha, \beta)$. Consider the theory

$$\Delta := \Lambda \cup \{Ec(x, y)c(x, z) \mid x, y, z \in \kappa\} \cup \{\neg Ec(x, 0)c(y, 0) \mid x, y \in \kappa, x \neq y\}.$$

Any finite subset F of Δ is satisfiable by a suitable expansion \mathbb{E}_F of \mathbb{E} . Then there exists an ultrafilter on the family of finite subsets of Δ such that the ultraproduct

$$\prod_{\substack{F \subset \Delta \\ \text{finite}}} \mathbb{E}_F / \mathcal{U}$$

satisfies Δ . Moreover, its reduct \mathbb{A} to the language of \mathbb{E} is an elementary extension of \mathbb{E} . By the downward Löwenheim-Skolem theorem, there exists a structure \mathbb{E}_0 of size κ such that $\mathbb{A} \succeq \mathbb{E}_0 \succeq \mathbb{E}$.

Now, repeat our preceding construction ω times to get an increasing chain

$$\mathbb{E} \preceq \mathbb{E}_0 \preceq \mathbb{E}_1 \preceq \mathbb{E}_2 \preceq \dots$$

of structures such that each $\text{dom}(\mathbb{E}_i)$ has cardinality κ . Note that \mathbb{E}_{κ} is an elementary extension of \mathbb{E} . Further, the domain of the direct limit $\mathbb{E}_{\kappa} := \bigcup_{i \in \omega} \mathbb{E}_i$ also has cardinality κ , so that \mathbb{E}_{κ} has exactly κ equivalence classes. Finally, for any $x \in \mathbb{E}_{\kappa}$, x belongs to some \mathbb{E}_n . Hence the equivalence class $[x]$ has size κ in \mathbb{E}_{n+1} and thus in \mathbb{E}_{κ} . It follows that every equivalence class in \mathbb{E}_{κ} has size κ . \square

Suppose, toward a contradiction, that there is a sentence φ in the language of \mathbb{D} such that neither φ nor $\neg\varphi$ belongs to $\text{Cn}(\Lambda)$. Then there are models \mathbb{A}^1 and \mathbb{A}^2 of Λ such that $\mathbb{A}^1 \models \neg\varphi$ and $\mathbb{A}^2 \models \varphi$. By the Löwenheim-Skolem theorem, we may assume that both of these have size $\kappa \geq \omega$. By Claim 1, we thus have two structures \mathbb{A}_{κ}^1 and \mathbb{A}_{κ}^2 such that $\mathbb{A}_{\kappa}^1 \models \neg\varphi$ and $\mathbb{A}_{\kappa}^2 \models \varphi$. But it’s easy to see that \mathbb{A}_{κ}^1 and \mathbb{A}_{κ}^2 must be isomorphic, which yields a contradiction.

- (2) Suppose, toward a contradiction, that there exist a formula $\theta(y, w_1, \dots, w_n)$ and an element $\bar{a} \in I$ such that $\theta[\mathbb{D}, \bar{a}]$ is both infinite and coinfinite. Adjoin to the language of \mathbb{D} new constant symbols $\bar{e} := (e_1, \dots, e_n)$, c , and d . For each $k \in \mathbb{Z}_{\geq 1}$, let $\lambda_k(x)$ denote the formula expressing that the equivalence class of x has cardinality $> k$. Now, consider the theory

$$\begin{aligned} \Gamma := \Lambda \cup \{ \lambda_k(c) \mid k \geq 1 \} \cup \{ \lambda_k(d) \mid k \geq 1 \} \\ \cup \{ \neg E e_i c \mid 1 \leq i \leq n \} \\ \cup \{ \neg E e_i d \mid 1 \leq i \leq n \} \\ \cup \{ \theta(c, \bar{e}), \neg \theta(d, \bar{e}) \} \end{aligned}$$

in our new language.

