Perry Hart
Boston University
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Abstract

This paper briefly describes the isometries of $\mathbb{C}^2$. In particular, it classifies five important groups of such maps in the category **Top** of topological spaces. Thanks to Steven Rosenberg for his guidance on this topic.

1 Isometries of $\mathbb{C}^2$ over $\mathbb{R}$

If M is a metric space, then let $\mathrm{Isom}(M)$ denote the set of all isometries of M. For now, let $\left(\mathbb{C}^2, \|\cdot\|\right)$ denote the normed vector space $\mathbb{C}^2$ over $\mathbb{R}$ where $\|\cdot\| : \mathbb{C}^2 \to [0, \infty)$ is given by

$$\|(z, w)\| = \sqrt{z\bar{z} + w\bar{w}}.$$

That is, $\|\cdot\|$ is exactly the norm induced by the (Euclidean) inner product $\langle (z, w), (z, w) \rangle$. Then $\mathbb{C}^2 \cong \mathbb{R}^4$ as normed vector spaces via the map $T : \mathbb{C}^2 \to \mathbb{R}^4$ given by

$$(a + bi, a' + b'i) \mapsto (a, a', b, b')\,. \tag{$*$}$$

Endow $\mathbb{C}^2$ and $\mathbb{R}^4$ with the standard Euclidean metrics d and d', respectively. Since $\|T(\vec{v})\| = \|\vec{v}\|$ and T is linear, we see that

$$d(\vec{v}, \vec{x}) = \|\vec{v} - \vec{x}\| = \|T(\vec{v}) - T(\vec{x})\| = d'(T(\vec{v}), T(\vec{x}))$$

for any $\vec{v}, \vec{x} \in \mathbb{C}^2$. Likewise, we see that

$$d(T^{-1}(\vec{y}), T^{-1}(\vec{z})) = \left\|T^{-1}(\vec{y}) - T^{-1}(\vec{z})\right\| = \|\vec{y} - \vec{z}\| = d'(\vec{y}, \vec{z})$$

for any $\vec{y}, \vec{z} \in \mathbb{R}^4$. Thus, the map $f \mapsto T \circ f \circ T^{-1}$ defines a group isomorphism $\mathrm{Isom}(\mathbb{C}^2) \xrightarrow{\cong} \mathrm{Isom}(\mathbb{R}^4)$, provided that both $\mathrm{Isom}(\mathbb{C}^2)$ and $\mathrm{Isom}(\mathbb{R}^4)$ are, in fact, groups under composition. Certainly they are closed under composition and contain the identity map. Also, every isometry f of a given metric space (X, ρ) must be injective. Indeed, if $x \neq y$ but $f(x) = f(y)$, then $\rho(x, y) \neq 0 = \rho(f(x), f(y))$, which is impossible. Since the inverse of f must also be an isometry, it just remains to show that f is surjective in order to prove that the two are groups. This is the content of Corollary 1.12 below.

We can form the group

$$\mathrm{O}(4) \coloneqq \left\{ f \in \mathrm{Isom}(\mathbb{R}^4) : f \text{ fixes } \vec{0} \right\}.$$

For each $\vec{v} \in \mathbb{R}^4$, define $T_{\vec{v}} : \mathbb{R}^4 \to \mathbb{R}^4$ by $\vec{x} \mapsto \vec{x} + \vec{v}$.

Lemma 1.1. *Any $A \in \mathrm{Isom}(\mathbb{R}^4)$ can be written uniquely as $T_{A(\vec{0})} \circ g$ for some $g \in \mathrm{O}(4)$.*

Proof. Define $g : \mathbb{R}^4 \to \mathbb{R}^4$ by $A(\vec{v}) - A(\vec{0})$. Then $g \in \mathrm{O}(4)$, and $A(\vec{v}) = T_{A(\vec{0})} \circ g(\vec{v})$ for any $\vec{v}$. Further, if $A = T_{A(\vec{0})} \circ k$ for some $k \in \mathrm{O}(4)$, then $g(\vec{v}) = A(\vec{v}) - A(\vec{0}) = k(\vec{v})$, thereby proving uniqueness. $\square$

Definition 1.2. A matrix $X \in \mathbb{M}^4(\mathbb{R})$ is *orthogonal* if its column vectors are orthonormal.

