

## Abstract

These notes are based on Davi Maximo's lectures for the course "Geometric Analysis and Topology I" at UPenn along with John Lee's *Introduction to Smooth Manifolds*, 2nd Ed. and Michael Spivak's *A Comprehensive Introduction to Differential Geometry, Vol. 1*. Any mistake in what follows is my own.

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# 1 Smooth manifolds

## 1.1 Lecture 1

We want to make precise our notion of a (topological) space that locally looks like  $\mathbb{R}^n$ .

**Definition 1.1.1.** A space  $M$  is a (topological)  $n$ -dimensional manifold (or  $n$ -manifold) if it is

- (i) Hausdorff,
- (ii) second-countable, and
- (iii) locally Euclidean of dimension  $n$ , i.e., for any  $x \in M$ , there exist an open set  $U \ni x$  and a homeomorphism  $\varphi : U \rightarrow V$  for some open subset  $V \subset \mathbb{R}^n$ .

Condition (iii) is equivalent to making  $U$  homeomorphic to an open ball in  $\mathbb{R}^n$  or to  $\mathbb{R}^n$  itself.

**Definition 1.1.2.** Let  $M$  be an  $n$ -manifold.

1. A *coordinate chart* on  $M$  is a pair  $(U, \varphi)$  where  $U \subset M$  is open and  $\varphi$  is a homeomorphism

$$U \xrightarrow[\text{open}]{\cong} W \subset \mathbb{R}^n.$$

If  $W$  is an open ball, then we call  $U$  a *coordinate ball*.

2. If  $(U, \varphi)$  is a coordinate chart and  $\pi_i : \mathbb{R}^n \rightarrow \mathbb{R}$  denotes the  $i$ -th projection map, then we call elements of the set  $\{(\pi_1(\varphi(p)), \dots, \pi_n(\varphi(p))) \mid p \in U\}$  *local coordinates on  $U$* .

*Notation.* We shall use the symbols  $x^i$  and  $x_i$  interchangeably for local coordinates.

**Definition 1.1.3.**

1. Given charts  $(U, \varphi)$ ,  $(V, \psi)$  with  $U \cap V \neq \emptyset$ , we say that the two are  $C^k$ -compatible if the *transition map*  $\psi \circ \varphi^{-1}$

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & \varphi(U \cap V) \\ & \searrow \psi & \downarrow \psi \circ \varphi^{-1} \\ & & \psi(U \cap V) \end{array}$$

is  $C^k$ .

2. A collection of charts  $(U_\alpha, \varphi_\alpha)$  which covers a smooth manifold  $M$  and is pairwise  $C^k$ -compatible is called a  $C^k$ -atlas for  $M$ .

**Example 1.1.4.** Consider the global charts  $(\mathbb{R}, x \mapsto x)$  and  $(\mathbb{R}, x \mapsto x^3)$ . Since  $x \mapsto x^{\frac{1}{3}}$  is not differentiable at 0, these charts fail to form a  $C^1$ -atlas on  $\mathbb{R}$ .

**Definition 1.1.5.** An atlas  $A$  is *maximal* if it contains every chart that is  $C^\infty$ - (or smoothly) compatible with every chart in  $A$ .

**Proposition 1.1.6.**

1. Every smooth atlas  $A$  is contained in a unique maximal atlas, namely the family of all charts that are smoothly compatible with every chart in  $A$ .
2. Two smooth atlases are contained in the same maximal atlas if and only if their union is also a smooth atlas.

This shows that an atlas is maximal if and only if it's maximal in the usual in the set-theoretic sense.

**Definition 1.1.7.** A manifold  $M$  is *smooth* if it admits a maximal smooth atlas, also known as a *smooth structure*.

By Proposition 1.1.6, it's enough to construct any smooth atlas for  $M$  to show that it's a smooth manifold.

An open problem is whether there is more than one smooth structure on  $\mathbb{S}^4$ . This is known for each  $n \neq 4$ . For example, Milnor (1958) gave an affirmative answer for  $\mathbb{S}^7$ .

## 1.2 Lecture 2

**Proposition 1.2.1.** *If  $M$  admits a smooth structure, then  $M$  admits uncountably many smooth structures.*

*Remark 1.2.2.*

1. There exists a 10-dimensional topological manifold that admits no smooth structure (Kervaire 1961).
2. Any 2- or 3-dimensional manifold admits a smooth structure.

Let us now look at several examples of smooth structures on topological manifolds.

**Example 1.2.3.**

- (1) Any (real) vector space  $V$  where of dimension  $n < \infty$  has a canonical smooth structure as follows. Endow  $V$  with any norm, since all norms on a finite-dimensional space are equivalent and hence generate the same topology. Pick any basis  $B := (b_1, \dots, b_n)$  of  $V$ . Define the isomorphism  $T : V \rightarrow \mathbb{R}^n$  by  $b_i \mapsto e_i$  where  $e_i$  denotes the  $i$ -th standard basis vector. This is also a diffeomorphism, implying that  $V$  is a topological manifold and that  $(V, T)$  is an atlas on  $V$ . If  $B'$  is any other basis of  $V$  and  $T'$  the corresponding isomorphism, then the transition map  $T' \circ T^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a linear isomorphism, hence a diffeomorphism. By Proposition 1.1.6(2), it follows that any two bases determine the same smooth structure on  $V$ .
- (2) The restriction of a smooth structure on a smooth manifold  $M$  to an open subset  $U \subset M$  yields a smooth structure on  $U$ , which is called an *open submanifold*.

Note that the general linear group  $\mathrm{GL}(n, \mathbb{F})$  is an open subset of  $M(n, \mathbb{F})$ , which is an  $n^2$ -manifold by Example 1.2.3(1). Indeed,  $\mathrm{GL}(n, \mathbb{F}) = \det^{-1}(\mathbb{F}^\times)$ , the preimage of an open set in  $\mathbb{F}$ . By Example 1.2.3(2),  $\mathrm{GL}(n, \mathbb{F})$  is an open submanifold.

**Example 1.2.4.**

- (1) Let  $U \subset \mathbb{R}^n$  be open and  $F : U \rightarrow \mathbb{R}^m$  be continuous. Let  $\Gamma(F)$  denote the graph of  $F$  and  $\pi_1 \upharpoonright_{\Gamma(F)}$  be the restriction of the projection map  $(x, y) \mapsto x$ . This is a homeomorphism  $\Gamma(F) \xrightarrow{\cong} U$  with inverse given by  $x \mapsto (x, f(x))$ . Hence  $(\Gamma(F), \pi_1 \upharpoonright_{\Gamma(F)})$  is a smooth atlas on  $\Gamma(F)$ .

- (2) For each  $i \in \{1, 2, \dots, n+1\}$ , let  $U_i^+ := \{\vec{x} \in \mathbb{R}^{n+1} : x_i > 0\}$ . Define  $U_i^-$  similarly, so that the  $U_i^\pm$  cover the  $n$ -sphere

$$\mathbb{S}^n := \{\vec{x} \in \mathbb{R}^{n+1} : |\vec{x}| = 1\}.$$

Define the map  $f : B_1(0) \subset \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f(\vec{u}) = \sqrt{1 - |\vec{u}|^2}$ . Define  $x_i : B_1(0) \rightarrow \mathbb{R}$  by  $f(x_1, \dots, \hat{x}_i, \dots, x_n)$ . Then  $\Gamma(x_i) = U_i^+ \cap \mathbb{S}^n$ , and  $\Gamma(-x_i) = U_i^- \cap \mathbb{S}^n$ . Thanks to (1), these graphs with their corresponding projections form a smooth structure on  $\mathbb{S}^n$ .

- (3) Let  $f : U \subset \mathbb{R}^m \rightarrow \mathbb{R}$  be smooth. For each  $c \in \mathbb{R}$ , let  $M_c := f^{-1}(c)$ . Assume that the total derivative  $\nabla f(a)$  is nonzero for each  $a \in M_c$ . Then  $f_{x_i}(a) \neq 0$  for some  $1 \leq i \leq m$ . By the implicit function theorem, there is some smooth function  $F : \mathbb{R}^{m-1} \rightarrow \mathbb{R}$  given by  $x_i = F(x_1, \dots, \hat{x}_i, \dots, x_m)$  on some neighborhood  $U_a \subset \mathbb{R}^m$  of  $a$  such that  $f^{-1}(c) \cap U_a$  equals the graph of  $F$ . This means that the open sets  $f^{-1}(c) \cap U_a$  together with their graph coordinates define a smooth atlas on  $M_c$ .

**Example 1.2.5 (Real projective space).** For each  $i \in \{1, 2, \dots, n+1\}$ , let  $\tilde{U}_i := \{\vec{x} \in \mathbb{R}^{n+1} : x_i \neq 0\}$ . Let  $\pi : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$  be the quotient map and  $U_i := \pi(\tilde{U}_i)$ . Since  $\tilde{U}_i$  is saturated and open, we know that  $\pi|_{\tilde{U}_i}$  is a quotient map.<sup>1</sup> Define  $f_i : U_i \rightarrow \mathbb{R}^n$  by

$$[x_1, \dots, x_{n+1}] \mapsto \left( \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i} \right),$$

whose inverse is given by  $(x_1, \dots, x_n) \mapsto [x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n]$ . Since  $f_i \circ \pi$  is continuous, so is  $f_i$ .<sup>2</sup> Hence  $f_i$  is a homeomorphism. It's easy to check that each transition  $f_i \circ f_j^{-1}$  is smooth. Thus,  $(U_i, f_i)$  defines a smooth atlas on  $\mathbb{RP}^n$ .

**Exercise 1.2.6.** Show that  $\mathbb{RP}^n$  is second countable and Hausdorff.

*Proof.* Recall that  $\mathbb{S}^n / \sim \cong \mathbb{RP}^n$  where  $x \sim y$  if  $y = -x$ . Thus it suffices to show these properties are true of  $P^n := \mathbb{S}^n / \sim$ .

To this end, let  $\mathcal{B} := \{V_n\}$  denote the usual countable basis of  $\mathbb{S}^n$  inherited from  $\mathbb{R}^{n+1}$ . If  $p \in U \subset P^n$  is open, then  $\pi^{-1}(U)$  is a neighborhood of  $\pi^{-1}(p)$ , which equals  $\{a, -a\}$  for some point  $a$  on the sphere. There exist  $q \in \mathbb{Q}$  and  $r \in \mathbb{Q}^{n+1}$  such that  $\mathcal{B} \ni B_q(r) \cap \mathbb{S}^n \ni a$ . In this case,  $\mathcal{B} \ni B_q(-r) \cap \mathbb{S}^n \ni -a$ . Note that the union of these two balls is contained in  $\pi^{-1}(U)$  and is saturated, hence is mapped to a neighborhood  $N \subset U$  of  $p$ . Thus  $\{\pi(V_n)\}_{n \in \mathbb{N}}$  is a countable basis of  $P^n$ .

Proving that  $\mathbb{RP}^n$  is Hausdorff is quite similar. □

**Example 1.2.7 (Product manifold).** Let  $M_1 \times \dots \times M_k$  be a product of  $n_i$ -dimensional smooth manifolds. Then this is a smooth manifold of dimension  $n_1 + \dots + n_k$ .

**Lemma 1.2.8 (Smooth manifold construction).** Let  $M$  be a set and let  $\{U_\alpha\}$  be a collection of subsets equipped with injections  $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}^n$  such that

(i) countably many  $U_\alpha$  cover  $M$ ,

(ii) each  $\varphi_\alpha(U_\alpha)$  is open,

<sup>1</sup>Munkres, James. *Topology*. Theorem 22.1.

<sup>2</sup>Ibid. Theorem 22.2.

- (iii) any set of the form  $\varphi_\alpha(U_\alpha \cap U_\beta)$  or  $\varphi_\beta(U_\alpha \cap U_\beta)$  is open,
- (iv) if  $U_\alpha \cap U_\beta \neq \emptyset$ , then  $\varphi_\alpha \circ \varphi_\beta^{-1}$  is smooth, and
- (v) if  $p, q \in M$  with  $p \neq q$ , then either both are in  $U_\alpha$  for some  $\alpha$  or they can be separated by sets in  $\{U_\alpha\}$ .

Then  $M$  has a unique smooth manifold structure with  $(U_\alpha, \varphi_\alpha)$  as charts.

*Notation.* The expression  $M^n$  means that  $M$  is an  $n$ -dimensional manifold.

**Definition 1.2.9.** If  $f : M^n \rightarrow \mathbb{R}$  is a function with  $M$  smooth, we say that  $f$  is *differentiable at  $p$*  if there is some chart  $(U_\alpha, \varphi_\alpha)$  such that the coordinate representation  $f \circ \varphi_\alpha^{-1} : \varphi(U_\alpha) \rightarrow \mathbb{R}$  is differentiable at  $p$ .

We must ensure that Definition 1.2.9 is coordinate-independent.

**Lemma 1.2.10.** If  $f \circ \varphi^{-1}$  is differentiable at  $\varphi(p)$  and  $\psi : V \rightarrow \mathbb{R}^n$  is another coordinate neighborhood of  $p \in M^n$ , then  $f \circ \psi^{-1} : \psi(V) \rightarrow \mathbb{R}$  is also differentiable at  $\varphi(p)$ .

*Proof.* This holds because

$$\begin{array}{ccc}
 U \cap V & \xrightarrow{f} & \mathbb{R} \\
 \varphi \downarrow & \searrow \scriptstyle f \circ \varphi^{-1} & \uparrow \scriptstyle f \circ \psi^{-1} \\
 \varphi(U \cap V) & \xleftarrow{\varphi \circ \psi^{-1}} & \psi(U \cap V)
 \end{array}$$

commutes. □

## 2 Smooth maps

### 2.1 Lecture 3

**Definition 2.1.1.** Let  $M^n$  and  $N^k$  be smooth manifolds. We say that  $F : M \rightarrow N$  is *smooth at  $p \in M$*  if there are charts  $(V, \varphi) \ni p$  and  $(V', \psi) \ni F(p)$  with  $F(V) \subset V'$  such that the coordinate representation  $\psi \circ F \circ \varphi^{-1}$  is smooth.

$$\begin{array}{ccc}
 V & \xrightarrow{F} & V' \\
 \varphi \downarrow & & \downarrow \psi \\
 \varphi(V) & \xrightarrow{\psi \circ F \circ \varphi^{-1}} & \psi(V')
 \end{array}$$

This definition is independent of coordinates. Indeed, if  $(U, \bar{\varphi})$  and  $(U', \bar{\psi})$  are other charts around  $p$  and  $F(p)$ , respectively, then

$$\begin{aligned}
 \bar{\psi} \circ F \circ \bar{\varphi}^{-1} &= (\bar{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \\
 \psi \circ F \circ \bar{\varphi}^{-1} &= (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ \bar{\varphi}^{-1}),
 \end{aligned}$$

which are smooth as composites of smooth maps.

**Lemma 2.1.2.** *Smoothness implies continuity.*

*Proof.* Using notation as in Definition 2.1.1, we see that for each  $p \in M$ , there is a neighborhood  $V$  of  $p$  such that  $F|_V = \psi^{-1} \circ \psi \circ F \circ \varphi^{-1} \circ \varphi$  is a composite of continuous maps (as we know smoothness implies continuity for maps between Euclidean spaces) and thus itself continuous. We can glue these restrictions together to conclude that  $F$  is continuous.  $\square$

**Note 2.1.3.** Being smooth is a local property of maps.

1. Given  $F : M \rightarrow N$ , if every  $p \in M$  has a neighborhood  $U_p$  so that  $F|_{U_p}$  is smooth, then  $F$  is smooth.
2. Conversely, the restriction of any smooth map to an open subset is smooth.

**Example 2.1.4.** The natural projection  $\pi : \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{RP}^n$  is smooth. Let  $v \in (\mathbb{R}^{n+1} \setminus \{0\}, \text{id})$ . Let  $(U_i, \varphi_i) \in A_n$  be a neighborhood of  $\pi(p)$ . Since  $\pi$  is continuous,  $S := \pi^{-1}(U_i) \cap (\mathbb{R}^{n+1} \setminus \{0\})$  is a neighborhood of  $v$ . Further,  $\varphi_i \circ \pi \circ \text{id} : S \rightarrow \varphi_i(U_i)$  is given by  $x \mapsto \frac{(x_1, \dots, \hat{x}_i, \dots, x_{n+1})}{x_i}$ , which is smooth.

**Definition 2.1.5.** A smooth map with a smooth inverse is a *diffeomorphism*.

This defines an equivalence relation  $\approx$  between smooth manifolds. Thanks to Lemma 2.1.2, any diffeomorphism is a homeomorphism, which gives us the following result.

**Theorem 2.1.6.** If  $M^n \approx N^k$ , then  $n = k$ .

**Example 2.1.7.**

1.  $(\mathbb{R}, \text{id}) \approx (\mathbb{R}, x \mapsto x^{\frac{1}{3}})$  via the mapping  $x \mapsto x^3$ .
2.  $F : \mathbb{B}^n \rightarrow \mathbb{R}^n$  given by  $F(x) = \frac{x}{\sqrt{1-|x|^2}}$  is a diffeomorphism with inverse  $G(y) = \frac{y}{\sqrt{1+|y|^2}}$ .
3.  $\mathbb{S}^n / \sim \approx \mathbb{RP}^n$ .
4. If  $M$  is a smooth manifold and  $(U, \varphi)$  is a chart, then  $\varphi : U \rightarrow \varphi(U)$  is a diffeomorphism.

At this point, we want to develop tools with which we can glue together already locally defined smooth functions  $U_\alpha \rightarrow \mathbb{R}$  to obtain a globally defined smooth function  $M \rightarrow \mathbb{R}$ .

**Definition 2.1.8.** If  $M$  is any space and  $f : M \rightarrow \mathbb{R}^n$  is continuous, then the *support* of  $f$  is

$$\text{supp } f := \text{cl}(\{x \in M : f(x) \neq 0\}).$$

**Lemma 2.1.9.** Given any  $0 < r_1 < r_2$ , there is some smooth function  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

- $H = 1$  on  $\bar{B}_{r_1}(0)$ ,
- $0 < H < 1$  on  $B_{r_2}(0) \setminus \bar{B}_{r_1}(0)$ , and
- $H = 1$  elsewhere.

*Proof.* We construct such an  $H$ . First recall that  $f : \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

is smooth. Now define  $h : \mathbb{R} \rightarrow \mathbb{R}$  by  $h(t) = \frac{f(r_2-t)}{f(r_2-t)+f(t-r_1)}$ . Finally, define  $H : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $H(x) = h(|x|)$ .  $\square$

## 2.2 Lecture 4

**Definition 2.2.1.** Let  $\mathcal{U}$  and  $\mathcal{V}$  be open covers of a space  $X$ .

1.  $\mathcal{V}$  is a *refinement* of  $\mathcal{U}$  if for every  $V \in \mathcal{V}$ , there is some  $U \in \mathcal{U}$  such that  $V \subset U$ .
2.  $\mathcal{U}$  is *locally finite* if each  $x \in X$  has some neighborhood that intersects only finitely many  $U \in \mathcal{U}$ .
3.  $X$  is *paracompact* if every open cover of  $X$  admits a locally finite refinement.

We are now ready to define our main tool for patching together local functions to obtain a global one.

**Definition 2.2.2.** Let  $M$  be a space and  $\mathcal{X} := (X_\alpha)_{\alpha \in A}$  be an open cover. A *partition of unity subordinate to  $\mathcal{X}$*  is a family  $(\psi_\alpha)_{\alpha \in A}$  of continuous functions  $\psi_\alpha : M \rightarrow \mathbb{R}$  with the following properties.

- (a)  $0 \leq \psi_\alpha(x) \leq 1$  for each  $\alpha$  and  $x$ .
- (b)  $\text{supp } \psi_\alpha \subset X_\alpha$  for each  $\alpha$ .
- (c) The family  $(\text{supp } \psi_\alpha)$  is locally finite, in the sense that every point  $p \in M$  has a neighborhood  $V_p$  such that  $V_p \cap \text{supp } \psi_\alpha \neq \emptyset$  for at most finitely many  $\alpha$ . In particular,  $M$  is paracompact.
- (d)  $\sum_{\alpha \in A} \psi_\alpha(x) \equiv \sup \left\{ \sum_{\alpha \in F} \psi_\alpha(x) : \substack{F \subset A \\ \text{finite}} \right\} = 1$  for each  $x$ .

**Lemma 2.2.3.** Every topological manifold  $M$  is paracompact.

Before proving this, let us recall that a subspace is *precompact* if its closure is compact.

*Proof.* Since  $M$  has a countable atlas, it has a countable basis  $\{B_n\}$  of precompact coordinate balls. (The continuous image of a precompact set into a Hausdorff space is also precompact.)

Step 1: By induction, we can build a countable covering  $\{U_n\}$  of precompact sets such that  $\text{cl}(U_{n-1}) \subset U_n$  and  $B_n \subset U_n$  for each  $n$ .

Step 2: We build a countable locally finite open cover  $\{V_n\}$ . Let

$$V_n = \begin{cases} \text{cl}(U_n) \setminus U_{n-2} & n > 2 \\ V_n = U_n & \text{otherwise} \end{cases}.$$

Note that every  $V_n$  intersects only finitely many other  $V_j$ , hence  $\{V_n\}$  is locally finite.

