

### Abstract

We present solutions to all exercises from Scott Weinstein’s “Model Theory” course lectures at UPenn. These are relatively self-contained and are meant to complement Weinstein’s written memoirs of our class meetings. The official reference for the course is David Marker’s *Model Theory: An Introduction*.

1. Let  $I$  be a countably infinite set. Let  $\mathbb{D} := \langle I, E \rangle$  be a structure where  $E$  is an equivalence relation for which there is exactly one equivalence class of size  $k$  for each  $k \in \mathbb{Z}_{\geq 1}$ .

- (1) Show that the set  $\Lambda$  of (first-order) sentences expressing that  $E$  is an equivalence relation with exactly one equivalence class of size  $k$  for each  $k \in \mathbb{Z}_{\geq 1}$  axiomatizes  $\mathbb{D}$ , i.e.,  $\text{Th}(\mathbb{D}) = \text{Cn}(\Lambda)$  where

$$\text{Cn}(\Lambda) := \{\varphi \in \text{FO}_{\mathbb{D}} \mid \Lambda \models \varphi\}.$$

- (2) Show that for every (first-order) formula  $\theta(y, \bar{w})$  and every  $\bar{a} \in I$ , the set

$$\theta[\mathbb{D}, \bar{a}] := \{x \in \text{dom}(\mathbb{D}) \mid \mathbb{D} \models \theta[x, \bar{a}]\}$$

is either finite or cofinite.

- (1) It suffices to prove that  $\Lambda$  is complete. For, in this case, any two models of  $\Lambda$  must be elementarily equivalent.

**Claim 1.** *Let  $\mathbb{E}$  be any model of  $\Lambda$  of size  $\kappa \geq \omega$ . There exists an elementary extension  $\mathbb{E}_{\kappa} \succeq \mathbb{E}$  of size  $\kappa$  such that  $\mathbb{E}_{\kappa}$  has exactly  $\kappa$  equivalence classes each of size  $\kappa$ .*

*Proof.* Let  $\lambda$  denote the cardinality of the set of all equivalence classes in  $\text{dom}(\mathbb{E})$ . Note that  $\lambda \leq \kappa$ . For every  $\alpha, \beta \in \kappa$ , adjoin to the language of  $\mathbb{E}$  a new constant symbol  $c(\alpha, \beta)$ . Consider the theory

$$\Delta := \Lambda \cup \{Ec(x, y)c(x, z) \mid x, y, z \in \kappa\} \cup \{\neg Ec(x, 0)c(y, 0) \mid x, y \in \kappa, x \neq y\}.$$

Any finite subset  $F$  of  $\Delta$  is satisfiable by a suitable expansion  $\mathbb{E}_F$  of  $\mathbb{E}$ . Then there exists an ultrafilter on the family of finite subsets of  $\Delta$  such that the ultraproduct

$$\prod_{\substack{F \subset \Delta \\ \text{finite}}} \mathbb{E}_F / \mathcal{U}$$

satisfies  $\Delta$ . Moreover, its reduct  $\mathbb{A}$  to the language of  $\mathbb{E}$  is an elementary extension of  $\mathbb{E}$ . By the downward Löwenheim-Skolem theorem, there exists a structure  $\mathbb{E}_0$  of size  $\kappa$  such that  $\mathbb{A} \succeq \mathbb{E}_0 \succeq \mathbb{E}$ .

Now, repeat our preceding construction  $\omega$  times to get an increasing chain

$$\mathbb{E} \preceq \mathbb{E}_0 \preceq \mathbb{E}_1 \preceq \mathbb{E}_2 \preceq \dots$$

of structures such that each  $\text{dom}(\mathbb{E}_i)$  has cardinality  $\kappa$ . Note that  $\mathbb{E}_{\kappa}$  is an elementary extension of  $\mathbb{E}$ . Further, the domain of the direct limit  $\mathbb{E}_{\kappa} := \bigcup_{i \in \omega} \mathbb{E}_i$  also has cardinality  $\kappa$ , so that  $\mathbb{E}_{\kappa}$  has exactly  $\kappa$  equivalence classes. Finally, for any  $x \in \mathbb{E}_{\kappa}$ ,  $x$  belongs to some  $\mathbb{E}_n$ . Hence the equivalence class  $[x]$  has size  $\kappa$  in  $\mathbb{E}_{n+1}$  and thus in  $\mathbb{E}_{\kappa}$ . It follows that every equivalence class in  $\mathbb{E}_{\kappa}$  has size  $\kappa$ .  $\square$

