

### Abstract

We continue doing higher Waldhausen  $K$ -theory. The main sources for this talk are the following.

- $n\text{Lab}$ .
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Ch. V.2.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

Recall that  $|wS_\bullet \mathcal{C}|$  is an  $H$ -space via the map

$$\coprod : |wS_\bullet \mathcal{C}| \times |wS_\bullet \mathcal{C}| \cong |wS_\bullet \mathcal{C} \times wS_\bullet \mathcal{C}| \rightarrow |wS_\bullet \mathcal{C}|.$$

This produces an  $H$ -space structure  $(K(\mathcal{C}), +)$ .

**Definition 1.** Let  $\mathcal{B}$  and  $\mathcal{C}$  be Waldhausen categories. We say that  $F' \rightarrowtail F \twoheadrightarrow F''$  is a *short exact sequence* or *cofiber sequence of exact functors* if every  $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$  is a cofiber sequence and  $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$  is a cofibration in  $\mathcal{C}$  for every  $A \rightarrowtail B$  in  $\mathcal{B}$ .

Let  $\mathcal{C}$  be a Waldhausen category. Let  $(\eta) : A \rightarrowtail B \twoheadrightarrow C$  be an object in  $S_2\mathcal{C}$ . Define the source  $s$ , target  $t$ , and quotient  $q$  functors  $S_2\mathcal{C} \rightarrow \mathcal{C}$  by  $s(\eta) = A$ ,  $t(\eta) = B$ , and  $q(\eta) = C$ . Then  $s \rightarrowtail t \twoheadrightarrow q$  is a cofiber sequence of functors. Since defining a cofiber sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$  is equivalent to defining an exact functor  $\mathcal{B} \rightarrow S_2\mathcal{C}$ , we may restrict our attention to  $s \rightarrowtail t \twoheadrightarrow q$  when proving things about a given cofiber sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$ . We say that  $S_2\mathcal{C}$  is universal in this sense.

**Theorem 1 (Extension theorem).** *Let  $\mathcal{C}$  be Waldhausen. The exact functor  $(s, q) : S_2\mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$  induces a homotopy  $K(S_2\mathcal{C}) \simeq K(\mathcal{C}) \times K(\mathcal{C})$ . The functor  $\coprod : (A, B) \rightarrow (A \rightarrowtail A \coprod B \twoheadrightarrow B)$  is a homotopy inverse.*

*Proof.* Let  $\mathcal{C}_m^w$  denote the category of  $m$ -length sequences of weak equivalences. For each  $n$ , define  $s_n \mathcal{C}_m^w$  as the commutative diagram

$$\begin{array}{ccccccc} X_1^0 & \rightarrowtail & X_2^0 & \rightarrowtail & \cdots & \rightarrowtail & X_n^0 \\ \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\ X_1^1 & \rightarrowtail & X_2^1 & \rightarrowtail & \cdots & \rightarrowtail & X_n^1 \\ \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\ \vdots & & \vdots & & & & \vdots \\ \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\ X_1^m & \rightarrowtail & X_2^m & \rightarrowtail & \cdots & \rightarrowtail & X_n^m \end{array}.$$

This is naturally isomorphic to an  $(m, n)$ -bisimplex in  $N_\bullet wS_\bullet \mathcal{C}$ , which is thus isomorphic to the bisimplicial set  $s_\bullet \mathcal{C}_{(-)}^w$ . One can show that the source  $s$  and quotient  $q$  functors  $S_2\mathcal{C} \rightarrow \mathcal{C}$  give a homotopy equivalence  $s \times q : s_\bullet S_2(\mathcal{C}_m^w) \rightarrow s_\bullet \mathcal{C}_m^w \times s_\bullet \mathcal{C}_m^w$  for each  $m$ . Thus, we get a homotopy equivalence

$$s_\bullet S_2(\mathcal{C}_{(-)}^w) \simeq s_\bullet \mathcal{C}_{(-)}^w \times s_\bullet \mathcal{C}_{(-)}^w$$

between bisimplicial sets. But we already have that  $s_\bullet \mathcal{C}_{(-)}^w \cong N_\bullet wS_\bullet \mathcal{C}$ , completing the proof.  $\square$

**Theorem 2 (The additivity theorem).** *Let  $F' \rightarrowtail F \twoheadrightarrow F''$  be a short exact sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$ . Then  $F_* \simeq F'_* + F''_*$  as maps  $K(\mathcal{B}) \rightarrow K(\mathcal{C})$ . Hence  $F_* = F'_* + F''_*$  as maps  $K_i(\mathcal{B}) \rightarrow K_i(\mathcal{C})$ .*

