

## Abstract

We continue looking at higher Waldhausen  $K$ -theory by presenting several of its key theorems. At the end, we see an encoding of Waldhausen  $K$ -theory as the infinite loop space of a sort of spectrum. The main sources for this talk are the following.

- $n\text{Lab}$ .
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Ch. V.2.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

## 1 Extension and additivity

Let  $\mathcal{B}$  and  $\mathcal{C}$  be Waldhausen categories. We say that  $F' \rightarrowtail F \twoheadrightarrow F''$  is a *short exact sequence* or *cofiber sequence of exact functors*  $\mathcal{B} \rightarrow \mathcal{C}$  if

- (i) every  $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$  is a cofiber sequence and
- (ii)  $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$  is a cofibration in  $\mathcal{C}$  for every  $A \rightarrowtail B$  in  $\mathcal{B}$ .

Let  $\eta : A \rightarrowtail B \twoheadrightarrow C$  be an object in  $S_2\mathcal{C}$ . Define the source  $s$ , target  $t$ , and quotient  $q$  functors  $S_2\mathcal{C} \rightarrow \mathcal{C}$  by  $s(\eta) = A$ ,  $t(\eta) = B$ , and  $q(\eta) = C$ , respectively. Then  $s \rightarrowtail t \twoheadrightarrow q$  is a cofiber sequence of functors.

Since defining a cofiber sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$  is equivalent to defining an exact functor  $\mathcal{B} \rightarrow S_2\mathcal{C}$ , we may restrict our attention to  $s \rightarrowtail t \twoheadrightarrow q$  when proving assertions about a given cofiber sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$ . (We say that  $S_2\mathcal{C}$  is *universal* in this sense.)

**Theorem 1.1 (Extension).** *The exact functor  $(s, q) : S_2\mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$  induces a homotopy equivalence  $K(S_2\mathcal{C}) \simeq K(\mathcal{C}) \times K(\mathcal{C})$ . The functor  $\coprod : (A, B) \rightarrow (A \rightarrowtail A \coprod B \twoheadrightarrow B)$  is a homotopy inverse.*

*Proof sketch.* Let  $\mathcal{C}_m^w$  denote the category of  $m$ -length sequences of weak equivalences. For each  $n$ , define  $s_n\mathcal{C}_m^w$  as the commutative diagram

$$\begin{array}{ccccccc}
 X_1^0 & \rightarrowtail & X_2^0 & \rightarrowtail & \cdots & \rightarrowtail & X_n^0 \\
 \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\
 X_1^1 & \rightarrowtail & X_2^1 & \rightarrowtail & \cdots & \rightarrowtail & X_n^1 \\
 \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\
 \vdots & & \vdots & & & & \vdots \\
 \sim \downarrow & & \sim \downarrow & & & & \sim \downarrow \\
 X_1^m & \rightarrowtail & X_2^m & \rightarrowtail & \cdots & \rightarrowtail & X_n^m
 \end{array} .$$

This is naturally isomorphic to an  $(m, n)$ -bisimplex in  $N_{\bullet}wS_{\bullet}\mathcal{C}$ , which is thus isomorphic to the bisimplicial set  $s_{\bullet}\mathcal{C}_{(-)}^w$ . One can show that the source  $s$  and quotient  $q$  functors  $S_2\mathcal{C} \rightarrow \mathcal{C}$  induce a homotopy equivalence  $s \times q : s_{\bullet}S_2(\mathcal{C}_m^w) \rightarrow s_{\bullet}\mathcal{C}_m^w \times s_{\bullet}\mathcal{C}_m^w$  for each  $m$ . Thus, we get a homotopy equivalence

$$s_{\bullet}S_2(\mathcal{C}_{(-)}^w) \simeq s_{\bullet}\mathcal{C}_{(-)}^w \times s_{\bullet}\mathcal{C}_{(-)}^w$$

between bisimplicial sets. But we already have that  $s_{\bullet}\mathcal{C}_{(-)}^w \cong N_{\bullet}wS_{\bullet}\mathcal{C}$ , thereby completing our proof.  $\square$