Proposition 1.3. *The following are equivalent.*

(a) X *is orthogonal.*

(b) $X \in \mathrm{GL}(4, \mathbb{R})$ *with* $X^T = X^{-1}$.

Corollary 1.4. *Any orthogonal matrix* $X \in \mathbb{M}^4(\mathbb{R})$ *preserves the inner product, i.e.,* $\langle X\vec{v}, X\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle$ *for any* $\vec{v}, \vec{w} \in \mathbb{R}^4$.

Proof. We have that $X\vec{v} \bullet X\vec{w} = \vec{v} \bullet X^T X\vec{w} = \vec{v} \bullet I\vec{w} = \vec{v} \bullet \vec{w}$. $\square$

Notation. The symbol $\bullet$ will denote the Euclidean inner product.

Corollary 1.5. *If* $X \in \mathbb{M}^4(\mathbb{R})$ *is orthogonal, then* $|\det(X)| = 1$.

Proof. We have that $1 = \det(I) = \det(XX^T) = \det(X)^2$. $\square$

Lemma 1.6. *If* $X \in \mathbb{M}^4(\mathbb{R})$ *is orthogonal, then* $X \in \mathrm{O}(4)$.

Proof. By Corollary 1.4, X preserves the inner product, which implies that

$$\begin{aligned} \|X\vec{v} - X\vec{w}\|^2 &= \|X\vec{v}\|^2 - 2X\vec{v} \bullet X\vec{w} + \|X\vec{w}\|^2 \\ &= \|\vec{v}\|^2 - 2\vec{v} \bullet \vec{w} + \|\vec{w}\|^2 \\ &= \|\vec{v} - \vec{w}\|^2 \end{aligned}$$

for any $\vec{v}, \vec{w} \in \mathbb{R}^4$. Thus, $d'(X\vec{v}, X\vec{w}) = d'(\vec{v}, \vec{w})$, and $X \in \mathrm{O}(4)$.

$\square$

Definition 1.7. An invertible linear operator T on a finite-dimensional vector space is *orientation-preserving* if $\det M_T > 0$ and *orientation-reversing* if $\det M_T < 0$ where M_T denotes the matrix of T.

Soon we shall prove that $\mathrm{O}(4) \subset \mathrm{GL}(4, \mathbb{R})$. Therefore, it makes sense to introduce the group

$$\mathrm{SO}(4) \coloneqq \left\{ f \in \mathrm{Isom}(\mathbb{R}^4) : f \text{ fixes } \vec{0} \text{ and is orientation-preserving} \right\}.$$

Let $\{\vec{e}_1, \dots, \vec{e}_4\}$ denote the standard basis of $\mathbb{R}^4$. We are now ready to establish one of our main results.

Theorem 1.8 (TRF-decomposition). *Let* $\mathcal{F} : \mathbb{R}^4 \to \mathbb{R}^4$ *be given either by the identity map or the reflection* $(a, b, c, d) \mapsto (a, b, c, -d)$. *Let* $A \in \mathrm{Isom}(\mathbb{R}^4)$. *Then we have that*

$$A = T_{A(\vec{0})} \circ R' \circ \mathcal{F}$$

for some $R' \in \mathrm{SO}(4)$.

Proof. By Lemma 1.1, we have that $A = T_{A(\vec{0})} \circ g$ for some $g \in \mathrm{O}(4)$. Since g is an isometry, we know that $\|\vec{x} - \vec{y}\|^2 = \|g(\vec{x}) - g(\vec{y})\|^2$ for any $\vec{x}, \vec{y} \in \mathbb{R}^4$. As g fixes $\vec{0}$, it follows that $\|g(\vec{v})\| = \vec{v}$ for any $\vec{v} \in \mathbb{R}^4$. We can apply the additivity of the inner product to get

$$\begin{aligned} \|g(\vec{v})\|^2 + \|g(\vec{w})\|^2 - 2\langle g(\vec{v}), g(\vec{w}) \rangle &= \langle g(\vec{v}) - g(\vec{w}), g(\vec{v}) - g(\vec{w}) \rangle \\ &= \langle \vec{v} - \vec{w}, \vec{v} - \vec{w} \rangle \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\langle \vec{v}, \vec{w} \rangle. \end{aligned}$$

We can cancel terms to find that g preserves the inner product. Note that our proof of this fact applies to any element of $\mathrm{O}(4)$.

