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Abstract

We introduce the concept of a universal property in category theory. The main sources for this talk are the following.

- nLab
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 4
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4
- Steve Awodey's *Category Theory*, Ch. 5.6

1 Universal arrows

Definition 1.1. Let $\mathscr{C}$ be a category.

1. An object X of $\mathscr{C}$ is *initial* if for each $Y \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $f : X \to Y$.
2. We say that X is *terminal* if for each $Z \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $g : Z \to X$.

Either condition is called a *universal property* of X.

Any property P of $\mathscr{C}$ has a dual property P^{op} of $\mathscr{C}^{\mathrm{op}}$ obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P. Then P is true of $\mathscr{C}$ iff P^{op} is true of $\mathscr{C}^{\mathrm{op}}$.

Example 1.2. Being initial and being terminal are dual properties.

Lemma 1.3. *Any two initial objects of $\mathscr{C}$ are unique up to unique isomorphism. The same holds for any two terminal objects of $\mathscr{C}$.*

Proof sketch. Let X and X' be two initial objects. Compose the two unique morphisms $X \to X'$ and $X' \to X$ to get an isomorphism between X and X'. Apply duality to this argument for the case of terminal objects. $\square$

We can think of a universal property as follows. Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $X \in \operatorname{ob}\mathscr{C}$. A *universal arrow from X to F* is an ordered pair (Y, f) with $Y \in \operatorname{ob}\mathscr{D}$ and $f : X \to F(Y)$ a morphism of $\mathscr{C}$ with the property that for any $X' \in \operatorname{ob}\mathscr{D}$ and morphism $f' : X \to F(X')$ of $\mathscr{C}$, there exists a unique morphism $\hat{f} : Y \to X'$ of $\mathscr{D}$ such that $F\left(\hat{f}\right) \circ f = f'$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & F(Y) \arrow[d, dashed, "F(\hat{f})"] \\ & F(X') \end{tikzcd}$$

Dually, a *universal arrow from F to X* is an ordered pair (Y, f) with $Y \in \operatorname{ob}\mathscr{D}$ and $f : F(Y) \to X$ of $\mathscr{C}$ with the property that for any $X' \in \operatorname{ob}\mathscr{D}$ and morphism $f' : F(X') \to X$, there exists a unique morphism $\hat{f} : X' \to Y$ such that $f' = f \circ F\left(\hat{f}\right)$.

$$\begin{tikzcd} F(X') \arrow[r, dashed, "F(\hat{f})"] \arrow[dr, "f'"'] & F(Y) \arrow[d, "f"] \\ & X \end{tikzcd}$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

Definition 1.4.

1. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor and $Y \in \operatorname{ob}\mathscr{D}$. The *slice* or *left fiber category*, denoted by (F/Y) or $(F \downarrow Y)$, has as objects pairs (X, f) where $f : F(X) \to Y$ and as morphisms from $f : F(X) \to Y$ to $f' : F(X') \to Y$ morphisms $g : X \to X'$ such that $f = f' \circ F(g)$.

2. The *coslice* or *right fiber category*, denoted by (Y/F) or $(Y \downarrow F)$, has as objects pairs (X, f) where $f : Y \to F(X)$ and as morphisms from $f : Y \to F(X)$ to $f' : Y \to F(X')$ morphisms $g : X \to X'$ such that $f' = F(g) \circ f$.

If $F^{\mathrm{op}} : C^{\mathrm{op}} \to D^{\mathrm{op}}$ is opposite to the functor $F : \mathscr{C} \to \mathscr{D}$ and $Y \in \operatorname{ob}\mathscr{D}$, then $(Y/F)^{\mathrm{op}} = F^{\mathrm{op}}/Y$. Thus, the left and right fiber categories are dual in the sense that $P(Y, F)$ is true of any right fiber category Y/F iff $P^{\mathrm{op}}(Y, F)$ is true of any left fiber category F/Y.

