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Abstract

This is an introduction to universal properties in category theory. The main sources for this talk are the following.

- nLab.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 4.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4.

1 Universal arrows

Definition 1.1. An object X of $\mathscr{C}$ is *initial* if for each $Y \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $f : X \to Y$. Moreover, we say that X is *terminal* if for each $Z \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $g : Z \to X$. Either condition is called a *universal property* of X.

Any property P of $\mathscr{C}$ has a dual property P^{op} of $\mathscr{C}^{\mathrm{op}}$ obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P. Then P is true of $\mathscr{C}$ iff P^{op} is true of $\mathscr{C}^{\mathrm{op}}$.

Example 1.2. Being initial and being terminal are dual properties.

Lemma 1.3. *Any two initial objects of $\mathscr{C}$ are canonically isomorphic. The same holds for any two terminal objects of $\mathscr{C}$.*

Proof. Let X and X' be two initial objects. Compose the two unique morphisms $X \to X'$ and $X' \to X$ to get an isomorphism between X and X'. Apply duality to this argument for the case of terminal objects. □

We can think of a universal property as follows. Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $X \in \operatorname{ob}\mathscr{C}$. A *universal arrow from X to F* is an ordered pair (Y, f) with $Y \in \operatorname{ob}\mathscr{D}$ and $f : X \to F(Y)$ a morphism of $\mathscr{C}$ with the property that for any $X' \in \operatorname{ob}\mathscr{D}$ and morphism $f' : X \to F(X')$ of $\mathscr{C}$, there exists a unique morphism $\hat{f} : Y \to X'$ of $\mathscr{D}$ such that $F\left(\hat{f}\right) \circ f = f'$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & F(Y) \arrow[d, dashed, "F(\hat{f})"] \\ & F(X') \end{tikzcd}$$

Dually, a *universal arrow from F to X* is an ordered pair (Y, f) with $Y \in \operatorname{ob}\mathscr{D}$ and $f : F(Y) \to X$ of $\mathscr{C}$ with the property that for any $X' \in \operatorname{ob}\mathscr{D}$ and morphism $f' : F(X') \to X$, there exists a unique morphism

$\hat{f} : X' \to Y$ such that $f' = f \circ F\left(\hat{f}\right)$.

$$\begin{tikzcd} F(X') \arrow[r, dashed, "F(\hat{f})"] \arrow[dr, "f'"'] & F(Y) \arrow[d, "f"] \\ & X \end{tikzcd}$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

Definition 1.4.

1. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor and $Y \in \operatorname{ob}\mathscr{D}$. The *slice* or *left fiber category*, denoted by (F/Y) or $(F \downarrow Y)$, has as objects pairs (X, f) where $f : F(X) \to Y$ and as morphisms from $f : F(X) \to Y$ to $f' : F(X') \to Y$ morphisms $g : X \to X'$ such that $f = f' \circ F(g)$.

2. The *coslice* or *right fiber category*, denoted by (Y/F) or $(Y \downarrow F)$, has as objects pairs (X, f) where $f : Y \to F(X)$ and as morphisms from $f : Y \to F(X)$ to $f' : Y \to F(X')$ morphisms $g : X \to X'$ such that $f' = F(g) \circ f$.

If $F^{\mathrm{op}} : C^{\mathrm{op}} \to D^{\mathrm{op}}$ is opposite to the functor $F : \mathscr{C} \to \mathscr{D}$ and $Y \in \operatorname{ob}\mathscr{D}$, then $(Y/F)^{\mathrm{op}} = F^{\mathrm{op}}/Y$. Thus, the left and right fiber categories are dual in the sense that $P(Y, F)$ is true of any right fiber category Y/F iff $P^{\mathrm{op}}(Y, F)$ is true of any left fiber category F/Y.