Now, it follows that $\|g(\vec{e}_i)\|^2 = \|\vec{e}_i\|^2 = 1$ for each $i = 1, 2, 3, 4$, so that $\|g(\vec{e}_i)\| = 1$. Similarly, we can deduce that $\langle g(\vec{e}_i), g(\vec{e}_j)\rangle = 0$ if $i \neq j$. Thus, $\{g(\vec{e}_i)\}_{i=1,2,3,4}$ is an orthonormal (hence linearly independent) set. Let

$$M := \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ g(\vec{e}_1) & g(\vec{e}_2) & g(\vec{e}_3) & g(\vec{e}_4) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Then $M^T M = M M^T = I$, so that M is invertible with $M^T = M^{-1}$. Lemma 1.6 implies that $M \in \mathrm{O}(4)$. The isometry $f := M^{-1} \circ g : \mathbb{R}^4 \to \mathbb{R}^4$ satisfies $f(\vec{0}) = \vec{0}$ and $f(\vec{e}_i) = \vec{e}_i$ for each i.

Since $f \in \mathrm{O}(4)$, it follow that

$$f(\vec{x}) \bullet f(\vec{e}_i) = \vec{x} \bullet \vec{e}_i = f(\vec{x}) \bullet \vec{e}_i = \vec{x} \bullet \vec{e}_i$$

for each i. Writing $\vec{x} = \sum_{i=1}^4 c_i \vec{e}_i$ for some $c_i \in \mathbb{R}$, we have that $f(\vec{x}) \bullet \vec{e}_i = \left(\sum_{i=1}^4 c_i \vec{e}_i\right) \bullet \vec{e}_i = c_i$, and thus $f(\vec{x}) = \vec{x}$. Hence $f = \mathrm{Id}_{\mathbb{R}^4}$, so that $M = g$. We deduce that any isometry of $\mathbb{R}^4$ that fixes $\vec{0}$ is given by an orthogonal matrix.

By Corollary 1.5, $\det(g) = \pm 1$. If $\det(g) = 1$, then $g \in \mathrm{SO}(4)$, and we're done. Assume that $\det(g) = -1$. Note that the reflection

$$\phi(a, b, c, d) \equiv (a, b, c, -d)$$

is given by the matrix

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since it's clear that $\phi \in \mathrm{O}(4)$, we see that $g \circ \phi \in \mathrm{O}(4)$. Also, $\det(gS) = \det(g)\det(S) = (-1)(-1) = 1$. Therefore, $g \circ \phi \in \mathrm{SO}(4)$. As $\phi = \phi^{-1}$, it follows that $(g \circ \phi) \circ \phi = g \circ (\phi^2) = g$. Now, set $R' = g \circ \phi$ and $\mathcal{F} = \phi$, thereby completing out proof.

□

By inspecting our proof of Theorem 1.8, we obtain several quick results.

Corollary 1.9. *If $X \in \mathbb{M}^4(\mathbb{R})$ preserves the inner product, then X is orthogonal.*

Corollary 1.10. *We have that*

$$\mathrm{O}(4) = \{X \in \mathrm{GL}(4, \mathbb{R}) : X \text{ is orthogonal}\}$$
$$\mathrm{SO}(4) = \{X \in \mathrm{GL}(4, \mathbb{R}) : X \text{ is orthogonal and } \det(X) = 1\}.$$

Corollary 1.11. *A function f is an element of $\mathrm{Isom}(\mathbb{R}^4)$ if and only if there exist $M \in \mathrm{O}(4)$ and $\vec{b} \in \mathbb{R}^4$ such that for any $\vec{x} \in \mathbb{R}^4$, $f(\vec{x}) = M\vec{x} + \vec{b}$. In this case, $M = R' \circ \mathcal{F}$ with notation as in Theorem 1.8.*