Proposition 1.5. *Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $x \in \operatorname{ob} C$. Then $u : x \to Fr$ is a universal arrow from x to F iff it is an initial object of the coslice $(x \downarrow F)$. Dually, $u' : Fr' \to x$ is a universal arrow from F to x iff it is a terminal object of the same category.*

Proof. Suppose that u is universal and $f : x \to Fy$ is another object of $(x \downarrow F)$. Then there exists a unique $\hat{f} : r \to y$ such that $F\left(\hat{f}\right) \circ u = f$. Thus $F\left(\hat{f}\right)$ is a unique morphism of the coslice.

Conversely, suppose that u is initial. Then for any object $f : x \to Fy$ of $(x \downarrow F)$, there exists a unique arrow $Sg : Fr \to Fy$ such that $Sg \circ u = f$. Hence setting $\hat{f} = g$ makes u a universal arrow. $\square$

Corollary 1.6. *Any two universal arrows from x to F can be canonically identified by Lemma 1.3.*

2 (Co)limits

Definition 2.1. A *zero object* of $\mathscr{C}$ is an object that is both initial and terminal.

Example 2.2. The unique initial object of **Set** is $\emptyset$, and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in iso(**Set**).

Given $X \in \operatorname{ob}\mathscr{C}$, the *undercategory* $X/\mathscr{C}$ has as objects morphisms in $\mathscr{C}$ of the form $i : X \to Y$ where X is fixed. Given $i : X \to Y$ and $i' : X \to Y'$ in $\operatorname{ob} X/\mathscr{C}$, define the set of morphisms from i to i' as the morphisms $f : Y \to Y'$ where

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i'"'] & Y \arrow[d, "f"] \\ & Y' \end{tikzcd}$$

commutes. (We call i the *structure morphism.*) Composition and identity carry over exactly from $\mathscr{C}$.

Likewise, given $x \in \operatorname{ob} \mathscr{C}$, the *overcategory* $\mathscr{C}/X$ has as objects morphisms in $\mathscr{C}$ of the form $i : Y \to X$ where X is fixed. Given $i : Y \to X$ and $i' : Y' \to X$ in $\operatorname{ob} \mathscr{C}/X$, define the set of morphisms from i to i' as the morphisms $f : Y \to Y'$ where

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "i"'] & Y' \arrow[d, "i'"] \\ & X \end{tikzcd}$$

commutes. Composition and identity carry over exactly from $\mathscr{C}$.

Remark 2.3. If $X \in \operatorname{ob} \mathscr{C}$, then $(X/\mathscr{C})^{\mathrm{op}} = \mathscr{C}^{\mathrm{op}}/X$. Thus, the under- and overcategories are dual in the sense that $P(X, \mathscr{C})$ is true of any undercategory $X/\mathscr{C}$ iff $P^{\mathrm{op}}(X, \mathscr{C})$ is true of any overcategory $\mathscr{C}/X$.

Lemma 2.4. *For any $X \in \mathscr{C}$, the identity morphism on X is an initial object $X/\mathscr{C}$. Dually, it is a terminal object in $\mathscr{C}/X$.*

Proof. Any morphism $i : X \to Y$ is itself the unique morphism from Id_X to i. □

Lemma 2.5. *Let X be an initial object of $\mathscr{C}$. The identity morphism on X is a zero object $\mathscr{C}/X$. Dually, if $Y \in \operatorname{ob} \mathscr{C}$ is terminal, then Id_Y is a zero object in $Y/\mathscr{C}$.*

Proof. We already know that Id_X is terminal. If $p : Y \to X$ is an object in $\mathscr{C}/X$, then there is a unique morphism $f : X \to Y$. Then $f \circ p$ must equal Id_X. □

Example 2.6. Let (X, x) be a pointed set with $X = \{x\}$. Let $\mathbf{Set}_*$ denotes the category of pointed sets with base point preserving functions. Since $\mathbf{Set}_* \cong X/\mathbf{Set}$, it follows that X is a zero object in $\mathbf{Set}_*$.