Proposition 1.5. *Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $x \in \operatorname{ob} C$. Then $u : x \to Fr$ is a universal arrow from x to F iff it is initial object of the coslice $(x \downarrow F)$. Dually, $u' : Fr' \to x$ is a universal arrow from F to x iff it is a terminal object of the same category.*

Proof. Suppose that u is universal and $f : x \to Fy$ is another object of $(x \downarrow F)$. Then there is some unique $\hat{f} : r \to y$ such that $F\left(\hat{f}\right) \circ u = f$. Thus $F\left(\hat{f}\right)$ is a unique morphism of the coslice.

Conversely, suppose that u is initial. Then for any object $f : x \to Fy$ of $(x \downarrow F)$, there is some unique arrow $Sg : Fr \to Fy$ such that $Sg \circ u = f$. Hence setting $\hat{f} = g$ makes u a universal arrow. □

Corollary 1.6. *Any two universal arrows from x to F can be canonically identified by Lemma 1.3.*

2 (Co)limits

Definition 2.1. A *zero object* of $\mathscr{C}$ is an object that is both initial and terminal.

Example 2.2. The unique initial object of **Set** is $\emptyset$, and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in iso(**Set**).

Given $X \in \operatorname{ob}\mathscr{C}$, the *undercategory* $X/\mathscr{C}$ has as objects morphisms in $\mathscr{C}$ of the form $i : X \to Y$ where X is fixed. Given $i : X \to Y$ and $i' : X \to Y'$ in $\operatorname{ob} X/\mathscr{C}$, define the set of morphisms from i to i' as the morphisms $f : Y \to Y'$ where

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i'"'] & Y \arrow[d, "f"] \\ & Y' \end{tikzcd}$$

commutes. (We call i the *structure morphism.*) Composition and identity carry over exactly from $\mathscr{C}$.

Likewise, given $x \in \operatorname{ob}\mathscr{C}$, the *overcategory* $\mathscr{C}/X$ has as objects morphisms in $\mathscr{C}$ of the form $i : Y \to X$ where X is fixed. Given $i : Y \to X$ and $i' : Y' \to X$ in $\operatorname{ob}\mathscr{C}/X$, define the set of morphisms from i to i' as the morphisms $f : Y \to Y'$ where

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "i"'] & Y' \arrow[d, "i'"] \\ & X \end{tikzcd}$$

commutes. Composition and identity carry over exactly from $\mathscr{C}$.

Remark 2.3. If $X \in \operatorname{ob}\mathscr{C}$, then $(X/\mathscr{C})^{\mathrm{op}} = \mathscr{C}^{\mathrm{op}}/X$. Thus, the under- and overcategories are dual in the sense that $P(X, \mathscr{C})$ is true of any undercategory $X/\mathscr{C}$ iff $P^{\mathrm{op}}(X, \mathscr{C})$ is true of any overcategory $\mathscr{C}/X$.

Lemma 2.4. *For any $X \in \mathscr{C}$, the identity morphism on X is an initial object $X/\mathscr{C}$. Dually, it is a terminal object in $\mathscr{C}/X$.*

Proof. Any $i : X \to Y$ is itself the unique morphism from Id_X to i. □

Lemma 2.5. *Let X be an initial object of $\mathscr{C}$. The identity morphism on X is a zero object $\mathscr{C}/X$. Dually, if $Y \in \operatorname{ob}\mathscr{C}$ is terminal, then Id_Y is a zero object in $Y/\mathscr{C}$.*

Proof. We already know that Id_X is terminal. If $p : Y \to X$ is an object in $\mathscr{C}/X$, then there is a unique morphism $f : X \to Y$. Then $f \circ p$ must equal Id_X. □

Example 2.6. Let (X, x) be a pointed set with $X = \{x\}$. Let $\mathbf{Set}_*$ denotes the category of pointed sets with base point preserving functions. Since $\mathbf{Set}_* \cong X/\mathbf{Set}$, it follows that X is a zero object in $\mathbf{Set}_*$.

