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Abstract

These notes are based on Davi Maximo's lectures for the course "Geometric Analysis and Topology I" at UPenn along with John Lee's *Introduction to Smooth Manifolds*, 2nd Ed. and Michael Spivak's *A Comprehensive Introduction to Differential Geometry, Vol. 1.* Any mistake in what follows is my own.

Contents

1 Smooth manifolds

1.1 Lecture 1

We want to make precise our notion of a (topological) space that locally looks like $\mathbb{R}^n$.

Definition 1.1.1. A space M is a *(topological) n-dimensional manifold* (or *n-manifold*) if it is

(i) Hausdorff,

(ii) second-countable, and

(iii) locally Euclidean of dimension n, i.e., for any $x \in M$, there exist an open set $U \ni x$ and a homeomorphism $\varphi : U \to V$ for some open subset $V \subset \mathbb{R}^n$.

Condition (iii) holds if and only if U is homeomorphic to an open ball in $\mathbb{R}^n$ or to $\mathbb{R}^n$ itself.

Terminology. A 2-dimensional manifold is called a *surface*.

Definition 1.1.2. Let M be an n-manifold.

1. A *coordinate chart on M* is a pair (U, φ) where $U \subset M$ is open and φ is a homeomorphism

$$U \xrightarrow{\cong} \underset{\text{open}}{W} \subset \mathbb{R}^n.$$

   If W is an open ball, then we call U a *coordinate ball.*

2. If (U, φ) is a coordinate chart and $\pi_i : \mathbb{R}^n \to \mathbb{R}$ denotes the i-th projection map, then we call elements of the set $\{(\pi_1(\varphi(p)), \ldots, \pi_n(\varphi(p))) \mid p \in U\}$ *local coordinates on U.*

Notation. We shall use the symbols x^i and x_i interchangeably for local coordinates.

Definition 1.1.3.

1. Given charts (U, φ), (V, ψ) with $U \cap V \neq \emptyset$, we say that the two are *C^k-compatible* if the *transition map $\psi \circ \varphi^{-1}$*

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[dr, "\psi"'] & \varphi(U \cap V) \arrow[d, "\psi\circ\varphi^{-1}"] \\ & \psi(U \cap V) \end{tikzcd}$$

   is C^k.

2. A collection of charts $(U_\alpha, \varphi_\alpha)$ which covers a smooth manifold M and is pairwise C^k-compatible is called a *C^k-atlas for M.*

Example 1.1.4. Consider the global charts $(\mathbb{R}, x \mapsto x)$ and $\big(\mathbb{R}, x \mapsto x^3\big)$. Since $x \mapsto x^{\frac{1}{3}}$ is not differentiable at 0, these charts fail to form a C^1-atlas on $\mathbb{R}$.

Definition 1.1.5. An atlas A is *maximal* if it contains every chart that is C^∞- (or smoothly) compatible with every chart in A.

Proposition 1.1.6.

1. *Every smooth atlas A is contained in a unique maximal atlas, namely the family of all charts that are smoothly compatible with every chart in A.*

2. *Two smooth atlases are contained in the same maximal atlas if and only if their union is also a smooth atlas.*

This shows that an atlas is maximal if and only if it's maximal in the usual in the set-theoretic sense.

Definition 1.1.7. A manifold M is *smooth* if it admits a maximal smooth atlas, also known as a *smooth structure.*

By Proposition 1.1.6, it's enough to construct any smooth atlas for M to show that it's a smooth manifold.

An open problem is whether there is more than one smooth structure on $\mathbb{S}^4$. This is known for each $n \neq 4$. For example, Milnor (1958) gave an affirmative answer for $\mathbb{S}^7$.

1.2 Lecture 2

Proposition 1.2.1. *If M admits a smooth structure, then M admits uncountably many smooth structures.*

Remark 1.2.2.

1. There exists a 10-dimensional topological manifold that admits no smooth structure (Kevaire 1961).

2. Any 2- or 3-dimensional manifold admits a smooth structure.

Let us now look at several examples of smooth structures on topological manifolds.

Example 1.2.3.

(1) Any (real) vector space V where of dimension $n < \infty$ has a canonical smooth structure as follows. Endow V with any norm, since all norms on a finite-dimensional space are equivalent and hence generate the same topology. Pick any basis $B := (b_1, \ldots, b_n)$ of V. Define the isomorphism $T : V \to \mathbb{R}^n$ by $b_i \mapsto e_i$ where e_i denotes the i-th standard basis vector. This is also a diffeomorphism, implying that V is a topological manifold and that (V, T) is an atlas on V. If B' is any other basis of V and T' the corresponding isomorphism, then the transition map $T' \circ T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism, hence a diffeomorphism. By Proposition 1.1.6(2), it follows that any two bases determine the same smooth structure on V.

(2) The restriction of a smooth structure on a smooth manifold M to an open subset $U \subset M$ yields a smooth structure on U, which is called an *open submanifold.*

Note that the general linear group $\mathrm{GL}(n, \mathbb{F})$ is an open subset of $M(n, \mathbb{F})$, which is an n^2-manifold by Example 1.2.3(1). Indeed, $\mathrm{GL}(n, \mathbb{F}) = \det^{-1}(\mathbb{F}^{-1})$, the preimage of an open set in $\mathbb{F}$. By Example 1.2.3(2), $\mathrm{GL}(n, \mathbb{F})$ is an open submanifold.

Example 1.2.4.

(1) Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^m$ be continuous. Let $\Gamma(F)$ denote the graph of F and $\pi_1 \restriction_{\Gamma(F)}$ be the restriction of the projection map $(x, y) \mapsto x$. This is a homeomorphism $\Gamma(F) \xrightarrow{\cong} U$ with inverse given by $x \mapsto (x, f(x))$. Hence $(\Gamma(F), \pi_1 \restriction_\Gamma (F))$ is a smooth atlas on $\Gamma(F)$.

(2) For each $i \in \{1, 2, \ldots, n+1\}$, let $U_i^+ := \left\{\vec{x} \in \mathbb{R}^{n+1} : x_i > 0\right\}$. Define U_i^- similarly, so that the $U_i^\pm$ cover the n-sphere

$$\mathbb{S}^n := \left\{\vec{x} \in \mathbb{R}^{n+1} : |\vec{x}| = 1\right\}.$$

Define the map $f : B_1(0) \subset \mathbb{R}^n \to \mathbb{R}$ by $f(\vec{u}) = \sqrt{1 - |\vec{u}|^2}$. Define $x_i : B_1(0) \to \mathbb{R}$ by $f(x_1, \ldots, \hat{x}_i, \ldots x_n)$. Then $\Gamma(x_i) = U_i^+ \cap \mathbb{S}^n$, and $\Gamma(-x_i) = U_i^- \cap \mathbb{S}^n$. Thanks to (1), these graphs with their corresponding projections form a smooth structure on $\mathbb{S}^n$.

(3) Let $f : \underset{\text{open}}{U} \subset \mathbb{R}^m \to \mathbb{R}$ be smooth. For each $c \in \mathbb{R}$, let $M_c := f^{-1}(c)$. Assume that the total derivative $\nabla f(a)$ is nonzero for each $a \in M_c$. Then $f_{x_i}(a) \neq 0$ for some $1 \leq i \leq m$. By the implicit function theorem, there is some smooth function $F : \mathbb{R}^{m-1} \to \mathbb{R}$ given by $x_i = F(x_1, \ldots, \hat{x}_i, \ldots, x_m)$ on some neighborhood $U_a \subset \mathbb{R}^m$ of a such that $f^{-1}(c) \cap U_a$ equals the graph of F. This means that the open sets $f^{-1}(c) \cap U_a$ together with their graph coordinates define a smooth atlas on M_c.

Example 1.2.5 (Real projective space). For each $i \in \{1, 2, \ldots, n+1\}$, let $\tilde{U_i} := \{\vec{x} \in \mathbb{R}^{n+1} : x_i \neq 0\}$. Let $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ be the quotient map and $U_i := \pi\left(\tilde{U_i}\right)$. Since $\tilde{U_i}$ is saturated and open, we know that $\pi \restriction_{\tilde{U_i}}$ is a quotient map.[1] Define $f_i : U_i \to \mathbb{R}^n$ by

$$[x_1, \ldots, x_{n+1}] \mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{x^{i-1}}{x_i}, \frac{x^{i+1}}{x_i}, \ldots \frac{x_{n+1}}{x_i}\right),$$

whose inverse if given by $(x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots x_n]$. Since $f_i \circ \pi$ is continuous, so is f_i.[2] Hence f_i is a homeomorphism. It's easy to check that each transition $f_i \circ f_j^{-1}$ is smooth. Thus, (U_i, f_i) defines a smooth atlas on $\mathbb{RP}^n$.

Exercise 1.2.6. *Show that $\mathbb{RP}^n$ is second countable and Hausdorff.*

Proof. Recall that $\mathbb{S}^n/\sim \cong \mathbb{RP}^n$ where $x \sim y$ if $y = -x$. Thus it suffices to show these properties are true of $P^n := \mathbb{S}^n/\sim$.

To this end, let $\mathcal{B} := \{V_n\}$ denote the usual countable basis of $\mathbb{S}^n$ inherited from $\mathbb{R}^{n+1}$. If $p \in U \subset P^n$ is open, then $\pi^{-1}(U)$ is a neighborhood of $\pi^{-1}(p)$, which equals $\{a, -a\}$ for some point a on the sphere. There exist $q \in \mathbb{Q}$ and $r \in \mathbb{Q}^{n+1}$ such that $\mathcal{B} \ni B_q(r) \cap \mathbb{S}^n \ni a$. In this case, $\mathcal{B} \ni B_q(-r) \cap \mathbb{S}^n \ni -a$. Note that the union of these two balls is contained in $\pi^{-1}(U)$ and is saturated, hence is mapped to a neighborhood $N \subset U$ of p. Thus $\{\pi(V_n)\}_{n \in \mathbb{N}}$ is a countable basis of P^n.

Proving that $\mathbb{RP}^n$ is Hausdorff is quite similar. □

Example 1.2.7 (Product manifold). Let $M_1 \times \cdots \times M_k$ be a product of n_i-dimensional smooth manifolds. Then this is a smooth manifold of dimension $n_1 + \cdots + n_k$.

Lemma 1.2.8 (Smooth manifold construction). *Let M be a set and let $\{U_\alpha\}$ be a collection of subsets equipped with injections $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ such that*

[1] Munkres, James. *Topology.* Theorem 22.1.
[2] Ibid. Theorem 22.2.

(i) countably many U_α cover M,

(ii) each $\varphi_\alpha(U_\alpha)$ is open,

(iii) any set of the form $\varphi_\alpha(U_\alpha \cap U_\beta)$ or $\varphi_\beta(U_\alpha \cap U_\beta)$ is open,

(iv) if $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\alpha \circ \varphi_\beta^{-1}$ is smooth, and

(v) if $p, q \in M$ with $p \neq q$, then either both are in U_α for some α or they can be separated by sets in $\{U_\alpha\}$.

Then M has a unique smooth manifold structure with $(U_\alpha, \varphi_\alpha)$ as charts.

Notation. The expression M^n means that M is an n-dimensional manifold.

Definition 1.2.9. If $f : M^n \to \mathbb{R}$ is a function with M smooth, we say that f is *differentiable at* p if there is some chart $(U_\alpha, \varphi_\alpha)$ such that the coordinate representation $f \circ \varphi_\alpha^{-1} : \varphi(U_\alpha) \to \mathbb{R}$ is differentiable at p.

We must ensure that Definition 1.2.9 is coordinate-independent.

Lemma 1.2.10. *If $f \circ \varphi^{-1}$ is differentiable at $\varphi(p)$ and $\psi : V \to \mathbb{R}^n$ is another coordinate neighborhood of $p \in M^n$, then $f \circ \psi^{-1} : \varphi(V) \to \mathbb{R}$ is also differentiable at $\varphi(p)$.*

Proof. This holds because

$$\begin{tikzcd}
U \cap V \arrow[r, "f"] \arrow[d, "\varphi"'] \arrow[dr, "\psi"] & \mathbb{R} \\
\varphi(U \cap V) \arrow[ur, "f \circ \varphi^{-1}"] & \psi(U \cap V) \arrow[l, "\varphi \circ \psi^{-1}"] \arrow[u, "f \circ \psi^{-1}"']
\end{tikzcd}$$

commutes. □

2 Smooth maps

2.1 Lecture 3

Definition 2.1.1. Let M^n and N^k be smooth manifolds. We say that $F : M \to N$ is *smooth at* $p \in M$ if there are charts $(V, \varphi) \ni p$ and $(V', \psi) \ni F(p)$ with $F(V) \subset V'$ such that the coordinate representation $\psi \circ F \circ \varphi^{-1}$ is smooth.

$$\begin{tikzcd}
V \arrow[r, "F"] \arrow[d, "\varphi"'] & V' \arrow[d, "\psi"] \\
\varphi(V) \arrow[r, "\psi \circ F \circ \varphi^{-1}"'] & \psi(V')
\end{tikzcd}$$

This definition is independent of coordinates. Indeed, if $(U, \bar{\varphi})$ and $\left(U', \bar{\psi}\right)$ are other charts around p and $F(p)$, respectively, then

$$\begin{aligned}
\bar{\psi} \circ F \circ \varphi^{-1} &= \left(\bar{\psi} \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \varphi^{-1}\right) \\
\psi \circ F \circ \bar{\varphi}^{-1} &= \left(\psi \circ F \circ \varphi^{-1}\right) \circ \left(\varphi \circ \bar{\varphi}^{-1}\right),
\end{aligned}$$

which are smooth as composites of smooth maps.

Lemma 2.1.2. *Smoothness implies continuity.*

Proof. Using notation as in Definition 2.1.1, we see that for each $p \in M$, there is a neighborhood V of p such that $F \restriction_V = \psi^{-1} \circ \psi \circ F \circ \varphi^{-1} \circ \varphi$ is a composite of continuous maps (as we know smoothness implies continuity for maps between Euclidean spaces) and thus itself continuous. We can glue these restrictions together to conclude that F is continuous. □

Note 2.1.3. Being smooth is a local property of maps.

1. Given $F : M \to N$, if every $p \in M$ has a neighborhood U_p so that $F \restriction_{U_p}$ is smooth, then F is smooth.
2. Conversely, the restriction of any smooth map to an open subset is smooth.

Example 2.1.4. The natural projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is smooth. Let $v \in \left(\mathbb{R}^{n+1} \setminus \{0\}, \mathrm{id}\right)$. Let $(U_i, \varphi_i) \in A_n$ be a neighborhood of $\pi(p)$. Since π is continuous, $S \coloneqq \pi^{-1}(U_i) \cap (\mathbb{R}^{n+1} \setminus \{0\})$ is a neighborhood of v. Further, $\varphi_i \circ \pi \circ \mathrm{id} : S \to \varphi_i(U_i)$ is given by $x \mapsto \frac{(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})}{x_i}$, which is smooth.

Definition 2.1.5. A smooth map with a smooth inverse is a *diffeomorphism*.

This defines an equivalence relation $\approx$ between smooth manifolds. Thanks to Lemma 2.1.2, any diffeomorphism is a homeomorphism, which gives us the following result.

Theorem 2.1.6. *If $M^n \approx N^k$, then $n = k$.*

Example 2.1.7.

1. $(\mathbb{R}, \mathrm{id}) \approx \left(\mathbb{R}, x \mapsto x^{\frac{1}{3}}\right)$ via the mapping $x \mapsto x^3$.
2. $F : \mathbb{B}^n \to \mathbb{R}^n$ given by $F(x) = \frac{x}{\sqrt{1-|x|^2}}$ is a diffeomorphism with inverse $G(y) = \frac{y}{\sqrt{1+|y|^2}}$.
3. $\mathbb{S}^n/\sim \, \approx \mathbb{RP}^n$.
4. If M is a smooth manifold and (U, φ) is a chart, then $\varphi : U \to \varphi(U)$ is a diffeomorphism.

At this point, we want to develop tools with which we can glue together already locally defined smooth functions $\underset{\text{covering } M}{U_\alpha} \to \mathbb{R}$ to obtain a globally defined smooth function $M \to \mathbb{R}$.

Definition 2.1.8. If M is any space and $f : M \to \mathbb{R}^n$ is continuous, then the *support of f* is

$$\operatorname{supp} f \coloneqq \operatorname{cl}\left(\{x \in M : f(x) \neq 0\}\right).$$

Lemma 2.1.9. *Given any $0 < r_1 < r_2$, there is some smooth function $H : \mathbb{R}^n \to \mathbb{R}$ such that*

- *$H = 1$ on $\bar{B}_{r_1}(0)$,*
- *$0 < H < 1$ on $B_{r_2}(0) \setminus \bar{B}_{r_1}(0)$, and*
- *$H = 1$ elsewhere.*

Proof. We construct such an H. First recall that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

is smooth. Now define $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = \frac{f(r_2 - t)}{f(r_2 - t) + f(t - r_1)}$. Finally, define $H : \mathbb{R}^n \to \mathbb{R}$ by $H(x) = h(|x|)$. □

2.2 Lecture 4

Definition 2.2.1. Let $\mathcal{U}$ and $\mathcal{V}$ be open covers of a space X.

1. $\mathcal{V}$ is a *refinement* of $\mathcal{U}$ if for every $V \in \mathcal{V}$, there is some $U \in \mathcal{U}$ such that $V \subset U$.
2. $\mathcal{U}$ is *locally finite* if each $x \in X$ has some neighborhood that intersects only finitely many $U \in \mathcal{U}$.
3. X is *paracompact* if every open cover of X admits a locally finite refinement.

We are now ready to define our main tool for patching together local functions to obtain a global one.

Definition 2.2.2. Let M be a space and $\mathcal{X} \coloneqq (X_\alpha)_{\alpha \in A}$ be an open cover. A *partition of unity subordinate to* $\mathcal{X}$ is a family $(\psi_\alpha)_{\alpha \in A}$ of continuous functions $\psi_\alpha : M \to \mathbb{R}$ with the following properties.

(a) $0 \leq \psi_\alpha(x) \leq 1$ for each α and x.

(b) $\operatorname{supp} \psi_\alpha \subset X_\alpha$ for each α.

(c) The family $(\operatorname{supp} \psi_\alpha)$ is locally finite, in the sense that every point $p \in M$ has a neighborhood V_p such that $V_p \cap \operatorname{supp} \psi_\alpha \neq \emptyset$ for at most finitely many α. In particular, M is paracompact.

(d) $\sum_{\alpha \in A} \psi_\alpha(x) \equiv \sup \left\{ \sum_{\alpha \in F} \psi(x) : \underset{\text{finite}}{F} \subset A \right\} = 1$ for each x.

Lemma 2.2.3. *Every topological manifold M is paracompact.*

Before proving this, let us recall that a subspace is *precompact* if its closure is compact.

Proof. Since M has a countable atlas, it has a countable basis $\{B_n\}$ of precompact coordinate balls. (The continuous image of a precompact set into a Hausdorff space is also precompact.)

Step 1: By induction, we can build a countable covering $\{U_n\}$ of precompact sets such that $\operatorname{cl}(U_{n-1}) \subset U_n$ and $B_n \subset U_n$ for each n.

Step 2: We build a countable locally finite open cover $\{V_n\}$. Let

$$V_n = \begin{cases} \operatorname{cl}(U_n) \setminus U_{n-2} & n > 2 \\ V_n = U_n & \text{otherwise} \end{cases}.$$

Note that every V_n intersects only finitely many other V_j, hence $\{V_n\}$ is locally finite.