Recall that  $|wS_{\bullet}\mathcal{C}|$  is an  $H$ -space via the map

$$\coprod : |wS_{\bullet}\mathcal{C}| \times |wS_{\bullet}\mathcal{C}| \cong |wS_{\bullet}\mathcal{C} \times wS_{\bullet}\mathcal{C}| \rightarrow |wS_{\bullet}\mathcal{C}|. \quad (\star)$$

This produces an  $H$ -space structure  $(K(\mathcal{C}), +)$ .

**Theorem 1.2 (Additivity).** *Let  $F' \rightarrowtail F \twoheadrightarrow F''$  be a short exact sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$ . Then  $F_* \simeq F'_* + F''_*$  as maps  $K(\mathcal{B}) \rightarrow K(\mathcal{C})$ , so that*

$$F_* = F'_* + F''_*$$

as maps  $K_i(\mathcal{B}) \rightarrow K_i(\mathcal{C})$ .

*Proof.* As  $S_2\mathcal{C}$  is universal, it suffices to prove that  $t_* \simeq s_* + q_*$ . Notice that the two composites

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\coprod} S_2\mathcal{C} \xrightarrow[s \coprod q]{t} \mathcal{C}$$

are the same. Theorem 1.1 implies that  $K(\coprod) : K(\mathcal{C}) \times K(\mathcal{C}) \rightarrow K(S_2\mathcal{C})$  is a homotopy equivalence. Since the  $H$ -space structure on  $K(\mathcal{C})$  is induced by  $\coprod$ , we get  $t_* \simeq s_* + q_*$ .  $\square$

**Definition 1.3.** We say that a sequence

$$* \rightarrow A_n \rightarrow \cdots \rightarrow A_0 \rightarrow *$$

is *admissibly exact* if each morphism in the sequences can be written as a cofiber sequence

$$A_{i+1} \twoheadrightarrow B_i \rightarrowtail A_i.$$

**Corollary 1.4.** *Suppose that*

$$* \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots \rightarrow F^n \rightarrow *$$

*is an admissibly exact sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$ . Then we have an equality*

$$\sum_i (-1)^i F_*^i = 0$$

*of maps  $K_i(\mathcal{B}) \rightarrow K_i(\mathcal{C})$ .*

**Corollary 1.5.** *Let  $F' \twoheadrightarrow F \twoheadrightarrow F''$  be a short exact sequence of exact functors  $\mathcal{B} \rightarrow \mathcal{C}$ . Then*

$$F''_* \simeq F_* - F_* \simeq 0.$$

This implies that the homotopy fiber of  $F''_* : K(\mathcal{B}) \rightarrow K(\mathcal{C})$  is homotopy equivalent to  $K(\mathcal{B}) \vee \Omega K(\mathcal{C})$ .

Let  $\mathcal{C}$  be a Waldhausen category. Recall the arrow category  $\text{Ar}(\mathcal{C})$  of  $\mathcal{C}$  consisting of morphisms in  $\mathcal{C}$  as objects and commutative squares as morphisms. Let  $s$  and  $t$  denote the source and target functors  $\text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$ , respectively.

**Definition 1.6.** A functor  $T : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$  is a (*mapping*) *cylinder functor* on  $\mathcal{C}$  if it comes equipped with natural transformations  $j_1 : s \Rightarrow T$ ,  $j_2 : t \Rightarrow T$ , and  $p : T \Rightarrow t$  such that for any  $f : A \rightarrow B$ , we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{j_1} & T(f) \xleftarrow{j_2} B \\ & \searrow f & \downarrow p \swarrow = \\ & & B \end{array} .$$

Moreover,  $T$  must satisfy the following axioms.