*Proof.* As  $S_2\mathcal{C}$  is universal, it suffices to prove that  $t_* \simeq s_* + q_*$ . Notice that the two compositions  $\mathcal{C} \times \mathcal{C} \xrightarrow{\coprod} S_2\mathcal{C} \xrightarrow[t_*]{s_*} \mathcal{C}$  are the same. The extension theorem implies that  $K(\coprod) : K(\mathcal{C}) \times K(\mathcal{C}) \rightarrow K(S_2\mathcal{C})$  is a homotopy equivalence. Since the  $H$ -space structure on  $K(\mathcal{C})$  is induced by  $\coprod$ , we get  $t_* \simeq s_* + q_*$ .  $\square$

**Definition 2.** Let  $\mathcal{C}$  be Waldhausen. We say that a sequence  $* \rightarrow A_n \rightarrow \cdots \rightarrow A_0 \rightarrow *$  is *admissibly exact* if each morphism in the sequences can be written as a cofiber sequence  $A_{i+1} \rightarrow B_i \rightarrow A_i$ .

**Corollary 3.** Suppose that  $* \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \rightarrow F^n \rightarrow *$  is an admissibly exact sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$ . Then  $\sum_i (-1)^i F_*^i = 0$  as maps  $K_i(\mathcal{B}) \rightarrow K_i(\mathcal{C})$ .

*Proof.* Induct on  $n$ .  $\square$

**Corollary 4.** Let  $F' \rightarrow F \rightarrow F''$  be a short exact sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$ . Then

$$F''_* \simeq F_* - F'_* \simeq 0.$$

This implies that the homotopy fiber of  $F''_* : K(\mathcal{B}) \rightarrow K(\mathcal{C})$  is homotopy equivalent to  $K(\mathcal{B}) \vee \Omega K(\mathcal{C})$ .

**Definition 3.** Let  $\mathcal{C}$  be a Waldhausen category. Recall the arrow category  $\text{Ar}(\mathcal{C})$  of  $\mathcal{C}$  consisting of morphisms in  $\mathcal{C}$  as objects and commutative squares as morphisms. Let  $s$  and  $t$  denote the source and target functors  $\text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$ , respectively.

A functor  $T : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$  is a (*mapping*) *cylinder functor* on  $\mathcal{C}$  if it comes equipped with natural transformations  $j_1 : s \Rightarrow T$ ,  $j_2 : t \Rightarrow T$ , and  $p : T \Rightarrow t$  such that for any  $f : A \rightarrow B$ , we have the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j_1} & T(f) \xleftarrow{j_2} B \\ & \searrow f & \downarrow p \swarrow = \\ & & B \end{array} .$$

Moreover,  $T$  must satisfy the following axioms.

1.  $T$  sends every initial morphism  $* \rightarrow A$  to  $A$  for any  $A \in \text{ob } \mathcal{C}$ .
2.  $j_1 \coprod j_2 : A \coprod B \rightarrow T(f)$  is a cofibration for any  $f : A \rightarrow B$ .
3. Given a morphism  $(a, b) : f \rightarrow f'$  in  $\text{Ar}(\mathcal{C})$ , if both  $a$  and  $b$  are w.e. in  $\mathcal{C}$ , then so is  $T(f) \rightarrow T(f')$ .
4. Given a morphism  $(a, b) : f \rightarrow f'$  in  $\text{Ar}(\mathcal{C})$ , if both  $a$  and  $b$  are cofibrations in  $\mathcal{C}$ , then so is  $T(f) \rightarrow T(f')$ . Also, the map  $A' \coprod_A T(f) \coprod_B B' \rightarrow T(f')$  induced by axiom 2 is a cofibration in  $\mathcal{C}$ .
5. (Cylinder Axiom) The map  $p : T(f) \rightarrow B$  is a w.e. in  $\mathcal{C}$ .

**Definition 4.** Let  $T$  be a cylinder functor on  $\mathcal{C}$ .

1. We call  $T(A \rightarrow *)$  the *cone* of  $A$ , denoted by  $\text{cone}(A)$ .
2. We call  $\text{cone}(A)/_A$  the *suspension* of  $A$ , denoted by  $\Sigma A$ .