Let F be any finite subset of Γ . Since both $\theta[\mathbb{D}, \bar{a}]$ and $\neg \theta[\mathbb{D}, \bar{a}]$ are infinite by assumption, we can find an expansion of \mathbb{D} that satisfies F by interpreting \bar{e} as \bar{a} and both c and d as members of large enough equivalence classes. By the compactness theorem, it follows that there is some model \mathbb{C} of Γ , which must be infinite. Let \mathbb{C}' denote the reduct of \mathbb{C} to the language of \mathbb{D} . Thanks to the Löwenheim-Skolem theorem, we may assume that $\text{dom}(\mathbb{C}')$ is countable. Thus, the equivalence classes $[c^{\mathbb{C}}]$ and $[d^{\mathbb{C}}]$ are countable. Note that $e_i^{\mathbb{C}} \notin [c^{\mathbb{C}}] \cup [d^{\mathbb{C}}]$ for each $1 \leq i \leq n$. Therefore, there is an automorphism of \mathbb{C}' sending $c^{\mathbb{C}}$ to $d^{\mathbb{C}}$ and fixing each $e_i^{\mathbb{C}}$. But this contradicts the fact that $\mathbb{C}' \models \theta[c^{\mathbb{C}}, \bar{e}^{\mathbb{C}}] \wedge \neg \theta[d^{\mathbb{C}}, \bar{e}^{\mathbb{C}}]$. ■

Definition 1 (Categoricity). For any cardinal κ , we say that a theory T is κ -categorical if any two models of T of size κ are isomorphic.

2. Show that a \mathcal{L} -structure \mathbb{A} is finite if and only if for any \mathcal{L} -structure \mathbb{B} ,

$$\mathbb{A} \equiv \mathbb{B} \iff \mathbb{A} \cong \mathbb{B}.$$

Remark. This shows that any complete theory with a finite model is κ -categorical for *any* cardinal κ .

(\implies)

It is always true that any two isomorphic structures are elementarily equivalent. Thus, it remains to show that $\mathbb{A} \equiv \mathbb{B} \implies \mathbb{A} \cong \mathbb{B}$.

First, assume that \mathcal{L} is finite. Consider the *atomic diagram* of \mathbb{A} , i.e., the set

$$D(\mathbb{A}) := \{ \varphi \mid \mathbb{A} \models \varphi, \varphi \text{ is either atomic or the negation of an atomic formula} \}$$

where \mathbb{A} denotes the expansion of \mathbb{A} obtained by adjoining a constant symbol c_a for each $a \in \text{dom}(\mathbb{A})$. Since both \mathcal{L} and $\text{dom}(\mathbb{A})$ are finite, we can encode $D(\mathbb{A})$ with a single sentence ψ . Therefore, the sentence

$$\psi_{\mathbb{A}} := \forall x \left(\bigvee_{a \in \text{dom}(\mathbb{A})} x = c_a \right) \wedge \psi$$

has the property that $\mathbb{B} \models \psi_{\mathbb{A}} \implies \mathbb{B} \cong \mathbb{A}$ for any other \mathcal{L} -structure \mathbb{B} . Now, if $\mathbb{A} \equiv \mathbb{B}$, then clearly both \mathbb{A} and \mathbb{B} satisfy $\psi_{\mathbb{A}}$, so that $\mathbb{B} \cong \mathbb{A}$.

Next, let \mathcal{L} be arbitrary and let $\mathbb{A} \equiv \mathbb{B}$. Suppose, toward a contradiction, that $\mathbb{A} \not\cong \mathbb{B}$. Then for any bijection $f : \text{dom}(\mathbb{A}) \rightarrow \text{dom}(\mathbb{B})$, there is some finite sublanguage \mathcal{L}_f of \mathcal{L} such that f is *not* an isomorphism $\mathbb{A}^{\mathcal{L}_f} \rightarrow \mathbb{B}^{\mathcal{L}_f}$ of reducts to \mathcal{L}_f . Consider the language

$$\mathcal{L}' := \bigcup_{\substack{f : \text{dom}(\mathbb{A}) \rightarrow \text{dom}(\mathbb{B}) \\ \text{bijection}}} \mathcal{L}_f,$$

which is finite as the finite union of finite sets. Thanks to our preceding discussion, we obtain an isomorphism $g : \mathbb{A}^{\mathcal{L}'} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}'}$. But $\mathcal{L}_g \subset \mathcal{L}'$ by our construction of \mathcal{L}' , and thus g induces an isomorphism $\mathbb{A}^{\mathcal{L}_g} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}_g}$, contrary to our choice of \mathcal{L}_g .