Given a morphism $\alpha : X \to Z$ in $\mathscr{C}$, define the *under-and-overcategory* $(X/\mathscr{C}/Z)_\alpha$ as having triples (Y, i, p) as objects where $i : X \to Y$ and $p : Y \to Z$ are morphisms in $\mathscr{C}$ such that $p \circ i = \alpha$. Define the set of morphisms from (Y, i, p) to (Y', u', p') as the set of morphisms $f : Y \to Y'$ such that

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "i"'] \arrow[dr, blue, "\alpha"] & Y' \arrow[d, "p'"] \\ Y \arrow[r, "p"'] \arrow[ur, red, "f"] & Z \end{tikzcd}$$

commutes. If $\alpha = \mathrm{Id}_X$, then we call $(X/\mathscr{C}/X)_{\mathrm{Id}_X}$ the category of *retractive* objects over X, with each triple (Y, i, p) being a retraction of Y onto X.

Example 2.7. If $F : \mathscr{C} \to \mathscr{C}$ is the identity functor, then the undercategory $Y/\mathscr{C}$ equals the right fiber category Y/F, and the overcategory $\mathscr{C}/Y$ equals the left fiber category F/Y.

Let $\mathscr{J}$ be a category. A *diagram of shape $\mathscr{J}$ in $\mathscr{C}$* is a functor $F : \mathscr{J} \to \mathscr{C}$.

Definition 2.8. Given a functor $F : \mathscr{J} \to \mathscr{C}$ and $X \in \operatorname{ob} \mathscr{C}$, a *cone over F* consists of an *apex* $X \in \operatorname{ob} \mathscr{C}$ and *legs* $f_j : X \to F(j)$ for each $j \in \operatorname{ob} \mathscr{J}$ such that for any morphism $\alpha : j \to j'$, the triangle

$$\begin{tikzcd} X \arrow[r, "f_j"] \arrow[dr, "f_{j'}"'] & F(j) \arrow[d, "F\alpha"] \\ & F(j') \end{tikzcd}$$

commutes.

This is simply a natural transformation $\Delta_{\mathscr{J}} X \Rightarrow F$ where $\Delta_{\mathscr{J}} X$ denotes the constant functor on $\mathscr{J}$ at X. If $\mathscr{J}$ is small, then $\Delta_{\mathscr{J}}$ is a functor from $\mathscr{C}$ to $\mathbf{Fun}(\mathscr{J}, \mathscr{C})$.

Definition 2.9.

1. The *category of cones over* F is the right fiber category X/F.
2. The *category of cocones under* F is the left fiber category F/X.

Definition 2.10 (Colimit). Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $g : Y \to Z$ a morphism in $\mathscr{D}$. Let $\Delta_{\mathscr{C}} g : \Delta_{\mathscr{C}} Y \Rightarrow \Delta_{\mathscr{C}} Z$ be the natural transformation with components $X \mapsto g$.

1. A *colimit* $\operatorname{colim}_{\mathscr{C}} F$ of the functor $F : \mathscr{C} \to \mathscr{D}$ is an object Y of $\mathscr{D}$ together with a natural transformation $i : F \Rightarrow \Delta_{\mathscr{C}} Y$ such that for any $Z \in \operatorname{ob} \mathscr{D}$ and any natural transformation $j : F \Rightarrow \Delta_{\mathscr{C}} Z$, there is a unique morphism $g : Y \to Z$ such that $j = \Delta_{\mathscr{C}} g \circ i$.
2. We say that $\mathscr{D}$ *admits/has* $\mathscr{C}$*-shaped colimits* if each functor $G : \mathscr{C} \to \mathscr{D}$ has a colimit.
3. We say that $\mathscr{D}$ is *cocomplete* if each functor $G : \mathscr{C} \to \mathscr{D}$ with $\mathscr{C}$ small has a colimit.

If $\mathscr{C}$ is small, then a colimit of $F : \mathscr{C} \to \mathscr{D}$ is just an initial object in the right fiber category $F/\Delta_{\mathscr{C}}$, which has as objects pairs $(Z, j : F \to \Delta Z)$ and as morphisms from (Y, i) to (Z, j) the morphisms $g : Y \to Z$ in $\mathscr{D}$ such that $\Delta g \circ i = j$.

