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Abstract

These notes are based on Julius Shaneson's lectures for the course "Algebraic Topology, Part I" at UPenn. Any mistake in what follows is my own.

Contents

1 Background material

1.1 Lecture 1

Here are the topics for the course:

- fiber bundles over cell complexes,

- spectral sequences,

- characteristic classes, and

- cobordism theory.

To start, let's review some basic concepts from homology theory.

Definition 1.1.1. A *(finite) cell complex* is a (topological) space X that can be written as $\bigcup_{n=0}^{K} X^n$ for some $K \in \mathbb{N}$ (called the *dimension of X*) where

- X^0 is chosen to be finite,

- $X^n = \frac{X^{n-1} \coprod D_1^n \coprod \cdots \coprod D_{k_n}^n}{x \sim \varphi_i(x)}$,

- $D_i^n \cong D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$ for each $i \in \{1, \ldots, k_n\}$, and

- $\varphi_i : \partial D_i^n = S^{n-1} \to X^{n-1}$, called an *attaching map*.

Terminology. Each D_i^n is called an *n-cell of X*.

Every attaching map $\varphi_i : \partial D_i^n \to X^{n-1}$ can be extended to a *characteristic map* given by the composition

$$D_i^n \hookrightarrow X^{n-1} \coprod D_1^n \coprod \cdots \coprod D_{k_n}^n \twoheadrightarrow X^n \hookrightarrow X.$$

Example 1.1.2. There are at least two ways of endowing S^2 with a cell structure.

1. $X^0 \equiv \{N, S\}$, $X^1 \equiv X^0 \cup_{\varphi_1} D_1^1 \cup_{\varphi_2} D_2^1$ where each φ_i is an embedding, and $X^2 \equiv X^1 \cup_{\varphi_1'} D_1^2 \cup_{\varphi_2'} D_2^2$ where each φ_i' is an embedding.

2. $\mathsf{pt} \cup_\varphi D^2$ where φ identifies the equator of the upper half-sphere with pt.

Definition 1.1.3. A cell complex X is *regular* if every characteristic map $D_i^n \to X$ is an embedding.

Definition 1.1.4. Given a family of functors $\{H_n : \mathbf{Top}^2 \to \mathbf{Ab}\}_{n \in \mathbb{N}}$ where $\mathbf{Top}^2$ denotes the category of (topological) pairs, we say that H_i is a *homology functor* if each of the following properties holds.

1. (LES) For any pair (X, A) of space, there is a natural long exact sequence

$$\cdots \longrightarrow H_i(A) \longrightarrow H_i(X) \longrightarrow H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \longrightarrow \cdots,$$

   where $H_i(Z) := H_i(Z, \emptyset)$ for any space Z.

2. (Excision) If $\mathrm{cl}(A) \subset \underset{open}{U} \subset X$, then $H_i(X \setminus A, U \setminus A) \cong H_i(X, U)$.

3. (Dimension) $H_i(\mathsf{pt}) = \begin{cases} 0 & i \neq 0 \\ \mathbb{Z} & i = 0 \end{cases}$.

4. (Homotopy) If f and g are homotopic, then $f_* = g_*$, where $h_* := H_i(h)$ for any map $h : (X, A) \to (Y, B)$.

Theorem 1.1.5. *There exists a family of homology functors.*

Example 1.1.6. In singular homology theory, we have that $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{otherwise} \end{cases}$.

Let X be a cell complex. Let $C_n(X)$ denote the free abelian group on the set of all n-cells of X. Define $\partial : C_n(X) \to C_{n-1}(X)$ by $\partial[D_i^n] = \sum_{j=1}^{k_{n-1}} \lambda_{ij}[D_j^{n-1}]$ where λ_{ij} is defined, up to sign, as follows. Consider the map

$$
S^{n-1} \xjoin \partial D_i^n \xrightarrow{\varphi_i} X^{n-1} \twoheadrightarrow \frac{X^{n-1}}{X^{n-2} \cup (\text{all cells of dim. } n-1 \text{ except } D_j^{n-1})} \xjoin D^{n-1}\big/\partial D_j^{n-1} \xjoin S^{n-1} .
$$

$$ \omega $$

Then let λ_{ij} satsify $\omega_*(x) = \lambda_{ij} x$ with x a chosen generator (i.e., orientation) of $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

Terminology. The integer λ_{ij} is called the *degree of ω*, denoted by $\deg(\omega)$.

Theorem 1.1.7. $\partial_n \partial_{n+1} = 0$, and $H_n(X) \cong \ker \partial_n \big/ \operatorname{im} \partial_{n+1}$, *which is independent of our choice of generator* x.

Example 1.1.8. Suppose that $f : S^n \to S^n$ is smooth. By Sard's theorem, we can find a regular value $x \in S^n$. There is some neighborhood U of x such that $f^{-1}(U) = U_1 \cup \cdots \cup U_n$ for some n. Using the inverse function theorem and the compactness of S^n, it follows that f^{-1} is of the form $\{x_1, \ldots, x_n\}$. Note that the differential $(df)_{x_i} : S_{x_i}^n \to S_x^n$ satisfies $\det (df)_{x_i} - \pm 1$. In fact,

$$
\deg(f) = \sum_{i=1}^n \det (df)_{x_i}.
$$

Exercise 1.1.9. *Prove that any finite cell complex $X = X^K$ is homotopy equivalent to a regular cell complex. (Hint: Consider the map $S^{n-1} \to X^{n-1} \times D^n$ given by $x \mapsto (\varphi(x), x)$ where φ denotes an attaching map of X.)*

Proof. Let us construct recursively a finite sequence $A^0, A^1, \ldots, A^K$ of spaces such that each A^i carries the stricture of a regular cell complex and is homotopy equivalent to X^i. For each $n \in \{1, \ldots, K\}$, let k_n denote the necessarily finite number of attaching maps $\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_{k_n}} : S^{n-1} \to X^{n-1}$ for the n-skeleton of X. Let

$$
A^0 = X^0 \times D_{\alpha_1}^1 \times \cdots D_{\alpha_{k_1}}^1,
$$

viewed as a product of finite cell-complexes. Note that the topology of A^0 is precisely the product topology. Thus, A^0 is homotopy equivalent to X^0 as D^1 is contractible. Now, suppose that $0 \le n \le K-1$ and that we have constructed our desired space A^n. This means that there is some homotopy equivalence $\gamma_n : X^n \to A^n$. Form A^{n+1} by attaching finitely many $(n+1)$-cells $e_{\alpha_1}^{n+1}, \ldots, e_{\alpha_{k_{n+1}}}^{n+1}$ to $Z_n \equiv A^n \times D_{\alpha_1}^{n+1} \times \cdots \times D_{\alpha_{k_{n+1}}}^{n+1}$ via the maps

$$
\psi_{\alpha_i} : S^n \to A^n \times D_{\alpha_1}^{n+1} \times \cdots \times D_{\alpha_{k_{n+1}}}^{n+1}
$$

$$
x \mapsto \left(\gamma_n \circ \varphi_i(x), 0, \ldots, 0, \underbrace{x}_{i\text{-th spot}}, 0, \ldots, 0 \right)
$$

where Z_n is viewed as a product of finite cell complexes (whose topology is precisely the product topology). It is easy to see that A^{n+1} is homotopy equivalent to X^{n+1}. Moreover, since each map ψ_{α_i} is an embedding and any n-disk has the structure of a regular cell complex, we see from our construction of (A^i) that A^K has the structure of a regular cell complex. By design, this space is homotopy equivalent to X^K, thereby completing our proof. $\square$

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1.2 Lecture 2

Example 1.2.1 (Real projective space). Recall that $\mathbb{RP}^n = S^n / x \sim -x$. Then $\mathbb{RP}^n = \mathbb{RP}^{n-1} \cup_{\pi_{n-1}} D^n$ where $\pi_{n-1} : S^{n-1} \to \mathbb{RP}^{n-1}$ denotes the canonical projection. Thus, $\mathbb{RP}^n$ is an n-dimension cell complex with $(\mathbb{RP}^n)^m = \mathbb{RP}^m$ for each integer $0 \le m \le n$.

Now, for each $0 \le m \le n$, we have that $C_m(\mathbb{RP}^n) \cong \mathbb{Z}$ with generator $[D^m]$. To determine $\partial[D^m] \in C_{m-1}(\mathbb{RP}^m)$, we must find the degree of the map

$$S^{m-1} \longrightarrow \mathbb{RP}^{m-1} \longrightarrow \mathbb{RP}^{m-1} / \mathbb{RP}^{m-1} \xMapsto{} D^{m-1} / \partial D^{m-1} \xMapsto{} S^{m-1}$$
$$\varphi$$

Assume, for convenience, that $m = 2$. Choose a regular value $p \in S^1$ so that $\varphi^{-1}(p) = \{N, S\}$. Let φ_T and φ_B denote the restrictions of φ to the top and bottom components of $S^1 \setminus \{(-1, 0), (1, 0)\}$, respectively. Note that both of these are homeomorphisms and thus have degrees equal to ± 1. If $a : S^{m-1} \to S^{m-1}$ denotes the antipodal map, we have that $\varphi_B \circ a = \varphi_T$. Hence $(d\varphi)_S \circ (da)_N = (d\varphi)_N$. Since $\deg(a) = \det(da) = (-1)^m$, it follows that

$$\deg(\varphi) = \begin{cases} \pm 2 & m \text{ even} \\ 0 & m \text{ odd} \end{cases}.$$

Thus, we get a chain complex

$$0 \longrightarrow C_n(\mathbb{RP}^n) \xrightarrow{\kappa_1} C_{n-1}(\mathbb{RP}^n) \xrightarrow{\kappa_2} \cdots \xrightarrow{0} C_2(\mathbb{RP}^n) \xrightarrow{\pm 2} C_1(\mathbb{RP}^n) \xrightarrow{0} C_0(\mathbb{RP}^n) \longrightarrow 0$$

where $\kappa_1 = \begin{cases} 0 & n \text{ odd} \\ \pm 2 & n \text{ even} \end{cases}$ and $\kappa_2 = \begin{cases} \pm 2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$.

This proves that

$$H_i(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i = 1 \\ 0 & i = 2 \\ \mathbb{Z}/2 & i_{odd} < n \\ 0 & i_{even} < n \\ 0 & i > n \\ \mathbb{Z} & i = n \text{ odd} \\ 0 & i = n \text{ even} \end{cases}.$$

Next, let's introduce some fundamental concepts from homotopy theory.

Definition 1.2.2. Let $M(X, Y)$ denote the set of maps $X \to Y$.

1. For any compact $C \subset X$ and open $U \subset Y$, let

$$N(C, U) = \{f : X \to Y \mid f(C) \subset U\}.$$

The *compact-open topology on* $M(X, Y)$ consists of all unions of finite intersections of subsets of the form $N(C, U)$. Under this topology, $M(X, Y)$ is called a *mapping space*.

2. The *n-th loop space* of a pointed space (X, x) is

$$\Omega^{n-1}(X, x) := M((D^{n-1}, \partial D^{n-1}), (X, x)),$$

which is a subset of $M(D^{n-1}, X)$.

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Definition 1.2.3 (Higher homotopy groups). If $n \geq 2$, then the *n-th homotopy group* of (X, x) is

$$\pi_n(X, x) := \pi_1(\Omega^{n-1}X, e_x).$$

Recall that $\pi_1(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Grp}$. Also, $\Omega^{n-1}(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Top}$ defined on morphisms $f : (X, x) \to (Y, y)$ by post-composition with f. Therefore, it's easy to see that $\pi_n(-)$ is a functor $\mathbf{Top}_* \to \mathbf{Grp}$ as well.

Notation. Let $f_* = \pi_n(f)$ for any $f : (X, x) \to (Y, y)$.

Proposition 1.2.4. *There is a homeomorphism $M(X \times Y, Z) \cong M(X, M(Y, Z))$ so long as Y is locally compact and Hausdorff.*

In particular, we have a composite

$$M((([0, 1], \{0, 1\}), (M((D^{n-1}, \partial), (X, x)), e_x)) \hookrightarrow M([0, 1], M(D^{n-1}, X)) \xrightarrow{\cong} M([0, 1] \times D^{n-1}, X),$$

whose image is precisely $M((D^n, \partial), (X, x)) \cong M((S^n, \mathsf{pt}), (X, x))$. This proves that $\pi_n(X, x)$ consists of all homotopy classes of maps $(I^n, \partial) \to (X, x)$ under the operation $[f] * [g] = [f * g]$ where

$$f * g(t_1, \ldots, t_n) = \begin{cases} f(2t_1, t_2, \ldots, t_n) & 0 \leq t_1 \leq \frac{1}{2} \\ f(2t_1 - 1, t_2, \ldots, t_n) & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Proposition 1.2.5. *If $n \geq 2$, then $\pi_n(X, x)$ is abelian.*

Remark 1.2.6. A map $f : S^{n-1} \to X$ is homotopic to the constant map if and only if there is some g such that

commutes.