Given a morphism $\alpha : X \to Z$ in $\mathscr{C}$, define the *under-and-overcategory* $(X/\mathscr{C}/Z)_\alpha$ as having triples (Y, i, p) as objects where $i : X \to Y$ and $p : Y \to Z$ are morphisms in $\mathscr{C}$ such that $p \circ i = \alpha$. Define the set of morphisms from (Y, i, p) to (Y', u', p') as the set of morphisms $f : Y \to Y'$ such that

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "i"'] \arrow[dr, "\alpha", blue] & Y' \arrow[d, "p'"] \\ Y \arrow[r, "p"'] \arrow[ur, "f", red] & Z \end{tikzcd}$$

commutes. If $\alpha = \mathrm{Id}_X$, then we call $(X/\mathscr{C}/X)_{\mathrm{Id}_X}$ the category of *retractive* objects over X, with each triple (Y, i, p) being a retraction of Y onto X.

Example 2.7. If $F : \mathscr{C} \to \mathscr{C}$ is the identity functor, then the undercategory $Y/\mathscr{C}$ equals the right fiber category Y/F, and the overcategory $\mathscr{C}/Y$ equals the left fiber category F/Y.

Definition 2.8. Let $\mathscr{J}$ be a category. A *diagram of shape $\mathscr{J}$ in $\mathscr{C}$* is a functor $F : \mathscr{J} \to \mathscr{C}$.

Definition 2.9. Given a functor $F : \mathscr{J} \to \mathscr{C}$ and $X \in \operatorname{ob}\mathscr{C}$, a *cone over* F consists of an *apex* $X \in \operatorname{ob}\mathscr{C}$ and *legs* $f_j : X \to F(j)$ for each $j \in \operatorname{ob}\mathscr{J}$ such that for any $\alpha : j \to j'$, the triangle

$$\begin{tikzcd} X \arrow[r, "f_j"] \arrow[dr, "f_{j'}"'] & F(j) \arrow[d, "F\alpha"] \\ & F(j') \end{tikzcd}$$

commutes.

This is simply a natural transformation $\Delta_{\mathscr{J}} X \Rightarrow F$ where $\Delta_{\mathscr{J}} X$ denotes the constant functor on $\mathscr{J}$ at X. If $\mathscr{J}$ is small, then $\Delta_{\mathscr{J}}$ is just a functor from $\mathscr{C}$ to $\mathbf{Fun}(\mathscr{J}, \mathscr{C})$.

Definition 2.10. The *category of cones over* F is the right fiber category X/F. The *category of cones under* F is the left fiber category F/X.

Definition 2.11. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $g : Y \to Z$ a morphism in $\mathscr{D}$. Let $\Delta_{\mathscr{C}} g : \Delta_{\mathscr{C}} Y \Rightarrow \Delta_{\mathscr{C}} Z$ be the natural transformation with components $X \mapsto g$.

1. A *colimit* $\operatorname{colim}_{\mathscr{C}} F$ of the functor $F : \mathscr{C} \to \mathscr{D}$ is an object Y of $\mathscr{D}$ together with a natural transformation $i : F \Rightarrow \Delta_{\mathscr{C}} Y$ such that for any $Z \in \operatorname{ob} \mathscr{D}$ and natural transformation $j : F \Rightarrow \Delta_{\mathscr{C}} Z$, there is a unique morphism $g : Y \to Z$ such that $j = \Delta_{\mathscr{C}} g \circ i$.

2. We say that $\mathscr{D}$ *admits/has* $\mathscr{C}$*-shaped colimits* if each functor $G : \mathscr{C} \to \mathscr{D}$ has a colimit.

3. We sat that $\mathscr{D}$ is *cocomplete* if each functor $G : \mathscr{C} \to \mathscr{D}$ with $\mathscr{C}$ small has a colimit.