Step 3: Let $\{X_\alpha\}$ be any open cover. For any $p \in M$, there is some α with $p \in X_\alpha$ as well as some neighborhood W_p that intersects V_j for only finitely many $j \in \mathbb{N}$. Set $\widetilde{W}_p = W_p \cap X_\alpha$. Then the $\widetilde{W}_p$ cover M. Since each V_j is precompact by construction, we know that V_j has a finite subcover $\widetilde{W}_{p_{j_{k_1}}}, \ldots, \widetilde{W}_{p_{j_{k_j}}}$. Then

$$V_j = \left(V_j \cap \widetilde{W}_{p_{j_{k_1}}}\right) \cup \cdots \cup \left(V_j \cap \widetilde{W}_{p_{j_{k_j}}}\right),$$

and thus $\left\{\left(V_j \cap \widetilde{W}_{p_{j_{k_1}}}\right), \ldots, \left(V_j \cap \widetilde{W}_{p_{j_{k_j}}}\right)\right\}_{j \in \mathbb{N}}$ is a locally finite refinement of $\{X_\alpha\}$, as desired. □

Remark 2.2.4. If X is connected, then X is paracompact if and only if it is second-countable.

Theorem 2.2.5 (Existence of partition of unity). *If M is a smooth manifold, then any open cover $\mathcal{X} \coloneqq \{X_\alpha\}_{\alpha \in A}$ of M admits a partition of unity.*

Proof. For each $\alpha \in A$, we can find a countable basis $\mathcal{C}_\alpha$ of precompact coordinate balls centered at 0 for X_α. Then $\mathcal{C} \coloneqq \bigcup_\alpha \mathcal{C}_\alpha$ is a basis for M. Since M is paracompact, $\mathcal{X}$ admits a locally finite refinement $\{C_i\}_{i \in \mathbb{I}}$ consisting of elements of $\mathcal{C}$. Note that the cover $\{\mathrm{cl}(B_i)\}$ is also locally finite. There are coordinate balls $C_i' \subset X_{\alpha_i}$ such that $C_i' \supset \mathrm{cl}(C_i)$. For each $i \in \mathbb{I}$, let $\varphi_i : C_i' \to \mathbb{R}^n$ be a smooth coordinate map so that $\varphi_i(C_i') \supset \varphi(C_i)$ and $\varphi(\mathrm{cl}(C_i)) = \mathrm{cl}(\varphi(C_i))$. Define $f_i : M \to \mathbb{R}$ by

$$f_i(x) = \begin{cases} H_i \circ \varphi_i & x \in C_i' \\ 0 & x \in M \setminus \mathrm{cl}(C_i) \end{cases}$$

where $H_i : \mathbb{R}^n \to \mathbb{R}$ is as in Lemma 2.1.9: a smooth function that is positive on $\varphi_i(C_i)$ and zero elsewhere. Note that f_i is well-defined because $f_i = 0$ on $C_i' \setminus \mathrm{cl}(C_i)$. Also, it is smooth by the point-set gluing lemma for open sets.

Define $f : M \to \mathbb{R}$ by $f(x) = \sum_i f_i(x)$, which is a finite sum and hence well-defined. We see that f is a smooth function and that $f(x) > 0$ for each $x \in M$. Then $g_i(x) \equiv \frac{f_i(x)}{f(x)}$ defines a smooth function $M \to \mathbb{R}$ for each i, so that $\sum_i g_i(x) = 1$ and $0 \leq g_i(x) \leq 1$ for each $x \in M$. Note that $\mathrm{supp}(g_i) = \mathrm{cl}(C_i)$.

For each $\alpha \in A$, define $\psi_\alpha : M \to \mathbb{R}$ by

$$\psi_\alpha(x) = \sum_{\substack{i \\ \alpha_i = \alpha}} g_i(x).$$

Interpret this as the zero function when there are no i such that $\alpha_i = \alpha$. Note that each ψ_α is smooth as a finite sum of smooth functions and satisfies $0 \leq \psi_\alpha \leq 1$. Moreover, we have that

$$\mathrm{supp}(\psi_\alpha) = \mathrm{cl}\left(\bigcup_{\substack{i \\ \alpha_i = \alpha}} C_i \right) = \bigcup_{\substack{i \\ \alpha_i = \alpha}} \mathrm{cl}(C_i).$$

Since $\{\mathrm{cl}(C_i)\}$ is locally finite, so is $\{\mathrm{supp}(\psi_\alpha)\}_{\alpha \in A}$. Finally, the fact that $\alpha_i \in A$ implies that

$$\sum_\alpha \psi_\alpha(x) = \sum_i g_i(x) = 1$$

for each $x \in M$. Therefore, we may take $\{\psi_\alpha\}$ as our desired partition of unity. □

Corollary 2.2.6 (Bump function). *If $A \subset U \subset M$ with A closed and U open in M, then there is a smooth function $f : M \to \mathbb{R}$ such that $f(x) = 1$ for each $x \in A$ and $f(x) = 0$ outside a neighborhood of A.*

Proof. Since $\{U, M \setminus A\}$ is an open cover of M, there is a partition of unity φ_1, φ_2 such that $\mathrm{supp}\, \varphi_1 \subset U$, $\mathrm{supp}\, \varphi_2 \subset M \setminus A$, and $\varphi_1 + \varphi_2 = 1$. Hence $\varphi_1 \restriction_A = 1 - 0 = 1$, and $\varphi_1 \restriction_{M \setminus U} = 0$. □

2.3 Lecture 5

Corollary 2.3.1 (Whitney). *Let M be a smooth manifold and $K \subset M$ be closed. Then there exists a non-negative smooth function $f : M \to \mathbb{R}$ such that $f^{-1}(0) = K$.*

This means that closed subsets of smooth manifolds are completely characterized as the 0-level sets of smooth maps. Being the 0-level set of analytic maps, such as polynomials, is much more special. Any object with such a property is called an *analytic submanifold* and is studied in algebraic geometry.

Proof. First assume that $M = \mathbb{R}^n$. We have that $M \setminus K$ is open, which is thus the union of countably many balls $B_{r_i}(x_i)$ with $r_i \leq 1$. Construct, as in Lemma 2.1.9, a smooth bump function $h : \mathbb{R}^n \to \mathbb{R}$ such that $h(x) = 1$ on $\bar{B}_{\frac{1}{2}}(0)$ and h is supported in $B_1(0)$. By our construction of h, we can verify that for each $i \in \mathbb{N}$, there is some $C_i \geq 1$ that bounds any of the partials of h up through order i.

Define $f : \mathbb{R}^n \to \mathbb{R}$ by

$$\sum_{i=1}^{\infty} \frac{r_i^i}{2^i C_i} h\left(\frac{x - x_i}{r_i}\right).$$

Each i-th term is bounded by $\frac{1}{2^i}$. Thanks to the Weierstrass M-test, f is well-defined and continuous. Since h is zero outside $B_1(0)$, we see that $f^{-1}(0) = K$.

To see that f is smooth, assume by induction that f is C^{k-1} for a given $k \geq 1$. By the chain rule and induction, we can write any k-th partial D_k of the i-th term of the series defining f as $\frac{(r_i)^{i-k}}{2^i C_i} D_k h(\frac{x - x_i}{r_i})$. As h is smooth, this expression is C^1. And since $r_i \leq 1$ and C_i bounds all partials up to order i, it is eventually bounded by $\frac{1}{2^i}$. Hence the series of these expressions converges uniformly to a continuous function. By Theorem C.31 (Lee), it follows that $D_k f$ exists and is continuous, thereby completing our induction.

Now, assume that M is arbitrary. Find a cover (B_α) of smooth coordinate balls for M. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to this cover. Note that each B_α is diffeomorphic to $\mathbb{R}^n$. Since the property of admitting a non-negative smooth function $f : M \to \mathbb{R}$ with $f^{-1}(0) = K$ can be stated in the language of smooth manifolds, it is invariant under diffeomorphism. Thus, there is some non-negative smooth function $f_\alpha : B_\alpha \to \mathbb{R}$ where $f^{-1}(0) = K \cap B_\alpha$ for each α. Then it's straightforward to check that $g \equiv \sum_\alpha \varphi_\alpha f_\alpha$ is as desired. $\square$

Corollary 2.3.2. *Let M be a smooth manifold and $K \subset M$ be closed. Let $c > 0$. Then there exists a non-negative smooth function $f : M \to \mathbb{R}$ such that $f^{-1}(c) = K$.*

Exercise 2.3.3. *Prove that the restriction of a smooth map on $\mathbb{R}^{n+1}$ to $\mathbb{S}^n$ is smooth.*

3 Tangent vectors

3.1 Lecture 6

We can view the tangent space $T_p\mathbb{S}^n$ of $\mathbb{S}^n$ at a point p as all of the directions from p with respect to which you can find the rate of change of a smooth map f provided that you're only allowed to roam through $\mathbb{S}^n$. We want to generalize our notion of a tangent space to arbitrary manifolds in order to do first-order calculus on them.

Notation. We shall denote the space of smooth functions $M \to \mathbb{R}$ by $C^\infty(M)$.

Definition 3.1.1. Given $a \in \mathbb{R}^n$, a map $\omega : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at a* if it

(i) is linear over $\mathbb{R}$ and

(ii) satisfies the *Leibniz rule*:

$$\omega(fg) = f(a)\omega(g) + g(a)\omega(f)$$

for any $f, g \in C^\infty(\mathbb{R}^n)$.

Let $T_a\mathbb{R}^n$ denote the vector space of derivations at a.

Note 3.1.2. If f is constant, then $\omega f = 0$ for any derivation ω.

Example 3.1.3. For any $u \in \mathbb{R}^n$, recall that the directional derivative of $f \in C^\infty(\mathbb{R}^n)$ in the direction u at a is

$$D_u f(a) \equiv \lim_{h\to 0} \frac{1}{h}(f(a+hu) - f(a)) = \frac{d}{dh}\Big|_{h=0} f(a+hu).$$

Then this is a derivation of f at a.

Notation. For any $a \in \mathbb{R}^n$, let $\mathbb{R}^n_a$ denote the (real) vector space $\{(a, v) \mid v \in \mathbb{R}^n\}$.

Theorem 3.1.4. *For each $a \in \mathbb{R}^n$, define $L_a : \mathbb{R}^n_a \to T_a\mathbb{R}^n$ by $v_a \mapsto D_v\big|_a$. This is an isomorphism.*

Proof. It is clear that L_a is linear. It remains to show that it is both injective and surjective.

Suppose that $u, v \in \mathbb{R}^n_a$ and $L_a(u) = L_a(v)$. Then by linearity $L_a(u - v) = 0$, yielding

$$\frac{d}{dt}\Big|_{t=0} f(a + t(u - v)) = 0$$

for any smooth function f. But if $u - v \neq 0$, then this says that for any f, the directional derivative of f at a in the direction of a certain nonzero vector vanishes, which is clearly false. Hence $u = v$, and L_a is injective.

Next, suppose that $\omega \in T_a\mathbb{R}^n$ and consider the coordinate projection $x^i : \mathbb{R}^n \to \mathbb{R}$ for each $i = 1, \ldots, n$. Set $v_i = \omega(x^i)$ and write $v = v_i e_i$. We claim that $L_a(v) = D_v\big|_a = \omega$. By Taylor's theorem, any $f \in C^\infty(\mathbb{R}^n)$ has an expansion

$$f(x) = f(a) + \sum_{i=1}^n f_{x_i}(a)(x_i - a_i) + c\sum_{i,j=1}^n (x_i - a_i)(x_j - a_j)\int_0^1 (1-t)\frac{\partial^2 f}{\partial x_i \partial x_j}(a + t(x-a))\,dt$$

for some $c > 0$. Each term of the second sum is the product of two smooth functions vanishing at a. We can apply the product rule along with linearity of ω to conclude that

$$\begin{aligned}
\omega f &= \omega\left(\sum_{i=1}^n f_{x_i}(a)(x_i - a_i)\right) \\
&= \sum_{i=1}^n \omega(f_{x_i}(a)(x_i - a_i)) \\
&= \sum_{i=1}^n f_{x_i}(a)(\omega(x_i) - \omega(a_i)) \\
&= \sum_{i=1}^n f_{x_i}(a)v_i \\
&= D_v\big|_a f.
\end{aligned}$$

□

Corollary 3.1.5. *We have* $\dim(T_a\mathbb{R}^n) = n$*, and the partial derivatives* $\left\{ \frac{\partial}{\partial x_i}\big|_a \right\}_{1\le i\le n}$ *form a basis of* $T_a\mathbb{R}^n$*.*

Definition 3.1.6. Let M be a smooth manifold and let $p \in M$.

1. An $\mathbb{R}$-linear map $v : C^\infty(M) \to \mathbb{R}$ is called a *derivation at* p if

$$v(fg) = f(p)v(g) + v(f)g(p)$$

   for any f and g.

2. The tangent space of M at p is the vector space

$$T_pM \equiv \{\omega : C^\infty(M) \to \mathbb{R} : \omega \text{ is a derivation of } M \text{ at } p\}.$$

   Any element of this space is called a *tangent vector*.

Definition 3.1.7 (Differential of a smooth map). Given smooth manifolds M and N, a smooth map $F : M \to N$, and $p \in M$, we define the *differential of* F *at* p as the map $dF_p : T_pM \to T_{F(p)}N$ given by

$$dF_p(v)(f) = v(f \circ F).$$

Terminology. We call $dF_p(v)$ the *pushforward of* v *by* dF.

Proposition 3.1.8. *Let* M*,* N*, and* P *be smooth manifolds,* $F : M \to N$ *and* $G : N \to P$ *be smooth maps, and* $p \in M$*.*

1. $dF_p : T_pM \to T_{F(p)}N$ *is linear.*
2. $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G(F(p))}P$*.*
3. $d(\mathrm{id}_M)_p = \mathrm{id} : T_pM \to T_pM$*.*
4. *If* F *is a diffeomorphism, then* dF_p *is an isomorphism with inverse* $d(F^{-1})_{F(p)}$*.*

Aside. This shows that mapping (M,p) to T_pM and $F : (M,p) \to (N, F(p))$ to dF_p defines a functor from $\mathbf{Diff}_*$ to $\mathbf{Vec}_\mathbb{R}$, known as the *tangent space functor*.

Lemma 3.1.9. *Let* $v \in T_pM$ *and* $f, g \in C^\infty(M)$*. Then if* f *and* g *agree on a neighborhood* N_p *of* p*, then* $vg = vf$*.*

Proof. Set $h = f - g$, so that h vanishes on N_p. We can find a smooth bump function $\varphi : M \to \mathbb{R}$ such that $\varphi \equiv 1$ on $\mathrm{supp}(h)$ and $\mathrm{supp}(\varphi) \subset M \setminus \{p\}$. Then $\varphi h(x) = h(x)$ for any $x \in M$. Since both φ and h vanish at p, it follows that $vf - vg = vh = v(\varphi h) = 0$. □

Proposition 3.1.10. *If* M *is an* n*-dimensional smooth manifold, then* $\dim(T_pM) = n$ *for every* $p \in M$*.*

In particular, we identify the standard basis $\{e_1, \ldots, e_n\}$ for $\mathbb{R}^n$ by $e_i \leftrightarrow \left(0, \ldots, 0, \frac{\partial}{\partial x_i}\big|_p, 0 \ldots, 0\right)$.

3.2 Lecture 7

Given a point $p \in M$, find a chart $(U, \varphi) \ni p$. Then $d\varphi_p : T_pM \cong T_pU \to T_{\varphi(p)}\varphi(U) \cong T_p\mathbb{R}^n$ is an isomorphism. This choice of chart yields a natural choice of basis for T_pM:

$$\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{1 \le i \le n}$$

where

$$\frac{\partial}{\partial x_i}\Big|_p \coloneqq (d\varphi_p)^{-1}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right) = \left(d\varphi^{-1}\right)_{\varphi(p)}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(p)}\right). \tag{$*$}$$

Let $F : M \to N$ be smooth with $M \subset \mathbb{R}^n$ and $N \subset \mathbb{R}^m$ open. Then by the chain rule we get

$$\begin{aligned} dF_p\left(\frac{\partial}{\partial x_i}\Big|_p\right) f &= \frac{\partial}{\partial x_i}\Big|_p (f \circ F) \\ &= \frac{\partial}{\partial x_i}\Big|_p (f(F_1, \dots, F_m)) \\ &= \sum_{j=1}^m \frac{\partial f}{\partial F_j}(F(p)) \frac{\partial F_j}{\partial x_i}(p) \\ &= \sum_{j=1}^m \frac{\partial F_j}{\partial x_i}(p) \left(\frac{\partial}{\partial y_j}\Big|_{F(p)}\right) f. \end{aligned}$$

Therefore, dF_p can be represented by the familiar $m \times n$ Jacobian matrix of F at p,

$$DF(p) \coloneqq \begin{bmatrix} \vdots & & \vdots \\ \frac{\partial F_j}{\partial x_1}(p) & \cdots & \frac{\partial F_j}{\partial x_n}(p) \\ \vdots & & \vdots \end{bmatrix},$$

which acts on $\mathbb{R}^n \cong T_pM$.

Now consider the general case $F : M \to N$ smooth between manifolds. For any $p \in M$, choose charts $(U, \varphi) \ni p$ and $(V, \psi) \ni F(p)$. Then the Euclidean map $\widehat{F} \coloneqq \psi \circ F \circ \varphi^{-1} : \varphi(F^{-1}(V) \cap U) \to \psi(V)$ is smooth. If $\hat{p} \coloneqq \varphi(p)$, it follows from ($*$) that $d\widehat{F}_{\hat{p}}$ is represented by the Jacobian of $\widehat{F}$ at $\hat{p}$. Noting that $F \circ \varphi^{-1} = \psi^{-1} \circ \widehat{F}$, we compute

$$\begin{aligned} dF_p\left(\frac{\partial}{\partial x_i}\Big|_p\right) &= dF_p\left(d(\varphi^{-1})\big|_{\hat{p}}\left(\frac{\partial}{\partial x_i}\Big|_{\hat{p}}\right)\right) \\ &= d(\psi^{-1})\big|_{\widehat{F}(\hat{p})}\left(d\widehat{F}\big|_{\hat{p}}\left(\frac{\partial}{\partial x_i}\Big|_{\hat{p}}\right)\right) \\ &= d(\psi^{-1})\big|_{\widehat{F}(\hat{p})}\left(\sum_{j=1}^m \frac{\partial \widehat{F}_j}{\partial x_i}(\hat{p}) \frac{\partial}{\partial y_j}\Big|_{\widehat{F}(\hat{p})}\right) \\ &= \sum_{j=1}^m \frac{\partial \widehat{F}_j}{\partial x_i}(\hat{p}) \frac{\partial}{\partial y_j}\Big|_{F(p)}. \end{aligned}$$

Therefore, dF_p can be represented by the Jacobian matrix of $\widehat{F}$ at $\hat{p}$.

Given any two pairs of coordinates for p and $F(p)$, the respective Jacobian matrices are related by the familiar change-of-basis matrix. In particular, they are similar.

Given a smooth manifold M, we define a notion of a smoothly varying tangent space as follows.

Definition 3.2.1. The *tangent bundle of M* is the set

$$TM \equiv \coprod_{p \in M} T_p M$$

endowed with a certain natural topology induced by the projection $\pi : TM \to M, \quad (\varphi, p) \mapsto p$.

Example 3.2.2. As $\mathbb{R}^n_a$ is canonically isomorphic to $\mathbb{R}^n$, we have $T\mathbb{R}^n \cong \coprod_{a \in \mathbb{R}^n} \{a\} \times \mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$.