Theorem 1.2.7 (Whitehead). *If $f : X \to Y$ is a map of connected cell complexes, then f is a homotopy equivalence if and only if $f_* : \pi_n(X, x) \to \pi_n(Y, y)$ is an isomorphism for each $n \in \mathbb{N}$.*

1.3 Lecture 3

Definition 1.3.1. If $x \in A \subset X$, then the *n-th relative homotopy group* $\pi_n(X, A, x)$ consists of all homotopy classes of maps $(D^n, S^{n-1}, x_0) \to (X, A, x)$.

We see that

$$M((D^n, S^{n-1}, x), (X, A, x_0)) \cong M((I^n, I^{n-1} \times \{1\}, \underbrace{\partial I^n \setminus \mathrm{Int}(I^{n-1} \times \{1\})}_{\partial_0 I^n}), (X, A, x_0))$$

by considering the homeomorphism $(I^n/\partial_0 I^n, \partial I^n/\partial_0 I^n) \cong (D^n, S^{n-1})$. Therefore, $\pi_n(X, A, x)$ can be viewed as consisting of all homotopy classes of maps $(I^n, \partial I^n, \partial_0 I^n) \to (X, A, x)$.

Definition 1.3.2. In order to interpret an exact sequence involving objects in the category of pointed sets, we define the *kernel of a function $f : (X, x) \to (Y, y)$ of pointed sets* as $\ker f \equiv f^{-1}(y)$.

Proposition 1.3.3.

1. *If $n \geq 2$, then $\pi_n(X, A, x)$ is, in fact, a group.*

2. *If $n \geq 3$, then $\pi_n(X, A, x)$ is abelian.*

3. We have a long exact sequence

$$\cdots \longrightarrow \pi_n(A,x) \longrightarrow \pi_n(X,x) \longrightarrow \pi_n(X,A,x) \xrightarrow{\;\partial\;} \pi_{n-1}(A,x)$$

$$\pi_{n-1}(X,x) \xleftarrow{\qquad} \cdots \longrightarrow \pi_0(A,x) \longrightarrow \pi_0(X,x) \longrightarrow 0$$

with $\partial[f] = [f \restriction_{I^{n-1}}]$.

Theorem 1.3.4 (Hurewicz). *Let $n \in \mathbb{N}_{\geq 2}$. If $\pi_i(X) = 0$ for each $i < n$, then $\pi_n(X) \cong H_n(X)$.*

Note 1.3.5. This result can't be improved in general. For example, $\pi_3(S^2) \cong \mathbb{Z}$, whereas $H_3(S^2) = 0$.

Let $A \subset X$ be a subcomplex. Recall that $H_i(X,A) \cong H_i(X/A.*)$ for each $i \geq 1$. But it is *not* the case that $\pi_i(X,A) \cong \pi_i(X/A.*)$, for otherwise $\pi_i(S^n) \cong \pi_i(D^n, S^{n-1}) \cong \pi_i(S^{n-1})$, which is known to be false exactly when $i > 2n - 2$.

Example 1.3.6. $\pi_4(S^3) \cong \mathbb{Z}/2 \not\cong \pi_4(S^4)$.

Finally, let's review the notion of a fibration of spaces.

Recall that if $p : E \to B$ is a covering projection, then TFAE.

1. For any $f : Z \to B$, there exists a unique $\hat{f} : Z \to E$ such that $p \circ \hat{f} = f$.

2. $f_*(\pi_1(Z)) \subset p_*(\pi_1(E))$.

The existence of $\hat{f}$ follows from the fact that any covering space satisfies the homotopy lifting property.

Definition 1.3.7 (Fibration). Suppose that $p : E \to B$ is any map. We say that p is a *fibration* if it satisfies the homotopy lifting property, i.e., given a commutative square

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\hat{f}_0} & E \\
\downarrow & & \downarrow{\scriptstyle p} \\
X \times [0,1] & \xrightarrow{f} & B
\end{array}
,
$$

where X is a cell complex, there is some G such that

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\hat{f}_0} & E \\
\downarrow & \nearrow{\scriptstyle G} & \downarrow{\scriptstyle p} \\
X \times [0,1] & \xrightarrow{f} & B
\end{array}
$$

commutes.

Theorem 1.3.8. *If $p : E \to B$ is a fibration with $e \in F := p^{-1}(b)$, then*

$$p_* : \pi_i(E,F,e) \xrightarrow{\;\cong\;} \pi_i(B,b,b) = \pi_i(B,b).$$

Proof. Let $f : (I^n, \partial I^n) \to (B,b)$. To prove that p_* is surjective, it suffices to find some $G : (I^n, \partial I^n) \to (E,F)$ such that

$$
\begin{array}{ccccccc}
\partial_0 I^n & \twoheadrightarrow & \{e\} & \hookrightarrow & F & \hookrightarrow & E \\
\downarrow & & & {\scriptstyle G} & & & \downarrow{\scriptstyle p} \\
I^{n-1} \times [0,1] & & & \xrightarrow{f} & & & B
\end{array}
$$

commutes, for in this case $[p \circ G'] = [f]$. Since p is a fibration, there is some G such that

$$
\begin{array}{ccccccc}
I^{n-1} \times \{0\} & \longrightarrow & \{e\} & \lhook\joinrel\longrightarrow & F & \lhook\joinrel\longrightarrow & E \\
\downarrow & & & G' & & & \downarrow p \\
I^{n-1} \times [0,1] & & \longrightarrow & & & f & B
\end{array}
$$

commutes. But $(I^n, \partial_0 I^n) \cong (I^n, I^{n-1} \times \{0\})$, and thus such a G' is enough. $\qquad\square$

Corollary 1.3.9. *We have a long exact sequence*

$$
\cdots \longrightarrow \pi_i(F, e) \longrightarrow \pi_i(E, e) \longrightarrow \pi_i(B, b) \xrightarrow{\ \partial\ } \pi_{i-1}(F, e) \longrightarrow \cdots .
$$

Example 1.3.10.

1. Suppose that

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\ \hat{f}\ } & B \times F \\
\downarrow & & \downarrow \pi_B \\
X \times [0,1] & \xrightarrow{\ f\ } & B
\end{array}
$$

   commutes. Then $\hat{f}(x,0) = \left(\hat{f}_1(x,0), \hat{f}_2(x,0) \right)$ where $\hat{f}_1(x,0) = f(x,0)$. Let $G(X,t) = (f(x,t), \hat{f}_2(x,0))$. Then

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\ \hat{f_0}\ } & B \times F \\
\downarrow & {\scriptstyle G}\nearrow & \downarrow \pi_B \\
X \times [0,1] & \xrightarrow{\ f\ } & B
\end{array}
$$

   commutes, so that π_B is a fibration. (Moreover, $\pi_n(B \times F) \cong \pi_n(B) \times \pi_n(F)$.)

2. Let $A \subset X$ be a subcomplex. The map $\varphi : M(X, Y) \to M(A, Y)$ defined by $f \mapsto f \restriction_A$ is a fibration.

3. Define the *Hopf fibration* as the quotient map

$$
S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \overline{z_1} + z_2 \overline{z_2} = 1\} \twoheadrightarrow \left. S^3 \middle/ {}_{x \sim -x} \right. = \mathbb{CP}^1 = S^2.
$$

Corollary 1.3.11. $\pi_3(S^3) \cong \pi_3(S^2)$.

Proof. Since we have an exact sequence

$$
\pi_3(S^1) \longrightarrow \pi_3(S^3) \longrightarrow \pi_3(S^2) \longrightarrow \pi_2(S^1) \ ,
$$

it suffices to show that both $\pi_3(S^1)$ and $\pi_2(S^1)$ are trivial. To this end, note that since $\pi_1(S^k) = 0$ for every $k > 1$, we can always find, for any $f : S^k \to S^1$, a map $\hat{f}$ such that

$$
\begin{array}{ccc}
 & & \mathbb{R} \\
 & {\scriptstyle \hat{f}}\nearrow & \downarrow {\scriptstyle e^{2\pi i x}} \\
S^k & \xrightarrow{\ f\ } & S^1
\end{array}
$$

commutes. Thus, f is homotopic to the constant map. Since f was arbitrary, our proof is complete. $\qquad\square$

Definition 1.3.12. A map $p : E \to B$ is *locally trivial* if for any $b \in B$, there exist a neighborhood $U \ni b$ in B, a space F, and a homeomorphism $\varphi : p^{-1}(U) \xrightarrow{\cong} U \times F$ such that $\pi_U \circ \varphi = p \restriction_{p^{-1}(U)}$.

Theorem 1.3.13. *Any locally trivial map $p : E \to B$ is a fibration whenever B is a cell complex.*

Exercise 1.3.14. *Prove that the Hopf fibration is locally trivial.*

Proof. For each $k \in \{0, 1\}$, let $U_k = \{[z_0, z_1] \in \mathbb{CP}^1 \mid z_k \neq 0\}$. Then U_0 and U_1 form an open cover of $\mathbb{CP}^1$. Note that the preimage of U_k under the Hopf fibration q is precisely $\{(z_0, z_1) \in S^3 \mid z_k \neq 0\}$. Define $f : q^{-1}(U_k) \to U_k \times S^1$ by

$$(z_0, z_1) \mapsto \left([z_0, z_1], \frac{z_k}{|z_k|} \right).$$

This is clearly continuous. Further, define the map $g : U_k \times S^1 \to q^{-1}(U_k)$ by

$$\left([z_0, z_1], e^{i\theta} \right) \mapsto \frac{e^{i\theta} |z_k|}{z_k |(z_0, z_1)|} (z_0, z_1).$$

Since U_k is a saturated open set, we have that the restriction of q to $q^{-1}(U_k)$ is a quotient map. But $g \circ q \restriction_{q^{-1}(U_k)}$ is continuous, so that g is also continuous by the characteristic property of quotient maps. Finally, it is easy to verify that g and f are inverses of each other and that $\pi_{U_I} \circ f = p \restriction_{q^{-1}(U_k)}$. $\qquad \square$

1.4 Lecture 4

Theorem 1.4.1. *Let $A \subset X$ be a subcomplex. Define $r : M(X, Y) \to M(A, Y)$ by $r(f) = f \restriction_A$. Then r is a fibration.*

Proof. We must fill any diagram of the form

$$
\begin{array}{ccc}
Z \times \{0\} & \xrightarrow{\hat{f}} & M(X, Y) \\
\downarrow & \overset{F}{\nearrow} & \downarrow{r} \\
Z \times [0, 1] & \xrightarrow{f} & M(A, Y)
\end{array} \quad .
$$

It suffices to find a map $\overline{F}$ such that

$$
\begin{array}{ccc}
Z \times \{0\} \times X & \xrightarrow{\hat{\hat{f}}} & Y \\
\downarrow & \overset{\overline{F}}{\nearrow} & \| \\
Z \times [0, 1] \times X & & Y \\
\uparrow & \overset{\bar{f}}{\nearrow} & \\
Z \times [0, 1] \times A & &
\end{array}
$$

commutes for, in this case, we can set $F(z, t)(x) = \overline{F}(z, t, x)$.

Note 1.4.2. Suppose that such an $\overline{F}$ exists. Define $g : Z \times X \to Y$ by $g(z, x) = \hat{\hat{f}}(z, 0, x)$. Define $h : Z \times X \times [0, 1] \to Y$ by $H(z, x, t) = \overline{F}(z, t, x)$. Then

$$
\begin{array}{c}
Z \times X \times \{0\} \\
\downarrow \quad \overset{g}{\searrow} \\
Z \times X \times [0, 1] \xrightarrow{H} Y \\
\uparrow \quad \overset{K}{\nearrow} \\
Z \times A \times [0, 1]
\end{array}
$$

commutes where $K(z, a, t) = \bar{f}(z, t, a)$. In the case where $Z = \mathsf{pt}$, this means that if $K : A \times [0, 1] \to Y$ is a homotopy from a map $f : A \to Y$ and g extends f to X, then there exists a homotopy $H : X \times [0, 1] \to Y$ such that $H \restriction_{A \times [0, 1]} = K$. In other words, the extension problem for cell complexes is a homotopy problem.

