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Homotopy and K-theory seminar
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Abstract

We begin low-dimensional K-theory, i.e., describe $K_0(-)$, $K_1(-)$, and $K_2(-)$, in various settings. The main sources for this talk are the following.

- nLab.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Chapters I and II.
- Eric M. Friedlander's *An Introduction to K-theory*, Chapter 1.

Recall that the forgetful functor $U : \mathbf{Ab} \to \mathbf{CMon}$ admits a left adjoint $K : \mathbf{CMon} \to \mathbf{Ab}$, called the *group completion* functor. Specifically, for any commutative monoid $(C, +)$, we call the abelian group $K(C)$ the *Grothendieck group of G*, which is constructed as follows.

Consider $S := {C \times C}/{\sim}$ where $(a_1, b_1) \sim (a_2, b_2)$ if

$$a_1 + b_2 + k = b_1 + a_2 + k$$

for some $k \in C$. Note that $\sim = \sim'$ where $(a_1, b_1) {\sim'} (a_2, b_2)$ if

$$(a_1 + k_1, b_1 + k_1) = (a_2 + k_2, b_2 + k_2)$$

for some $(k_1, k_2) \in C \times C$. Then set $K(C) = (S, +)$, where $+$ is inherited from C and acts componentwise on equivalence classes. The definition of $\sim'$ makes it clear that $[a_1, b_1]^{-1} = [b_1, a_1]$.

Proposition 1. *The inclusion $C \hookrightarrow K(C)$ given by $x \mapsto [x] := [x, 0]$ is injective iff C is a cancellation monoid.*

Lemma 2 (Universal property of the Grothendieck group). *Let B be an abelian group and $f : A \to B$ a monoid homomorphism. Then we have*

$$\begin{tikzcd} A \arrow[d, hook] \arrow[dr, "f"] & \\ K(A) \arrow[r, dashed, "\exists! \tilde{f}"'] & B \end{tikzcd}.$$

Proof. Define $\tilde{f}$ by $[a_1, b_1] \mapsto f(a_1) - f(b_1)$. □

Lemma 3. *$K(C_1 \times C_2) \cong K(C_1) \times K(C_2)$.*

Definition 1. A submonoid L of C is *cofinal* if for any $c \in C$, there is some $c' \in C$ such that $c + c' \in L$.

Proposition 4. *Let L be cofinal in commutative C.*

1. *Any element of $K(C)$ can be written as $[m] - [n]$ for some $m, n \in C$.*
2. *$K(L) \leq K(C)$.*
3. *Any element of $K(C)$ can be written as $[m] - [l]$ for some $m \in C$ and $l \in L$.*
4. *If $[m] = [m']$, then $m + l = m' + l$ for some $l \in L$.*

Example 2.

1. $K(\mathbb{N}) \cong \mathbb{Z}$ via $[a_1, b_1] \mapsto a_1 - b_1$.
2. $K(\mathbb{Z}^\times) \cong \mathbb{Q}^\times$ via $[a_1, b_1] \mapsto \frac{a_1}{b_1}$.

Definition 3. Let R be a unital ring. Let $(\mathbf{P}(R), \oplus, \otimes_R)$ denote the semiring of (isomorphism classes of) finitely generated projective R-modules. Then we define $K_0(R) = K(\mathbf{P}(R))$.

Lemma 5. *$\mathbf{P}(R_1 \times R_2) \cong \mathbf{P}(R_1) \times \mathbf{P}(R_2)$. Therefore, K_0 can be computed componentwise by Lemma 3.*

Remark 1. $K_0(-)$ defines a functor from **Ring** to **Ab**. Let $f : R \to S$ be a ring homomorphism and P be a finitely generated projective R-module. The assignment of f under $K_0(-)$ goes as follows.

1. Construct $S \otimes_R P$, the base extension of P. This is the *unique* S-module $(s', s \otimes p) \mapsto s's \times p$ compatible with the R-module structure on S induced by f. This is also an R-module with $f(r) \cdot t \coloneqq r \cdot t$ for $t \in S \otimes_R P$. We know that $P \oplus Q$ is free for some R-module Q. Since $S \otimes_R (P \oplus Q) \cong_S (S \otimes_R P) \oplus (S \otimes_R Q)$ and $P \oplus Q$ is free over S via f, it follows that $S \otimes_R P$ is a finitely generated projective S-module.