Example 2.11. If $\mathscr{C}$ is the empty category, then the empty functor $F : \mathscr{C} \to \mathscr{D}$ satisfies $F/\Delta_{\mathscr{C}} \cong \mathscr{D}$, so that the colimit of F is exactly the initial object of $\mathscr{D}$.

Proposition 2.12. *There is a natural bijection* $\mathscr{D}(Y, Z) \cong \mathbf{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta Z)$ *if and only if* $Y = \operatorname{colim}_{\mathscr{C}} F$.

Lemma 2.13. *Any two colimits are unique up to unique isomorphism.*

Proof. When $\mathscr{C}$ is small, this follows immediately from Lemma 1.3. Notice, however, that our proof of Lemma 1.3 does *not* require that $\mathscr{C}$ be locally small (a property which Rognes stipulates of any category). $\square$

Remark 2.14. Assume that $\mathscr{D}$ has $\mathscr{C}$-shaped colimits and that $\mathscr{C}$ is small. Then a (possibly global) choice function $\operatorname{colim}_{\mathscr{C}} : \mathbf{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor $\Delta_{\mathscr{C}} : \mathscr{D} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$. Indeed, for any functor $F : \mathscr{C} \to \mathscr{D}$, there is a bijection $\mathscr{D}(\operatorname{colim}_{\mathscr{C}} F, Z) \cong \mathbf{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta_{\mathscr{C}} Z)$.

Definition 2.15 (Limit). A *limit* of the functor $F : \mathscr{C} \to \mathscr{D}$ is a colimit of $F^{\mathrm{op}} : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$.

Explicitly, a limit for $F : \mathscr{C} \to \mathscr{D}$ is an object Z of $\mathscr{D}$ along with a natural transformation $p : \Delta_{\mathscr{C}} Z \Rightarrow F$ such that for any $Y \in \operatorname{ob} \mathscr{D}$ and any natural transformation $q : \Delta_{\mathscr{C}} Y \Rightarrow F$, there is a unique morphism $g : Y \to Z$ such that $q = p \circ \Delta_{\mathscr{C}} g$.

Note that the colimit of a functor F is exactly the limit of F^{op}. Hence *limit* and *colimit* are dual properties, and our results so far for colimits can be dualized for limits.

Definition 2.16 ((Co)product). Let $\mathscr{J}$ be a discrete small category. Consider a diagram $\{A_i\}_{i\in \mathrm{ob}\,\mathscr{J}}$ of shape $\mathscr{J}$.

1. The limit of $\{A_i\}_i$ is called the *product* $\prod_i A_i$, equipped with projections $\pi_i : \prod_i A_i \to A_i$ such that for every $f_i : U \to A_i$ there exists a unique map $f \coloneqq (f_i) : U \to \prod_i A_i$ satisfying $\pi_i \circ f = f_i$.

2. The colimit of $\{A_i\}_i$ is called the *coproduct* $\coprod_i A_i$, equipped with inclusions $u_i : A_i \to \coprod_i A_i$ such that for any $f_i : A_i \to Y$, there exists a unique map $f \coloneqq (f_i) : \coprod_i A_i \to Y$ satisfying $f_i = f \circ u_i$.

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Example 2.17.

(1) Consider any small diagram $F : \mathscr{J} \to \mathbf{Set}$. On the one hand,

$$\operatorname*{colim}_j F_j \cong \left(\coprod_{j\in \mathrm{ob}\,\mathscr{J}} F_j \right) \Big/ \sim$$

where $\sim$ is the smallest equivalence relation such that $F_j \ni f_j \sim f_{j'} \in F_{j'}$ whenever $F(\psi)(f_j) = f_{j'}$ for some $\psi : j \to j'$.

On the other hand,

$$\lim_j F_j \cong \left\{ (f_j)_j \in \prod_{j\in \mathrm{ob}\,\mathscr{J}} F_j \mid \forall \psi : j \to j' \text{ in } \mathscr{J},\ F(\psi)(f_j) = f_{j'} \right\}.$$

We have shown that **Set** is both complete and cocomplete.