Let's return to proving our theorem. By induction, it suffices to consider just the case where $X = A \cup_\varphi D^n$, with characteristic map $\chi : D^n \to X$. Thus, it suffices to find a map w such that

$$
\begin{array}{c}
Z \times D^n \times \{0\} \\
\downarrow \\
Z \times D^n \times [0,1] \xrightarrow{\quad w \quad} Y \\
\uparrow \\
Z \times S^{n-1} \times [0,1] \xrightarrow[\mathrm{id}_Z \times \varphi \times \mathrm{id}_{[0,1]}]{} Z \times A \times [0,1]
\end{array}
$$

with arrows $\mathrm{id}_Z \times (g \circ \chi)$ and K into Y.

commutes for, in this case, we can set $H(z,x,t) = g \cup_\varphi w$, thereby making

$$
\begin{array}{c}
Z \times D^n \times \{0\} \\
\downarrow \\
Z \times D^n \times [0,1] \xrightarrow[\mathrm{id}_Z \times \chi \times \mathrm{id}_{[0,1]}]{} Z \times X \times [0,1] \xrightarrow{\ H\ } Y \\
\uparrow \\
Z \times S^{n-1} \times [0,1] \\
\xrightarrow[\mathrm{id}_Z \times \varphi \times \mathrm{id}_{[0,1]}]{} \\
Z \times A \times [0,1]
\end{array}
$$

with arrows $\mathrm{id}_Z \times (g \circ \chi)$, w, and K.

commute. To this end, define the retraction $u : D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]$ by picking a point $*$ directly above the cylinder $D^n \times [0,1]$ and then sending any point x in the cylinder to the unique point where $D^n \times \{0\} \cup S^{n-1} \times [0,1]$ intersects the line containing $*$ and x. Now, define w so that

$$
\begin{array}{c}
Z \times (D^n \times [0,1]) \xrightarrow{\quad w \quad} Y \\
\mathrm{id}_Z \times u \Big\downarrow \qquad \nearrow{\scriptstyle \mathrm{id}_Z \times \left(g \circ \chi \cup K \circ (\varphi \times \mathrm{id}_{[0,1]})\right)} \\
Z \times (D^n \times \{0\} \cup S^{n-1} \times [0,1])
\end{array}
$$

commutes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Exercise 1.4.3. *Let $x \in X$. Consider the loop space $\Omega(X, x) \equiv M((S^1, \mathsf{pt}), (X, x))$. Prove that $\pi_n(\Omega X) \cong \pi_{n+1}(X)$.*

Proof. Consider the *path space* $PX \equiv \{\gamma : [0,1] \to X \mid \gamma(0) = x\}$ of (X, x), equipped with the compact-open topology. We claim that PX is contractible. Indeed, define $K : PX \times [0,1] \to PX$ by

$$
(\gamma, t) \mapsto (s \mapsto \gamma(s(1 - t))) .
$$

Then K is a homotopy from id_{PX} to the constant map at the constant path at x.

Define the map $p : PX \to X$ by $\gamma \mapsto \gamma(1)$. Then $p^{-1}(x) = \Omega(X)$. By Corollary 1.3.9, it suffices to show that p is a fibration. To this end, suppose that the square

$$
\begin{array}{ccc}
Y \times \{0\} & \xrightarrow{\ \hat{f}\ } & PX \\
\downarrow & & \downarrow{\scriptstyle p} \\
Y \times [0,1] & \xrightarrow{\ f\ } & X
\end{array}
$$

commutes. Define $H : Y \times [0,1] \to PX$ by $(y, t) \mapsto H(y, t)$ where

$$
H(y,t)(s) = \begin{cases} \hat{f}(y)\left((1+t)s\right) & 0 \leq s \leq \frac{1}{1+t} \\ f(y, (1+t)s - 1) & \frac{1}{1+t} \leq s \leq 1 \end{cases} .
$$

We see that H is continuous when viewed as a function of (y, t, s) and thus is continuous. It is easy to check that

$$
\begin{array}{ccc}
Y \times \{0\} & \xrightarrow{\hat{f}} & PX \\
\downarrow & \nearrow^{H} & \downarrow{p} \\
Y \times [0,1] & \xrightarrow{f} & X
\end{array}
$$

commutes, as desired. $\qquad\square$

Let $p : E \to B$ be a map. Recall that the pullback of p along $f : X \to B$ is given explicitly as

$$
f^* E \equiv \{(x, e) \in X \times E \mid f(x) = p(e)\}.
$$

Let $f^* p$ denote the map $\pi_X \restriction_{f^* E}$.

Proposition 1.4.4. *If p is a fibration, then so is $f^* p$.*

Lemma 1.4.5. *If p is locally trivial, then so is $f^* p$.*

Proof. Let $a \in X$. Since p is locally trivial by assumption, we can find a neighborhood U of $f(a)$ in B and a homeomorphism $\varphi : p^{-1}(U) \to U \times F$. Observe that

$$
(f^* p)^{-1}(f^{-1}(U)) = \{(x, e) \mid f(x) = p(e), \; f(x) \in U\} \subset f^{-1}(U) \times p^{-1}(U).
$$

Further, we have a map $\psi : f^{-1}(U) \to p^{-1}(U) \to f^{-1}(U) \times F$ given by $(x, e) \mapsto (x, \pi_F(\varphi(e)))$. Define $\lambda : f^{-1}(U) \times F \to (f^* p)^{-1}(f^{-1}(U))$ by $(x, y) \mapsto (x, \varphi^{-1}(f(x), y))$. Using the fact that

$$
\begin{array}{ccc}
p^{-1}(U) & \xrightarrow{\varphi} & U \times F \\
 & \searrow_{p} & \downarrow{\pi_U} \\
 & & U
\end{array}
$$

commutes, it is easy to check that ψ and λ are inverses of each other. $\qquad\square$

1.5 Lecture 5

Theorem 1.5.1. *Let B be a cell complex and let $p : E \to B$ be locally trivial. Then p is a fibration.*

Proof. It suffices to prove the following claim:

> If $h : Z \to X \times [0,1]$ is locally trivial, $X = \bigcup_{i=0}^n X^i$ is a cell complex, and $\sigma_0 : X \times \{0\} \to Z$ satisfies $h \circ \sigma_0 = \mathrm{id}_{X \times \{0\}}$, then there is some map $\sigma : X \times [0,1] \to Z$ such that $\sigma_{X \times \{0\}} = \sigma_0$ and $h \circ \sigma = \mathrm{id}_{X \times [0,1]}$.

For, in this case, Lemma 1.4.5 implies that given any commutative square

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\hat{f}} & E \\
\downarrow & & \downarrow{p} \\
X \times [0,1] & \xrightarrow{f} & B
\end{array}
,
$$

we can find some σ such that

$$
\begin{array}{ccc}
 & f^* E & \longrightarrow E \\
\nearrow^{\sigma_0} & \uparrow\downarrow{\sigma} & \downarrow{p} \\
X \times \{0\} \hookrightarrow & X \times [0,1] & \xrightarrow{f} B
\end{array}
$$

commutes where $\sigma_0(x, 0) = (x, 0, \hat{f}(x, 0))$.

For induction, let us assume that our claim is true for each $X^0, X^1, \ldots, X^{n-1}$. We may assume, wlog, that $X = D^n$. It suffices to find a map $\tau : S^{n-1} \times [0,1] \to Z$ such that $h \circ \tau = \mathrm{id}_{S^{n-1} \times [0,1]}$ and

$$
\begin{array}{ccccc}
 & & Z & & \\
 & \sigma_0 \nearrow & \big\downarrow h & \nwarrow \tau & \\
D^n \times \{0\} & \longrightarrow & D^n \times [0,1] & \longleftarrow & S^{n-1} \times [0,1] \\
 & \nwarrow & & \nearrow & \\
 & & S^{n-1} \times \{0\} & &
\end{array}
$$

commutes since there is a retraction $D^n \times [0,1] \to D^n \times \{0\} \cup S^{n-1} \times [0,1]$. Fix a positive integer m. For any $i \in \mathbb{N}$, let $a_i = \frac{i}{m}$ and let $I_j = [a_j, a_{j+1}]$. By making m large enough, we can ensure that $p \restriction_{p^{-1}(I_{j_1} \times \cdots I_{j_{n+1}})}$ is trivial.

Claim. $p \restriction_{p^{-1}(I_{j_1} \times I_{j_n} \times \cdots [0,1])}$ *is also trivial.*

Proof. ?? $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

?? $\qquad$ □

2 Fiber bundles

Definition 2.0.1. A *topological group* G is a group such that both multiplication $G \times G \xrightarrow{\mu} G$ and inversion $G \xrightarrow{(-)^{-1}} G$ are continuous.

Definition 2.0.2 (Fiber bundle). Let G be a topological group.

1. A *fiber F of G* is a space equipped with a faithful (i.e., injective) group action $\rho : G \to \mathrm{Homeo}(F) \subset M(F,F)$.

2. An *atlas for the structure of a (fiber) bundle with group G and fiber F on a map $p : E \to B$* consists of

   (a) a family $(U_\alpha, h_\alpha)_{\alpha \in A}$ where each U_α is open and each h_α is a homeomorphism $p^{-1}(U_\alpha) \to U_\alpha \times F$ and

   (b) a family of continuous *transition functions* $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}_{\alpha,\beta \in A}$

   such that

      i $B = \bigcup_{\alpha \in A} U_\alpha$,

     ii $\pi_{U_\alpha} \circ h_\alpha = p \restriction_{p^{-1}(U_\alpha)}$, and

    iii $x \in U_\alpha \cap U_\beta \implies h_\beta \circ h_\alpha^{-1}(x, f) = (x, h_{\beta\alpha}(x) \cdot f)$

3. Two atlases are *compatible* if their union is an atlas.

4. A *bundle structure on B* is a maximal atlas on p.

Terminology. If B is equipped with a bundle structure, then we say that p is a (fiber) bundle.

Example 2.0.3.

1. The tangent bundle $\pi : TM \to M$ of a smooth n-manifold M is a bundle with group $\mathrm{GL}(n, \mathbb{R})$.

Proof. Let (U, φ) be any coordinate chart for M with coordinate functions (x^i). Define $h : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by

$$v^i \frac{\partial}{\partial x^i}(p) \mapsto \left(p, \left(v^1, \ldots, v^n\right)\right).$$

It is clear that $\pi_U(h(p)) = \pi(c)$ for any $c \in \pi^{-1}(U)$. To see that h is a homeomorphism, note that the composite $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n$ is given by

$$v^i \frac{\partial}{\partial x^i}(p) \mapsto \left(x^1(p), \ldots, x^n(p), v^1, \ldots, v^n\right),$$

the inverse of which is given by $\left(x^1, \ldots, x^n, v^1, \ldots, v^n\right) \mapsto v^i \frac{\partial}{\partial x^i}\left(\varphi^{-1}(x)\right)$. Therefore, $(\varphi \times \mathrm{id}_{\mathbb{R}^n}) \circ h$ is given locally by

$$\left(x^1, \ldots, x^n, v^1, \ldots, v^n\right) \mapsto \left(\tilde{x}^1(x), \ldots, \tilde{x}^n(x), \frac{\partial \tilde{x}^1}{\partial x^j}(x)v^j, \ldots, \frac{\partial \tilde{x}^n}{\partial x^j}(x)v^j\right),$$

which is smooth. Thus, h is a diffeomorphism as the composition of two diffeomorphisms. In particular, h is a homeomorphism.

It remains to describe the transition functions $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{R})\}$ for TM. Note that

$$U_{\alpha\beta} \times \mathbb{R}^n \xleftarrow{h_\alpha} \pi^{-1}(U_{\alpha\beta}) \xrightarrow{h_\beta} U_{\beta\alpha} \times \mathbb{R}^n$$
$$\pi_1 \searrow \quad \downarrow \pi \quad \swarrow \pi_1$$
$$U_{\alpha\beta}$$

commutes. In particular, $\pi_1 \circ h_\beta \circ h_\alpha^{-1} = \pi_1$, which implies that $h_\beta \circ h_\alpha^{-1}(u, v) = (u, f(u, v))$ for some smooth map $f : U_{\alpha\beta} \times \mathbb{R}^n \to \mathbb{R}^n$. This must be a linear isomorphism when restricted to $\{u\} \times \mathbb{R}^n$ for any $u \in U_{\alpha\beta}$, which is uniquely determined by an element $h_{\beta\alpha}(u)$ of $\mathrm{GL}(n, \mathbb{R})$ (provided that we have fixed a basis of $\mathbb{R}^n$). Hence

$$h_\beta \circ h_\alpha^{-1}(u, v) = (u, h_{\beta\alpha}(u)v).$$

Since the map $h_{\beta\alpha} : U_{\alpha\beta} \to \mathrm{GL}(n, \mathbb{R})$ is continuous, our proof is complete. $\qquad\square$

2. Let $p : E \to B$ be any bundle with group $\{e\}$. Then p is the trivial bundle, i.e., is isomorphic to the projection map.