**Corollary 5.** The induced suspension map  $\Sigma : K(\mathcal{C}) \rightarrow K(\mathcal{C})$  is a homotopy inverse for the  $H$ -space  $K(\mathcal{C})$ .

*Proof.* Note that axiom 3 gives us a cofiber sequence  $A \rightarrow \text{cone}(A) \rightarrow \Sigma A$ . Therefore,  $1 \rightarrow \text{cone} \rightarrow \Sigma$  is an exact sequence of functors. By the cylinder axiom, we know that  $\text{cone}$  is null-homotopic. It follows by the additivity theorem that  $\Sigma_* + 1 = \text{cone}_* = *$ .  $\square$

Let  $\mathcal{C}$  be a category with cofibrations. Equip it with two Waldhausen subcategories  $v(\mathcal{C})$  and  $w(\mathcal{C})$  of weak equivalences such that  $v(\mathcal{C}) \subset w(\mathcal{C})$ . Assume that  $(\mathcal{C}, w)$  admits a cylinder functor. Suppose that  $w(\mathcal{C})$  is saturated and closed under extensions.

*Notation.* Let  $\mathcal{C}^w$  denote the Waldhausen subcategory of  $(\mathcal{C}, v)$  consisting of any  $A$  where  $* \rightarrow A$  is in  $w(\mathcal{C})$ .

[[Are the initial morphisms the only w.e.?]]

**Theorem 6 (Waldhausen localization theorem).** *The sequence*

$$K(A^w) \rightarrow K(\mathcal{C}, v) \rightarrow K(\mathcal{C}, w)$$

*is a homotopy fibration sequence.*

*Proof.* Recall that a small bicategory is a bisimplicial set such that each row/column is the nerve of a category. Note that  $v_{(-)}w_{(-)}\mathcal{C}$  is a bicategory whose bimorphisms are commutative squares of the form

$$\begin{array}{ccc} (-) & \xrightarrow{w'} & (-) \\ v \downarrow & & \downarrow v' \\ (-) & \xrightarrow{w} & (-) \end{array} .$$

It turns out that treating  $w\mathcal{C}$  as a bicategory with a single vertical morphism proves that  $w\mathcal{C} \simeq v_{(-)}w_{(-)}\mathcal{C}$ . This gives  $wS_n\mathcal{C} \simeq v_{(-)}w_{(-)}S_n\mathcal{C}$  for each  $n$ .

Now, let  $v_{(-)}\text{cow}_{(-)}\mathcal{C}$  denote the subcategory of the above squares where the horizontal maps are also cofibrations. One can show that the inclusion  $v_{(-)}\text{cow}_{(-)}\mathcal{C} \subset v_{(-)}w_{(-)}\mathcal{C}$  is a homotopy equivalence. Since each  $S_n\mathcal{C}$  inherits a cylinder functor from  $\mathcal{C}$ , we simplicial bi-subcategory  $v_{(-)}\text{cow}_{(-)}S_{\bullet}\mathcal{C}$  such that the inclusion into  $v_{(-)}w_{(-)}S_{\bullet}\mathcal{C}$  is a homotopy equivalence. We have now obtained the following diagram.

$$\begin{array}{ccccc} vS_{\bullet}C^w & \longrightarrow & vS_{\bullet}C & \longrightarrow & v_{(-)}\text{cow}_{(-)}S_{\bullet}C \\ & & \downarrow & & \downarrow \simeq \\ & & wS_{\bullet}C & \xrightarrow{\simeq} & v_{(-)}w_{(-)}S_{\bullet}C \end{array}$$

It therefore suffices to show that the top row is a fibration. [[What about the left vertical morphism?]] You can do this by using the relative  $K$ -theory space construction. See Weibel, IV.8.5.3 and V.2.1.  $\square$

**Definition 5.** Let  $\mathcal{A}$  be an exact category embedded in an abelian category  $\mathcal{B}$  and let  $\mathbf{Ch}^b(\mathcal{A})$  denote the category of bounded chain complexes in  $\mathcal{A}$ . One can verify that  $\mathbf{Ch}^b(\mathcal{A})$  is Waldhausen where the cofibrations  $A_{\bullet} \rightarrow B_{\bullet}$  are precisely the degree-wise admissible monomorphisms (i.e., those giving a short exact sequence  $A_n \rightarrow B_n \rightarrow B_n/A_n$  in  $\mathcal{A}$  for each  $n$ ) and the w.e. are precisely the chain maps which are quasi-isomorphisms of complexes in  $\mathbf{Ch}(\mathcal{B})$ .