(2) Let A be any set. Define the *cumulative hierarchy* $V_n(A)$ of rank $n < \omega$ over A along with a countable sequence

$$V_0 \xrightarrow{v_0} V_1 \xrightarrow{v_1} V_2 \longrightarrow \cdots \longrightarrow V_n \xrightarrow{v_n} V_{n+1} \longrightarrow \cdots$$

of maps recursively by

$$\begin{aligned} V_0(A) &= A \\ V_{n+1}(A) &= A \coprod \mathcal{P}(V_n(A)) \end{aligned}$$

$$\begin{aligned} v_0 : A &\hookrightarrow A \coprod \mathcal{P}(A), & a &\mapsto a \\ v_{n+1} : A \coprod \mathcal{P}(V_n(A)) &\to A \coprod \mathcal{P}(V_{n+1}(A)), & &(\mathrm{Id}_A, \mathcal{P}(V_n(A))). \end{aligned}$$

Let $V_\omega(A) = \mathrm{colim}_{n<\omega} V_n(A)$, which exists by part (1). Then $V_\omega(\emptyset)$ is exactly the set of all hereditarily finite sets. To see that $V_\omega(-)$ is a functor $\mathbf{Set} \to \mathbf{Set}$, let $f : A \to B$ be a function. Then we can build a cocone

$$\begin{tikzcd}
V_0(A) \arrow[r, "v_0"] \arrow[d, "f_0 \equiv f"'] & V_1(A) \arrow[r] \arrow[d, "(f,\mathcal{P}(f))"] & \cdots \arrow[r] & V_n(A) \arrow[r, "v_n"] \arrow[d, "(f,\mathcal{P}(f_{n-1}))"] & V_{n+1}(A) \arrow[d, "(f,\mathcal{P}(f_n))"] & \\
V_0(B) \arrow[r] & V_1(B) \arrow[r] & \cdots \arrow[r] & V_n(B) \arrow[r] & V_{n+1}(B) \arrow[r] & V_\omega(B)
\end{tikzcd}$$

under $\{V_n(A)\}_n$ recursively. By the universal property of colimits, there exists a unique map $V_\omega(A) \to V_\omega(B)$, so that $V_\omega(-)$ is functorial.

Let $\mathscr{J}$ be a category of the form $\bullet \rightrightarrows \bullet$. Then a diagram D of shape $\mathscr{J}$ looks like $A \underset{g}{\overset{f}{\rightrightarrows}} B$. A cone over D with apex C and legs $f_1 : C \to A$ and $f_2 : C \to B$ satisfies $f \circ f_1 = f_2 = g \circ f_1$.

Definition 2.18 ((Co)equalizer).

1. If such an object C together with f_1 is the limit of D, then we say it is the *equalizer* of f and g.

2. The colimit of D is called the *coequalizer* of f and g.

Example 2.19. The equalizer in **Set** of $f, g : X \to Y$ is the subset $X' \coloneqq \{x \in X : f(x) = g(x)\}$ together with the inclusion function $X' \hookrightarrow X$.

The coequalizer of (f, g) is precisely ${}^{Y}\!/\!_{\sim}$ together with the quotient map on B where $\sim$ is the smallest equivalence relation under which $f(x) \sim g(x)$ for every x.

It is easy to check that any equalizer $f : C \to A$ must be monic. Further, if f is split epic, i.e., has a section $g : A \to C$, as well, then f is an isomorphism. For, in this case, $f \circ (g \circ f) = \mathrm{Id}_A \circ f = f \circ \mathrm{Id}_C$. As f is monic, we have that $g \circ f = \mathrm{Id}_C$, so that g is an inverse of f.

Next, let $\mathscr{J}$ be a category of the form $\bullet \to \bullet \leftarrow \bullet$. Then a diagram of this shape looks like $B \xrightarrow{f} C \xleftarrow{g} A$, and a cone over this diagram looks like

$$\begin{tikzcd}
E \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, "\alpha"'] & A \arrow[d, "g"] \\
B \arrow[r, "f"'] & C
\end{tikzcd} .$$

Definition 2.20 (Pullback). If such an object E together with i and j is the limit of this diagram, then we call it the *pullback* of f and g, denoted by $B \times_C A$.