- (1)  $T$  sends every initial morphism  $* \rightarrow A$  to  $A$  for any  $A \in \text{ob } \mathcal{C}$ .
- (2)  $j_1 \amalg j_2 : A \amalg B \rightarrow T(f)$  is a cofibration for any  $f : A \rightarrow B$ .
- (3) Given a morphism  $(a, b) : f \rightarrow f'$  in  $\text{Ar}(\mathcal{C})$ , if both  $a$  and  $b$  are weak equivalences in  $\mathcal{C}$ , then so is  $T(f) \rightarrow T(f')$ .
- (4) Given a morphism  $(a, b) : f \rightarrow f'$  in  $\text{Ar}(\mathcal{C})$ , if both  $a$  and  $b$  are cofibrations in  $\mathcal{C}$ , then so is  $T(f) \rightarrow T(f')$ . Also, the map  $A' \amalg_A T(f) \amalg_B B' \rightarrow T(f')$  induced by axiom (2) is a cofibration in  $\mathcal{C}$ .
- (5) (*Cylinder axiom*) The map  $p : T(f) \rightarrow B$  is a weak equivalence in  $\mathcal{C}$ .

*Terminology.* Let  $T$  be a cylinder functor on  $\mathcal{C}$ .

1. We call  $T(A \rightarrow *)$  the *cone* of  $A$ , denoted by  $\text{cone}(A)$ .
2. We call  $\text{cone}(A)/_A$  the *suspension* of  $A$ , denoted by  $\Sigma A$ .

**Corollary 1.7.** *The induced suspension map  $\Sigma : K(\mathcal{C}) \rightarrow K(\mathcal{C})$  is a homotopy inverse for the  $H$ -space structure  $(\star)$ .*

*Proof.* Note that axiom (3) gives us a cofiber sequence  $A \rightarrow \text{cone}(A) \rightarrow \Sigma A$ . Therefore,  $1 \rightarrow \text{cone} \rightarrow \Sigma$  is an exact sequence of functors. By the cylinder axiom, we know that  $\text{cone}$  is null-homotopic. It follows by Theorem 1.2 that  $\Sigma_* + 1 = \text{cone}_* = *$ .  $\square$

## 2 Localization

Let  $\mathcal{C}$  be a category with cofibrations. Equip it with two Waldhausen subcategories  $v(\mathcal{C})$  and  $w(\mathcal{C})$  of weak equivalences such that  $v(\mathcal{C}) \subset w(\mathcal{C})$ . Assume that  $(\mathcal{C}, w)$  admits a cylinder functor. Suppose that  $w(\mathcal{C})$  is saturated and closed under extensions.

Let  $\mathcal{C}^w$  denote the Waldhausen subcategory of  $(\mathcal{C}, v)$  consisting of all  $A$  such that  $* \rightarrow A$  belongs to  $w(\mathcal{C})$ .

**Theorem 2.1 (Waldhausen localization).** *The sequence*

$$K(\mathcal{C}^w) \rightarrow K(\mathcal{C}, v) \rightarrow K(\mathcal{C}, w)$$

*is a homotopy fibration sequence.*

*Proof sketch.* Recall that a small bicategory is a bisimplicial set such that each row/column is the nerve of a category. Note that  $v_{(-)}w_{(-)}\mathcal{C}$  is a bicategory whose bimorphisms are commutative squares of the form

$$\begin{array}{ccc} (-) & \xrightarrow{w'} & (-) \\ v \downarrow & & \downarrow v' \\ (-) & \xrightarrow{w} & (-) \end{array} \quad (\star)$$

Treating  $w\mathcal{C}$  as a bicategory with a single vertical morphism reveals that

$$w\mathcal{C} \simeq v_{(-)}w_{(-)}\mathcal{C}.$$

This yields  $wS_n\mathcal{C} \simeq v_{(-)}w_{(-)}S_n\mathcal{C}$  for each  $n$ .