Suppose, toward a contradiction, that there is a sentence  $\varphi$  in the language of  $\mathbb{D}$  such that neither  $\varphi$  nor  $\neg\varphi$  belongs to  $\text{Cn}(\Lambda)$ . Then there are models  $\mathbb{A}^1$  and  $\mathbb{A}^2$  of  $\Lambda$  such that  $\mathbb{A}^1 \models \neg\varphi$  and  $\mathbb{A}^2 \models \varphi$ . By the Löwenheim-Skolem theorem, we may assume that both of these have size  $\kappa \geq \omega$ . By Claim 1, we thus have two structures  $\mathbb{A}_{\kappa}^1$  and  $\mathbb{A}_{\kappa}^2$  such that  $\mathbb{A}_{\kappa}^1 \models \neg\varphi$  and  $\mathbb{A}_{\kappa}^2 \models \varphi$ . But it’s easy to see that  $\mathbb{A}_{\kappa}^1$  and  $\mathbb{A}_{\kappa}^2$  must be isomorphic, which yields a contradiction.

- (2) Suppose, toward a contradiction, that there exist a formula  $\theta(y, w_1, \dots, w_n)$  and an element  $\bar{a} \in I$  such that  $\theta[\mathbb{D}, \bar{a}]$  is both infinite and coinfinite. Adjoin to the language of  $\mathbb{D}$  new constant symbols  $\bar{e} := (e_1, \dots, e_n)$ ,  $c$ , and  $d$ . For each  $k \in \mathbb{Z}_{\geq 1}$ , let  $\lambda_k(x)$  denote the formula expressing that the equivalence class of  $x$  has cardinality  $> k$ . Now, consider the theory

$$\begin{aligned} \Gamma := \Lambda \cup \{ \lambda_k(c) \mid k \geq 1 \} &\cup \{ \lambda_k(d) \mid k \geq 1 \} \\ &\cup \{ \neg E e_i c \mid 1 \leq i \leq n \} \\ &\cup \{ \neg E e_i d \mid 1 \leq i \leq n \} \\ &\cup \{ \theta(c, \bar{e}), \neg \theta(d, \bar{e}) \} \end{aligned}$$

in our new language.

Let  $F$  be any finite subset of  $\Gamma$ . Since both  $\theta[\mathbb{D}, \bar{a}]$  and  $\neg \theta[\mathbb{D}, \bar{a}]$  are infinite by assumption, we can find an expansion of  $\mathbb{D}$  that satisfies  $F$  by interpreting  $\bar{e}$  as  $\bar{a}$  and both  $c$  and  $d$  as members of large enough equivalence classes. By the compactness theorem, it follows that there is some model  $\mathbb{C}$  of  $\Gamma$ , which must be infinite. Let  $\mathbb{C}'$  denote the reduct of  $\mathbb{C}$  to the language of  $\mathbb{D}$ . Thanks to the Löwenheim-Skolem theorem, we may assume that  $\text{dom}(\mathbb{C}')$  is countable. Thus, the equivalence classes  $[c^{\mathbb{C}}]$  and  $[d^{\mathbb{C}}]$  are countable. Note that  $e_i^{\mathbb{C}} \notin [c^{\mathbb{C}}] \cup [d^{\mathbb{C}}]$  for each  $1 \leq i \leq n$ . Therefore, there is an automorphism of  $\mathbb{C}'$  sending  $c^{\mathbb{C}}$  to  $d^{\mathbb{C}}$  and fixing each  $e_i^{\mathbb{C}}$ . But this contradicts the fact that  $\mathbb{C}' \models \theta[c^{\mathbb{C}}, \bar{e}^{\mathbb{C}}] \wedge \neg \theta[d^{\mathbb{C}}, \bar{e}^{\mathbb{C}}]$ . ■

**Definition 1 (Categoricity).** For any cardinal  $\kappa$ , we say that a theory  $T$  is  $\kappa$ -categorical if any two models of  $T$  of size  $\kappa$  are isomorphic.

2. Show that a  $\mathcal{L}$ -structure  $\mathbb{A}$  is finite if and only if for any  $\mathcal{L}$ -structure  $\mathbb{B}$ ,

$$\mathbb{A} \equiv \mathbb{B} \iff \mathbb{A} \cong \mathbb{B}.$$

*Remark.* This shows that any complete theory with a finite model is  $\kappa$ -categorical for *any* cardinal  $\kappa$ .

( $\implies$ )

It is always true that any two isomorphic structures are elementarily equivalent. Thus, it remains to show that  $\mathbb{A} \equiv \mathbb{B} \implies \mathbb{A} \cong \mathbb{B}$ .