   Proof. We have that $h_\beta = h_\alpha$ on $p^{-1}(U_\alpha \cap U_\beta) = p^{-1}(U_\alpha) \cap p^{-1}(U_\beta)$, so that $h \equiv \bigcup_{\alpha \in A} h_\alpha$ is a well-defined homeomorphism $E \cong B \times F$. $\qquad\square$

2.1 Lecture 6

Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure with group G and fiber F on $p : E \to B$. Let $U = U_\alpha \cap U_\beta \cap U_\gamma$. Consider the commutative diagram

$$\begin{array}{c} p^{-1}(U) \\ h_\alpha^{-1} \nearrow \qquad \searrow h_\gamma \\ U \times F \xrightarrow[h_\alpha^{-1}]{} p^{-1}(U) \xrightarrow[h_\beta]{} U \times F \xrightarrow[h_\beta^{-1}]{} p^{-1}(U) \xrightarrow[h_\gamma]{} U \times F \end{array}.$$

The bottom row is given by $(u, f) \mapsto (u, h_{\beta\alpha}(u) \cdot f) \mapsto (u, h_{\gamma\beta}(u) \cdot (h_{\beta\alpha}(u) \cdot f)) = (u, (h_{\gamma\beta}(u)h_{\beta\alpha}(u)) \cdot f)$, and the top composite is given by $(u, f) \mapsto (u, h_{\gamma\alpha}(u) \cdot f)$. It follows that

$$h_{\gamma\beta}(u)h_{\beta\alpha}(u) = h_{\gamma\alpha}(u)$$

for each $u \in U$. This property is known as the *cocycle condition*.

Theorem 2.1.1. *Let G be a topological group acting on a space F. Suppose that $\{U_\alpha\}$ is an open cover of B and $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is a family of continuous functions satisfying the cocycle condition. Then there exists a bundle $p : E \to B$ with group G, fiber F, and transition functions $h_{\beta\alpha}$.*

Proof sketch. Let $E = \coprod_\alpha U_\alpha \times F \big/ {\sim}$ where $(u, f)_\alpha \sim (u, h_{\beta\alpha}(u) \cdot f)_\beta$. Define $p : E \to B$ by $(u, f) \mapsto u$. $\qquad\square$

Definition 2.1.2 (Bundle map). A *morphism of bundles p_1 and p_2 with group G and fiber F* is a commutative square of the form

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\hat{g}} & E_2 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\
B_1 & \xrightarrow{g} & B_2
\end{array}
\;.
$$

Suppose that $(\hat{g}, g)$ is a bundle map $p_1 \to p_2$. Let $\{(U_\alpha, h_\alpha)\}$ and $\{(V_\beta, k_\beta)\}$ be bundle structures on B_2 and B_1, respectively. We have a commutative diagram

so that $d_{\alpha\beta}(x, f) = (g(x), \lambda_{\alpha\beta}(x) \cdot f)$ for some continuous map $\lambda_{\alpha\beta} : g^{-1}(U_\alpha) \cap V_\beta \to G$. Letting $W = g^{-1}(U_\alpha \cap U_{\alpha'}) \cap (V_\beta \cap V_{\beta'})$, we have that

$$
h_{\alpha'\alpha}(w)\lambda_{\alpha\beta}(w)k_{\beta\beta'}(w) = \lambda_{\alpha'\beta'}(w) \tag{$\dagger$}
$$

for every $w \in W$.

Exercise 2.1.3 (Pullback bundle). *Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure on $p : E \to B$ with group G and consider the pullback diagram*

$$
\begin{array}{ccc}
g^*E & \longrightarrow & E \\
{\scriptstyle g^*p}\downarrow & & \downarrow{\scriptstyle p} \\
X & \xrightarrow{g} & B
\end{array}
\;.
$$

Define $h'_{\beta\alpha} : g^{-1}(U_\alpha) \cap g^{-1}(U_\beta) \to G$ as the composite $h_{\beta\alpha} \circ g$ restricted to $g^{-1}(U_\alpha \cap U_\beta)$. Show that the family $\{h'_{\beta\alpha}\}$ induces a bundle structure on g^*p.

Theorem 2.1.4. *Every bundle map*

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\hat{g}} & E_2 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\
B_1 & \xrightarrow{g} & B_2
\end{array}
$$

factors as

$$
\begin{array}{ccccc}
E_1 & \xrightarrow{\tau} & g^*E_2 & \xrightarrow{\bar{g}} & E_2 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle g^*p_2} & & \downarrow{\scriptstyle p_2} \\
B_1 & \xrightarrow{\mathrm{id}_{B_1}} & B_1 & \xrightarrow{g} & B_2
\end{array}
$$

where $\tau(e) = (p_1(e), \hat{g}(e))$ for any $e \in E_1$.

2.2 Lecture 7

Note 2.2.1. If $\{h_{\beta\alpha} : U_\alpha \cap U_\beta \to G\}$ is a family of transition functions, then

$$h_{\alpha\beta}(x) = (h_{\beta\alpha}(x))^{-1}$$

for any $x \in U_\alpha \cap U_\beta$. In particular, $h_{\alpha\alpha}(x) = (h_{\alpha\alpha}(x))^{-1}$.

Theorem 2.2.2. *Any bundle map of the form*

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ \hat{g}\ } & E_2 \\
 & {\scriptstyle p_1}\searrow & \downarrow{\scriptstyle p_2} \\
 & & B
\end{array}
$$

is an isomorphism.

Proof. Note that

$$
\begin{array}{ccccc}
 & & p_2^{-1}(U_\alpha \cap U_\beta) & & \\
 & {\scriptstyle h_\beta}\nearrow & \uparrow{\scriptstyle \hat{g}} & \nwarrow{\scriptstyle h_\alpha} & \\
(U_\alpha \cap U_\beta) \times F & \xleftarrow{\ k_\beta\ } & p_1^{-1}(U_\alpha \cap U_\beta) & \xleftarrow{\ k_\alpha^{-1}\ } & (U_\alpha \cap U_\beta) \times F \\
 & \searrow & \downarrow & \swarrow & \\
 & & U_\alpha \cap U_\beta & &
\end{array}
$$

commutes. We have that $h_\beta \circ \hat{g} \circ k_\alpha^{-1}(x, f) = (x, \lambda_{\beta\alpha}(x) \cdot f)$. Thus, if $h_\alpha(e) = (x, f)$, then $h_\alpha(\hat{g}(e)) = (x, \lambda_{\alpha\alpha}(x) \cdot d)$. Let

$$(\hat{g})^{-1}(e) = k_\alpha^{-1}\left(x, \lambda_{\alpha\alpha}(x)^{-1} \cdot f\right)$$

where $(x, f) = h_\alpha(e)$. If this is well-defined on E_2 (??), then it indeed equals the inverse of $\hat{g}$. Moreover, by Note 2.2.1, it is easy to check that $d_{\alpha'\beta'}(x)^{-1}$ satisfies (†), and thus it can be shown that $(\hat{g})^{-1}$ is a bundle map. $\qquad\square$

Corollary 2.2.3. *Every bundle $E \to X$ is isomorphic to the pullback of E by id_X.*

Let $\{(U_\alpha, h_\alpha)\}$ be a bundle structure with group G and fiber G on $p : E \to X$. In particular,

$$
\begin{array}{ccc}
U_\alpha \times G & \xleftarrow{\ h_\alpha\ } & p^{-1}(U_\alpha) \\
{\scriptstyle \pi_1}\downarrow & \swarrow{\scriptstyle p} & \\
U_\alpha & &
\end{array}
$$

commutes. Define the free action $E \times G \to E$ by

$$e \cdot g = h_\alpha^{-1}\left(h_\alpha(e) \cdot g\right).$$

where $p(e) \in U_\alpha$ and $(u, h) \cdot g \equiv (u, hg)$. This is well-defined because it does not depend on our choice of α. Indeed, suppose that $p(e)$ also belongs to U_β. We have that $h_\alpha(e) = (p(e), h)$ and $h_\beta(e) = (p(e), h')$ for some $h, h' \in G$. Then $e \cdot g = h_\alpha^{-1}(p(e), hg)$, and we must show that this equals $h_\beta^{-1}(p(e), h'g)$. Note that $h_\beta(e \cdot g) = (p(e), h_{\beta\alpha}(p(e))hg)$. But

$$(p(e), h_{\beta\alpha}(p(e))h) = h_\beta\left(h_\alpha^{-1}(p(e), h)\right) = (p(e), h'),$$

so that $h_{\beta\alpha}(p(e))h = h'$, and thus $h_\beta(e \cdot g) = (p(e), h'g)$, as desired.

Note 2.2.4. $E\big/G \cong \{p^{-1}(x) \mid x \in X\} \cong X$.

Definition 2.2.5 (Balanced product). Let F be a space. The *balanced product* $E \times_G F$ *of* E *and* F is the quotient space $E \times F \big/ {\sim}$ where

$$(e, f) \sim \left(eg, g^{-1}f\right)$$

for any $e \in E$ and $f \in F$.

By the universal property of the quotient space, there is a unique map $\bar{p}$ such that

$$
\begin{array}{ccc}
E \times F & \longrightarrow\!\!\!\!\to & E \times_G F \\
{\scriptstyle p \circ \pi_E} \downarrow & \swarrow {\scriptstyle \bar{p}} & \\
X & &
\end{array}
\qquad . \tag{$\star$}
$$

Notation. Let $\mathcal{B}(X, G, \rho, F)$ denote the set of all isomorphism classes of bundles over X with group G and fiber F.

Lemma 2.2.6. $\bar{p}$ *is a bundle with group* G *and fiber* F.

Proof. As $(g, f) \sim (e_G, gf)$, we see that $(U \times G) \times_G F \cong U \times F$. Thus, we can endow $\bar{p}$ with local trivializations and transition functions that are exactly similar to those for p. $\qquad\square$

Proposition 2.2.7. *The function* $p \mapsto \bar{p}$ *defines a set isomorphism* $\mathcal{B}(X, G, \rho, G) \xrightarrow{\cong} \mathcal{B}(X, G, \rho, F)$.

Let $p_1 : E \to B_1$ and $p_2 : E \to B_2$ be bundles. Let $e_1 \in E_1$, $e_2 \in E_2$, and $b_1 \in B_1$.

Question. Can we find a bundle map

$$
\begin{array}{ccc}
E_1 & \dashrightarrow & E_2 \\
{\scriptstyle p_1} \downarrow & & \downarrow {\scriptstyle p_2} \\
B_1 & \dashrightarrow & B_2
\end{array}
$$

such that $e_1 \mapsto e_2$ and $e_1 \mapsto b_1$?

Define the action $G \times E_2 \to E_2$ by $g * e_2 = e_2 \cdot g^{-1}$. From this, we obtain a bundle

$$\psi : \underbrace{E_1 \times_G E_2}_{(E_1 \times E_2)/G} \to E_1 \times_G \mathsf{pt} \cong B_1$$

with fiber E_2.

Lemma 2.2.8. *There is a one-to-one correspondence between bundle maps* $p_1 \to p_2$ *and sections of* ψ.

Proof. Suppose that σ is a section of ψ. As G acts freely on $E_1 \times E_2$, we see that for any $e \in E_1$, there exists a unique $\tilde{e}$ such that $\sigma(p(e)) = [(e, \tilde{e})]$. Define $\hat{g} : E_1 \to E_2$ by $e \mapsto \tilde{e}$. This respects the action of G and thus must be a bundle map. $\qquad\square$

Now, let $A \subset B_1$ and suppose that

$$
\begin{array}{ccc}
p_1^{-1}(A) & \longrightarrow & E_2 \\
\downarrow & {\scriptstyle \alpha} & \downarrow {\scriptstyle p_2} \\
A & \longrightarrow & B_2
\end{array}
$$

is a bundle map. Then α extends when ??. Also, the corresponding section

$$\sigma : A \to p^{-1}(A) \times_G E_2 \subset E_1 \times_G E_2$$

extends.