**Theorem 7 (Gillet-Waldhausen).** *Let  $\mathcal{A}$  be an exact category closed under kernels of surjections. Then the exact inclusion  $\mathcal{A} \rightarrow \mathbf{Ch}^b(\mathcal{A})$  induces a homotopy equivalence  $K(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A})$ . Hence*

$$K_i(\mathcal{A}) = K_i\mathbf{Ch}^b(\mathcal{A})$$

*for every  $i$ .*

*Proof.* Apply the localization theorem. See Weibel, V.2.2.  $\square$

**Definition 6.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an exact functor between Waldhausen categories. We say that  $F$  satisfies the *approximate lifting property* if for any map  $b : F(A) \rightarrow B$  in  $\mathcal{B}$ , there is some map  $a : A \rightarrow A'$  in  $\mathcal{A}$  and some w.e.  $b' : F(A') \simeq B$  in  $\mathcal{B}$  so that

$$\begin{array}{ccc} F(A') & \overset{\sim}{\dashrightarrow} & B \\ \uparrow F(a) & \nearrow b & \\ F(A) & & \end{array} .$$

commutes. In this way, we can lift to a w.e.

**Proposition 8.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an exact functor between Waldhausen categories such that the following hold.

1.  $F$  satisfies the approximate lifting property.
2.  $\mathcal{A}$  admits a cylinder functor.
3. A morphism  $f$  in  $\mathcal{A}$  is a w.e. iff  $F(f)$  is a w.e. in  $\mathcal{B}$ .

Then  $wF : w\mathcal{A} \rightarrow w\mathcal{B}$  is a homotopy equivalence.

**Corollary 9 (Waldhausen approximation theorem).** With the same conditions as before, we have

$$K(\mathcal{A}) \simeq K(\mathcal{B}).$$

*Proof.* One can show that each functor  $S_n\mathcal{A} \rightarrow S_n\mathcal{B}$  is exact and also has the approximate lifting property. The previous proposition thus gives degree-wise homotopy equivalence between the bisimplicial map  $wS_\bullet\mathcal{A} \rightarrow wS_\bullet\mathcal{B}$ , which is enough.  $\square$

**Definition 7.** Let  $\mathcal{A}$  be an abelian category  $\mathbf{Ch}(\mathcal{A})$  denote the category of chain complexes over  $\mathcal{A}$ . We say that a complex  $C_\bullet$  is *homologically bounded* if only finitely many  $H_i(C_j)$  are nonzero. Let  $\mathbf{Ch}_\pm^{hb}$  denote the subcategory of bounded below (respectively, bounded above) complexes.

**Example 8.** Let  $\mathcal{A}$  be an abelian category. By homology theory, we have that  $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}_-^{hb}(\mathcal{A})$  and  $\mathbf{Ch}_+^{hb}(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$  have the approximate lifting property. We also have that  $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}_+^{hb}(\mathcal{A})$  and  $\mathbf{Ch}_+^{hb}(\mathcal{A}) \subset \mathbf{Ch}^{hb}(\mathcal{A})$  satisfy the dual of the approximate lifting property. Thus, we can apply the approximation theorem and Gillet-Waldhausen to see that

$$K(\mathcal{A}) \simeq K\mathbf{Ch}^b(\mathcal{A}) \simeq K\mathbf{Ch}_-^{hb} \simeq K\mathbf{Ch}_+^{hb}(\mathcal{A}) \simeq K\mathbf{Ch}^{hb}(\mathcal{A}).$$

**Definition 9.** A *symmetric spectrum*  $\mathbf{X}$  in topological spaces in a sequence of based  $\Sigma_n$ -spaces  $(X_n)$  endowed with structure maps  $\sigma : X_n \wedge S^1 \rightarrow X_{n+1}$  such that  $\sigma^k : X_n \wedge S^k \rightarrow X_{n+k}$  is  $(\Sigma_n \times \Sigma_k)$ -equivariant for any  $n, k \geq 0$ , where  $S^k := \underbrace{S^1 \wedge \cdots \wedge S^1}_{k\text{-times}}$ . A map  $\mathbf{f} : \vec{x} \rightarrow \mathbf{Y}$  of symmetric spectra is a sequence  $(f_n : X_n \rightarrow Y_n)$  of based  $\Sigma_n$ -equivariant maps such that for each  $n \geq 0$ , the square

$$\begin{array}{ccc} X_n \wedge S^1 & \xrightarrow{f_n \wedge \text{Id}} & Y_n \wedge S^1 \\ \sigma \downarrow & & \downarrow \sigma \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

commutes. Let  $\text{Sp}^\Sigma$  denote the category of symmetric spectra in topological spaces.