2. We've just defined a monoid homomorphism $\tilde{f} : \mathbf{P}(R) \to \mathbf{P}(S)$.

3. Apply the universal property of K to find the filling

$$\begin{tikzcd} \mathbf{P}(R) \arrow[r, "\tilde{f}"] \arrow[d, hook] & \mathbf{P}(S) \arrow[d, hook] \\ K(\mathbf{P}(R)) \arrow[r, dotted, "f_*"] & K(\mathbf{P}(S)) \end{tikzcd},$$

where we set $K_0(f) = f_*$.

Remark 2 (Eilenberg Swindle). Suppose $P \oplus Q = R^n$ as R-modules. Then

$$P \oplus R^\infty \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \cong R^\infty.$$

Therefore, if we added R^∞ to $\mathbf{P}(R)$, then we would have $[P] = 0$ for each finitely generated projective P.

Example 4. If $R = F$ is a field, then $\mathbf{P}(R) \cong \mathbb{N}$ and, by Example 2, $K_0(R) \cong \mathbb{Z}$.

We can generalize this phenomenon a bit.

Definition 5. A ring R has the *invariant basis property (IBP)* if $R^n \ncong R^m$ when $n \neq m$. Note that any commutative ring has the IBP.

Definition 6. An R-module P is *stably free* of rank $m - n$ if $P \oplus R^m \cong R^n$ for some m and n.

Lemma 6. *The map $f : \mathbb{N} \to \mathbf{P}(R)$ defined by $n \mapsto R^n$ induces a homomorphism $\phi : \mathbb{Z} \to K_0(R)$.*

1. *ϕ is injective iff R has the IBP.*

2. *Suppose R has IBP. Then $K_0(R) \cong \mathbb{Z}$ iff every finitely generated projective R-module is stably free.*

Proof.

1. By Proposition 4(4), we know that $[P] = [Q]$ in $K_0(R)$ iff $P \oplus R^m \cong Q \oplus R^m$ for some m.

2. $[P] = [R^n]$ iff P is stably free.

□

Example 7. Suppose that R is commutative. There is a ring homomorphism $R \to F$ with F a field. Then the induced map $K_0(R) \to K_0(F) \cong \mathbb{Z}$ sends $[R]$ to 1. Also, the map $\phi : \mathbb{Z} \to K_0(R)$ is injective by Lemma 6. Letting $K \coloneqq \ker(K_0(R) \to \mathbb{Z})$, we get a split exact sequence of abelian groups, so that $K_0(R) \cong \mathbb{Z} \oplus K$.

$$1 \longrightarrow K \longrightarrow K_0(R) \longrightarrow \mathbb{Z} \longrightarrow 1$$

Example 8. A ring R is a *flasque* if there is an R-bimodule M which is also a finitely generated projective on one side along with a bimodule isomorphism $R \oplus M \cong M$. Then since $P \oplus (P \otimes_R M) \cong P \otimes_R (R \oplus M) \cong P \otimes_R M$, we see that $K_0(R) = 0$.

Example 9. A module is *semisimple* if it is the direct sum of simple modules. A ring R is called semisimple if it a semisimple R-module. Notice that any semisimple module is both Noetherian and Artinian and that any module over a semisimple ring is semisimple.

Suppose R is semisimple with summands $V_1, \ldots, V_m$. Then any finitely generated R-module is $\bigoplus_{i=1}^m V_i^{l_i}$, where the l_i are uniquely determined by Krull-Remak-Schmidt. Hence $\mathbf{P}(R) \cong \mathbb{N}^m$, and $K_0(R) \cong \mathbb{Z}^m$.

Example 10. A ring R is *von Neumann regular* if

$$(\forall r \in R)(\exists x_r \in R)(r x_r r = r).$$

It turns out that any one-sided ideal in R is generated by an idempotent element. Let $E/\sim$ denote the set of idempotent elements in R under the equivalence $e_1 \sim e_2$ if the two generate the same ideal. Then $E/\sim$ forms a lattice where the join and meet correspond to ideal addition and intersection, respectively.

Kaplansky (1998) proved that any projective R-module is some direct sum of (e) with e idempotent. It follows that $E/\sim$ determines $K_0(R)$.