(\Leftarrow)

Suppose that \mathbb{A} is infinite. We must find a structure \mathbb{B} such that $\mathbb{A} \equiv \mathbb{B}$ but $\mathbb{A} \not\equiv \mathbb{B}$. But this follows at once from the Löwenheim-Skolem theorem, which implies that $\text{Th}(\mathbb{A})$ has a model of any infinite size. ■

Definition 2 (Ehrenfeucht-Fraïssé game). Suppose that \mathcal{L} is a finite language without function symbols. Let \mathbb{D} and \mathbb{E} be two \mathcal{L} -structures. Let $n \in \omega$. The *Ehrenfeucht-Fraïssé game* $\text{EF}_n(\mathbb{D}, \mathbb{E})$ of length n on \mathbb{D} and \mathbb{E} is a game of perfect information played as follows.

- (a) There are exactly two players, the *spoiler* and the *duplicator*.
- (b) There are exactly n rounds.
- (c) The spoiler begins round $k \leq n$ by picking an element (sometimes called a pebble) of either $\text{dom}(\mathbb{D})$ or $\text{dom}(\mathbb{E})$. Next, the duplicator picks an element of the other domain.
- (d) This yields two sequences (d_1, \dots, d_n) and (e_1, \dots, e_n) such that $d_i \in \text{dom}(\mathbb{D})$ and $e_i \in \text{dom}(\mathbb{E})$ for each $i = 1, \dots, n$. If the mapping $d_i \mapsto e_i$ defines an isomorphism of finite substructures, then we say that the duplicator has won $\text{EF}_n(\mathbb{D}, \mathbb{E})$. Otherwise, we say that the spoiler has won.

Theorem 3 (Fraïssé). *The duplicator has a winning strategy in $\text{EF}_n(\mathbb{D}, \mathbb{E})$ for each $n \in \omega$ if and only if $\mathbb{D} \equiv \mathbb{E}$.*

3. Let $\mathbb{N}^* = \langle \omega, < \rangle$. Show that for any infinite cardinal κ , $\text{Th}(\mathbb{N}^*)$ is *not* κ -categorical.

Expand the language of \mathbb{N}^* by adjoining countably many constants $\{c_i\}_{i \in \mathbb{Z}}$. Consider the theory

$$T := \text{Th}(\mathbb{N}^*) \cup \{c_i > c_{i+1} \mid i \in \mathbb{Z}\}. \quad (\star)$$

in our new language. Any finite subset of T is satisfied by an expansion of \mathbb{N}^* suitably interpreting the c_i since \mathbb{N}^* has descending chains of all finite lengths. By the compactness theorem, it follows that there is some model \mathbb{A} of T , which must be infinite. If $|\mathbb{A}| > \aleph_0$, then apply the Löwenheim-Skolem theorem to get a model \mathbb{B} of T such that $|\mathbb{B}| = \aleph_0$. Let

$$\mathbb{A}' = \begin{cases} \mathbb{B} & |\mathbb{A}| > \aleph_0 \\ \mathbb{A} & |\mathbb{A}| = \aleph_0 \end{cases}.$$

Note that $\mathbb{A}' \models T$. Since the property of being a linearly ordered set is expressible by a first-order sentence, we see that \mathbb{A}' is linearly ordered by $<$. Further, we see that \mathbb{A}' has an infinite descending chain, which means that \mathbb{A}' is not well-ordered by $<$. But $(\omega, <)$ is a well-ordered set, and thus the reduct of \mathbb{A}' to the language of \mathbb{N}^* is not isomorphic to \mathbb{N}^* . It does, however, satisfy $\text{Th}(\mathbb{N}^*)$. This shows that $\text{Th}(\mathbb{N}^*)$ is not \aleph_0 -categorical.

Unfortunately, it's unclear that this method can be adapted to show that $\text{Th}(\mathbb{N}^*)$ is not κ -categorical when κ is uncountable. In this case, we instead shall employ two binary operations on the class of all linear orderings. Let L_1 and L_2 be linearly ordered sets.

- L_1^{op} refers to L_1 equipped with the inverse order.
- $L_1 \otimes L_2$ refers to $L_1 \times L_2$ equipped with the lexicographic order.
- $L_1 \oplus L_2$ refers to L_1 with its ordering followed by L_2 with its ordering.