**Definition 10.** Let  $(\mathcal{C}, w\mathcal{C})$  be a Waldhausen category. The *external  $n$ -fold  $S_\bullet$ -construction* on  $\mathcal{C}$  is the  $n$ -multisimplicial Waldhausen category

$$(S_\bullet \cdots S_\bullet \mathcal{C}, wS_\bullet \cdots S_\bullet \mathcal{C}).$$

It multidegree  $(q_1, \dots, q_n)$ , it has as objects the diagrams  $X : \text{Ar}[q_1] \times \cdots \times \text{Ar}[q_n] \rightarrow \mathcal{C}$  such that

1.  $X((i_1, j_1), \dots, (i_n, j_n)) = *$  if  $i_k = j_k$  for some  $1 \leq k \leq n$ .
2.  $X(\dots, (i_t, j_t), \dots) \twoheadrightarrow X(\dots, (i_t, k_t), \dots) \twoheadrightarrow X(\dots, (j_t, k_t), \dots)$  is a cofiber sequence in the  $(n-1)$ -fold iterated  $S_\bullet$ -construction for any  $i_t \leq j_t \leq k_t$  in  $[q_t]$ .

**Definition 11.** Let  $(\mathcal{C}, w\mathcal{C})$  be a Waldhausen category. The *internal  $n$ -fold  $S_\bullet$ -construction* on  $\mathcal{C}$  is the simplicial Waldhausen category

$$(S_\bullet^{(n)} \mathcal{C}, wS_\bullet^{(n)} \mathcal{C}).$$

It has as  $q$ -simplices the functor categories  $(S_q \cdots S_q \mathcal{C}, wS_q \cdots S_q \mathcal{C})$  whose objects are the  $(\text{Ar}[q])^n$ -shaped diagrams  $X : (\text{Ar}[q])^n \rightarrow \mathcal{C}$  such that

1.  $X((i_1, j_1), \dots, (i_n, j_n)) = *$  if  $i_k = j_k$  for some  $1 \leq k \leq n$ .
2.  $X(\dots, (i_t, j_t), \dots) \rightarrow X(\dots, (i_t, k_t), \dots) \twoheadrightarrow X(\dots, (j_t, k_t), \dots)$  is a cofiber sequence in the  $(n-1)$ -fold iterated  $S_\bullet$ -construction for any  $i_t \leq j_t \leq k_t$  in  $[q]$ .

Note that  $\Sigma_n$  acts on  $S_\bullet^{(n)}\mathcal{C}$  by  $(\pi \cdot X)(\dots, (i_t, j_t), \dots) = X(\dots, (i_{\pi^{-1}(t)}, j_{\pi^{-1}(t)}), \dots)$ .

**Definition 12.** The (symmetric) algebraic  $K$ -theory spectrum  $\mathbf{K}(\mathcal{C}, w)$  of a small Waldhausen category  $(\mathcal{C}, w\mathcal{C})$  has  $n$ -th space  $K(\mathcal{C}, w)_n = |wS_\bullet^{(n)}\mathcal{C}|$  based at  $*$ . There is a  $\Sigma_n$ -action on  $K(\mathcal{C}, w)_n$  induced by permuting the order of the internal  $S_\bullet$ -constructions. Moreover, we have

$$|wS_\bullet^{(n)}\mathcal{C}| \wedge S^1 \cong |wS_\bullet^{(n)}S_\bullet\mathcal{C}|^{(1)} \subset |wS_\bullet^{(n)}S_\bullet\mathcal{C}| \cong |wS_\bullet^{(n+1)}\mathcal{C}|$$

, where  $^{(1)}$  denotes the 1-skeleton with respect to the last simplicial direction. This determines the structure map  $\sigma$ . Then  $\sigma^k$  is  $(\Sigma_n \times \Sigma_k)$ -invariant.

**Theorem 10.** For any  $i \geq 0$ , we have that  $K_i(\mathcal{C}, w) = \pi_{i+1}K(\mathcal{C}, w)_1 \cong \pi_i\mathbf{K}(\mathcal{C}, w)$ .

*Proof.* See Rognes, Lemma 8.7.4. □

In this way, we encode our algebraic  $K$ -theory in an infinite loop space.