Step 3: Let  $\{X_\alpha\}$  be any open cover. For any  $p \in M$ , there is some  $\alpha$  with  $p \in X_\alpha$  as well as some neighborhood  $W_p$  that intersects  $V_j$  for only finitely many  $j \in \mathbb{N}$ . Set  $\widetilde{W}_p = W_p \cap X_\alpha$ . Then the  $\widetilde{W}_p$  cover  $M$ . Since each  $V_j$  is precompact by construction, we know that  $V_j$  has a finite subcover  $\widetilde{W}_{p_{j_{k_1}}}, \dots, \widetilde{W}_{p_{j_{k_j}}}$ . Then

$$V_j = \left( V_j \cap \widetilde{W}_{p_{j_{k_1}}} \right) \cup \dots \cup \left( V_j \cap \widetilde{W}_{p_{j_{k_j}}} \right),$$

and thus  $\left\{ \left( V_j \cap \widetilde{W}_{p_{j_{k_1}}} \right), \dots, \left( V_j \cap \widetilde{W}_{p_{j_{k_j}}} \right) \right\}_{j \in \mathbb{N}}$  is a locally finite refinement of  $\{X_\alpha\}$ , as desired.  $\square$

*Remark 2.2.4.* If  $X$  is connected, then  $X$  is paracompact if and only if it is second-countable.



**Theorem 2.2.5 (Existence of partition of unity).** *If  $M$  is a smooth manifold, then any open cover  $\mathcal{X} := \{X_\alpha\}_{\alpha \in A}$  of  $M$  admits a partition of unity.*

*Proof.* For each  $\alpha \in A$ , we can find a countable basis  $\mathcal{C}_\alpha$  of precompact coordinate balls centered at 0 for  $X_\alpha$ . Then  $\mathcal{C} := \bigcup_\alpha \mathcal{C}_\alpha$  is a basis for  $M$ . Since  $M$  is paracompact,  $\mathcal{X}$  admits a locally finite refinement  $\{C_i\}_{i \in \mathbb{I}}$  consisting of elements of  $\mathcal{C}$ . Note that the cover  $\{\text{cl}(B_i)\}$  is also locally finite. There are coordinate balls  $C'_i \subset X_{\alpha_i}$  such that  $C'_i \supset \text{cl}(C_i)$ . For each  $i \in \mathbb{I}$ , let  $\varphi_i : C'_i \rightarrow \mathbb{R}^n$  be a smooth coordinate map so that  $\varphi_i(C'_i) \supset \varphi(C_i)$  and  $\varphi(\text{cl}(C_i)) = \text{cl}(\varphi(C_i))$ . Define  $f_i : M \rightarrow \mathbb{R}$  by

$$f_i(x) = \begin{cases} H_i \circ \varphi_i & x \in C'_i \\ 0 & x \in M \setminus \text{cl}(C_i) \end{cases}$$

where  $H_i : \mathbb{R}^n \rightarrow \mathbb{R}$  is as in Lemma 2.1.9: a smooth function that is positive on  $\varphi_i(C_i)$  and zero elsewhere. Note that  $f_i$  is well-defined because  $f_i = 0$  on  $C'_i \setminus \text{cl}(C_i)$ . Also, it is smooth by the point-set gluing lemma for open sets.

Define  $f : M \rightarrow \mathbb{R}$  by  $f(x) = \sum_i f_i(x)$ , which is a finite sum and hence well-defined. We see that  $f$  is a smooth function and that  $f(x) > 0$  for each  $x \in M$ . Then  $g_i(x) \equiv \frac{f_i(x)}{f(x)}$  defines a smooth function  $M \rightarrow \mathbb{R}$  for each  $i$ , so that  $\sum_i g_i(x) = 1$  and  $0 \leq g_i(x) \leq 1$  for each  $x \in M$ . Note that  $\text{supp}(g_i) = \text{cl}(C_i)$ .

For each  $\alpha \in A$ , define  $\psi_\alpha : M \rightarrow \mathbb{R}$  by

$$\psi_\alpha(x) = \sum_{\substack{i \\ \alpha_i = \alpha}} g_i(x).$$

Interpret this as the zero function when there are no  $i$  such that  $\alpha_i = \alpha$ . Note that each  $\psi_\alpha$  is smooth as a finite sum of smooth functions and satisfies  $0 \leq \psi_\alpha \leq 1$ . Moreover, we have that

$$\text{supp}(\psi_\alpha) = \text{cl} \left( \bigcup_{\substack{i \\ \alpha_i = \alpha}} C_i \right) = \bigcup_{\substack{i \\ \alpha_i = \alpha}} \text{cl}(C_i).$$

Since  $\{\text{cl}(C_i)\}$  is locally finite, so is  $\{\text{supp}(\psi_\alpha)\}_{\alpha \in A}$ . Finally, the fact that  $\alpha_i \in A$  implies that

$$\sum_\alpha \psi_\alpha(x) = \sum_i g_i(x) = 1$$

for each  $x \in M$ . Therefore, we may take  $\{\psi_\alpha\}$  as our desired partition of unity.  $\square$

**Corollary 2.2.6 (Bump function).** *If  $A \subset U \subset M$  with  $A$  closed and  $U$  open in  $M$ , then there is a smooth function  $f : M \rightarrow \mathbb{R}$  such that  $f(x) = 1$  for each  $x \in A$  and  $f(x) = 0$  outside a neighborhood of  $A$ .*

*Proof.* Since  $\{U, M \setminus A\}$  is an open cover of  $M$ , there is a partition of unity  $\varphi_1, \varphi_2$  such that  $\text{supp } \varphi_1 \subset U$ ,  $\text{supp } \varphi_2 \subset M \setminus A$ , and  $\varphi_1 + \varphi_2 = 1$ . Hence  $\varphi_1 \upharpoonright_A = 1 - 0 = 1$ , and  $\varphi_1 \upharpoonright_{M \setminus U} = 0$ .  $\square$

## 2.3 Lecture 5

**Corollary 2.3.1 (Whitney).** *Let  $M$  be a smooth manifold and  $K \subset M$  be closed. Then there exists a non-negative smooth function  $f : M \rightarrow \mathbb{R}$  such that  $f^{-1}(0) = K$ .*

This means that closed subsets of smooth manifolds are completely characterized as the 0-level sets of smooth maps. Being the 0-level set of analytic maps, such as polynomials, is much more special. Any object with such a property is called an *analytic submanifold* and is studied in algebraic geometry.

*Proof.* First assume that  $M = \mathbb{R}^n$ . We have that  $M \setminus K$  is open, which is thus the union of countably many balls  $B_{r_i}(x_i)$  with  $r_i \leq 1$ . Construct, as in Lemma 2.1.9, a smooth bump function  $h : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $h(x) = 1$  on  $\bar{B}_{\frac{1}{2}}(0)$  and  $h$  is supported in  $B_1(0)$ . By our construction of  $h$ , we can verify that for each  $i \in \mathbb{N}$ , there is some  $C_i \geq 1$  that bounds any of the partials of  $h$  up through order  $i$ .

Define  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$\sum_{i=1}^{\infty} \frac{r_i^i}{2^i C_i} h\left(\frac{x - x_i}{r_i}\right).$$

Each  $i$ -th term is bounded by  $\frac{1}{2^i}$ . Thanks to the Weierstrass M-test,  $f$  is well-defined and continuous. Since  $h$  is zero outside  $B_1(0)$ , we see that  $f^{-1}(0) = K$ .

To see that  $f$  is smooth, assume by induction that  $f$  is  $C^{k-1}$  for a given  $k \geq 1$ . By the chain rule and induction, we can write any  $k$ -th partial  $D_k$  of the  $i$ -th term of the series defining  $f$  as  $\frac{(r_i)^{i-k}}{2^i C_i} D_k h\left(\frac{x - x_i}{r_i}\right)$ . As  $h$  is smooth, this expression is  $C^1$ . And since  $r_i \leq 1$  and  $C_i$  bounds all partials up to order  $i$ , it is eventually bounded by  $\frac{1}{2^i}$ . Hence the series of these expressions converges uniformly to a continuous function. By Theorem C.31 (Lee), it follows that  $D_k f$  exists and is continuous, thereby completing our induction.

Now, assume that  $M$  is arbitrary. Find a cover  $(B_\alpha)$  of smooth coordinate balls for  $M$ . Let  $\{\varphi_\alpha\}$  be a partition of unity subordinate to this cover. Note that each  $B_\alpha$  is diffeomorphic to  $\mathbb{R}^n$ . Since the property of admitting a non-negative smooth function  $f : M \rightarrow \mathbb{R}$  with  $f^{-1}(0) = K$  can be stated in the language of smooth manifolds, it is invariant under diffeomorphism. Thus, there is some non-negative smooth function  $f_\alpha : B_\alpha \rightarrow \mathbb{R}$  where  $f_\alpha^{-1}(0) = K \cap B_\alpha$  for each  $\alpha$ . Then it's straightforward to check that  $g \equiv \sum_\alpha \varphi_\alpha f_\alpha$  is as desired.  $\square$

**Corollary 2.3.2.** *Let  $M$  be a smooth manifold and  $K \subset M$  be closed. Let  $c > 0$ . Then there exists a non-negative smooth function  $f : M \rightarrow \mathbb{R}$  such that  $f^{-1}(c) = K$ .*

**Exercise 2.3.3.** *Prove that the restriction of a smooth map on  $\mathbb{R}^{n+1}$  to  $\mathbb{S}^n$  is smooth.*

## 3 Tangent vectors

### 3.1 Lecture 6

We can view the tangent space  $T_p \mathbb{S}^n$  of  $\mathbb{S}^n$  at a point  $p$  as all of the directions from  $p$  with respect to which you can find the rate of change of a smooth map  $f$  provided that you're only allowed to roam through  $\mathbb{S}^n$ . We want to generalize our notion of a tangent space to arbitrary manifolds in order to do first-order calculus on them.

*Notation.* We shall denote the space of smooth functions  $M \rightarrow \mathbb{R}$  by  $C^\infty(M)$ .

**Definition 3.1.1.** Given  $a \in \mathbb{R}^n$ , a map  $\omega : C^\infty(\mathbb{R}^n) \rightarrow \mathbb{R}$  is called a *derivation at  $a$*  if it

- (i) is linear over  $\mathbb{R}$  and

(ii) satisfies the *Leibniz rule*:

$$\omega(fg) = f(a)\omega(g) + g(a)\omega(f)$$

for any  $f, g \in C^\infty(\mathbb{R}^n)$ .

Let  $T_a\mathbb{R}^n$  denote the vector space of derivations at  $a$ .

**Note 3.1.2.** If  $f$  is constant, then  $\omega f = 0$  for any derivation  $\omega$ .

**Example 3.1.3.** For any  $u \in \mathbb{R}^n$ , recall that the directional derivative of  $f \in C^\infty(\mathbb{R}^n)$  in the direction  $u$  at  $a$  is

$$D_u f(a) \equiv \lim_{h \rightarrow 0} \frac{1}{h} (f(a + hu) - f(a)) = \left. \frac{d}{dh} \right|_{h=0} f(a + hu).$$

Then this is a derivation of  $f$  at  $a$ .

*Notation.* For any  $a \in \mathbb{R}^n$ , let  $\mathbb{R}_a^n$  denote the (real) vector space  $\{(a, v) \mid v \in \mathbb{R}^n\}$ .

**Theorem 3.1.4.** For each  $a \in \mathbb{R}^n$ , define  $L_a : \mathbb{R}_a^n \rightarrow T_a\mathbb{R}^n$  by  $v_a \mapsto D_v|_a$ . This is an isomorphism.

*Proof.* It is clear that  $L_a$  is linear. It remains to show that it is both injective and surjective.

Suppose that  $u, v \in \mathbb{R}_a^n$  and  $L_a(u) = L_a(v)$ . Then by linearity  $L_a(u - v) = 0$ , yielding

$$\left. \frac{d}{dt} \right|_{t=0} f(a + t(u - v)) = 0$$

for any smooth function  $f$ . But if  $u - v \neq 0$ , then this says that for any  $f$ , the directional derivative of  $f$  at  $a$  in the direction of a certain nonzero vector vanishes, which is clearly false. Hence  $u = v$ , and  $L_a$  is injective.

Next, suppose that  $\omega \in T_a\mathbb{R}^n$  and consider the coordinate projection  $x^i : \mathbb{R}^n \rightarrow \mathbb{R}$  for each  $i = 1, \dots, n$ . Set  $v_i = \omega(x^i)$  and write  $v = v_i e_i$ . We claim that  $L_a(v) = D_v|_a = \omega$ . By Taylor's theorem, any  $f \in C^\infty(\mathbb{R}^n)$  has an expansion

$$f(x) = f(a) + \sum_{i=1}^n f_{x_i}(a)(x_i - a_i) + c \sum_{i,j=1}^n (x_i - a_i)(x_j - a_j) \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(a + t(x - a)) dt$$

for some  $c > 0$ . Each term of the second sum is the product of two smooth functions vanishing at  $a$ . We can apply the product rule along with linearity of  $\omega$  to conclude that

$$\begin{aligned} \omega f &= \omega \left( \sum_{i=1}^n f_{x_i}(a)(x_i - a_i) \right) \\ &= \sum_{i=1}^n \omega(f_{x_i}(a)(x_i - a_i)) \\ &= \sum_{i=1}^n f_{x_i}(a)(\omega(x_i) - \omega(a_i)) \\ &= \sum_{i=1}^n f_{x_i}(a)v_i \\ &= D_v|_a f. \end{aligned}$$

□

**Corollary 3.1.5.** *We have  $\dim(T_a\mathbb{R}^n) = n$ , and the partial derivatives  $\left\{\frac{\partial}{\partial x_i}\big|_a\right\}_{1 \leq i \leq n}$  form a basis of  $T_a\mathbb{R}^n$ .*

**Definition 3.1.6.** Let  $M$  be a smooth manifold and let  $p \in M$ .

1. An  $\mathbb{R}$ -linear map  $v : C^\infty(M) \rightarrow \mathbb{R}$  is called a *derivation at  $p$*  if

$$v(fg) = f(p)v(g) + v(f)g(p)$$

for any  $f$  and  $g$ .

2. The tangent space of  $M$  at  $p$  is the vector space

$$T_pM \equiv \{\omega : C^\infty(M) \rightarrow \mathbb{R} : \omega \text{ is a derivation of } M \text{ at } p\}.$$

Any element of this space is called a *tangent vector*.

**Definition 3.1.7 (Differential of a smooth map).** Given smooth manifolds  $M$  and  $N$ , a smooth map  $F : M \rightarrow N$ , and  $p \in M$ , we define the *differential of  $F$  at  $p$*  as the map  $dF_p : T_pM \rightarrow T_{F(p)}N$  given by

$$dF_p(v)(f) = v(f \circ F).$$

*Terminology.* We call  $dF_p(v)$  the *pushforward of  $v$  by  $dF$* .

**Proposition 3.1.8.** *Let  $M$ ,  $N$ , and  $P$  be smooth manifolds,  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be smooth maps, and  $p \in M$ .*

1.  $dF_p : T_pM \rightarrow T_{F(p)}N$  is linear.
2.  $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \rightarrow T_{G(F(p))}P$ .
3.  $d(\text{id}_M)_p = \text{id} : T_pM \rightarrow T_pM$ .
4. If  $F$  is a diffeomorphism, then  $dF_p$  is an isomorphism with inverse  $d(F^{-1})_{F(p)}$ .

*Aside.* This shows that mapping  $(M, p)$  to  $T_pM$  and  $F : (M, p) \rightarrow (N, F(p))$  to  $dF_p$  defines a functor from  $\mathbf{Diff}_*$  to  $\mathbf{Vec}_{\mathbb{R}}$ , known as the *tangent space functor*.

**Lemma 3.1.9.** *Let  $v \in T_pM$  and  $f, g \in C^\infty(M)$ . Then if  $f$  and  $g$  agree on a neighborhood  $N_p$  of  $p$ , then  $vg = vf$ .*

*Proof.* Set  $h = f - g$ , so that  $h$  vanishes on  $N_p$ . We can find a smooth bump function  $\varphi : M \rightarrow \mathbb{R}$  such that  $\varphi \equiv 1$  on  $\text{supp}(h)$  and  $\text{supp}(\varphi) \subset M \setminus \{p\}$ . Then  $\varphi h(x) = h(x)$  for any  $x \in M$ . Since both  $\varphi$  and  $h$  vanish at  $p$ , it follows that  $vf - vg = vh = v(\varphi h) = 0$ .  $\square$

**Proposition 3.1.10.** *If  $M$  is an  $n$ -dimensional smooth manifold, then  $\dim(T_pM) = n$  for every  $p \in M$ .*

In particular, we identify the standard basis  $\{e_1, \dots, e_n\}$  for  $\mathbb{R}^n$  by  $e_i \leftrightarrow \left(0, \dots, 0, \frac{\partial}{\partial x_i}\big|_p, 0, \dots, 0\right)$ .

### 3.2 Lecture 7

Given a point  $p \in M$ , find a chart  $(U, \varphi) \ni p$ . Then  $d\varphi_p : T_p M \cong T_p U \rightarrow T_{\varphi(p)} \varphi(U) \cong T_p \mathbb{R}^n$  is an isomorphism. This choice of chart yields a natural choice of basis for  $T_p M$ :

$$\left\{ \frac{\partial}{\partial x_i} \Big|_p \right\}_{1 \leq i \leq n}$$

where

$$\frac{\partial}{\partial x_i} \Big|_p := (d\varphi_p)^{-1} \left( \frac{\partial}{\partial x_i} \Big|_{\varphi(p)} \right) = (d\varphi^{-1})_{\varphi(p)} \left( \frac{\partial}{\partial x_i} \Big|_{\varphi(p)} \right). \quad (*)$$

Let  $F : M \rightarrow N$  be smooth with  $M \subset \mathbb{R}^n$  and  $N \subset \mathbb{R}^m$  open. Then by the chain rule we get

$$\begin{aligned} dF_p \left( \frac{\partial}{\partial x_i} \Big|_p \right) f &= \frac{\partial}{\partial x_i} \Big|_p (f \circ F) \\ &= \frac{\partial}{\partial x_i} \Big|_p (f(F_1, \dots, F_m)) \\ &= \sum_{j=1}^m \frac{\partial f}{\partial F_j} (F(p)) \frac{\partial F_j}{\partial x_i} (p) \\ &= \sum_{j=1}^m \frac{\partial F_j}{\partial x_i} (p) \left( \frac{\partial}{\partial y_j} \Big|_{F(p)} \right) f. \end{aligned}$$

Therefore,  $dF_p$  can be represented by the familiar  $m \times n$  Jacobian matrix of  $F$  at  $p$ ,

$$DF(p) := \begin{bmatrix} \vdots & & \vdots \\ \frac{\partial F_j}{\partial x_1}(p) & \cdots & \frac{\partial F_j}{\partial x_n}(p) \\ \vdots & & \vdots \end{bmatrix},$$

which acts on  $\mathbb{R}^n \cong T_p M$ .

Now consider the general case  $F : M \rightarrow N$  smooth between manifolds. For any  $p \in M$ , choose charts  $(U, \varphi) \ni p$  and  $(V, \psi) \ni F(p)$ . Then the Euclidean map  $\hat{F} := \psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \rightarrow \psi(V)$  is smooth. If  $\hat{p} := \varphi(p)$ , it follows from  $(*)$  that  $d\hat{F}_{\hat{p}}$  is represented by the Jacobian of  $\hat{F}$  at  $\hat{p}$ . Noting that  $F \circ \varphi^{-1} = \psi^{-1} \circ \hat{F}$ , we compute

$$\begin{aligned} dF_p \left( \frac{\partial}{\partial x_i} \Big|_p \right) &= dF_p \left( d(\varphi^{-1}) \Big|_{\hat{p}} \left( \frac{\partial}{\partial x_i} \Big|_{\hat{p}} \right) \right) \\ &= d(\psi^{-1}) \Big|_{\hat{F}(\hat{p})} \left( d\hat{F}_{\hat{p}} \left( \frac{\partial}{\partial x_i} \Big|_{\hat{p}} \right) \right) \\ &= d(\psi^{-1}) \Big|_{\hat{F}(\hat{p})} \left( \sum_{j=1}^m \frac{\partial \hat{F}_j}{\partial x_i} (\hat{p}) \frac{\partial}{\partial y_j} \Big|_{\hat{F}(\hat{p})} \right) \\ &= \sum_{j=1}^m \frac{\partial \hat{F}_j}{\partial x_i} (\hat{p}) \frac{\partial}{\partial y_j} \Big|_{F(p)}. \end{aligned}$$

Therefore,  $dF_p$  can be represented by the Jacobian matrix of  $\hat{F}$  at  $\hat{p}$ .

Given any two pairs of coordinates for  $p$  and  $F(p)$ , the respective Jacobian matrices are related by the familiar change-of-basis matrix. In particular, they are similar.

Given a smooth manifold  $M$ , we define a notion of a smoothly varying tangent space as follows.