Now, let  $v_{(-)}\operatorname{co} w_{(-)}\mathcal{C}$  denote the subcategory of all squares like  $(\star)$  where the horizontal maps are also cofibrations. One can show that the inclusion  $v_{(-)}\operatorname{co} w_{(-)}\mathcal{C} \subset v_{(-)}w_{(-)}\mathcal{C}$  is a homotopy equivalence. Since each  $S_n\mathcal{C}$  inherits a cylinder functor from  $\mathcal{C}$ , we obtain a simplicial bi-subcategory  $v_{(-)}\operatorname{co} w_{(-)}S_{\bullet}\mathcal{C}$  such that the inclusion into  $v_{(-)}w_{(-)}S_{\bullet}\mathcal{C}$  is a homotopy equivalence. This yields a commutative diagram

$$\begin{array}{ccccc} vS_{\bullet}C^w & \longrightarrow & vS_{\bullet}C & \longrightarrow & v_{(-)}\operatorname{co} w_{(-)}S_{\bullet}C \\ & & \downarrow & & \downarrow \simeq \\ & & wS_{\bullet}C & \xrightarrow{\simeq} & v_{(-)}w_{(-)}S_{\bullet}C \end{array}.$$

It therefore suffices to show that the top row is a fibration. One can do this by using the relative  $K$ -theory space construction. See IV.8.5.3 and V.2.1 (Weibel).  $\square$

Now, let  $\mathcal{A}$  be an exact category embedded in an abelian category  $\mathcal{B}$  and let  $\mathbf{Ch}^b(\mathcal{A})$  denote the category of bounded chain complexes in  $\mathcal{A}$ . One can verify that  $\mathbf{Ch}^b(\mathcal{A})$  is Waldhausen where the cofibrations  $A_{\bullet} \rightarrow B_{\bullet}$  are precisely the degree-wise admissible monomorphisms (i.e., those admitting a short exact sequence  $A_n \rightarrow B_n \rightarrow B_n/A_n$  in  $\mathcal{A}$  for each  $n$ ) and the weak equivalences are precisely the chain maps which are quasi-isomorphisms of complexes in  $\mathbf{Ch}(\mathcal{B})$ .

Our next result is a consequence of Theorem 2.1 and can be found in V.2.2 (Weibel).

**Theorem 2.2 (Gillet-Waldhausen).** *Let  $\mathcal{A}$  be an exact category closed under kernels of surjections. Then the exact inclusion  $\mathcal{A} \rightarrow \mathbf{Ch}^b(\mathcal{A})$  induces a homotopy equivalence  $K(\mathcal{A}) \simeq K \mathbf{Ch}^b(\mathcal{A})$ . Hence*

$$K_i(\mathcal{A}) = K_i \mathbf{Ch}^b(\mathcal{A})$$

for every  $i$ .

**Definition 2.3.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an exact functor between Waldhausen categories. We say that  $F$  satisfies the *approximate lifting property* if for any map  $b : F(A) \rightarrow B$  in  $\mathcal{B}$ , there exist a map  $a : A \rightarrow A'$  in  $\mathcal{A}$  and a weak equivalence  $b' : F(A') \simeq B$  in  $\mathcal{B}$  such that

$$\begin{array}{ccc} F(A') & \xrightarrow{\sim} & B \\ \uparrow F(a) & \nearrow b & \\ F(A) & & \end{array} .$$

commutes.

This means that  $F$  has the approximate lifting property just in case we can always lift it to a weak equivalence.

**Proposition 2.4.** *Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an exact functor between Waldhausen categories with the following properties.*

1.  *$F$  satisfies the approximate lifting property.*
2.  *$\mathcal{A}$  admits a cylinder functor.*
3. *A morphism  $f$  in  $\mathcal{A}$  is a weak equivalence iff  $F(f)$  is a weak equivalence in  $\mathcal{B}$ .*

Then  $wF : w\mathcal{A} \rightarrow w\mathcal{B}$  is a homotopy equivalence.