First, assume that  $\mathcal{L}$  is finite. Consider the *atomic diagram* of  $\mathbb{A}$ , i.e., the set

$$D(\mathbb{A}) := \{ \varphi \mid \mathbb{A} \models \varphi, \varphi \text{ is either atomic or the negation of an atomic formula} \}$$

where  $\mathbb{A}$  denotes the expansion of  $\mathbb{A}$  obtained by adjoining a constant symbol  $c_a$  for each  $a \in \text{dom}(\mathbb{A})$ . Since both  $\mathcal{L}$  and  $\text{dom}(\mathbb{A})$  are finite, we can encode  $D(\mathbb{A})$  with a single sentence  $\psi$ . Therefore, the sentence

$$\psi_{\mathbb{A}} := \forall x \left( \bigvee_{a \in \text{dom}(\mathbb{A})} x = c_a \right) \wedge \psi$$

has the property that  $\mathbb{B} \models \psi_{\mathbb{A}} \implies \mathbb{B} \cong \mathbb{A}$  for any other  $\mathcal{L}$ -structure  $\mathbb{B}$ . Now, if  $\mathbb{A} \equiv \mathbb{B}$ , then clearly both  $\mathbb{A}$  and  $\mathbb{B}$  satisfy  $\psi_{\mathbb{A}}$ , so that  $\mathbb{B} \cong \mathbb{A}$ .

Next, let  $\mathcal{L}$  be arbitrary and let  $\mathbb{A} \equiv \mathbb{B}$ . Suppose, toward a contradiction, that  $\mathbb{A} \not\cong \mathbb{B}$ . Then for any bijection  $f : \text{dom}(\mathbb{A}) \rightarrow \text{dom}(\mathbb{B})$ , there is some finite sublanguage  $\mathcal{L}_f$  of  $\mathcal{L}$  such that  $f$  is *not* an isomorphism  $\mathbb{A}^{\mathcal{L}_f} \rightarrow \mathbb{B}^{\mathcal{L}_f}$  of reducts to  $\mathcal{L}_f$ . Consider the language

$$\mathcal{L}' := \bigcup_{\substack{f : \text{dom}(\mathbb{A}) \rightarrow \text{dom}(\mathbb{B}) \\ \text{bijection}}} \mathcal{L}_f,$$

which is finite as the finite union of finite sets. Thanks to our preceding discussion, we obtain an isomorphism  $g : \mathbb{A}^{\mathcal{L}'} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}'}$ . But  $\mathcal{L}_g \subset \mathcal{L}'$  by our construction of  $\mathcal{L}'$ , and thus  $g$  induces an isomorphism  $\mathbb{A}^{\mathcal{L}_g} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}_g}$ , contrary to our choice of  $\mathcal{L}_g$ .

( $\Leftarrow$ )

Suppose that  $\mathbb{A}$  is infinite. We must find a structure  $\mathbb{B}$  such that  $\mathbb{A} \equiv \mathbb{B}$  but  $\mathbb{A} \not\equiv \mathbb{B}$ . But this follows at once from the Löwenheim-Skolem theorem, which implies that  $\text{Th}(\mathbb{A})$  has a model of any infinite size.  $\blacksquare$

**Definition 2 (Ehrenfeucht-Fraïssé game).** Suppose that  $\mathcal{L}$  is a finite language without function symbols. Let  $\mathbb{D}$  and  $\mathbb{E}$  be two  $\mathcal{L}$ -structures. Let  $n \in \omega$ . The *Ehrenfeucht-Fraïssé game*  $\text{EF}_n(\mathbb{D}, \mathbb{E})$  of length  $n$  on  $\mathbb{D}$  and  $\mathbb{E}$  is a game of perfect information played as follows.

- (a) There are exactly two players, the *spoiler* and the *duplicator*.
- (b) There are exactly  $n$  rounds.
- (c) The spoiler begins round  $k \leq n$  by picking an element (sometimes called a pebble) of either  $\text{dom}(\mathbb{D})$  or  $\text{dom}(\mathbb{E})$ . Next, the duplicator picks an element of the other domain.
- (d) This yields two sequences  $(d_1, \dots, d_n)$  and  $(e_1, \dots, e_n)$  such that  $d_i \in \text{dom}(\mathbb{D})$  and  $e_i \in \text{dom}(\mathbb{E})$  for each  $i = 1, \dots, n$ . If the mapping  $d_i \mapsto e_i$  defines an isomorphism of finite substructures, then we say that the duplicator has won  $\text{EF}_n(\mathbb{D}, \mathbb{E})$ . Otherwise, we say that the spoiler has won.

**Theorem 3 (Fraïssé).** *The duplicator has a winning strategy in  $\text{EF}_n(\mathbb{D}, \mathbb{E})$  for each  $n \in \omega$  if and only if  $\mathbb{D} \equiv \mathbb{E}$ .*

**3.** Let  $\mathbb{N}^* = \langle \omega, < \rangle$ . Show that for any infinite cardinal  $\kappa$ ,  $\text{Th}(\mathbb{N}^*)$  is *not*  $\kappa$ -categorical.