The universal property of a pullback square states that for any commutative diagram of the form

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[dr, bend right] & & \\
& B \times_C A \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd} ,$$

there is a unique *mediating map* $Z \to B \times_C A$ fitting into it.

If we perform a dual construction for $\mathscr{J}^{\mathrm{op}}$, then the colimit of the resulting diagram is called the *pushout*, denoted by $B \cup_C A$. The universal property of a pullback square states that for any commutative diagram of the form

$$\begin{tikzcd}
B \times_C A \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \arrow[ddr, bend left] & \\
B \arrow[r, "g"'] \arrow[drr, bend right] & C & \\
& & Z
\end{tikzcd} ,$$

there is a unique mediating map $B \cup_C A \to Z$ fitting into it.

Example 2.21.

1. The pullback in **Set** of $f : X \to Z$ and $g : Y \to Z$ is precisely $\{(x, y) \in X \times Y : f(x) = g(y)\}$, called the *fibered product* of X and Y over Z.

2. The pushout in **Set** of $f : Z \to X$ and $g : Z \to Y$ is precisely the quotient of $X \coprod Y$ by the equivalence relation $\sim$ generated by the formula $(\forall z \in Z)\,(f(z) \sim g(z))$. We call ${X \coprod Y}/{\sim}$ the *fibered sum* of X and Y under Z.

Example 2.22. Let $\mathbf{FinSet}_{\text{mono}}$ denote the category of all finite sets with injective functions as arrows. The category of *nominal sets* consists of all pullback-preserving functors $\mathbf{FinSet}_{\text{mono}} \to \mathbf{Set}$ with natural transformations as arrows. These correctly encode the syntax of functional programming languages modulo renaming of bound variables (which is necessary for substitution).

Proposition 2.23. *The pullback of a monomorphism in a category $\mathscr{C}$ is again a monomorphism in $\mathscr{C}$.*

Proof. Consider any pullback square

$$\begin{tikzcd} B \times_C A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

in $\mathscr{C}$ where f is monic. We must show that π_1 is monic. Let $h_1, h_2 : B' \to B \times_C A$ be morphisms in $\mathscr{C}$ such that

$$\pi_1 \circ h_1 = \pi_1 \circ h_2$$
$$\Downarrow$$
$$f \circ \pi_2 \circ h_1 = g \circ \pi_1 \circ h_1 = g \circ \pi_1 \circ h_2 = f \circ \pi_2 \circ h_2.$$

Since f is monic by assumption, it follows that $\pi_2 \circ h_1 = \pi_2 \circ h_2$. As a result, the universal property of pullbacks implies that $h_1 = h_2$, as required. $\square$

Our next two results are quite useful and follow directly from the universal property of pullback (dually, pushout) squares.

Proposition 2.24. *Let $f : X \to Y$ be a morphism in a category $\mathscr{C}$.*

1. *The commutative square*

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback if and only if f is a monomorphism.

2. *The commutative square*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, equal] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

is a pushout if and only if f is an epimorphism.

Proposition 2.25 (Pasting law). *Consider a commutative diagram of the form*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \end{tikzcd}$$

in a category $\mathscr{C}$.

1. *Suppose that the righthand square is a pullback. Then the total rectangle is a pullback if and only if the lefthand square is one.*
2. *Suppose that the lefthand square is a pushout. Then the total rectangle is a pushout if and only if the righthand square is one.*

Corollary 2.26. *The operations of forming pullbacks and forming pushouts are associative up to isomorphism.*

All coequalizers $A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{h} C$ can be obtained from taking binary coproducts and pushouts as follows.

$$\begin{tikzcd} A \coprod A \arrow[r, "{(f,g)}"] \arrow[d, "{(\mathrm{Id}_A,\mathrm{Id}_A)}"'] & B \arrow[d, "h"] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & C \end{tikzcd}$$

Therefore, any category with binary coproducts and pushouts has coequalizers.

