

1. Let I be a countably infinite set. Let $\mathbb{D} := \langle I, E \rangle$ be a structure where E is an equivalence relation for which there is exactly one equivalence class of size k for each $k \in \mathbb{Z}_{\geq 1}$.

- (1) Show that the set Λ of (first-order) sentences expressing that E is an equivalence relation with exactly one equivalence class of size k for each $k \in \mathbb{Z}_{\geq 1}$ axiomatizes \mathbb{D} , i.e., $\text{Th}(\mathbb{D}) = \text{Cn}(\Lambda)$ where

$$\text{Cn}(\Lambda) := \{\varphi \in \text{FO}_{\mathbb{D}} \mid \Lambda \models \varphi\}.$$

- (2) Show that for every (first-order) formula $\theta(y, \bar{w})$ and every $\bar{a} \in I$, the set

$$\theta[\mathbb{D}, \bar{a}] := \{x \in \text{dom}(\mathbb{D}) \mid \mathbb{D} \models \theta[x, \bar{a}]\}$$

is either finite or cofinite.

- (1) It suffices to prove that Λ is complete. For, in this case, any two models of Λ must be elementarily equivalent.

Claim 1. *Let \mathbb{E} be any model of Λ of size $\kappa \geq \omega$. There exists an elementary extension $\mathbb{E}_{\kappa} \succeq \mathbb{E}$ of size κ such that \mathbb{E}_{κ} has exactly κ equivalence classes each of size κ .*

Proof. Let λ denote the cardinality of the set of all equivalence classes in $\text{dom}(\mathbb{E})$. Note that $\lambda \leq \kappa$. For every $\alpha, \beta \in \kappa$, adjoin to the language of \mathbb{E} a new constant symbol $c(\alpha, \beta)$. Consider the theory

$$\Delta := \Lambda \cup \{Ec(x, y)c(x, z) \mid x, y, z \in \kappa\} \cup \{\neg Ec(x, 0)c(y, 0) \mid x, y \in \kappa, x \neq y\}.$$

Any finite subset F of Δ is satisfiable by a suitable expansion \mathbb{E}_F of \mathbb{E} . Then there exists an ultrafilter on the family of finite subsets of Δ such that the ultraproduct

$$\prod_{\substack{F \subset \Delta \\ \text{finite}}} \mathbb{E}_F / \mathcal{U}$$

satisfies Δ . Moreover, its reduct \mathbb{A} to the language of \mathbb{E} is an elementary extension of \mathbb{E} . By the downward Löwenheim-Skolem theorem, there exists a structure \mathbb{E}_0 of size κ such that $\mathbb{A} \succeq \mathbb{E}_0 \succeq \mathbb{E}$.

Now, repeat our preceding construction ω times to get an increasing chain

$$\mathbb{E} \preceq \mathbb{E}_0 \preceq \mathbb{E}_1 \preceq \mathbb{E}_2 \preceq \cdots$$

of structures such that each $\text{dom}(\mathbb{E}_i)$ has cardinality κ . Note that \mathbb{E}_{κ} is an elementary extension of \mathbb{E} . Further, the universe of $\mathbb{E}_{\kappa} := \bigcup_{i \in \omega} \mathbb{E}_i$ also has cardinality κ , so that \mathbb{E}_{κ} has exactly κ equivalence classes. Finally, for any $x \in \mathbb{E}_{\kappa}$, x belongs to some \mathbb{E}_n . Hence the equivalence class $[x]$ has size κ in \mathbb{E}_{n+1} and thus in \mathbb{E}_{κ} . It follows that every equivalence class in \mathbb{E}_{κ} has size κ . \square

Suppose, toward a contradiction, that there is a sentence φ in the language of \mathbb{D} such that neither φ nor $\neg\varphi$ belongs to $\text{Cn}(\Lambda)$. Then there are models \mathbb{A}^1 and \mathbb{A}^2 of Λ such that $\mathbb{A}^1 \models \neg\varphi$ and $\mathbb{A}^2 \models \varphi$. By the Löwenheim-Skolem theorem, we may assume that both of these have size $\kappa \geq \omega$. By Claim 1, we thus have two structures \mathbb{A}_{κ}^1 and \mathbb{A}_{κ}^2 such that $\mathbb{A}_{\kappa}^1 \models \neg\varphi$ and $\mathbb{A}_{\kappa}^2 \models \varphi$. But it's easy to see that \mathbb{A}_{κ}^1 and \mathbb{A}_{κ}^2 must be isomorphic, which yields a contradiction.