**Definition 3.2.1.** The *tangent bundle of  $M$*  is the set

$$TM \equiv \coprod_{p \in M} T_p M$$

endowed with a certain natural topology induced by the projection  $\pi : TM \rightarrow M$ ,  $(\varphi, p) \mapsto p$ .

**Example 3.2.2.** As  $\mathbb{R}_a^n$  is canonically isomorphic to  $\mathbb{R}^n$ , we have  $T\mathbb{R}^n \cong \coprod_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ .

### 3.3 Lecture 8

**Lemma 3.3.1.** For any smooth  $n$ -dimensional manifold  $M$ , the tangent bundle  $TM$  has a natural topology and smooth structure such that

- $TM$  is a  $2n$ -dimensional smooth manifold and
- the projection  $\pi : TM \rightarrow M$  is smooth.

*Proof.* Given a chart  $(U, \varphi)$ , define  $\tilde{\varphi} : \pi^{-1}(U) \rightarrow \mathbb{R}^n$  by

$$v_i \frac{\partial}{\partial x_i} \Big|_p \mapsto (x^1(p), \dots, x^n(p), v_1, \dots, v_n)$$

where  $\varphi = (x^1, \dots, x^n)$ .<sup>3</sup> This is continuous with  $\text{Im } \tilde{\varphi} = \varphi(U) \times \mathbb{R}^n$ , which is open. Further,  $\tilde{\varphi}^{-1}$  is given by  $(x_1, \dots, x_n, v_1, \dots, v_n) \mapsto v_i \frac{\partial}{\partial x_i} \Big|_{\varphi^{-1}(x)}$  on  $\varphi(U) \times \mathbb{R}^n$ . Take  $\{(\pi^{-1}(U), \tilde{\varphi})\}$  to be charts on  $TM$ . Given two such charts  $(\pi^{-1}(U), \tilde{\varphi})$  and  $(\pi^{-1}(V), \tilde{\psi})$ , it's straightforward to check that  $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \rightarrow \psi(U \cap V) \times \mathbb{R}^n$  is smooth.

Next, notice that if we take a countable cover  $\{U_i\}$  of  $M$  by smooth coordinate domains, then  $\{\pi^{-1}(U_i)\}$  satisfies the conditions of Lemma 1.2.8.

Finally, to see that  $\pi : TM \rightarrow M$  is smooth, notice that its coordinate representation at every point is given by the projection  $\pi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^n$ ,  $(x, v) \mapsto x$ .  $\square$

*Terminology.* We call the  $\tilde{\varphi}((f, p))$  the *natural coordinates* on  $TM$ .

Given  $F : M \rightarrow N$  is smooth, define the *global differential*  $dF : TM \rightarrow TN$  of  $F$  by  $dF(\varphi, p) = dF_p(\varphi)$ .

**Proposition 3.3.2.** The global differential  $dF : TM \rightarrow TN$  is smooth.

*Aside.* This shows that mapping  $M$  to  $TM$  and  $F$  to  $dF$  defines a functor from **Diff** to itself, known as the *tangent functor*.

**Note 3.3.3.** If  $F$  is a diffeomorphism, then so is  $dF$  with  $d(F^{-1}) = (df)^{-1}$ .

**Definition 3.3.4.** Given a smooth curve  $\gamma : J \rightarrow M$  and  $t_0 \in J$ , the *velocity of  $\gamma$  at  $t_0$*  is

$$\gamma'(t_0) \equiv d\gamma \left( \frac{d}{dt} \Big|_{t_0} \right) \in T_{\gamma(t_0)} M.$$

---

<sup>3</sup>The expression  $v_i \frac{\partial}{\partial x_i} \Big|_p$  is secretly a summation, in accordance with the *Einstein summation convention*.

**Note 3.3.5.** Let  $(U, \varphi) \ni \gamma(t_0)$  be a chart on  $M$ . Then  $\gamma'(t_0) = \frac{d\gamma^i}{dt} \frac{\partial}{\partial x_i} \Big|_{\gamma(t_0)}$ .

**Lemma 3.3.6.** Every  $v \in T_p M$  is the velocity of some smooth curve  $\gamma : J \rightarrow M$  at 0 such that  $\gamma(0) = p$ .

*Proof.* Let  $(U, \varphi)$  be a chart centered at  $p$ . Write  $v = v_i \frac{\partial}{\partial x_i} \Big|_p$ . For any  $\epsilon > 0$  small, define  $\gamma : (-\epsilon, \epsilon) \rightarrow U$  by  $\gamma(t) = \varphi^{-1}(tv_1, \dots, tv_n)$ . Note 3.3.5 implies that  $\gamma'(0) = v$ .  $\square$

**Proposition 3.3.7.** Let  $v \in T_p M$ . Then  $dF_p(v) = (F \circ \gamma)'(0)$  for any smooth map  $\gamma : J \rightarrow M$  satisfying  $\gamma(0) = p$  and  $\gamma'(0) = v$ .

*Aside.* A smooth function element on  $M$  is a pair  $(f, U)$  with  $U \subset M$  open and  $f : M \rightarrow \mathbb{R}$  smooth. Say that  $(f, U) \sim (g, V)$  if  $p \in U \cap V$  and  $f = g$  on some neighborhood of  $p$ . The equivalence class  $[f]_p := [(f, U)]$  is called the *germ of  $f$  at  $p$* . The set of such classes is denoted by  $C_p^\infty(M)$ . This is an associative algebra over  $\mathbb{R}$ .

Define a *derivation of  $C_p^\infty(M)$*  as a linear map  $v : C_p^\infty(M) \rightarrow \mathbb{R}$  satisfying  $v[fg]_p = f(p)v[g]_p + g(p)v[f]_p$ . The tangent space  $\mathcal{D}_p M$  of such derivations serves as an equivalent (in the sense of isomorphism) definition of the tangent space of  $M$  at  $p$ .

### 3.4 Lecture 9

**Theorem 3.4.1 (Inverse function).** If  $F : M \rightarrow N$  is smooth and  $dF_p$  is invertible, then there are connected neighborhoods  $U_0$  of  $p$  and  $V_0$  of  $F(p)$  such that  $F|_{U_0} : U_0 \rightarrow V_0$  is a diffeomorphism.

*Proof.* Notice that  $M$  and  $N$  have equal dimension (say  $n$ ) because  $dF_p$  is invertible. Choose charts  $(U, f)$  centered at  $p$  and  $(V, g)$  centered at  $F(p)$  such that  $F(U) \subset V$ . Then  $\widehat{F} := g \circ F \circ f^{-1}$  is smooth map from  $f(U) \subset \mathbb{R}^n$  to  $g(V) \subset \mathbb{R}^n$  with  $\widehat{F}(0) = 0$ . Now  $d\widehat{F}_0$  is invertible as the composite of three invertible maps. The inverse function theorem for Euclidean space implies that there are open balls  $B_r(0)$  and  $B_s(0)$  such that  $\widehat{F} : B_r(0) \rightarrow B_s(0)$  is a diffeomorphism. Thus, we can take  $F : f^{-1}(B_r(0)) \rightarrow g^{-1}(B_s(0))$  as our desired diffeomorphism.  $\square$

**Corollary 3.4.2.** If  $dF_p$  is nonsingular at each  $p \in M$ , then  $F$  is a local diffeomorphism.

**Proposition 3.4.3.**

1. The finite product of local diffeomorphisms is a local diffeomorphism.
2. The composite of two local diffeomorphisms is a local diffeomorphism.
3. Any bijective local diffeomorphism is a diffeomorphism.
4. A map  $F$  is a local diffeomorphism if and only if each point in  $\text{dom}(F)$  has a neighborhood where  $F$ 's coordinate representation is a local diffeomorphism.

**Definition 3.4.4.** The rank of a smooth map  $F$  at a point  $p$  is the rank of  $dF_p$ . If the rank of  $F$  is the same at each point, then we say  $F$  has constant rank.

**Theorem 3.4.5 (Constant rank).** Let  $F : M^m \rightarrow N^n$  be smooth with constant rank  $r \leq m, n$ . Then for each  $p \in M$ , there are charts  $(U, f)$  centered at  $p$  and  $(V, g)$  centered at  $F(p)$  such that  $F(U) \subset V$  and the coordinate representation of  $F$  is given by

$$\widehat{F}(x_1, \dots, x_r, x_{r+1}, \dots, x_m) = (x_1, \dots, x_r, 0, \dots, 0).$$

Before proving this, we should mention a couple of things:

- If  $m = n = r$ , then this follows immediately from the inverse function theorem.
- The global condition on the rank of  $F$  cannot be weakened, as the space of  $n \times m$  matrices of rank  $r$  need *not* be open. For example, consider the map  $A(t) \equiv \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$ , which has rank 2 when  $t \neq 1$  and rank 1 otherwise.

*Proof.* Since our statement is local, we may assume that  $M \subset \mathbb{R}^m$  and  $N \subset \mathbb{R}^n$  are open subsets. Since  $DF(p)$  has rank  $r$ , it has some invertible  $r \times r$  sub-matrix, which we may assume is the upper left sub-matrix  $\left(\frac{\partial F^i}{\partial x^j}\right)_{i,j \in [r]}$ . Write  $(x, y) = (x^1, \dots, x^r, y^1, \dots, y^{m-r})$  and  $(v, w) = (v^1, \dots, v^r, w^1, \dots, w^{n-r})$  for the standard coordinates on  $\mathbb{R}^m$  and  $\mathbb{R}^n$ , respectively. By applying suitable translations, we may assume that  $p = (0, 0)$  and  $F(p) = (0, 0)$ . We have  $F(x, y) = (Q(x, y), R(x, y))$  for some smooth map  $Q : M \rightarrow \mathbb{R}^r$  and  $R : M \rightarrow \mathbb{R}^{n-r}$ . Then the Jacobian matrix  $\left(\frac{\partial Q^i}{\partial x^j}\right)$  is invertible at  $(0, 0)$  by hypothesis.

Define  $f : M \rightarrow \mathbb{R}^m$  by  $(x, y) \mapsto (Q(x, y), y)$ . Define the *Kronecker delta* symbol  $\delta_i^j$  by

$$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then

$$D[f](0, 0) \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}(0, 0) & \frac{\partial Q^i}{\partial y^j}(0, 0) \\ 0 & \delta_j^i \end{bmatrix}.$$

Since

$$\det(D[f](0, 0)) = \det\left(\frac{\partial Q^i}{\partial x^j}(0, 0)\right) \cdot \det(\delta_j^i) = \det\left(\frac{\partial Q^i}{\partial x^j}(0, 0)\right) \neq 0,$$

it follows that  $D[f]$  is invertible at  $(0, 0)$ .

Thus, we can apply the inverse function theorem to get a connected open set  $U_0 \ni (0, 0)$  and an open cube  $\tilde{U}_0 \ni f(0, 0) = (0, 0)$  such that  $f : U_0 \rightarrow \tilde{U}_0$  is a diffeomorphism. Let  $f^{-1}(x, y) = (A(x, y), B(x, y))$ . Then  $(x, y) = f(A(x, y), B(x, y)) = (Q(A(x, y), B(x, y)), B(x, y))$ , so that  $y = B(x, y)$ . Hence

$$f^{-1}(x, y) = (A(x, y), y).$$

Additionally,  $Q(A(x, y), y) = x$  since  $f \circ f^{-1} = \text{id}_{\tilde{U}_0}$ . If  $\tilde{R} : \tilde{U}_0 \rightarrow \mathbb{R}^{n-r}$  is defined by  $(x, y) \mapsto R(A(x, y), y)$ , then

$$F \circ f^{-1}(x, y) = (x, \tilde{R}(x, y)).$$

Therefore,

$$D[F \circ f^{-1}](x, y) = \begin{bmatrix} \delta_j^i & 0 \\ \frac{\partial \tilde{R}^i}{\partial x^j}(x, y) & \frac{\partial \tilde{R}^i}{\partial y^j}(x, y) \end{bmatrix}$$

for any  $(x, y) \in \tilde{U}_0$ . It's clear that the first  $r$  columns of this matrix are linearly independent. But since  $f^{-1}$  is a diffeomorphism, it has rank  $r$  on  $\tilde{U}_0$ . It follows that  $\frac{\partial \tilde{R}^i}{\partial y^j}(x, y) = 0$  for each  $(x, y) \in \tilde{U}_0$ . But  $\tilde{U}_0$  was chosen to be an open cube, so that  $\tilde{R}(x, y) = \tilde{R}(x, 0)$ . If  $S(x) := \tilde{R}(x, 0)$ , then  $F \circ f^{-1}(x, y) = (x, S(x))$ .

Now, let

$$V_0 = \left\{ (v, w) \in N \mid (v, 0) \in \tilde{U}_0 \right\},$$



which is a neighborhood of  $(0, 0)$  in  $N$ . Since  $\tilde{U}_0$  is a cube, we see that  $F \circ f^{-1}(\tilde{U}_0) \subset V_0$ . Hence  $F(U_0) \subset V_0$ . Define  $g : V_0 \rightarrow \mathbb{R}^n$  by  $(v, w) \mapsto (v, w - S(v))$ , which is smooth with inverse  $g^{-1}(s, t) = (s, t + S(s))$ . Then

$$\hat{F}(x, y) = g \circ F \circ f^{-1}(x, y) = (x, S(x) - S(x)) = (x, 0),$$

as desired.  $\square$

### 3.5 Lecture 10

**Definition 3.5.1.** Consider a smooth map  $F : M \rightarrow N$ .

1. It is a (*smooth*) *submersion* if it has constant rank equal to  $\dim(N)$ .
2. It is a (*smooth*) *immersion* if it has constant rank equal to  $\dim(M)$ .

**Definition 3.5.2.** A *topological embedding* is a continuous map  $F : M \rightarrow N$  which is a homeomorphism onto  $F(M)$ .

**Example 3.5.3.**

1. The map  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$  defined by  $t \mapsto (t^3, 0)$  is a smooth topological embedding but not an immersion, since  $\gamma'(0) = 0$ .
2. The curve  $f : (-\pi, \pi) \rightarrow \mathbb{R}^2$  defined by  $f(t) = (\sin 2t, \sin t)$  is known as a *lemniscate*, describing a figure-eight curve. This is not a topological embedding, because the figure-eight is compact whereas  $(-\pi, \pi)$  is not. But it is a smooth immersion as  $f'$  never vanishes.

**Definition 3.5.4.** A map is a *smooth embedding* if it both a topological embedding and a smooth immersion.

**Example 3.5.5.**

1. There is a smooth embedding of  $\mathbb{RP}^2$  into  $\mathbb{R}^4$  but not into  $\mathbb{R}^3$ .
2. If  $U \subset M$  is open, then the inclusion  $U \hookrightarrow M$  is a smooth embedding.

**Definition 3.5.6.** A manifold  $S \subset M$  in the subspace topology is an *embedded submanifold* if it has a smooth structure such that the inclusion  $S \hookrightarrow M$  is a smooth embedding.

**Note 3.5.7.** The image of a smooth embedding is an embedded submanifold.

*Terminology.* If  $S \subset M$  is an embedded submanifold, then  $\dim(M) - \dim(S)$  is called the *codimension* of  $S$  in  $M$ .

**Proposition 3.5.8.** Let  $U \subset M^m$  be open and  $f : U \rightarrow N$  be smooth. The graph  $\Gamma(f)$  of  $f$  is an embedded  $m$ -dimensional submanifold of  $M \times N$ .

*Proof.* Define  $\gamma_f(x) : U \rightarrow M \times N$  by  $\gamma_f(x) = (x, f(x))$ . It's easy to check this is a smooth embedding.  $\square$

Our next notion is a local version of the standard embedding  $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$  where  $k \leq n$  but works for any submanifold.

**Definition 3.5.9.** We say that a subset  $S \subset M$  has the *local  $k$ -slice condition* if for each  $p \in S$ , there is a chart  $(U, \varphi) \ni p$  for  $M$  such that

$$\varphi(U \cap S) = \underbrace{\{x \in \varphi(U) : x^{k+1} = \dots = x^n = 0\}}_{k\text{-slice of } \varphi(U)}, \quad n \equiv \dim(M).$$

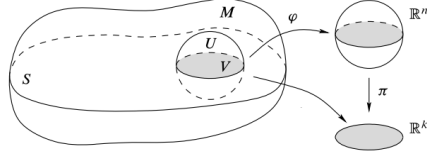


Figure 1: Copied from Lee (102)

$k$ -slice condition with  $V \equiv U \cap S$

**Theorem 3.5.10.** Let  $M^n$  be a smooth manifold. Let  $S \subset M$ . If  $S$  is an embedded manifold with  $\dim(S) = k$ , then  $S$  has the local  $k$ -slice condition.

Conversely, if  $S$  has the local  $k$ -slice condition, then  $S$  is a smooth manifold in the subspace topology and has a smooth structure making it an embedded submanifold of dimension  $k$ .

*Proof.*

( $\implies$ )

Let  $p \in S$ . In particular, the inclusion  $i : S \hookrightarrow M$  is a smooth immersion and thus has constant rank  $k$ . By the constant rank theorem, we can find charts  $(U, \varphi)$  and  $(V, \psi)$  centered at  $p$  for  $S$  and  $M$ , respectively, for which  $i$  has coordinate representation

$$(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0).$$

This means that  $i(U)$  is a  $k$ -slice for  $S$  in  $V$ . We have that  $U = W \cap S$  for some open set  $W$  in  $M$ . Let  $V' = W \cap V$ , which is neighborhood of  $p$  in  $M$ . Then  $(V', \psi|_{V'})$  is a chart on  $M$  such that  $V' \cap S = i(U)$ , so that  $V'$  is slice for  $S$  in  $M$ .

( $\impliedby$ )

See Theorem 5.8 (Lee).

□

**Example 3.5.11.** For any  $n$ ,  $\mathbb{S}^n \subset \mathbb{R}^{n+1}$  is an embedded hypersurface because it is locally the graph of smooth map and thus has the local  $n$ -slice condition.

**Theorem 3.5.12.** Let  $F : M^m \rightarrow N^n$  be smooth with constant rank  $r$ . Each level set of  $F$  is an embedded submanifold of codimension  $r$  in  $M$ .

*Proof.* Set  $k = m - r$ . Let  $c \in N$  and  $p \in F^{-1}(c)$ . By the constant rank theorem, there are charts  $(U, f)$  centered at  $p$  and  $(V, g)$  centered at  $F(p) = c$  for which  $F$  has coordinate representation given by

$$(x_1, \dots, x_r, x_{r+1}, \dots, x_m) \mapsto (x_1, \dots, x_r, 0, \dots, 0),$$

which must send each point in  $f(F^{-1}(c) \cap U)$  to 0. Thus,  $f(F^{-1}(c) \cap U)$  equals the  $k$ -slice

$$\{x \in \mathbb{R}^m : x_1 = \cdots = x_r = 0\}.$$

By Theorem 3.5.10,  $S$  is an embedded submanifold of dimension  $k$ . □

### 3.6 Lecture 11

*Question.* Can  $M^n$  with  $n \geq 1$  be homeo-/diffeomorphic to  $M \setminus \{p\}$ ?

*Remark 3.6.1.* We can generalize Theorem 3.5.12 to maps that are not necessarily of constant rank.

**Definition 3.6.2.** Let  $\varphi : M \rightarrow N$  be smooth. We say that  $p \in M$  is

- a *regular point* of  $\varphi$  if  $d\varphi_p$  is surjective and
- a *critical point* of  $\varphi$  otherwise.

**Definition 3.6.3.** Let  $\varphi : M \rightarrow N$  be smooth. We say that  $c \in N$  is

- a *regular value* of  $\varphi$  if each point in  $\varphi^{-1}(c)$  is regular and
- a *critical value* of  $\varphi$  otherwise.

We say that  $S \subset M$  is a *regular level set* of  $\varphi$  if it has the form  $\varphi^{-1}(c)$  with  $c$  a regular value.

**Theorem 3.6.4.** Every regular level set  $S$  of a smooth map  $F : M^m \rightarrow N^n$  is an embedded submanifold of codimension  $n$ .

*Proof.* Let  $S = F^{-1}(c)$ . Note that the subspace of full-rank matrices is open due to continuity of the det. As a result, the set  $U$  of points  $p \in M$  where  $dF_p$  is surjective is open in  $M$ . Hence  $F|_U : U \rightarrow N$  is a smooth submersion. In particular, it has constant rank  $n$ . Thanks to Theorem 3.5.12, it follows that  $F^{-1}(c)$  is an embedded submanifold of  $U$  with codimension  $n$ , where  $U$  itself is an open submanifold of  $M$ . □

**Example 3.6.5.**  $\mathbb{S}^n$  is a regular level set of the smooth function  $\vec{x} \mapsto |\vec{x}|^2$ .

**Theorem 3.6.6 (Sard).** If  $F : M \rightarrow N$  is smooth, then the set of all critical values of  $F$  has measure zero in  $N$ .

**Proposition 3.6.7.** Suppose  $M$  is smooth and  $S \subset M$  is embedded. Then for any  $f \in C^\infty(S)$ , there is some neighborhood  $U$  of  $S$  in  $M$  along with some  $\hat{f} \in C^\infty(U)$  such that  $\hat{f}|_S = f$ .

