

### Abstract

We begin low-dimensional  $K$ -theory, i.e., describe  $K_0(-)$ ,  $K_1(-)$ , and  $K_2(-)$ , in various settings. The main sources for this talk are the following.

- *nLab*.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*. Chapters I and II.
- Eric M. Friedlander's *An Introduction to K-theory*, Chapter 1.

**Definition 1.** Recall that the forgetful functor  $U : \mathbf{Ab} \rightarrow \mathbf{CMon}$  admits a left adjoint  $K : \mathbf{CMon} \rightarrow \mathbf{Ab}$ , called the *group completion* functor. For any commutative monoid  $(C, +)$ , we call the abelian group  $K(C)$  the *Grothendieck group* of  $C$ , which is constructed as follows.

Consider  $S := C \times C / \sim$  where  $(a_1, b_1) \sim (a_2, b_2)$  if

$$(a_1 + b_2 + k = b_1 + a_2 + k)$$

for some  $k \in C$ . Note that  $\sim = \sim'$  where  $(a_1, b_1) \sim' (a_2, b_2)$  if

$$(a_1 + k_1, b_1 + k_1) = (a_2 + k_2, b_2 + k_2)$$

for some  $(k_1, k_2) \in C \times C$ . Then set  $K(C) = (S, +)$ , where  $+$  is inherited from  $C$  and acts componentwise on equivalence classes. Notice that  $\sim'$  makes it clear that  $[a_1, b_1]^{-1} = [b_1, a_1]$ .

**Proposition 1.** *The inclusion  $C \hookrightarrow K(C)$  given by  $x \mapsto [x] := [x, 0]$  is injective iff  $C$  is a cancellation monoid.*

**Lemma 2.** *(Universal property of the Grothendieck group) Let  $B$  be an abelian group and  $f : A \rightarrow B$  a monoid homomorphism. Then we have*

$$\begin{array}{ccc} A & & \\ \downarrow & \searrow f & \\ K(A) & \xrightarrow{\exists! \tilde{f}} & B \end{array}$$

*Proof.* Define  $\tilde{f}$  by  $[a_1, b_1] \mapsto f(a_1) - f(b_1)$ . □

**Lemma 3.**  $K(C_1 \times C_2) \cong K(C_1) \times K(C_2)$ .

**Definition 2.** A submonoid  $L$  of  $C$  is *cofinal* if for any  $c \in C$ , there is some  $c' \in C$  such that  $c + c' \in L$ .

**Proposition 4.** *Let  $L$  be cofinal in commutative  $C$ .*

1. Any element of  $K(C)$  can be written as  $[m] - [n]$  for some  $m, n \in C$ .
2.  $K(L) \leq K(C)$ .
3. Any element of  $K(C)$  can be written as  $[m] - [l]$  for some  $m \in C$  and  $l \in L$ .
4. If  $[m] = [m']$ , then  $m + l = m' + l$  for some  $l \in L$ .

**Example 3.**

1.  $K(\mathbb{N}) \cong \mathbb{Z}$  via  $[a_1, b_1] \mapsto a_1 - b_1$ .
2.  $K(\mathbb{Z}^\times) \cong \mathbb{Q}^\times$  via  $[a_1, b_1] \mapsto \frac{a_1}{b_1}$ .

**Definition 4.** Let  $R$  be a unital ring. Let  $(\mathbf{P}(R), \oplus, \otimes_R)$  denote the semiring of (isomorphism classes of) finitely generated projective  $R$ -modules. Then we define  $K_0(R) = K(\mathbf{P}(R))$ .

**Lemma 5.**  $\mathbf{P}(R_1 \times R_2) \cong \mathbf{P}(R_1) \times \mathbf{P}(R_2)$ . Therefore,  $K_0$  can be computed componentwise by Lemma 4.

*Remark 1.*  $K_0(-)$  defines a functor from **Ring** to **Ab**. Let  $f : R \rightarrow S$  be a ring homomorphism and  $P$  be a finitely generated projective  $R$ -module. The assignment of  $f$  under  $K_0(-)$  goes as follows.