Now, consider the following linearly ordered structures:

$$\begin{aligned} & \mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa) \\ & \mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa)), \end{aligned}$$

both of which have cardinality κ . These orderings possess bottom elements and are *discrete* in the sense that both structures satisfy the sentences

$$\begin{aligned} & \forall x \exists y (x < y \wedge \neg \exists z (x < z \wedge z < y)) \\ & \forall x (\exists w (w < x) \rightarrow \exists y (y < x \wedge \neg \exists z (y < z \wedge z < x))). \end{aligned} \tag{1}$$

(Informally, we can view y here as the *successor/predecessor* of x .) Note that, on the one hand, $\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa)$ cannot possess an descending chain of length ω^2 , for otherwise κ , which is well-ordered, would possess an infinite descending chain. On the other hand, $\mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa))$ does possess such a chain since ω^* (the order type of $\mathbb{Z}_{<0}$) can be embedded in \mathbb{Q} . Therefore,

$$\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa) \not\equiv \mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa)).$$

Claim 2. Suppose that $(\mathbb{E}, <)$ is a discrete linear ordering with a bottom element but no top element. Then $\mathbb{E} \equiv \mathbb{N}^*$.

Proof sketch. Consider the Ehrenfeucht-Fraïssé game $\text{EF}_n(\mathbb{E}, \mathbb{N}^*)$. The duplicator has a winning strategy in $\text{EF}_n(\mathbb{E}, \mathbb{N}^*)$ by adhering to the following rules.

- (i) If, in round m , the spoiler chooses an element of one of the structures that is connected to a previously chosen element or the bottom element by a path of successors of length $k < \infty$, then choose the corresponding element of the other structure in round m .
- (ii) Otherwise, make sure that any chosen element of $\text{dom}(\mathbb{N}^*)$ is always separated by at least $n+1$ elements from any previously chosen element of $\text{dom}(\mathbb{N}^*)$ while preserving the required order of your choices.

In this case, choose first a natural number separated by more than 3^n elements from the greatest previously chosen element of $\text{dom}(\mathbb{N}^*)$.

□

Thanks to Theorem 3, it follows that both $\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa)$ and $\mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa))$ are elementarily equivalent to \mathbb{N}^* and thus models of $\text{Th}(\mathbb{N}^*)$. Hence $\text{Th}(\mathbb{N}^*)$ is not κ -categorical. ■

4. Show that any set definable over \mathbb{N}^* is either finite or cofinite.

Remark. This shows that \mathbb{N}^* is *o-minimal* in the sense that every definable set over \mathbb{N}^* is a finite union of points and intervals in ω .

Note that any set definable over \mathbb{N}^* is 0-definable because any natural number n is uniquely determined by the first-order property

$$\begin{cases} \text{“}n \text{ is less than any other element”} & n = 0 \\ \text{“there are exactly } n - 1 \text{ elements between 0 (the bottom element) and } n \text{”} & n > 1 \end{cases}$$

Suppose, toward a contradiction, that there exist a formula $\theta(y)$ such that $\theta[\mathbb{N}^*]$ is both infinite and coinfinite. Consider, again, the theory (\star) . Let

$$T' = T \cup \{\theta(c_0), \neg\theta(c_1)\}.$$

Since both $\theta[\mathbb{N}^*]$ and $\neg\theta[\mathbb{N}^*]$ are infinite by assumption, we can find an expansion of \mathbb{N}^* that satisfies any finite subset of T' . By the compactness theorem together with the Löwenheim-Skolem theorem, we thus can find a countable model \mathbb{D} of T' and take its reduct \mathbb{C} to the language of \mathbb{N}^* . Note that $(\text{dom}(\mathbb{C}), <)$ is a

countable linear ordering with an infinite descending and ascending chain χ on which both $c_0^{\mathbb{D}}$ and $c_1^{\mathbb{D}}$ lie. Moreover, this ordering is discrete in the sense of (1). Therefore, we may assume that χ has the form

$$\cdots < x_{m-1} < x_m < x_{m+1} < \cdots$$

where x_{m+1} denotes the immediate successor of x_m . There is an automorphism of \mathbb{C} mapping $c_0^{\mathbb{D}}$ to $c_1^{\mathbb{D}}$ by suitably shifting χ finitely many places to the left and fixing all elements outside χ . But this contradicts the fact that $\mathbb{C} \models \theta[c_0^{\mathbb{D}}] \wedge \neg\theta[c_1^{\mathbb{D}}]$. \blacksquare

5. Consider the theory DLO of the dense linear ordering without endpoints. For any uncountable cardinal κ , show that there are 2^κ many models of DLO up to isomorphism.