Proposition 7. *Let R be commutative. It can be shown that the following are equivalent.*

1. *R_{red} is a commutative von Neumann regular ring.*
2. *R has (Krull) dimension 0.*
3. *$\operatorname{Spec}(R)$ is compact, Hausdorff, and totally disconnected. (This is a very strong condition.)*

Lemma 8. *If $I \subset R$ is nilpotent, then it's not hard to show that $\mathbf{P}\left(R/I\right) \cong \mathbf{P}(R)$, hence $K_0(R) \cong K_0\left(R/I\right)$.*

Definition 11. Let R be a commutative ring. The *rank* of a finitely generated projective R-module P at a prime ideal $\mathfrak{p}$ is the function

$$\operatorname{rk} : \operatorname{Spec}(R) \to \mathbb{N}, \quad \mathfrak{p} \mapsto \dim_{R_{\mathfrak{p}}}(P \otimes R_{\mathfrak{p}}).$$

Proposition 9. *The rank of a finitely generated projective module is*

1. *continuous.*
2. *a semiring homomorphism.*

Definition 12. An R-module M is a *componentwise free module* if we have $R = \prod_{i=1}^n R_i$ and $M \cong \prod_{i=1}^n R_i^{c_i}$ for some integers c_i. Note that M must be projective in this case.

Lemma 10. *Let R be commutative. The monoid L of finitely generated componentwise free R-modules has is isomorphic to $[\operatorname{Spec}(R), \mathbb{N}]$.*

Proof. Let $f : \operatorname{Spec}(R) \to \mathbb{N}$ be continuous. By some point-set topology, we see that $\operatorname{im} f$ is finite, say $\{n_1, \ldots, n_c\}$. It's also possible to write $R = R_1 \times \cdots \times R_c$. Then $R^f := R_1^{n_1} \times \cdots \times R_c^{n_c}$ is a finitely generated componentwise free R-module. Moreover, $f \mapsto R^f$ has inverse rk restricted to componentwise free modules. □

Theorem 11 (Pierce). *If R is a 0-dimensional commutative ring, then*

$$K_0(R) \cong [\operatorname{Spec}(R), \mathbb{Z}],$$

where $[X, Y]$ denotes the semiring of continuous maps $f : X \to Y$.

Proof. We have that R_{red} is a commutative von Neumann regular ring by Proposition 7. Any ideal (d) in R_{red} where d is idempotent is componentwise free. By Kaplansky, every object X of $\mathbf{P}(R)$ is therefore componentwise free. Therefore, $\mathbf{P}(R_{\text{red}}) \cong [\operatorname{Spec}(R_{\text{red}}), \mathbb{N}]$, giving $K_0(R_{\text{red}}) \cong [\operatorname{Spec}(R_{\text{red}}), \mathbb{Z}]$. By Lemma 8 and the fact that $\operatorname{Spec}(R_{\text{red}})$ is homeomorphic to $\operatorname{Spec}(R)$, it follows that $K_0(R) \cong [\operatorname{Spec}(R_{\text{red}}), \mathbb{Z}] \cong [\operatorname{Spec}(R), \mathbb{Z}]$. □

Remark 3. When R is commutative, let $H_0(R) := [\mathrm{Spec}(R), \mathbb{Z}]$. If R is Noetherian, then $H_0(R) \cong \mathbb{Z}^c$ where $c < \infty$ denotes the number of components of $H_0(R)$. If R is a domain, then $H_0(R)$ is connected, implying $H_0(R) \cong \mathbb{Z}$.

The submonoid $L \subset \mathbf{P}(R)$ of componentwise free modules is cofinal, so that $K(L) \leq K_0(R)$. Moreover, $K(L) \cong H_0(R)$ by Lemma 10.

The rank of a projective module induces a homomorphism rank : $K_0(R) \to H_0(R)$. Since $\mathrm{rank}(R^f) = f$ for any $R^f \in L$, we see that

$$1 \longrightarrow H_0(R) \cong K(L) \hookrightarrow K_0(R) \xrightarrow{\text{rank}} H_0(R) \longrightarrow 1$$

splits. This implies that

$$K_0(R) \cong H_0(R) \oplus \widetilde{K}_0(R),$$

where $\widetilde{K}_0(R)$ denotes ker(rank).

