

### Abstract

This project briefly describes the isometries of  $\mathbb{C}^2$ . In particular, it classifies five important groups of such maps in the category **Top** of topological spaces. Thanks to Steven Rosenberg for his guidance on this topic.

## 1 Isometries of $\mathbb{C}^2$ over $\mathbb{R}$

If  $M$  is a metric space, then let  $\text{Isom}(M)$  denote the set of all isometries of  $M$ . For now, let  $(\mathbb{C}^2, \|\cdot\|)$  denote the normed vector space  $\mathbb{C}^2$  over  $\mathbb{R}$  where  $\|\cdot\| : \mathbb{C}^2 \rightarrow [0, \infty)$  is given by

$$\|(z, w)\| = \sqrt{z\bar{z} + w\bar{w}}.$$

That is,  $\|\cdot\|$  is exactly the norm induced by the (Euclidean) inner product  $\langle (z, w), (z, w) \rangle$ . Then  $\mathbb{C}^2 \cong \mathbb{R}^4$  as normed vector spaces via the map  $T : \mathbb{C}^2 \rightarrow \mathbb{R}^4$  given by

$$(a + bi, a' + b'i) \mapsto (a, a', b, b'). \quad (*)$$

Endow  $\mathbb{C}^2$  and  $\mathbb{R}^4$  with the standard Euclidean metrics  $d$  and  $d'$ , respectively. Since  $\|T(\vec{v})\| = \|\vec{v}\|$  and  $T$  is linear, we see that

$$d(\vec{v}, \vec{x}) = \|\vec{v} - \vec{x}\| = \|T(\vec{v}) - T(\vec{x})\| = d'(T(\vec{v}), T(\vec{x}))$$

for any  $\vec{v}, \vec{x} \in \mathbb{C}^2$ . Likewise, we see that

$$d(T^{-1}(\vec{y}), T^{-1}(\vec{z})) = \|T^{-1}(\vec{y}) - T^{-1}(\vec{z})\| = \|\vec{y} - \vec{z}\| = d'(\vec{y}, \vec{z})$$

for any  $\vec{y}, \vec{z} \in \mathbb{R}^4$ . Thus, the map  $f \mapsto T \circ f \circ T^{-1}$  defines a group isomorphism  $\text{Isom}(\mathbb{C}^2) \xrightarrow{\cong} \text{Isom}(\mathbb{R}^4)$ , provided that both  $\text{Isom}(\mathbb{C}^2)$  and  $\text{Isom}(\mathbb{R}^4)$  are, in fact, groups under composition. Certainly they are closed under composition and contain the identity map. Also, every isometry  $f$  of a given metric space  $(X, \rho)$  must be injective. Indeed, if  $x \neq y$  but  $f(x) = f(y)$ , then  $\rho(x, y) \neq 0 = \rho(f(x), f(y))$ , which is impossible. Since the inverse of  $f$  must also be an isometry, it just remains to show that  $f$  is surjective in order to prove that the two are groups. This is the content of Corollary 1.12 below.

We can form the group

$$\text{O}(4) := \{f \in \text{Isom}(\mathbb{R}^4) : f \text{ fixes } \vec{0}\}.$$

For each  $\vec{v} \in \mathbb{R}^4$ , define  $T_{\vec{v}} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  by  $\vec{x} \mapsto \vec{x} + \vec{v}$ .

**Lemma 1.1.** *Any  $A \in \text{Isom}(\mathbb{R}^4)$  can be written uniquely as  $T_{A(\vec{0})} \circ g$  for some  $g \in \text{O}(4)$ .*

*Proof.* Define  $g : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  by  $A(\vec{v}) - A(\vec{0})$ . Then  $g \in \text{O}(4)$ , and  $A(\vec{v}) = T_{A(\vec{0})} \circ g(\vec{v})$  for any  $\vec{v}$ . Further, if  $A = T_{A(\vec{0})} \circ k$  for some  $k \in \text{O}(4)$ , then  $g(\vec{v}) = A(\vec{v}) - A(\vec{0}) = k(\vec{v})$ , thereby proving uniqueness.  $\square$

**Definition 1.2.** A matrix  $X \in \mathbb{M}^4(\mathbb{R})$  is *orthogonal* if its column vectors are orthonormal.

**Proposition 1.3.** *The following are equivalent.*

(a)  $X$  is orthogonal.