3.3 Lecture 8

Lemma 3.3.1. *For any smooth n-dimensional manifold M, the tangent bundle TM has a natural topology and smooth structure such that*

- *TM is a $2n$-dimensional smooth manifold and*
- *the projection $\pi : TM \to M$ is smooth.*

Proof. Given a chart (U, φ), define $\tilde{\varphi} : \pi^{-1}(U) \to \mathbb{R}^n$ by

$$v_i \frac{\partial}{\partial x_i}\Big|_p \mapsto \left(x^1(p), \ldots, x^n(p), v_1, \ldots, v_n\right)$$

where $\varphi = \left(x^1, \ldots, x^n\right)$.[3] This is continuous with $\operatorname{Im} \tilde{\varphi} = \varphi(U) \times \mathbb{R}^n$, which is open. Further, $\tilde{\varphi}^{-1}$ is given by $(x_1, \ldots, x_n, v_1, \ldots, v_n) \mapsto v_i \frac{\partial}{\partial x_i}\big|_{\varphi^{-1}(x)}$ on $\varphi(U) \times \mathbb{R}^n$. Take $\left\{\left(\pi^{-1}(U), \tilde{\varphi}\right)\right\}$ to be charts on TM. Given two such charts $\left(\pi^{-1}(U), \tilde{\varphi}\right)$ and $\left(\pi^{-1}(V), \tilde{\psi}\right)$, it's straightforward to check that $\tilde{\psi} \circ \tilde{\varphi}^{-1} : \varphi(U \cap V) \times \mathbb{R}^n \to \psi(U \cap V) \times \mathbb{R}^n$ is smooth.

Next, notice that if we take a countable cover $\{U_i\}$ of M by smooth coordinate domains, then $\left\{\pi^{-1}(U_i)\right\}$ satisfies the conditions of Lemma 1.2.8.

Finally, to see that $\pi : TM \to M$ is smooth, notice that its coordinate representation at every point is given by the projection $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n, \quad (x, v) \mapsto x$. ∎

Terminology. We call the $\tilde{\varphi}\left((f, p)\right)$ the *natural coordinates* on TM.

Given $F : M \to N$ is smooth, define the *global differential* $dF : TM \to TN$ of F by $dF(\varphi, p) = dF_p(\varphi)$.

Proposition 3.3.2. *The global differential $dF : TM \to TN$ is smooth.*

Aside. This shows that mapping M to TM and F to dF defines a functor from **Diff** to itself, known as the *tangent functor*.

Note 3.3.3. If F is a diffeomorphism, then so is dF with $d(F^{-1}) = (df)^{-1}$.

Definition 3.3.4. Given a smooth curve $\gamma : J \to M$ and $t_0 \in J$, the *velocity of γ at t_0* is

$$\gamma'(t_0) \equiv d\gamma \left(\frac{d}{dt}\Big|_{t_0}\right) \in T_{\gamma(t_0)} M.$$

[3] The expression $v_i \frac{\partial}{\partial x_i}\big|_p$ is secretly a summation, in accordance with the *Einstein summation convention*.

Note 3.3.5. Let $(U, \varphi) \ni \gamma(t_0)$ be a chart on M. Then $\gamma'(t_0) = \frac{d\gamma^i}{dt} \frac{\partial}{\partial x_i}\big|_{\gamma(t_0)}$.

Lemma 3.3.6. *Every $v \in T_pM$ is the velocity of some smooth curve $\gamma : J \to M$ at 0 such that $\gamma(0) = p$.*

Proof. Let (U, φ) be a chart centered at p. Write $v = v_i \frac{\partial}{\partial x_i}\big|_p$. For any $\epsilon > 0$ small, define $\gamma : (-\epsilon, \epsilon) \to U$ by $\gamma(t) = \varphi^{-1}(tv_1, \ldots, tv_n)$. Note 3.3.5 implies that $\gamma'(0) = v$. □

Proposition 3.3.7. *Let $v \in T_pM$. Then $dF_p(v) = (F \circ \gamma)'(0)$ for any smooth map $\gamma : J \to M$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$.*

Aside. A *smooth function element on M* is a pair (f, U) with $U \subset M$ open and $f : M \to \mathbb{R}$ smooth. Say that $(f, U) \sim (g, V)$ if $p \in U \cap V$ and $f = g$ on some neighborhood of p. The equivalence class $[f]_p \coloneqq [(f, U)]$ is called the *germ of f at p*. The set of such classes is denoted by $C_p^\infty(M)$. This is an associative algebra over $\mathbb{R}$.

Define a *derivation of $C_p^\infty(M)$* as a linear map $v : C_p^\infty(M) \to \mathbb{R}$ satisfying $v[fg]_p = f(p)v[g]_p + g(p)v[f]_p$. The tangent space $\mathcal{D}_pM$ of such derivations serves as an equivalent (in the sense of isomorphism) definition of the tangent space of M at p.

3.4 Lecture 9

Theorem 3.4.1 (Inverse function). *If $F : M \to N$ is smooth and dF_p is invertible, then there are connected neighborhoods U_0 of p and V_0 of $F(p)$ such that $F \restriction_{U_0} : U_0 \to V_0$ is a diffeomorphism.*

Proof. Notice that M and N have equal dimension (say n) because dF_p is invertible. Choose charts (U, f) centered at p and (V, g) centered at $F(p)$ such that $F(U) \subset V$. Then $\widehat{F} \coloneqq g \circ F \circ f^{-1}$ is smooth map from $f(U) \subset \mathbb{R}^n$ to $g(V) \subset \mathbb{R}^n$ with $\widehat{F}(0) = 0$. Now $d\widehat{F}_0$ is invertible as the composite of three invertible maps. The inverse function theorem for Euclidean space implies that there are open balls $B_r(0)$ and $B_s(0)$ such that $\widehat{F} : B_r(0) \to B_s(0)$ is a diffeomorphism. Thus, we can take $F : f^{-1}(B_r(0)) \to g^{-1}(B_s(0))$ as our desired diffeomorphism . □

Corollary 3.4.2. *If dF_p is nonsingular at each $p \in M$, then F is a local diffeomorphism.*

Proposition 3.4.3.

1. *The finite product of local diffeomorphisms is a local diffeomorphism.*
2. *The composite of two local diffeomorphisms is a local diffeomorphism.*
3. *Any bijective local diffeomorphism is a diffeomorphism.*
4. *A map F is a local diffeomorphism if and only if each point in $\mathrm{dom}(F)$ has a neighborhood where F's coordinate representation is a local diffeomorphism.*

Definition 3.4.4. The *rank of a smooth map F at a point p* is the rank of dF_p. If the rank of F is the same at each point, then we say *F has constant rank*.

Theorem 3.4.5 (Constant rank). *Let $F : M^m \to N^n$ be smooth with constant rank $r \leq m, n$. Then for each $p \in M$, there are charts (U, f) centered at p and (V, g) centered at $F(p)$ such that $F(U) \subset V$ and the coordinate representation of F is given by*

$$\widehat{F}(x_1, \ldots, x_r, x_{r+1}, \ldots x_m) = (x_1, \ldots, x_r, 0, \ldots, 0) \,.$$

Before proving this, we should mention a couple of things:

- If $m = n = r$, then this follows immediately from the inverse function theorem.
- The global condition on the rank of F cannot be weakened, as the space of $n \times m$ matrices of rank r need *not* be open. For example, consider the map $A(t) \equiv \begin{bmatrix} 1 & t \\ 1 & 1 \end{bmatrix}$, which has rank 2 when $t \neq 1$ and rank 1 otherwise.

Proof. Since our statement is local, we may assume that $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ are open subsets. Since $DF(p)$ has rank r, it has some invertible $r \times r$ sub-matrix, which we may assume is the upper left sub-matrix $\left(\frac{\partial F^i}{\partial x^j}\right)_{i,j \in [r]}$. Write $(x, y) = \left(x^1, \ldots, x^r, y^1, \ldots, y^{m-r}\right)$ and $(v, w) = \left(v^1, \ldots, v^r, w^1, \ldots, w^{n-r}\right)$ for the standard coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. By applying suitable translations, we may assume that $p = (0, 0)$ and $F(p) = (0, 0)$. We have $F(x, y) = (Q(x, y), R(x, y))$ for some smooth map $Q : M \to \mathbb{R}^r$ and $R : M \to \mathbb{R}^{n-r}$. Then the Jacobian matrix $\left(\frac{\partial Q^i}{\partial x^j}\right)$ is invertible at $(0, 0)$ by hypothesis.

Define $f : M \to \mathbb{R}^m$ by $(x, y) \mapsto (Q(x, y), y)$. Define the *Kronecker delta* symbol δ_i^j by

$$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Then

$$D[f]\,(0, 0) \begin{bmatrix} \frac{\partial Q^i}{\partial x^j}\,(0, 0) & \frac{\partial Q^i}{\partial y^j}\,(0, 0) \\ 0 & \delta_j^i \end{bmatrix}.$$

Since

$$\det(D[f]\,(0, 0)) = \det\left(\frac{\partial Q^i}{\partial x^j}\,(0, 0)\right) \cdot \det(\delta_j^i) = \det\left(\frac{\partial Q^i}{\partial x^j}\,(0, 0)\right) \neq 0,$$

it follows that $D[f]$ is invertible at $(0, 0)$.

Thus, we can apply the inverse function theorem to get a connected open set $U_0 \ni (0, 0)$ and an open cube $\widetilde{U}_0 \ni f\,(0, 0) = (0, 0)$ such that $f : U_0 \to \widetilde{U}_0$ is a diffeomorphism. Let $f^{-1}(x, y) = (A(x, y), B(x, y))$. Then $(x, y) = f(A(x, y), B(x, y)) = (Q(A(x, y), B(x, y)), B(x, y))$, so that $y = B(x, y)$. Hence

$$f^{-1}(x, y) = (A(x, y), y)\,.$$

Additionally, $Q(A(x, y), y) = x$ since $f \circ f^{-1} = \mathrm{id}_{\widetilde{U}_0}$. If $\widetilde{R} : \widetilde{U}_0 \to \mathbb{R}^{n-r}$ is defined by $(x, y) \mapsto R(A(x, y), y)$, then

$$F \circ f^{-1}(x, y) = \left(x, \widetilde{R}(x, y)\right).$$

Therefore,

$$D[F \circ f^{-1}](x, y) = \begin{bmatrix} \delta_j^i & 0 \\ \frac{\partial \widetilde{R}^i}{\partial x^j}(x, y) & \frac{\partial \widetilde{R}^i}{\partial y^j}(x, y) \end{bmatrix}$$

for any $(x, y) \in \widetilde{U}_0$. It's clear that the first r columns of this matrix are linearly independent. But since f^{-1} is a diffeomorphism, it has rank r on $\widetilde{U}_0$. It follows that $\frac{\partial \widetilde{R}^i}{\partial y^j}(x, y) = 0$ for each $(x, y) \in \widetilde{U}_0$. But $\widetilde{U}_0$ was chosen to be an open cube, so that $\widetilde{R}(x, y) = \widetilde{R}(x, 0)$. If $S(x) \coloneqq \widetilde{R}(x, 0)$, then $F \circ f^{-1}(x, y) = (x, S(x))$.

Now, let

$$V_0 = \left\{(v, w) \in N \mid (v, 0) \in \widetilde{U}_0\right\},$$

which is a neighborhood of $(0,0)$ in N. Since $\widetilde{U}_0$ is a cube, we see that $F \circ f^{-1}(\widetilde{U}_0) \subset V_0$. Hence $F(U_0) \subset V_0$. Define $g : V_0 \to \mathbb{R}^n$ by $(v,w) \mapsto (v, w - S(v))$, which is smooth with inverse $g^{-1}(s,t) = (s, t + S(s))$. Then

$$\widehat{F}(x,y) = g \circ F \circ f^{-1}(x,y) = (x, S(x) - S(x)) = (x, 0)\,,$$

as desired. □

3.5 Lecture 10

Definition 3.5.1. Consider a smooth map $F : M \to N$.

1. It is a *(smooth) submersion* if it has constant rank equal to $\dim(N)$.
2. It is a *(smooth) immersion* if it has constant rank equal to $\dim(M)$.

Definition 3.5.2. A *topological embedding* is a continuous map $F : M \to N$ which is a homeomorphism onto $F(M)$.

Example 3.5.3.

1. The map $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by $t \mapsto \left(t^3, 0\right)$ is a smooth topological embedding but not an immersion, since $\gamma'(0) = 0$.
2. The curve $f : (-\pi, \pi) \to \mathbb{R}^2$ defined by $f(t) = (\sin 2t, \sin t)$ is known as a *lemniscate*, describing a figure-eight curve. This is not a topological embedding, because the figure-eight is compact whereas $(-\pi, \pi)$ is not. But it is a smooth immersion as f' never vanishes.

Definition 3.5.4. A map is a *smooth embedding* if it both a topological embedding and a smooth immersion.

Example 3.5.5.

1. There is a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^4$ but not into $\mathbb{R}^3$
2. If $U \subset M$ is open, then the inclusion $U \hookrightarrow M$ is a smooth embedding.

Definition 3.5.6. A manifold $S \subset M$ in the subspace topology is an *embedded submanifold* if it has a smooth structure such that the inclusion $S \hookrightarrow M$ is a smooth embedding.

Note 3.5.7. The image of a smooth embedding is an embedded submanifold.

Terminology. If $S \subset M$ is an embedded submanifold, then $\dim(M) - \dim(S)$ is called the *codimension of S in M*.

Proposition 3.5.8. *Let $U \subset M^m$ be open and $f : U \to N$ be smooth. The graph $\Gamma(f)$ of f is an embedded m-dimensional submanifold of $M \times N$.*

Proof. Define $\gamma_f(x) : U \to M \times N$ by $\gamma_f(x) = (x, f(x))$. It's easy to check this is a smooth embedding. □

Our next notion is a local version of the standard embedding $\mathbb{R}^k \hookrightarrow \mathbb{R}^n$ where $k \le n$ but works for any submanifold.

Definition 3.5.9. We say that a subset $S \subset M$ has the *local k-slice condition* if for each $p \in S$, there is a chart $(U, \varphi) \ni p$ for M such that

$$\varphi(U \cap S) = \underbrace{\left\{x \in \varphi(U) : x^{k+1} = \cdots = x^n = 0\right\}}_{k\text{-slice of } \varphi(U)}, \qquad n \equiv \dim(M).$$

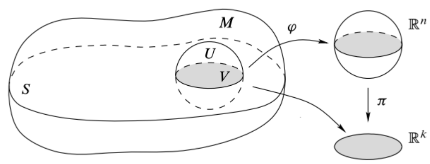

Figure 1: Copied from Lee (102)
k-slice condition with $V \equiv U \cap S$

Theorem 3.5.10. *Let M^n be a smooth manifold. Let $S \subset M$. If S is an embedded manifold with* $\dim(S) = k$*, then S has the local k-slice condition.*

Conversely, if S has the local k-slice condition, then S is a smooth manifold in the subspace topology and has a smooth structure making it an embedded submanifold of dimension k.

Proof.

($\Longrightarrow$)

Let $p \in S$. In particular, the inclusion $i : S \hookrightarrow M$ is a smooth immersion and thus has constant rank k. By the constant rank theorem, we can find charts (U, φ) and (V, ψ) centered at p for S and M, respectively, for which i has coordinate representation

$$\left(x^1, \ldots, x^k\right) \mapsto \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

This means that $i(U)$ is a k-slice for S in V. We have that $U = W \cap S$ for some open set W in M. Let $V' = W \cap V$, which is neighborhood of p in M. Then $(V', \psi \restriction_{V'})$ is a chart on M such that $V' \cap S = i(U)$, so that V' is slice for S in M.

($\Longleftarrow$)

See Theorem 5.8 (Lee).

□

Example 3.5.11. For any n, $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is an embedded hypersurface because it is locally the graph of smooth map and thus has the local n-slice condition.

Theorem 3.5.12. *Let $F : M^m \to N^n$ be smooth with constant rank r. Each level set of F is an embedded submanifold of codimension r in M.*

Proof. Set $k = m - r$. Let $c \in N$ and $p \in F^{-1}(c)$. By the constant rank theorem, there are charts (U, f) centered at p and (V, g) centered at $F(p) = c$ for which F has coordinate representation given by

$$(x_1, \ldots, x_r, x_{r+1}, \ldots, x_m) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0),$$

which must send each point in $f(F^{-1}(c) \cap U)$ to 0. Thus, $f(F^{-1}(c) \cap U)$ equals the k-slice

$$\{x \in \mathbb{R}^m : x_1 = \cdots = x_r = 0\}.$$

By Theorem 3.5.10, S is an embedded submanifold of dimension k. □

3.6 Lecture 11

Question. Can M^n with $n \geq 1$ be homeo-/diffeomorphic to $M \setminus \{p\}$?

Remark 3.6.1. We can generalize Theorem 3.5.12 to maps that are not necessarily of constant rank.

Definition 3.6.2. Let $\varphi : M \to N$ be smooth. We say that $p \in M$ is

- a *regular point of* φ if $d\varphi_p$ is surjective and
- a *critical point of* φ otherwise.

Definition 3.6.3. Let $\varphi : M \to N$ be smooth. We say that $c \in N$ is

- a *regular value of* φ if each point in $\varphi^{-1}(c)$ is regular and
- a *critical value of* φ otherwise.

We say that $S \subset M$ is a *regular level set of* φ if it has the form $\varphi^{-1}(c)$ with c a regular value.

Theorem 3.6.4. *Every regular level set S of a smooth map $F : M^m \to N^n$ is an embedded submanifold of codimension n.*

Proof. Let $S = F^{-1}(c)$. Note that the subspace of full-rank matrices is open due to continuity of the det. As a result, the set U of points $p \in M$ where dF_p is surjective is open in M. Hence $F\restriction_U : U \to N$ is a smooth submersion. In particular, it has constant rank n. Thanks to Theorem 3.5.12, it follows that $F^{-1}(c)$ is an embedded submanifold of U with codimension n, where U itself is an open submanifold of M. □

Example 3.6.5. $\mathbb{S}^n$ is a regular level set of the smooth function $\vec{x} \mapsto |\vec{x}|^2$.

Theorem 3.6.6 (Sard). *If $F : M \to N$ is smooth, then the set of all critical values of F has measure zero in N.*

Proposition 3.6.7. *Suppose M is smooth and $S \subset M$ is embedded. Then for any $f \in C^\infty(S)$, there is some neighborhood U of S in M along with some $\hat{f} \in C^\infty(U)$ such that $\hat{f}\restriction_S = f$.*

Proposition 3.6.8. *The tangent space of a submanifold $S \subset M$ at $p \in S$ is precisely the image of the injective canonical map $di_p : T_pS \to T_pM$ where i denotes inclusion, i.e.,*

$$A := \{\gamma'(0) \in T_pM : \gamma : (-\epsilon, \epsilon) \to S \text{ and } \gamma(0) = p\}.$$

Proof. Let $v \in T_pS$. We know that $v = \gamma'(0)$ for some curve γ in S. Then $i \circ \gamma$ is a curve in M with $(i \circ \gamma)' = di_p(v)$.

Conversely, let $v := w'(0) \in A$. We have $w = j \circ w$ where $j : i(S) \to S$ is the reverse inclusion. Since $(j \circ w)'(0) = dj_p(v) \in T_pS$, it follows that $d_i((j \circ w)'(0)) = v$. □

At this point, we begin developing the theory of differential forms. Let $F : \mathbb{R}^n \to \mathbb{R}$ be smooth. The gradient ∇F has two main properties.