Moreover, any colimit of a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \tag{$*$}$$

is precisely the coequalizer of

$$\coprod_n X_n \underset{(u_{n+1}\circ f_n)_n}{\overset{\mathrm{Id}}{\rightrightarrows}} \coprod_n X_n.$$

Therefore, any category with coequalizers and small coproducts has colimits of diagrams like ($*$). This fact can be generalized as follows.

Theorem 2.27 (Freyd).

(i) If $\mathscr{C}$ has equalizers and small (resp. finite) products, then it has small (resp. finite) limits.

(ii) If $\mathscr{C}$ has pullbacks and a terminal object, then it has finite limits.

Proof sketch.

1. Let $F : \mathscr{J} \to \mathscr{C}$ be any diagram with $\mathscr{J}$ small. Consider the following two morphisms in $\mathscr{C}$:

$$f, g : \prod_{j \in \mathrm{ob}\,\mathscr{J}} F_j \to \prod_{\alpha: i \to j} F_j$$

$$\pi_{\alpha:i\to j} \circ f \equiv \pi_j$$

$$\pi_{\alpha:i\to j} \circ g \equiv F(\alpha) \circ \pi_i.$$

   Then $\lim_{\mathscr{J}} F$ is precisely the equalizer of f and g.

2. Thanks to part (i), it suffices to show that $\mathscr{C}$ has equalizers and finite products. By assumption, there is some terminal object 1. Then any product $A_1 \times A_2$ can be realized as the pullback of $A_1 \to 1 \leftarrow A_2$. By induction, it follows that $\mathscr{C}$ has finite products. Moreover, for any morphisms $f, g : A \to B$, note that any cone over the diagram

$$A \xrightarrow{(\mathrm{Id}_A, g)} A \times B \xleftarrow{(\mathrm{Id}_A, f)} A$$

yields morphisms $h : A \to C$ and $k : C \to A$ such that $h = k$ and $fk = gh$. As a result, the pullback for this cospan is an equalizer of f and g, and thus our proof is complete.

$\square$

We may view Example 2.17(1) as an instance of Theorem 2.27.

Next, let us show that adjoints interact nicely with (co)limits under mild conditions.

Proposition 2.28 (Left adjoints preserve colimits). *Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be functors such that (F, G) is an adjoint pair. Let $\mathscr{E}$ be small category. If $X : \mathscr{E} \to \mathscr{C}$ is a functor whose colimit exists, then*

$$\operatorname*{colim}_{\mathscr{E}}(F \circ X) \cong F\left(\operatorname*{colim}_{\mathscr{E}} X\right).$$

Dually, if $Y : \mathscr{E} \to \mathscr{D}$ is a functor whose limit exists, then

$$\lim_{\mathscr{E}}(G \circ Y) \cong G\left(\lim_{\mathscr{E}} Y\right).$$

Proof. We have the following chain of natural bijections in $Y \in \operatorname{ob}\mathscr{D}$:

$$\begin{aligned}
\mathscr{D}\left(F\left(\operatorname*{colim}_{\mathscr{E}} X\right), Y\right) &\cong \mathscr{C}\left(\operatorname*{colim}_{\mathscr{E}} X, G(Y)\right) \\
&\cong \lim_{\mathscr{E}} \mathscr{C}(X(-), G(Y)) \\
&\cong \lim_{\mathscr{E}} \mathscr{D}(F(X(-)), Y) \\
&\cong \mathbf{Fun}(\mathscr{E}, \mathscr{D})(F \circ X, \Delta Y).
\end{aligned}$$

The second bijection exists because both sets can be identified with the components of all natural transformations $X \Rightarrow \Delta G(Y)$. $\square$

3 Fibers and Fibrations

Definition 3.1. Suppose $\mathscr{C}$ has a terminal object 1. Let $f : X \to Y$ be a morphism in $\mathscr{C}$.

1. Given a global element $p : 1 \to Y$ of Y, the *fiber* $f^{-1}(p)$ *of* f at p is the pullback of the cospan $1 \to Y \leftarrow X$.