If $\mathscr{C}$ is small, then a colimit of $F : \mathscr{C} \to \mathscr{D}$ is just an initial object in the right fiber category $F/\Delta_{\mathscr{C}}$, which has as objects pairs $(Z, j : F \to \Delta Z)$ and as morphisms from (Y, i) to (Z, j) the morphisms $g : Y \to Z$ in $\mathscr{D}$ such that $\Delta g \circ i = j$.

Remark 2.12. There is a natural bijection $\mathscr{D}(Y, Z) \cong \mathbf{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta Z)$ iff $Y = \operatorname{colim}_{\mathscr{C}} F$.

Proposition 2.13. *Any two colimits are canonically isomorphic.*

Proof. When $\mathscr{C}$ is small, this is immediate from Lemma 1.3. But note that the proof of Lemma 1.3 does *not* require that $\mathscr{C}$ be locally small (a property which Rognes stipulates of any category). $\square$

Assume that $\mathscr{D}$ has $\mathscr{C}$-shaped colimits and that $\mathscr{C}$ is small. Then a (possibly global) choice function $\operatorname{colim}_{\mathscr{C}} : \mathbf{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor $\Delta_{\mathscr{C}} : \mathscr{D} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$. Indeed, for any functor $F : \mathscr{C} \to \mathscr{D}$, there is a bijection $\mathscr{D}(\operatorname{colim}_{\mathscr{C}} F, Z) \cong \mathbf{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta_{\mathscr{C}} Z)$.

Definition 2.14. A *limit* of the functor $F : \mathscr{C} \to \mathscr{D}$ is the colimit of $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$.

Explicitly, a limit for $F : \mathscr{C} \to \mathscr{D}$ is an object Z of $\mathscr{D}$ along with a natural transformation $p : \Delta_{\mathscr{C}} Z \Rightarrow F$ such that for any $Y \in \operatorname{ob} \mathscr{D}$ and natural transformation $q : \Delta_{\mathscr{C}} Y \Rightarrow F$, there is a unique morphism $g : Y \to Z$ such that $q = p \circ \Delta_{\mathscr{C}} g$.

Remark 2.15. The colimit of a functor F is the limit of F^{op}. Hence *limit* and *colimit* are dual properties, and our results so far for colimits can be dualized for limits.

Example 2.16. If $\mathscr{C}$ is the empty category, then the empty functor $F : \mathscr{C} \to \mathscr{D}$ satisfies $F/\Delta_{\mathscr{C}} \cong \mathscr{D}$, so that the colimit is an initial object of $\mathscr{D}$.

Definition 2.17. Let $\mathscr{J}$ be a discrete small category. Consider a diagram $\{A_i\}_{i \in \operatorname{ob} \mathscr{J}}$ of shape $\mathscr{J}$.

1. The limit of this diagram is called the *product* $\prod_i A_i$, equipped with projections $\pi_i : \prod_i A_i \to A_i$ such that for every $f_i : U \to A_i$ there is some unique map $f \coloneqq (f_i) : U \to \prod_i A_i$ satisfying $\pi_i \circ f = f_i$.

2. Dually, the colimit of the diagram is called the *coproduct* $\coprod_i A_i$, equipped with inclusions $u_i : A_i \to \coprod_i A_i$ such that for any $f_i : A_i \to Y$, there is some unique map $f := (f_i) : \coprod_i A_i \to Y$ satisfying $f_i = f \circ u_i$.

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Let $\mathscr{J}$ be the category $\bullet \rightrightarrows \bullet$. Then a diagram of shape $\mathscr{J}$ looks like $A \overset{f}{\underset{g}{\rightrightarrows}} B$. A cone over this with apex C and legs $f_1 : C \to A$ and $f_2 : C \to B$ satisfies $ff_1 = f_2 = gf_1$.

Definition 2.18. If such an object C together with f_1 is the limit of the diagram, then we say it is the *equalizer* of f and g. Dually, the colimit is called the *coequalizer* of f and g.