1. It is orthogonal to the level sets of F.

2. $dF_p(v) = \langle \nabla F_p, v \rangle$.

But given a smooth manifold M, we don't necessarily have an inner product on M unless M is a *Riemannian manifold*, which by definition has a smoothly varying inner product. Instead, we shall view dF_p as a so-called 1-form.

3.7 Lecture 12

Recall that if $\pi : M \to N$ is a continuous map, then a *section of* π is a continuous right inverse of π.

Definition 3.7.1. A *(smooth) vector field* X is a smooth section of the projection map $\pi : TM \to M$, i.e., $X_p := F(p) \in T_pM$ for each $p \in M$.

Notation. Let $\mathscr{X}(M)$ denote the vector space of all smooth vector fields in M.

Note that $\mathscr{X}(M)$ is a module over $C^\infty(M)$ under the action $f \cdot X \equiv (p \mapsto f(p)X_p)$.

Given a chart U on M^n, if $p \in U$, then we can write $X_p = \sum_{i=1}^n r_i \frac{\partial}{\partial x_i}\big|_p$ for some unique real coefficients r_i. Define $X^i : U \to \mathbb{R}$ by $X_i(p) = r_i$ for each $i = 1, \ldots, n$. Then

$$X_p = \sum_i X_i(p) \frac{\partial}{\partial x_i}\Big|_p.$$

We call such X_i the *component functions of* X for the chart U.

Proposition 3.7.2. *A vector field X is smooth if and only if each component function in any given chart is smooth.*

Lemma 3.7.3. *If S is a closed subset of M and X a smooth vector field along S, then there is an extension of X to a smooth vector field on M.*

Definition 3.7.4. Let $U \subset M^n$ be open and $X_1, \ldots, X_k \in \mathscr{X}(M)$.

1. $X_1, \ldots, X_k$ are *linearly independent* if for any $p \in U$, we have that $\{X_1(p), \ldots, X_k(p)\}$ is linearly independent in T_pM.

2. If $k = n$ and $X_1, \ldots, X_k$ are linearly independent, then $\{X_1, \ldots, X_k\}$ is a *local frame* in U.

Example 3.7.5. The basis vectors $p \mapsto \frac{\partial}{\partial x_i}\big|_p$ form a local frame for a given chart U around p, called the *coordinate frame.*

Definition 3.7.6. A local frame for U is called a *global frame* if $U = M$. If such a frame exists, then M is called *parallelizable.*

Example 3.7.7. $\mathbb{R}^n$ is parallelizable via the standard coordinate vector fields.

Lemma 3.7.8. *M is parallelizable if and only if $TM \approx M \times \mathbb{R}^n$, i.e., its tangent bundle is trivial.*

Theorem 3.7.9 (Kervaire). *$\mathbb{S}^n$ is parallelizable if and only if $n \in \{0, 1, 3, 7\}$.*

Definition 3.7.10 (Lie group). A *Lie group* is a group G equipped with a smooth structure such that both $\times : G \times G \to G$ and $(-)^{-1} : G \to G$ are smooth maps.

Example 3.7.11. Any Lie group is parallelizable.

Note that $\mathscr{X}(M)$ acts on $C^\infty(U)$ for any $U \subset M$ with the action $X \cdot f \equiv (p \mapsto X_p(f))$. Given $X \in \mathscr{X}(M)$, this induces a linear map $X : C^\infty(U) \to C^\infty(U)$ satisfying the product rule

$$X(fg) = fXg + gXf.$$

We call such a map a *derivation* of $C^\infty(U)$.

Moreover, if $F : M \to N$ is smooth, then $dF_p X(p) \in T_{F(p)}N$ for each $p \in M$. Yet, this may *not* define a vector field on N, since F may not be surjective.

Example 3.7.12. Let $X, Y \in \mathscr{X}(M)$. Then $X(Yf)$ need *not* be a derivation. Indeed, let $M = \mathbb{R}^2$, $X = \frac{\partial}{\partial x}$, and $Y = x\frac{\partial}{\partial y}$. If $f(x, y) = x$ and $g(x, y) = y$, then $XY(fg) = 2x$ whereas $fXY(g) + gXY(f) = x$, so that $XY(f)$ is not a derivation.

Definition 3.7.13. Let $X, Y \in \mathscr{X}(M)$. The *Lie bracket of X and Y* is

$$[X, Y] \equiv XY - YX : C^\infty(M) \to C^\infty(M).$$

Proposition 3.7.14 (Clairaut). *If $X_i = \frac{\partial}{\partial x_i} \in \mathscr{X}(M)$, then $[X_i, X_j] = 0$ for any $1 \le i, j \le n$.*

Lemma 3.7.15. *A map $D : C^\infty(M) \to C^\infty(M)$ is a derivation if and only if there is some $X \in \mathscr{X}(M)$ such that $Df = Xf$ for any f.*

Proof. We have established the ($\Longleftarrow$) direction. Conversely, assume that D is a derivation. Define $X : M \to TM$ by $X_p(f) = (Df)(p)$. Since $Df = Xf$ is smooth for each X, it follows that X is smooth thanks to Proposition 8.14 (Lee). $\square$

Lemma 3.7.16. *Any Lie bracket $[X, Y]$ is a smooth vector field.*

Proof. By Lemma 3.7.15, it suffices to show that $[X, Y]$ is a derivation. Let f, g be smooth functions on M. Then

$$\begin{aligned}
[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fYg + gYf) - Y(fXg + gXf) \\
&= XfYg + XgYf - YfXg - YgXf \\
&= fXYg + YgXf + gXYf + YfXg \\
&\quad - fYXg - XgYf - gYXf - XfYg \\
&= fXYg + gXYf - fYXg - gYXf \\
&= f[X, Y]g + g[X, Y]f.
\end{aligned}$$

$\square$

3.8 Lecture 13

Consider two smooth vector fields X and Y on M. Define $[X,Y] : M \to TM$ by $p \mapsto (f \mapsto X_p(Yf) - Y_p(Xf))$.

Proposition 3.8.1. *Write $X = X^i \frac{\partial}{\partial x_i}$ and $Y = Y^j \frac{\partial}{\partial x_j}$ in local coordinates. Then*

$$[X,Y] = \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.$$

Proof. Since $[X,Y]$ is a vector field, we see that $([X,Y]\,f) \restriction_U = [X,Y]\,(f \restriction_U)$ for any open subset $U \subset M$. Therefore, we may compute, say, Xf in a local coordinate expression for X. To this end, let us apply the product rule together with Clairaut's theorem to get

$$\begin{aligned}
[X,Y]\,f &= X^i \frac{\partial}{\partial x_i} \left(Y^j \frac{\partial f}{\partial y_j} \right) - Y^j \frac{\partial}{\partial x_j} \left(X^i \frac{\partial f}{\partial x_i} \right) \\
&= X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial f}{\partial x_j} + X^i Y^j \frac{\partial^2 f}{\partial x_i x_j} - Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial f}{\partial x_i} - Y^j X^i \frac{\partial^2 f}{\partial x_j x_i} \\
&= X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial f}{\partial x_j} - Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial f}{\partial x_i} \\
&= \sum_{i,j} \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i} \right) \frac{\partial}{\partial x_j}.
\end{aligned}$$

□

Remark 3.8.2. If $X_1, \ldots, X_n \in \mathscr{X}(U)$ satisfy $[X_i, X_j] = 0$, then there are local coordinates $x^i : V \to \mathbb{R}$ such that $X_i = \frac{\partial}{\partial x^i}$. This is a converse of Clairaut's theorem.

Proposition 3.8.3.

1. *(Bilinearity) For any $a, b \in \mathbb{R}$,*

$$[aX + bY, Z] = a[X,Z] + b[Y,Z]$$
$$[Z, aX + bY] = a[Z,X] + b[Z,Y].$$

2. *(Antisymmetry)*

$$[X,Y] = -[Y,X].$$

3. *(Jacobi Identity)*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

4. *For any $f, g \in C^\infty(M)$,*

$$[fX, gY] = fg\,[X,Y] + (fXg)\,Y - (gYf)\,X,$$

where fX denotes the module action $f \cdot X$.

Now, let $X \in \mathscr{X}(M)$ and $Y \in \mathscr{X}(N)$. Let $F : M \to N$ be a diffeomorphism. The *pushforward of X by F*, denoted by F_*X, is the vector field on N given by

$$q \mapsto dF_{F^{-1}(q)} \left(X_{F^{-1}(q)} \right).$$

We say X and Y are *F-related* if $Y = F_*X$.

Note 3.8.4. $X(f \circ F) = (Yf) \circ F$ if and only if X and Y are F-related.

Theorem 3.8.5 (Naturality of the Lie bracket). $F_* [X, Y] = [F_* X, F_* Y]$.

Proof. Let $f \in C^\infty(M)$. By Note 3.8.4, we see that $XY(f \circ F) = X(F_* Y f \circ F) = F_* X(F_* Y f) \circ F$, and likewise $YX(f \circ F) = F_* Y(F_* X f) \circ F$. Thus,

$$[X, Y] (f \circ F) = F_* X(F_* Y f) \circ F - F_* Y(F_* X f) \circ F = ([F_* X, F_* Y] f) \circ F.$$

We conclude by again applying Note 3.8.4. □

Corollary 3.8.6. *Let $S \subset M$ be a submanifold. If $X, Y \in \mathscr{X}(M)$ satisfy $X_p, Y_p \in T_p(S)$ for each $p \in S$, then $[X, Y]_p \in T_p(S)$ as well.*

Proof. Let $i : S \to M$ denote inclusion. Then there are $X', Y' \in \mathscr{X}(S)$ with X' i-related to $X \restriction_S$ and Y' i-related to $Y \restriction_S$. This implies that $[X', Y']$ is i-related to $[X, Y] \restriction_S$, which in turn implies that $[X, Y]_p \in T_p(S)$ for any $p \in S$. □

4 Vector bundles

Definition 4.0.1. Let M be a space. A *(real) vector bundle of rank k over M* is a space E together with a continuous surjection $\pi : E \to M$ having the following properties.

(I) For each $p \in M$, the fiber $E_p \coloneqq \pi^{-1}(p)$ is a k-dimensional vector space.

(II) For each $p \in M$, there is a neighborhood U_p in M together with a homeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a *local trivialization*) such that

  (a) $\pi_U \circ \varphi = \pi \restriction_{\pi^{-1}(U)}$, where $\pi_U : U \times \mathbb{R}^k \to U$ denotes the projection and

  (b) for each $q \in U$, $\varphi \restriction_{E_q}$ is a linear isomorphism $E_q \xrightarrow{\cong} \{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

If M and E are smooth manifolds and each local trivialization is smooth, then E is called a *smooth vector bundle*.

Note 4.0.2. Any vector bundle $\pi : E \to M$ is an open map. Indeed, suppose that V is open in E. Then

$$\pi(V) = \bigcup_{x \in \pi(V)} \pi \restriction_{\pi^{-1}(U_x)} (V \cap \pi^{-1}(U_x)).$$

But each map $\pi \restriction_{\pi^{-1}(U_x)}$ is open as the composite $\pi_{U_x} \circ \varphi_{U_x}$ of open maps. Hence $\pi(V)$ is open as the union of open sets.

Example 4.0.3. The Möbius strip and $\mathbb{S}^1 \times \mathbb{R}$ are distinct vector bundles of rank 1 over $\mathbb{S}^1$.

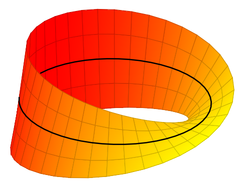

Figure 2: Möbius strip

We can always construct a global section of a smooth vector bundle by using partitions of unity. But we cannot always ensure that it is non-vanishing, as shown by the hairy ball theorem (Corollary 7.3.6) for bundles over $\mathbb{S}^2$.

4.1 Lecture 14

Lemma 4.1.1 (Vector bundle construction). *Let M^n be a smooth manifold and suppose that for any $p \in M$, there is some vector space E_p of dimension k. Let $E = \coprod_{p\in M} E_p$ and let $\pi : E \to M$ be the projection map. Further, suppose we have the following data:*

(a) an open cover $\{U_\alpha\}$,

(b) for each α, a bijection $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^k$ whose restriction to each E_p is a linear isomorphism to $\{p\} \times \mathbb{R}^k$, and

(c) for each $U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k, \mathbb{R})$ such that $\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) = (p, \tau_{\alpha\beta}(p)v)$.

Then E has a unique topology and smooth structure making (E, π) into a smooth vector bundle of rank k over M.

The matrices $\tau_{\alpha\beta}(p)$ are called the *transition functions* of the vector bundle E. They satisfy the so-called cocycle condition:

$$\tau_{\alpha\alpha}(p) = I_k \qquad \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)\tau_{\gamma\alpha}(p) = I_k.$$

Definition 4.1.2 (Bundle map). Let $p_1 : E_1 \to M_1$ and $p_2 : E_2 \to M_2$ be two vector bundles of rank k. A *homomorphism* $p_1 \to p_2$ is a commutative square

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ M_1 \arrow[r, "g"'] & M_2 \end{tikzcd}$$

in the category of spaces such that each map $f\restriction_{p_1^{-1}(x)}$ is linear.

Note that g is uniquely determined by f because p_1 is surjective.

Let us now explore a specific kind of vector bundle. To this end, consider any vector space V as well as its *dual space*

$$V^* \equiv \operatorname{Hom}(V, \mathbb{R}),$$

which consists of all linear maps $V \to \mathbb{R}$, known as *covectors on* V. If $A : V \to W$ is linear, then let A^* denote the linear map $W^* \to V^*$ defined by $w \mapsto (v \mapsto w(Av))$, called the dual map of A.

Let $\{v_1, \ldots, v_n\}$ be a basis for V. The *dual basis* (or *cobasis*) consists of those linear functionals $\varphi_i : V \to \mathbb{R}$ given by

$$\varphi_i(v_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}.$$

for each $i = 1, \ldots, n$.

Proposition 4.1.3.

(1) If $\dim(V) = n$*, then* $\dim(V^*) = n$.

Proof. Pick a basis $b_1, \ldots, b_n$ for V. Consider its dual basis $\left\{b^1, \ldots, b^n\right\}$. It is easy to check that this is linearly independent. Further, for any $T \in V^*$, we see that

$$T = T_1 b^1 + \cdots + T_n b^n, \qquad T_i \equiv T(b_i).$$

This means that the b^i span $\operatorname{Hom}(V, \mathbb{R})$ as well. □

Remark 4.1.4. The induced isomorphism $V \to V^*$ is *not* unique, for it depends on our chosen basis of V.

(2) The mapping $v \mapsto \underbrace{(\varphi \mapsto \varphi(v))}_{\mathrm{ev}_v}$ *defines a canonical isomorphism*

$$V \xrightarrow{\cong} (V^*)^* = \operatorname{Hom}(V^*, \mathbb{R}).$$

Definition 4.1.5. Let M^n be a smooth manifold.

1. Define the *cotangent space at* p as $T_p^* M$.
2. Define the *cotangent bundle of* M as $T^*M \equiv \coprod_p T_p^* M$.

Lemma 4.1.6. *T^*M is a smooth n-vector bundle over M.*

Proof. Let (U, φ) be a smooth chart on M. Define $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by $a_i \lambda^i\big|_p \mapsto (p, a_1, \ldots, a_n)$ where $\left\{\lambda^i\big|_p\right\}$ is a chosen dual basis for $T_p M$. Now we apply the vector bundle construction lemma. See Proposition 11.9 (*Lee*). □

Let $\left(U, x^i\right)$ be smooth coordinates for M^n. Then the map $\psi : a_i \lambda^i\big|_p \mapsto \left(x^1(p), \ldots, x^n(p), a_1, \ldots, a_n\right)$ makes $\left(\pi^{-1}(U), \psi\right)$ a chart on T^*M.

A smooth section of T^*M is called a *covector field* (or *(differential/smooth) 1-form*) on M. The vector space of such sections will be denoted by $\Gamma(T^*M)$.

Moreover, if U is a chart on M, then a tuple $\left(\epsilon^1, \ldots, \epsilon^k\right)$ of covector fields on M is a *local coframe* if $\left\{\epsilon^1\big|_p, \ldots, \epsilon^k\big|_p\right\}$ is a basis of $T_p^* U$ for each $p \in U$.

Aside. Let $\pi : E \to M$ be a smooth vector bundle. The *jet bundle* $J^k E \to M$ *of order* k is the smooth vector bundle whose fiber at $p \in M$ consists of all *order-k jets* of smooth sections of π, i.e., equivalence classes of smooth sections of π where two sections are declared equivalent if their first k partial derivatives agree on a neighborhood of p. Note that a germ is precisely an order-1 jet.

We have a sequence of maps

$$\cdots J^3 E \twoheadrightarrow J^2 E \twoheadrightarrow J^1 E \twoheadrightarrow E,$$

whose limit is called the *infinite jet bundle* $J^\infty E$.

5 Differential forms

5.1 Lecture 15

Definition 5.1.1 (Differential of a smooth function). Define $C^\infty(M) \to \Gamma(T^* M)$ by $f \mapsto (p \mapsto df_p)$ where

$$df_p(v) \equiv vf$$

for every $v \in T_p M$. We call df the *differential of* f.

Let $\left(U, x^i\right)$ be local coordinates for M. Let $\left(dx^i\right)$ denote the corresponding coordinate coframe. We have $df_p = A_i(p) dx^i\big|_p$ for some functions $A_i : U \to \mathbb{R}$. Then

$$A_i(p) = df_p\left(\frac{\partial}{\partial x^i}\Big|_p\right) = \frac{\partial f}{\partial x^i}(p)$$
$$\Downarrow$$
$$df_p = \frac{\partial f}{\partial x^i}(p) dx^i\Big|_p.$$

In this way, the differential of f generalizes the gradient of a smooth function on $\mathbb{R}^n$.

Proposition 5.1.2. *If M is connected, then f is constant if and only if $df = 0$.*

Proof. Since $vf = 0$ for any derivation v and constant function f, the forward direction is clear. Conversely, suppose that $df = 0$ and let $p \in M$. Set $C = \{q \in M : f(q) = f(p)\}$. We must show that $C = M$. Provided that M is connected, it suffices to show that C is clopen. For any $q \in C$, choose a coordinate ball $U \ni p$. Then since $0 = df = \frac{\partial f}{\partial x^i} dx^i$, it follows that $\frac{\partial f}{\partial x^i} = 0$ for each i. Elementary calculus reveals that f must be constant on U. Hence C is open. Since $C = f^{-1}(p)$, it is also closed. $\square$

Note 5.1.3 (Transition functions for changing coordinates). Let $p \in M$ and suppose that $\left(x^i\right)_{1 \le i \le n}$ and $\left(y^i\right)_{1 \le i \le n}$ are two coordinate charts around p. The chain rule for partial derivatives states that

$$\frac{\partial}{\partial x^j}\Big|_p = \sum_k \frac{\partial y^k}{\partial x^j}(\hat{p}) \frac{\partial}{\partial y^k}\Big|_p$$

where $\hat{p} \coloneqq \big(x^1(p), \ldots, x^n(p)\big)$. Dually, for each $i \in \{1, \ldots, n\}$, we have that

$$dx^i\big|_p = \sum_\ell A^i_\ell dy^\ell\big|_p$$

for some $A^i_\ell \in \mathbb{R}$, $l = 1, \ldots, n$. It follows that

$$\begin{aligned}
\delta^j_i &= dx^i\big|_p \left(\frac{\partial}{\partial x^j}\Big|_p\right) \\
&= dx^i\big|_p \left(\sum_k \frac{\partial y^k}{\partial x^j}\frac{\partial}{\partial y^k}\Big|_p\right) \\
&= \sum_k \frac{\partial y^k}{\partial x^j} dx^i\big|_p \left(\frac{\partial}{\partial y^k}\Big|_p\right) \\
&= \sum_k \frac{\partial y^k}{\partial x^j} \sum_\ell A^i_\ell dy^\ell\big|_p \left(\frac{\partial}{\partial y^k}\Big|_p\right) \\
&= \sum_k \frac{\partial y^k}{\partial x^j} \sum_\ell A^i_\ell \delta^k_\ell \\
&= \sum_k A^i_k \frac{\partial y^k}{\partial x^j}.
\end{aligned}$$

Therefore, if A denotes the $n \times n$ matrix $\big(A^i_\ell\big)$ and J denotes the Jacobian of $\big(y^1, \ldots, y^n\big)$ at $\hat{p}$, then $I_n = JA$, i.e., $A = J^{-1}$.