Expand the language of  $\mathbb{N}^*$  by adjoining countably many constants  $\{c_i\}_{i \in \mathbb{Z}}$ . Consider the theory

$$T := \text{Th}(\mathbb{N}^*) \cup \{c_i > c_{i+1} \mid i \in \mathbb{Z}\}. \quad (\star)$$

in our new language. Any finite subset of  $T$  is satisfied by an expansion of  $\mathbb{N}^*$  suitably interpreting the  $c_i$  since  $\mathbb{N}^*$  has descending chains of all finite lengths. By the compactness theorem, it follows that there is some model  $\mathbb{A}$  of  $T$ , which must be infinite. If  $|\mathbb{A}| > \aleph_0$ , then apply the Löwenheim-Skolem theorem to get a model  $\mathbb{B}$  of  $T$  such that  $|\mathbb{B}| = \aleph_0$ . Let

$$\mathbb{A}' = \begin{cases} \mathbb{B} & |\mathbb{A}| > \aleph_0 \\ \mathbb{A} & |\mathbb{A}| = \aleph_0 \end{cases}.$$

Note that  $\mathbb{A}' \models T$ . Since the property of being a linearly ordered set is expressible by a first-order sentence, we see that  $\mathbb{A}'$  is linearly ordered by  $<$ . Further, we see that  $\mathbb{A}'$  has an infinite descending chain, which means that  $\mathbb{A}'$  is not well-ordered by  $<$ . But  $(\omega, <)$  is a well-ordered set, and thus the reduct of  $\mathbb{A}'$  to the language of  $\mathbb{N}^*$  is not isomorphic to  $\mathbb{N}^*$ . It does, however, satisfy  $\text{Th}(\mathbb{N}^*)$ . This shows that  $\text{Th}(\mathbb{N}^*)$  is not  $\aleph_0$ -categorical.

Unfortunately, it's unclear that this method can be adapted to show that  $\text{Th}(\mathbb{N}^*)$  is not  $\kappa$ -categorical when  $\kappa$  is uncountable. In this case, we instead shall employ two binary operations on the class of all linear orderings. Let  $L_1$  and  $L_2$  be linearly ordered sets.

- $L_1^{\text{op}}$  refers to  $L_1$  equipped with the inverse order.
- $L_1 \otimes L_2$  refers to  $L_1 \times L_2$  equipped with the lexicographic order.
- $L_1 \oplus L_2$  refers to  $L_1$  with its ordering followed by  $L_2$  with its ordering.

Now, consider the following linearly ordered structures:

$$\begin{aligned} & \mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa) \\ & \mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa)), \end{aligned}$$

both of which have cardinality  $\kappa$ . These orderings possess bottom elements and are *discrete* in the sense that both structures satisfy the sentences

$$\begin{aligned} & \forall x \exists y (x < y \wedge \neg \exists z (x < z \wedge z < y)) \\ & \forall x (\exists w (w < x) \rightarrow \exists y (y < x \wedge \neg \exists z (y < z \wedge z < x))). \end{aligned} \tag{1}$$

(Informally, we can view  $y$  here as the *successor/predecessor* of  $x$ .) Note that, on the one hand,  $\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa)$  cannot possess an descending chain of length  $\omega^2$ , for otherwise  $\kappa$ , which is well-ordered, would possess an infinite descending chain. On the other hand,  $\mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa))$  does possess such a chain since  $\omega^*$  (the order type of  $\mathbb{Z}_{<0}$ ) can be embedded in  $\mathbb{Q}$ . Therefore,

$$\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa) \not\equiv \mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa)).$$

**Claim 2.** Suppose that  $(\mathbb{E}, <)$  is a discrete linear ordering with a bottom element but no top element. Then  $\mathbb{E} \equiv \mathbb{N}^*$ .

*Proof sketch.* Consider the Ehrenfeucht-Fraïssé game  $\text{EF}_n(\mathbb{E}, \mathbb{N}^*)$ . The duplicator has a winning strategy in  $\text{EF}_n(\mathbb{E}, \mathbb{N}^*)$  by adhering to the following rules.

- (i) If, in round  $m$ , the spoiler chooses an element of one of the structures that is connected to a previously chosen element or the bottom element by a path of successors of length  $k < \infty$ , then choose the corresponding element of the other structure in round  $m$ .
- (ii) Otherwise, make sure that any chosen element of  $\text{dom}(\mathbb{N}^*)$  is always separated by at least  $n+1$  elements from any previously chosen element of  $\text{dom}(\mathbb{N}^*)$  while preserving the required order of your choices.

In this case, choose first a natural number separated by more than  $3^n$  elements from the greatest previously chosen element of  $\text{dom}(\mathbb{N}^*)$ .