Example 2.19. The equalizer in **Set** of $f, g : X \to Y$ is the subset $X' := \{x \in X : f(x) = g(x)\}$ together with the inclusion function $X' \hookrightarrow X$. The coequalizer of (f, g) is $Y/\sim$ together with the quotient map on B where $\sim$ is the smallest equivalence relation under which $f(x) \sim g(x)$ for every x.

Example 2.20. The equalizer in **Grp** is defined as in Example 2.19 except that the relation $\sim$ becomes a certain minimal normal subgroup.

Now, let $\mathscr{J}$ be the category $\bullet \to \bullet \leftarrow \bullet$. Then a diagram of this shape looks like $B \xrightarrow{f} D \xleftarrow{g} A$, and a cone over this diagram looks like

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, "\alpha"'] & A \arrow[d, "g"] \\ B \arrow[r, "f"'] & D \end{tikzcd} .$$

Definition 2.21. If such an object C together with i and j is the limit of this diagram, then we call it the *pullback* of f and g, denoted by $B \times_D A$.

We can perform an analogous construction for $\mathscr{J}^{\mathrm{op}}$. Then the colimit of the resulting diagram is called the *pushout*, denoted by $B \cup_D A$.

Example 2.22. The pullback in **Set** of $f : X \to Z$ and $g : Y \to Z$ is the subset $\{(x, y) \in X \times Y : f(x) = g(y)\}$, called the *fiber product* of X and Y over Z.

All coequalizers $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{h} C$ can be obtained from taking binary coproducts and pushouts as follows.

$$\begin{tikzcd} A \coprod A \arrow[r, "{(f,g)}"] \arrow[d, "{(\mathrm{Id}_A, \mathrm{Id}_A)}"'] & B \arrow[d, "h"] \\ A \arrow[r] \arrow[ur, phantom, near end, "\ulcorner"] & C \end{tikzcd}$$

Therefore, any category with binary coproducts and pushouts has coequalizers.

Moreover, any colimit of a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \tag{$*$}$$

is precisely the coequalizer of

$$\coprod_n X_n \overset{\mathrm{Id}}{\underset{(u_{n+1}\circ f_n)}{\rightrightarrows}} \coprod_n X_n.$$

Therefore, any category with coequalizers and small coproducts has colimits of diagrams like ($*$). This fact can be generalized as follows.

Theorem 2.23 (Freyd).

(i) If $\mathscr{C}$ has equalizers and small (resp. finite) products, then it has small (resp. finite) limits.

(ii) If $\mathscr{C}$ has pullbacks and a terminal object, then it has finite limits.

Proof.

1. See Theorem 4.9 (Johnstone).

2. Thanks to part (i), it suffices to show that $\mathscr{C}$ has equalizers and finite products. By assumption, there is some terminal object 1. Then any product $A_1 \times A_2$ can be realized as the pullback of $A_1 \to 1 \leftarrow A_2$. By induction, $\mathscr{C}$ has finite products. Moreover, for morphisms $f, g : A \to B$, note that any cone over the diagram

$$A \xrightarrow{(\mathrm{Id}_A, g)} A \times B \xleftarrow{(\mathrm{Id}_A, f)} A$$

yields morphisms $h : A \to C$ and $k : C \to A$ such that $h = k$ and $fk = gh$. As a result, the pullback for this diagram is an equalizer of f and g, and thus our proof is complete.

□

Corollary 2.24. *Both* **Set** *and* **Grp** *are complete and cocomplete (or bicomplete).*

It turns out that adjoints interact nicely with (co)limits under mild conditions.