(b)  $X \in \text{GL}(4, \mathbb{R})$  with  $X^T = X^{-1}$ .

**Corollary 1.4.** *Any orthogonal matrix  $X \in \mathbb{M}^4(\mathbb{R})$  preserves the inner product, i.e.,  $\langle X\vec{v}, X\vec{w} \rangle = \langle \vec{v}, \vec{w} \rangle$  for any  $\vec{v}, \vec{w} \in \mathbb{R}^4$ .*

*Proof.* We have that  $X\vec{v} \bullet X\vec{w} = \vec{v} \bullet X^T X \vec{w} = \vec{v} \bullet I \vec{w} = \vec{v} \bullet \vec{w}$ . □

*Notation.* The symbol  $\bullet$  will denote the Euclidean inner product.

**Corollary 1.5.** *If  $X \in \mathbb{M}^4(\mathbb{R})$  is orthogonal, then  $|\det(X)| = 1$ .*

*Proof.* We have that  $1 = \det(I) = \det(XX^T) = \det(X)^2$ . □

**Lemma 1.6.** *If  $X \in \mathbb{M}^4(\mathbb{R})$  is orthogonal, then  $X \in \text{O}(4)$ .*

*Proof.* By Corollary 1.4,  $X$  preserves the inner product, which implies that

$$\begin{aligned} \|X\vec{v} - X\vec{w}\|^2 &= \|X\vec{v}\|^2 - 2X\vec{v} \bullet X\vec{w} + \|X\vec{w}\|^2 \\ &= \|\vec{v}\|^2 - 2\vec{v} \bullet \vec{w} + \|\vec{w}\|^2 \\ &= \|\vec{v} - \vec{w}\|^2 \end{aligned}$$

for any  $\vec{v}, \vec{w} \in \mathbb{R}^4$ . Thus,  $d'(X\vec{v}, X\vec{w}) = d'(\vec{v}, \vec{w})$ , and  $X \in \text{O}(4)$ . □

**Definition 1.7.** An invertible linear operator  $T$  on a finite-dimensional vector space is *orientation-preserving* if  $\det M_T > 0$  and *orientation-reversing* if  $\det M_T < 0$  where  $M_T$  denotes the matrix of  $T$ .

Soon we shall prove that  $\text{O}(4) \subset \text{GL}(4, \mathbb{R})$ . Therefore, it makes sense to introduce the group

$$\text{SO}(4) := \{f \in \text{Isom}(\mathbb{R}^4) : f \text{ fixes } \vec{0} \text{ and is orientation-preserving}\}.$$

Let  $\{\vec{e}_1, \dots, \vec{e}_4\}$  denote the standard basis of  $\mathbb{R}^4$ . We are now ready to establish one of our main results.

**Theorem 1.8 (TRF-decomposition).** *Let  $\mathcal{F} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  be given either by the identity map or the reflection  $(a, b, c, d) \mapsto (a, b, c, -d)$ . Let  $A \in \text{Isom}(\mathbb{R}^4)$ . Then we have that*

$$A = T_{A(\vec{0})} \circ R' \circ \mathcal{F}$$

for some  $R' \in \text{SO}(4)$ .

*Proof.* By Lemma 1.1, we have that  $A = T_{A(\vec{0})} \circ g$  for some  $g \in \text{O}(4)$ . Since  $g$  is an isometry, we know that  $\|\vec{x} - \vec{y}\|^2 = \|g(\vec{x}) - g(\vec{y})\|^2$  for any  $\vec{x}, \vec{y} \in \mathbb{R}^4$ . As  $g$  fixes  $\vec{0}$ , it follows that  $\|g(\vec{v})\| = \|\vec{v}\|$  for any  $\vec{v} \in \mathbb{R}^4$ . We can apply the additivity of the inner product to get

$$\begin{aligned} \|g(\vec{v})\|^2 + \|g(\vec{w})\|^2 - 2\langle g(\vec{v}), g(\vec{w}) \rangle &= \langle g(\vec{v}) - g(\vec{w}), g(\vec{v}) - g(\vec{w}) \rangle \\ &= \langle \vec{v} - \vec{w}, \vec{v} - \vec{w} \rangle \\ &= \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\langle \vec{v}, \vec{w} \rangle. \end{aligned}$$

We can cancel terms to find that  $g$  preserves the inner product. Note that our proof of this fact applies to any element of  $O(4)$ .