□

Thanks to Theorem 3, it follows that both  $\mathbb{N}^* \oplus (\mathbb{Z} \otimes \kappa)$  and  $\mathbb{N}^* \oplus (\mathbb{Z} \otimes (\mathbb{Q} \oplus \kappa))$  are elementarily equivalent to  $\mathbb{N}^*$  and thus models of  $\text{Th}(\mathbb{N}^*)$ . Hence  $\text{Th}(\mathbb{N}^*)$  is not  $\kappa$ -categorical. ■

**4.** Show that any set definable over  $\mathbb{N}^*$  is either finite or cofinite.

*Remark.* This shows that  $\mathbb{N}^*$  is *o-minimal* in the sense that every definable set over  $\mathbb{N}^*$  is a finite union of points and intervals in  $\omega$ .

Note that any set definable over  $\mathbb{N}^*$  is 0-definable because any natural number  $n$  is uniquely determined by the first-order property

$$\begin{cases} \text{“}n \text{ is less than any other element”} & n = 0 \\ \text{“there are exactly } n - 1 \text{ elements between 0 (the bottom element) and } n \text{”} & n > 1 \end{cases}$$

Suppose, toward a contradiction, that there exist a formula  $\theta(y)$  such that  $\theta[\mathbb{N}^*]$  is both infinite and coinfinite. Consider, again, the theory  $(\star)$ . Let

$$T' = T \cup \{\theta(c_0), \neg\theta(c_1)\}.$$

Since both  $\theta[\mathbb{N}^*]$  and  $\neg\theta[\mathbb{N}^*]$  are infinite by assumption, we can find an expansion of  $\mathbb{N}^*$  that satisfies any finite subset of  $T'$ . By the compactness theorem together with the Löwenheim-Skolem theorem, we thus can find a countable model  $\mathbb{D}$  of  $T'$  and take its reduct  $\mathbb{C}$  to the language of  $\mathbb{N}^*$ . Note that  $(\text{dom}(\mathbb{C}), <)$  is a

countable linear ordering with an infinite descending and ascending chain  $\chi$  on which both  $c_0^{\mathbb{D}}$  and  $c_1^{\mathbb{D}}$  lie. Moreover, this ordering is discrete in the sense of (1). Therefore, we may assume that  $\chi$  has the form

$$\cdots < x_{m-1} < x_m < x_{m+1} < \cdots$$

where  $x_{m+1}$  denotes the immediate successor of  $x_m$ . There is an automorphism of  $\mathbb{C}$  mapping  $c_0^{\mathbb{D}}$  to  $c_1^{\mathbb{D}}$  by suitably shifting  $\chi$  finitely many places to the left and fixing all elements outside  $\chi$ . But this contradicts the fact that  $\mathbb{C} \models \theta[c_0^{\mathbb{D}}] \wedge \neg\theta[c_1^{\mathbb{D}}]$ .  $\blacksquare$

**5.** Consider the theory DLO of the dense linear ordering without endpoints. For any uncountable cardinal  $\kappa$ , show that there are  $2^\kappa$  many models of DLO up to isomorphism.

*Remark.* This shows that DLO is *not*  $\kappa$ -categorical even though it is  $\aleph_0$ -categorical.

Consider the linear orderings

$$\begin{aligned} L_1 &:= \mathbb{Q} \otimes (\omega_1^{\text{op}} \oplus \omega_1) \\ L_2 &:= \mathbb{Q} \otimes (1 \oplus \omega_1^{\text{op}} \oplus \omega_1). \end{aligned}$$

Now, by replacing each  $\alpha \in \kappa$  with a choice of  $L_1$  or  $L_2$ , we obtain  $2^\kappa$  many dense linear orderings  $\{P_\beta\}_{\beta < 2^\kappa}$  without endpoints such that  $|P_\beta| = \kappa$  for ever  $\beta$ . It remains to show that these are pairwise non-isomorphic.

To this end, suppose that there is an isomorphism  $f : P_\beta \xrightarrow{\cong} P_{\beta'}$ . By construction, both  $P_\beta$  and  $P_{\beta'}$  consist of  $\kappa$ -sequences

$$\begin{aligned} L_{i_0} &< L_{i_1} < \cdots < L_{i_\alpha} < \cdots \\ L_{i'_0} &< L_{i'_1} < \cdots < L_{i'_\alpha} < \cdots, \end{aligned}$$

respectively, where  $i_\alpha, i'_\alpha \in \{1, 2\}$ . Since any isomorphism of well-ordered sets is unique, we see that the function  $f \upharpoonright_{L_{i_\alpha}}$  is an isomorphism  $L_{i_\alpha} \xrightarrow{\cong} L_{i'_\alpha}$  for any  $\alpha \in \kappa$ .