Corollary 1.12. *Every $f \in \mathrm{Isom}(\mathbb{R}^4)$ and every $g \in \mathrm{Isom}(\mathbb{C}^2)$ are invertible, so that both $\mathrm{Isom}(\mathbb{C}^2)$ and $\mathrm{Isom}(\mathbb{R}^4)$ are groups under composition.*

Proof. Thanks to Corollary 1.11, we can write $f(\vec{x}) = M\vec{x} + \vec{b}$. Now it's easy to verify that $f^{-1}(\vec{x}) = M^{-1}\vec{x} - M^{-1}\vec{b}$.

Moreover, with T given by $(*)$, we find that $g = T \circ h \circ T^{-1}$ for some $h \in \mathrm{Isom}(\mathbb{R}^4)$. Hence g is the composite of three invertible functions and thus is invertible. $\square$

Note 1.13. The decomposition of A given in Theorem 1.8 is unique.

Proof. Suppose $A(\vec{x}) = M\vec{x} + \vec{b} = M'\vec{x} + \vec{b}'$ for every $\vec{x} \in \mathbb{R}^4$. Then $\vec{b} = \vec{b}'$, so that $M = M'$. Moreover, if $M = T \circ \mathcal{F}$ for some $T \in \mathrm{SO}(4)$, then $T = M \circ \mathcal{F}$. This shows that the decomposition $A = T_{A(\vec{0})} \circ g \circ \mathcal{F}$ given in Theorem 1.8 is, indeed, unique. $\square$

2 Isometries of $\mathbb{C}^2$ over $\mathbb{C}$

Now, view $\mathbb{C}^2$ as a two-dimensional vector space over $\mathbb{C}$. Recall that the Hermitian inner product $H : \mathbb{C}^2 \times \mathbb{C}^2 \to [0, \infty)$ is defined by $H(x, y) = x_1\bar{y}_1 + x_2\bar{y}_2$.

Definition 2.1. For any $n \in \mathbb{N}$, a matrix $X \in \mathbb{M}^n(\mathbb{C})$ is *unitary* if its column vectors are orthonormal with respect to H.

Let $\mathrm{U}(n)$ denote the set of all unitary matrices. Lemma 2.5 below indicates that these are isometries of $\mathbb{C}^2$.

Proposition 2.2. *The following are equivalent.*

(a) $X \in \mathrm{U}(2)$.

(b) $X \in \mathrm{GL}(2, \mathbb{C})$ *with* $X^* = X^{-1}$*, where* X^* *denotes the conjugate transpose of* X*.*

Corollary 2.3. $\mathrm{U}(n)$ *is a group under composition for each* $n = 1, 2$*.*

Proof. First, note that $\mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\} = S^1$, which is a group because the complex modulus is multiplicative and $|z| = 1 \implies \left|z^{-1}\right| = \frac{|\bar{z}|}{|z|^2} = 1$. Next, consider $\mathrm{U}(2)$. It suffices to verify closure. If $A, B \in \mathrm{U}(2)$, then

$$(AB)^* \, (AB) = B^*A^*AB = B^*B = I,$$

and thus $AB \in \mathrm{U}(2)$. $\square$

Note that $\mathrm{U}(2)$ is nonabelian. Indeed, let $A := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. These are unitary, but $0 \neq AB = -BA$.

Corollary 2.4. *Every* 2×2 *unitary matrix* X *has* $|\det(X)| = 1$*, where* $|\cdot|$ *denote the complex modulus.*

Proof. We have that

$$\begin{aligned} 1 &= \det(I) \\ &= \det(XX^*) \\ &= \det(X)\det(X^*) \\ &= \det(X)\overline{\det(X)} \\ &= |\det(X)|\,. \end{aligned}$$