1. Construct  $S \otimes_R P$ , the base extension of  $P$ . This is the *unique*  $S$ -module  $(s', s \otimes p) \mapsto s' s \times p$  compatible with the  $R$ -module structure on  $S$  induced by  $f$ . This is also an  $R$ -module with  $f(r) \cdot t := r \cdot t$  for  $t \in S \otimes_R P$ . We know that  $P \oplus Q$  is free for some  $R$ -module  $Q$ . Since  $S \otimes_R (P \oplus Q) \cong_S (S \otimes_R P) \oplus (S \otimes_R Q)$  and  $P \oplus Q$  is free over  $S$  via  $f$ , it follows that  $S \otimes_R P$  is a finitely generated projective  $S$ -module.
2. We've just defined a monoid homomorphism  $\tilde{f} : \mathbf{P}(R) \rightarrow \mathbf{P}(S)$ .
3. Apply the universal property of  $K$  to find the filling

$$\begin{array}{ccc} \mathbf{P}(R) & \xrightarrow{\tilde{f}} & \mathbf{P}(S) \\ \downarrow & & \downarrow \\ K(\mathbf{P}(R)) & \xrightarrow{f_*} & K(\mathbf{P}(S)) \end{array},$$

where we set  $K_0(f) = f_*$ .

*Remark 2.* (Eilenberg Swindle) Suppose  $P \oplus Q = R^n$  as  $R$ -modules. Then

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong R^\infty.$$

Therefore, if we added  $R^\infty$  to  $\mathbf{P}(R)$ , then we would have  $[P] = 0$  for each finitely generated projective  $P$ .

**Example 5.** If  $R = F$  is a field, then  $\mathbf{P}(R) \cong \mathbb{N}$  and, by Example 7,  $K_0(R) \cong \mathbb{Z}$ .

We can generalize this phenomenon a bit.

**Definition 6.** A ring  $R$  has the *invariant basis property (IBP)* if  $R^n \not\cong R^m$  when  $n \neq m$ . Note that any commutative ring has the IBP.

**Definition 7.** An  $R$ -module  $P$  is *stably free* of rank  $m - n$  if  $P \oplus R^m \cong R^n$  for some  $m$  and  $n$ .

**Lemma 6.** The map  $f : \mathbb{N} \rightarrow \mathbf{P}(R)$  defined by  $n \mapsto R^n$  induces a homomorphism  $\phi : \mathbb{Z} \rightarrow K_0(R)$ .

1.  $\phi$  is injective iff  $R$  has the IBP.
2. Suppose  $R$  has IBP. Then  $K_0(R) \cong \mathbb{Z}$  iff every finitely generated projective  $R$ -module is stably free.

*Proof.*

1. By Proposition 6(4), we know that  $[P] = [Q]$  in  $K_0(R)$  iff  $P \oplus R^m \cong Q \oplus R^m$  for some  $m$ .
2.  $[P] = [R^n]$  iff  $P$  is stably free.

□

**Example 8.** Suppose that  $R$  is commutative. There is a ring homomorphism  $R \rightarrow F$  with  $F$  a field. Then the induced map  $K_0(R) \rightarrow K_0(F) \cong \mathbb{Z}$  sends  $[R]$  to 1. Also, the map  $\phi : \mathbb{Z} \rightarrow K_0(R)$  is injective by Lemma 15. Letting  $K := \ker(K_0(R) \rightarrow \mathbb{Z})$ , we get a split exact sequence of abelian groups, so that  $K_0(R) \cong \mathbb{Z} \oplus K$ .

$$1 \longrightarrow K \longrightarrow K_0(R) \longrightarrow \mathbb{Z} \longrightarrow 1$$

**Example 9.** A ring  $R$  is a *flasque* if there is an  $R$ -bimodule  $M$  which is also a finitely generated projective on one side along with a bimodule isomorphism  $R \oplus M \cong M$ . Then since  $P \oplus (P \otimes_R M) \cong P \otimes_R (R \oplus M) \cong P \otimes_R M$ , we see that  $K_0(R) = 0$ .