- (2) Suppose, toward a contradiction, that there exist a formula $\theta(y, w_1, \dots, w_n)$ and an element $\bar{a} \in I$ such that $\theta[\mathbb{D}, \bar{a}]$ is both infinite and coinfinite. Adjoin to the language of \mathbb{D} new constant symbols $\bar{e} := (e_1, \dots, e_n)$, c , and d . For each $k \in \mathbb{Z}_{\geq 1}$, let $\lambda_k(x)$ denote the formula expressing that the equivalence class of x has cardinality $> k$. Now, consider the theory

$$\begin{aligned} \Gamma := \Lambda \cup \{ \lambda_k(c) \mid k \geq 1 \} \cup \{ \lambda_k(d) \mid k \geq 1 \} \\ \cup \{ \neg E e_i c \mid 1 \leq i \leq n \} \\ \cup \{ \neg E e_i d \mid 1 \leq i \leq n \} \\ \cup \{ \theta(c, \bar{e}), \neg \theta(d, \bar{e}) \} \end{aligned}$$

in our new language.

Let F be any finite subset of Γ . Since both $\theta[\mathbb{D}, \bar{a}]$ and $\neg \theta[\mathbb{D}, \bar{a}]$ are infinite, we can find an expansion of \mathbb{D} that satisfies F by interpreting \bar{e} as \bar{a} and both c and d as members of large enough equivalence classes. By the compactness theorem, it follows that there is some model \mathbb{C} of Γ , which must be infinite. Let \mathbb{C}' denote the reduct of \mathbb{C} to the language of \mathbb{D} . Thanks to the Löwenheim-Skolem theorem, we may assume that $\text{dom}(\mathbb{C}')$ is countable. Thus, the equivalence classes $[c^{\mathbb{C}}]$ and $[d^{\mathbb{C}}]$ are countable. Note that $e_i^{\mathbb{C}} \notin [c^{\mathbb{C}}] \cup [d^{\mathbb{C}}]$ for each $1 \leq i \leq n$. Therefore, there is an automorphism of \mathbb{C}' sending $c^{\mathbb{C}}$ to $d^{\mathbb{C}}$ and fixing each $e_i^{\mathbb{C}}$. But this contradicts the fact that $\mathbb{C}' \models \theta[c^{\mathbb{C}}, \bar{e}^{\mathbb{C}}] \wedge \neg \theta[d^{\mathbb{C}}, \bar{e}^{\mathbb{C}}]$. ■

2. Show that a \mathcal{L} -structure \mathbb{A} is finite if and only if for any \mathcal{L} -structure \mathbb{B} ,

$$\mathbb{A} \equiv \mathbb{B} \iff \mathbb{A} \cong \mathbb{B}.$$

(\implies)

It is always true that any two isomorphic structures are elementarily equivalent. Thus, it remains to show that $\mathbb{A} \equiv \mathbb{B} \implies \mathbb{A} \cong \mathbb{B}$.

First, assume that \mathcal{L} is finite. Consider the *atomic diagram* of \mathbb{A} , i.e., the set

$$D(\mathbb{A}) := \{ \varphi \mid \mathbb{A} \models \varphi, \varphi \text{ is either atomic or the negation of an atomic formula} \}$$

where \mathbb{A} denotes the expansion of \mathbb{A} obtained by adjoining a constant symbol c_a for each $a \in \text{dom}(\mathbb{A})$. Since both \mathcal{L} and $\text{dom}(\mathbb{A})$ are finite, we can encode $D(\mathbb{A})$ with a single sentence ψ . Therefore, the sentence

$$\psi_{\mathbb{A}} := \forall x \left(\bigvee_{a \in \text{dom}(\mathbb{A})} x = c_a \right) \wedge \psi$$

has the property that $\mathbb{B} \models \psi_{\mathbb{A}} \implies \mathbb{B} \cong \mathbb{A}$ for any other \mathcal{L} -structure \mathbb{B} . Now, if $\mathbb{A} \equiv \mathbb{B}$, then clearly both \mathbb{A} and \mathbb{B} satisfy $\psi_{\mathbb{A}}$, so that $\mathbb{B} \cong \mathbb{A}$.

Next, let \mathcal{L} be arbitrary and let $\mathbb{A} \equiv \mathbb{B}$. Suppose, toward a contradiction, that $\mathbb{A} \not\cong \mathbb{B}$. Then for any bijection $f : \text{dom}(\mathbb{A}) \rightarrow \text{dom}(\mathbb{B})$, there is some finite sublanguage \mathcal{L}_f of \mathcal{L} such that f is *not* an isomorphism $\mathbb{A}^{\mathcal{L}_f} \rightarrow \mathbb{B}^{\mathcal{L}_f}$ of reducts to \mathcal{L}_f . Consider the language

$$\mathcal{L}' := \bigcup_{\substack{f : \text{dom}(\mathbb{A}) \rightarrow \text{dom}(\mathbb{B}) \\ \text{bijection}}} \mathcal{L}_f,$$

which is finite as the finite union of finite sets. Thanks to our preceding discussion, we obtain an isomorphism $g : \mathbb{A}^{\mathcal{L}'} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}'}$. But $\mathcal{L}_g \subset \mathcal{L}'$ by our construction of \mathcal{L}' , and thus g induces an isomorphism $\mathbb{A}^{\mathcal{L}_g} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}_g}$, contrary to our choice of \mathcal{L}_g .