Definition 2.2.9 (Principal bundle). Let G be a topological group. A *principal* G-*bundle* is a fiber bundle with group G and fiber G with G acting on itself by left translation.

Theorem 2.2.10. *Let* f *and* g *be homotopic maps* $X \to Y$. *Let* $p : E \to Y$ *be any bundle with group* G *and fiber* F. *Then* $f^*p \cong g^*p$.

2.3 Lecture 8

Before proving this, we wish to determine when, given any two bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ and any map $g : B_1 \to B_2$, we can find a map $\hat{g}$ such that

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\hat{g}} & E_2 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\
B_1 & \xrightarrow{g} & B_2
\end{array}
$$

commutes.

Define the *diagonal action* ΔG of G on $E_1 \times E_2$ by

$$(e_1, e_2) \cdot h = (e_1 \cdot h, e_2 \cdot h),$$

so that $E_1 \times_G E_2 = {E_1 \times E_2}/{\Delta G}$. By $(\star)$, we can find a unique map τ such that

$$
\begin{array}{ccc}
E_1 \times_G E_2 & & \\
\downarrow & \searrow{\scriptstyle \tau} & \\
B_1 & \xleftarrow[\pi_1]{} & B_1 \times B_2
\end{array}
$$

commutes.

Exercise 2.3.1. *Show that $\hat{g}$ exists if and only if there is some $\lambda : B_1 \to E_1 \times_G E_2$ such that $\tau\left(\lambda\left(b_1\right)\right) = (b_1, g(b_1))$.*

Proof.
($\Longleftarrow$) As G acts freely on $E_1 \times E_2$, we see that $(e, e') \sim (e, e'') \implies e' = e''$ for any $e', e'' \in E_2$. Hence for any $e \in E_1$, there exists a unique $\hat{e} \in E_2$ such that $\lambda\left(p_1(e)\right) = [(e, \hat{e})]$. Let $\hat{g}(e) = \hat{e}$. Then $\hat{g}$ is clearly continuous and G-equivariant, and thus $(\hat{g}, g)$ is a bundle map.

($\Longrightarrow$) Consider the homeomorphism $\varphi : B_1 \xrightarrow{\cong} {E_1}/{G}$ with $\varphi(b) = p_1^{-1}(b)$. Let $b \in B_1$. Let $\varphi(b) = [e]$. Define $\lambda : B_1 \to E_1 \times_G E_2$ by $\lambda(b) = [(e, \hat{g}(e))]$. Since $\hat{g}$ is G-equivariant, we see that λ is well-defined. Further, λ is continuous as the quotient of the map

$$f : E_1 \to E_1 \times E_2, \quad f(x) = (x, \hat{g}(x))$$

by G. Finally, it is easy to check that $\tau\left(\lambda\left(b_1\right)\right) = (b_1, g(b_1))$ for any $b_1 \in B_1$. $\qquad\square$

Lemma 2.3.2. τ *is locally trivial, hence a fibration.*

Proof. Locally, we have that $E_1 \cong U \times G$ and $E_2 \cong V \times G$, so that $E_1 \times E_2 \cong U \times V \times G \times G$. It follows that, locally, $E_1 \times_G E_2 \cong U_1 \times U_2 \times {G \times G}/{\Delta G}$ where $\Delta G \equiv \{(g, g) \mid g \in G\}$. $\qquad\square$

Remark 2.3.3. In fact, τ is a bundle with fiber ${G \times G}/{\Delta G} \cong G$.

Proof of Theorem 2.2.10. Due to Proposition 2.2.7, we may assume that p is a principal G-bundle. By assumption, there is some homotopy $H : X \times I \to Y$ from f to g. Let $\omega = H^*p$. Then

$$f^*p = \omega \restriction_{\omega^{-1}(X \times \{0\})} : \omega^{-1}(X \times \{0\}) \to X \times \{0\} \cong X$$
$$g^*p = \omega \restriction_{\omega^{-1}(X \times \{1\})} : \omega^{-1}(X \times \{1\}) \to X \times \{1\} \cong X.$$

Therefore, it suffices to show that $f^*p \times \mathrm{id}_I \cong \omega$ such that the diagram

$$
\begin{array}{ccccc}
f^*E \times I & \xrightarrow{\cong} & H^*E & \longrightarrow & E \\
{\scriptstyle f^*p \times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle p} \\
X \times I & =\!\!=\!\!= & X \times I & \xrightarrow{H} & Y
\end{array}
$$

commutes. For, in this case, our isomorphism restricts over $X \times \{1\}$, i.e., $g^*p = \omega \restriction_{X \times \{1\}} \cong f^*p$. It thus suffices to exhibit a bundle map $f^*p \times I \to \omega$ over $\mathrm{id}_{X \times I}$ that equals the identity over $\omega \restriction_{X \times \{0\}} = f^*p$.

Remark 2.3.4. It is easy to show that there is some bundle map $f^*p \times \mathrm{id}_I \to \omega$. Indeed, by the homotopy lifting property, we obtain a section σ fitting into the commutative diagram

$$
\begin{array}{ccc}
& & (f^*E \times I) \times_G H^*E \\
& \overset{\lambda_0}{\nearrow} & \Big\downarrow \sigma \\
X \times \{0\} & \longrightarrow & X \times I
\end{array} \quad ,
$$

in which case we obtain our desired map by Lemma 2.2.8. As mentioned, however, we want a bundle map that equals the identity over f^*p.

To get such a map, we must find a section λ such that

$$
\begin{array}{ccccc}
& & (f^*E \times I) \times_G H^*E \\
& \overset{\lambda_0}{\nearrow} & \Big\downarrow \lambda & \overset{\tau}{\searrow} \\
X \times \{0\} & \lhook\joinrel\longrightarrow & X \times I & \underset{\Delta}{\longrightarrow} & (X \times I) \times (X \times I)
\end{array}
$$

commutes. But λ must exist since τ is a fibration by virtue of Lemma 2.3.2. $\qquad\square$

Corollary 2.3.5. *Any bundle over a contractible space B is trivial.*

Proof. Let $i : \mathsf{pt} \to B$ and $\pi : B \to \mathsf{pt}$ denote inclusion and projection, respectively. Then

$$
\begin{aligned}
p &\cong (\mathrm{id})^* p \\
&\cong (i\pi)^* p \\
&\cong \pi^* \underbrace{i^*p}_{\text{trivial}},
\end{aligned}
$$

which is trivial since the pullback of a trivial bundle is trivial. $\qquad\square$

Corollary 2.3.6. *Every bundle p over $X \times I$ is isomorphic to $\left(p \restriction_{p^{-1}(X \times \{0\})}\right) \times \mathrm{id}_I$.*

Example 2.3.7. Consider $S^1 \subset \mathbb{R}^2$ with center the origin. Let $p : E \to S^1$ be a bundle with group G and fiber F. Cover S^1 with the open intervals $I_1 := S^1 \setminus \{-1\}$ and $I_2 := S^1 \setminus \{1\}$. We may assume that $F = p^{-1}(-1)$. Then $E = E_1 \cup E_2$ where $E_i \cong I_i \times F$ via, say, φ_i for each $i = 1, 2$. By Corollary 2.3.6, we see that

$$
\varphi_1 \restriction_{\varphi_1^{-1}(\{1\} \times F)} = \varphi_2 \restriction_{\varphi_2^{-1}(\{-1\} \times F)} = \mathrm{id}_F .
$$

Moreover, the transition function $\varphi_2^{-1} \circ \varphi_1 \restriction_{p^{-1}(1)} : F \to F$ is given by multiplication by some $g \in G$. Hence the map $G \to \mathcal{B}\left(S^1, G, F\right)$ is surjective. In fact, it can be shown that this maps descends to an isomorphism

$$
\pi_0(G) \cong {}^{G}\!\big/\!_{G_0} \overset{\cong}{\longrightarrow} \mathcal{B}\left(S^1, G, F\right)
$$

where G_0 denotes the connected component of e_G.

For example, if $G = F = \mathrm{GL}(n, \mathbb{R})$, then $\pi_0(G)$ consists of the set of matrices with positive determinant and the set of matrices with negative determinant, so that $\mathcal{B}\left(S^1, G, F\right) \cong \mathbb{Z}/2$.

Example 2.3.8. The set $\mathcal{B}\left(S^2, G, F\right)$ is isomorphic to the set of homotopy classes of maps $S^1 \to G$, As it turns out, we can ignore base points, so that $\mathcal{B}\left(S^2, G, F\right) \cong \pi_1(G)$.

For example, if $G = F = \mathrm{SO}(2)$, then $G \cong S^1$, so that $\mathcal{B}\left(S^2, G, F\right) \cong \mathbb{Z}$.

2.4 Lecture 9

Theorem 2.4.1. *Let X be a cell complex with $\dim X \leq n$. Let $A \subset X$ be a subcomplex. Let $p : E \to X$ be a bundle with fiber F such that $\pi_i(F, f) = 0$ for each $i \leq n-1$. Suppose that $\sigma_0 : A \to E$ satisfies $p \circ \sigma_0(a) = a$ for each $a \in A$. Then σ_0 extends to a section $\sigma : X \to E$ of p.*

$$
\begin{array}{ccc}
 & & E \\
 & \overset{\sigma_0}{\nearrow} & p \Big\downarrow \, \sigma \\
A & \lhook\joinrel\longrightarrow & X
\end{array}
$$

Proof. First, assume that X is a regular complex. Since X is finite, we may assume that $X = A \cup_{S^{k-1}} D^k$ where $k \leq n$. Further, we may assume, wlog, that $X = D^k$. Thus, we must find a section σ such that

$$
\begin{array}{ccc}
 & & E \\
 & \overset{\sigma_0 \upharpoonright_{S^{k-1}}}{\nearrow} & p \Big\downarrow \, \sigma \\
S^{k-1} & \lhook\joinrel\longrightarrow & D^k
\end{array}
$$

commutes. Since D^k is contractible, we have that $E \cong D^k \times F$. Then $\sigma_0(x) = (x, \tilde{\sigma}_0(x))$ for each $x \in S^{k-1}$. But $\tilde{\sigma}_0(x) : S^{k-1} \to F$ extends to a map $\tilde{\sigma} : D^k \to F$ because $\pi_{k-1}(F) = 0$. Hence we can take σ to be the map defined by $x \mapsto (x, \tilde{\sigma}(x))$.

Next, drop the assumption that X is regular. Using Exercise 1.1.9, we get a homotopy equivalence

$$
(X, A) \underset{g}{\overset{h}{\rightleftarrows}} \underbrace{(\overline{X}, \overline{A})}_{\text{regular}}
$$

of pairs. Define $\overline{A} \to g^*E$ by $\bar{\sigma}_0(a) = (a, \sigma_0(g(a)))$. By our preceding discussion, this extends to a section $\bar{\sigma}$ on $\overline{X}$.[1] We wish to find σ such that

$$
\begin{array}{ccc}
 & g^*E \longrightarrow E & \\
\bar{\sigma}_0 \Bigg(\begin{array}{cc} g^*E \\ \overline{X} \xrightarrow{g} X \\ \overline{A} \xrightarrow{g} A \end{array} \Bigg) \sigma_0 &
\end{array}
$$

commutes. But since $p \cong h^* g^* p$, we have a commutative diagram

$$
\begin{array}{ccccc}
g^*E & \longleftarrow & h^*g^*E & \xrightarrow{\cong} & E \\
\bar{\sigma} \Big\uparrow \Big\downarrow g^*p & & h^*g^*p \Big\downarrow & \swarrow p & \\
\overline{X} & \longleftarrow_{h} & X & &
\end{array} \quad ,
$$

from which we obtain our desired section σ. $\qquad\square$

Notation. $[X, Y] \coloneqq$ (homotopy classes of maps $X \to Y$).

Corollary 2.4.2. *Let $p : E \to B$ be a principal G-bundle and suppose that $\pi_i(E) = 0$ for any $i \leq n-1$. The function $\chi_X : [X, B] \to \mathcal{B}(X, G, G)$ given by $f \mapsto f^*p$ is bijective.*

---

[1] As $\dim \overline{X} > \dim X$, we tacitly rely on the fact that $\pi_i(F)$ is trivial for large enough i.

Proof.