**Proposition 3.6.8.** The tangent space of a submanifold  $S \subset M$  at  $p \in S$  is precisely the image of the injective canonical map  $di_p : T_p S \rightarrow T_p M$  where  $i$  denotes inclusion, i.e.,

$$A := \{\gamma'(0) \in T_p M : \gamma : (-\epsilon, \epsilon) \rightarrow S \text{ and } \gamma(0) = p\}.$$

*Proof.* Let  $v \in T_p S$ . We know that  $v = \gamma'(0)$  for some curve  $\gamma$  in  $S$ . Then  $i \circ \gamma$  is a curve in  $M$  with  $(i \circ \gamma)' = di_p(v)$ .

Conversely, let  $v := w'(0) \in A$ . We have  $w = j \circ w$  where  $j : i(S) \rightarrow S$  is the reverse inclusion. Since  $(j \circ w)'(0) = dj_p(v) \in T_p S$ , it follows that  $d_i((j \circ w)'(0)) = v$ . □

At this point, we begin developing the theory of differential forms. Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  be smooth. The gradient  $\nabla F$  has two main properties.

1. It is orthogonal to the level sets of  $F$ .
2.  $dF_p(v) = \langle \nabla F_p, v \rangle$ .

But given a smooth manifold  $M$ , we don't necessarily have an inner product on  $M$  unless  $M$  is a *Riemannian manifold*, which by definition has a smoothly varying inner product. Instead, we shall view  $dF_p$  as a so-called 1-form.

### 3.7 Lecture 12

Recall that if  $\pi : M \rightarrow N$  is a continuous map, then a *section of  $\pi$*  is a continuous right inverse of  $\pi$ .

**Definition 3.7.1.** A (smooth) *vector field*  $X$  is a smooth section of the projection map  $\pi : TM \rightarrow M$ , i.e.,  $X_p := F(p) \in T_p M$  for each  $p \in M$ .

*Notation.* Let  $\mathcal{X}(M)$  denote the vector space of all smooth vector fields in  $M$ .

Note that  $\mathcal{X}(M)$  is a module over  $C^\infty(M)$  under the action  $f \cdot X \equiv (p \mapsto f(p)X_p)$ .

Given a chart  $U$  on  $M^n$ , if  $p \in U$ , then we can write  $X_p = \sum_{i=1}^n r_i \frac{\partial}{\partial x_i} \Big|_p$  for some unique real coefficients  $r_i$ . Define  $X^i : U \rightarrow \mathbb{R}$  by  $X_i(p) = r_i$  for each  $i = 1, \dots, n$ . Then

$$X_p = \sum_i X_i(p) \frac{\partial}{\partial x_i} \Big|_p.$$

We call such  $X_i$  the *component functions of  $X$*  for the chart  $U$ .

**Proposition 3.7.2.** A vector field  $X$  is smooth if and only if each component function in any given chart is smooth.

**Lemma 3.7.3.** If  $S$  is a closed subset of  $M$  and  $X$  a smooth vector field along  $S$ , then there is an extension of  $X$  to a smooth vector field on  $M$ .

**Definition 3.7.4.** Let  $U \subset M^n$  be open and  $X_1, \dots, X_k \in \mathcal{X}(M)$ .

1.  $X_1, \dots, X_k$  are *linearly independent* if for any  $p \in U$ , we have that  $\{X_1(p), \dots, X_k(p)\}$  is linearly independent in  $T_p M$ .
2. If  $k = n$  and  $X_1, \dots, X_k$  are linearly independent, then  $\{X_1, \dots, X_k\}$  is a *local frame* in  $U$ .

**Example 3.7.5.** The basis vectors  $p \mapsto \frac{\partial}{\partial x_i} \Big|_p$  form a local frame for a given chart  $U$  around  $p$ , called the *coordinate frame*.

**Definition 3.7.6.** A local frame for  $U$  is called a *global frame* if  $U = M$ . If such a frame exists, then  $M$  is called *parallelizable*.

**Example 3.7.7.**  $\mathbb{R}^n$  is parallelizable via the standard coordinate vector fields.

**Lemma 3.7.8.**  $M$  is parallelizable if and only if  $TM \approx M \times \mathbb{R}^n$ , i.e., its tangent bundle is trivial.

**Theorem 3.7.9 (Kervaire).**  $\mathbb{S}^n$  is parallelizable if and only if  $n \in \{0, 1, 3, 7\}$ .

**Definition 3.7.10 (Lie group).** A Lie group is a group  $G$  equipped with a smooth structure such that both  $\times : G \times G \rightarrow G$  and  $(-)^{-1} : G \rightarrow G$  are smooth maps.

**Example 3.7.11.** Any Lie group is parallelizable.

Note that  $\mathcal{X}(M)$  acts on  $C^\infty(U)$  for any  $U \subset M$  with the action  $X \cdot f \equiv (p \mapsto X_p(f))$ . Given  $X \in \mathcal{X}(M)$ , this induces a linear map  $X : C^\infty(U) \rightarrow C^\infty(U)$  satisfying the product rule

$$X(fg) = fXg + gXf.$$

We call such a map a *derivation* of  $C^\infty(U)$ .

Moreover, if  $F : M \rightarrow N$  is smooth, then  $dF_p X(p) \in T_{F(p)} N$  for each  $p \in M$ . Yet, this may *not* define a vector field on  $N$ , since  $F$  may not be surjective.

**Example 3.7.12.** Let  $X, Y \in \mathcal{X}(M)$ . Then  $X(Yf)$  need *not* be a derivation. Indeed, let  $M = \mathbb{R}^2$ ,  $X = \frac{\partial}{\partial x}$ , and  $Y = x \frac{\partial}{\partial y}$ . If  $f(x, y) = x$  and  $g(x, y) = y$ , then  $XY(fg) = 2x$  whereas  $fXY(g) + gXY(f) = x$ , so that  $XY(f)$  is not a derivation.

**Definition 3.7.13.** Let  $X, Y \in \mathcal{X}(M)$ . The Lie bracket of  $X$  and  $Y$  is

$$[X, Y] \equiv XY - YX : C^\infty(M) \rightarrow C^\infty(M).$$

**Proposition 3.7.14 (Clairaut).** If  $X_i = \frac{\partial}{\partial x_i} \in \mathcal{X}(M)$ , then  $[X_i, X_j] = 0$  for any  $1 \leq i, j \leq n$ .

**Lemma 3.7.15.** A map  $D : C^\infty(M) \rightarrow C^\infty(M)$  is a derivation if and only if there is some  $X \in \mathcal{X}(M)$  such that  $Df = Xf$  for any  $f$ .

*Proof.* We have established the ( $\Leftarrow$ ) direction. Conversely, assume that  $D$  is a derivation. Define  $X : M \rightarrow TM$  by  $X_p(f) = (Df)(p)$ . Since  $Df = Xf$  is smooth for each  $X$ , it follows that  $X$  is smooth thanks to Proposition 8.14 (Lee).  $\square$

**Lemma 3.7.16.** Any Lie bracket  $[X, Y]$  is a smooth vector field.

*Proof.* By Lemma 3.7.15, it suffices to show that  $[X, Y]$  is a derivation. Let  $f, g$  be smooth functions on  $M$ . Then

$$\begin{aligned} [X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\ &= X(fYg + gYf) - Y(fXg + gXf) \\ &= XfYg + XgYf - YfXg - YgXf \\ &= fXYg + YgXf + gXYf + YfXg \\ &\quad - fYXg - XgYf - gYXf - XfYg \\ &= fXYg + gXYf - fYXg - gYXf \\ &= f[X, Y]g + g[X, Y]f. \end{aligned}$$

$\square$

### 3.8 Lecture 13

Consider two smooth vector fields  $X$  and  $Y$  on  $M$ . Define  $[X, Y] : M \rightarrow TM$  by  $p \mapsto (f \mapsto X_p(Yf) - Y_p(Xf))$ .

**Proposition 3.8.1.** Write  $X = X^i \frac{\partial}{\partial x_i}$  and  $Y = Y^j \frac{\partial}{\partial x_j}$  in local coordinates. Then

$$[X, Y] = \sum_{i,j} \left( X^i \frac{\partial Y^j}{\partial x_i} - Y^j \frac{\partial X^i}{\partial x_j} \right) \frac{\partial}{\partial x_j}.$$

*Proof.* Since  $[X, Y]$  is a vector field, we see that  $([X, Y]f) \upharpoonright_U = [X, Y](f \upharpoonright_U)$  for any open subset  $U \subset M$ . Therefore, we may compute, say,  $Xf$  in a local coordinate expression for  $X$ . To this end, let us apply the product rule together with Clairaut's theorem to get

$$\begin{aligned} [X, Y]f &= X^i \frac{\partial}{\partial x_i} \left( Y^j \frac{\partial f}{\partial x_j} \right) - Y^j \frac{\partial}{\partial x_j} \left( X^i \frac{\partial f}{\partial x_i} \right) \\ &= X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial f}{\partial x_j} + X^i Y^j \frac{\partial^2 f}{\partial x_i \partial x_j} - Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial f}{\partial x_i} - Y^j X^i \frac{\partial^2 f}{\partial x_j \partial x_i} \\ &= X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial f}{\partial x_j} - Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial f}{\partial x_i} \\ &= \sum_{i,j} \left( X^i \frac{\partial Y^j}{\partial x_i} - Y^j \frac{\partial X^i}{\partial x_j} \right) \frac{\partial}{\partial x_j}. \end{aligned}$$

□

*Remark 3.8.2.* If  $X_1, \dots, X_n \in \mathcal{X}(U)$  satisfy  $[X_i, X_j] = 0$ , then there are local coordinates  $x^i : V \rightarrow \mathbb{R}$  such that  $X_i = \frac{\partial}{\partial x^i}$ . This is a converse of Clairaut's theorem.

**Proposition 3.8.3.**

1. (Bilinearity) For any  $a, b \in \mathbb{R}$ ,

$$[aX + bY, Z] = a[X, Z] + b[Y, Z]$$

$$[Z, aX + bY] = a[Z, X] + b[Z, Y].$$

2. (Antisymmetry)

$$[X, Y] = -[Y, X].$$

3. (Jacobi Identity)

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

4. For any  $f, g \in C^\infty(M)$ ,

$$[fX, gY] = fg[X, Y] + (fXg)Y - (gYf)X,$$

where  $fX$  denotes the module action  $f \cdot X$ .

Now, let  $X \in \mathcal{X}(M)$  and  $Y \in \mathcal{X}(N)$ . Let  $F : M \rightarrow N$  be a diffeomorphism. The *pushforward* of  $X$  by  $F$ , denoted by  $F_*X$ , is the vector field on  $N$  given by

$$q \mapsto dF_{F^{-1}(q)}(X_{F^{-1}(q)}).$$

We say  $X$  and  $Y$  are *F-related* if  $Y = F_*X$ .

**Note 3.8.4.**  $X(f \circ F) = (Yf) \circ F$  if and only if  $X$  and  $Y$  are  $F$ -related.

**Theorem 3.8.5 (Naturality of the Lie bracket).**  $F_*[X, Y] = [F_*X, F_*Y]$ .

*Proof.* Let  $f \in C^\infty(M)$ . By Note 3.8.4, we see that  $XY(f \circ F) = X(F_*Yf \circ F) = F_*X(F_*Yf) \circ F$ , and likewise  $YX(f \circ F) = F_*Y(F_*Xf) \circ F$ . Thus,

$$[X, Y](f \circ F) = F_*X(F_*Yf) \circ F - F_*Y(F_*Xf) \circ F = ([F_*X, F_*Y]f) \circ F.$$

We conclude by again applying Note 3.8.4. □

**Corollary 3.8.6.** Let  $S \subset M$  be a submanifold. If  $X, Y \in \mathcal{X}(M)$  satisfy  $X_p, Y_p \in T_p(S)$  for each  $p \in S$ , then  $[X, Y]_p \in T_p(S)$  as well.

*Proof.* Let  $i : S \rightarrow M$  denote inclusion. Then there are  $X', Y' \in \mathcal{X}(S)$  with  $X'$   $i$ -related to  $X \upharpoonright_S$  and  $Y'$   $i$ -related to  $Y \upharpoonright_S$ . This implies that  $[X', Y']$  is  $i$ -related to  $[X, Y] \upharpoonright_S$ , which in turn implies that  $[X, Y]_p \in T_p(S)$  for any  $p \in S$ . □

## 4 Vector bundles

**Definition 4.0.1.** Let  $M$  be a space. A (real) vector bundle of rank  $k$  over  $M$  is a space  $E$  endowed with the following structure.

- (I) A surjective continuous map  $\pi : E \rightarrow M$ .
- (II) For each  $p \in M$ ,  $E_p := \pi^{-1}(p)$  is a  $k$ -dimensional vector space.
- (III) For each  $p \in M$ , there is a neighborhood  $U_p$  in  $M$  together with a homeomorphism  $\varphi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$  (called a *local trivialization*) such that
  - (a)  $\pi_U \circ \varphi = \pi \upharpoonright_{\pi^{-1}(U)}$ , where  $\pi_U : U \times \mathbb{R}^k \rightarrow U$  denotes the projection and
  - (b) for each  $q \in U$ ,  $\varphi \upharpoonright_{E_q}$  is a linear isomorphism  $E_q \xrightarrow{\cong} \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$ .

If  $M$  and  $E$  are smooth manifolds and each local trivialization is smooth, then  $E$  is called a *smooth vector bundle*.

**Example 4.0.2.** The Möbius strip and  $\mathbb{S}^1 \times \mathbb{R}$  are distinct vector bundles of rank 1 over  $\mathbb{S}^1$ .

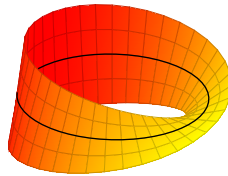


Figure 2: Möbius strip

We can always construct a global section of a smooth vector bundle by using partitions of unity. But we cannot always ensure that it is non-vanishing, as shown by the hairy ball theorem (Corollary 7.3.6) for bundles over  $\mathbb{S}^2$ .

### 4.1 Lecture 14

**Lemma 4.1.1 (Vector bundle construction).** *Let  $M^n$  be a smooth manifold and suppose that for any  $p \in M$ , there is some vector space  $E_p$  of dimension  $k$ . Let  $E := \coprod_{p \in M} E_p$  and  $\pi : E \rightarrow M$  be the projection map. Further, suppose we have the following data:*

- (a) *an open cover  $\{U_\alpha\}$ ,*
- (b) *for each  $\alpha$ , a bijection  $\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$  whose restriction to each  $E_p$  is a linear isomorphism to  $\{p\} \times \mathbb{R}^k$ , and*
- (c) *for each  $U_\alpha \cap U_\beta \neq \emptyset$ , a smooth map  $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(k, \mathbb{R})$  such that  $\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$ .*

*Then  $E$  has a unique topology and smooth structure making it into a smooth vector bundle of rank  $k$  over  $M$ .*

The matrices  $\tau_{\alpha\beta}(p)$  are called the *transition functions* of the vector bundle  $E$ . They satisfy the so-called cocycle condition:

$$\tau_{\alpha\alpha}(p) = I_k \quad \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)\tau_{\gamma\alpha}(p) = I_k.$$

**Definition 4.1.2 (Bundle map).** Let  $p_1 : E_1 \rightarrow M_1$  and  $p_2 : E_2 \rightarrow M_2$  be two vector bundles of rank  $k$ . A *homomorphism*  $p_1 \rightarrow p_2$  is a commutative square

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ p_1 \downarrow & & \downarrow p_2 \\ M_1 & \xrightarrow{g} & M_2 \end{array}$$

in the category of spaces such that each map  $f|_{p_1^{-1}(x)}$  is linear.

Note that  $g$  is uniquely determined by  $f$  because  $p_1$  is surjective.

Let us now explore a specific kind of vector bundle. To this end, consider any vector space  $V$  as well as its *dual space*

$$V^* \equiv \text{Hom}(V, \mathbb{R}),$$

which consists of all linear maps  $V \rightarrow \mathbb{R}$ , known as *covectors on  $V$* . If  $A : V \rightarrow W$  is linear, then let  $A^*$  denote the linear map  $W^* \rightarrow V^*$  defined by  $w \mapsto (v \mapsto w(Av))$ , called the dual map of  $A$ .

Let  $\{v_1, \dots, v_n\}$  be a basis for  $V$ . The *dual basis* (or *cobasis*) consists of those linear functionals  $\varphi_i : V \rightarrow \mathbb{R}$  given by

$$\varphi_i(v_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}.$$

for each  $i = 1, \dots, n$ .

**Proposition 4.1.3.**

- (1) *If  $\dim(V) = n$ , then  $\dim(V^*) = n$ .*



*Proof.* Pick a basis  $b_1, \dots, b_n$  for  $V$ . Consider its dual basis  $\{b^1, \dots, b^n\}$ . It is easy to check that this is linearly independent. Further, for any  $T \in V^*$ , we see that

$$T = T_1 b^1 + \dots + T_n b^n, \quad T_i \equiv T(b_i).$$

This means that the  $b^i$  span  $\text{Hom}(V, \mathbb{R})$  as well.  $\square$

*Remark 4.1.4.* The induced isomorphism  $V \rightarrow V^*$  is *not* unique, for it depends on our chosen basis of  $V$ .

(2) The mapping  $v \mapsto \underbrace{(\varphi \mapsto \varphi(v))}_{\text{ev}_v}$  defines a canonical isomorphism

$$V \xrightarrow{\cong} (V^*)^* = \text{Hom}(V^*, \mathbb{R}).$$

**Definition 4.1.5.** Let  $M^n$  be a smooth manifold.

1. Define the *cotangent space* at  $p$  as  $T_p^*M$ .
2. Define the *cotangent bundle* of  $M$  as  $T^*M \equiv \coprod_p T_p^*M$ .

**Lemma 4.1.6.**  $T^*M$  is a smooth  $n$ -vector bundle over  $M$ .

*Proof.* Let  $(U, \varphi)$  be a smooth chart on  $M$ . Define  $\varphi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  by  $a_i \lambda^i|_p \mapsto (p, a_1, \dots, a_n)$  where  $\{\lambda^i|_p\}$  is a chosen dual basis for  $T_p^*M$ . Now we apply the vector bundle construction lemma. See Proposition 11.9 (Lee).  $\square$

Let  $(U, x^i)$  be smooth coordinates for  $M^n$ . Then the map  $\psi : a_i \lambda^i|_p \mapsto (x^1(p), \dots, x^n(p), a_1, \dots, a_n)$  makes  $(\pi^{-1}(U), \psi)$  a chart on  $T^*M$ .

A smooth section of  $T^*M$  is called a *covector field* (or *(differential/smooth) 1-form*) on  $M$ . The vector space of such sections will be denoted by  $\Gamma(T^*M)$ .

Moreover, if  $U$  is a chart on  $M$ , then a tuple  $(\epsilon^1, \dots, \epsilon^k)$  of covector fields on  $M$  is a *local coframe* if  $\{\epsilon^1|_p, \dots, \epsilon^k|_p\}$  is a basis of  $T_p^*U$  for each  $p \in U$ .

*Aside.* Let  $\pi : E \rightarrow M$  be a smooth vector bundle. The *jet bundle*  $J^k E \rightarrow M$  of order  $k$  is the smooth vector bundle whose fiber at  $p \in M$  consists of all *order- $k$  jets* of smooth sections of  $\pi$ , i.e., equivalence classes of smooth sections of  $\pi$  where two sections are declared equivalent if their first  $k$  partial derivatives agree on a neighborhood of  $p$ . Note that a germ is precisely an order-1 jet.

We have a sequence of maps

$$\dots J^3 E \twoheadrightarrow J^2 E \twoheadrightarrow J^1 E \twoheadrightarrow E,$$

whose limit is called the *infinite jet bundle*  $J^\infty E$ .

## 5 Differential forms

### 5.1 Lecture 15

**Definition 5.1.1 (Differential of a smooth function).** Define  $C^\infty(M) \rightarrow \Gamma(T^*M)$  by  $f \mapsto (p \mapsto df_p)$  where

$$df_p(v) \equiv vf$$

for every  $v \in T_pM$ . We call  $df$  the *differential of  $f$* .

Let  $(U, x^i)$  be local coordinates for  $M$ . Let  $(dx^i)$  denote the corresponding coordinate coframe. We have  $df_p = A_i(p)dx^i|_p$  for some functions  $A_i : U \rightarrow \mathbb{R}$ . Then

$$\begin{aligned} A_i(p) &= df_p \left( \frac{\partial}{\partial x^i} \Big|_p \right) = \frac{\partial f}{\partial x^i}(p) \\ &\Downarrow \\ df_p &= \frac{\partial f}{\partial x^i}(p) dx^i|_p. \end{aligned}$$

In this way, the differential of  $f$  generalizes the gradient of a smooth function on  $\mathbb{R}^n$ .