Proposition 2.25. *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors such that (F, G) is an adjoint pair. Let $\mathscr{E}$ be small category. If $X : \mathscr{E} \to \mathscr{C}$ is a functor whose colimit exists, then*

$$\mathrm{colim}_{\mathscr{E}}(F \circ X) = F(\mathrm{colim}_{\mathscr{E}} X).$$

Dually, if $Y : \mathscr{E} \to \mathscr{D}$ is a functor whose limit exists, then

$$\lim_{\mathscr{E}}(G \circ Y) = G(\lim_{\mathscr{E}} Y).$$

Proof. We have the following chain of bijections natural in $Y \in \mathscr{D}$:

$$\begin{aligned}
\mathscr{D}(F(\mathrm{colim}_{\mathscr{E}}), Y) &\cong \mathscr{C}(\mathrm{colim}_{\mathscr{E}} X, G(Y)) \\
&\cong \lim_{\mathscr{E}} \mathscr{C}(X(-), G(Y)) \\
&\cong \lim_{\mathscr{E}} \mathscr{D}(F(X(-)), Y) \\
&\cong \mathbf{Fun}(\mathscr{E}, \mathscr{D})(F \circ X, \Delta Y).
\end{aligned}$$

The second bijection exists because both sets can be identified with the components of all natural transformations $X \Rightarrow \Delta G(Y)$. □

3 Fibrations

Definition 3.1. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. The *fiber category* $F^{-1}(Y)$ is the full subcategory of $\mathscr{C}$ generated by the objects X with $F(X) = Y$.

Definition 3.2. Suppose $\mathscr{C}$ has a terminal object 1.

1. Given a morphism $p : 1 \to Y$, the *fiber* of f at p is the pullback $f^{-1}(p)$ of $1 \to Y \leftarrow X$.
2. The *cofiber* of a morphism $f : X \to Y$ is the pushout Y/X of $1 \leftarrow X \to Y$.

Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. For each $Y \in \operatorname{ob}\mathscr{D}$, there is a full and faithful functor $F^{-1}(Y) \to F/Y$ given by $X \mapsto (X, \mathrm{Id}_Y)$.

Definition 3.3. We say that $\mathscr{C}$ is a *precofibered category* over $\mathscr{D}$ if F admits a left adjoint given by $(Z, g : F(Z) \to Y) \mapsto g_*(Z)$.

Likewise, there is a full and faithful functor $F^{-1}(Y) \to Y/F$. We say that $\mathscr{C}$ is a *prefibered category* over $\mathscr{D}$ if this functor admits a right adjoint given by $(Z, g : Y \to F(Z)) \mapsto g_*(Z)$.

Definition 3.4. Let $F; \mathscr{C} \to \mathscr{D}$ be a functor.

1. Let $f : c' \to c$ be a morphism in $\mathscr{C}$. We say f is *cartesian* if for any morphism $f' : c'' \to c$ in $\mathscr{C}$ and any morphism $g : F(c'') \to F(c')$ in $\mathscr{D}$ such that $Ff \circ g = Ff'$, there exists a unique $\phi : c'' \to c$ such that $f' = f \circ \phi$ and $F\phi = g$.

   In other words, any filler of

$$\begin{tikzcd} c'' \arrow[r, dashed, "\exists!"] \arrow[dr, "f'"'] & c' \arrow[d, "f"] \\ & c \end{tikzcd}$$

   can be lifted to a filler in $\mathscr{D}$.

2. We say that F is a *fibration* if for any $c \in \mathscr{C}$ and morphism $f : d \to Fc$, there is a cartesian $\phi : c' \to c$ such that $F\phi = f$. Such an ϕ is called a *cartesian lifting* of f to c.

Example 3.5. Let the category **Mod** consist of pairs (R, M) as objects where R is a ring and M is a left R-module and pairs $\left(f, \tilde{f}\right)$ as morphisms where $f : R \to R'$ is a ring homomorphism and $\tilde{f} : M \to M'$ is an R-linear map with M' viewed as an R-module via f. Then the forgetful functor $U : \mathbf{Mod} \to \mathbf{Ring}$ is a fibration.