Remark. This shows that DLO is *not* κ -categorical even though it is \aleph_0 -categorical.

Consider the linear orderings

$$\begin{aligned} L_1 &:= \mathbb{Q} \otimes (\omega_1^{\text{op}} \oplus \omega_1) \\ L_2 &:= \mathbb{Q} \otimes (1 \oplus \omega_1^{\text{op}} \oplus \omega_1). \end{aligned}$$

Now, by replacing each $\alpha \in \kappa$ with a choice of L_1 or L_2 , we obtain 2^κ many dense linear orderings $\{P_\beta\}_{\beta < 2^\kappa}$ without endpoints such that $|P_\beta| = \kappa$ for ever β . It remains to show that these are pairwise non-isomorphic.

To this end, suppose that there is an isomorphism $f : P_\beta \xrightarrow{\cong} P_{\beta'}$. By construction, both P_β and $P_{\beta'}$ consist of κ -sequences

$$\begin{aligned} L_{i_0} &< L_{i_1} < \cdots < L_{i_\alpha} < \cdots \\ L_{i'_0} &< L_{i'_1} < \cdots < L_{i'_\alpha} < \cdots, \end{aligned}$$

respectively, where $i_\alpha, i'_\alpha \in \{1, 2\}$. Since any isomorphism of well-ordered sets is unique, we see that the function $f \upharpoonright_{L_{i_\alpha}}$ is an isomorphism $L_{i_\alpha} \xrightarrow{\cong} L_{i'_\alpha}$ for any $\alpha \in \kappa$.

Claim 3. $L_1 \not\cong L_2$.

Proof. On the one hand, L_1 has a suborder isomorphic to ω_1^{op} with no lower bound in L_1 . On the other hand, any such suborder of L_2 has a lower bound in L_2 . Hence there is no isomorphism from L_1 to L_2 . \square

It follows that $L_{i_\alpha} = L_{i'_\alpha}$ for every $\alpha \in \kappa$, which completes our proof. \blacksquare

Definition 4. Let T be a theory and let $\Gamma(\bar{x})$ be a set of formulas in free variables x_1, \dots, x_n . We say that Γ is an n -type over T if for any finite subset $\Delta \subset \Gamma$, the expanded theory

$$T \cup \{(\exists \bar{x}) \bigwedge \Delta\}$$

is satisfiable.

Notation. Let \mathbb{M} be an \mathcal{L} -structure and let $A \subset \text{dom}(\mathbb{M})$. Let $\mathcal{L}_A = \mathcal{L} \cup \{c_a \mid a \in A\}$ and let \mathbb{M}_A denote the \mathcal{L}_A -structure induced by \mathbb{M} . Then $\mathbb{S}_n^{\mathbb{M}}(A)$ refers to the set of all complete n -types over $\text{Th}_A(\mathbb{M}) := \text{Th}(\mathbb{M}_A)$.

Definition 5 (Stability). Let T be a complete theory in \mathcal{L} and let κ be an infinite cardinal. We say that T is κ -stable if whenever $\mathbb{M} \models T$, $A \subset \text{dom}(\mathbb{M})$, and $|A| = \kappa$, we have that $|\mathbb{S}_n^{\mathbb{M}}(A)| = \kappa$.

6. Let \mathbb{A} be a structure and $\theta(x, y)$ be a formula in the language of \mathbb{A} . Suppose that $B \subset \text{dom}(\mathbb{A})$ is an infinite set on which $\theta[A]$ is a linear order \prec . Show that $\text{Th}(\mathbb{A})$ is *not* ω -stable (i.e., \aleph_0 -stable).