**Proposition 5.1.2.** *If  $M$  is connected, then  $f$  is constant if and only if  $df = 0$ .*

*Proof.* Since  $vf = 0$  for any derivation  $v$  and constant function  $f$ , the forward direction is clear. Conversely, suppose that  $df = 0$  and let  $p \in M$ . Set  $C = \{q \in M : f(q) = f(p)\}$ . We must show that  $C = M$ . Provided that  $M$  is connected, it suffices to show that  $C$  is clopen. For any  $q \in C$ , choose a coordinate ball  $U \ni p$ . Then since  $0 = df = \frac{\partial f}{\partial x^i} dx^i$ , it follows that  $\frac{\partial f}{\partial x^i} = 0$  for each  $i$ . Elementary calculus reveals that  $f$  must be constant on  $U$ . Hence  $C$  is open. Since  $C = f^{-1}(f(p))$ , it is also closed.  $\square$

**Note 5.1.3 (Transition functions for changing coordinates).** Let  $p \in M$  and suppose that  $(x^i)_{1 \leq i \leq n}$  and  $(y^i)_{1 \leq i \leq n}$  are two coordinate charts around  $p$ . The chain rule for partial derivatives states that

$$\frac{\partial}{\partial x^j} \Big|_p = \sum_k \frac{\partial y^k}{\partial x^j}(\hat{p}) \frac{\partial}{\partial y^k} \Big|_p$$

where  $\hat{p} := (x^1(p), \dots, x^n(p))$ . Dually, for each  $i \in \{1, \dots, n\}$ , we have that

$$dx^i|_p = \sum_\ell A_\ell^i dy^\ell|_p$$

for some  $A_\ell^i \in \mathbb{R}$ ,  $l = 1, \dots, n$ . It follows that

$$\begin{aligned}
 \delta_i^j &= dx^i|_p \left( \frac{\partial}{\partial x^j} \Big|_p \right) \\
 &= dx^i|_p \left( \sum_k \frac{\partial y^k}{\partial x^j} \frac{\partial}{\partial y^k} \Big|_p \right) \\
 &= \sum_k \frac{\partial y^k}{\partial x^j} dx^i|_p \left( \frac{\partial}{\partial y^k} \Big|_p \right) \\
 &= \sum_k \frac{\partial y^k}{\partial x^j} \sum_\ell A_\ell^i dy^\ell|_p \left( \frac{\partial}{\partial y^k} \Big|_p \right) \\
 &= \sum_k \frac{\partial y^k}{\partial x^j} \sum_\ell A_\ell^i \delta_\ell^k \\
 &= \sum_k A_k^i \frac{\partial y^k}{\partial x^j}.
 \end{aligned}$$

Therefore, if  $A$  denotes the  $n \times n$  matrix  $(A_\ell^i)$  and  $J$  denotes the Jacobian of  $(y^1, \dots, y^n)$  at  $\hat{p}$ , then  $I_n = JA$ , i.e.,  $A = J^{-1}$ .

**Definition 5.1.4.** Let  $F : M \rightarrow N$  be smooth. Let  $\omega \in \Gamma(T^*N)$ . Define the *pullback*  $F^*\omega$  of  $\omega$  as the element of  $\Gamma(T^*M)$  given by

$$F^*\omega|_p \left( X|_p \right) \equiv \omega|_{F(p)} \left( F_*|_p X_p \right).$$

Note that, unlike the pushforward, the pullback requires merely that  $F$  be smooth.

**Lemma 5.1.5.** Let  $F : M \rightarrow N$  be smooth,  $\alpha, \beta \in \Gamma(T^*N)$  and  $f, g \in C^\infty(N)$ . Then

$$F^*(f\alpha + g\beta) = (f \circ F)F^*\alpha + (g \circ F)F^*\beta.$$

*Proof.* Let  $X \in \mathcal{X}(M)$ . We have that

$$\begin{aligned}
 F^*(f\alpha + g\beta)|_p(X_p) &= (f\alpha + g\beta)|_{F(p)} \left( F_*|_p X_p \right) \\
 &= f(F(p)) \alpha_{F(p)} \left( F_*|_p X_p \right) + g(F(p)) \beta_{F(p)} \left( F_*|_p X_p \right) \\
 &= [(f \circ F)F^*\alpha]_p(X_p) + [(g \circ F)F^*\beta]_p(X_p).
 \end{aligned}$$

□

Let  $\gamma : J \subset \mathbb{R} \rightarrow M$  be a smooth curve in  $M$ . Note that  $\Gamma(T^*\mathbb{R}) = \{f(t)dt \mid f : T \rightarrow \mathbb{R}\}$ . Then

$$\omega \in \Gamma(T^*M) \implies \gamma^*\omega \in \Gamma(T^*\mathbb{R}) \implies \gamma^*\omega = f(t)dt$$

for some curve  $f$  along  $J$ . This enables us to modestly generalize our notion of integration.

**Definition 5.1.6.** The *integral of  $\omega$  along  $\gamma$*  is

$$\int_\gamma \omega \equiv \int_J \gamma^*\omega.$$

**Proposition 5.1.7.** Suppose that  $\varphi$  is a positive reparameterization of  $\gamma$  (i.e., one with positive derivative). Then  $\int_{\gamma} \omega = \int_{\gamma \circ \varphi} \omega$ .<sup>4</sup>

**Definition 5.1.8.** A differential 1-form  $\omega$  on a smooth manifold  $M$  is *closed* if the equation

$$\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0$$

holds for any  $i, j$  in any chart on  $M$ .

**Exercise 5.1.9.** Show that being closed is a well-defined property.

**Example 5.1.10.** By Clairaut's theorem,  $df$  is closed for any  $f \in C^\infty(M)$ .

## 5.2 Lecture 16

Recall that a map  $T : V_1 \times \cdots \times V_k \rightarrow W$  of vector spaces is *multilinear* if it is linear in each argument, i.e.,

$$T(v_1, \dots, ax + by, \dots, v_k) = aT(v_1, \dots, x, \dots, v_k) + bT(v_1, \dots, y, \dots, v_k)$$

for any  $a, b \in \mathbb{R}$ .

**Theorem 5.2.1 (Universal property of the tensor product).** Let  $V_1, \dots, V_k$  be vector spaces. There exists a vector space  $V_1 \otimes \cdots \otimes V_k$  together with a map  $\otimes : V_1 \times \cdots \times V_k$  so that for any multilinear map  $T : V_1 \times \cdots \times V_k \rightarrow W$ , there is some unique linear map  $\tilde{T} : V_1 \otimes \cdots \otimes V_k \rightarrow W$  such that

$$\begin{array}{ccc} V_1 \times \cdots \times V_k & \xrightarrow{T} & W \\ \otimes \downarrow & \nearrow \tilde{T} & \\ V_1 \otimes \cdots \otimes V_k & & \end{array}$$

commutes.

*Terminology.*  $V_1 \otimes \cdots \otimes V_k$  is called the *tensor product* of the  $V_i$ .

*Proof.* Let us just prove this when  $k = 2$ , for then we're done by induction. Let  $\mathbb{R}\langle V_1 \times V_2 \rangle$  denote the free vector space on  $V_1 \times V_2$ , which consists of all finite formal linear combinations of  $V_1 \times V_2$ . Let

$$G = \langle (av_1, v_2) - a(v_1, v_2), (v_1, av_2) - a(v_1, v_2), (v_1 + w_1, v_2) - (v_1, v_2) - (w_1, v_2), (v_1, w_2 + v_2) - (v_1, w_2) - (v_1, v_2) \rangle.$$

Given a multilinear map  $T : V_1 \times V_2 \rightarrow W$ , define  $\tilde{T} : \mathbb{R}\langle V_1 \times V_2 \rangle \rightarrow W$  by

$$\sum a_{(v_1, v_2)}(v_1, v_2) \mapsto \sum a_{(v_1, v_2)} T(v_1, v_2).$$

Since  $T$  is multilinear,  $G \subset \ker \tilde{T}$ . Therefore, the vector space  $V_1 \otimes V_2 := \mathbb{R}\langle V_1 \times V_2 \rangle / G$  fits in a commutative triangle

$$\begin{array}{ccc} \mathbb{R}\langle V_1 \times V_2 \rangle & \xrightarrow{\tilde{T}} & W \\ \pi \downarrow & \nearrow \tilde{\tilde{T}} & \\ V_1 \otimes V_2 & & \end{array}.$$

---

<sup>4</sup>Proposition 11.31 (Lee).

Thus, if  $i : V_1 \times V_2 \rightarrow \mathbb{R}\langle V_1 \times V_2 \rangle$  denotes inclusion, then  $\tilde{T} \circ \pi \circ i = \tilde{T} \circ i$ , which induces our desired diagram. We see that  $\tilde{T}$  is unique because it is uniquely determined by elements of the form

$$v_1 \otimes v_2 := [(v_1, v_2)]$$

under  $T$  and every element of  $V_1 \otimes V_2$  can be written as some linear combination of such elements.  $\square$

A basic property of the tensor product is that its generic elements are bilinear in the following sense.

**Proposition 5.2.2.** *If  $a, b \in \mathbb{R}$ , then  $(av_1 + bw_1) \otimes v_2 = a(v_1 \otimes v_2) + b(w_1 \otimes v_2)$ .*

**Proposition 5.2.3.**

1.  $(\mathbf{Vect}_{\mathbb{R}}, \oplus, \otimes)$  is a semiring.
2.  $V \otimes W \cong W \otimes V$ .
3.  $V \otimes \mathbb{R} \cong V$ .
4.  $(V \otimes W)^* \cong V^* \otimes W^*$ .

Let  $B(V, W)$  denote the space of bilinear maps  $V \times W \rightarrow \mathbb{R}$ .

**Lemma 5.2.4.** *There is a canonical isomorphism  $V^* \otimes W^* \cong B(V, W)$ .*

*Proof.* Define  $\Phi : V^* \times W^* \rightarrow B(V, W)$  by  $(\omega, \eta) \mapsto ((v, w) \mapsto \omega(v)\eta(w))$ . This is linear and hence induces a commutative diagram

$$\begin{array}{ccc} V^* \times W^* & \xrightarrow{\Phi} & B(V, W) \\ \pi \downarrow & \nearrow \tilde{\Phi} & \\ V^* \otimes W^* & & \end{array}.$$

To see that  $\tilde{\Phi}$  is an isomorphism, pick bases  $\{f_1, \dots, f_n\}$  and  $\{g_1, \dots, g_n\}$  for  $V$  and  $W$ , respectively. Consider their respective dual bases  $\{\xi\}$  and  $\{\eta\}$ . Then  $\{\xi^i \otimes \eta^j : 1 \leq i, j \leq n\}$  is a basis for  $V^* \otimes W^*$ . Define the linear map  $\Psi : B(V, W) \rightarrow V^* \otimes W^*$  by

$$b \mapsto \sum_{i,j} b(f_i, g_j) \xi^i \otimes \eta^j.$$

It is straightforward to check that  $\Psi$  is the inverse of  $\tilde{\Phi}$ .  $\square$

We can generalize Theorem 7.2.3 to obtain an isomorphism

$$V_1^* \otimes \dots \otimes V_k^* \cong L(V_1, \dots, V_k; \mathbb{R}).$$

**Definition 5.2.5 (Tensor type).** We say that an element of

$$V_\ell^k := \underbrace{V^* \otimes \dots \otimes V^*}_{k \text{ copies}} \otimes \underbrace{V \otimes \dots \otimes V}_{\ell \text{ copies}}$$

is a  $(k, \ell)$ -tensor.

*Terminology.*

1. A  $(k, 0)$ -tensor is called *covariant*.
2. A  $(0, \ell)$ -tensor is called *contravariant*.

Let  $M$  be a smooth manifold. Define the  $(k, \ell)$ -tensor bundle as

$$T_\ell^k M \equiv \coprod_{p \in M} (T_p)^k_\ell M.$$

In particular,  $T^1 M = T^* M$ , and  $T_1 M = TM$ .

**Exercise 5.2.6.** Find the dimension of  $T_\ell^k M$ .

Let us examine the form of a generic  $(k, 0)$ -tensor. Suppose that  $(x^i)$  and  $(y^i)$  are two local coordinate systems around a point  $p \in M$ . Then

$$\begin{aligned} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} &= \left( \frac{\partial x^{i_1}}{\partial y^{\ell_1}} dy^{p_1} \right) \otimes \cdots \otimes \left( \frac{\partial x^{i_k}}{\partial y^{\ell_k}} dy^{p_k} \right) \\ &= \sum_{p_1, \dots, p_k} \frac{\partial x^{i_1}}{\partial y^{\ell_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{\ell_k}} dy^{p_1} \otimes \cdots \otimes dy^{p_k}. \end{aligned}$$

**Definition 5.2.7.** A  $(k, \ell)$ -tensor field is a (smooth) section of  $T_\ell^k M$ .

Let  $\mathcal{T}_\ell^k(M)$  denote the space  $\Gamma(T_\ell^k M)$  of all such sections.

### 5.3 Lecture 17

Let  $(U, x^i)$  be local coordinates for  $M$ . Then any  $A \in \mathcal{T}_\ell^k(M)$  can be written in  $U$  as

$$A|_p = A_{i_1 \dots i_k}^{j_1 \dots j_\ell} dx^{i_1} \big|_p \otimes \cdots \otimes dx^{i_k} \big|_p \otimes \frac{\partial}{\partial x^{j_1}} \big|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{j_\ell}} \big|_p,$$

summed over  $n^k n^\ell$  many tensors.

**Example 5.3.1.** Let  $\sigma = \delta_j^i dx^j \otimes \frac{\partial}{\partial x^i}$ ,  $X = X^k \frac{\partial}{\partial x^k}$ , and  $w = w_\ell dx^\ell$ . Then

$$\begin{aligned} \sigma(X, w) &= \delta_j^i dx^j \otimes \frac{\partial}{\partial x^i} (X^k \frac{\partial}{\partial x^k}, w_\ell dx^\ell) \\ &= \delta_j^i dx^j (X^k \frac{\partial}{\partial x^k}) \frac{\partial}{\partial x^i} w_\ell dx^\ell \\ &= \delta_j^i \delta_k^j X^k w_\ell \delta_i^\ell \\ &= w_k X^k \\ &= w(X). \end{aligned}$$

We say that  $\sigma$  is *invariant* in this case.

**Example 5.3.2.** Show that the tensor  $\delta_i^j dx^i \otimes dx^j$  is *not* invariant.

**Proposition 5.3.3.**

1. Any  $\sigma \in \mathcal{T}_\ell^k(M)$  induces a  $C^\infty(M)$ -multilinear map

$$\begin{aligned} \hat{\sigma} : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{k \text{ copies}} \times \underbrace{\mathcal{X}^*(M) \times \cdots \times \mathcal{X}^*(M)}_{\ell \text{ copies}} &\longrightarrow C^\infty(M) \\ (X_1, \dots, X_k, w_1, \dots, w_\ell) &\mapsto \left( p \mapsto \sigma \left( X_1|_p, \dots, X_k|_p, w_1|_p, \dots, w_\ell|_p \right) \right). \end{aligned} \quad (*)$$

2. Any multilinear map over  $C^\infty(M)$  is of the form (1) for some  $(k, \ell)$ -tensor field.

Notice that the smooth function  $\hat{\sigma}_p$  induced by  $\sigma$  of Example 5.3.1 is determined completely by the values  $X_1(p), \dots, X_k(p), w_1(p), \dots, w_\ell(p)$ .

**Note 5.3.4.** The Lie bracket is *not* multilinear over  $C^\infty(M)$ , for

$$[fX + gY, Z] = f[X, Y] + g[Y, Z] - Z(f)X - Z(g)Y.$$

**Definition 5.3.5.** A covariant  $k$ -tensor  $T$  is *alternating* if for any vectors  $Y, X_1, \dots, X_{k-1}$ , it follows that

$$T(X_1, X_2, \dots, Y, \dots, Y, \dots, X_{k-1}) = 0.$$

In this case,  $T$  is also called an *exterior form*.

**Example 5.3.6.** If  $\sigma$  is a 0-tensor or a 1-tensor, then it is alternating.

**Proposition 5.3.7.** *TFAE.*

1.  $T$  is alternating.
2.  $T(X_1, \dots, X_k) = 0$  whenever  $\{X_1, \dots, X_k\}$  is linearly dependent.
3.  $T(X_1, \dots, X_i, X_{i+1}, \dots, X_k) = -T(X_1, \dots, X_{i+1}, X_i, \dots, X_k)$ .

*Notation.* The expression  $\bigwedge^k(V)$  will denote the subspace of  $T^k(V)$  consisting of alternating covariant  $k$ -tensors.

**Definition 5.3.8.** Given  $T \in T^k(V)$ , the *alternation*  $\text{Alt}(T)$  of  $T$  is the multilinear map defined by

$$(V_1, \dots, V_k) \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(V_{\sigma(1)}, \dots, V_{\sigma(k)}).$$

**Example 5.3.9.**

1.  $\text{Alt}(T)(X, Y) = \frac{1}{2} (T(X, Y) - T(Y, X))$ .
2.  $\text{Alt}(T)(X, Y, Z) = \frac{1}{6} (T(X, Y, Z) + T(Y, Z, X) + T(Z, X, Y) - T(Y, X, Z) - T(Z, Y, X) - T(X, Z, Y))$ .

**Example 5.3.10.** Suppose that  $\{w^1, \dots, w^n\}$  is the cobasis of the standard basis  $\{e_1, \dots, e_n\}$  for the vector space  $V$ . Then

$$\begin{aligned} &\text{Alt}(w^1 \otimes \cdots \otimes w^n)(e_1, \dots, e_n) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) w^1 \otimes \cdots \otimes w^n(e_{\sigma(1)}, \dots, e_{\sigma(n)}) \\ &= \frac{1}{n!} \text{sgn}(\text{id}_n) w^1 \otimes \cdots \otimes w^n(e_1, \dots, e_n) \\ &= \frac{1}{n!}. \end{aligned}$$

**Proposition 5.3.11.**

1.  $\text{Alt}(T) \in \bigwedge^k(V)$ .
2.  $\text{Alt}(T) = T \iff T \in \bigwedge^k(V)$ .
3. The induced map  $\text{Alt} : T^k(V) \rightarrow \bigwedge^k(V)$  is linear.

**5.4 Lecture 18**

**Lemma 5.4.1.** Let  $V$  be a vector space of dimension  $k < \infty$ . Let  $\{w^1, \dots, w^n\}$  be a cobasis for  $V$ . Let  $k \leq n$ . Then

$$A := \{\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_k}) : 1 \leq i_1 < \dots < i_k \leq n\}$$

is a basis for  $\bigwedge^k(V)$ .

*Proof.* It's clear from Proposition 5.3.11 that  $A$  spans  $\bigwedge^k(V)$ . It remains to show that  $A$  is linearly independent.

**Claim.**

- (a) If the integers  $i_1, \dots, i_k$  are not pairwise distinct, then  $\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_k}) = 0$ .
- (b)  $\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_j} \otimes \omega^{i_{j+1}} \otimes \dots \otimes \omega^{i_k}) = -\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_{j+1}} \otimes \omega^{i_j} \otimes \dots \otimes \omega^{i_k})$ .

As a consequence,  $\text{span}(A) = \text{span}\{\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_k}) : 1 \leq i_1 < \dots < i_k \leq n\}$ .

**Exercise 5.4.2.** Show that this implies that  $A$  is linearly independent.

□

**Corollary 5.4.3.** If  $\dim(V) = n$ , then  $\dim\left(\bigwedge^k(V)\right) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ .

**Definition 5.4.4.** Define the *wedge product* as the map

$$\wedge : \bigwedge^k(V) \times \bigwedge^\ell(V) \rightarrow \bigwedge^{k+\ell}(V) \quad (w, q) \mapsto w \wedge q \equiv \frac{(k+\ell)!}{k!\ell!} \text{Alt}(w \otimes q).$$

This is like the tensor product.

**Example 5.4.5.** With notation as in Example 5.3.10, we have that  $\omega^1 \wedge \dots \wedge \omega^n(e_1, \dots, e_n) = 1$ .

**Lemma 5.4.6.** The set  $\{\omega^{i_1} \wedge \dots \wedge \omega^{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$  is a basis for  $\bigwedge^k(V)$ .

*Proof sketch.* For each  $k$ -tuple  $(i_1, \dots, i_k)$ , one can show that  $\omega^{i_1} \wedge \dots \wedge \omega^{i_k}$  and  $\text{Alt}(\omega^{i_1} \otimes \dots \otimes \omega^{i_k})$  differ precisely by a real factor. This is enough thanks to Lemma 5.4.1. □

Consider the standard basis  $B := \{e_1, \dots, e_n\}$  for  $V$ . Note that  $\det \in \bigwedge_B^n(V)$  by Proposition 5.3.11. But  $\bigwedge_B^n(V) = 1$ , so that  $\det = c(\omega^1 \wedge \dots \wedge \omega^n)$ . But evaluating both sides at  $(e_1, \dots, e_n)$  yields the equation  $1 = c(1) = c$ . Thus,

$$\det_B = \omega^1 \wedge \dots \wedge \omega^n.$$



**Proposition 5.4.7.** *Suppose that  $\omega$ ,  $\omega'$ ,  $\eta$ , and  $\eta'$  are exterior forms. The following are properties of the wedge product.*

(1) (Bilinearity) *If  $a, a' \in \mathbb{R}$ , then*

$$\begin{aligned} (a\omega + a'\omega') \wedge \eta &= a(\omega \wedge \eta) + a'(\omega' \wedge \eta) \\ \eta \wedge (a\omega + a'\omega') &= a(\eta \wedge \omega) + a'(\eta \wedge \omega'). \end{aligned}$$

(2) (Associativity)

$$(\eta \wedge \omega) \wedge \omega' = \eta \wedge (\omega \wedge \omega').$$

(3) (Anticommutativity) *If  $\omega \in \bigwedge^k(V)$  and  $\eta \in \bigwedge^\ell(V)$ , then*

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega.$$

**Corollary 5.4.8.** *If  $\omega$  is a 1-form, then  $\omega \wedge \omega = 0$ .*

(4) *If  $\omega^1, \dots, \omega^k \in \bigwedge^1(V)$ , then*

$$\omega^1 \wedge \dots \wedge \omega^k(v_1, \dots, v_k) = \det(\omega^j(v_i)).$$

**Definition 5.4.9.** Let  $M^n$  be a smooth manifold. Define the *alternating bundle of rank  $k$*  as

$$\bigwedge^k(M) \equiv \coprod_{p \in M} \bigwedge^k(T_p M).$$

A smooth section of  $\bigwedge^k(M)$  is called a (differential)  $k$ -form.