**Claim 3.**  $L_1 \not\cong L_2$ .

*Proof.* On the one hand,  $L_1$  has a suborder isomorphic to  $\omega_1^{\text{op}}$  with no lower bound in  $L_1$ . On the other hand, any such suborder of  $L_2$  has a lower bound in  $L_2$ . Hence there is no isomorphism from  $L_1$  to  $L_2$ .  $\square$

It follows that  $L_{i_\alpha} = L_{i'_\alpha}$  for every  $\alpha \in \kappa$ , which completes our proof.  $\blacksquare$

**Definition 4.** Let  $T$  be a theory and let  $\Gamma(\bar{x})$  be a set of formulas in free variables  $x_1, \dots, x_n$ . We say that  $\Gamma$  is an *n-type over  $T$*  if for any finite subset  $\Delta \subset \Gamma$ , the expanded theory

$$T \cup \{(\exists \bar{x}) \bigwedge \Delta\}$$

is satisfiable.

*Notation.* Let  $\mathbb{M}$  be an  $\mathcal{L}$ -structure and let  $A \subset \text{dom}(\mathbb{M})$ . Let  $\mathcal{L}_A = \mathcal{L} \cup \{c_a \mid a \in A\}$  and let  $\mathbb{M}_A$  denote the  $\mathcal{L}_A$ -structure induced by  $\mathbb{M}$ . Then  $\mathbb{S}_n^{\mathbb{M}}(A)$  refers to the set of all complete  $n$ -types over  $\text{Th}_A(\mathbb{M}) := \text{Th}(\mathbb{M}_A)$ .

**Definition 5 (Stability).** Let  $T$  be a complete theory in  $\mathcal{L}$  and let  $\kappa$  be an infinite cardinal. We say that  $T$  is  $\kappa$ -stable if whenever  $\mathbb{M} \models T$ ,  $A \subset \text{dom}(\mathbb{M})$ , and  $|A| = \kappa$ , we have that  $|\mathbb{S}_n^{\mathbb{M}}(A)| = \kappa$ .

**6.** Let  $\mathbb{A}$  be a structure and  $\theta(x, y)$  be a formula in the language of  $\mathbb{A}$ . Suppose that  $B \subset \text{dom}(\mathbb{A})$  is an infinite set on which  $\theta[A]$  is a linear order  $\prec$ . Show that  $\text{Th}(\mathbb{A})$  is *not*  $\omega$ -stable (i.e.,  $\aleph_0$ -stable).

Thanks to the axiom of dependent choice, we can find a countably infinite chain of at least one of the following two forms.

$$\begin{aligned} a_0 < b_0 < a_1 < b_1 < a_2 < b_2 < \dots \\ \dots < b_2 < a_2 < b_1 < a_1 < b_0 < a_0 \end{aligned}$$

with  $a_i, b_i \in B$  for each  $i = 0, 1, 2, \dots$ . Without loss of generality, assume that we can find the former kind of chain and that  $\theta$  has the form  $x < y$ . In this case,

$$\mathbb{A} \models \theta[a_i, b_j] \iff i \leq j. \quad (*)$$

**Claim 4.** *There exist sequences  $(a_x)_{x \in 2^{\aleph_0}}$  and  $(b_x)_{x \in 2^{\aleph_0}}$  along with a model  $\mathbb{A}'$  of  $\text{Th}(\mathbb{A})$  such that*

$$\mathbb{A}' \models \theta[a_x, b_y] \iff x \leq y.$$

*Proof.* Adjoin to the language of  $\mathbb{A}$  two new constant symbols  $c_x$  and  $d_y$  for every  $x, y \in 2^{\aleph_0}$ . Consider the theory