**Example 10.** A module is *semisimple* if it is the direct sum of simple modules. A ring  $R$  is called semisimple if it is a semisimple  $R$ -module. Notice that any semisimple module is both Noetherian and Artinian and that any module over a semisimple ring is semisimple.

Suppose  $R$  is semisimple with summands  $V_1, \dots, V_m$ . Then any finitely generated  $R$ -module is  $\bigoplus_{i=1}^m V_i^{l_i}$ , where the  $l_i$  are uniquely determined by Krull-Remak-Schmidt. Hence  $\mathbf{P}(R) \cong \mathbb{N}^m$ , and  $K_0(R) \cong \mathbb{Z}^m$ .

**Example 11.** A ring  $R$  is *von Neumann regular* if  $(\forall r \in R)(\exists x_r \in R)(rx_r r = r)$ . It turns out that any one-sided ideal in  $R$  is generated by an idempotent element. Let  $E/\sim$  denote the set of idempotent elements in  $R$  under the equivalence  $e_1 \sim e_2$  if the two generate the same ideal. Then  $E/\sim$  forms a lattice where the join and meet correspond to ideal addition and intersection, respectively.

Kaplansky (1998) proved that any projective  $R$ -module is some direct sum of  $(e)$  with  $e$  idempotent. It follows that  $E/\sim$  determines  $K_0(R)$ .

**Proposition 7.** *Let  $R$  be commutative. It can be shown that the following are equivalent.*

1.  $R_{\text{red}}$  is a commutative von Neumann regular ring.
2.  $R$  has (Krull) dimension 0.
3.  $\text{Spec}(R)$  is compact, Hausdorff, and totally disconnected. (This is a very strong condition.)

**Lemma 8.** *If  $I \subset R$  is nilpotent, then it's not hard to show that  $\mathbf{P}(R/I) \cong \mathbf{P}(R)$ , hence  $K_0(R) \cong K_0(R/I)$ .*

**Definition 12.** Let  $R$  be a commutative ring. The *rank* of a finitely generated projective  $R$ -module  $P$  at a prime ideal  $\mathfrak{p}$  is the function

$$\text{rk} : \text{Spec}(R) \rightarrow \mathbb{N} \quad \mathfrak{p} \mapsto \dim_{R_{\mathfrak{p}}}(P \otimes R_{\mathfrak{p}}).$$

**Proposition 9.** *The rank of a finitely generated projective module is*

1. continuous.
2. a semiring homomorphism.

**Definition 13.** An  $R$ -module  $M$  is a *componentwise free module* if we have  $R = \prod_{i=1}^n R_i$  and  $M \cong \prod_{i=1}^n R_i^{c_i}$  for some integers  $c_i$ . Note that  $M$  must be projective in this case.

**Lemma 10.** *Let  $R$  be commutative. The monoid  $L$  of finitely generated componentwise free  $R$ -modules has isomorphism to  $[\text{Spec}(R), \mathbb{N}]$ .*

*Proof.* Let  $f : \text{Spec}(R) \rightarrow \mathbb{N}$  be continuous. By some point-set topology, we see that  $\text{im } f$  is finite, say  $\{n_1, \dots, n_c\}$ . It's also possible to write  $R = R_1 \times \dots \times R_c$ . Then  $R^f := R_1^{n_1} \times \dots \times R_c^{n_c}$  is a finitely generated componentwise free  $R$ -module. Moreover,  $f \mapsto R^f$  has inverse  $\text{rk}$  restricted to componentwise free modules.  $\square$

**Theorem 11.** (Pierce) *If  $R$  is a 0-dimensional commutative ring, then*

$$K_0(R) \cong [\text{Spec}(R), \mathbb{Z}],$$

where  $[X, Y]$  denotes the semiring of continuous maps  $f : X \rightarrow Y$ .