<u>Surjective:</u> Let $p_1 : E_1 \to X$ be a bundle. Due to Theorem 2.1.4, it suffices to find a bundle map $\left(\hat{f}, f\right)$ such that

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\ \hat{f}\ } & E \\
{\scriptstyle p_1}\downarrow & & \downarrow \\
X & \xrightarrow{\ f\ } & B
\end{array}
$$

commutes. Such a map can be found precisely when there exists a section of the bundle $E_1 \times_G E \to X$, which holds by applying Theorem 2.4.1 to the case where $A = \emptyset$.

<u>Injective:</u> Suppose that $\chi_X(f) = \chi_X(g)$. We must show that $f \simeq g$, i.e., that there is some bundle map $\left(\hat{H}, H\right)$ such that

$$
\begin{array}{ccc}
f^*p \times \{0,1\} \longrightarrow f^*p \times I \xdashrightarrow{\ \hat{H}\ } E \\
\end{array}
$$

$$
f^*p \cup g^*p
$$

$$
X \times I \xdashrightarrow{\ H\ } B
$$

$$
X \times \{0,1\}
$$

commutes. This is equivalent to finding a section λ such that

$$
\begin{array}{ccc}
(X \times \{0,1\}) \times B & \xleftarrow{\ \tau\ } & (f^*p \times I) \times_G E \\
{\scriptstyle \gamma}\uparrow & \nearrow{\scriptstyle \lambda_0} & \uparrow{\scriptstyle \lambda} \\
X \times \{0,1\} & \longrightarrow & X \times I
\end{array}
$$

commutes where

$$
\gamma(x,t) = \begin{cases} (x,t,f(x)) & t = 0 \\ (x,t,g(x)) & t = 1 \end{cases}.
$$

But this exists by Theorem 2.4.1 because $\pi_i(E) = 0$ by assumption. $\qquad\square$

Definition 2.4.3 (Classifying space). A *classifying space for principal G-bundles* is a space B such that χ_X is bijective for every cell complex X.

Example 2.4.4. Let $G = \{\pm 1\}$. Then any principal G-bundle over X is a two-fold covering space of X, i.e., a subgroup of index two in $\pi(X)$, i.e., a nontrivial homomorphism $\pi_1 X \to G$.

For example, let $\{U_i\}$ denote the usual open covering of $\mathbb{RP}^n = {S^n}\big/_G$. Let $\pi : S^n \to \mathbb{RP}^n$ denote the projection map. We have that $\pi^{-1}(U_i) \cong U_i \times G$. Indeed, define $h_i : \pi^{-1}(U_i) \to U_i \times G$ by

$$
(x_0, \ldots, x_n) \mapsto \left([x_0, \ldots, x_n], \frac{x_i}{|x_i|}\right),
$$

the inverse of which is given by

$$
(y_0, \ldots y_n) \mapsfrom ([x_0, \ldots, x_n], \epsilon)
$$

$$
y_k \equiv \epsilon x_k \cdot \frac{|x_i|}{x_i}.
$$

Note that any transition function $h_{ji} : U_i \cap U_j \to G$ is given by $h_{ji}(x) = -1$.

Using the fact that π_1 is the abelianization of H_1 along with the universal coefficient theorem for cohomology, one can prove the following.

Proposition 2.4.5. $\mathcal{B}\left(X, \mathbb{Z}_2, F\right) \cong [X, \mathbb{RP}^n] \cong \mathrm{Hom}\left(\pi_1\left(X\right), \mathbb{Z}/2\right) \cong H^1\left(X, \mathbb{Z}/2\right)$.

Let $w_1 \in H^1\left(\mathbb{RP}^n, \mathbb{Z}/2\right) \cong \mathbb{Z}_2$ be nonzero. Let $p_1 : E \to X$ be a $\mathbb{Z}/2$-bundle. We call $w_1\left(p_1\right) := f^* w_1 \in H^1\left(X, \mathbb{Z}/2\right)$ the *first Stiefel-Whitney class of p*.

2.5   Lecture 10

Example 2.5.1. Let $n \in \mathbb{N}$. Recall that $\mathbb{CP}^n$, by definition, consists of all the complex lines in $\mathbb{C}^{n+1}$. Let $G = S^1$. Then G acts on $\mathbb{C}^{n+1}$ by $g \cdot (z_0, \ldots, z_n) = (gz_0, \ldots, gz_n)$. We have that $\mathbb{CP}^n \cong S^{2n+1}/\sim$ where $z \sim \zeta \cdot z$ for any $\zeta \in S^1$. Consider the projection map $\pi : S^{2n+1} \twoheadrightarrow \mathbb{CP}^n$. For each $i \in \{0, \ldots, n\}$, let $H_i = \{z \in \mathbb{CP}^n \mid z_i = 0\} \cong \mathbb{CP}^{n-1}$ and let $U_i = \mathbb{CP}^n \setminus H_i$. Then the U_i form an open cover of $\mathbb{CP}^n$. Define $h_i : \pi^{-1}\left(U_i\right) \to U_i \times S^1$ by $(z_0, \ldots, z_n) \mapsto \left([z_0, \ldots, z_n], \frac{z_i}{|z_i|}\right)$.

Exercise 2.5.2.

1. Prove that h_i is a homeomorphism.

2. Find the transition functions $h_{ij} : U_j \cap U_i \to S^1$.

Proof.

1. It is obvious that h_i is continuous. Define $g_i : U_i \times S^1 \to \pi^{-1}\left(U_i\right)$ by

$$([z_0, \ldots, z_n], \epsilon) \mapsto (y_0, \ldots, y_n)$$

$$y_k \equiv \epsilon z_k \cdot \frac{|z_i|}{z_i}, \ \ k = 0, \ldots, n.$$

   It is easy to check that this is well-defined and that g_i is the inverse of h_i. It remains to show that g_i is continuous. Consider the quotient map $q := \pi \times \mathrm{id}_{S^1} : S^{2n+1} \times S^1 \to \mathbb{CP}^n \times S^1$. Let $\widetilde{U}_i = \{z \in S^{2n+1} \mid z_i \neq 0\}$. Note that $g_i \circ q \restriction_{\widetilde{U}_i \times S^1}$ is clearly continuous. But $\widetilde{U}_i \times S^1$ is both open in $S^{2n+1} \times S^1$ and saturated with respect to q. Hence $\restriction_{\widetilde{U}_i \times S^1}$ is a quotient map, so that g_i is continuous.

2. Note that

$$h_i \circ h_j^{-1}\left([z_0, \ldots, z_n], \epsilon\right) = \left([z_0, \ldots, z_n], \epsilon \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}\right)$$

   for any $[z_0, \ldots, z_n] \in U_i \cap U_j$. This implies that

$$h_{ij}\left([z_0, \ldots, z_n]\right) = \frac{|z_j|}{z_j} \cdot \frac{z_i}{|z_i|}.$$

$\square$

It follows that π is a principal S^1-bundle. Since each homotopy group $\pi_i\left(S^{2n+1}\right)$ is trivial, Corollary 2.4.2 implies that

$$\mathcal{B}\left(X, S^1, F\right) \cong [X, \mathbb{CP}^n],$$

which for large enough n, is isomorphic to $[X, \mathbb{CP}^\infty]$ where X denotes and any cell complex and

$$\mathbb{CP}^\infty \equiv \bigcup_{k \in \mathbb{N}} \mathbb{CP}^k$$

equipped with the weak topology.

Definition 2.5.3. An *Eilenberg-MacLane space of type* $K\left(G, n\right)$ is a space satisfying

$$\begin{cases} \pi_i K = 0 & i \neq n \\ \pi_i K \cong G & i = n \end{cases}.$$

Theorem 2.5.4. *If X is a cell complex, then $[X, K(G, n)] \cong H^n(X, G)$.*

Example 2.5.5. By inspecting the long exact sequence

$$\cdots \longrightarrow \pi_2\left(S^{2n+1}\right) \longrightarrow \pi_2\left(\mathbb{CP}^n\right)$$
$$\underbrace{\pi_1\left(S^1\right)}_{\mathbb{Z}} \longrightarrow \pi_1\left(S^{2n+1}\right) \longrightarrow \cdots \quad ,$$

we see that $\mathbb{CP}^n$ is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2)$. Moreover, there is a commutative triangle

$$\mathbb{CP}^\infty \longleftarrow \mathbb{CP}^n$$
$$\uparrow \nearrow$$
$$S^i$$

for any $i \in \mathbb{N}$. Thus, $\pi_i\left(\mathbb{CP}^\infty\right) = \pi_i\left(\mathbb{CP}^n\right)$ when n is large enough. This means that $\mathbb{CP}^\infty$ is also an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2)$. By Theorem 2.5.4, we have that

$$\mathcal{B}\left(X, S^1, F\right) \cong H^2(X, \mathbb{Z})$$

whenever X is a cell complex.

For us, a CW complex refers to a cell complex X for which there may be infinitely many attaching maps of any dimension. In this name, "C" stands for the property *closure-finite*, i.e., every open cell e^i is contained in a finite subcomplex of X. Further, "W" stands for the weak topology, with which X is equipped.

Remark 2.5.6. Each of our results holds even if we assume that a certain space is merely a CW complex rather than a cell complex.

Note 2.5.7 (Milnor construction). There exists a functor **TopGrp** $\to$ **PrinBund** that maps each topological group G to a principal G-bundle

$$E_G \xrightarrow{p_G} B_G$$

such that B_G is a CW complex and $\pi_i(E_G) = 0$. This means that B_G is a classifying space for principal G-bundles.

By applying our LES on homotopy groups to p_G, we see that $\pi_i(B_G) \cong \pi_{i-1}(G)$.

Alternatively, one can use the Brown representability theorem (nLab article) to obtain a classifying space B'_G (not necessarily a CW complex) because the pullback functor satisfies

- homotopy invariance,

- excision, and

- Mayer-Vietoris.

Lemma 2.5.8. *Let $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ be classifying spaces for principal G-bundles. Then $B_1 \simeq B_2$.*

Proof. By Corollary 2.4.2, there is some map $f : B_1 \to B_2$ such that $f^* p_2 \cong p_1$. Likewise, there is some map $g : B_2 \to B_1$ such that $g^* p_1 \cong p_2$. Therefore,

$$(f \circ g)^* p_2 \cong g^* f^* p_2$$
$$\cong g^* p_1$$
$$\cong p_2$$
$$\cong \operatorname{id}_{B_2}^* p_2.$$

Therefore, $f \circ g \simeq \operatorname{id}_{B_2}$. Similarly, $g \circ f \simeq \operatorname{id}_{B_1}$. $\qquad\square$

In particular, $B_G \simeq B'_G$.

Example 2.5.9. $B_{S^1} = \mathbb{CP}^\infty$.

Let $H \leq G$. Consider the commutative square

$$
\begin{array}{ccc}
E_G & \overset{q}{\twoheadrightarrow} & E_G/H \\
{\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle r} \\
B_G & \!\!=\!\!=\!\! & E_G/G
\end{array} \quad .
$$

Note that, locally, r looks like the trivial map with fiber G/H. Thus, q locally looks like the map

$$ U \times G \to U \times G/H. $$

This shows that if the natural projection $G \to G/H$ is a principal H-bundle, then so is q. In this case, we have that $B_H \simeq E_G/H$ by Corollary 2.4.2 together with Lemma 2.5.8.

Theorem 2.5.10. *If G is a Lie group and H is a closed subgroup of G, then the natural projection $G \to G/H$ is a principal H-bundle.*

Definition 2.5.11. The *orthogonal group* $\mathrm{O}(n, \mathbb{R})$ is the group of $n \times n$ real matrices A such that $AA^t = A^t A = I_n$, equivalently, $Av \bullet Aw = v \bullet w$ for any $v, w \in \mathbb{R}^n$. We call such an A *orthogonal*.

In particular, if A is orthogonal, then $\|Av\| = \|v\|$ for any $v \in \mathbb{R}^n$.