**Corollary 2.5 (Waldhausen approximation).** *With the same hypotheses as in Proposition 2.4, we have*

$$K(\mathcal{A}) \simeq K(\mathcal{B}).$$

*Proof sketch.* One can show that each functor  $S_n \mathcal{A} \rightarrow S_n \mathcal{B}$  is exact and also has the approximate lifting property. Proposition 2.4 thus gives degree-wise homotopy equivalence between the bisimplicial map  $wS_\bullet \mathcal{A} \rightarrow wS_\bullet \mathcal{B}$ , which is enough.  $\square$

**Definition 2.6.** Let  $\mathcal{A}$  be an abelian category  $\mathbf{Ch}(\mathcal{A})$  denote the category of chain complexes over  $\mathcal{A}$ . We say that a complex  $C_\bullet$  is *homologically bounded* if only finitely many  $H_i(C_j)$  are nonzero.

*Notation.* Let  $\mathbf{Ch}_\pm^{\text{hb}}$  denote the subcategory of bounded below (respectively, bounded above) complexes.

**Example 2.7.** Let  $\mathcal{A}$  be an abelian category. One can show that the inclusions  $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}_-^{\text{hb}}(\mathcal{A})$  and  $\mathbf{Ch}_+^{\text{hb}}(\mathcal{A}) \subset \mathbf{Ch}^b(\mathcal{A})$  have the approximate lifting property. Also, the inclusions  $\mathbf{Ch}^b(\mathcal{A}) \subset \mathbf{Ch}_+^{\text{hb}}(\mathcal{A})$  and  $\mathbf{Ch}_-^{\text{hb}}(\mathcal{A}) \subset \mathbf{Ch}^b(\mathcal{A})$  satisfy the dual of the approximate lifting property. Thus, we can apply Corollary 2.5 along with Theorem 2.2 to find that

$$K(\mathcal{A}) \simeq K \mathbf{Ch}^b(\mathcal{A}) \simeq K \mathbf{Ch}_-^{\text{hb}} \simeq K \mathbf{Ch}_+^{\text{hb}}(\mathcal{A}) \simeq K \mathbf{Ch}^{\text{hb}}(\mathcal{A}).$$

### 3 $K$ -theory spectrum

**Definition 3.1.** A *symmetric spectrum*  $\mathbf{X}$  in topological spaces in a sequence of based  $\Sigma_n$ -spaces  $(X_n)$  endowed with structure maps  $\sigma : X_n \wedge S^1 \rightarrow X_{n+1}$  such that  $\sigma^k : X_n \wedge S^k \rightarrow X_{n+k}$  is  $(\Sigma_n \times \Sigma_k)$ -equivariant for any  $n, k \geq 0$ , where  $S^k \equiv \underbrace{S^1 \wedge \cdots \wedge S^1}_{k \text{ times}}$ .

A map  $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$  of symmetric spectra is a sequence  $(f_n : X_n \rightarrow Y_n)$  of based  $\Sigma_n$ -equivariant maps such that for each  $n \geq 0$ , the square

$$\begin{array}{ccc} X_n \wedge S^1 & \xrightarrow{f_n \wedge \text{Id}} & Y_n \wedge S^1 \\ \sigma \downarrow & & \downarrow \sigma \\ X_{n+1} & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

commutes. Let  $\text{Sp}^\Sigma$  denote the category of symmetric spectra in topological spaces.