Definition 5.1.4. Let $F : M \to N$ be smooth. Let $\omega \in \Gamma(T^*N)$. Define the *pullback $F^*\omega$ of ω* as the element of $\Gamma(T^*M)$ given by

$$F^*\omega\big|_p \left(X\big|_p\right) \equiv \omega\big|_{F(p)} \left(F_*\big|_p X_p\right).$$

Note that, unlike the pushforward, the pullback requires merely that F be smooth.

Lemma 5.1.5. *Let $F : M \to N$ be smooth, $\alpha, \beta \in \Gamma(T^*N)$ and $f, g \in C^\infty(N)$. Then*

$$F^*(f\alpha + g\beta) = (f \circ F)\, F^*\alpha + (g \circ F) F^*\beta.$$

Proof. Let $X \in \mathscr{X}(M)$. We have that

$$\begin{aligned}
F^*(f\alpha + g\beta)\big|_p(X_p) &= (f\alpha + g\beta)\big|_{F(p)} \left(F_*\big|_p X_p\right) \\
&= f\left(F(p)\right) \alpha_{F(p)} \left(F_*\big|_p X_p\right) + g\left(F(p)\right) \beta_{F(p)} \left(F_*\big|_p X_p\right) \\
&= \left[(f \circ F) F^*\alpha\right]_p (X_p) + \left[(g \circ F) F^*\beta\right]_p (X_p).
\end{aligned}$$

□

Let $\gamma : J \subset \mathbb{R} \to M$ be a smooth curve in M. Note that $\Gamma(T^*\mathbb{R}) = \{f(t)dt \mid f : T \to \mathbb{R}\}$. Then

$$\omega \in \Gamma(T^*M) \implies \gamma^*\omega \in \Gamma(T^*\mathbb{R}) \implies \gamma^*\omega = f(t)dt$$

for some curve f along J. This enables us to modestly generalize our notion of integration.

Definition 5.1.6. The *integral of ω along γ* is

$$\int_\gamma \omega \equiv \int_J \gamma^* \omega.$$

Proposition 5.1.7. *Suppose that φ is a positive reparameterization of γ (i.e., one with positive derivative). Then $\int_\gamma \omega = \int_{\gamma\circ\varphi} \omega$.*[4]

Definition 5.1.8. A differential 1-form ω on a smooth manifold M is *closed* if the equation

$$\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0$$

holds for any i, j in any chart on M.

Exercise 5.1.9. *Show that being closed is a well-defined property.*

Example 5.1.10. By Clairaut's theorem, df is closed for any $f \in C^\infty(M)$.

5.2 Lecture 16

Recall that a map $T : V_1 \times \cdots \times V_k \to W$ of vector spaces is *multilinear* if it is linear in each argument, i.e.,

$$T(v_1, \ldots, ax + by, \ldots v_k) = aT(v_1, \ldots, x, \ldots, v_k) + bT(v_1, \ldots, y, \ldots, v_k)$$

for any $a, b \in \mathbb{R}$.

Theorem 5.2.1 (Universal property of the tensor product). *Let $V_1, \ldots, V_k$ be vector spaces. There exists a vector space $V_1 \otimes \cdots \otimes V_k$ together with a map $: \otimes : V_1 \times \cdots \times V_k$ so that for any multilinear map $T : V_1 \times \cdots \times V_k \to W$, there is some unique linear map $\widetilde{T} : V \otimes \cdots \otimes V_k \to W$ such that*

$$\begin{tikzcd} V_1 \times \cdots \times V_k \arrow[r, "T"] \arrow[d, "\otimes"'] & W \\ V_1 \otimes \cdots \otimes V_k \arrow[ur, dashed, "\widetilde{T}"'] & \end{tikzcd}$$

commutes.

Terminology. $V_1 \otimes \cdots \otimes V_k$ is called the *tensor product of the V_i*.

Proof. Let us just prove this when $k = 2$, for then we're done by induction. Let $\mathbb{R}\langle V_1 \times V_2 \rangle$ denote the free vector space on $V_1 \times V_2$, which consists of all finite formal linear combinations of $V_1 \times V_2$. Let

$$G = \langle (av_1, v_2) - a(v_1, v_2), (v_1, av_2) - a(v_1, v_2), (v_1 + w_1, v_2) - (v_1, v_2) - (w_1, v_2), (v_1, w_2 + v_2) - (v_1, w_2) - (v_1, v_2) \rangle.$$

Given a multilinear map $T : V_1 \times V_2 \to W$, define $\widetilde{T} : \mathbb{R}\langle V_1 \times V_2 \rangle \to W$ by

$$\sum a_{(v_1,v_2)}(v_1, v_2) \mapsto \sum a_{(v_1,v_2)} T(v_1, v_2).$$

Since T is multilinear, $G \subset \ker \widetilde{T}$. Therefore, the vector space $V_1 \otimes V_2 := \mathbb{R}\langle V_1 \times V_2 \rangle / G$ fits in a commutative triangle

$$\begin{tikzcd} \mathbb{R}\langle V_1 \times V_2 \rangle \arrow[r, "\tilde{T}"] \arrow[d, "\pi"'] & W \\ V_1 \otimes V_2 \arrow[ur, dashed, "\tilde{\tilde{T}}"'] & \end{tikzcd} .$$

[4] Proposition 11.31 (Lee).

Thus, if $i : V_1 \times V_2 \to \mathbb{R}\langle V_1 \times V_2\rangle$ denotes inclusion, then $\widetilde{\widetilde{T}} \circ \pi \circ i = \widetilde{T} \circ i$, which induces our desired diagram. We see that $\widetilde{\widetilde{T}}$ is unique because it is uniquely determined by elements of the form

$$v_1 \otimes v_2 \coloneqq [(v_1, v_2)]$$

under T and every element of $V_1 \otimes V_2$ can be written as some linear combination of such elements. □

A basic property of the tensor product is that its generic elements are bilinear in the following sense.

Proposition 5.2.2. *If $a, b \in \mathbb{R}$, then $(av_1 + bw_1) \otimes v_2 = a(v_1 \otimes v_2) + b(w_1 \otimes v_2)$.*

Proposition 5.2.3.

1. $(\mathbf{Vect}_{\mathbb{R}}, \oplus, \otimes)$ *is a semiring.*
2. $V \otimes W \cong W \otimes V$.
3. $V \otimes \mathbb{R} \cong V$.
4. $(V \otimes W)^* \cong V^* \otimes W^*$.

Let $B(V, W)$ denote the space of bilinear maps $V \times W \to \mathbb{R}$.

Lemma 5.2.4. *There is a canonical isomorphism $V^* \otimes W^* \cong B(V, W)$.*

Proof. Define $\Phi : V^* \times W^* \to B(V, W)$ by $(\omega, \eta) \mapsto ((v, w) \mapsto \omega(v)\eta(w))$. This is linear and hence induces a commutative diagram

$$\begin{tikzcd} V^* \times W^* \arrow[r, "\Phi"] \arrow[d, "\pi"'] & B(V,W) \\ V^* \otimes W^* \arrow[ur, dashed, "\widetilde{\Phi}"'] & \end{tikzcd}.$$

To see that $\tilde{\Phi}$ is an isomorphism, pick bases $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_n\}$ for V and W, respectively. Consider their respective dual bases $\{\xi\}$ and $\{\eta\}$. Then $\left\{\xi^i \otimes \eta^j : 1 \le i, j \le n\right\}$ is a basis for $V^* \otimes W^*$. Define the linear map $\Psi : B(V, W) \to V^* \otimes W^*$ by

$$b \mapsto \sum_{i,j} b(f_i, g_j)\xi^i \otimes \eta^j.$$

It is straightforward to check that Ψ is the inverse of $\tilde{\Phi}$. □

We can generalize Theorem 7.2.3 to obtain an isomorphism

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \ldots, V_k; \mathbb{R}).$$

Definition 5.2.5 (Tensor type). We say that an element of

$$V_\ell^k \coloneqq \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ copies}} \otimes \underbrace{V \otimes \cdots \otimes V}_{\ell \text{ copies}}$$

is a (k, ℓ)*-tensor*.

Terminology.

1. A $(k,0)$-tensor is called *covariant.*
2. A $(0,\ell)$-tensor is called *contravariant.*

Let M be a smooth manifold. Define the (k,ℓ)-*tensor bundle* as

$$T^k_\ell M \equiv \coprod_{p\in M} (T_p)^k_\ell\, M.$$

In particular, $T^1M = T^*M$, and $T_1M = TM$.

Exercise 5.2.6. *Find the dimension of $T^k_\ell M$.*

Let us examine the form of a generic $(k,0)$-tensor. Suppose that $\left(x^i\right)$ and $\left(y^i\right)$ are two local coordinate systems around a point $p \in M$. Then

$$\begin{aligned} dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_k} &= \left(\frac{\partial x^{i_1}}{\partial y^{\ell_1}} dy^{p_1}\right) \otimes \cdots \otimes \left(\frac{\partial x^{i_k}}{\partial y^{l_k}} dy^{p_k}\right) \\ &= \sum_{p_1,\ldots,p_k} \frac{\partial x^{i_1}}{\partial y^{\ell_1}} \cdots \frac{\partial x^{i_k}}{\partial y^{\ell_k}} \otimes dy^{p_1} \otimes \cdots \otimes dy^{p_k}. \end{aligned}$$

Definition 5.2.7. A (k,ℓ)-*tensor field* is a (smooth) section of $T^k_\ell M$.

Let $\mathcal{T}^k_\ell(M)$ denote the space $\Gamma(T^k_\ell M)$ of all such sections.

5.3 Lecture 17

Let $\left(U, x^i\right)$ be local coordinates for M. Then any $A \in \mathcal{T}^\ell_k(M)$ can be written in U as

$$A\big|_p = A^{j_1\ldots j_\ell}_{i_1\ldots i_k} dx^{i_1}\big|_p \otimes \cdots \otimes dx^{i_k}\big|_p \otimes \frac{\partial}{\partial x^{j_1}}\big|_p \otimes \cdots \otimes \frac{\partial}{\partial x^{j_\ell}}\big|_p,$$

summed over $n^k n^\ell$ many tensors.

Example 5.3.1. Let $\sigma = \delta^i_j dx^j \otimes \frac{\partial}{\partial x^i}$, $X = X^k \frac{\partial}{\partial x^k}$, and $w = w_\ell dx^\ell$. Then

$$\begin{aligned} \sigma(X,w) &= \delta^i_j dx^j \otimes \frac{\partial}{\partial x^i}\left(X^k \frac{\partial}{\partial x^k}, w_\ell dx^\ell\right) \\ &= \delta^i_j dx^j \left(X^k \frac{\partial}{\partial x^k}\right) \frac{\partial}{\partial x^i} w_\ell dx^\ell \\ &= \delta^i_j \delta^j_k X^k w_\ell \delta^\ell_i \\ &= w_k X^k \\ &= w(X). \end{aligned}$$

We say that σ is *invariant* in this case.

Example 5.3.2. Show that the tensor $\delta^j_i dx^i \otimes dx^j$ is *not* invariant.

Proposition 5.3.3.

1. *Any $\sigma \in \mathcal{T}^k_\ell(M)$ induces a $C^\infty(M)$-multilinear map*

$$\hat{\sigma} : \underbrace{\mathscr{X}(M) \times \cdots \times \mathscr{X}(M)}_{k \text{ copies}} \times \underbrace{\mathscr{X}^*(M) \times \cdots \times \mathscr{X}^*(M)}_{\ell \text{ copies}} \longrightarrow C^\infty(M)$$
$$(X_1, \ldots, X_k, w_1, \ldots, w_\ell) \mapsto \left(p \mapsto \sigma\left(X_1\big|_p, \ldots, X_k\big|_p, w_1\big|_p, \ldots, w_\ell\big|_p\right)\right). \qquad (*)$$

2. *Any multilinear map over $C^\infty(M)$ is of the form (1) for some (k, ℓ)-tensor field.*

Notice that the smooth function $\hat{\sigma}_p$ induced by σ of Example 5.3.1 is determined completely by the values $X_1(p)$, …, $X_k(p)$, $w_1(p)$, …, $w_\ell(p)$.

Note 5.3.4. The Lie bracket is *not* multilinear over $C^\infty(M)$, for

$$[fX + gY, Z] = f\,[X, Y] + g[Y, Z] - Z(f)X - Z(g)Y.$$

Definition 5.3.5. A covariant k-tensor T is *alternating* if for any vectors $Y, X_1, \ldots, X_{k-1}$, it follows that

$$T(X_1, X_2, \ldots, Y, \ldots, Y, \ldots, X_{k-1}) = 0.$$

In this case, T is also called an *exterior form.*

Example 5.3.6. If σ is a 0-tensor or a 1-tensor, then it is alternating.

Proposition 5.3.7. *TFAE.*

1. *T is alternating.*

2. *$T(X_1, \ldots, X_k) = 0$ whenever $\{X_1, \ldots, X_k\}$ is linearly dependent.*

3. *$T(X_1, \ldots, X_i, X_{i+1}, \ldots, X_k) = -T(X_1, \ldots, X_{i+1}, X_i, \ldots, X_k)$.*

Notation. The expression $\bigwedge^k(V)$ will denote the subspace of $T^k(V)$ consisting of alternating covariant k-tensors.

Definition 5.3.8. Given $T \in T^k(V)$, the *alternation* $\mathrm{Alt}(T)$ *of* T is the multilinear map defined by

$$(V_1, \ldots, V_k) \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) T\left(V_{\sigma(1)}, \ldots, V_{\sigma(k)}\right).$$

Example 5.3.9.

1. $\mathrm{Alt}(T)(X, Y) = \frac{1}{2}\left(T(X, Y) - T(Y, X)\right)$.

2. $\mathrm{Alt}(T)(X, Y, Z) = \frac{1}{6}\left(T(X, Y, Z) + T(Y, Z, X) + T(Z, X, Y) - T(Y, X, Z) - T(Z, Y, X) - T(X, Z, Y)\right)$.

Example 5.3.10. Suppose that $\{w^1, \ldots, w^n\}$ is the cobasis of the standard basis $\{e_1, \ldots, e_n\}$ for the vector space V. Then

$$\begin{aligned}
&\mathrm{Alt}(w^1 \otimes \cdots \otimes w^n)(e_1, \ldots, e_n) \\
&= \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) w^1 \otimes \cdots \otimes w^n \left(e_{\sigma(1)}, \ldots, e_{\sigma(n)}\right) \\
&= \frac{1}{n!} \mathrm{sgn}\left(\mathrm{id}_n\right) w^1 \otimes \cdots \otimes w^n \left(e_1, \ldots, e_n\right) \\
&= \frac{1}{n!}.
\end{aligned}$$

Proposition 5.3.11.

1. $\operatorname{Alt}(T) \in \bigwedge^k(V)$.
2. $\operatorname{Alt}(T) = T \iff T \in \bigwedge^k(V)$.
3. *The induced map* $\operatorname{Alt} : T^k(V) \to \bigwedge^k(V)$ *is linear.*

5.4 Lecture 18

Lemma 5.4.1. *Let V be a vector space of dimension $k < \infty$. Let $\{w^1, \dots, w^n\}$ be a cobasis for V. Let $k \leq n$. Then*

$$A := \left\{ \operatorname{Alt}(\omega^{i_1} \otimes \cdots \otimes \omega^{i_k}) : 1 \leq i_1 < \cdots < i_k \leq n \right\}$$

is a basis for $\bigwedge^k(V)$.

Proof. It's clear from Proposition 5.3.11 that A spans $\bigwedge^k(V)$. It remains to show that A is linearly independent.

Claim.

(a) *If the integers $i_1, \dots, i_k$ are not pairwise distinct, then* $\operatorname{Alt}(\omega^{i_1} \otimes \cdots \otimes \omega^{i_k}) = 0$.

(b) $\operatorname{Alt}(\omega^{i_1} \otimes \cdots \otimes \omega^{i_j} \otimes \omega^{i_{j+1}} \otimes \cdots \otimes \omega^{i_k}) = -\operatorname{Alt}(\omega^{i_1} \otimes \cdots \otimes \omega^{i_{j+1}} \otimes \omega^{i_j} \otimes \cdots \otimes \omega^{i_k})$.

As a consequence, $\operatorname{span}(A) = \operatorname{span}\left\{ \operatorname{Alt}(\omega^{i_1} \otimes \cdots \otimes \omega^{i_k}) : 1 \leq i_1 \leq \cdots \leq i_k \leq n \right\}$.

Exercise 5.4.2. *Show that this implies that A is linearly independent.*

□

Corollary 5.4.3. *If* $\dim(V) = n$, *then* $\dim\left(\bigwedge^k(V)\right) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Definition 5.4.4. Define the *wedge product* as the map

$$\wedge : \bigwedge^k(V) \times \bigwedge^\ell(V) \to \bigwedge^{k+\ell}(V) \qquad (w, q) \mapsto w \wedge q \equiv \frac{(k+\ell)!}{k!\ell!} \operatorname{Alt}(w \otimes q).$$

This is like the tensor product.

Example 5.4.5. With notation as in Example 5.3.10, we have that $\omega^1 \wedge \cdots \wedge \omega^n(e_1, \dots, e_n) = 1$.

Lemma 5.4.6. *The set* $\left\{ \omega^{i_1} \wedge \cdots \wedge \omega^{i_k} : 1 \leq i_1 < \cdots < i_k \leq n \right\}$ *is a basis for* $\bigwedge^k(V)$.

Proof sketch. For each k-tuple $(i_1, \dots, i_k)$, one can show that $\omega^{i_1} \wedge \cdots \wedge \omega^{i_k}$ and $\operatorname{Alt}(\omega^{i_1} \otimes \cdots \otimes \omega^{i_k})$ differ precisely by a real factor. This is enough thanks to Lemma 5.4.1. □

Consider the standard basis $B := \{e_1, \dots, e_n\}$ for V. Note that $\det_B \in \bigwedge^n(V)$ by Proposition 5.3.11. But $\bigwedge^n(V) = 1$, so that $\det_B = c(\omega^1 \wedge \cdots \wedge \omega^n)$. But evaluating both sides at $(e_1, \dots, e_n)$ yields the equation $1 = c(1) = c$. Thus,

$$\det_B = \omega^1 \wedge \cdots \wedge \omega^n.$$

Proposition 5.4.7. *Suppose that ω, ω, η, and η' are exterior forms. The following are properties of the wedge product.*

(1) (Bilinearity) If $a, a' \in \mathbb{R}$, then

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta)$$
$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta \wedge \omega').$$

(2) (Associativity)

$$(\eta \wedge \omega) \wedge \omega' = \eta \wedge (\omega \wedge \omega').$$

(3) (Anticommutativity) If $\omega \in \bigwedge^k(V)$ and $\eta \in \bigwedge^\ell(V)$, then

$$\omega \wedge \eta = (-1)^{kl}\, \eta \wedge \omega.$$

Corollary 5.4.8. *If ω is a 1-form, then $\omega \wedge \omega = 0$.*

(4) If $\omega^1, \dots, \omega^k \in \bigwedge^1(V)$, then

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1, \dots, v_k) = \det(\omega^j(v_i)).$$

Definition 5.4.9. Let M^n be a smooth manifold. Define the *alternating bundle of rank k* as

$$\bigwedge^k(M) \equiv \coprod_{p \in M} \bigwedge^k(T_pM).$$

A smooth section of $\bigwedge^k(M)$ is called a *(differential) k-form.*

Let both $\Omega^k(M)$ and $\mathcal{A}^k(M)$ stand for the infinite-dimensional vector space of differential k-forms on the manifold M. We also have a graded associative algebra $(\Omega^*(M), \wedge)$ over $\mathbb{R}$.