Now, it follows that  $\|g(\vec{e}_i)\|^2 = \|\vec{e}_i\|^2 = 1$  for each  $i = 1, 2, 3, 4$ , so that  $\|g(\vec{e}_i)\| = 1$ . Similarly, we can deduce that  $\langle g(\vec{e}_i), g(\vec{e}_j) \rangle = 0$  if  $i \neq j$ . Thus,  $\{g(\vec{e}_i)\}_{i=1,2,3,4}$  is an orthonormal (hence linearly independent) set. Let

$$M := \begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ g(\vec{e}_1) & g(\vec{e}_2) & g(\vec{e}_3) & g(\vec{e}_4) \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Then  $M^T M = M M^T = I$ , so that  $M$  is invertible with  $M^T = M^{-1}$ . Lemma 1.6 implies that  $M \in O(4)$ . The isometry  $f := M^{-1} \circ g : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  satisfies  $f(\vec{0}) = \vec{0}$  and  $f(\vec{e}_i) = \vec{e}_i$  for each  $i$ .

Since  $f \in O(4)$ , it follow that

$$f(\vec{x}) \bullet f(\vec{e}_i) = \vec{x} \bullet \vec{e}_i = f(\vec{x}) \bullet \vec{e}_i = \vec{x} \bullet \vec{e}_i$$

for each  $i$ . Writing  $\vec{x} = \sum_{i=1}^4 c_i \vec{e}_i$  for some  $c_i \in \mathbb{R}$ , we have that  $f(\vec{x}) \bullet \vec{e}_i = \left(\sum_{i=1}^4 c_i \vec{e}_i\right) \bullet \vec{e}_i = c_i$ , and thus  $f(\vec{x}) = \vec{x}$ . Hence  $f = \text{Id}_{\mathbb{R}^4}$ , so that  $M = g$ . We deduce that any isometry of  $\mathbb{R}^4$  that fixes  $\vec{0}$  is given by an orthogonal matrix.

By Corollary 1.5,  $\det(g) = \pm 1$ . If  $\det(g) = 1$ , then  $g \in \text{SO}(4)$ , and we're done. Assume that  $\det(g) = -1$ . Note that the reflection

$$\phi(a, b, c, d) \equiv (a, b, c, -d)$$

is given by the matrix

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Since it's clear that  $\phi \in O(4)$ , we see that  $g \circ \phi \in O(4)$ . Also,  $\det(gS) = \det(g) \det(S) = (-1)(-1) = 1$ . Therefore,  $g \circ \phi \in \text{SO}(4)$ . As  $\phi = \phi^{-1}$ , it follows that  $(g \circ \phi) \circ \phi = g \circ (\phi^2) = g$ . Now, set  $R' = g \circ \phi$  and  $\mathcal{F} = \phi$ , thereby completing out proof.

□

By inspecting our proof of Theorem 1.8, we obtain several quick results.

**Corollary 1.9.** *If  $X \in \text{M}^4(\mathbb{R})$  preserves the inner product, then  $X$  is orthogonal.*

**Corollary 1.10.** *We have that*

$$O(4) = \{X \in \text{GL}(4, \mathbb{R}) : X \text{ is orthogonal}\}$$

$$\text{SO}(4) = \{X \in \text{GL}(4, \mathbb{R}) : X \text{ is orthogonal and } \det(X) = 1\}.$$

**Corollary 1.11.** *A function  $f$  is an element of  $\text{Isom}(\mathbb{R}^4)$  if and only if there exist  $M \in O(4)$  and  $\vec{b} \in \mathbb{R}^4$  such that for any  $\vec{x} \in \mathbb{R}^4$ ,  $f(\vec{x}) = M\vec{x} + \vec{b}$ . In this case,  $M = R' \circ \mathcal{F}$  with notation as in Theorem 1.8.*

**Corollary 1.12.** *Every  $f \in \text{Isom}(\mathbb{R}^4)$  and every  $g \in \text{Isom}(\mathbb{C}^2)$  are invertible, so that both  $\text{Isom}(\mathbb{C}^2)$  and  $\text{Isom}(\mathbb{R}^4)$  are groups under composition.*

*Proof.* Thanks to Corollary 1.11, we can write  $f(\vec{x}) = M\vec{x} + \vec{b}$ . Then it's easy to verify that  $f^{-1}(\vec{x}) = M^{-1}\vec{x} - M^{-1}\vec{b}$ .