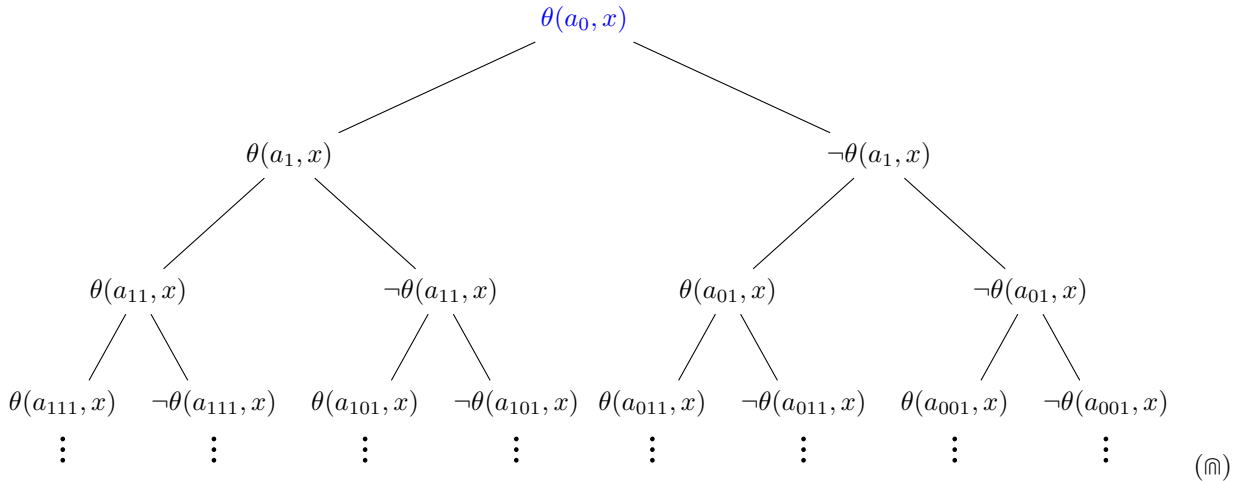
$$\Gamma := \text{Th}(\mathbb{A}) \cup \{\theta(c_x, d_y) \mid x, y \in 2^{\aleph_0}, x \leq y\} \cup \{\neg\theta(c_x, d_y) \mid x, y \in 2^{\aleph_0}, x > y\}.$$

in our expanded language. In light of  $(*)$ , any finite subset of  $\Gamma$  is satisfiable by a suitable expansion of  $\mathbb{A}$ . Thus, by the compactness theorem,  $\Gamma$  has a model  $\mathbb{B}$ . Finally, let  $\mathbb{A}'$  denote the reduct of  $\mathbb{B}$  to the language of  $\mathbb{A}$ .  $\square$

Instead of indexing the sequences  $(a_x)$  and  $(b_x)$  by  $(2^{\aleph_0}, \leq)$ , let us index them by the set of all  $2^{\aleph_0}$ -indexed binary strings  $\sigma$  under the string order  $<$ . We have that

$$\mathbb{A}' \models \theta[a_\sigma, b_{\sigma'}] \iff \sigma \leq \sigma'.$$

Consider the countably infinite subset  $X := \{a_\sigma \mid \sigma \in 2^{\aleph_0}\}$  of  $\text{dom}(\mathbb{A}')$ . By recursion, we can build a binary tree of the form



with height  $\omega$ . We call nodes of the form  $\theta(a_\sigma, x)$  *positive* and those of the form  $-\theta(a_\sigma, x)$  *negative*. Let  $U$  denote any branch of  $(\text{m})$ . Let  $U_p$  denote the set of all strings  $\sigma \in 2^{\aleph_0}$  such that  $a_\sigma$  occurs in a positive node of  $U$ . Since  $U_p$  is countable, it has an upper bound in  $(2^{\aleph_0}, <)$ . Since  $(2^{\aleph_0}, <)$  is a complete order and  $2^{\aleph_0}$  is a limit ordinal, it follows that  $U_p$  has a supremum  $\tau$  in  $2^{\aleph_0}$ . By construction of  $(\text{m})$ , if  $\theta(a_\sigma, x)$  is a positive node of  $U$  and  $-\theta(a_{\sigma'}, x)$  is a negative one, then  $\sigma' > \sigma$ . Hence  $\tau \leq \sigma'$  for any  $\sigma'$  occurring in a negative node of  $U$ . As a result, we see that  $\mathbb{A}' \models \varphi[a_\sigma, b_\tau]$  for any node  $\varphi$  of  $U$ .

Therefore, every branch of  $(\text{m})$  determines a unique 1-type over  $\text{Th}_Y(\mathbb{A}')$  where

$$Y := \{x \in X \mid x \text{ occurs in a node of } (\text{m})\}.$$

This shows that  $|\mathbb{S}_1^{\mathbb{A}'}(Y)| = 2^{\aleph_0} > \aleph_0$ . But  $(\mathfrak{M})$  has exactly

$$\left| \bigcup_{n \in \omega} 2^n \right| = \aleph_0$$

many nodes, so that  $|Y| = \aleph_0$ . Hence  $\text{Th}(\mathbb{A})$  is not  $\omega$ -stable. ■

Informally, an *abstract logic*  $L$  consists of a set of  $L$ -sentences together with a satisfaction relation  $\models_L$  between structures and  $L$ -sentences.

**Definition 6 (Löwenheim-Skolem property).** We say that  $L$  has the *Löwenheim-Skolem property* if any countable satisfiable  $L$ -theory has a model of size  $\leq \aleph_0$ .