In local coordinates we have a basis $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}_{1 \le i \le n}$ for T_pM as well as a corresponding dual basis $\{dx^i\}$. Then for any $\omega \in \bigwedge^k(M)$, we can write

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{1}$$

locally at p. Let $I = \{i_1 < \cdots < i_k\}$. Since

$$dx^{i_1} \wedge \cdots \wedge dx^{i_\ell} \left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}}\right) = \delta^I_J$$

where $\delta^I_J = 1$ if and only if $I = J$ as sets, it follows that

$$\omega_{i_1,\dots,i_k} = \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right). \tag{2}$$

We abbreviate (1) by writing

$$\omega = \omega_I dx^I,$$

where we tacitly sum over the I. In this case, for any other ordered set of indices $J := \{j_1 < \cdots < j_k\}$, we have

$$\omega\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}}\right) = \omega_I dx^I\left(\frac{\partial}{\partial x^{j_1}}, \dots, \frac{\partial}{\partial x^{j_k}}\right) = \omega_I \delta^I_J.$$

Note 5.4.10. Let $w = w_I dx^I$ and $w = \tilde{w}_J d\tilde{x}^J$ be two coordinate representations of w. Observe that

$$\begin{aligned}
\tilde{\omega}_J &= \omega\left(\frac{\partial}{\partial \tilde{x}^{j_1}}, \ldots, \frac{\partial}{\partial \tilde{x}^{j_k}}\right) && ((2))\\
&= \omega\left(\sum_t \frac{\partial x^{i_t}}{\partial \tilde{x}^{j_1}} \frac{\partial}{\partial x^{i_t}}, \ldots, \sum_t \frac{\partial x^{i_t}}{\partial \tilde{x}^{j_k}} \frac{\partial}{\partial x^{i_t}}\right) && \text{(chain rule)}\\
&= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \frac{\partial x^{i_1}}{\partial \tilde{x}^{j_1}} \cdots \frac{\partial x^{i_k}}{\partial \tilde{x}^{j_k}} \omega\left(\frac{\partial}{\partial \tilde{x}^{\sigma(i_1)}}, \ldots, \frac{\partial}{\partial \tilde{x}^{\sigma(i_k)}}\right) && \text{(multilinearity of } \omega)\\
&= \det\left(k \times k \text{ minor of } \frac{\partial x}{\partial \tilde{x}} \text{ relative to } i_1, \ldots, i_k \text{ and } j_1, \ldots, j_k\right). && \text{(Proposition 5.4.7(4))}
\end{aligned}$$

5.5 Lecture 19

The following notion generalizes Definition 5.1.4 to differential forms of arbitrary degree.

Definition 5.5.1 (Pullback). Let $F : M \to N$ be smooth and $\omega \in \bigwedge^k(N)$. The *pullback* $F^*\omega$ *of* ω *by* F is the differential k-form on M given pointwise by

$$F^*\omega\big|_p(v_1, \ldots, v_k) = \omega_{F(p)}\left(dF_p(v_1), \ldots, dF_p(v_k)\right).$$

Note that $F^*(-)$ is a linear map $\Omega^k(N) \to \Omega^k(M)$ over $\mathbb{R}$.

Lemma 5.5.2 (Naturality of the pullback). $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$.

Proof. This is easily seen from Definition 5.5.1 together with Definition 5.4.4. □

Lemma 5.5.3. *In any local coordinates, we have that*

$$F^*\left(\sum_I \omega_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) = \sum_I (\omega_I \circ F)\, d\left(y^{i_1} \circ F\right) \wedge \cdots \wedge d\left(y^{i_k} \circ F\right).$$

Proof. It is easy to check that $F^*\omega(X_1, \ldots, X_k) = \sum_I w_I \circ F dy^I(F_*X_1, \ldots, F_*X_k)$. Hence it suffices to show that

$$d\left(y^{i_1} \circ F\right) \wedge \cdots \wedge d\left(y^{i_k} \circ F\right)(X_1, \ldots, X_k) = dy^I(F_*X_1, \ldots, F_*X_k).$$

For this, it suffices to show that $d\left(y^i \circ F\right)(X) = dy^i(F_*X)$ for each $i \in \{i_1, \ldots, i_k\}$. Let $\left(x^i\right)$ denote local coordinates on M. On the one hand, thanks to Definition 5.1.1, we see that

$$d\left(y^i \circ F\right)(X) = X\left(y^i \circ F\right) = X^j \frac{\partial F^i}{\partial x^j}.$$

On the other hand, we see that

$$\begin{aligned}
dy^i(F_*X) &= dy^i\left(X^j \frac{\partial F^r}{\partial x^j} \frac{\partial}{\partial y^r}\right)\\
&= X^j \frac{\partial F^i}{\partial x^j}.
\end{aligned}$$

□

Example 5.5.4. Consider the change of variables to polar coordinates $\mathbb{R}^2 \to \mathbb{R}^2$:

$$(r, \theta) \mapsto (r\cos\theta, r\sin\theta).$$

Locally, this is precisely the identity map with the domain endowed with one atlas and the codomain endowed with another. Lemma 5.5.3 together with certain computational properties of $\wedge$ yields

$$\begin{aligned}
dx \wedge dy &= d(r\cos\theta) \wedge d(r\sin\theta) \\
&= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta) \\
&= (\cos\theta dr - r\sin\theta d\theta) \wedge \sin\theta dr + (\cos\theta dr - r\sin\theta d\theta) \wedge r\cos\theta d\theta \\
&= (\cos\theta dr \wedge \sin\theta dr) - (r\sin\theta d\theta \wedge \sin\theta dr) + (\cos\theta dr \wedge r\cos\theta d\theta) - (r\sin\theta d\theta \wedge r\cos\theta d\theta) \\
&= -(r\sin\theta d\theta \wedge \sin\theta dr) + (\cos\theta dr \wedge r\cos\theta d\theta) \\
&= r\sin^2\theta(dr \wedge d\theta) + r\cos^2\theta(dr \wedge d\theta) \\
&= r dr \wedge d\theta.
\end{aligned}$$

Now, let us begin defining a differential operator on smooth forms that generalizes Definition 5.1.1. Let ω be a 1-form on a smooth manifold M. For this to arise as the differential of a smooth function df, each component function ω_i must have the form $\frac{\partial f}{\partial x^i}$. By Clairaut's theorem, this means that ω is closed in the sense of Definition 5.1.8, i.e.,

$$\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 0 \qquad (*)$$

in any chart on M. This is property is actually coordinate-independent by Lee (Proposition 11.45). Therefore, we want to express $(*)$ as the ij-component of a 2-form, namely

$$d\omega \equiv \sum_{j<i} \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i.$$

Notice that ω is closed if and only if $d\omega = 0$ in any chart on M.

5.6 Lecture 20

Let $\omega \in \mathcal{A}^k(M)$ with local coordinate representation $\omega_I dx^I$. The *exterior derivative* of ω is the $(k+1)$-form

$$d\omega \equiv d\omega_I \wedge dx^I.$$

We refer to the operation $d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ as *exterior differentiation.*

Note 5.6.1. $d\omega = \sum_I \sum_j \frac{\partial}{\partial x^j} \omega_I dx^j \wedge dx^I$.

Aside. If we view $\Omega^k : \mathbf{Diff}^{\mathrm{op}} \to \mathbf{Vec}_{\mathbb{R}}$ as the functor sending each smooth map f to the pullback f^*, then the exterior derivative becomes a natural transformation $\Omega^k \Rightarrow \Omega^{k+1}$.

Definition 5.6.2. Let $\omega \in \mathcal{A}^k(M)$.

1. We say that ω is *closed* if $d\omega = 0$.

2. We say that ω is *exact* if $\omega = d\eta$ for some $\eta \in \mathcal{A}^{k-1}(M)$.

Lemma 5.6.3. *Suppose that $M = \mathbb{R}^n$, equivalently, that M has a global chart.*

(1) d is linear over $\mathbb{R}$.

(2) $d(F^*\omega) = F^*(d\omega)$.

(3) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

(4) $d \circ d = 0$.

Proof. Statement (1) is obvious. For (2), by linearity, it suffices to consider the case where $\omega = u dx^I$. Using Lemma 5.5.3, we compute

$$\begin{aligned} F^*\left(d\left(u dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\right) &= F^*\left(du \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \\ &= d(u \circ F) \wedge d\left(x^{i_1} \circ F\right) \wedge \cdots \wedge d\left(x^{i_k} \circ F\right) \end{aligned}$$

$$\begin{aligned} d\left(F^*\left(u dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right)\right) &= d\left((u \circ F) d\left(x^{i_1} \circ F\right) \wedge \cdots \wedge d\left(x^{i_k} \circ F\right)\right) \\ &= d(u \circ F) \wedge d\left(x^{i_1} \circ F\right) \wedge \cdots \wedge d\left(x^{i_k} \circ F\right) \end{aligned}$$

For (3), let $\eta = v dx^J$. Again, by linearity, it suffices to compute $d(u dx^I \wedge v dx^J)$.

$$\begin{aligned} d(u dx^I \wedge v dx^J) &= d(uv dx^I \wedge dx^J) \\ &= (v du + u dv) \wedge dx^I \wedge dx^J \\ &= \left(du \wedge dx^I\right) \wedge \left(v dx^J\right) \wedge \left(dv \wedge u dx^I\right) \wedge dx^J \\ &= \left(du \wedge dx^I\right) \wedge \left(v dx^J\right) \wedge (-1)^k \left(u dx^I\right) \wedge \left(dv \wedge dx^J\right) \\ &= d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

To prove (4), first observe that so long as $k = 1$ and $\omega = \omega_j dx^j$, we have that

$$\begin{aligned} d\omega &= \sum_{i,j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i>j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right) dx^i \wedge dx^j. \end{aligned}$$

This together with Clairaut's theorem implies that

$$d(du) = \frac{\partial^2 u}{\partial x^i \partial x^j} dx^i \wedge dx^j = \sum_{i<j} \left(\frac{\partial^2 u}{\partial x^i \partial x^j} - \frac{\partial^2 u}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0.$$

Now, drop the assumption that $k = 1$. Then expanding $d(d\omega)$ yields a sum of two summations of wedge products. One of which contains the term $d(d\omega_J)$, and the other contains the term $d(dx^{j_i})$. These both equal zero, and thus the entire expression $d(d\omega)$ vanishes. □

Corollary 5.6.4 (Naturality of the exterior derivative). *If F is a smooth map, then*

$$d(F^*\omega) = F^*(d\omega).$$

Corollary 5.6.5. *The exterior derivative is well-defined.*

Proof. Let (U, φ) be a chart on M. Notice that

$$d\omega = \varphi^* d\left(\varphi^{-1^*}\omega\right)$$

on U. Let (V, ψ) be another chart. Then

$$\left(\varphi \circ \psi^{-1}\right)^* d\left(\varphi^{-1^*}\omega\right) = d\left(\left(\varphi \circ \psi^{-1}\right)^* \varphi^{-1^*}\omega\right).$$

Since $\left(\varphi \circ \psi^{-1}\right)^* = \psi^{-1^*} \circ \varphi^*$ and $F^* \circ F^{-1^*} = \mathrm{id}$ for any diffeomorphism F, it follows that

$$\psi^{-1^*} \circ \varphi^* d\left(\varphi^{-1^*}\omega\right) = d\left(\psi^{-1^*}\omega\right).$$
$$\Downarrow$$
$$\varphi^* d\left(\varphi^{-1^*}\omega\right) = \psi^* d\left(\psi^{-1^*}\omega\right).$$

□

Corollary 5.6.6. *Any exact form is closed.*

It is *not* the case, however, that any closed form is exact. Let $M = \mathbb{R}^2 \setminus \{0\}$. Define the 1-form $\omega : M \to T^*M$ by

$$(x, y) \mapsto \frac{xdy - ydx}{x^2 + y^2}.$$

On the one hand, a straightforward computation shows that $d\omega = 0$. On the other hand, recall from basic calculus that ω is exact on a connected open subset $\omega \subset M$ if and only if $\int_c \omega = 0$ for any closed curve $c \subset \omega$. But if $\gamma : [0, 2\pi] \to M$ is given by $(\cos\theta, \sin\theta)$, then

$$\int_\gamma \omega = \int_0^{2\pi} d\theta = 2\pi \neq 0, \tag{†}$$

which means that ω is not exact.

Theorem 5.6.7 (Unique differentiation). *The exterior derivative is the unique linear map $\bar{d} : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}$ such that*

(i) $\bar{d}(\omega \wedge \eta) = \bar{d}\omega \wedge \eta + (-1)^k \omega \wedge \bar{d}\eta$,

(ii) $\bar{d}f(X) = Xf$ *for any* $f \in C^\infty(M)$, *and*

(iii) $\bar{d} \circ \bar{d} = 0$.

For example, consider the linear map $\bar{d} : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M)$ given by

$$\begin{aligned}\bar{d}\omega\left(X_1, \ldots, X_{k+1}\right) = &\sum_{i=1}^{n+1} (-1)^{k+1} X_i \left(w\left(X_1, \ldots, \widehat{X}_i, \ldots, X_{k+1}\right)\right) \\ &- \sum_{i,j} (-1)^{i+j} w\left(\left[X_i, X_j\right], X_1, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{k+1}\right).\end{aligned}$$

This satisfies conditions (i), (ii), and (iii) of Theorem 5.6.7, and thus $\bar{d} = d$.

To conclude this lecture, let's look at a particular dual operation to exterior differentiation, which will be useful for our discussion of orientation.

Let V be a finite-dimensional vector space. For each vector $v \in V$, define *interior multiplication by* v as the linear map $i_v : \bigwedge^k(V) \to \bigwedge^{k-1}(V)$ given by

$$i_v\omega(w_1, \ldots, w_{k-1}) = \omega(v, w_1, \ldots, w_{k-1}).$$

Let $v \lrcorner \omega \coloneqq i_v\omega$.

Extend interior multiplication as follows. For each $X \in \mathscr{X}(M)$ and $\omega \in \mathcal{A}^k(M)$, define the $(k-1)$-form $X \lrcorner \omega$ by $p \mapsto X_p \lrcorner \omega_p$.

5.7 Lecture 21

Definition 5.7.1. Let V be a finite-dimensional vector space. Suppose that E and E' are two bases for V. We say that E and E' are *co-oriented* if the change-of-basis matrix from E to E' has positive determinant.

This notion provides us with exactly two equivalence classes of bases for V, which we call the *orientations for* V. If $[E_1, \ldots, E_n]$ is a chosen orientation for V, then we call any basis in it *(positively) oriented* and any basis not in it *negatively oriented*.

Definition 5.7.2 (Orientation). An *orientation* on a smooth manifold M is a continuous choice of orientation for T_pM as p varies over M.

Equivalently, if $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ denotes the smooth structure on M, we say that M is *orientable* if the Jacobian $D\left[\varphi_\beta \circ \varphi_\alpha^{-1}\right]$ has positive determinant on $\varphi_\alpha(U_\alpha \cap U_\beta)$ for any $\alpha, \beta \in A$.

Example 5.7.3. $\mathbb{S}^n$ is orientable for any $n \geq 1$. For each $p \in \mathbb{S}^n$, say that $(v_1, \ldots, v_n)$ is positively oriented on $T_p\mathbb{S}^n$ if $(p, v_1, \ldots, v_n)$ is positively oriented on $\mathbb{R}^{n+1}$, i.e., is co-oriented with the standard basis for $\mathbb{R}^{n+1}$.

Lemma 5.7.4. *Let $\pi : E \to M$ be a smooth vector bundle and $V \subset E$ be open. If V_p is a convex subspace of E_p for every $p \in M$, then there is some $\sigma \in \Gamma(E)$ such that $\sigma_p \in V_p$ for every p.*

Proof. Find a cover of E by local trivializations U_α over M along with smooth sections σ_α of them. There is some partition of unity ψ_α subordinate to (U_α). Define $\sigma : M \to E$ as $\sum_\alpha \psi_\alpha \sigma_\alpha$, so that $\sigma \in \Gamma(E)$. Then σ_p belongs to V_p by convexity. □

Proposition 5.7.5. *Suppose that M is a smooth n-manifold. Any nowhere vanishing n-form on M gives rise to a unique orientation on M.*

Conversely, any orientation on M gives rise to a nowhere vanishing n-form on M.

Proof.

$(\Longrightarrow)$

Let $\omega \in \mathcal{A}^n(M)$ be nowhere vanishing. For each $p \in M$, we see that ω_p defines an orientation O_M^p on M by declaring that $[e_1, \ldots, e_n] \in O_M^p$ if and only if $\omega_p(e_1, \ldots, e_n) > 0$. It remains to show that if $p \in M$, then we can find some chart U_p around p and some local frame $(E_1, \ldots, E_n)_p$ on U_p such that $\omega_q(E_1\big|_q, \ldots, E_n\big|_q) > 0$

for every $q \in U_p$. To see this, pick any U_p and local frame $(E_1, \ldots, E_n)_p$ on U_p. Write $\omega = f dE^1 \wedge \cdots \wedge dE^n$ locally for some smooth function $f : U_p \to \mathbb{R}$. Since ω is nowhere vanishing, it follows that

$$\omega(E_1, \ldots, E_n) = f \neq 0.$$

Since f is continuous and M connected, we see that $f > 0$ or $f < 0$. We may assume that $f > 0$ for otherwise we can choose $(-E_1, \ldots, -E_n)_p$ instead.

($\Longleftarrow$)

Given $p \in M$ and an orientation O_M^p on T_pM, say that $w \in \bigwedge^n(T_pM)$ is positively oriented if and only if $w(e_1, \ldots, e_n) > 0$ for any $[e_1, \ldots, e_n] \in O_M^p$. Then the subspace $\bigwedge^n_+(T_pM)$ is open and convex. By Lemma 5.7.4, we are done. $\square$

Definition 5.7.6. A diffeomorphism $F : M \to N$ between two oriented manifolds is *orientation-preserving* if the isomorphism dF_p maps positively oriented bases for T_pM to positively oriented bases for $T_{F(p)}N$ for each $p \in M$. It is *orientation-reversing* if it maps positively oriented bases to negatively oriented ones.