Thanks to the axiom of dependent choice, we can find a countably infinite chain of at least one of the following two forms.

$$\begin{aligned} a_0 \prec b_0 \prec a_1 \prec b_1 \prec a_2 \prec b_2 \prec \dots \\ \dots \prec b_2 \prec a_2 \prec b_1 \prec a_1 \prec b_0 \prec a_0 \end{aligned}$$

with $a_i, b_i \in B$ for each $i = 0, 1, 2, \dots$. Without loss of generality, assume that we can find the former kind of chain and that θ has the form $x \prec y$. In this case,

$$\mathbb{A} \models \theta[a_i, b_j] \iff i \leq j. \quad (*)$$

Claim 4. *There exist sequences $(a_x)_{x \in 2^{\aleph_0}}$ and $(b_x)_{x \in 2^{\aleph_0}}$ along with a model \mathbb{A}' of $\text{Th}(\mathbb{A})$ such that*

$$\mathbb{A}' \models \theta[a_x, b_y] \iff x \leq y.$$

Proof. Adjoin to the language of \mathbb{A} two new constant symbols c_x and d_y for every $x, y \in 2^{\aleph_0}$. Consider the theory

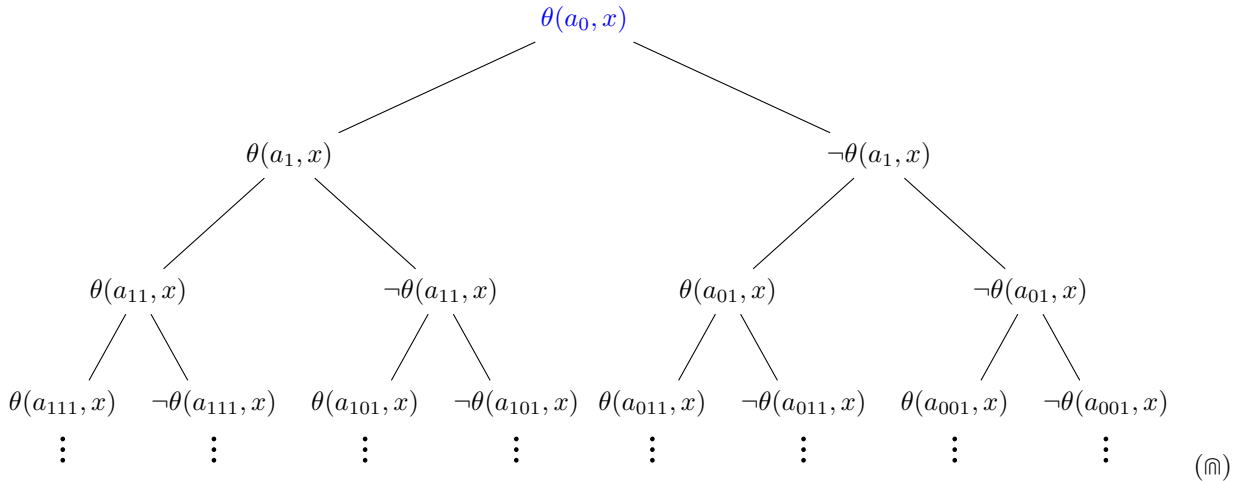
$$\Gamma := \text{Th}(\mathbb{A}) \cup \{\theta(c_x, d_y) \mid x, y \in 2^{\aleph_0}, x \leq y\} \cup \{\neg\theta(c_x, d_y) \mid x, y \in 2^{\aleph_0}, x > y\}.$$

in our expanded language. In light of $(*)$, any finite subset of Γ is satisfiable by a suitable expansion of \mathbb{A} . Thus, by the compactness theorem, Γ has a model \mathbb{B} . Finally, let \mathbb{A}' denote the reduct of \mathbb{B} to the language of \mathbb{A} . \square

Instead of indexing the sequences (a_x) and (b_x) by $(2^{\aleph_0}, \leq)$, let us index them by the set of all 2^{\aleph_0} -indexed binary strings σ under the string order $<$. We have that

$$\mathbb{A}' \models \theta[a_\sigma, b_{\sigma'}] \iff \sigma \leq \sigma'.$$

Consider the countably infinite subset $X := \{a_\sigma \mid \sigma \in 2^{\aleph_0}\}$ of $\text{dom}(\mathbb{A}')$. By recursion, we can build a binary tree of the form