$\square$

From a linear-algebraic perspective, we see that U(2) is the complex analogue of O(4). Group-theoretically, however, we can construct an embedding $F : \mathrm{U}(2) \hookrightarrow \mathrm{SO}(4)$ as follows.[1] For each $M \in \mathrm{U}(2)$, write

$$M = \begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + i \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = A + iB$$

and set $F(M) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$. It's easy to verify that $F(M)$ is orthogonal. Also, note that

$$\begin{aligned}
\det(F(M)) &= 1 \cdot \det\left(\begin{bmatrix} A & -B \\ B & A \end{bmatrix}\right) \cdot 1 \\
&= \det\left(\begin{bmatrix} I & 0 \\ iI & I \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} I & 0 \\ -iI & I \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} A + iB & -B \\ 0 & A - iB \end{bmatrix}\right) \\
&= \det(A + iB) \det(A - iB) \\
&= \det(A)^2 + \det(B)^2 \\
&= |\det(M)|^2 = 1.
\end{aligned}$$

Therefore, F is well-defined. To verify that F is a homomorphism, note that if $N = C + Di$, then $MN = (AC - BD) + (AD + BC)i$. In this case

$$F(MN) = \begin{bmatrix} AC - BD & -AD - BC \\ AD + BC & AC - BD \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} C & -D \\ D & C \end{bmatrix} = F(M)F(N).$$

Furthermore, if $F(M) \in \ker(F)$, then $A = I_2$ and $B = 0_2$, i.e., $M = I_2$. Hence $\ker(F)$ is trivial, and thus F is an injective homomorphism, as desired.

In fact, the 2×2 unitary matrices are precisely those elements of SO(4) which preserve the Hermitian inner product H. This provides us with a geometric distinction between U(2) and SO(4).

Lemma 2.5. *A map $R \in \mathbb{M}^2(\mathbb{C})$ satisfies $H(R(x), R(y)) = H(x, y)$ for any $x, y \in \mathbb{C}^2$ if and only if $R \in \mathrm{U}(2)$.*

Proof. Note that $H(x, y) = \bar{x}^T y$. Hence

$$\begin{aligned}
H(Rx, Ry) = H(x, y) &\iff \overline{Rx}^T Ry = \bar{x}^T y \\
&\iff \bar{x}^T (\overline{R}^T R) y = \bar{x}^T y \\
&\iff \overline{R}^T R = I.
\end{aligned}$$

$\square$

[1] As a result, SO(4) is nonabelian and hence not isomorphic to SO(2).

Let us look now at the complex analogue of SO(4). The map $D : \mathrm{U}(2) \to \mathrm{U}(1)$ given by $D(X) = \det(X)$ is well-defined by Corollary 2.4. As det is multiplicative, it is also a homomorphism. For any $e^{i\theta} \in \mathbb{C}$, we see that $M := \begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \in \mathrm{U}(2)$ and $D(M) = e^{i\theta}$, which means that D is surjective. Now note that

$$\ker D = K := \{X \in \mathrm{U}(2) : \det(X) = 1\}.$$

This yields an isomorphism $\mathrm{U}(2)/K \cong \mathrm{U}(1)$ in the category **Grp** of groups.

Let $\mathrm{SU}(2) = \ker(D)$. Then SU(2) consists precisely of those 2×2 unitary matrices which are orientation-preserving. Let $W \in \mathrm{SU}(2)$ and write $W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Since $\det(W) = 1$, we find that $W^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Since $W^* = W^{-1}$, it follows that $d = \bar{a}$ and $-\bar{b} = c$. Therefore, $\det(W) = \|(a, c)\|^2 = a\bar{a} + c\bar{c} = 1$, and $W = \begin{bmatrix} a & c \\ -\bar{c} & \bar{a} \end{bmatrix}$. Conversely, the column vectors of such a matrix are orthonormal. Hence

$$\mathrm{SU}(2) = \left\{ X \in \mathbb{M}^2(\mathbb{C}) : X = \begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix} \text{ with } x\bar{x} + y\bar{y} = 1 \right\}.$$

Theorem 2.6. $\mathrm{U}(2) \cong (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ *in* **Grp**.