Moreover, with  $T$  given by (\*), we find that  $g = T \circ h \circ T^{-1}$  for some  $h \in \text{Isom}(\mathbb{R}^4)$ . Hence  $g$  is the composite of three invertible functions and thus is invertible.  $\square$

**Note 1.13.** The decomposition of  $A$  given in Theorem 1.8 is unique.

*Proof.* Suppose  $A(\vec{x}) = M\vec{x} + \vec{b} = M'\vec{x} + \vec{b}'$  for every  $\vec{x} \in \mathbb{R}^4$ . Then  $\vec{b} = \vec{b}'$ , so that  $M = M'$ . Moreover, if  $M = T \circ \mathcal{F}$  for some  $T \in \text{SO}(4)$ , then  $T = M \circ \mathcal{F}$ . This shows that the decomposition  $A = T_{A(\vec{0})} \circ g \circ \mathcal{F}$  given in Theorem 1.8 is, indeed, unique.  $\square$

## 2 Isometries of $\mathbb{C}^2$ over $\mathbb{C}$

Now, view  $\mathbb{C}^2$  as a two-dimensional vector space over  $\mathbb{C}$ . Recall that the Hermitian inner product  $H : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow [0, \infty)$  is defined by  $H(x, y) = x_1\bar{y}_1 + x_2\bar{y}_2$ .

**Definition 2.1.** For any  $n \in \mathbb{N}$ , a matrix  $X \in \mathbb{M}^n(\mathbb{C})$  is *unitary* if its column vectors are orthonormal with respect to  $H$ .

Let  $U(n)$  denote the set of all unitary matrices. Lemma 2.5 below indicates that these are isometries of  $\mathbb{C}^2$ .

**Proposition 2.2.** *The following are equivalent.*

- (a)  $X \in U(2)$ .
- (b)  $X \in \text{GL}(2, \mathbb{C})$  with  $X^* = X^{-1}$ , where  $X^*$  denotes the conjugate transpose of  $X$ .

**Corollary 2.3.**  $U(n)$  is a group under composition for each  $n = 1, 2$ .

*Proof.* First, note that  $U(1) = \{z \in \mathbb{C} : |z| = 1\} = S^1$ , which is a group because the complex modulus is multiplicative and  $|z| = 1 \implies |z^{-1}| = \frac{|z|}{|z|^2} = 1$ . Next, consider  $U(2)$ . It suffices to verify closure. If  $A, B \in U(2)$ , then

$$(AB)^*(AB) = B^*A^*AB = B^*B = I,$$

and thus  $AB \in U(2)$ .  $\square$

Note that  $U(2)$  is nonabelian. Indeed, let  $A := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$  and  $B := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ . These are unitary, but  $0 \neq AB = -BA$ .

**Corollary 2.4.** Every  $2 \times 2$  unitary matrix  $X$  has  $|\det(X)| = 1$ , where  $|\cdot|$  denote the complex modulus.

*Proof.* We have that

$$\begin{aligned} 1 &= \det(I) \\ &= \det(XX^*) \\ &= \det(X)\det(X^*) \\ &= \det(X)\overline{\det(X)} \\ &= |\det(X)|. \end{aligned}$$

□

From a linear-algebraic perspective, we see that  $U(2)$  is the complex analogue of  $O(4)$ . Group-theoretically, however, we can construct an embedding  $F : U(2) \hookrightarrow SO(4)$  as follows.<sup>1</sup> For each  $M \in U(2)$ , write

$$M = \begin{bmatrix} a_1 + b_1 i & a_2 + b_2 i \\ a_3 + b_3 i & a_4 + b_4 i \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + i \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = A + iB$$

and set  $F(M) = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ . It's easy to verify that  $F(M)$  is orthogonal. Also, note that

$$\begin{aligned} \det(F(M)) &= 1 \cdot \det \left( \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \right) \cdot 1 \\ &= \det \left( \begin{bmatrix} I & 0 \\ iI & I \end{bmatrix} \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} I & 0 \\ -iI & I \end{bmatrix} \right) \\ &= \det \left( \begin{bmatrix} A + iB & -B \\ 0 & A - iB \end{bmatrix} \right) \\ &= \det(A + iB) \det(A - iB) \\ &= \det(A)^2 + \det(B)^2 \\ &= |\det(M)|^2 = 1. \end{aligned}$$