*Proof.* We have that  $R_{\text{red}}$  is a commutative von Neumann regular ring by Proposition 20. Any ideal  $(d)$  in  $R_{\text{red}}$  where  $d$  is idempotent is componentwise free. By Kaplansky, every object  $X$  of  $\mathbf{P}(R)$  is therefore componentwise free. Therefore,  $\mathbf{P}(R_{\text{red}}) \cong [\text{Spec}(R_{\text{red}}), \mathbb{N}]$ , giving  $K_0(R_{\text{red}}) \cong [\text{Spec}(R_{\text{red}}), \mathbb{Z}]$ . By Lemma 21 and the fact that  $\text{Spec}(R_{\text{red}})$  is homeomorphic to  $\text{Spec}(R)$ , it follows that  $K_0(R) \cong [\text{Spec}(R_{\text{red}}), \mathbb{Z}] \cong [\text{Spec}(R), \mathbb{Z}]$ .  $\square$

*Remark 3.* When  $R$  is commutative, let  $H_0(R) := [\text{Spec}(R), \mathbb{Z}]$ . If  $R$  is Noetherian, then  $H_0(R) \cong \mathbb{Z}^c$  where  $c < \infty$  denotes the number of components of  $H_0(R)$ . If  $R$  is a domain, then  $H_0(R)$  is connected, implying  $H_0(R) \cong \mathbb{Z}$ .

The submonoid  $L \subset \mathbf{P}(R)$  of componentwise free modules is cofinal, so that  $K(L) \leq K_0(R)$ . Moreover,  $K(L) \cong H_0(R)$  by Lemma 25.

The rank of a projective module induces a homomorphism  $\text{rank} : K_0(R) \rightarrow H_0(R)$ . Since  $\text{rank}(R^f) = f$  for any  $R^f \in L$ , we see that

$$1 \longrightarrow H_0(R) \cong K(L) \hookrightarrow K_0(R) \xrightarrow{\text{rank}} H_0(R) \longrightarrow 1$$

splits. This implies that

$$K_0(R) \cong H_0(R) \oplus \tilde{K}_0(R),$$

where  $\tilde{K}_0(R)$  denotes  $\ker(\text{rank})$ .

**Example 14.** The *Whitehead group* of a group  $G$  is the quotient  $Wh_0(G) = K_0(\mathbb{Z}[G])/\mathbb{Z}$ , where  $\mathbb{Z}[G]$  denotes the group ring. The augmentation map  $f : \mathbb{Z}[G] \rightarrow \mathbb{Z}$  induces a split exact sequence

$$1 \longrightarrow Wh_0(G) \longrightarrow K_0(\mathbb{Z}[G]) \longrightarrow K_0(\mathbb{Z}) = \mathbb{Z} \longrightarrow 1.$$

Hence  $K_0(\mathbb{Z}[G]) \cong \mathbb{Z} \oplus Wh_0(G)$ . We know due to Swan that if  $G$  is finite, then  $Wh_0(G) \cong \tilde{K}_0(\mathbb{Z}[G])$  and  $\mathbb{Z} \cong H_0(\mathbb{Z})$ .

**Definition 15.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *additive* if  $F : \mathcal{C}(X, Y) \rightarrow \mathcal{D}(FX, FY)$  is a homomorphism of abelian groups for any  $X, Y \in \text{ob } \mathcal{C}$ .

**Definition 16.** The rings  $R$  and  $S$  are *Morita equivalent* if there exists an additive equivalence between  $\mathbf{Mod}_R R$  and  $\mathbf{Mod}_S$ .

**Theorem 12.** If  $R$  and  $S$  are Morita equivalent, then  $K_0(R) \cong K_0(S)$ .

*Proof.* Click [here](#) for a self-contained proof. □

[[We move from algebraic to topological  $K$ -theory.]]