with height ω . We call nodes of the form $\theta(a_\sigma, x)$ *positive* and those of the form $-\theta(a_\sigma, x)$ *negative*. Let U denote any branch of (m) . Let U_p denote the set of all strings $\sigma \in 2^{\aleph_0}$ such that a_σ occurs in a positive node of U . Since U_p is countable, it has an upper bound in $(2^{\aleph_0}, <)$. Since $(2^{\aleph_0}, <)$ is a complete order and 2^{\aleph_0} is a limit ordinal, it follows that U_p has a supremum τ in 2^{\aleph_0} . By construction of (m) , if $\theta(a_\sigma, x)$ is a positive node of U and $-\theta(a_{\sigma'}, x)$ is a negative one, then $\sigma' > \sigma$. Hence $\tau \leq \sigma'$ for any σ' occurring in a negative node of U . As a result, we see that $\mathbb{A}' \models \varphi[a_\sigma, b_\tau]$ for any node φ of U .

Therefore, every branch of (m) determines a unique 1-type over $\text{Th}_Y(\mathbb{A}')$ where

$$Y := \{x \in X \mid x \text{ occurs in a node of } (\text{m})\}.$$

This shows that $|\mathbb{S}_1^{\mathbb{A}'}(Y)| = 2^{\aleph_0} > \aleph_0$. But (\mathfrak{M}) has exactly

$$\left| \bigcup_{n \in \omega} 2^n \right| = \aleph_0$$

many nodes, so that $|Y| = \aleph_0$. Hence $\text{Th}(\mathbb{A})$ is not ω -stable. ■

Informally, an *abstract logic* L consists of a set of L -sentences together with a satisfaction relation \models_L between structures and L -sentences.

Definition 6 (Löwenheim-Skolem property). We say that L has the *Löwenheim-Skolem property* if any countable satisfiable L -theory has a model of size $\leq \aleph_0$.

7. Consider the extension $L(Q_0)$ of first-order logic by the *generalized quantifier* $\exists^{<\omega}$ signifying “there are finitely many.” Formally,

$$\mathbb{A} \models (Q_0 x) \varphi(x) \iff |\{a \in \text{dom}(\mathbb{A}) \mid \mathbb{A} \models \varphi[a]\}| < \aleph_0.$$

Show that $L(Q_0)$ has the Löwenheim-Skolem property.

Without loss of generality, consider $L(Q_0)$ with $\exists^{<\omega}$ replaced by $\exists^\infty := \neg \exists^{<\omega}$. We have the following version of the Tarski-Vaught elementary submodel criterion.

Claim 5. Let \mathbb{B} be a structure for $L(Q_0)$ and \mathbb{A} be a submodel of \mathbb{B} . Suppose that for any formula $\varphi(\bar{x}, y)$ and any $\bar{a} \in \text{dom}(\mathbb{A})$,

$$\begin{aligned} \{b \in \text{dom}(\mathbb{B}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\} \neq \emptyset &\implies \{b \in \text{dom}(\mathbb{A}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\} \neq \emptyset \\ |\{b \in \text{dom}(\mathbb{B}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\}| \geq \aleph_0 &\implies |\{b \in \text{dom}(\mathbb{A}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\}| \geq \aleph_0 \end{aligned}$$

Then $\mathbb{A} \preceq_{L(Q_0)} \mathbb{B}$.

Proof sketch. This is easily proved by induction on the complexity of formulas just as it is for first-order logic. □

Now, suppose that Γ is a countable $L(Q_0)$ -theory with an infinite model \mathbb{M} . It suffices to show that for any $X \subset \text{dom}(\mathbb{M})$, there is an elementary submodel \mathbb{M}' of \mathbb{M} such that $X \subset \text{dom}(\mathbb{M}')$ and $|\mathbb{M}'| = |X| + \aleph_0$. To this end, inductively construct an ω -sequence

$$X := X_0 \subset X_1 \subset X_2 \subset \dots$$

of subsets of $\text{dom}(\mathbb{M})$ such that $|X_i| = |X| + \aleph_0$ for every $i \in \omega$ as follows. Suppose that we have already defined X_i as desired. For every formula $\varphi(\bar{x}, y)$ and any $\bar{a} \in X_i$, consider the set

$$F_{\varphi, \bar{a}} := \{b \in \text{dom}(\mathbb{M}) \mid \mathbb{M} \models \varphi[\bar{a}, b]\}.$$

By the axiom of choice, we can find a set of the form

$$\tilde{F}_{\varphi, \bar{a}} := \begin{cases} F_{\varphi, \bar{a}} & F_{\varphi, \bar{a}} \text{ is finite} \\ \text{a chosen countably infinite subset of } F_{\varphi, \bar{a}} & \text{otherwise} \end{cases}.$$

Now, let

$$X_{i+1} = X_i \cup \bigcup_{\varphi, \bar{a}} \tilde{F}_{\varphi, \bar{a}}.$$

Since there are countably many formulas and, by induction, $|X| + \aleph_0$ many $\bar{a} \in X_i$, we deduce that X_{i+1} has cardinality $|X| + \aleph_0$.