Therefore,  $F$  is well-defined. To verify that  $F$  is a homomorphism, note that if  $N = C + Di$ , then  $MN = (AC - BD) + (AD + BC)i$ . In this case

$$F(MN) = \begin{bmatrix} AC - BD & -AD - BC \\ AD + BC & AC - BD \end{bmatrix} = \begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} C & -D \\ D & C \end{bmatrix} = F(M)F(N).$$

Furthermore, if  $F(M) \in \ker(F)$ , then  $A = I_2$  and  $B = 0_2$ , i.e.,  $M = I_2$ . Hence  $\ker(F)$  is trivial, and thus  $F$  is an injective homomorphism, as desired.

In fact, the  $2 \times 2$  unitary matrices are precisely those elements of  $SO(4)$  which preserve the Hermitian inner product  $H$ . This provides us with a geometric distinction between  $U(2)$  and  $SO(4)$ .

**Lemma 2.5.** *A map  $R \in M^2(\mathbb{C})$  satisfies  $H(R(x), R(y)) = H(x, y)$  for any  $x, y \in \mathbb{C}^2$  if and only if  $R \in U(2)$ .*

*Proof.* Note that  $H(x, y) = \bar{x}^T y$ . Then

$$\begin{aligned} H(Rx, Ry) = H(x, y) &\iff \overline{Rx}^T Ry = \bar{x}^T y \\ &\iff \bar{x}^T (\overline{R}^T R) y = \bar{x}^T y \\ &\iff \overline{R}^T R = I. \end{aligned}$$

□

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<sup>1</sup>As a result,  $SO(4)$  is nonabelian and hence not isomorphic to  $SO(2)$ .

Let us look now at the complex analogue of  $\text{SO}(4)$ . The map  $D : \text{U}(2) \rightarrow \text{U}(1)$  given by  $D(X) = \det(X)$  is well-defined by Corollary 2.4. As  $\det$  is multiplicative, it is also a homomorphism. For any  $e^{i\theta} \in \mathbb{C}$ , we see that  $M := \begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \in \text{U}(2)$  and  $D(M) = e^{i\theta}$ , which means that  $D$  is surjective. Now note that

$$\ker D = K := \{X \in \text{U}(2) : \det(X) = 1\}.$$

This yields an isomorphism  $\text{U}(2)/K \cong \text{U}(1)$  in the category **Grp** of groups.

Let  $\text{SU}(2) = \ker(D)$ . Then  $\text{SU}(2)$  consists precisely of those  $2 \times 2$  unitary matrices which are orientation-preserving. Let  $W \in \text{SU}(2)$  and write  $W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Since  $\det(W) = 1$ , we find that  $W^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ . Since  $W^* = W^{-1}$ , it follows that  $d = \bar{a}$  and  $-\bar{b} = c$ . Therefore,  $\det(W) = \|(a, c)\|^2 = a\bar{a} + c\bar{c} = 1$ , and  $W = \begin{bmatrix} a & c \\ -\bar{c} & \bar{a} \end{bmatrix}$ . Conversely, the column vectors of such a matrix are orthonormal. Hence

$$\text{SU}(2) = \left\{ X \in \mathbb{M}^2(\mathbb{C}) : X = \begin{bmatrix} x & y \\ -\bar{y} & \bar{x} \end{bmatrix} \text{ with } x\bar{x} + y\bar{y} = 1 \right\}.$$