**Definition 17.** Let  $f : F \rightarrow X$  and  $g : G \rightarrow X$  be vector bundles. The *Whitney sum* of  $f$  and  $g$  is the vector bundle  $F \oplus G$  on  $X$  whose fiber at  $x \in X$  is  $F_x \oplus G_x$ . The *tensor product bundle*  $F \otimes G$  is defined similarly.

**Definition 18.** A *vector bundle homomorphism* between  $\phi : E_1 \rightarrow X_1$  and  $\psi : E_2 \rightarrow X_2$  is a pair of maps  $f : E_1 \rightarrow E_2$  and  $g : X_1 \rightarrow X_2$  such that the following conditions holds.

1.

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ \phi \downarrow & & \downarrow \psi \\ X_1 & \xrightarrow{g} & X_2 \end{array}$$

2. For each  $x \in X_1$ , the map  $f \upharpoonright_{\phi^{-1}(x)} : \phi^{-1}(x) \rightarrow \psi^{-1}(g(x))$  is a linear map.

**Definition 19.** Let  $(\text{Vect}_{\mathbb{F}}(X), \oplus)$  denote the abelian monoid of (isomorphism classes of)  $\mathbb{F}$ -vector bundles on the paracompact space  $X$ . We define

$$KU(X) = K(\text{Vect}_{\mathbb{C}}(X)) \quad KO(X) = K(\text{Vect}_{\mathbb{R}}(X)).$$

Note that these are commutative rings with identity. **We apply the notation  $K_{\text{top}}(-)$  on topological spaces when we wish to omit the base field.**

*Remark 4.*  $KU(-)$  and  $KO(-)$  define contravariant functors  $\mathbf{Top} \rightarrow \mathbf{Ab}$ . Let  $f : Y \rightarrow X$  be a map of spaces and  $\phi : E \rightarrow X$  be a vector bundle. Define the subspace  $f^*E = \{(y, e) \in Y \times E : f(y) = \phi(e)\}$ . Define the vector bundle  $f^*(\phi) : f^*E \rightarrow Y$  as the restriction of the projection map  $\pi : Y \times E \rightarrow Y$ . Hence we have a morphism  $\phi \mapsto f^*(\phi) \rightarrow \text{Vect}_{\mathbb{F}}(X)$  to  $\text{Vect}_{\mathbb{F}}(Y)$  of monoids. The universal property of  $K$  induces a unique morphism  $f^* : K_{\text{top}}(X) \rightarrow K_{\text{top}}(Y)$ .

**Lemma 13.** *If  $X$  and  $Y$  are homotopy equivalent, then  $K(X) \cong K(Y)$ .*

*Proof.* Apply the homotopy invariance theorem (HIT), which states that if  $Y$  is paracompact and  $f, g : Y \rightarrow X$  are homotopic, then  $f^*E \cong g^*E$  for any vector bundle  $E$  over  $X$ .  $\square$

**Example 20.**

1.  $K_{\text{top}}(*) = \mathbb{Z}$ .
2. If  $X$  is contractible, then the HIT implies  $KO(X) = KU(X) = \mathbb{Z}$
3. We compute the following groups. See I.4.9 of *The K-book* for a justification.

$$\begin{aligned} KO(S^1) &\cong \mathbb{Z} \times C_2 & KU(S^1) &\cong \mathbb{Z} \\ KO(S^2) &\cong \mathbb{Z} \times C_2 & KU(S^2) &\cong \mathbb{Z} \times \mathbb{Z} \\ KO(S^3) &\cong KU(S^3) &&\cong \mathbb{Z} \\ KO(S^4) &\cong KU(S^4) &&\cong \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

**Definition 21.** The *dimension* of bundle  $E$  over  $X$  is the continuous homomorphism  $\widehat{\dim}(E) : X \rightarrow \mathbb{N}$  given by  $x \mapsto \dim(E_x)$ .

**Definition 22.** A vector bundle  $p : E \rightarrow X$  is a *componentwise trivial bundle* if we can write  $X = \coprod X_i$  such that each  $X_i$  is a component of  $X$  and  $p|_{p^{-1}(X_i)}$  is trivial.