**7.** Consider the extension  $L(Q_0)$  of first-order logic by the *generalized quantifier*  $\exists^{<\omega}$  signifying “there are finitely many.” Formally,

$$\mathbb{A} \models (Q_0 x) \varphi(x) \iff |\{a \in \text{dom}(\mathbb{A}) \mid \mathbb{A} \models \varphi[a]\}| < \aleph_0.$$

Show that  $L(Q_0)$  has the Löwenheim-Skolem property.

Without loss of generality, consider  $L(Q_0)$  with  $\exists^{<\omega}$  replaced by  $\exists^\infty := \neg \exists^{<\omega}$ . We have the following version of the Tarski-Vaught elementary submodel criterion.

**Claim 5.** Let  $\mathbb{B}$  be a structure for  $L(Q_0)$  and  $\mathbb{A}$  be a submodel of  $\mathbb{B}$ . Suppose that for any formula  $\varphi(\bar{x}, y)$  and any  $\bar{a} \in \text{dom}(\mathbb{A})$ ,

$$\begin{aligned} \{b \in \text{dom}(\mathbb{B}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\} \neq \emptyset &\implies \{b \in \text{dom}(\mathbb{A}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\} \neq \emptyset \\ |\{b \in \text{dom}(\mathbb{B}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\}| \geq \aleph_0 &\implies |\{b \in \text{dom}(\mathbb{A}) \mid \mathbb{B} \models \varphi[\bar{a}, b]\}| \geq \aleph_0 \end{aligned}$$

Then  $\mathbb{A} \preceq_{L(Q_0)} \mathbb{B}$ .

*Proof sketch.* This is easily proved by induction on the complexity of formulas just as it is for first-order logic. □

Now, suppose that  $\Gamma$  is a countable  $L(Q_0)$ -theory with an infinite model  $\mathbb{M}$ . It suffices to show that for any  $X \subset \text{dom}(\mathbb{M})$ , there is an elementary submodel  $\mathbb{M}'$  of  $\mathbb{M}$  such that  $X \subset \text{dom}(\mathbb{M}')$  and  $|\mathbb{M}'| = |X| + \aleph_0$ . To this end, inductively construct an  $\omega$ -sequence

$$X := X_0 \subset X_1 \subset X_2 \subset \dots$$

of subsets of  $\text{dom}(\mathbb{M})$  such that  $|X_i| = |X| + \aleph_0$  for every  $i \in \omega$  as follows. Suppose that we have already defined  $X_i$  as desired. For every formula  $\varphi(\bar{x}, y)$  and any  $\bar{a} \in X_i$ , consider the set

$$F_{\varphi, \bar{a}} := \{b \in \text{dom}(\mathbb{M}) \mid \mathbb{M} \models \varphi[\bar{a}, b]\}.$$

By the axiom of choice, we can find a set of the form

$$\tilde{F}_{\varphi, \bar{a}} := \begin{cases} F_{\varphi, \bar{a}} & F_{\varphi, \bar{a}} \text{ is finite} \\ \text{a chosen countably infinite subset of } F_{\varphi, \bar{a}} & \text{otherwise} \end{cases}.$$

Now, let

$$X_{i+1} = X_i \cup \bigcup_{\varphi, \bar{a}} \tilde{F}_{\varphi, \bar{a}}.$$

Since there are countably many formulas and, by induction,  $|X| + \aleph_0$  many  $\bar{a} \in X_i$ , we deduce that  $X_{i+1}$  has cardinality  $|X| + \aleph_0$ .

It is easy to see that the union  $\mathbb{M}' := \bigcup_{i \in \omega} X_i$  forms an elementary submodel of  $\mathbb{M}$  thanks to Claim 5. Further, we have that  $|\mathbb{M}'| = |X| + \aleph_0 + \aleph_0 = |X| + \aleph_0$ , as desired. ■

Let  $X \subset \omega$ . We write  $\phi_e^X : W_e^X \rightarrow \{0, 1\}$  for the partial function computed by the Turing machine with index  $e$  and access to an oracle for  $X$ .

**Definition 7 (Computability).** Let  $Y \subset \omega$ .

1. We say that  $Y$  is *computable/recursive relative to  $X$*  if the characteristic function  $\chi_Y$  equals  $\phi_e^X$  for some index  $e$ .
2. We say that  $Y$  is *computably/recursively enumerable relative to  $X$*  if  $Y = W_e^X$  for some index  $e$ .

If  $Y = \emptyset$ , then we omit “relative to  $X$ .”

Let  $\mathcal{C}$  denote any collection of computably enumerable sets. We say that a set  $B \subset \{0, 1\}^*$  of binary strings is a *weak index set for  $\mathcal{C}$*  if  $\mathcal{C} = \{W_e \mid e \in B\}$ .

**8.** Let REC denote the collection of all recursive sets. Show that REC has a computably enumerable weak index set.

■