(\impliedby)

Suppose that \mathbb{A} is infinite. We must find a structure \mathbb{B} such that $\mathbb{A} \equiv \mathbb{B}$ but $\mathbb{A} \not\cong \mathbb{B}$. But this follows at once from the Löwenheim-Skolem theorem, which implies that $\text{Th}(\mathbb{A})$ has a model of any infinite size. ■

3. Let $\mathbb{N}^* = \langle \omega, < \rangle$. Show that for any infinite cardinal κ , $\text{Th}(\mathbb{N}^*)$ is not κ -categorical.

Expand the language of \mathbb{N}^* by adjoining countably many constants $\{c_i\}_{i \in \mathbb{Z}}$. Consider the theory

$$T := \text{Th}(\mathbb{N}^*) \cup \{c_i > c_{i+1} \mid i \in \mathbb{Z}\}. \quad (\star)$$

in our new language. Any finite subset of T is satisfied by an expansion of \mathbb{N}^* suitably interpreting the c_i since \mathbb{N}^* has descending chains of all finite lengths. By the compactness theorem, it follows that there is some model \mathbb{A} of T , which must be infinite. If $|\mathbb{A}| > \aleph_0$, then apply the Löwenheim-Skolem theorem to get a model \mathbb{B} of T such that $|\mathbb{B}| = \aleph_0$. Let

$$\mathbb{A}' = \begin{cases} \mathbb{B} & |\mathbb{A}| > \aleph_0 \\ \mathbb{A} & |\mathbb{A}| = \aleph_0 \end{cases}.$$

Note that $\mathbb{A}' \models T$. Since the property of being a linearly ordered set is expressible by a first-order sentence, we see that \mathbb{A}' is linearly ordered by $<$. Further, we see that \mathbb{A}' has an infinite descending chain, which means that \mathbb{A}' is not well-ordered by $<$. But $(\omega, <)$ is a well-ordered set, and thus the reduct of \mathbb{A}' to the language of \mathbb{N}^* is not isomorphic to \mathbb{N}^* . It does, however, satisfy $\text{Th}(\mathbb{N}^*)$. This shows that $\text{Th}(\mathbb{N}^*)$ is not \aleph_0 -categorical.

Unfortunately, it's unclear that this method can be adapted to show that $\text{Th}(\mathbb{N}^*)$ is not κ -categorical when κ is uncountable. In this case, we instead shall employ two binary operations on the class of all linear orderings. Let L_1 and L_2 be linearly ordered sets.

- $L_1 \cdot L_2$ refers to $L_1 \times L_2$ equipped with the lexicographic order.
- $L_1 + L_2$ refers to L_1 with its ordering followed by L_2 with its ordering.

Now, consider the following linearly ordered structures:

$$\begin{aligned} &\mathbb{N}^* + (\mathbb{Z} \cdot \kappa) \\ &\mathbb{N}^* + (\mathbb{Z} \cdot (\mathbb{Q} + \kappa)), \end{aligned}$$

both of which have cardinality κ . These orderings possess minimal elements and are *discrete* in the sense that both structures satisfy the sentences

$$\begin{aligned} &\forall x \exists y (x < y \wedge \neg \exists z (x < z \wedge z < y)) \\ &\forall x (\exists w (w < x) \rightarrow \exists y (y < x \wedge \neg \exists z (y < z \wedge z < x))). \end{aligned} \quad (1)$$

Note that, on the one hand, $\mathbb{N}^* + (\mathbb{Z} \cdot \kappa)$ cannot possess an descending chain of length ω^2 , for otherwise κ , which is well-ordered, would possess an infinite descending chain. On the other hand, $\mathbb{N}^* + (\mathbb{Z} \cdot (\mathbb{Q} + \kappa))$ does possess such a chain since ω^* (the order type of $\mathbb{Z}_{<0}$) can be embedded in \mathbb{Q} . Therefore,

$$\mathbb{N}^* + (\mathbb{Z} \cdot \kappa) \not\cong \mathbb{N}^* + (\mathbb{Z} \cdot (\mathbb{Q} + \kappa)).$$

Definition 1. Suppose that \mathcal{L} is a finite language without function symbols. Let \mathbb{D} and \mathbb{E} be two \mathcal{L} -structures. Let $n \in \omega$. The *Ehrenfeucht-Fraïssé game* $G_n(\mathbb{D}, \mathbb{E})$ of length n on \mathbb{D} and \mathbb{E} is a game of perfect information played as follows.