**Definition 3.2.** Let  $(\mathcal{C}, w\mathcal{C})$  be a Waldhausen category. The *external  $n$ -fold  $S_\bullet$ -construction* on  $\mathcal{C}$  is the  $n$ -multisimplicial Waldhausen category

$$(S_\bullet \cdots S_\bullet \mathcal{C}, wS_\bullet \cdots S_\bullet \mathcal{C}).$$

In multidegree  $(q_1, \dots, q_n)$ , it has as objects the diagrams  $X : \text{Ar}[q_1] \times \cdots \times \text{Ar}[q_n] \rightarrow \mathcal{C}$  such that

- (i)  $X((i_1, j_1), \dots, (i_n, j_n)) = *$  when  $i_k = j_k$  for some  $1 \leq k \leq n$  and
- (ii)  $X(\dots, (i_t, j_t), \dots) \rightrightarrows X(\dots, (i_t, k_t), \dots) \rightarrow X(\dots, (j_t, k_t), \dots)$  is a cofiber sequence in the  $(n-1)$ -fold iterated  $S_\bullet$ -construction for any  $i_t \leq j_t \leq k_t$  in  $[q_t]$ .

**Definition 3.3.** Let  $(\mathcal{C}, w\mathcal{C})$  be a Waldhausen category. The *internal  $n$ -fold  $S_\bullet$ -construction* on  $\mathcal{C}$  is the simplicial Waldhausen category

$$(S_\bullet^{(n)} \mathcal{C}, wS_\bullet^{(n)} \mathcal{C}).$$

It has as  $q$ -simplices the functor categories  $(S_q \cdots S_q \mathcal{C}, wS_q \cdots S_q \mathcal{C})$  whose objects are precisely the  $(\text{Ar}[q])^n$ -shaped diagrams  $X : (\text{Ar}[q])^n \rightarrow \mathcal{C}$  such that

- (i)  $X((i_1, j_1), \dots, (i_n, j_n)) = *$  when  $i_k = j_k$  for some  $1 \leq k \leq n$ .
- (ii)  $X(\dots, (i_t, j_t), \dots) \rightrightarrows X(\dots, (i_t, k_t), \dots) \rightarrow X(\dots, (j_t, k_t), \dots)$  is a cofiber sequence in the  $(n-1)$ -fold iterated  $S_\bullet$ -construction for any  $i_t \leq j_t \leq k_t$  in  $[q]$ .

Note that  $\Sigma_n$  acts on  $S_\bullet^{(n)} \mathcal{C}$  by the relation  $(\pi \cdot X)(\dots, (i_t, j_t), \dots) = X(\dots, (i_{\pi^{-1}(t)}, j_{\pi^{-1}(t)}), \dots)$ .

The *(symmetric) algebraic  $K$ -theory spectrum*  $\mathbf{K}(\mathcal{C}, w)$  of a small Waldhausen category  $(\mathcal{C}, w\mathcal{C})$  has  $n$ -th space

$$K(\mathcal{C}, w)_n \equiv \left| wS_\bullet^{(n)} \mathcal{C} \right|$$

based at  $*$ . There is a  $\Sigma_n$ -action on  $K(\mathcal{C}, w)_n$  induced by permuting the order of the internal  $S_\bullet$ -constructions. Moreover, we have that

$$\left| wS_\bullet^{(n)}\mathcal{C} \right| \wedge S^1 \cong \left| wS_\bullet^{(n)}S_\bullet\mathcal{C} \right|^{(1)} \subset \left| wS_\bullet^{(n)}S_\bullet\mathcal{C} \right| \cong \left| wS_\bullet^{(n+1)}\mathcal{C} \right|$$

where  $^{(1)}$  denotes the 1-skeleton with respect to the last simplicial direction. This determines the structure map  $\sigma$ .

**Note 3.4.**  $\sigma^k$  is  $(\Sigma_n \times \Sigma_k)$ -invariant.

**Theorem 3.5.** *For any  $i \geq 0$ , we have that  $K_i(\mathcal{C}, w) = \pi_{i+1}K(\mathcal{C}, w)_1 \cong \pi_i\mathbf{K}(\mathcal{C}, w)$ .*<sup>1</sup>

This enables us to encode our algebraic  $K$ -theory in an infinite loop space.

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<sup>1</sup>Lemma 8.7.4 (Rognes).