**Lemma 14.** *The submonoid of componentwise trivial bundles over  $X$  is isomorphic to  $[X, \mathbb{N}]$ .*

*Proof.* Send a given map  $f : X \rightarrow \mathbb{N}$  to  $T^f := \coprod_{i \in \mathbb{N}} (f^{-1}(i) \times \mathbb{F})$ . Conversely, if  $E$  be a componentwise trivial bundle, then  $E \cong T^{\widehat{\dim}(E)}$ .  $\square$

*Remark 5.* Thus, the sub-monoid of trivial bundles and the sub-monoid of componentwise trivial bundles are naturally isomorphic to  $\mathbb{N}$  and  $[X, \mathbb{N}]$ , respectively. When  $X$  is compact, these are cofinal in  $\text{Vect}_{\mathbb{F}}(X)$  by the subbundle theorem (proven using Riemannian geometry), giving  $\mathbb{Z} \leq [X, \mathbb{Z}] \leq K_{\text{top}}(X)$ .

*Remark 6.* We get a split exact sequence.

$$1 \longrightarrow \widetilde{K}_{\text{top}}(X) \longrightarrow K_{\text{top}}(X) \xrightarrow[\widehat{\dim}]{\quad \quad} [X, \mathbb{Z}] \longrightarrow 1 ,$$

where  $\widetilde{K}_{\text{top}}(X)$  denotes  $\ker(\widehat{\dim})$ .

*Remark 7.* The map of monoids  $\text{Vect}_{\mathbb{R}}(X) \rightarrow \text{Vect}_{\mathbb{C}}(X)$  given by  $[E] \mapsto [E \otimes \mathbb{C}]$  extends by universality to a homomorphism  $KO(X) \rightarrow KU(X)$ . Likewise, the forgetful functor  $\text{Vect}_{\mathbb{C}}(X) \rightarrow \text{Vect}_{\mathbb{R}}(X)$  extends to a homomorphism  $KU(X) \rightarrow KO(X)$ .

**Theorem 15.** (*Swan*) *Here is a nice early connection between algebraic and topological K-theory. Let  $X$  be a compact Hausdorff space and  $\mathcal{C}(X, \mathbb{F})$  denote the ring of continuous functions  $X \rightarrow \mathbb{F}$ . For any  $E \in \text{Vect}_{\mathbb{F}}(X)$ , set  $\Gamma(X, E) = \{s : X \rightarrow E : p \circ s = \text{Id}_X\}$ , the vector space of global sections of  $E$ . Then the map  $E \mapsto \Gamma(X, E)$  induces isomorphisms  $KO(X) \cong K_0(\mathcal{C}(X, \mathbb{R}))$  and  $KU(X) \cong K_0(\mathcal{C}(X, \mathbb{C}))$ .*

**Definition 23.** Our results thus far can be extended to symmetric monoidal categories because these come equipped with a notion of direct sum that enabled our Grothendieck construction. A *symmetric monoidal category*  $S$  is equipped with a functor  $\square : S \times S \rightarrow S$ , a base object  $e$ , and four natural isomorphisms expressing commutativity, associativity, and that  $e$  acts as an identity. These four must also satisfy coherence properties.

**Example 24.** The following are examples of symmetric monoidal category .

1.  $k$ -vector spaces with  $\otimes_k$ .
2. Any category with finite coproducts where  $s \sqcup t := s \amalg t$ .
3. The category of pointed topological spaces where  $s \sqcap t := s \wedge t$  and  $e := S^0$ .

**Definition 25.** Suppose that the class of isomorphism classes of objects of a category  $S$  is a set, called  $S^{\text{iso}}$ . If  $S$  is symmetric monoidal, then  $(S^{\text{iso}}, \sqcup)$  is an abelian monoid with identity element  $e$ . Then we define the *Grothendieck group* of  $S$  as  $K_0(S)$ .