Proof. Define $\psi : \mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{U}(2)$ by $(A, k) \mapsto kA$. This map is certainly a well-defined homomorphism. Moreover, for any $X \in \mathrm{U}(2)$, note that $\sqrt{\det(X)} \in \mathrm{U}(1)$ and $\frac{1}{\sqrt{\det(X)}} X \in \mathrm{SU}(2)$, so that

$$\psi\left(\frac{1}{\sqrt{\det(X)}} X, \sqrt{\det(X)} \right) = X.$$

Thus, ψ is surjective. Finally, notice that $\ker \psi = \{\pm(I, 1)\} \cong \mathbb{Z}_2$. By the first isomorphism theorem, we get an isomorphism $\tilde{\psi} : \mathrm{U}(2) \xrightarrow{\cong} (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$, as desired.

□

It turns out that SU(2) is the same as the group of unit quaternions.

Theorem 2.7. $\mathrm{SU}(2) \cong S^3$ *in* **Grp**.

Proof. For any $x := (x_1, x_2, x_3, x_4) \in S^3$, let $z = x_1 + x_2 i \in \mathbb{C}$ and $w = x_3 + x_4 i \in \mathbb{C}$. Then $x = z + wj$. Define the map $f : S^3 \to \mathrm{SU}(2)$ by

$$f(x) = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}.$$

We see that $|x|^2 = |z|^2 + |w|^2 = \det \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$. Hence $x \in S^3$ if and only if $\det \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} = 1$, which establishes a bijection. It remains to check that f is a homomorphism. Let $y \in S^3$ so that $y = p + qj$. Since $jw = \bar{w}j$ and $jz = \bar{z}j$, we obtain

$$xy = pz + pwj + q(jz) + p(jw)j = (pz - p\bar{w}) + pw + q\bar{z})j.$$

Finally, we compute

$$\begin{aligned}
f(yx) &= \begin{bmatrix} pz - q\bar{w} & pw + q\bar{z} \\ \overline{-pw + q\bar{z}} & \overline{pz - q\bar{w}} \end{bmatrix} \\
&= \begin{bmatrix} pz - q\bar{w} & pw + q\bar{z} \\ -\bar{p}\bar{w} - \bar{q}z & \bar{p}\bar{z} - \bar{q}w \end{bmatrix} \\
&= \begin{bmatrix} p & q \\ -\bar{q} & \bar{p} \end{bmatrix} \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \\
&= f(y)f(x).
\end{aligned}$$

□

3 Topology of $\mathrm{Isom}(\mathbb{C}^2)$

Let us turn our attention to providing the groups

- $\mathrm{SU}(2)$
- $\mathrm{U}(2)$
- $\mathrm{SO}(4)$
- $\mathrm{O}(4)$
- $\mathrm{Isom}(\mathbb{R}^4)$

with *topological* characterizations, having treated them only as algebraic objects thus far. The first four of these groups are topological spaces as subsets of normed vector spaces. The last group, $\mathrm{Isom}(\mathbb{R}^4)$, has the metric topology induced by

$$d(f, g) \equiv \max \left\{ |f(x) - g(x)| : x \in \mathbb{R}^4,\ |x| \leq 1 \right\},$$

which is a modest generalization of the metric induced by the familiar operator norm in the theory of finite-dimensional vector spaces.

Remark 3.1. All five groups are Lie groups.

Theorem 3.2. $\mathrm{SU}(2) \cong S^3$ *in* **Top**.

Proof. We claim that the map f from Theorem 2.7 is a homeomorphism. Indeed, note that as S^3 is a closed and bounded subset of Euclidean space, it is compact. Also, $\mathrm{SU}(2)$ is Hausdorff as a topological group. Thus, it suffices to show that f is continuous. By identifying each matrix in f's codomain with a vector in $\mathbb{C}^4$, we find that continuity follows from the fact that complex conjugation is continuous along with the fact that continuity is preserved by addition and multiplication. □

Corollary 3.3. $\mathrm{SU}(2)$ *is simply connected.*

Theorem 3.4. $\mathrm{U}(2) \cong (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2$ *in* **Top**.