Example 13. The *Whitehead group* of a group G is the quotient $Wh_0(G) = {K_0(\mathbb{Z}[G])}/{\mathbb{Z}}$, where $\mathbb{Z}[G]$ denotes the group ring. The augmentation map $f : \mathbb{Z}[G] \to \mathbb{Z}$ induces a split exact sequence

$$1 \longrightarrow Wh_0(G) \longrightarrow K_0(\mathbb{Z}[G]) \longrightarrow K_0(\mathbb{Z}) = \mathbb{Z} \longrightarrow 1 \ .$$

Hence $K_0(\mathbb{Z}[G]) \cong \mathbb{Z} \oplus Wh_0(G)$. We know due to Swan that if G is finite, then $Wh_0(G) \cong \widetilde{K}_0(\mathbb{Z}[G])$ and $\mathbb{Z} \cong H_0(\mathbb{Z})$.

Definition 14. A functor $F : \mathscr{C} \to \mathscr{D}$ is *additive* if $F : \mathscr{C}(X, Y) \to \mathscr{D}(FX, FY)$ is a homomorphism of abelian groups for any $X, Y \in \mathrm{ob}\,\mathscr{C}$.

Definition 15. The rings R and S are *Morita equivalent* if there exists an additive equivalence between $\mathbf{Mod}_R R$ and $\mathbf{Mod}_S$.

Theorem 12. *If R and S are Morita equivalent, then $K_0(R) \cong K_0(S)$.*

Proof. Click here for a self-contained proof. □

Let's move from algebraic to topological K-theory.

Definition 16. Let $f : F \to X$ and $g : G \to X$ be vector bundles. The *Whitney sum* of f and g is the vector bundle $F \oplus G$ on X whose fiber at $x \in X$ is $F_x \oplus G_x$. The *tensor product bundle* $F \oplus G$ is defined similarly.

Definition 17. A *vector bundle homomorphism* between $\phi : E_1 \to X_1$ and $\psi : E_2 \to X_2$ is a pair of maps $f : E_1 \to E_2$ and $g : X_1 \to X_2$ such that the following conditions holds.

1. $$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\phi"'] & E_2 \arrow[d, "\psi"] \\ X_1 \arrow[r, "g"'] & X_2 \end{tikzcd}$$

2. For each $x \in X_1$, the map $f \restriction_{\phi^{-1}(x)} : \phi^{-1}(x) \to \psi^{-1}(g(x))$ is a linear map.

Definition 18. Let $(\mathbf{Vect}_{\mathbb{F}}(X), \oplus)$ denote the abelian monoid of (isomorphism classes of) $\mathbb{F}$-vector bundles on the paracompact space X. We define

$$KU(X) = K(\mathbf{Vect}_{\mathbb{C}}(X)) \qquad KO(X) = K(\mathbf{Vect}_{\mathbb{R}}(X)).$$

Note that these are commutative rings with identity. **We apply the notation $K_{\mathrm{top}}(-)$ on topological spaces when we wish to omit the base field.**

Remark 4. $KU(-)$ and $KO(-)$ define contravariant functors $\mathbf{Top} \to \mathbf{Ab}$. Let $f : Y \to X$ be a map of spaces and $\phi : E \to X$ be a vector bundle. Define the subspace $f^*E = \{(y, e) \in Y \times E : f(y) = \phi(e)\}$. Define the vector bundle $f^*(\phi) : f^*E \to Y$ as the restriction of the projection map $\pi : Y \times E \to Y$. Hence we have a morhism $\phi \mapsto f^*(\phi) \to \mathbf{Vect}_{\mathbb{F}}(X)$ to $\mathbf{Vect}_{\mathbb{F}}(Y)$ of monoids. The universal property of K induces a unique morphism $f^* : K_{\text{top}}(X) \to K_{\text{top}}(Y)$.

Lemma 13. *If X and Y are homotopy equivalent, then $K(X) \cong K(Y)$.*

Proof. Apply the homotopy invariance theorem (HIT), which states that if Y is paracompact and $f, g : Y \to X$ are homotopic, then $f^*E \cong g^*E$ for any vector bundle E over X. □

Example 19.