Let both  $\Omega^k(M)$  and  $\mathcal{A}^k(M)$  stand for the infinite-dimensional vector space of differential  $k$ -forms on the manifold  $M$ . We also have a graded associative algebra  $(\Omega^*(M), \wedge)$  over  $\mathbb{R}$ .

In local coordinates we have a basis  $\left\{ \frac{\partial}{\partial x^i} \Big|_p \right\}_{1 \leq i \leq n}$  for  $T_p M$  as well as a corresponding dual basis  $\{dx^i\}$ . Then for any  $\omega \in \bigwedge^k(M)$ , we can write

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} \omega_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad (1)$$

locally at  $p$ . Let  $I = \{i_1 < \dots < i_k\}$ . Since

$$dx^{i_1} \wedge \dots \wedge dx^{i_k} \left( \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \delta_J^I$$

where  $\delta_J^I = 1$  if and only if  $I = J$  as sets, it follows that

$$\omega_{i_1, \dots, i_k} = \omega \left( \frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}} \right). \quad (2)$$

We abbreviate (1) by writing

$$\omega = \omega_I dx^I,$$

where we tacitly sum over the  $I$ . In this case, for any other ordered set of indices  $J := \{j_1 < \dots < j_k\}$ , we have

$$\omega \left( \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \omega_I dx^I \left( \frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}} \right) = \omega_I \delta_J^I.$$

**Note 5.4.10.** Let  $w = w_I dx^I$  and  $w = \tilde{w}_J d\tilde{x}^J$  be two coordinate representations of  $w$ . Observe that

$$\begin{aligned}
 \tilde{\omega}_J &= \omega \left( \frac{\partial}{\partial \tilde{x}^{j_1}}, \dots, \frac{\partial}{\partial \tilde{x}^{j_k}} \right) & ((2)) \\
 &= \omega \left( \sum_t \frac{\partial x^{i_t}}{\partial \tilde{x}^{j_1}} \frac{\partial}{\partial x^{i_t}}, \dots, \sum_t \frac{\partial x^{i_t}}{\partial \tilde{x}^{j_k}} \frac{\partial}{\partial x^{i_t}} \right) & (\text{chain rule}) \\
 &= \sum_{\sigma \in S_k} \text{sgn}(\sigma) \frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial \tilde{x}^{j_k}} \omega \left( \frac{\partial}{\partial x^{\sigma(i_1)}}, \dots, \frac{\partial}{\partial x^{\sigma(i_k)}} \right) & (\text{multilinearity of } \omega) \\
 &= \det \left( k \times k \text{ minor of } \frac{\partial x}{\partial \tilde{x}} \text{ relative to } i_1, \dots, i_k \text{ and } j_1, \dots, j_k \right). & (\text{Proposition 5.4.7(4)})
 \end{aligned}$$

## 5.5 Lecture 19

The following notion generalizes Definition 5.1.4 to differential forms of arbitrary degree.

**Definition 5.5.1 (Pullback).** Let  $F : M \rightarrow N$  be smooth and  $\omega \in \bigwedge^k(N)$ . The *pullback*  $F^*\omega$  of  $\omega$  by  $F$  is the differential  $k$ -form on  $M$  given pointwise by

$$F^*\omega|_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)).$$

Note that  $F^*(-)$  is a linear map  $\Omega^k(N) \rightarrow \Omega^k(M)$  over  $\mathbb{R}$ .

**Lemma 5.5.2 (Naturality of the pullback).**  $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$ .

*Proof.* This is easily seen from Definition 5.5.1 together with Definition 5.4.4. □

**Lemma 5.5.3.** *In any local coordinates, we have that*

$$F^* \left( \sum_I \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k} \right) = \sum_I (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F).$$

*Proof.* It is easy to check that  $F^*\omega(X_1, \dots, X_k) = \sum_I \omega_I \circ F dy^I(F_*X_1, \dots, F_*X_k)$ . Hence it suffices to show that

$$d(y^{i_1} \circ F) \wedge \cdots \wedge d(y^{i_k} \circ F)(X_1, \dots, X_k) = dy^I(F_*X_1, \dots, F_*X_k).$$

For this, it suffices to show that  $d(y^i \circ F)(X) = dy^i(F_*X)$  for each  $i \in \{i_1, \dots, i_k\}$ . Let  $(x^i)$  denote local coordinates on  $M$ . On the one hand, thanks to Definition 5.1.1, we see that

$$d(y^i \circ F)(X) = X(y^i \circ F) = X^j \frac{\partial F^i}{\partial x^j}.$$

On the other hand, we see that

$$\begin{aligned}
 dy^i(F_*X) &= dy^i \left( X^j \frac{\partial F^r}{\partial x^j} \frac{\partial}{\partial y^r} \right) \\
 &= X^j \frac{\partial F^i}{\partial x^j}.
 \end{aligned}$$

□

**Example 5.5.4.** Consider the change of variables to polar coordinates  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ :

$$(r, \theta) \mapsto (r \cos \theta, r \sin \theta).$$

Locally, this is precisely the identity map with the domain endowed with one atlas and the codomain endowed with another. Lemma 5.5.3 together with certain computational properties of  $\wedge$  yields

$$\begin{aligned} dx \wedge dy &= d(r \cos \theta) \wedge d(r \sin \theta) \\ &= (\cos \theta dr - r \sin \theta d\theta) \wedge (\sin \theta dr + r \cos \theta d\theta) \\ &= (\cos \theta dr - r \sin \theta d\theta) \wedge \sin \theta dr + (\cos \theta dr - r \sin \theta d\theta) \wedge r \cos \theta d\theta \\ &= (\cos \theta dr \wedge \sin \theta dr) - (r \sin \theta d\theta \wedge \sin \theta dr) + (\cos \theta dr \wedge r \cos \theta d\theta) - (r \sin \theta d\theta \wedge r \cos \theta d\theta) \\ &= -(r \sin \theta d\theta \wedge \sin \theta dr) + (\cos \theta dr \wedge r \cos \theta d\theta) \\ &= r \sin^2 \theta (dr \wedge d\theta) + r \cos^2 \theta (dr \wedge d\theta) \\ &= r dr \wedge d\theta. \end{aligned}$$

Now, let us begin defining a differential operator on smooth forms that generalizes Definition 5.1.1. Let  $\omega$  be a 1-form on a smooth manifold  $M$ . For this to arise as the differential of a smooth function  $df$ , each component function  $\omega_i$  must have the form  $\frac{\partial f}{\partial x^i}$ . By Clairaut's theorem, this means that  $\omega$  is closed in the sense of Definition 5.1.8, i.e.,

$$\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0 \quad (*)$$

in any chart on  $M$ . This property is actually coordinate-independent by Lee (Proposition 11.45). Therefore, we want to express  $(*)$  as the  $ij$ -component of a 2-form, namely

$$d\omega \equiv \sum_{j < i} \left( \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i.$$

Notice that  $\omega$  is closed if and only if  $d\omega = 0$  in any chart on  $M$ .

## 5.6 Lecture 20

Let  $\omega \in \mathcal{A}^k(M)$  with local coordinate representation  $\omega_I dx^I$ . The *exterior derivative* of  $\omega$  is the  $(k+1)$ -form

$$d\omega \equiv d\omega_I \wedge dx^I.$$

We refer to the operation  $d : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k+1}(M)$  as *exterior differentiation*.

**Note 5.6.1.**  $d\omega = \sum_I \sum_j \frac{\partial}{\partial x^j} \omega_I dx^j \wedge dx^I$ .

*Aside.* If we view  $\Omega^k : \mathbf{Diff}^{\text{op}} \rightarrow \mathbf{Vec}_{\mathbb{R}}$  as the functor sending each smooth map  $f$  to the pullback  $f^*$ , then the exterior derivative becomes a natural transformation  $\Omega^k \Rightarrow \Omega^{k+1}$ .

**Definition 5.6.2.** Let  $\omega \in \mathcal{A}^k(M)$ .

1. We say that  $\omega$  is *closed* if  $d\omega = 0$ .
2. We say that  $\omega$  is *exact* if  $\omega = d\eta$  for some  $\eta \in \mathcal{A}^{k-1}(M)$ .

**Lemma 5.6.3.** *Suppose that  $M = \mathbb{R}^n$ , equivalently, that  $M$  has a global chart.*

- (1)  $d$  is linear over  $\mathbb{R}$ .
- (2)  $d(F^*\omega) = F^*(d\omega)$ .
- (3)  $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ .
- (4)  $d \circ d = 0$ .

*Proof.* Statement (1) is obvious. For (2), by linearity, it suffices to consider the case where  $\omega = u dx^I$ . Using Lemma 5.5.3, we compute

$$\begin{aligned} F^*(d(u dx^{i_1} \wedge \cdots \wedge dx^{i_k})) &= F^*(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \\ &= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) \\ d(F^*(u dx^{i_1} \wedge \cdots \wedge dx^{i_k})) &= d((u \circ F) d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F)) \\ &= d(u \circ F) \wedge d(x^{i_1} \circ F) \wedge \cdots \wedge d(x^{i_k} \circ F) \end{aligned}$$

For (3), let  $\eta = v dx^J$ . Again, by linearity, it suffices to compute  $d(u dx^I \wedge v dx^J)$ .

$$\begin{aligned} d(u dx^I \wedge v dx^J) &= d(uv dx^I \wedge dx^J) \\ &= (v du + u dv) \wedge dx^I \wedge dx^J \\ &= (du \wedge dx^I) \wedge (v dx^J) \wedge (dv \wedge u dx^I) \wedge dx^J \\ &= (du \wedge dx^I) \wedge (v dx^J) \wedge (-1)^k (u dx^I) \wedge (dv \wedge dx^J) \\ &= d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

To prove (4), first observe that so long as  $k = 1$  and  $\omega = \omega_j dx^j$ , we have that

$$\begin{aligned} d\omega &= \sum_{i,j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i < j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i > j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i < j} \left( \frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j. \end{aligned}$$

This together with Clairaut's theorem implies that

$$d(du) = \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i \wedge dx^j = \sum_{i < j} \left( \frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j = 0.$$

Now, drop the assumption that  $k = 1$ . Then expanding  $d(d\omega)$  yields a sum of two summations of wedge products. One of which contains the term  $d(d\omega_j)$ , and the other contains the term  $d(dx^{j_i})$ . These both equal zero, and thus the entire expression  $d(d\omega)$  vanishes.  $\square$

**Corollary 5.6.4 (Naturality of the exterior derivative).** *If  $F$  is a smooth map, then*

$$d(F^*\omega) = F^*(d\omega).$$

**Corollary 5.6.5.** *The exterior derivative is well-defined.*

*Proof.* Let  $(U, \varphi)$  be a chart on  $M$ . Notice that

$$d\omega = \varphi^* d(\varphi^{-1*} \omega)$$

on  $U$ . Let  $(V, \psi)$  be another chart. Then

$$(\varphi \circ \psi^{-1})^* d(\varphi^{-1*} \omega) = d((\varphi \circ \psi^{-1})^* \varphi^{-1*} \omega).$$

Since  $(\varphi \circ \psi^{-1})^* = \psi^{-1*} \circ \varphi^*$  and  $F^* \circ F^{-1*} = \text{id}$  for any diffeomorphism  $F$ , it follows that

$$\begin{aligned} \psi^{-1*} \circ \varphi^* d(\varphi^{-1*} \omega) &= d(\psi^{-1*} \omega). \\ \Downarrow \\ \varphi^* d(\varphi^{-1*} \omega) &= \psi^* d(\psi^{-1*} \omega). \end{aligned}$$

□

**Corollary 5.6.6.** *Any exact form is closed.*

It is *not* the case, however, that any closed form is exact. Let  $M = \mathbb{R}^2 \setminus \{0\}$ . Define the 1-form  $\omega : M \rightarrow T^*M$  by

$$(x, y) \mapsto \frac{xdy - ydx}{x^2 + y^2}.$$

On the one hand, a straightforward computation shows that  $d\omega = 0$ . On the other hand, recall from basic calculus that  $\omega$  is exact on a connected open subset  $\omega \subset M$  if and only if  $\int_c \omega = 0$  for any closed curve  $c \subset \omega$ . But if  $\gamma : [0, 2\pi] \rightarrow M$  is given by  $(\cos \theta, \sin \theta)$ , then

$$\int_\gamma \omega = \int_0^{2\pi} d\theta = 2\pi \neq 0, \tag{†}$$

which means that  $\omega$  is not exact.

**Theorem 5.6.7 (Unique differentiation).** *The exterior derivative is the unique linear map  $\bar{d} : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k+1}$  such that*

- (i)  $\bar{d}(\omega \wedge \eta) = \bar{d}\omega \wedge \eta + (-1)^k \omega \wedge \bar{d}\eta$ ,
- (ii)  $\bar{d}f(X) = Xf$  for any  $f \in C^\infty(M)$ , and
- (iii)  $\bar{d} \circ \bar{d} = 0$ .

For example, consider the linear map  $\bar{d} : \mathcal{A}^k(M) \rightarrow \mathcal{A}^{k+1}(M)$  given by

$$\begin{aligned} \bar{d}\omega(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{n+1} (-1)^{k+1} X_i \left( w(X_1, \dots, \widehat{X}_i, \dots, X_{k+1}) \right) \\ &\quad - \sum_{i,j} (-1)^{i+j} w([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

This satisfies conditions (i), (ii), and (iii) of Theorem 5.6.7, and thus  $\bar{d} = d$ .

To conclude this lecture, let's look at a particular dual operation to exterior differentiation, which will be useful for our discussion of orientation.

Let  $V$  be a finite-dimensional vector space. For each vector  $v \in V$ , define *interior multiplication by  $v$*  as the linear map  $i_v : \bigwedge^k(V) \rightarrow \bigwedge^{k-1}(V)$  given by

$$i_v \omega(w_1, \dots, w_{k-1}) = \omega(v, w_1, \dots, w_{k-1}).$$

Let  $v \lrcorner \omega := i_v \omega$ .

Extend interior multiplication as follows. For each  $X \in \mathcal{X}(M)$  and  $\omega \in \mathcal{A}^k(M)$ , define the  $(k-1)$ -form  $X \lrcorner \omega$  by  $p \mapsto X_p \lrcorner \omega_p$ .

## 5.7 Lecture 21

**Definition 5.7.1.** Let  $V$  be a finite-dimensional vector space. Suppose that  $E$  and  $E'$  are two bases for  $V$ . We say that  $E$  and  $E'$  are *co-oriented* if the change-of-basis matrix from  $E$  to  $E'$  has positive determinant.

This notion provides us with exactly two equivalence classes of bases for  $V$ , which we call the *orientations* for  $V$ . If  $[E_1, \dots, E_n]$  is a chosen orientation for  $V$ , then we call any basis in it *(positively) oriented* and any basis not in it *negatively oriented*.

**Definition 5.7.2 (Orientation).** An *orientation* on a smooth manifold  $M$  is a continuous choice of orientation for  $T_p M$  as  $p$  varies over  $M$ .

Equivalently, if  $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$  denotes the smooth structure on  $M$ , we say that  $M$  is *orientable* if the Jacobian  $D[\varphi_\beta \circ \varphi_\alpha^{-1}]$  has positive determinant on  $\varphi_\alpha(U_\alpha \cap U_\beta)$  for any  $\alpha, \beta \in A$ .

**Example 5.7.3.**  $\mathbb{S}^n$  is orientable for any  $n \geq 1$ . For each  $p \in \mathbb{S}^n$ , say that  $(v_1, \dots, v_n)$  is positively oriented on  $T_p \mathbb{S}^n$  if  $(p, v_1, \dots, v_n)$  is positively oriented on  $\mathbb{R}^{n+1}$ , i.e., is co-oriented with the standard basis for  $\mathbb{R}^{n+1}$ .

**Lemma 5.7.4.** Let  $\pi : E \rightarrow M$  be a smooth vector bundle and  $V \subset E$  be open. If  $V_p$  is a convex subspace of  $E_p$  for every  $p \in M$ , then there is some  $\sigma \in \Gamma(E)$  such that  $\sigma_p \in V_p$  for every  $p$ .

*Proof.* Find a cover of  $E$  by local trivializations  $U_\alpha$  over  $M$  along with smooth sections  $\sigma_\alpha$  of them. There is some partition of unity  $\psi_\alpha$  subordinate to  $(U_\alpha)$ . Define  $\sigma : M \rightarrow E$  as  $\sum_\alpha \psi_\alpha \sigma_\alpha$ , so that  $\sigma \in \Gamma(E)$ . Then  $\sigma_p$  belongs to  $V_p$  by convexity.  $\square$

**Proposition 5.7.5.** Suppose that  $M$  is a smooth  $n$ -manifold. Any nowhere vanishing  $n$ -form on  $M$  gives rise to a unique orientation on  $M$ .

*Conversely, any orientation on  $M$  gives rise to a nowhere vanishing  $n$ -form on  $M$ .*

*Proof.*

( $\implies$ )

Let  $\omega \in \mathcal{A}^n(M)$  be nowhere vanishing. For each  $p \in M$ , we see that  $\omega_p$  defines an orientation  $O_M^p$  on  $M$  by declaring that  $[e_1, \dots, e_n] \in O_M^p$  if and only if  $\omega_p(e_1, \dots, e_n) > 0$ . It remains to show that if  $p \in M$ , then we can find some chart  $U_p$  around  $p$  and some local frame  $(E_1, \dots, E_n)_p$  on  $U_p$  such that  $\omega_q(E_1|_q, \dots, E_n|_q) > 0$

for every  $q \in U_p$ . To see this, pick any  $U_p$  and local frame  $(E_1, \dots, E_n)_p$  on  $U_p$ . Write  $\omega = f dE^1 \wedge \dots \wedge dE^n$  locally for some smooth function  $f : U_p \rightarrow \mathbb{R}$ . Since  $\omega$  is nowhere vanishing, it follows that

$$\omega(E_1, \dots, E_n) = f \neq 0.$$

Since  $f$  is continuous and  $M$  connected, we see that  $f > 0$  or  $f < 0$ . We may assume that  $f > 0$  for otherwise we can choose  $(-E_1, \dots, -E_n)_p$  instead.

( $\Leftarrow$ )

Given  $p \in M$  and an orientation  $O_M^p$  on  $T_p M$ , say that  $w \in \bigwedge^n(T_p M)$  is positively oriented if and only if  $w(e_1, \dots, e_n) > 0$  for any  $[e_1, \dots, e_n] \in O_M^p$ . Then the subspace  $\bigwedge_+^n(T_p M)$  is open and convex. By Lemma 5.7.4, we are done.  $\square$

**Definition 5.7.6.** A diffeomorphism  $F : M \rightarrow N$  between two oriented manifolds is *orientation-preserving* if the isomorphism  $dF_p$  maps positively oriented bases for  $T_p M$  to positively oriented bases for  $T_{F(p)} N$  for each  $p \in M$ . It is *orientation-reversing* if it maps positively oriented bases to negatively oriented ones.

We see that

$$\begin{aligned} F \text{ is orientation-preserving} &\iff \det(dF_p) > 0 \text{ for each } p \in M \\ &\iff F^* \omega \text{ is positively oriented for any positively oriented form } \omega. \end{aligned}$$

**Lemma 5.7.7.** The antipodal map  $\alpha : \mathbb{S}^n \rightarrow \mathbb{S}^n$  is orientation-preserving if and only if  $n$  is odd.