**Theorem 2.6.**  $\text{U}(2) \cong (\text{SU}(2) \times \text{U}(1)) / \mathbb{Z}_2$  in **Grp**.

*Proof.* Define  $\psi : \text{SU}(2) \times \text{U}(1) \rightarrow \text{U}(2)$  by  $(A, k) \mapsto kA$ . This map is certainly a well-defined homomorphism. Moreover, for any  $X \in \text{U}(2)$ , note that  $\sqrt{\det(X)} \in \text{U}(1)$  and  $\frac{1}{\sqrt{\det(X)}}X \in \text{SU}(2)$ , so that

$$\psi\left(\frac{1}{\sqrt{\det(X)}}X, \sqrt{\det(X)}\right) = X.$$

Thus,  $\psi$  is surjective. Finally, notice that  $\ker \psi = \{\pm(I, 1)\} \cong \mathbb{Z}_2$ . By the first isomorphism theorem, we get an isomorphism  $\tilde{\psi} : \text{U}(2) \xrightarrow{\cong} (\text{SU}(2) \times \text{U}(1)) / \mathbb{Z}_2$ , as desired. □

It turns out that  $\text{SU}(2)$  is the same as the group of unit quaternions.

**Theorem 2.7.**  $\text{SU}(2) \cong S^3$  in **Grp**.

*Proof.* For any  $x := (x_1, x_2, x_3, x_4) \in S^3$ , let  $z = x_1 + x_2i \in \mathbb{C}$  and  $w = x_3 + x_4i \in \mathbb{C}$ . Then  $x = z + wj$ . Define the map  $f : S^3 \rightarrow \text{SU}(2)$  by

$$f(x) = \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}.$$

We see that  $|x|^2 = |z|^2 + |w|^2 = \det \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$ . Hence  $x \in S^3$  if and only if  $\det \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} = 1$ , which establishes a bijection. It remains to check that  $f$  is a homomorphism. Let  $y \in S^3$  so that  $y = p + qj$ . Then since  $jw = \bar{w}j$  and  $jz = \bar{z}j$ , we obtain

$$xy = pz + pwj + q(jz) + p(jw)j = (pz - p\bar{w}) + pw + q\bar{z}j.$$

Finally, we compute

$$\begin{aligned}
f(yx) &= \begin{bmatrix} pz - q\bar{w} & pw + q\bar{z} \\ -\bar{p}w + \bar{q}\bar{z} & \bar{p}z - \bar{q}\bar{w} \end{bmatrix} \\
&= \begin{bmatrix} pz - q\bar{w} & pw + q\bar{z} \\ -\bar{p}\bar{w} - \bar{q}z & \bar{p}\bar{z} - \bar{q}w \end{bmatrix} \\
&= \begin{bmatrix} p & q \\ -\bar{q} & \bar{p} \end{bmatrix} \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \\
&= f(y)f(x).
\end{aligned}$$

□

### 3 Topology of $\text{Isom}(\mathbb{C}^2)$

Let us turn our attention to providing the groups

- $\text{SU}(2)$
- $\text{U}(2)$
- $\text{SO}(4)$
- $\text{O}(4)$
- $\text{Isom}(\mathbb{R}^4)$

with *topological* characterizations, having treated them only as algebraic objects thus far. The first four of these groups are topological spaces as subsets of normed vector spaces. The last group,  $\text{Isom}(\mathbb{R}^4)$ , has the metric topology induced by

$$d(f, g) \equiv \max \{ |f(x) - g(x)| : x \in \mathbb{R}^4, |x| \leq 1 \},$$

which is a modest generalization of the metric induced by the familiar operator norm in the theory of finite-dimensional vector spaces.

*Remark 3.1.* All five groups are Lie groups.

**Theorem 3.2.**  $\text{SU}(2) \cong S^3$  in **Top**.

*Proof.* We claim that the map  $f$  from Theorem 2.7 is a homeomorphism. Indeed, note that as  $S^3$  is a closed and bounded subset of Euclidean space, it is compact. Also,  $\text{SU}(2)$  is Hausdorff as a topological group. Thus, it suffices to show that  $f$  is continuous. By identifying each matrix in  $f$ 's codomain with a vector in  $\mathbb{C}^4$ , we find that continuity follows from the fact that complex conjugation is continuous along with the fact that continuity is preserved by addition and multiplication. □