- (a) There are exactly two players, the *spoiler* and the *duplicator*.
- (b) There are exactly n rounds.
- (c) The spoiler begins round $k \leq n$ by picking an element of either $\text{dom}(\mathbb{D})$ or $\text{dom}(\mathbb{E})$. Next, the duplicator picks an element of the other domain.

- (d) This yields two sequences (d_1, \dots, d_n) and (e_1, \dots, e_n) such that $d_i \in \text{dom}(\mathbb{D})$ and $e_i \in \text{dom}(\mathbb{E})$ for each $i = 1, \dots, n$. If the mapping $d_i \mapsto e_i$ defines an isomorphism of finite substructures, then we say that the duplicator has won $G_n(\mathbb{D}, \mathbb{E})$. Otherwise, we say that the spoiler has won.

Theorem 2 (Fraïssé). *The duplicator has a winning strategy in $G_n(\mathbb{D}, \mathbb{E})$ for each $n \in \omega$ if and only if $\mathbb{D} \equiv \mathbb{E}$.*

Claim 2. *Suppose that $(\mathbb{E}, <)$ is a discrete linear ordering with a minimal element but no maximal element. Then $\mathbb{E} \equiv \mathbb{N}^*$.*

Proof sketch. Consider the Ehrenfeucht-Fraïssé game $G_n(\mathbb{E}, \mathbb{N}^*)$ of length $n \in \omega$ on \mathbb{E} and \mathbb{N}^* . The duplicator has a winning strategy in $G_n(\mathbb{E}, \mathbb{N}^*)$ by adhering to the following rules.

- (i) If, in round m , the spoiler chooses an element of one of the structures that is connected to a previously chosen element or the minimal element by a path of successors of length $k < \infty$, then choose the corresponding element of the other structure in round m .
 - (ii) Otherwise, make sure that any chosen element of $\text{dom}(\mathbb{N}^*)$ is always separated by at least $n+1$ elements from any previously chosen element of $\text{dom}(\mathbb{N}^*)$ while preserving the required order of your choices.
- In this case, choose first a natural number separated by more than 3^n elements from the greatest previously chosen element of $\text{dom}(\mathbb{N}^*)$.

□

Thanks to Theorem 2, it follows that both $\mathbb{N}^* + (\mathbb{Z} \cdot \kappa)$ and $\mathbb{N}^* + (\mathbb{Z} \cdot (\mathbb{Q} + \kappa))$ are elementarily equivalent to \mathbb{N}^* and thus models of $\text{Th}(\mathbb{N}^*)$. Hence $\text{Th}(\mathbb{N}^*)$ is not κ -categorical. ■

4. Show that any set definable over \mathbb{N}^* is either finite or cofinite.

Remark. This shows that \mathbb{N}^* is *o-minimal* in the sense that every definable set over \mathbb{N}^* is a finite union of points and intervals in ω .

Note that any set definable over \mathbb{N}^* is 0-definable because any natural number n is uniquely determined by the first-order property

$$\begin{cases} \text{"}n \text{ is less than any other element"} & n = 0 \\ \text{"there are exactly } n - 1 \text{ elements between 0 and } n" & n > 1 \end{cases}.$$

Suppose, toward a contradiction, that there exist a formula $\theta(y)$ such that $\theta[\mathbb{N}^*]$ is both infinite and coinfinite. Consider, again, the theory (\star) . Let

$$T' = T \cup \{\theta(c_0), \neg\theta(c_1)\}.$$

Since both $\theta[\mathbb{N}^*]$ and $\neg\theta[\mathbb{N}^*]$ are infinite, we can find an expansion of \mathbb{N}^* that satisfies any finite subset of T' . By the compactness theorem together with the Löwenheim-Skolem theorem, we thus can find a countable model \mathbb{D} of T' and take its reduct \mathbb{C} to the language of \mathbb{N}^* . Note that $(\text{dom}(\mathbb{C}), <)$ is a countable linear ordering with an infinite descending and ascending chain χ on which both $c_0^{\mathbb{D}}$ and $c_1^{\mathbb{D}}$ lie. Moreover, this ordering is discrete in the sense of (1). Therefore, we may assume that χ has the form

$$\dots < x_{m-1} < x_m < x_{m+1} < \dots$$

where x_{m+1} denotes the immediate successor of x_m . There is an automorphism of \mathbb{C} mapping $c_0^{\mathbb{D}}$ to $c_1^{\mathbb{D}}$ by suitably shifting χ finitely many places to the left and fixing all elements outside χ . But this contradicts the fact that $\mathbb{C} \models \theta[c_0^{\mathbb{D}}] \wedge \neg\theta[c_1^{\mathbb{D}}]$. ■