*Proof.* Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{\alpha} & \mathbb{S}^n \\ \downarrow & & \downarrow \\ \mathbb{R}^{n+1} & \xrightarrow{\hat{\alpha}} & \mathbb{R}^{n+1} \end{array}$$

where  $\hat{\alpha}(\vec{x}) \equiv -\vec{x}$ . Note that the Jacobian of  $\hat{\alpha}$  is precisely the identity matrix  $I_{n+1}$ . Since  $\det(I_{n+1}) = (-1)^{n+1}$ , we see that  $\hat{\alpha}$  is orientation-preserving if and only if  $n$  is odd. Thus, the restriction  $\alpha$  of  $\hat{\alpha}$  to  $\mathbb{S}^n$  has the same property.  $\square$

**Corollary 5.7.8.**  $\mathbb{R}P^n$  is not orientable when  $n$  is even.

*Proof.* Let  $n$  be even. Suppose, toward a contradiction, that  $\mathbb{R}P^n$  admits an orientation. Apply Proposition 5.7.5 to obtain a nowhere vanishing  $n$ -form  $\omega$  on  $\mathbb{R}P^n$ . If  $\pi : \mathbb{S}^n \rightarrow \mathbb{R}P^n$  denotes the natural projection, then we also obtain the nowhere vanishing  $n$ -form  $\pi^* \omega$  on  $\mathbb{S}^n$ . Applying Proposition 5.7.5 again shows that this determines the usual orientation on  $\mathbb{S}^n$ .

Note that  $\pi \circ \alpha = \pi$ , so that  $\alpha^* \pi^* \mathbb{S}^n = \pi^* \mathbb{S}^n$ . But this implies that  $\alpha$  preserves the orientation of  $\mathbb{S}^n$ , contrary to Lemma 5.7.7.  $\square$

The converse of Corollary 5.7.8 is also true, although we omit a proof of it.

Before moving to integration, we should look at a modest variant of our notion of *manifold*. Consider the intersection of  $\mathbb{R}^n$  with a half-plane

$$\mathbb{H}^n := \{(x^1, \dots, x^n) \in \mathbb{R}^n : x^n \geq 0\}.$$

**Definition 5.7.9 (Manifold with boundary).**

1. An  $n$ -dimensional manifold with boundary  $M$  is a second-countable Hausdorff space that is locally homeomorphic to either an open Euclidean ball or an open subset of  $\mathbb{H}^n$ .
2. Any point  $p \in M$  is an *interior point* if it belongs to a chart homeomorphic to an open ball.
3. The point  $p$  is a *boundary point* if it belongs to a chart that sends  $p$  to a point in  $\partial\mathbb{H}^n$ .

Note that every point in  $M$  is either an interior or a boundary point, but not both.

**Proposition 5.7.10.** *The set of boundary points  $\partial M$  is an  $(n - 1)$ -dimensional embedded submanifold of  $M$ .*

Moreover,  $\partial M$  inherits an orientation from  $M$  when  $M$  is oriented. This is called the *induced* or *Stokes orientation*. Indeed, we may construct a smooth outward-pointing vector field  $N$  along  $\partial M$ , which is nowhere tangent to  $\partial M$ . Therefore, if  $\omega$  denotes the orientation form for  $M$ , then the form  $i_{\partial M}^*(N \lrcorner \omega)$  is an orientation form for  $\partial M$ .

**Example 5.7.11.**  $\mathbb{S}^n$  is orientable as the boundary of the closed unit ball.

## 6 Integration

### 6.1 Lecture 22

**Definition 6.1.1.** Let  $A_0^k(\mathbb{R}^k)$  denote the space of  $k$ -forms with compact support. Let  $\omega \in A_0^k(\mathbb{R}^k)$  and  $\omega = f dx^1 \wedge \cdots \wedge dx^k$ . Define

$$\int_{\mathbb{R}^k} \omega = \int_{\mathbb{R}^k} f(x^1, \dots, x^k) dx^1 \cdots dx^k.$$

**Exercise 6.1.2.** *Given another coordinate representation  $\omega = gy^1 \wedge \cdots \wedge y^k$  with  $\det\left(\frac{\partial x}{\partial y}\right) > 0$ , show that*

$$\int_{\mathbb{R}^k} f(x^1, \dots, x^k) dx^1 \cdots dx^k = \int_{\mathbb{R}^k} g(y^1, \dots, y^k) dy^1 \cdots dy^k.$$

In other words, Definition 6.1.1 makes sense.

A *singular  $k$ -cell* on  $M^n$  is a smooth map  $\sigma : [0, 1]^k \rightarrow M$ . Note that 0-cells are precisely points in  $M$  and 1-cells are precisely smooth curves in  $M$ . Let  $\omega \in \mathcal{A}^k(M)$  and  $\sigma$  be a singular  $k$ -cell on  $M$ . Define

$$\int_{\sigma} \omega = \int_{[0,1]^k} \sigma^* \omega.$$

**Proposition 6.1.3.** *Let  $p : [0, 1]^k \rightarrow [0, 1]^k$  be a diffeomorphism.*

1. *If  $p$  is orientation-preserving, then  $\int_{\sigma} \omega = \int_{\sigma \circ p} \omega$ .*
2. *If  $p$  is orientation-reversing, then  $\int_{\sigma} \omega = -\int_{\sigma \circ p} \omega$ .*

**Definition 6.1.4.**



1. A *singular  $k$ -chain on  $M$*  is a formal finite  $\mathbb{R}$ -combination  $\sigma = \sum_{i=1}^N a_i \sigma_i$  of singular  $k$ -cells on  $M$ . Define

$$\int_{\sigma} \omega = \sum_{i=1}^N a_i \int_{\sigma_i} \omega.$$

2. Let  $\sigma$  be a singular  $k$ -cell on  $M$ . Let  $i = 1, \dots, 2k$  and  $\alpha = 0, 1$ . Define the  $(i, \alpha)$ -*face* of  $\sigma$  as the smooth map  $\sigma_{(i, \alpha)}$  given by

$$\sigma_{(i, \alpha)}(x^1, \dots, x^k) = \sigma(x^1, \dots, x^{i-1}, \alpha, x^i, \dots, x^k).$$

Moreover, define the *boundary* of  $\sigma$  as the  $(k-1)$ -chain

$$\partial\sigma \equiv \sum_{i=1}^k (-1)^{i+1} (\sigma_{(i, 1)} - \sigma_{(i, 0)}).$$

3. If  $\sigma := \sum_{i=1}^N a_i \sigma_i$  is a singular  $k$ -chain, then define the *boundary* of  $\sigma$  as the  $(k-1)$ -chain

$$\partial\sigma \equiv \sum_{i=1}^N a_i \partial\sigma_i.$$

Note that  $\int_{\partial\sigma} \omega = \sum_{i=1}^N a_i \int_{\partial\sigma_i} \omega$ .

**Definition 6.1.5.** A singular  $k$ -chain  $\sigma$  is a *closed* if  $\partial\sigma = 0$ .

**Exercise 6.1.6.** Show that if  $\sigma$  is any singular  $k$ -chain, then  $\partial\sigma$  is closed.

**Theorem 6.1.7 (Stokes's theorem for chains).** Let  $\sigma$  be a  $k$ -chain and  $\omega \in \mathcal{A}^{k-1}(M)$ . Then

$$\int_{\sigma} d\omega = \int_{\partial\sigma} \omega.$$

*Proof.* For now, assume that  $M = \mathbb{R}^k$  and  $\sigma = I^k$ . As the smooth structure on  $\mathbb{R}^k$  is global, we may write  $\omega = f dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$  for some distinguished  $1 \leq i \leq k$  and some smooth function  $f : \mathbb{R}^k \rightarrow \mathbb{R}$ . We compute

$$\begin{aligned} d\omega &= df \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k \\ &= \left( \sum_{j=1}^k \frac{\partial f}{\partial x^j} dx^j \right) \wedge dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k \\ &= (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^k. \end{aligned}$$

Now, apply Fubini's theorem together with the fundamental theorem of calculus (FTC) to obtain

$$\begin{aligned} \int_{\sigma} d\omega &= (-1)^{i-1} \int_{[0,1]^k} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^k \\ &= (-1)^{i-1} \int_0^1 \dots \int_0^1 \left( \int_0^1 \frac{\partial f}{\partial x^i} dx^i \right) dx^1 \dots \widehat{dx^i} \dots dx^k \\ &= (-1)^{i-1} \int_0^1 \dots \int_0^1 (f(x^1, \dots, \underbrace{1}_{i\text{-th position}}, \dots, x^k) - f(x^1, \dots, \underbrace{0}_{i\text{-th position}}, \dots, x^k)) dx^1 \dots \widehat{dx^i} \dots dx^k \\ &= (-1)^{i-1} \left( \int_{[0,1]^{k-1}} f(x^1, \dots, 1, \dots, x^k) dx^1 \dots \widehat{dx^i} \dots dx^k - \int_{[0,1]^{k-1}} f(x^1, \dots, 0, \dots, x^k) dx^1 \dots \widehat{dx^i} \dots dx^k \right) \\ &= (-1)^{i-1} \left( \int_{\sigma_{(i, 1)}} \omega - \int_{\sigma_{(i, 0)}} \omega \right). \end{aligned}$$

Moreover, we compute

$$\int_{\partial\sigma} \omega = \sum_{j=1}^k (-1)^{j-1} \left( \int_{\sigma_{(j,1)}} \omega - \int_{\sigma_{(j,0)}} \omega \right).$$

Since  $x^j$  is constant along the  $(j, \alpha)$ -face for each  $\alpha = 0, 1$ , it follows that  $dx^j = 0$ . Therefore,

$$\int_{\partial\sigma} \omega = (-1)^{i-1} \left( \int_{\sigma_{(i,1)}} \omega - \int_{\sigma_{(i,0)}} \omega \right) = \int_{\sigma} d\omega.$$

Finally, assume that  $M$  is arbitrary and  $\sigma$  is an arbitrary  $k$ -cell on  $M$ . By the special case just proved, we have that

$$\int_{\sigma} d\omega = \int_{[0,1]^k} \sigma^*(d\omega) = \int_{[0,1]^k} d(\sigma^*\omega) = \int_{\partial[0,1]^k} \sigma^*\omega = \int_{\partial\sigma} \omega.$$

This clearly remains true if  $\sigma$  is a  $k$ -chain on  $M$ .  $\square$

The FTC occurs precisely when  $\sigma = I^1$  and  $\omega = f$ . This shows that Theorem 6.1.7 is equivalent to the FTC.

## 6.2 Lecture 23

Let  $M$  be an orientable manifold. Let  $\omega \in \mathcal{A}^n(M)$ . Let  $\sigma_1$  and  $\sigma_2$  be singular  $n$ -cells on  $M$  that can be extended to diffeomorphisms on (open) neighborhoods of  $[0, 1]^n$ . Suppose that both are orientation-preserving.

**Lemma 6.2.1.** *If  $\text{supp } \omega \subset \sigma_1([0, 1]^n) \cap \sigma_2([0, 1]^n)$ , then  $\int_{\sigma_1} \omega = \int_{\sigma_2} \omega$ .*

*Proof.* Since  $\text{supp } \omega \subset \sigma_1([0, 1]^n) \cap \sigma_2([0, 1]^n)$ , Proposition 6.1.3 implies that

$$\int_{\sigma_1} \omega = \int_{\sigma_2 \circ (\sigma_2^{-1} \circ \sigma_1)} \omega = \int_{\sigma_2} \omega.$$

$\square$

Let  $\omega \in \mathcal{A}^n(M)$ . Let  $\sigma$  be an orientation-preserving singular  $n$ -cell on  $M$ . If  $\text{supp } \omega \subset \sigma([0, 1]^n)$ , then Lemma 6.2.1 allows us to define

$$\int_M \omega = \int_{\sigma} \omega.$$

In general, there exists an open cover  $(U_{\alpha})$  of  $M$  such that  $U_{\alpha} \subset \sigma_{\alpha}([0, 1]^n)$  for each  $\alpha$  where  $\sigma_{\alpha}$  is some orientation-preserving singular  $n$ -cell on  $M$ . Find a partition of unity  $(\varphi_{\alpha})$  subordinate to this cover. Note that each  $\varphi_{\alpha}\omega$  belongs to  $\mathcal{A}^n(M)$  and is supported in  $U_{\alpha}$ . If  $\omega$  is compactly supported, then  $\text{supp } \omega$  intersects at most finitely many  $\text{supp } \varphi_{\alpha}$ . In this case, we define

$$\int_M \omega = \sum_{\alpha} \int_M \varphi_{\alpha} \omega,$$

which is finite. It remains to check that this definition makes sense.

**Lemma 6.2.2.** *If  $(V_{\beta}, \psi_{\beta})$  is another such partition of unity, then  $\sum_{\beta} \int_M \psi_{\beta} \omega = \sum_{\alpha} \int_M \varphi_{\alpha} \omega$ .*

*Proof.*

$$\begin{aligned}
\sum_{\alpha} \int_M \varphi_{\alpha} \omega &= \sum_{\alpha} \int_M \varphi_{\alpha} \sum_{\beta} \psi_{\beta} \omega \\
&= \sum_{\alpha} \sum_{\beta} \int_M \varphi_{\alpha} \psi_{\beta} \omega \\
&= \sum_{\beta} \sum_{\alpha} \int_M \psi_{\beta} \varphi_{\alpha} \omega \\
&= \sum_{\beta} \int_M \psi_{\beta} \sum_{\alpha} \varphi_{\alpha} \omega \\
&= \sum_{\beta} \int_M \psi_{\beta} \omega.
\end{aligned}$$

□

**Note 6.2.3.** If  $\omega$  is not assumed to be compact, then  $\int_M \omega$  may be infinite but is still well-defined.

**Theorem 6.2.4 (Stokes).** *Let  $M$  be an oriented compact  $n$ -manifold with boundary. If  $\omega \in \mathcal{A}^{n-1}(M)$ , then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

*Proof.* There are three cases to consider.

Case 1: Suppose that there is some orientation-preserving  $n$ -cell  $\sigma$  on  $M$  such that  $\text{supp } \omega \subset \text{Int}(\text{im } \sigma)$  and  $\text{im } \sigma \cap \partial M = \emptyset$ . By Theorem 6.1.7, it follows that

$$\int_M d\omega = \int_{\sigma} d\omega = \int_{\partial\sigma} \omega = 0 = \int_{\partial M} \omega.$$

Case 2: Suppose that there is some orientation-preserving  $n$ -cell  $\sigma$  on  $M$  such that  $\text{supp } \omega \subset \text{im } \sigma$ ,  $\text{im } \sigma \cap \partial M = \sigma_{(n,0)}([0, 1]^{n-1})$ , and  $\text{supp } \omega \cap \text{im } \partial\sigma \subset \sigma_{(n,0)}$ . By Theorem 6.1.7, it follows that

$$\int_M d\omega = \int_{\sigma} d\omega = \int_{\partial\sigma} \omega = (-1)^n \int_{\sigma_{(n,0)}} \omega.$$

Note that if  $\mu$  denotes the usual orientation on  $\mathbb{H}^n$ , then the induced orientation on the boundary  $\partial\mathbb{H}^n$  is equal to  $(-1)^n \mu$ . Therefore,  $\sigma_{(n,0)} : [0, 1]^{n-1} \rightarrow \partial M$  is orientation-preserving if and only if  $n$  is even. In either situation, we have that

$$(-1)^n \int_{\sigma_{(n,0)}} \omega = \int_{\partial M} \omega,$$

which completes our present case.

Case 3: In general, there exist an open cover  $(U_{\alpha})$  of  $M$  and a partition of unity  $(\varphi_{\alpha})$  subordinate to it such that each  $\varphi_{\alpha} \omega$  is an  $(n-1)$ -form of the kind in Case 1 or Case 2. Since  $\sum_{\alpha} \varphi_{\alpha}$  is constant, we see that

$$0 = d\left(\sum_{\alpha} \varphi_{\alpha}\right) = \sum_{\alpha} d\varphi_{\alpha}.$$

Hence  $\sum_{\alpha} d\varphi_{\alpha} \wedge \omega = 0$ , so that  $\sum_{\alpha} \int_M d\varphi_{\alpha} \wedge \omega = 0$ . From this we compute

$$\begin{aligned}
 \int_M d\omega &= \int_M \sum_{\alpha} \varphi_{\alpha} d\omega \\
 &= \sum_{\alpha} \int_M \varphi_{\alpha} d\omega \\
 &= \sum_{\alpha} \int_M d\varphi_{\alpha} \wedge \omega + \varphi_{\alpha} d\omega \\
 &= \sum_{\alpha} \int_M d(\varphi_{\alpha} \omega) \\
 &= \sum_{\alpha} \int_{\partial M} \varphi_{\alpha} \omega \\
 &= \int_{\partial M} \omega.
 \end{aligned}$$

□

## 7 De Rham cohomology

### 7.1 Lecture 24

Given a smooth manifold  $M^n$  and integer  $k \geq 1$ , consider the vector spaces

$$\begin{aligned}
 Z^k(M) &:= \{\omega \in \mathcal{A}^k(M) : d\omega = 0\} \\
 B^k(M) &:= \{d\eta : \eta \in \mathcal{A}^{k-1}(M)\}.
 \end{aligned}$$

Since  $B^k(M) \subset Z^k(M)$ , we may form the quotient space

$$H_{\text{dR}}^k(M) := Z^k(M) / B^k(M),$$

called the  $k$ -th de Rham cohomology group of  $M$ .

*Remark 7.1.1.* This is the same as the singular cohomology group over  $\mathbb{R}$ .

$H_{\text{dR}}^k(M)$  can be thought of as a quantitative measure of the number of submanifolds of  $M$  over which we can't integrate certain closed forms to find a potentials for them. In this sense, the failure of a closed form to be exact indicates holes in  $M$ .

**Theorem 7.1.2.** *If  $M$  and  $N$  are continuously homotopy equivalent, then  $H_{\text{dR}}^k(M) \cong H_{\text{dR}}^k(N)$  for each  $k \geq 1$ .*

Recall that a space  $X$  is *contractible* if  $\text{id}_X$  is smoothly homotopic to the constant map at some point in  $X$ .

**Lemma 7.1.3 (Poincaré).** *If  $M$  is contractible, then  $H_{\text{dR}}^k(M) = 0$  for each  $k \geq 1$ .*

*Proof.* For simplicity, assume that  $k = 1$ . For each  $t \in [0, 1]$ , define  $\iota_t : M \rightarrow M \times [0, 1]$  by  $p \mapsto (p, t)$ .

**Claim.** *If  $\omega$  is any closed 1-form on  $M \times [0, 1]$ , then  $\iota_1^* \omega - \iota_0^* \omega$  is exact.*

*Proof.* If  $\pi_M : M \times [0, 1] \rightarrow M$  denotes the projection and  $(U, x^i)$  denotes local coordinates on  $M$ , then  $(\pi_M^{-1}(U), (\bar{x}^i, t))$  is a coordinate chart on  $M \times [0, 1]$  where  $\bar{x}^i := x^i \circ \pi_M$ . We thus have that  $\omega = w_i d\bar{x}^i + f dt$ . For each  $\alpha \in \{0, 1\}$ , we see that

$$\iota_\alpha^* \omega = \iota_\alpha^* (w_i d\bar{x}^i + f dt) = w_i(-, \alpha) dx^i + 0.$$

Moreover,

$$\begin{aligned} 0 &= d\omega \\ &= dw_i \wedge d\bar{x}^i + df \wedge dt \\ &= (\text{terms not involving } dt) \\ &\quad + \frac{\partial w_i}{\partial t} dt \wedge d\bar{x}^i + \frac{\partial f}{\partial \bar{x}^i} d\bar{x}^i \wedge dt. \end{aligned}$$

This implies that  $\frac{\partial w_i}{\partial t} = \frac{\partial f}{\partial \bar{x}^i}$  for each  $i$ . For each  $p \in U$ , we compute the sum

$$w_i(p, 1) - w_i(p, 0) = \int_0^1 \frac{\partial w_i}{\partial t}(p, t) dt = \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p, t) dt.$$

As a result,

$$\iota_1^* \omega - \iota_0^* \omega = \left( \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p, t) dt \right) dx^i.$$

Now, define  $g : U \rightarrow \mathbb{R}$  by  $\int_0^1 f(p, t) dt$ , so that

$$\frac{\partial g}{\partial x^i} = \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p, t) dt.$$

It follows that  $\iota_1^* \omega - \iota_0^* \omega = \frac{\partial g}{\partial x^i} dx^i = dg$ . Since the pullback is coordinate-independent,  $g$  must be as well. This completes our proof.  $\square$

By assumption, there is some smooth map  $H : M \times [0, 1] \rightarrow M$  such that  $H \circ \iota_1 = \text{id}_M$  and  $H \circ \iota_0 = e_{p_0}$  where  $p_0 \in M$ . Let  $\omega$  be a closed 1-form on  $M$ . Then  $H^* \omega$  is closed since pullbacks commute with exterior derivatives. Recall that the pullback is a contravariant functor, giving us

$$\iota_k^* H^* \omega = (H \circ \iota_k)^* \omega$$

for each  $k = 0, 1$ . By our claim, it follows that

$$\iota_1^* H^* \omega - \iota_0^* H^* \omega = \omega - 0 = \omega$$

is closed.  $\square$

The generalization of this result to any positive integer  $k$  proceeds as follows.