Example 2.5.12. The orthogonal group $\mathrm{O}(n, \mathbb{R})$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ because $\mathrm{O}(n, \mathbb{R}) = f^{-1}(I_n)$ where $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is given by $X \mapsto XX^t$. Let $\gamma : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{O}(n, \mathbb{R})$ denote the map given by the Gram-Schmidt procedure. Let $i : \mathrm{O}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ denote the inclusion map. Then γ and i are homotopy inverses of each other, so that

$$ \mathrm{GL}(n, \mathbb{R}) \simeq \mathrm{O}(n, \mathbb{R}). $$

Since $\pi : \mathrm{GL}(n, \mathbb{R}) \to \underbrace{\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n, \mathbb{R})}_{M}$ is an $\mathrm{O}(n, \mathbb{R})$-bundle by Theorem 2.5.10, our LES on homotopy groups applied to π shows that $\pi_i(M) = 0$ for each $i \in \mathbb{N}$. Further, our LES applied to the M-bundle $r : B_{\mathrm{O}(n, \mathbb{R})} \to B_{\mathrm{GL}(n, \mathbb{R})}$ shows that

$$ \pi_i\left(B_{\mathrm{O}(n, \mathbb{R})}\right) \cong \pi_i\left(B_{\mathrm{GL}(n, \mathbb{R})}\right) $$

for each i. By Theorem 1.2.7, it follows that

$$ B_{\mathrm{O}(n, \mathbb{R})} \simeq B_{\mathrm{GL}(n, \mathbb{R})}. $$

An exactly similar argument proves that $B_{\mathrm{U}(n, \mathbb{C})} \simeq B_{\mathrm{GL}(n, \mathbb{C})}$.

Eventually, we want to describe $H^*(B_G)$. This will lead us to the notion of a spectral sequence.

2.6 Lecture 11

Before moving to spectral sequences, let us look at a couple more examples of fiber bundles.

Example 2.6.1. Let $\{e_i\}_{1 \leq i \leq n}$ denote the standard basis of $\mathbb{R}^n$. Consider the map $\rho : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^n \setminus \{0\}$ given by $A \mapsto Ae_n$ and its restriction $\tau : \mathrm{O}(n, \mathbb{R}) \to S^{n-1}$. Note that $\rho^{-1}(e_n)$ consists of all $n \times n$ matrices of the form

$$ \left(\begin{array}{c|c} B & 0 \\ \hline * & 1 \end{array} \right) $$

where B denotes an invertible $(n-1) \times (n-1)$ matrix. This means that $\rho^{-1}(e_n) \simeq \mathrm{GL}(n-1, \mathbb{R})$. Similarly, we see that $\tau^{-1}(e_n) \simeq \mathrm{O}(n-1, \mathbb{R})$. Moreover, both ρ and τ are locally trivial. In particular, this yields a LES

$$
\begin{array}{cccc}
& & \cdots \longrightarrow \pi_{i+1}(S^{n=1}) & \\
& & & \\
\pi_i(\mathrm{O}(n-1)) \longrightarrow & \pi_i(\mathrm{O}(n)) \longrightarrow & \pi_i(S^{n-1}) & . \\
& & & \\
\pi_{i-1}(\mathrm{O}(n-1)) \longrightarrow & \cdots & &
\end{array}
$$

Since $\pi_i(S^{n-1})$ is trivial for any $0 \le i \le n-2$, we see that the map $\pi_i(\mathrm{O}(n-1)) \to \pi_i(\mathrm{O}(n))$ is an isomorphism for any $i \le n-3$ and an epimorphism when $i = n-2$. The same result holds with $\mathrm{O}(n)$ replaced by $\mathrm{GL}(n, \mathbb{R})$.

Example 2.6.2. Consider the *Stiefel manifold* $V_{n+k,k}$ consisting of orthonormal k-frames (i.e., k-tuples) in $\mathbb{R}^{n+k}$. If we view the standard basis of $\mathbb{R}^k$ as the "zero element" of $V_{n+k,k}$, then we have a "short exact sequence"

$$
0 \longrightarrow \mathrm{O}(n) \overset{i}{\hookrightarrow} \mathrm{O}(n+k) \overset{p_1}{\twoheadrightarrow} V_{n+k,k} \longrightarrow 0
$$

where i is given by $A \mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & I \end{array}\right)$ and p_1 is given by $A \mapsto (Ae_{n+1}, \ldots, Ae_{n+k})$. In this case,

$$
V_{n+k,k} \cong \frac{\mathrm{O}(n+k)}{\mathrm{O}(n)},
$$

a coset space. Note that i induces an isomorphism $\pi_i(\mathrm{O}(n)) \overset{\cong}{\longrightarrow} \mathrm{O}(n+k)$ for each $i \le n-2$ and an epimorphism when $i = n-1$.

Claim. *The map p_1 is a fiber bundle.*

Proof. Let $F \in V_{n+k,k}$ and choose any orthonormal basis B of the n-plane orthogonal to F. For any n-plane near B, take the orthogonal projection of B onto B' and then apply the Gram-Schmidt process to the new basis to obtain an orthonormal basis $\underline{B'}$ of B'. The assignment $B \mapsto \underline{B'}$ is continuous, and the space of all n-planes orthogonal to any $(n+k)$-plane near F is identifiable with $V_n(\mathbb{R}^n) \cong \mathrm{O}(n)$. Therefore, we get a trivialization around F, which was arbitrary. $\qquad\square$

Using the LES obtained from Corollary 1.3.9, we see that $\pi_i(V_{n+k,k}) = 0$ for each $i \le n-1$. Consider now the *Grassmann manifold*

$$
G_{n+k,k} \equiv \frac{\mathrm{O}(n+k)}{\mathrm{O}(n) \times \mathrm{O}(k)}
$$

where each pair $(A, B) \in \mathrm{O}(n) \times \mathrm{O}(k)$ is identified with the orthogonal $(n+k) \times (n+k)$ matrix $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$.

Note that $G_{n_k,k}$ may be viewed as the space of all k-dimensional planes in $\mathbb{R}^{n+k}$.

Remark 2.6.3.

- The space $E_{\mathrm{O}(k)}$ consists of all orthonormal k-frames in $\mathbb{R}^\infty$.

- The *Grassmannian* $B_{\mathrm{O}(k)} \equiv B_{\mathrm{GL}(k)} \equiv G_{\infty,k}$ consists of all k-planes in $\mathbb{R}^\infty$.

- Similarly, the space $B_{U(k)}$ consists of all k-planes in $\mathbb{C}^\infty$.

Define $p_2 : V_{n+k,k} \to G_{n+k,k}$ by sending each $v \in V_{n+k,k}$ to the subspace of $\mathbb{R}^{n+k}$ spanned by v.

Claim. *The map p_2 is a principal $\mathrm{O}(k)$-bundle.*

Proof. This follows from the fact that $\mathrm{O}(n+k) \to G_{n+k,l}$ is a principal $\mathrm{O}(n) \times \mathrm{O}(k)$-bundle. $\qquad\square$

It follows that $\pi_i(G_{n+k,k}) = 0$ for each $i \le n-2$.

3 Spectral sequences

We are given a fibration:

$$
\begin{array}{c}
F \\
\downarrow \\
E \\
\downarrow {\scriptstyle \pi} \\
X
\end{array}
$$

where X is a connected cell complex and $F = \pi^{-1}(x)$ for some distinguished point x.

Question. What is $H_n(E)$ if we know $H_n(F)$ and $H_n(X)$?

Recall that $H_n(X) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$ where ∂_n is defined as the composite

$$
\overbrace{H_n(X^n, X^{n-1})}^{C_n(X)} \longrightarrow H_{n-1}(X^{n-1}) \longrightarrow \overbrace{H_{n-1}(X^{n-1}, X^{n-2})}^{C_{n-1}}(X) \;,
$$

$$
\partial_n
$$

where $H_i(X^n, X^{n-1}) = 0$ for any $i \neq n$. Furthermore, letting $E_n = \pi^{-1}(X_n)$, we have that $H_*(E_n, E_{n-1}) = C_*(X) \otimes H_*(F)$.

At this point, it is useful to generalize our situation by developing the theory of spectral sequences. For each $r \in \mathbb{Z}_{\geq 0}$, let $\left\{E_{p,q}^r\right\}_{p,q \in \mathbb{Z}}$ be a family of abelian groups and let $\left\{d_r^{p,q} : E_{p,q}^r \to E_{p-r,q+r-1}^r\right\}_{p,q \in \mathbb{Z}}$ be a family of maps (called *differentials*) such that

(a) $d_r^{p,q} \circ d_r^{p+r,q-r+1} = 0$ and

(b) $E_{p,q}^{r+1} = \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p+r,q-r+1}}$.

Such a sequence $(E^r, d_r)_{r \in \mathbb{Z}_{\geq 0}}$ of pairs is called a *homological spectral sequence*, and each double complex (E^r, d_r) is called the *r-th page* of the sequence.

Note 3.0.1. $E^{r+1} = H_*(E^r, d_r)$.

We shall consider only *first-quadrant* spectral sequences, i.e., those for which $E_{p,q}^r = 0$ unless $p, q \geq 0$.
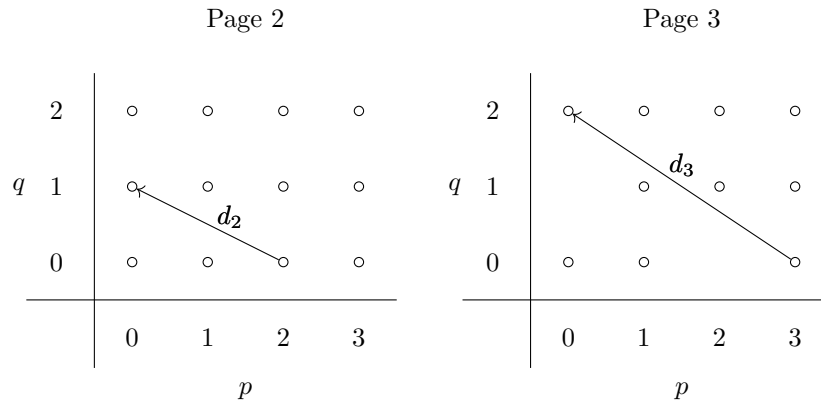
As a result, there is some $k \in \mathbb{N}$ such that $E^r = E^{r+1}$ for any $r \geq k$.

Notation. $E^\infty := E^k$.

Definition 3.0.2 (Convergence). We say that a spectral sequence $E^* := (E^r, d_r)$ *converges* to a sequence of abelian groups $\{A_n\}_{n \in \mathbb{Z}_{\geq 0}}$, written as

$$E^* \Rightarrow \{A_n\},$$

if for each n, there exists a filtration

$$\cdots \subset A_{-1,n+1} = \{0\} \subset A_{0,n} \subset \cdots A_{n-1,1} \subset A_{n,0} = A_n$$

of A_n such that $\frac{A_{p,q}}{A_{p-1,q+1}} \cong E_{p,q}^\infty$.

Theorem 3.0.3. *Let B be a simply connected, path connected cell complex with n-skeleton B^n and suppose that $\pi : E \to B$ is a fibration with fiber F. There exists a (first-quadrant) spectral sequence (E^r, d_r) that*

(a) *converges to $\{H_n(E)\}_{n \in \mathbb{Z}_{\geq 0}}$ and*

(b) *satisfies $E_{p,q}^2 \cong H_p(B; H_q(F))$.*

The filtration $D_{p,q} := (H_n(E))_{p+q=n}$ witnessing this convergence is given by $\mathrm{im}(H_n(\pi^{-1}(B^p)) \to H_n(E))$.

Remark 3.0.4. This holds without the hypothesis that B is a cell complex.