We see that

$$\begin{aligned} F \text{ is orientation-preserving} \quad &\Longleftrightarrow \quad \det(dF_p) > 0 \text{ for each } p \in M \\ &\Longleftrightarrow \quad F^*\omega \text{ is positively oriented for any positively oriented form } \omega. \end{aligned}$$

Lemma 5.7.7. *The antipodal map $\alpha : \mathbb{S}^n \to \mathbb{S}^n$ is orientation-preserving if and only if n is odd.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{\alpha} & \mathbb{S}^n \\ \downarrow & & \downarrow \\ \mathbb{R}^{n+1} & \xrightarrow{\hat{\alpha}} & \mathbb{R}^{n+1} \end{array}$$

where $\hat{\alpha}(\vec{x}) \equiv -\vec{x}$. Note that the Jacobian of $\hat{\alpha}$ is precisely the identity matrix I_{n+1}. Since $\det(I_{n+1}) = (-1)^{n+1}$, we see that $\hat{\alpha}$ is orientation-preserving if and only if n is odd. Thus, the restriction α of $\hat{\alpha}$ to $\mathbb{S}^n$ has the same property. $\square$

Corollary 5.7.8. *$\mathbb{RP}^n$ is not orientable when n is even.*

Proof. Let n be even. Suppose, toward a contradiction, that $\mathbb{RP}^n$ admits an orientation. Apply Proposition 5.7.5 to obtain a nowhere vanishing n-form ω on $\mathbb{RP}^n$. If $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ denotes the natural projection, then we also obtain the nowhere vanishing n-form $\pi^*\omega$ on $\mathbb{S}^n$. Applying Proposition 5.7.5 again shows that this determines the usual orientation on $\mathbb{S}^n$.

Note that $\pi \circ \alpha = \pi$, so that $\alpha^*\pi^*\mathbb{S}^n = \pi^*\mathbb{S}^n$. But this implies that α preserves the orientation of $\mathbb{S}^n$, contrary to Lemma 5.7.7. $\square$

The converse of Corollary 5.7.8 is also true, although we omit a proof of it.

Before moving to integration, we should look at a modest variant of our notion of *manifold*. Consider the intersection of $\mathbb{R}^n$ with a half-plane

$$\mathbb{H}^n := \left\{ \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n : x^n \geq 0 \right\}.$$

Definition 5.7.9 (Manifold with boundary).

1. An *n-dimensional manifold with boundary M* is a second-countable Hausdorff space that is locally homeomorphic to either an open Euclidean ball or an open subset of $\mathbb{H}^n$.

2. Any point $p \in M$ is an *interior point* if it belongs to a chart homeomorphic to an open ball.

3. The point p is a *boundary point* if it belongs to a chart that sends p to a point in $\partial\mathbb{H}^n$.

Note that every point in M is either an interior or a boundary point, but not both.

Proposition 5.7.10. *The set of boundary points ∂M is an $(n-1)$-dimensional embedded submanifold of M.*

Moreover, ∂M inherits an orientation from M when M is oriented. This is called the *induced* or *Stokes orientation*. Indeed, we may construct a smooth outward-pointing vector field N along ∂M, which is nowhere tangent to ∂M. Therefore, if ω denotes the orientation form for M, then the form $i^*_{\partial M}(N \lrcorner \omega)$ is an orientation form for ∂M.

Example 5.7.11. $\mathbb{S}^n$ is orientable as the boundary of the closed unit ball.

6 Integration

6.1 Lecture 22

Definition 6.1.1. Let $A^k_0(\mathbb{R}^k)$ denote the space of k-forms with compact support. Let $\omega \in A^k_0(\mathbb{R}^k)$ and $\omega = f dx^1 \wedge \cdots \wedge dx^k$. Define

$$\int_{\mathbb{R}^k} \omega = \int_{\mathbb{R}^k} f(x^1, \dots, x^k) dx^1 \cdots dx^k.$$

Exercise 6.1.2. *Given another coordinate representation $\omega = g y^1 \wedge \cdots \wedge y^k$ with $\det\left(\frac{\partial x}{\partial y}\right) > 0$, show that*

$$\int_{\mathbb{R}^k} f(x^1, \dots, x^k) dx^1 \cdots dx^k = \int_{\mathbb{R}^k} g(x^1, \dots, x^k) dy^1 \cdots dy^k.$$

In other words, Definition 6.1.1 makes sense.

A *singular k-cell on M^n* is a smooth map $\sigma : [0,1]^k \to M$. Note that 0-cells are precisely points in M and 1-cells are precisely smooth curves in M. Let $\omega \in \mathcal{A}^k(M)$ and σ be a singular k-cell on M. Define

$$\int_\sigma \omega = \int_{[0,1]^k} \sigma^* \omega.$$

Proposition 6.1.3. *Let $p : [0,1]^k \to [0,1]^k$ be a diffeomorphism.*

1. *If p is orientation-preserving, then $\int_\sigma \omega = \int_{\sigma \circ p} \omega$.*

2. *If p is orientation-reversing, then $\int_\sigma \omega = -\int_{\sigma \circ p} \omega$.*

Definition 6.1.4.

1. A *singular k-chain on M* is a formal finite $\mathbb{R}$-combination $\sigma = \sum_{i=1}^{N} a_i \sigma_i$ of singular k-cells on M. Define
$$\int_\sigma \omega = \sum_{i=1}^{N} a_i \int_{\sigma_i} \omega.$$
2. Let σ be a singular k-cell on M. Let $i = 1, \ldots, 2k$ and $\alpha = 0, 1$. Define the *(i, α)-face of σ* as the smooth map $\sigma_{(i,\alpha)}$ given by
$$\sigma_{(i,\alpha)}(x^1, \ldots, x^k) = \sigma(x^1, \ldots, x^{i-1}, \alpha, x^i, \ldots, x^k).$$
Moreover, define the *boundary of σ* as the $(k-1)$-chain
$$\partial\sigma \equiv \sum_{i=1}^{k} (-1)^{i+1} \left(\sigma_{(i,1)} - \sigma_{(i,0)}\right).$$
3. If $\sigma \coloneqq \sum_{i=1}^{N} a_i \sigma_i$ is a singular k-chain, then define the *boundary of σ* as the $(k-1)$-chain
$$\partial\sigma \equiv \sum_{i=1}^{N} a_i \partial\sigma_i.$$

Note that $\int_{\partial\sigma} \omega = \sum_{i=1}^{N} a_i \int_{\partial\sigma_i} \omega$.

Definition 6.1.5. A singular k-chain σ is a *closed* if $\partial\sigma = 0$.

Exercise 6.1.6. *Show that if σ is any singular k-chain, then $\partial\sigma$ is closed.*

Theorem 6.1.7 (Stokes's theorem for chains). *Let σ be a k-chain and $\omega \in \mathcal{A}^{k-1}(M)$. Then*
$$\int_\sigma d\omega = \int_{\partial\sigma} \omega.$$

Proof. For now, assume that $M = \mathbb{R}^k$ and $\sigma = I^k$. As the smooth structure on $\mathbb{R}^k$ is global, we may write $\omega = f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k$ for some distinguished $1 \leq i \leq k$ and some smooth function $f : \mathbb{R}^k \to \mathbb{R}$. We compute
$$\begin{aligned} d\omega &= df \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k \\ &= \left(\sum_{j=1}^{k} \frac{\partial f}{\partial x^j} dx^j \right) \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k \\ &= (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge dx^k. \end{aligned}$$

Now, apply Fubini's theorem together with the fundamental theorem of calculus (FTC) to obtain

$$\begin{aligned}
\int_\sigma d\omega &= (-1)^{i-1}\int_{[0,1]^k}\frac{\partial f}{\partial x^i}dx^1\wedge\cdots\wedge dx^k \\
&= (-1)^{i-1}\int_0^1\cdots\int_0^1\left(\int_0^1\frac{\partial f}{\partial x^i}dx^i\right)dx^1\cdots\widehat{dx^i}\cdots dx^k \\
&= (-1)^{i-1}\int_0^1\cdots\int_0^1(f(x^1,\dots,\underbrace{1}_{i\text{-th position}},\dots,x^k)-f(x^1,\dots,\underbrace{0}_{i\text{-th position}},\dots,x^k))dx^1\cdots\widehat{dx^i}\cdots dx^k \\
&= (-1)^{i-1}\left(\int_{[0,1]^{k-1}}f(x^1,\dots,1,\dots,x^k)dx^1\cdots\widehat{dx^i}\cdots dx^k-\int_{[0,1]^{k-1}}f(x^1,\dots,0,\dots,x^k)dx^1\cdots\widehat{dx^i}\cdots dx^k\right) \\
&= (-1)^{i-1}\left(\int_{\sigma_{(i,1)}}\omega-\int_{\sigma_{(i,0)}}\omega\right).
\end{aligned}$$

Moreover, we compute

$$\int_{\partial\sigma}\omega=\sum_{j=1}^k(-1)^{j-1}\left(\int_{\sigma_{(j,1)}}\omega-\int_{\sigma_{(j,0)}}\omega\right).$$

Since x^j is constant along the (j,α)-face for each $\alpha=0,1$, it follows that $dx^j=0$. Therefore,

$$\int_{\partial\sigma}\omega=(-1)^{i-1}\left(\int_{\sigma_{(i,1)}}\omega-\int_{\sigma_{(i,0)}}\omega\right)=\int_\sigma d\omega.$$

Finally, assume that M is arbitrary and σ is an arbitrary k-cell on M. By the special case just proved, we have that

$$\int_\sigma d\omega=\int_{[0,1]^k}\sigma^*(d\omega)=\int_{[0,1]^k}d(\sigma^*\omega)=\int_{\partial[0,1]^k}\sigma^*\omega=\int_{\partial\sigma}\omega.$$

This clearly remains true if σ is a k-chain on M. □

The FTC occurs precisely when $\sigma=I^1$ and $\omega=f$. This shows that Theorem 6.1.7 is equivalent to the FTC.

6.2 Lecture 23

Let M be an orientable manifold. Let $\omega\in\mathcal{A}^n(M)$. Let σ_1 and σ_2 be singular n-cells on M that can be extended to diffeomorphisms on (open) neighborhoods of $[0,1]^n$. Suppose that both are orientation-preserving.

Lemma 6.2.1. *If* $\operatorname{supp}\omega\subset\sigma_1([0,1]^n)\cap\sigma_2([0,1]^n)$, *then* $\int_{\sigma_1}\omega=\int_{\sigma_2}\omega$.

Proof. Since $\operatorname{supp}\omega\subset\sigma_1([0,1]^n)\cap\sigma_2([0,1]^n)$, Proposition 6.1.3 implies that

$$\int_{\sigma_1}\omega=\int_{\sigma_2\circ(\sigma_2^{-1}\circ\sigma_1)}\omega=\int_{\sigma_2}\omega.$$

□

Let $\omega\in\mathcal{A}^n(M)$. Let σ be an orientation-preserving singular n-cell on M. If $\operatorname{supp}\omega\subset\sigma([0,1]^n)$, then Lemma 6.2.1 allows us to define

$$\int_M\omega=\int_\sigma\omega.$$

In general, there exists an open cover (U_α) of M such that $U_\alpha \subset \sigma_\alpha([0,1]^n)$ for each α where σ_α is some orientation-preserving singular n-cell on M. Find a partition of unity (φ_α) subordinate to this cover. Note that each $\varphi_\alpha \omega$ belongs to $\mathcal{A}^n(M)$ and is supported in U_α. If ω is compactly supported, then $\operatorname{supp}\omega$ intersects at most finitely many $\operatorname{supp}\varphi_\alpha$. In this case, we define

$$\int_M \omega = \sum_\alpha \int_M \varphi_\alpha \omega,$$

which is finite. It remains to check that this definition makes sense.

Lemma 6.2.2. *If (V_β, ψ_β) is another such partition of unity, then $\sum_\beta \int_M \psi_\beta \omega = \sum_\alpha \int_M \varphi_\alpha \omega$.*

Proof.

$$\begin{aligned}\sum_\alpha \int_M \varphi_\alpha \omega &= \sum_\alpha \int_M \varphi_\alpha \sum_\beta \psi_\beta \omega \\ &= \sum_\alpha \sum_\beta \int_M \varphi_\alpha \psi_\beta \omega \\ &= \sum_\beta \sum_\alpha \int_M \psi_\beta \varphi_\alpha \omega \\ &= \sum_\beta \int_M \psi_\beta \sum_\alpha \varphi_\alpha \omega \\ &= \sum_\beta \int_M \psi_\beta \omega.\end{aligned}$$

□

Note 6.2.3. If ω is not assumed to be compact, then $\int_M \omega$ may be infinite but is still well-defined.

Theorem 6.2.4 (Stokes). *Let M be an oriented compact n-manifold with boundary. If $\omega \in \mathcal{A}^{n-1}(M)$, then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. There are three cases to consider.

Case 1: Suppose that there is some orientation-preserving n-cell σ on M such that $\operatorname{supp}\omega \subset \operatorname{Int}(\operatorname{im}\sigma)$ and $\operatorname{im}\sigma \cap \partial M = \emptyset$. By Theorem 6.1.7, it follows that

$$\int_M d\omega = \int_\sigma d\omega = \int_{\partial\sigma} \omega = 0 = \int_{\partial M} \omega.$$

Case 2: Suppose that there is some orientation-preserving n-cell σ on M such that $\operatorname{supp}\omega \subset \operatorname{im}\sigma$, $\operatorname{im}\sigma \cap \partial M = \sigma_{(n,0)}\left([0,1]^{n-1}\right)$, and $\operatorname{supp}\omega \cap \operatorname{im}\partial\sigma \subset \sigma_{(n,0)}$. By Theorem 6.1.7, it follows that

$$\int_M d\omega = \int_\sigma d\omega = \int_{\partial\sigma} \omega = (-1)^n \int_{\sigma_{(n,0)}} \omega.$$

Note that if μ denotes the usual orientation on $\mathbb{H}^n$, then the induced orientation on the boundary $\partial\mathbb{H}^n$ is equal to $(-1)^n \mu$. Therefore, $\sigma_{(n,0)} : [0,1]^{n-1} \to \partial M$ is orientation-preserving if and only if n is even. In either situation, we have that

$$(-1)^n \int_{\sigma_{(n,0)}} \omega = \int_{\partial M} \omega,$$

which completes our present case.

Case 3: In general, there exist an open cover (U_α) of M and a partition of unity (φ_α) subordinate to it such that each $\varphi_\alpha\omega$ is an $(n-1)$-form of the kind in Case 1 or Case 2. Since $\sum_\alpha \varphi_\alpha$ is constant, we see that

$$0 = d\left(\sum_\alpha \varphi_\alpha\right) = \sum_\alpha d\varphi_\alpha.$$

Hence $\sum_\alpha d\varphi_\alpha \wedge \omega = 0$, so that $\sum_\alpha \int_M d\varphi_\alpha \wedge \omega = 0$. From this we compute

$$\begin{aligned}
\int_M d\omega &= \int_M \sum_\alpha \varphi_\alpha d\omega \\
&= \sum_\alpha \int_M \varphi_\alpha d\omega \\
&= \sum_\alpha \int_M d\varphi_\alpha \wedge \omega + \varphi_\alpha d\omega \\
&= \sum_\alpha \int_M d(\varphi_\alpha \omega) \\
&= \sum_\alpha \int_{\partial M} \varphi_\alpha \omega \\
&= \int_{\partial M} \omega.
\end{aligned}$$

□

7 De Rham cohomology

7.1 Lecture 24

Given a smooth manifold M^n and integer $k \geq 1$, consider the vector spaces

$$\begin{aligned}
Z^k(M) &:= \{\omega \in \mathcal{A}^k(M) : d\omega = 0\} \\
B^k(M) &:= \{d\eta : \eta \in \mathcal{A}^{k-1}(M)\}.
\end{aligned}$$

Since $B^k(M) \subset Z^k(M)$, we may form the quotient space

$$H^k_{\mathrm{dR}}(M) := {}^{Z^k(M)}\!\big/_{B^k(M)},$$

called the *k-th de Rham cohomology group of M*.

Remark 7.1.1. This is the same as the singular cohomology group over $\mathbb{R}$.

$H^k_{\mathrm{dR}}(M)$ can be thought of as a quantitative measure of the number of submanifolds of M over which we can't integrate certain closed forms to find a potentials for them. In this sense, the failure of a closed form to be exact indicates holes in M.

Theorem 7.1.2. *If M and N are continuously homotopy equivalent, then $H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$ for each $k \geq 1$.*

Recall that a space X is *contractible* if id_X is smoothly homotopic to the constant map at some point in X.

Lemma 7.1.3 (Poincaré). *If M is contractible, then $H^k_{\mathrm{dR}}(M) = 0$ for each $k \geq 1$.*

Proof. For simplicity, assume that $k = 1$. For each $t \in [0,1]$, define $\iota_t : M \to M \times [0,1]$ by $p \mapsto (p,t)$.

Claim. *If ω is any closed 1-form on $M \times [0,1]$, then $\iota_1^*\omega - \iota_0^*\omega$ is exact.*

Proof. If $\pi_M : M \times [0,1] \to M$ denotes the projection and $\left(U, x^i\right)$ denotes local coordinates on M, then $\left(\pi_M^{-1}(U), (\bar{x}^i, t)\right)$ is a coordinate chart on $M \times [0,1]$ where $\bar{x}^i \coloneqq x^i \circ \pi_M$. We thus have that $\omega = w_i d\bar{x}^i + f dt$. For each $\alpha \in \{0,1\}$, we see that

$$\iota_\alpha^*\omega = \iota_\alpha^*(w_i d\bar{x}^i + f dt) = w_i(-,\alpha) dx^i + 0.$$

Moreover,

$$\begin{aligned} 0 &= d\omega \\ &= dw_i \wedge d\bar{x}^i + df \wedge dt \\ &= (\text{terms not involving } dt) \\ &+ \frac{\partial w_i}{\partial t} dt \wedge d\bar{x}^i + \frac{\partial f}{\partial \bar{x}^i} d\bar{x}^i \wedge dt. \end{aligned}$$

This implies that $\frac{\partial w_i}{\partial t} = \frac{\partial f}{\partial \bar{x}^i}$ for each i. For each $p \in U$, we compute the sum

$$w_i(p,1) - w_i(p,0) = \int_0^1 \frac{\partial w_i}{\partial t}(p,t) dt = \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p,t) dt.$$

As a result,

$$\iota_1^*\omega - \iota_0^*\omega = \left(\int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p,t) dt\right) dx^i.$$

Now, define $g : U \to \mathbb{R}$ by $\int_0^1 f(p,t) dt$, so that

$$\frac{\partial g}{\partial x^i} = \int_0^1 \frac{\partial f}{\partial \bar{x}^i}(p,t) dt.$$

It follows that $\iota_1^*\omega - \iota_0^*\omega = \frac{\partial g}{\partial x^i} dx^i = dg$. Since the pullback is coordinate-independent, g must be as well. This completes our proof. □

By assumption, there is some smooth map $H : M \times [0,1] \to M$ such that $H \circ \iota_1 = \mathrm{id}_M$ and $H \circ \iota_0 = e_{p_0}$ where $p_0 \in M$. Let ω be a closed 1-form on M. Then $H^*\omega$ is closed since pullbacks commute with exterior derivatives. Recall that the pullback is a contravariant functor, giving us

$$\iota_k^* H^*\omega = \left(H \circ \iota_k\right)^* \omega$$

for each $k = 0, 1$. By our claim, it follows that

$$\iota_1^* H^*\omega - \iota_0^* H^*\omega = \omega - 0 = \omega$$

is closed. □

The generalization of this result to any positive integer k proceeds as follows.