We have the decomposition

$$T_{(p,t)} M \times [0, 1] = \ker d\pi|_{(p,t)} \oplus \ker d\pi_M|_{(p,t)}$$

where  $\pi : M \times [0, 1] \rightarrow [0, 1]$  denotes projection. Then any 1-form  $\omega$  on  $M \times [0, 1]$  may be written uniquely as  $\omega = \omega_1 + \omega_2$  such that  $\omega_i(v_1 + v_2) = \omega(v_i)$  for each  $i = 1, 2$ . Hence there is some unique map  $f : M \times [0, 1] \rightarrow \mathbb{R}$

such that  $\omega_2 = f dt$ . In general, one can show that if  $\omega$  is a  $k$ -form on  $M \times [0, 1]$ , then we can write  $\omega$  uniquely as

$$\omega = \omega_1 + (dt \wedge \eta)$$

where  $\omega_1(v_1, \dots, v_k) = 0$  if some  $v_i$  belongs to  $\ker d\pi_M|_{(p,t)}$  and  $\eta$  is a  $(k-1)$ -form with the analogous property.

**Lemma 7.1.4.** *Define the  $(k-1)$ -form  $I\omega$  on  $M$  by*

$$I\omega|_p(v_1, \dots, v_{k-1}) = \int_0^1 \eta(p, t) \left( d\iota_t|_{(p,t)}(v_1), \dots, d\iota_t|_{(p,t)}(v_{k-1}) \right) dt.$$

*Then  $\iota_1^*\omega - \iota_0^*\omega = d(I\omega) + I(d\omega)$ . In particular,  $\iota_1^*\omega - \iota_0^*\omega$  is exact whenever  $d\omega = 0$ .*

*Proof.* For an argument similar to our case where  $k = 1$ , see Theorem 7.17 (Spivak). In particular,  $I\omega$  and  $\eta$  correspond to our  $g$  and  $f$ , respectively.  $\square$

**Corollary 7.1.5.** *Recalling (†), we see that  $\mathbb{R}^2 \setminus \{0\}$  is not contractible.*

This proves that  $\mathbb{R}^2 \setminus \{0\} \not\approx \mathbb{R}^2$ .

## 7.2 Lecture 25

**Corollary 7.2.1.** *If  $M$  is closed (i.e., compact without boundary) and orientable, then  $M$  is not contractible.*

*Proof.* There is some positively oriented orientation form  $\omega$  on  $M$ . Then  $d\omega = 0$ , and  $\int_M \omega > 0$ . But if  $\omega = d\eta$  for some form  $\eta$ , then  $\int_M \omega = \int_{\partial M} \eta = 0$  thanks to Theorem 6.2.4, a contradiction. Hence  $H^n(M) \neq 0$ .  $\square$

**Example 7.2.2.**  $\mathbb{S}^n$  is not contractible.

**Theorem 7.2.3.** *If  $M$  is a (connected) orientable  $n$ -manifold, then we have an isomorphism*

$$\underbrace{H_c^n(M)}_{\text{compactly supported}} \xrightarrow{\cong} \mathbb{R}, \quad [\omega] \mapsto \int_M \omega.$$

*Proof.* Assume that this statement holds when  $M = \mathbb{R}^n$ . There is some compactly supported orientation form  $\omega$  on  $M$  such that  $\int_M \omega \neq 0$  and  $\text{supp } \omega \subset U \subset M$ , where  $U$  is open. Let  $\omega'$  be a compactly supported  $n$ -form on  $M$ . Pick a partition of unity  $(\varphi_\alpha)$  on  $M$ . Then  $\omega' = \varphi_1 \omega' + \dots + \varphi_k \omega'$ . Thus, we may assume that  $\text{supp } \omega' \subset V$  where  $V \approx \mathbb{R}^n$ . We want to show that  $\omega' = c\omega + d\eta$  for some  $c \in \mathbb{R}$  and some  $\eta \in \mathcal{A}^{n-1}(M)$ . Since  $M$  is connected, there is some sequence

$$U = V_1, V_2, \dots, V_r = V$$

of open sets such that  $V_i \approx \mathbb{R}^n$  and  $V_i \cap V_{i+1} \neq \emptyset$  for each  $i = 1, \dots, r-1$ . We can find a family  $\{\omega_i\}_{1 \leq i \leq r-1}$  of forms on  $M$  such  $\int_M \omega_i \neq 0$  and  $\text{supp } \omega_i \subset V_i \cap V_{i+1}$ . It follows that

$$\begin{aligned} \omega_1 &= c_1 \omega + d\eta_1 \\ \omega_2 &= c_2 \omega_1 + d\eta_2 \\ &\vdots \\ \omega' &= c_r \omega_{r-1} + d\eta_r, \end{aligned}$$

as desired.  $\square$

If  $M$  and  $N$  are closed orientable  $n$ -manifolds and  $f : M \rightarrow N$  is smooth, then the pullback  $f^*$  induces a linear map  $f^* : H_{\text{dR}}^n(N) \rightarrow H_{\text{dR}}^n(M)$ . In light of Theorem 7.2.3, we get a linear map  $f^* : \mathbb{R} \rightarrow \mathbb{R}$ , which shows that there is a unique real number  $a$  such that

$$\int_M f^* \omega = a \int_N \omega$$

for every  $\omega \in H_{\text{dR}}^n(N)$ . The scalar  $a$  is called the *degree* of  $f$ .

### 7.3 Lecture 26

Let  $M$  and  $N$  be closed orientable  $n$ -manifolds and  $f : M \rightarrow N$  be smooth. By Theorem 3.6.6, find some regular value  $q$  of  $f$ . For each  $p \in f^{-1}(q)$ , let

$$\text{sgn}_p f = \begin{cases} 1 & df_p \text{ orientation-preserving} \\ -1 & df_p \text{ orientation-reversing} \end{cases}.$$

**Theorem 7.3.1.**

$$\deg f = \sum_{p \in f^{-1}(q)} \text{sgn}_p f$$

where  $\deg f \equiv 0$  if  $f^{-1}(q) = \emptyset$ . In particular,  $\deg f$  is always an integer.

*Proof.* Since  $f$  has constant rank  $n$  and  $\{q\} \subset N$  is compact, we see that  $f^{-1}(q)$  is a compact 0-dimensional submanifold of  $M$  by Theorem 3.6.4 and thus must be finite. Let  $f^{-1}(q) = \{p_1, \dots, p_k\}$ . Find charts  $U_1, \dots, U_k$  which are pairwise disjoint so that each  $u_i \in U_i$  is a regular point of  $f$ . Find a chart  $(V, y^i)$  around  $q$  such that the components of  $f^{-1}(V)$  are precisely the  $U_i$ . Let  $\omega = g dy^1 \wedge \dots \wedge dy^n$  where  $g$  is nonnegative and compactly supported in  $V$ . This implies that  $f^* \omega \subset f^{-1}(V) = U_1 \sqcup \dots \sqcup U_k$ . Therefore,

$$\int_M f^* \omega = \sum_{i=1}^k \int_{U_i} f^* \omega.$$

Since each  $f|_{U_i} : U_i \rightarrow V$  is a diffeomorphism, we have that

$$\int_{U_i} f^* \omega = \begin{cases} \int_V \omega & f|_{U_i} \text{ orientation-preserving} \\ -\int_V \omega & f|_{U_i} \text{ orientation-reversing} \end{cases}.$$

As a result,

$$\int_M f^* \omega = \left( \sum_{p \in f^{-1}(q)} \text{sgn}_p f \right) \int_V \omega = \left( \sum_{p \in f^{-1}(q)} \text{sgn}_p f \right) \int_M \omega.$$

□

**Example 7.3.2.** Let  $A_n : \mathbb{S}^n \rightarrow \mathbb{S}^n$  denote the antipodal map. Choose  $p_0 \in \mathbb{S}^n$ , which is a regular value of  $A_n$ . Hence  $\deg A_n = (-1)^{n-1}$ .

**Theorem 7.3.3.** Suppose that  $f$  and  $g$  are smoothly homotopic maps  $M \rightarrow N$ . Then  $f^* = g^*$  as linear maps.

*Proof.* By assumption, there exists a smooth map  $H : M \times [0, 1] \rightarrow M$  such that  $H \circ \iota_0 = f$  and  $H \circ \iota_1 = g$ . Let  $\omega \in Z^k(N)$ . We apply Lemma 7.1.4 (including its notation) to compute

$$\begin{aligned} g^*\omega - f^*\omega &= (H \circ \iota_1)^*\omega - (H \circ \iota_0)^*\omega \\ &= \iota_1^*(H^*\omega) - \iota_0^*(H^*\omega) \\ &= d(IH^*\omega) + I(dH^*\omega) = d(IH^*\omega). \end{aligned}$$

This implies that  $f^*([\omega]) = g^*([\omega])$ , as desired.  $\square$

**Corollary 7.3.4.** *If  $f$  and  $g$  are smoothly homotopic, then  $\int_M f^*\omega = \int_M g^*\omega$  for any closed  $n$ -form  $\omega$ .*

*Proof.* By Theorem 7.3.3,  $f^*\omega = g^*\omega + d\eta$  for some  $(n-1)$ -form  $\eta$ . Since  $M$  is closed by hypothesis, applying  $\int$  to both sides and then invoking Stokes's theorem finishes our proof.  $\square$

**Corollary 7.3.5.** *If  $f$  and  $g$  are smoothly homotopic, then  $\deg f = \deg g$ .*

**Corollary 7.3.6 (Hairy ball).** *If  $n \in \mathbb{N}$  is even, then there is no non-vanishing vector field on  $\mathbb{S}^n$ .*

*Proof.* The identity map  $\text{id}_{\mathbb{S}^n}$  has degree 1 and thus is not homotopic to the antipodal map  $A_n$ . Suppose, toward a contradiction, that there is some non-vanishing  $X \in \mathcal{X}(\mathbb{S}^n)$ . For each  $p \in \mathbb{S}^n$ , there is a unique great semicircle  $\gamma_p$  traveling from  $p$  to  $A(p)$  whose tangent vector at  $p$  equals  $cX_p$  for some  $c \in \mathbb{R}$ . The smooth map  $H(p, t) \equiv \gamma_p(t)$  defines a homotopy between  $\text{id}_{\mathbb{S}^n}$  and  $A_n$ , a contradiction.  $\square$

## 8 Integral curves and flows

### 8.1 Lecture 27

**Definition 8.1.1.** Let  $M$  be a smooth manifold and  $X \in \mathcal{X}(M)$ . We say that a differentiable curve  $\gamma : J \rightarrow M$  is an *integral curve for  $X$*  if  $\gamma'(t) = X_{\gamma(t)}$  for any  $t \in J$ .

*Terminology.* If  $0 \in J$ , then  $\gamma(0)$  is called the *starting point of  $\gamma$* .

**Example 8.1.2.** Let  $M = \mathbb{R}^2$ ,  $X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ , and  $\gamma(t) = (x(t), y(t))$ . Then  $\gamma'(t) = x'(t) \frac{\partial}{\partial x} + y'(t) \frac{\partial}{\partial y}$ . The system

$$\begin{cases} x(t) = x'(t) \\ y(t) = y'(t) \end{cases}$$

determines that  $\gamma(t) = e^t(x(0), y(0))$ .

In general, define the vector field  $x^i \frac{\partial}{\partial x^i}$  on a chart  $(U, x^i)$  for the  $n$ -manifold  $M$ . Then given an integral curve  $\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t))$  for  $X$  where  $\gamma^i = \gamma \circ x^i$ , we obtain the system

$$\gamma'^i(t) = X^i(\gamma^1(t), \dots, \gamma^n(t)).$$

Given that  $\gamma(0) = p$ , we have an initial value problem, to which we can always find a *local* solution.



**Theorem 8.1.3 (Fundamental theorem for autonomous ODEs).** *Let  $U \subset \mathbb{R}^n$  be open and  $X : U \rightarrow \mathbb{R}^n$  be a smooth vector field. Consider the initial value problem*

$$\begin{cases} \gamma'^i(t) = X^i(\gamma^1(t), \dots, \gamma^n(t)) \\ \gamma(t_0) = (c^1, \dots, c^n) \end{cases} \quad (1)$$

(a) (Existence) *Let  $t_0 \in \mathbb{R}$  and  $x_0 \in U$ . There exist some interval  $J_0 \ni t_0$  and open subset  $U_0 \subset U$  such that for each  $c \in U_0$ , there is some  $C^1$  curve  $\gamma : J_0 \rightarrow U_0$  that solves Eq. (1).*

(b) (Uniqueness) *Any two differentiable solutions to Eq. (1) agree on their common domain.*

(c) (Smoothness) *Let  $J_0$  and  $U_0$  be as in (a). Define  $\theta : J_0 \times U_0 \rightarrow U$  by  $(t, x) \mapsto \gamma_x(t)$  where  $\gamma_x : J_0 \rightarrow U$  uniquely solves Eq. (1) with initial condition  $\gamma(t_0) = x$ . Then  $\theta$  is smooth.*

**Example 8.1.4.** For any compact manifold  $M$ , we may stipulate that the  $U_0$  form a finite cover  $\{U_1, \dots, U_k\}$  of  $M$ . Make  $J_0$  smaller than any of the corresponding intervals  $J_1, \dots, J_k$ . This yields a smooth map  $\theta : J \times \mathbb{S}^n \rightarrow \mathbb{S}^n$  defined by  $(t, p) \mapsto \gamma_p^i(t)$ .

**Corollary 8.1.5.** *Let  $X$  be a smooth vector field on  $M$  and  $p \in M$ . There is some  $\epsilon > 0$  along with a smooth curve  $\gamma : (-\epsilon, \epsilon) \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma$  is an integral curve for  $X$ .*

**Definition 8.1.6.** Let  $\theta : \mathbb{R} \times M \rightarrow M$  be a group action on  $M$ .

1. We call  $\theta$  a *global flow on  $M$*  if it is smooth, i.e.,  $\theta^p(t) := \theta(t, p) : \mathbb{R} \rightarrow M$  is smooth for every  $p \in M$ .
2. We call the vector field  $p \mapsto (\theta^p)'(0)$  the *infinitesimal generator of  $\theta$* .

*Question.* When is a smooth vector field an infinitesimal generator of a global flow?

**Example 8.1.7.** Define  $X = x^3 \frac{\partial}{\partial x}$  on  $\mathbb{R}^2$ . Then any integral curve  $\gamma(t) = (x(t), y(t))$  for  $X$  must satisfy

$$\begin{aligned} \frac{dx}{dt} &= x^3 \implies dx = x^3 dt \\ &\implies -\frac{1}{2x^2} = t + c \\ &\implies x(t) = \frac{1}{\sqrt{c - 2t}}, \end{aligned}$$

which is not smooth on  $\mathbb{R}$ . Hence  $X$  fails to generate a global flow.

**Lemma 8.1.8 (Escape).** *Let  $X \in \mathcal{X}(M)$  and  $\gamma$  be an integral curve for  $X$ . If the domain of  $\gamma$  is not equal to  $\mathbb{R}$ , then  $\text{im } \gamma$  is not contained in any compact set.*

*Remark 8.1.9.* If  $M$  is compact, then every smooth vector field on  $M$  generates a global flow.

**Definition 8.1.10.** A *flow domain* for  $M$  is an open subset  $D \subset \mathbb{R} \times M$  such that for every  $p \in M$ , the set  $\{t \in \mathbb{R} \mid (t, p) \in D\}$  is an open interval containing 0

**Theorem 8.1.11 (Fundamental theorem on flows).** *Let  $M$  be a smooth manifold and  $X \in \mathcal{X}(M)$ . There exist some unique maximal flow domain  $\mathcal{D} \subset \mathbb{R} \times M$  and unique flow  $\varphi : \mathcal{D} \rightarrow M$  such that  $X$  generates  $\varphi$ .*

*Terminology.* We call  $\varphi$  the *flow of  $X$* .

**Corollary 8.1.12.** *If  $M$  is a closed manifold, then  $\mathcal{D} = \mathbb{R} \times M$ .*

## 8.2 Lecture 28

Let  $M$  be a smooth manifold without boundary. Let  $V \in \mathcal{X}(M)$  and let  $\theta$  denote the flow of  $V$ . For any  $W \in \mathcal{X}(M)$ , define the section of  $TM$  by

$$(\mathcal{L}_V W)_p \equiv \lim_{t \rightarrow 0} \frac{d(\theta_{-t})_{\theta_t(p)} (W_{\theta_t(p)}) - W_p}{t},$$

which always exists. This is called the *Lie derivative of  $W$  with respect to  $V$* .

**Proposition 8.2.1.**  $\mathcal{L}_V W \in \mathcal{X}(M)$ .

We can view the Lie derivative at a point  $p$  as the rate of change of  $W$  along the tangent vector  $V|_p$ .

**Theorem 8.2.2.** If  $V, W \in \mathcal{X}(M)$ , then  $\mathcal{L}_V W = [V, W]$ .

*Proof.* Let  $\mathcal{R}(M)$  denote the set of points  $p \in M$  such that  $V_p \neq 0$ . Note that  $\text{cl}(\mathcal{R}(M)) = \text{supp } V$ . Let  $p \in M$ . We have three cases to consider.

- (i) Suppose that  $p \in \mathcal{R}(M)$ . We can find smooth coordinates  $(U, u^i)$  near  $p$  such that  $V = \frac{\partial}{\partial u^1}$ . In these coordinates we thus have that  $\theta_t(u) = (u^1 + t, u^2, \dots, u^n)$ . The Jacobian of  $\theta_{-t}$  at each  $t$  equals the identity. For any  $u \in U$ , it follows that

$$\begin{aligned} & d(\theta_{-t})_{\theta_t(u)} (W_{\theta_t(u)}) \\ &= d(\theta_{-t})_{\theta_t(u)} \left( W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_{\theta_t(u)} \right) \\ &= W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u. \end{aligned}$$

From this we compute

$$\begin{aligned} (\mathcal{L}_V W)_p &= \frac{d}{dt} \Big|_{t=0} W^j(u^1 + t, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u \\ &= \frac{\partial}{\partial u^1} W^j(u^1, u^2, \dots, u^n) \frac{\partial}{\partial u^j} \Big|_u \\ &= [V, W]_u. \end{aligned}$$

- (ii) Suppose that  $p \in \text{supp } V \setminus \mathcal{R}(M)$ . Since  $\text{supp } V$  is dense in  $M$  and  $TM$  is Hausdorff, it follows that

$$(\mathcal{L}_V W)_p = [V, W]_p.$$

- (iii) If  $p \in M \setminus \text{supp } V$ , then  $V$  vanishes on some neighborhood  $H$  of  $p$ . This implies that  $\theta_t = \text{id}_H$ , so that

$$d(\theta_{-t})_{\theta_t(p)} (W_{\theta_t(p)}) = W_p. \text{ Hence } (\mathcal{L}_V W)_p = 0 = [V, W]_p.$$

□

**Definition 8.2.3.** A smooth local frame  $(X_1, \dots, X_n)$  is called a *commuting* or *holonomic frame* if  $[X_i, X_j] = 0$  for any  $1 \leq i, j \leq n$ .

**Theorem 8.2.4.** Let  $M$  be a smooth  $n$ -manifold. Let  $(X_1, \dots, X_k)$  be a linearly independent  $k$ -tuple of smooth commuting vector fields defined on an open set  $W \subset M$ . For any  $p \in W$ , there is some chart  $(U, x^i)$  around  $p$  such that

$$X_i = \frac{\partial}{\partial x^i}$$

on  $U$  for each  $i = 1, \dots, k$ .

*Proof sketch.* As this statement is local, we may assume that  $M = \mathbb{R}^n$  and  $p = 0$ . Since the  $X_i$  are linearly independent, we can find coordinates  $(V, t^i)$  around 0 such that  $X_i|_0 = \frac{\partial}{\partial t^i}|_0$  for each  $i$ . Let  $\theta^i$  denote the flow of  $X_i$ . By making  $V$  a sufficiently small neighborhood of 0 in  $\mathbb{R}^k \times \mathbb{R}^{n-k} \approx \mathbb{R}^n$ , define  $\Psi : V \rightarrow \mathbb{R}^n$  by

$$\Psi(t^1, \dots, t^n) = \theta_{t^1}^1 \circ \dots \circ \theta_{t^k}^k (0, \dots, 0, t^{k+1}, \dots, t^n).$$

Since the  $X_i$  are commuting, one can show that

$$d\Psi_0 = \begin{cases} X_i|_0 & i = 1, \dots, k \\ \frac{\partial}{\partial t^i}|_0 & i = k+1, \dots, n. \end{cases}$$

This is invertible, and thus  $\Psi$  is a local diffeomorphism by the inverse function theorem. This gives us our desired local coordinates.  $\square$

## 9 Distributions

**Definition 9.0.1.** Let  $M$  be a smooth manifold. A  $k$ -distribution on  $M$  is a rank- $k$  smooth subbundle of  $TM$ .

In particular, 1-distributions are precisely vector fields.

**Definition 9.0.2.** Let  $N \subset M$  be a nonempty submanifold and

$$D := \coprod_{p \in M} D_p$$

be a distribution on  $M$ . Then  $N$  is called an *integral manifold of  $D$*  if  $D_p = T_p N$  for each  $p \in N$ . Moreover, we say that  $D$  is *integrable* if each  $p \in M$  is contained in an integrable manifold of  $D$ .

**Definition 9.0.3.** We say that a distribution  $D$  is *involutive* if  $[X, Y] \in D$  whenever  $X, Y \in D$ .

**Proposition 9.0.4.** *If  $D$  is integrable, then it is involutive.*

**Theorem 9.0.5 (Frobenius).** *If  $D$  is involutive, then it is integrable.*