2. The *cofiber* Y/X *of* f is the pushout of the span $1 \leftarrow X \to Y$.

For any functor $F : \mathscr{C} \to \mathscr{D}$, the *fiber category* $F^{-1}(Y)$ is the full subcategory of $\mathscr{C}$ generated by those objects X such that $F(X) = Y$.

For each $Y \in \operatorname{ob} \mathscr{D}$, there is a full and faithful functor $F^{-1}(Y) \to F/Y$ given by $X \mapsto (X, \mathrm{Id}_Y)$. We say that $\mathscr{C}$ is a *precofibered category* over $\mathscr{D}$ if F has a left adjoint given by

$$(Z, g : F(Z) \to Y) \mapsto g_*(Z).$$

Further, there is a full and faithful functor $F^{-1}(Y) \to Y/F$. We say that $\mathscr{C}$ is a *prefibered category* over $\mathscr{D}$ if this functor has a right adjoint given by $(Z, g : Y \to F(Z)) \mapsto g_*(Z)$.

Definition 3.2. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor.

1. Let $f : c' \to c$ be a morphism in $\mathscr{C}$. We say f is *cartesian* if for any morphism $f' : c'' \to c$ in $\mathscr{C}$ and any morphism $g : F(c'') \to F(c')$ in $\mathscr{D}$ such that $Ff \circ g = Ff'$, there exists a unique morphism $\phi : c'' \to c$ such that $f' = f \circ \phi$ and $F\phi = g$.

   In pictures,

$$\begin{tikzcd} F(c'') \arrow[r, dashed, "g"] \arrow[dr, "Ff'"'] & F(c') \arrow[d, "Ff"] \\ & F(c) \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} c'' \arrow[r, dashed, "\exists!\phi"] \arrow[dr, "f'"'] & c' \arrow[d, "f"] \\ & c \end{tikzcd}, \qquad \phi \overset{F}{\longmapsto} g.$$

2. We say that F is a *fibration* if for any $c \in \operatorname{ob} \mathscr{C}$ and morphism $f : d \to Fc$ in $\mathscr{D}$, there is a cartesian morphism $\phi_f : c' \to c$ such that $F\phi_f = f$. Such a ϕ_f is called a *cartesian lifting* of f to c.

   In this case, assuming the axiom of choice, we obtain a mapping $f \mapsto \phi_f$, known as a *cleavage* of F. If this respects the identity map and composition, then we call F a *normal* and *split* fibration, respectively.

Intuitively, if F is a fibration, then the fibers $F^{-1}(Y)$ depend functorially on $Y \in \operatorname{ob} \mathscr{D}$.

Example 3.3.

1. Let the category **Mod** consist of pairs (R, M) as objects where R is a ring and M is a left R-module and pairs $\left(f, \tilde{f}\right)$ as morphisms where $f : R \to R'$ is a ring homomorphism and $\tilde{f} : M \to M'$ is an R-linear map with M' viewed as an R-module via f. Then the forgetful functor $U : \mathbf{Mod} \to \mathbf{Ring}$ is a fibration.

2. For any category $\mathscr{C}$ with pullbacks, consider the arrow category $\mathrm{Ar}(\mathscr{C})$ along with the codomain functor $\mathsf{cod} : \mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$ defined by

$$a \xrightarrow{f} b \quad \mapsto \quad b$$

$$\begin{tikzcd} a \arrow[r] \arrow[d] & a' \arrow[d] \\ b \arrow[r] & b' \end{tikzcd} \quad \mapsto \quad b \to b'.$$

   This is a fibration. Indeed, for any object $x \to y$ in $\mathrm{Ar}(\mathscr{C})$ and any morphism $z \to y$ in $\mathscr{C}$, the cartesian lifting of $z \to y$ to $x \to y$ is given by the pullback square

$$\begin{tikzcd} z \times_y x \arrow[r] \arrow[d] & x \arrow[d] \\ z \arrow[r] & y \end{tikzcd}.$$