We have the decomposition

$$T_{(p,t)}M \times [0,1] = \ker d\pi\big|_{(p,t)} \oplus \ker d\pi_M\big|_{(p,t)}$$

where $\pi : M \times [0,1] \to [0,1]$ denotes projection. Then any 1-form ω on $M \times [0,1]$ may be written uniquely as $\omega = \omega_1 + \omega_2$ such that $\omega_i(v_1+v_2) = \omega(v_i)$ for each $i = 1, 2$. Hence there is some unique map $f : M \times [0,1] \to \mathbb{R}$ such that $\omega_2 = f dt$. In general, one can show that if ω is a k-form on $M \times [0,1]$, then we can write ω uniquely as

$$\omega = \omega_1 + (dt \wedge \eta)$$

where $\omega_1(v_1, \ldots, v_k) = 0$ if some v_i belongs to $\ker d\pi_M\big|_{(p,t)}$ and η is a $(k-1)$-form with the analogous property.

Lemma 7.1.4. *Define the $(k-1)$-form $I\omega$ on M by*

$$I\omega\big|_p(v_1, \ldots, v_{k-1}) = \int_0^1 \eta(p,t)\left(d\iota_t\big|_{(p,t)}(v_1), \ldots, d\iota_t\big|_{(p,t)}(v_{k-1})\right) dt.$$

Then $\iota_1^*\omega - \iota_0^*\omega = d(I\omega) + I(d\omega)$. In particular, $\iota_1^*\omega - \iota_0^*\omega$ is exact whenever $d\omega = 0$.

Proof. For an argument similar to our case where $k = 1$, see Theorem 7.17 (Spivak). In particular, $I\omega$ and η correspond to our g and f, respectively. $\square$

Corollary 7.1.5. *Recalling (†), we see that $\mathbb{R}^2 \setminus \{0\}$ is not contractible.*

This proves that $\mathbb{R}^2 \setminus \{0\} \not\approx \mathbb{R}^2$.

7.2 Lecture 25

Corollary 7.2.1. *If M is closed (i.e., compact without boundary) and orientable, then M is not contractible.*

Proof. There is some positively oriented orientation form ω on M. Then $d\omega = 0$, and $\int_M \omega > 0$. But if $\omega = d\eta$ for some form η, then $\int_M \omega = \int_{\partial M} \eta = 0$ thanks to Theorem 6.2.4, a contradiction. Hence $H^n(M) \neq 0$. $\square$

Example 7.2.2. $\mathbb{S}^n$ is not contractible.

Theorem 7.2.3. *If M is a (connected) orientable n-manifold, then we have an isomorphism*

$$\underbrace{H_c^n(M)}_{\textit{compactly supported}} \xrightarrow{\cong} \mathbb{R}, \qquad [\omega] \mapsto \int_M \omega.$$

Proof. Assume that this statement holds when $M = \mathbb{R}^n$. There is some compactly supported orientation form ω on M such that $\int_M \omega \neq 0$ and $\operatorname{supp} \omega \subset \underset{\text{open}}{U} \subset M$. Let ω' be a compactly supported n-form on M. Pick a partition of unity (φ_α) on M. Then $\omega' = \varphi_1\omega' + \cdots + \varphi_k\omega'$, Thus, we may assume that $\operatorname{supp} \omega' \subset V$ where $V \approx \mathbb{R}^n$. We want to show that $\omega' = c\omega + d\eta$ for some $c \in \mathbb{R}$ and some $\eta \in \mathcal{A}^{n-1}(M)$. Since M is connected, there is some sequence

$$U = V_1, V_2, \ldots, V_r = V$$

of open sets such that $V_i \approx \mathbb{R}^n$ and $V_i \cap V_{i+1} \neq \emptyset$ for each $i = 1, \ldots, r-1$. We can find a family $\{\omega_i\}_{1 \leq i \leq r-1}$ of forms on M such $\int_M \omega_i \neq 0$ and $\operatorname{supp} \omega_i \subset V_i \cap V_{i+1}$. It follows that

$$\begin{aligned} \omega_1 &= c_1\omega + d\eta_1 \\ \omega_2 &= c_2\omega_1 + d\eta_2 \\ &\vdots \\ \omega' &= c_r\omega_{r-1} + d\eta_r, \end{aligned}$$

as desired. ☐

If M and N are closed orientable n-manifolds and $f : M \to N$ is smooth, then the pullback f^* induces a linear map $f^* : H^n_{\mathrm{dR}}(N) \to H^n_{\mathrm{dR}}(M)$. In light of Theorem 7.2.3, we get a linear map $f^* : \mathbb{R} \to \mathbb{R}$, which shows that there is a unique real number a such that

$$\int_M f^*\omega = a \int_N \omega$$

for every $\omega \in H^n_{\mathrm{dR}}(N)$. The scalar a is called the *degree of* f.

7.3 Lecture 26

Let M and N be closed orientable n-manifolds and $f : M \to N$ be smooth. By Theorem 3.6.6, find some regular value q of f. For each $p \in f^{-1}(q)$, let

$$\operatorname{sgn}_p f = \begin{cases} 1 & df_p \text{ orientation-preserving} \\ -1 & df_p \text{ orientation-reversing} \end{cases}.$$

Theorem 7.3.1.

$$\deg f = \sum_{p \in f^{-1}(q)} \operatorname{sgn}_p f$$

where $\deg f \equiv 0$ *if* $f^{-1}(q) = \emptyset$. *In particular,* $\deg f$ *is always an integer.*

Proof. Since f has constant rank n and $\{q\} \subset N$ is compact, we see that $f^{-1}(q)$ is a compact 0-dimensional submanifold of M by Theorem 3.6.4 and thus must be finite. Let $f^{-1}(q) = \{p_1, \ldots, p_k\}$. Find charts $U_1, \ldots, U_k$ which are pairwise disjoint so that each $u_i \in U_i$ is a regular point of f. Find a chart (V, y^i) around q such that the components of $f^{-1}(V)$ are precisely the U_i. Let $\omega = g dy^1 \wedge \cdots \wedge dy^n$ where g is nonnegative and compactly supported in V. This implies that $f^*\omega \subset f^{-1}(V) = U_1 \sqcup \cdots \sqcup U_k$. Therefore,

$$\int_M f^*\omega = \sum_{i=1}^k \int_{U_i} f^*\omega.$$

Since each $f\restriction_{U_i} : U_i \to V$ is a diffeomorphism, we have that

$$\int_{U_i} f^*\omega = \begin{cases} \int_V \omega & f\restriction_{U_i} \text{ orientation-preserving} \\ -\int_V \omega & f\restriction_{U_i} \text{ orientation-reversing} \end{cases}.$$

As a result,

$$\int_M f^*\omega = \left(\sum_{p \in f^{-1}(q)} \operatorname{sgn}_p f \right) \int_V \omega = \left(\sum_{p \in f^{-1}(q)} \operatorname{sgn}_p f \right) \int_M \omega.$$

☐

Example 7.3.2. Let $A_n : \mathbb{S}^n \to \mathbb{S}^n$ denote the antipodal map. Choose $p_0 \in \mathbb{S}^n$, which is a regular value of A_n. Hence $\deg A_n = (-1)^{n-1}$.

Theorem 7.3.3. *Suppose that f and g are smoothly homotopic maps $M \to N$. Then $f^* = g^*$ as linear maps.*

Proof. By assumption, there exists a smooth map $H : M \times [0,1] \to M$ such that $H \circ \iota_0 = f$ and $H \circ \iota_1 = g$. Let $\omega \in Z^k(N)$. We apply Lemma 7.1.4 (including its notation) to compute

$$\begin{aligned}
&g^*\omega - f^*\omega \\
&= (H \circ \iota_1)^* \omega - (H \circ \iota_0)^* \omega \\
&= \iota_1^*(H^*\omega) - \iota_0^*(H^*\omega) \\
&= d(IH^*\omega) + I(dH^*\omega) = d(IH^*\omega).
\end{aligned}$$

This implies that $f^*([\omega]) = g^*([\omega])$, as desired. ☐

Corollary 7.3.4. *If f and g are smoothly homotopic, then $\int_M f^*\omega = \int_M g^*\omega$ for any closed n-form ω.*

Proof. By Theorem 7.3.3, $f^*\omega = g^*\omega + d\eta$ for some $(n-1)$-form η. Since M is closed by hypothesis, applying $\int$ to both sides and then invoking Stokes's theorem finishes our proof. ☐

Corollary 7.3.5. *If f and g are smoothly homotopic, then* $\deg f = \deg g$.

Corollary 7.3.6 (Hairy ball). *If $n \in \mathbb{N}$ is even, then there is no non-vanishing vector field on $\mathbb{S}^n$.*

Proof. The identity map $\mathrm{id}_{\mathbb{S}^n}$ has degree 1 and thus is not homotopic to the antipodal map A_n. Suppose, toward a contradiction, that there is some non-vanishing $X \in \mathscr{X}(\mathbb{S}^n)$. For each $p \in \mathbb{S}^n$, there is a unique great semicircle γ_p traveling from $\underset{t=0}{p}$ to $\underset{t=1}{A(p)}$ whose tangent vector at p equals cX_p for some $c \in \mathbb{R}$. The smooth map $H(p,t) \equiv \gamma_p(t)$ defines a homotopy between $\mathrm{id}_{\mathbb{S}^n}$ and A_n, a contradiction. ☐

8 Integral curves and flows

8.1 Lecture 27

Definition 8.1.1. Let M be a smooth manifold and $X \in \mathscr{X}(M)$. We say that a differentiable curve $\gamma : J \to M$ is an *integral curve for X* if $\gamma'(t) = X_{\gamma(t)}$ for any $t \in J$.

Terminology. If $0 \in J$, then $\gamma(0)$ is called the *starting point of* γ.

Example 8.1.2. Let $M = \mathbb{R}^2$, $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, and $\gamma(t) = (x(t), y(t))$. Then $\gamma'(t) = x'(t)\frac{\partial}{\partial x} + y'(t)\frac{\partial}{\partial y}$. The system

$$\begin{cases} x(t) = x'(t) \\ y(t) = y'(t) \end{cases}$$

determines that $\gamma(t) = e^t(x(0), y(0))$.

In general, define the vector field $x^i \frac{\partial}{\partial x^i}$ on a chart $\left(U, x^i\right)$ for the n-manifold M. Then given an integral curve $\gamma(t) = \left(\gamma^1(t), \ldots, \gamma^n(t)\right)$ for X where $\gamma^i = \gamma \circ x^i$, we obtain the system

$$\gamma'^i(t) = X^i\left(\gamma^1(t), \ldots, \gamma^n(t)\right).$$

Given that $\gamma(0) = p$, we have an initial value problem, to which we can always find a *local* solution.

Theorem 8.1.3 (Fundamental theorem for autonomous ODEs). *Let $U \subset \mathbb{R}^n$ be open and $X : U \to \mathbb{R}^n$ be a smooth vector field. Consider the initial value problem*

$$\begin{cases} \gamma'^i(t) = X^i\left(\gamma^1(t), \ldots, \gamma^n(t)\right) \\ \gamma(t_0) = (c^1, \ldots, c^n) \end{cases}. \tag{1}$$

(a) (Existence) Let $t_0 \in \mathbb{R}$ and $x_0 \in U$. There exist some interval $J_0 \ni t_0$ and open subset $U_0 \subset U$ such that for each $c \in U_0$, there is some C^1 curve $\gamma : J_0 \to U_0$ that solves Eq. (1).

(b) (Uniqueness) Any two differentiable solutions to Eq. (1) agree on their common domain.

(c) (Smoothness) Let J_0 and U_0 be as in (a). Define $\theta : J_0 \times U_0 \to U$ by $(t, x) \mapsto \gamma_x(t)$ where $\gamma_x : J_0 \to U$ uniquely solves Eq. (1) with initial condition $\gamma(t_0) = x$. Then θ is smooth.

Example 8.1.4. For any compact manifold M, we may stipulate that the U_0 form a finite cover $\{U_1, \ldots, U_k\}$ of M. Make J_0 smaller than any of the corresponding intervals $J_1, \ldots, J_k$. This yields a smooth map $\theta : J \times \mathbb{S}^n \to \mathbb{S}^n$ defined by $(t, p) \mapsto \gamma_p^i(t)$.

Corollary 8.1.5. *Let X be a smooth vector field on M and $p \in M$. There is some $\epsilon > 0$ along with a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and γ is an integral curve for X.*

Definition 8.1.6. Let $\theta : \mathbb{R} \times M \to M$ be a group action on M.

1. We call θ a *global flow on M* if it is smooth, i.e., $\theta^p(t) \coloneqq \theta(t, p) : \mathbb{R} \to M$ is smooth for every $p \in M$.
2. We call the vector field $p \mapsto (\theta^p)'(0)$ the *infinitesimal generator of θ*.

Question. When is a smooth vector field an infinitesimal generator of a global flow?

Example 8.1.7. Define $X = x^3 \frac{\partial}{\partial x}$ on $\mathbb{R}^2$. Then any integral curve $\gamma(t) = (x(t), y(t))$ for X must satisfy

$$\begin{aligned} \frac{dx}{dt} = x^3 &\implies dx = x^3 dt \\ &\implies -\frac{1}{2x^2} = t + c \\ &\implies x(t) = \frac{1}{\sqrt{c - 2t}}, \end{aligned}$$

which is not smooth on $\mathbb{R}$. Hence X fails to generate a global flow.

Lemma 8.1.8 (Escape). *Let $X \in \mathscr{X}(M)$ and γ be an integral curve for X. If the domain of γ is* not *equal to $\mathbb{R}$, then $\operatorname{im} \gamma$ is not contained in any compact set.*

Remark 8.1.9. If M is compact, then every smooth vector field on M generates a global flow.

Definition 8.1.10. A *flow domain* for M is an open subset $D \subset \mathbb{R} \times M$ such that for every $p \in M$, the set $\{t \in \mathbb{R} \mid (t,p) \in D\}$ is an open interval containing 0

Theorem 8.1.11 (Fundamental theorem on flows). *Let M be a smooth manifold and $X \in \mathscr{X}(M)$. There exist some unique maximal flow domain $\mathcal{D} \subset \mathbb{R} \times M$ and unique flow $\varphi : \mathcal{D} \to M$ such that X generates φ.*

Terminology. We call φ the *flow of X*.

Corollary 8.1.12. *If M is a closed manifold, then $\mathcal{D} = \mathbb{R} \times M$.*

8.2 Lecture 28

Let M be a smooth manifold without boundary. Let $V \in \mathscr{X}(M)$ and let θ denote the flow of V. For any $W \in \mathscr{X}(M)$, define the section of TM by

$$(\mathcal{L}_V W)_p \equiv \lim_{t \to 0} \frac{d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) - W_p}{t},$$

which always exists. This is called the *Lie derivative of W with respect to V*.

Proposition 8.2.1. *$\mathcal{L}_V W \in \mathscr{X}(M)$.*

We can view the Lie derivative at a point p as the rate of change of W along the tangent vector $V\big|_p$.

Theorem 8.2.2. *If $V, W \in \mathscr{X}(M)$, then $\mathcal{L}_V W = [V, W]$.*

Proof. Let $\mathcal{R}(M)$ denote the set of points $p \in M$ such that $V_p \neq 0$. Note that $\operatorname{cl}(\mathcal{R}(M)) = \operatorname{supp} V$. Let $p \in M$. We have three cases to consider.

(i) Suppose that $p \in \mathcal{R}(M)$. We can find smooth coordinates $\left(U, u^i\right)$ near p such that $V = \frac{\partial}{\partial u^1}$. In these coordinates we thus have that $\theta_t(u) = \left(u^1 + t, u^2, \dots, u^n\right)$. The Jacobian of θ_{-t} at each t equals the identity. For any $u \in U$, it follows that

$$\begin{aligned}
&d(\theta_{-t})_{\theta_t(u)}\left(W_{\theta_{t(u)}}\right) \\
&= d(\theta_{-t})_{\theta_t(x)}\left(W^j(u^1 + t, u^2, \dots, u^n)\frac{\partial}{\partial u^j}\Big|_{\theta_t(u)}\right) \\
&= W^j(u^1 + t, u^2, \dots, u^n)\frac{\partial}{\partial u^j}\Big|_u.
\end{aligned}$$

From this we compute

$$\begin{aligned}
(\mathcal{L}_V W)_p &= \frac{d}{dt}\Big|_{t=0} W^j(u^1 + t, u^2, \dots, u^n)\frac{\partial}{\partial u^j}\Big|_u \\
&= \frac{\partial}{\partial u^1} W^j(u^1, u^2, \dots, u^n)\frac{\partial}{\partial u^j}\Big|_u \\
&= [V, W]_u .
\end{aligned}$$

(ii) Suppose that $p \in \operatorname{supp} V \setminus \mathcal{R}(M)$. Since $\operatorname{supp} V$ is dense in M and TM is Hausdorff, it follows that $(\mathcal{L}_V W)_p = [V, W]_p$.

(iii) If $p \in M \setminus \operatorname{supp} V$, then V vanishes on some neighborhood H of p. This implies that $\theta_t = \mathrm{id}_H$, so that $d(\theta_{-t})_{\theta_t(p)}\left(W_{\theta_t(p)}\right) = W_p$. Hence $\left(\mathcal{L}_V W\right)_p = 0 = [V, W]_p$.

$\square$

Definition 8.2.3. A smooth local frame $(X_1, \dots X_n)$ is called a *commuting* or *holonomic frame* if $[X_i, X_j] = 0$ for any $1 \le i, j \le n$.

Theorem 8.2.4. *Let M be a smooth n-manifold. Let $(X_1, \dots, X_k)$ be a linearly independent k-tuple of smooth commuting vector fields defined on an open set $W \subset M$. For any $p \in W$, there is some chart $\left(U, x^i\right)$ around p such that*

$$X_i = \frac{\partial}{\partial x^i}$$

on U for each $i = 1, \dots, k$.

Proof sketch. As this statement is local, we may assume that $M = \mathbb{R}^n$ and $p = 0$. Since the X_i are linearly independent, we can find coordinates $\left(V, t^i\right)$ around 0 such that $X_i\big|_0 = \frac{\partial}{\partial t^i}\big|_0$ for each i. Let θ^i denote the flow of X_i. By making V a sufficiently small neighborhood of 0 in $\mathbb{R}^k \times \mathbb{R}^{n-k} \approx \mathbb{R}^n$, define $\Psi : V \to \mathbb{R}^n$ by

$$\Psi(t^1, \dots, t^n) = \theta^1_{t^1} \circ \dots \circ \theta^k_{t^k}\left(0, \dots, 0, t^{k+1}, \dots, t^n\right).$$

Since the X_i are commuting, one can show that

$$d\Psi_0 = \begin{cases} X_i\big|_0 & i = 1, \dots, k \\ \frac{\partial}{\partial t^i}\big|_0 & i = k+1, \dots, n. \end{cases}$$

This is invertible, and thus Ψ is a local diffeomorphism by the inverse function theorem. This gives us our desired local coordinates. $\square$

9 Distributions

Definition 9.0.1. Let M be a smooth manifold. A *k-distribution on M* is a rank-k smooth subbundle of TM.

In particular, 1-distributions are precisely vector fields.

Definition 9.0.2. Let $N \subset M$ be a nonempty submanifold and

$$D \coloneqq \coprod_{p \in M} D_p$$

be a distribution on M. Then N is called an *integral manifold of D* if $D_p = T_p N$ for each $p \in N$. Moreover, we say that D is *integrable* if each $p \in M$ is contained in an integrable manifold of D.

Definition 9.0.3. We say that a distribution D is *involutive* if $[X, Y] \in D$ whenever $X, Y \in D$.

Proposition 9.0.4. *If D is integrable, then it is involutive.*

Theorem 9.0.5 (Frobenius). *If D is involutive, then it is integrable.*