1. $K_{\text{top}}(*) = \mathbb{Z}$.
2. If X is contractible, then the HIT implies that $KO(X) = KU(X) = \mathbb{Z}$
3. According to I.4.9 of *The K-book*, we have

$$\begin{aligned} &KO(S^1) \cong \mathbb{Z} \times C_2 \\ &KU(S^1) \cong \mathbb{Z} \\ &KO(S^2) \cong \mathbb{Z} \times C_2 \\ &KU(S^2) \cong \mathbb{Z} \times \mathbb{Z} \\ &KO(S^3) \cong KU(S^3) \cong \mathbb{Z} \\ &KO(S^4) \cong KU(S^4) \cong \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

Definition 20. The *dimension* of bundle E over X is the continuous homomorphism $\widehat{\dim}(E) : X \to \mathbb{N}$ given by $x \mapsto \dim(E_x)$.

Definition 21. A vector bundle $p : E \to X$ is a *componentwise trivial bundle* if we can write $X = \coprod X_i$ such that each X_i is a component of X and $p \restriction_{p^{-1}(X_i)}$ is trivial.

Lemma 14. *The submonoid of componentwise trivial bundles over X is isomorphic to $[X, \mathbb{N}]$.*

Proof. Send a given map $f : X \to \mathbb{N}$ to $T^f \coloneqq \coprod_{i \in \mathbb{N}}(f^{-1}(i) \times \mathbb{F})$. Conversely, if E be a componentwise trivial bundle, then $E \cong T^{\widehat{\dim}(E)}$. □

Remark 5. Thus, the sub-monoid of trivial bundles and the sub-monoid of componentwise trivial bundles are naturally isomorphic to $\mathbb{N}$ and $[X, \mathbb{N}]$, respectively. When X is compact, these are cofinal in $\mathbf{Vect}_{\mathbb{F}}(X)$ by the subbundle theorem (proven using Riemannian geometry), yielding $\mathbb{Z} \leq [X, \mathbb{Z}] \leq K_{\text{top}}(X)$.

Note 22.

1. We get a split exact sequence.

$$\begin{tikzcd} 1 \arrow[r] & \widetilde{K}_{\text{top}}(X) \arrow[r] & K_{\text{top}}(X) \arrow[r, "\widehat{\dim}"'] & {[X, \mathbb{Z}]} \arrow[l, bend right] \arrow[r] & 1 \end{tikzcd},$$

   where $\widetilde{K}_{\text{top}}(X)$ denotes $\ker\left(\widehat{\dim}\right)$.

2. The map of monoids $\mathbf{Vect}_{\mathbb{R}}(X) \to \mathbf{Vect}_{\mathbb{C}}(X)$ given by $[E] \mapsto [E \otimes \mathbb{C}]$ extends by universality to a homomorphism $KO(X) \to KU(X)$. Likewise, the forgetful functor $\mathbf{Vect}_{\mathbb{C}}(X) \to \mathbf{Vect}_{\mathbb{R}}(X)$ extends to a homomorphism $KU(X) \to KO(X)$.

Here is a nice early connection between algebraic and topological K-theory. Let X be a compact Hausdorff space and $\mathcal{C}(X, \mathbb{F})$ denote the ring of continuous functions $X \to \mathbb{F}$. For any $E \in \mathbf{Vect}_{\mathbb{F}}(X)$, set $\Gamma(X, E) = \{s : X \to E : p \circ s = \text{Id}_X\}$, the vector space of global sections of E.

Theorem 15 (Swan). *The map $E \mapsto \Gamma(X, E)$ induces isomorphisms $KO(X) \cong K_0(\mathcal{C}(X, \mathbb{R}))$ and $KU(X) \cong K_0(\mathcal{C}(X, \mathbb{C}))$.*

Our results thus far can be extended to symmetric monodical categories because these come equipped with a notion of direct sum that enabled our Grothendieck construction.

Definition 23. A *symmetric monoidal category* S is equipped with a functor $\square : S \times S \to S$, a base object e, and four natural isomorphisms expressing commutativity, associativity, and that e acts as an identity. These four isomorphisms must also satisfy certain coherence properties.

Example 24. The following are examples of symmetric monoidal category .

1. k-vector spaces with $\otimes_k$.
2. Any category with finite coproducts where $s \square t := s \amalg t$.
3. The category of pointed topological spaces where $s \square t := s \wedge t$ and $e := S^0$.

Suppose that the class of isomorphism classes of objects of a category S is a set, called S^{iso}. If S is symmetric monoidal, then $\left(S^{\mathrm{iso}}, \square\right)$ is an abelian monoid with identity element e. Then we define the *Grothendieck group* of S as $K_0(S)$.