Proof. We claim that the map $\tilde{\psi}$ from Theorem 2.6 is a homeomorphism. Indeed, it is continuous due to the universal property of quotient spaces. Moreover, its inverse is given by

$$X \mapsto \left[\left(X \frac{1}{\sqrt{\det X}}, \sqrt{\det X}\right)\right],$$

which is continuous because both $\sqrt{\cdot}$ and $\det(\cdot)$ are continuous. $\square$

Proposition 3.5. *For any quaternions x and y, we have that $\overline{xy} = \bar{y}\bar{x}$.*

Corollary 3.6. $|xy| = |x|\,|y|$, *where* $|x| \equiv \sqrt{x\bar{x}}$

Theorem 3.7. $\mathrm{SO}(4) \cong S^3 \times \mathrm{SO}(3)$ *in* **Top**.

Proof. We may identify $\mathbb{R}^4$ with the group of quaternions. For each $q \in S^3$, the map $\alpha_q : \mathbb{R}^4 \to \mathbb{R}^4$ given by $a \mapsto aq$ satisfies $|aq| = |a|\,|q| = |a|$ thanks to Corollary 3.6. Hence for any $a, b \in \mathbb{R}^4$, we see that

$$|a - b| = |\alpha_q(a - b)| = |aq - bq|,$$

so that $\alpha_q \in \mathrm{Isom}(\mathbb{R}^4)$. Further, since $\alpha_q(0) = 0$, it belongs to $\mathrm{O}(4)$. Hence it preserves the Euclidean inner product.

We construct a continuous embedding $E : \mathrm{O}(3) \hookrightarrow \mathrm{O}(4)$ as follows. Let $X \in \mathrm{O}(3)$ and $X = \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}$ where $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$. Now set

$$E(X) = (1, x, y, z) \coloneqq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \vdots & \vdots & \vdots \\ 0 & \vec{x} & \vec{y} & \vec{z} \\ 0 & \vdots & \vdots & \vdots \end{bmatrix},$$

which is an element of $\mathrm{O}(4)$. Now, define $f : S^3 \times \mathrm{O}(3) \to \mathrm{O}(4)$ by $(q, (1, x, y, z)) \mapsto (q, xq, yq, zq)$. As α_q preserves the norm and the inner product, it preserves orthonormality. This means that f is well-defined. It's clear that f is continuous. Moreover, f is invertible with continuous inverse $(v, u, r, s) \mapsto \left(v, \left(1, uv^{-1}, rv^{-1}, sv^{-1}\right)\right)$. Note that, in fact, $\left(1, uv^{-1}, rv^{-1}, sv^{-1}\right) \in \mathrm{O}(3)$ because $\alpha_{v^{-1}}$ preserves orthonormality, so that in particular vv^{-1} must be orthogonal to each of the other three column vectors. Hence the first row vector must be $(1, 0, 0, 0)$, as required.

Finally, the restriction of f to $S^3 \times \mathrm{SO}(3)$ yields a homeomorphism $S^3 \times \mathrm{SO}(3) \xrightarrow{\cong} \mathrm{SO}(4)$. $\square$

Corollary 3.8. $\mathrm{SO}(4) \cong S^3 \times \mathbb{RP}^3$.

Corollary 3.9. $\mathrm{O}(4) \cong S^3 \times \mathrm{O}(3)$.

Our final result classifies the entire space $\mathrm{Isom}(\mathbb{R}^4)$.

Theorem 3.10. $\mathrm{Isom}(\mathbb{R}^4) \cong \mathrm{O}(4) \times \mathbb{R}^4$ *in* **Top**.

Proof. With notation as in Corollary 1.11, define $F : \mathrm{Isom}(\mathbb{R}^4) \to \mathrm{O}(4) \times \mathbb{R}^4$ by $f \mapsto \left(M, \vec{b}\right)$. Note 1.13 implies that F is well-defined, and Corollary 1.9 implies that it is a bijection. Note that $F_1(f) = M = T_{-\vec{b}} \circ f$, which is a composite of continuous functions. Further, $F_2(f) = \vec{b} = f(\vec{0})$. Hence each component map of F is continuous. It's clear that the inverse $\left(M, \vec{b}\right) \to \left(\vec{x} \mapsto M\vec{x} + \vec{b}\right)$ is also continuous. Thus, F is a homeomorphism. $\square$