It is easy to see that the union $\mathbb{M}' := \bigcup_{i \in \omega} X_i$ forms an elementary submodel of \mathbb{M} thanks to Claim 5. Further, we have that $|\mathbb{M}'| = |X| + \aleph_0 + \aleph_0 = |X| + \aleph_0$, as desired. ■

Let $X \subset \omega$. We write $\phi_e^X : W_e^X \rightarrow \{0, 1\}$ for the partial function computed by the Turing machine with index e and access to an oracle for X .

Definition 7 (Computability). Let $Y \subset \omega$.

1. We say that Y is *computable/recursive relative to X* if the characteristic function χ_Y equals ϕ_e^X for some index e .
2. We say that Y is *computably/recursively enumerable relative to X* if $Y = W_e^X$ for some index e .

If $Y = \emptyset$, then we omit “relative to X .”

Let \mathcal{C} denote any collection of computably enumerable sets. We say that a set $B \subset \{0, 1\}^*$ of binary strings is a *weak index set for \mathcal{C}* if $\mathcal{C} = \{W_e \mid e \in B\}$.

8. Let REC denote the collection of all recursive sets. Show that REC has a computably enumerable weak index set.

By mapping all invalid encodings of Turing machines to a distinguished trivial Turing machine, we may assume that our binary representation scheme

$$\langle - \rangle : \{a \mid a \text{ is a TM}\} \rightarrow \{0, 1\}^*$$

of Turing machines is surjective, i.e., every binary string represents a Turing machine. Therefore, we may computably enumerate all Turing machines

$$M_1 < M_2 < M_3 < \dots$$

according to the string order $<$. With this in mind, construct an enumerator E that prints, for each Turing machine M_i , the binary representation of a new Turing machine M'_i given as follows.

Algorithm 1: pseudocode describing M'_i

Input: the binary string x

```

1 run  $U_{\text{TM}}$  on  $\langle M_i, x \rangle$ ;
2 if  $U_{\text{TM}}$  rejects then
3   | reject
4 else
5   | run  $U_{\text{TM}}$  on  $\langle M_i, y \rangle$  for all strings  $y$  such that  $|y| \leq |x|$ ;
6   | if  $U_{\text{TM}}$  halts for each such  $y$  then
7   |   | accept
8   | else
9   |   | reject
10  | end
11 end
```

It is easy to see that

$$L(M'_i) = \begin{cases} L(M_i) & M_i \text{ is total} \\ \text{a finite set} & \text{otherwise} \end{cases}.$$

We claim that the language $\{\langle M'_i \rangle \mid i \in \mathbb{Z}_{\geq 1}\}$ enumerated by E is a weak index set for REC, i.e.,

$$\text{REC} = \{L(M'_i) \mid i \in \mathbb{Z}_{\geq 1}\}.$$

Indeed, if Y is recursive, then there is some Turing machine M_k deciding it, in which case $Y = L(M_k) = L(M'_k)$. Conversely, for any language of the form $L(M'_k)$, M_k is either total or non-total. If it is total, then $L(M'_k) = L(M_k)$ and $L(M_k)$ is recursive. If it is non-total, then $L(M'_k)$ is finite and thus recursive.

Remark. We could *not* have used the set $S := \{D_1, D_2, D_3, \dots\}$ of all deciders as our weak index set for Rec, for S is not computably enumerable. Indeed, suppose, toward a contradiction, that S is computably enumerable. Enumerate all binary strings

$$w_1 < w_2 < w_3 < \dots .$$

We now can construct a Turing machine N such that for each integer $i \geq 1$, N accepts w_i if D_i rejects it and rejects w_i if D_i accepts it. Then $L(N)$ is a decidable language, but $N \notin S$ by construction, a contradiction. ■