Example 3.0.5. Consider the path space fibration

$$
\begin{array}{c}
\Omega X \\
\downarrow \\
PX \quad \cdot \\
\downarrow \\
X
\end{array}
$$

Recall that PX is contractible. Let $n \geq 2$ and $X = S^n$. Then

$$E_{p,q}^2 \cong H_p(S^n; H_q(\Omega S^n)) = \begin{cases} H_q(\Omega S^n) & p = 0, n \\ 0 & \text{otherwise} \end{cases},$$

and $(E^r, d_r) \Rightarrow \{\mathbb{Z}, 0, 0, \ldots\}$. This means that $d_k = 0$ for any $k \neq n$, so that

$$E^2 = E^3 = \cdots = E^n$$
$$E^{n+1} = E^{n+2} = \cdots = E^\infty.$$

As a result, each differential $d_n^{p,q}$ is an isomorphism provided that $(p,q) \neq (n, 1-n)$ for, otherwise, $E_{p,q}^{n+1}$ is nontrivial, which is impossible. Hence the n-th page looks like

$$
\begin{array}{c|ccc}
3(n-1) & {}_{,-3}(\Omega S^n) & & H_{3n-3}(\Omega S^n) \\
& & \nwarrow \text{\textit{iso.}} & \\
2(n-1) & {}_{,-2}(\Omega S^n) & & H_{2n-2}(\Omega S^n) \\
& & \nwarrow \text{\textit{iso.}} & \\
n-1 & {}_{,-1}(\Omega S^n) & & H_{n-1}(\Omega S^n) \\
& & \nwarrow \text{\textit{iso.}} & \\
0 & \mathbb{Z} & & \mathbb{Z} \\
\hline
& 0 & & n
\end{array}
$$

This implies that $H_q(\Omega S^n) \cong H_{q+n-1}(\Omega X)$ for any $q \in \mathbb{Z}_{\geq 0}$. But ΩS^n is path connected since S^n is simply connected. By induction, it follows that

$$H_q(\Omega X) \cong \begin{cases} \mathbb{Z} & q \equiv 0 \mod (n-1) \\ 0 & \text{otherwise} \end{cases}.$$

3.1 Lecture 12

Suppose that

$$\begin{array}{c} F \\ \downarrow \\ E \\ \downarrow{\scriptstyle \pi} \\ B \end{array}$$

is a fibration with B simply connected and F path connected. Thanks to Theorem 3.2.2, we have the inclusion

$$E_{0,n}^\infty \cong \frac{D_{0,n}}{D_{-1,n+1}} = D_{0,n} \subset H_n(E)$$

as well as a commutative (??) diagram

of abelian groups. Let i denote the inclusion map $i : F \cong p^{-1}(b) \to E$ where b is any chosen element of B. This induces a map i_* in homology

Now, consider the commutative diagram

$$
\begin{array}{ccc}
\pi_n(B) & \xrightarrow{\ \partial\ } & \pi_{n-1}(F) \\
h_n \downarrow & & \downarrow h_{n-1} \\
H_n(B) & \xrightarrow{\ \partial\ } & H_{n-1}(F) \\
\cong \uparrow & & \downarrow \\
E^2_{n,0} & & E^3_{0,n-1} \\
\uparrow & & \downarrow \\
\vdots & & \vdots \\
\uparrow & & \downarrow \\
E^n_{n,0} & \xrightarrow{\ d_n\ } & E^n_{0,n-1}
\end{array}
$$

where h_n denotes the *Hurewicz homomorphism*, defined for an arbitrary path connected space X as follows. Let $\gamma \coloneqq [f]$ be any element of $\pi_n(X, x)$, so that f is a map $(S^n, x_0) \to (X, x)$. Choose any generator $\tau \in H_n(S^n) \cong \mathbb{Z}$ and let

$$h(\gamma) = f_*(\tau) \in H_n(X).$$

Likewise, we can define the *relative Hurewicz homomorphism* $\tilde{h} : \pi_n(X, A) \to H_n(X, A)$ by

$$\left[f : \left(D^n, S^{n-1}, x_0 \right) \to (X, A, \mathsf{pt}) \right] \mapsto f_*(\sigma)$$

where σ is any chosen generator of $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$.

Theorem 3.1.1 (Hurewicz). *Let $n \in \mathbb{Z}_{\geq 2}$. If $\pi_i(X) = 0$ for each $1 \leq i \leq n-1$, then h_n is an isomorphism and h_{n+1} is surjective.*

Theorem 3.1.2 (Relative Hurewicz). *Let $n \in \mathbb{Z}_{\geq 2}$. If both X and A are simply connected and $\pi_i(X, A) = 0$ for each $i \leq n-1$, then $\tilde{h}_n$ is an isomorphism and $\tilde{h}_{n+1}$ is surjective.*

Proof of Hurewicz theorem. Suppose that $\pi_i(X) = 0$ for each $1 \leq i \leq n-1$. For induction, assume that h_{n-1} is an isomorphism for any path connected space. From Example 3.0.5, we gather that the n-th page of the spectral sequence induced by the path space fibration $\Omega X \to PX \to X$ looks like

where d_n is an isomorphism. Thanks to our inductive hypothesis together with Exercise 1.4.3, we have now a commutative square of the form

$$
\begin{array}{ccc}
\pi_n(X) & \xrightarrow[\cong]{\ \partial\ } & \pi_{n-1}(\Omega X) \\
h_n \downarrow & & \downarrow h_{n-1} \ {\scriptstyle\cong} \\
H_n(X) & \xrightarrow[\cong]{\ d_n\ } & H_{n-1}(\Omega X)
\end{array}
\qquad (*)
$$

This implies that h_n is an isomorphism. It remains to verify our base case. Note that $\pi_1(\Omega X)$ is isomorphic to $\pi_2(X)$ and thus abelian. It can be shown directly that h_1 factors as a composite

$$\pi_1(\Omega X) \xrightarrow{\cong} \pi_1(\Omega X)^{\mathrm{ab}} \xrightarrow{\cong} H_1(\Omega X)$$
$$\underbrace{\phantom{\pi_1(\Omega X) \xrightarrow{\cong} \pi_1(\Omega X)^{\mathrm{ab}} \xrightarrow{\cong} H_1(\Omega X)}}_{h_1}$$

of isomorphisms. Hence h_2 must be an ismorphism by $(*)$. $\qquad\square$

Question. Does a similar argument work for the relative Hurewicz theorem?

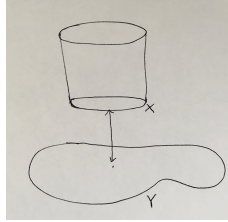**Corollary 3.1.3.** *Let X be path connected.*

1. *$H_1(X) \cong \pi_1^{\mathrm{ab}}(X)$.*

2. *If X is simply connected and $H_i(X) = 0$ for every $1 \leq i \leq n-1$, then $\pi_i(X) = 0$ for every $1 \leq i \leq n-1$.*

3. *If $\pi_i(X) = 0$ for each $0 \leq i \leq n-1$, then $\widetilde{H}_i(X) = 0$ for each $0 \leq i \leq n-1$.*

Let $n \geq 2$ and pick any generator $[f]$ of $\pi_{n-1}(\Omega S^n) \cong \pi_n(S^n) \cong \mathbb{Z}$. By Theorem 3.1.1, the induced map $f_* : H_{n-1}(S^{n-1}) \to H_{n-1}(\Omega S^n)$ is an isomorphism.

Remark 3.1.4. Let $g : X \to Y$ be any map of spaces. Recall the mapping cylinder

$$\mathrm{Cyl}(g) \equiv \frac{(X \times I) \coprod Y}{(x,0) \sim g(x)}$$

of g.

This is precisely the pushout of the span $X \times I \xleftarrow{\sigma_0} X \xrightarrow{g} Y$. As it turns out, g factors as

$$X \xhookrightarrow{\iota} \mathrm{Cyl}(g) \xrightarrow{h} Y$$
$$\overbrace{\phantom{X \xhookrightarrow{\iota} \mathrm{Cyl}(g) \xrightarrow{h} Y}}^{g}$$

for some deformation retraction h. Further, ι is a so-called *cofibration*, the dual notion to a fibration.

Consider the subspace of ΩS^n consisting of all great circles passing through, say, the north pole. This is

clearly homeomorphic to S^{n-1}. Thus, we get a LES in homology

$$
\cdots \longrightarrow \overbrace{H_{2n-2}(S^{n-1})}^{0} \longrightarrow \overbrace{H_{2n-2}(\Omega S^n)}^{\mathbb{Z}} \overset{\cong}{\longrightarrow} H_{2n-2}(\Omega S^n, S^{n-1})
$$

$$
\underbrace{H_{2n-3}(S^{n-1})}_{0} \longrightarrow H_{2n-3}(\Omega S^n) \overset{\cong}{\longrightarrow} H_{2n-3}(\Omega S^n, S^{n-1}) \longrightarrow \cdots
$$

$$
\underbrace{H_{n+1}(S^{n-1})}_{0} \longrightarrow H_{n+1}(\Omega S^n) \overset{\cong}{\longrightarrow} H_{n+1}(\Omega S^n, S^{n-1})
$$

$$
\underbrace{H_n(S^{n-1})}_{0} \longrightarrow H_n(\Omega S^n) \overset{\cong}{\longrightarrow} H_n(\Omega S^n, S^{n-1})
$$

$$
H_{n-1}(S^{n-1}) \xrightarrow[\underset{\overline{f_*}}{\cong}]{} H_{n-1}(\Omega S^n) \xrightarrow[0]{} H_{n-1}(\Omega S^n, S^{n-1})
$$

$$
\underbrace{H_{n-2}(S^{n-1})}_{0} \longrightarrow \cdots
$$

From this, we deduce that

$$
H_i(\Omega S^n, S^{n-1}) = \begin{cases} 0 & i \le 2n-3 \\ \mathbb{Z} & i = 2n-2 \end{cases}.
$$

By Corollary 3.1.3(2), this means that

$$
\pi_i(\Omega S^n, S^{n-1}) = \begin{cases} 0 & i \le 2n-3 \\ \mathbb{Z} & i = 2n-2 \end{cases}.
$$

This yields a LES in homotopy

$$
\cdots \longrightarrow \pi_{2n-2}(\Omega S^n) \longrightarrow \pi_{2n-2}(\Omega S^n, S^{n-1})
$$

$$
\pi_{2n-3}(S^{n-1}) \longrightarrow \pi_{2n-3}(\Omega S^n) \longrightarrow \overbrace{\pi_{2n-3}(\Omega S^n, S^{n-1})}^{0}
$$

$$
\pi_{2n-4}(S^{n-1}) \overset{\cong}{\longrightarrow} \underbrace{\pi_{2n-4}(\Omega S^n)}_{\pi_{2n-3}(S^n)} \longrightarrow \underbrace{\pi_{2n-4}(\Omega S^n, S^{n-1})}_{0} \longrightarrow \cdots
$$

,

which proves the following statement.

Theorem 3.1.5 (Suspension theorem). *If $0 \le i \le 2n-4$, then $\pi_i(S^{n-1}) \cong \pi_{i+1}(S^n)$.*

3.2   Lecture 13

As expected, spectral sequences have exact analogues in cohomology. Specifically, a cohomological spectral sequence consists of the following data:

- for each $r \in \mathbb{Z}_{\geq 0}$, a family of abelian groups $\{E_r^{p,q}\}_{p,q\in\mathbb{Z}}$ and

- a family of maps $\{d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\}_{p,q\in\mathbb{Z}}$ (called *differentials*) such that

- $d_r^{p,q} \circ d_r^{p-r,q+r-1} = 0$ and

- $E_{r+1}^{p,q} = \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}$.

Again, we shall consider only *first-quadrant* spectral sequences, i.e., those for which $E_r^{p,q} = 0$ unless $p, q \geq 0$. As a result, there is some $k \in \mathbb{N}$ such that $E_r = E_{r+1}$ for any $r \geq k$.

Notation. $E_\infty := E_k$.

Definition 3.2.1 (Convergence). We say that a spectral sequence $E_* := (E_r, d_r)$ *converges* to a sequence of abelian groups $\{D^n\}_{n\in\mathbb{Z}_{\geq 0}}$, written as
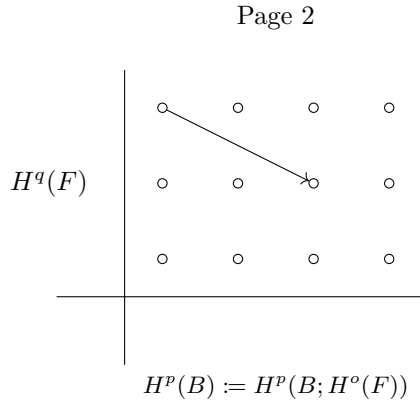$$E_* \Rightarrow \{D^n\},$$
if for each n, there exists a filtration
$$\cdots \subset D^{n+1,-1} = \{0\} \subset D^{n,0} \subset \cdots D^{1,n-1} \subset D^{0,n} = D^n$$
of D^n such that $\frac{D^{p,q}}{D^{p+1,q-1}} \cong E_\infty^{p,q}$.

Theorem 3.2.2. *Let B be simply connected and path connected and suppose that $\pi : E \to B$ is a fibration with fiber F. There exists a (first-quadrant) spectral sequence (E^r, d_r) that*

(a) *converges to $\{H^n(E)\}_{n\in\mathbb{Z}_{\geq 0}}$ and*

(b) *satisfies $E_2^{p,q} \cong H^p(B; H^q(F))$.*

In pictures, we have

Page 2

$H^q(F)$

$H^p(B) := H^p(B; H^o(F))$

$$H^q(E) == D^{0,q} \twoheadrightarrow E_\infty^{0,q} \hookrightarrow \cdots \hookrightarrow E_2^{0,q} == H^q(F)$$

i^*

$$H^p(B) == E_2^{p,0} \twoheadrightarrow E_3^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{p,0} \hookrightarrow H^p(E)$$

π^*

.

4   Chern classes

To do.

5   Cobordism theory

To do.