**Corollary 3.3.**  $\text{SU}(2)$  is simply connected.

**Theorem 3.4.**  $U(2) \cong (SU(2) \times U(1)) / \mathbb{Z}_2$  in **Top**.

*Proof.* We claim that the map  $\tilde{\psi}$  from Theorem 2.6 is a homeomorphism. Indeed, it is continuous due to the universal property of quotient spaces. Moreover, its inverse is given by

$$X \mapsto \left[ \left( X \frac{1}{\sqrt{\det X}}, \sqrt{\det X} \right) \right],$$

which is continuous because both  $\sqrt{\cdot}$  and  $\det(\cdot)$  are continuous.  $\square$

**Proposition 3.5.** For any quaternions  $x$  and  $y$ , we have that  $\overline{xy} = \bar{y}\bar{x}$ .

**Corollary 3.6.**  $|xy| = |x| |y|$ , where  $|x| \equiv \sqrt{x\bar{x}}$

**Theorem 3.7.**  $SO(4) \cong S^3 \times SO(3)$  in **Top**.

*Proof.* We may identify  $\mathbb{R}^4$  with the group of quaternions. For each  $q \in S^3$ , the map  $\alpha_q : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  given by  $a \mapsto aq$  satisfies  $|aq| = |a| |q| = |a|$  thanks to Corollary 3.6. Hence for any  $a, b \in \mathbb{R}^4$ , we see that

$$|a - b| = |\alpha_q(a - b)| = |aq - bq|,$$

so that  $\alpha_q \in \text{Isom}(\mathbb{R}^4)$ . Further, since  $\alpha_q(0) = 0$ , it belongs to  $O(4)$ . Hence it preserves the Euclidean inner product.

We construct a continuous embedding  $E : O(3) \hookrightarrow O(4)$  as follows. Let  $X \in O(3)$  and  $X = \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \end{bmatrix}$  where  $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ . Then set

$$E(X) = (1, x, y, z) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \vdots & \vdots & \vdots \\ 0 & \vec{x} & \vec{y} & \vec{z} \\ 0 & \vdots & \vdots & \vdots \end{bmatrix},$$

which is an element of  $O(4)$ . Now, define  $f : S^3 \times O(3) \rightarrow O(4)$  by  $(q, (1, x, y, z)) \mapsto (q, xq, yq, zq)$ . As  $\alpha_q$  preserves the norm and the inner product, it preserves orthonormality. This means that  $f$  is well-defined. It's clear that  $f$  is continuous. Moreover,  $f$  is invertible with continuous inverse  $(v, u, r, s) \mapsto (v, (1, uv^{-1}, rv^{-1}, sv^{-1}))$ . Note that, in fact,  $(1, uv^{-1}, rv^{-1}, sv^{-1}) \in O(3)$  because  $\alpha_{v^{-1}}$  preserves orthonormality, so that in particular  $vv^{-1}$  must be orthogonal to each of the other three column vectors. Hence the first row vector must be  $(1, 0, 0, 0)$ , as required.

Finally, the restriction of  $f$  to  $S^3 \times SO(3)$  yields a homeomorphism  $S^3 \times SO(3) \xrightarrow{\cong} SO(4)$ .  $\square$

**Corollary 3.8.**  $SO(4) \cong S^3 \times \mathbb{RP}^3$ .

**Corollary 3.9.**  $O(4) \cong S^3 \times O(3)$ .

Our final result classifies the entire space  $\text{Isom}(\mathbb{R}^4)$ .

**Theorem 3.10.**  $\text{Isom}(\mathbb{R}^4) \cong O(4) \times \mathbb{R}^4$  in **Top**.

*Proof.* With notation as in Corollary 1.11, define  $F : \text{Isom}(\mathbb{R}^4) \rightarrow O(4) \times \mathbb{R}^4$  by  $f \mapsto (M, \vec{b})$ . Note 1.13 implies that  $F$  is well-defined, and Corollary 1.9 implies that it is a bijection. Note that  $F_1(f) = M = T_{-\vec{b}} \circ f$ , which is a composite of continuous functions. Further,  $F_2(f) = \vec{b} = f(\vec{0})$ . Hence each component map of  $F$  is continuous. It's clear that the inverse  $(M, \vec{b}) \rightarrow (\vec{x} \mapsto M\vec{x} + \vec{b})$  is also continuous. Thus,  $F$  is a homeomorphism.  $\square$