

### Abstract

We continue to look at low-dimensional  $K$ -theory, finishing our description of  $K_0(-)$  and then defining  $K_1(-)$ , and  $K_2(-)$  for rings. The main sources for this talk are the following.

- $n\text{Lab}$ .
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Chapters II and III.
- Eric M. Friedlander's *An Introduction to K-theory*, Chapter 1.
- <http://people.math.harvard.edu/~lurie/281notes/Lecture3-Whitehead.pdf>.

## 1 $K_0$ of a Waldhausen category

**Definition 1.1.** Let  $\mathcal{C}$  be a category equipped with a “subcategory”  $\text{co}\mathcal{C}$  of morphisms called *cofibrations*. The pair  $(\mathcal{C}, \text{co})$  is a *category with cofibrations* if the following conditions hold.

1. (W0) Every isomorphism in  $\mathcal{C}$  is a cofibration.
2. (W1) There is a base point 0 in  $\mathcal{C}$  such that the unique morphism  $0 \rightarrowtail A$  for every  $A \in \text{ob}\mathcal{C}$ .
3. (W2) We have

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & & \downarrow \\ C & \rightarrowtail & B \cup_A C \end{array} .$$

Note that  $B \amalg C$  always exists as the pushout  $B \cup_0 C$  and that the cokernel of any  $i : A \rightarrowtail B$  exists as  $B \cup_A 0$  along  $A \rightarrow 0$ . We call  $A \rightarrowtail B \twoheadrightarrow B/A$  a *cofibration sequence*.

**Definition 1.2.** A *Waldhausen category*  $\mathcal{C}$  is a category with cofibrations together with a subcategory  $w(\mathcal{C})$  of morphisms called *weak equivalences* such that every isomorphism in  $\mathcal{C}$  is a w.e. and the following “glueing axiom” holds.

1. (W3) For any diagram

$$\begin{array}{ccccc} C & \longleftarrow & A & \rightarrowtail & B \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ C' & \longleftarrow & A' & \rightarrowtail & B' \end{array} ,$$

the induced map  $B \cup_A C \rightarrow B' \cup_{A'} C'$  is a w.e.

**Definition 1.3.** Let  $\mathcal{C}$  be a Waldhausen category. Define  $K_0(\mathcal{C})$  as the abelian group generated by  $[C]$  for each object  $C$  of  $\mathcal{C}$  such that

1.  $[C] = [C']$  if there some w.e. from  $C$  to  $C'$
2.  $[C] = [B] + \left[ \frac{C}{B} \right]$  for every  $B \rightarrowtail C \twoheadrightarrow C/B$

3. The weak equivalence classes of objects in  $\mathcal{C}$  is a set.

**Proposition 1.4.**

1.  $[0] = 0$ .
2.  $[B] \coprod [C] = [B] + [C]$ .
3.  $[B \cup_A C] = [B] + [C] - [A]$ .
4.  $[C] = 0$  whenever  $0 \simeq C$ .

**Example 1.5.** Let  $\mathcal{R}_f(*)$  denote the category of finite CW complexes. Here, cofibrations and weak equivalences correspond to cellular inclusions and weak homotopy equivalences, respectively. It is known that  $K_0(\mathcal{R}_f) \cong \mathbb{Z}$ .

**Definition 1.6.** if  $\mathcal{C}$  and  $\mathcal{D}$  are Waldhausen, then a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *exact* if

- (a) preserves base points, cofibrations, and weak equivalences and
- (b) for any  $A \rightarrowtail B$ ,  $FB \cup_{FA} FC \rightarrow F(B \cup_A C)$  is an isomorphism.

Note that  $F$  induces a group map  $K_0(F) : K_0(\mathcal{C}) \rightarrow K_0(\mathcal{D})$ .

**Theorem 1.7.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be an exact functor. Assume the following.

- (1) A morphism  $f$  is a w.e. iff  $F(f)$  is a w.e.
- (2) For any morphism  $b : FA \rightarrow B$  in  $\mathcal{B}$ , there is some  $a : A \rightarrowtail A'$  in  $\mathcal{A}$  and a w.e.  $b' : FA' \xrightarrow{\sim} B$  in  $\mathcal{B}$  such that  $b = b' \circ F(a)$ . Moreover, we may choose  $a$  to be a w.e. whenever  $b$  is a w.e.

Then  $F$  induces an isomorphism  $K_0(\mathcal{A}) \cong K_0(\mathcal{B})$ .

*Proof.* Apply condition (2) to any  $0 \rightarrowtail B$  to get  $FA' \xrightarrow{\sim} B$ . If this is a w.e., then there is some  $A \xrightarrow{\sim} A'$ . Hence there is a bijection between the set  $W$  of w.e. classes of objects of  $\mathcal{A}$  and that in  $\mathcal{B}$ .

The group  $K_0(\mathcal{B})$  is given by the free abelian group  $\mathbb{Z}[W]$  modulo the relation

$$[C] = [B] + \left[ \frac{C}{B} \right].$$

Let  $FA \xrightarrow{\sim} B$ . Then applying condition (2) yields the diagram

$$\begin{array}{ccccc} 0 & \longleftarrow & FA & \twoheadrightarrow & FA' \\ \downarrow & & \sim \downarrow & & \sim \downarrow \\ 0 & \longleftarrow & B & \twoheadrightarrow & C \end{array}.$$

Apply the glueing axiom to see that  $F\left(\frac{A'}{A}\right) \rightarrow \frac{C}{B}$  is a w.e. Hence  $[C] = [B] + \left[ \frac{C}{B} \right]$  holds iff  $[A'] = [A] + \left[ \frac{A'}{A} \right]$  holds.  $\square$

## 2 $K_1$ for rings

Let  $R$  be a unital ring. Recall that direct limits in  $\mathbf{Mod}_R$  always exist. Let

$$K_1(R) = \mathrm{GL}(R)^{\mathrm{ab}}$$

where  $\mathrm{GL}(R) \equiv \mathrm{colim}_{n \in \mathbb{N}} \mathrm{GL}(n, R)$ .

**Note 2.1 (Universal property of  $K$ ).** The universal property of  $\mathrm{ab} : \mathbf{Grp} \rightarrow \mathbf{Ab}$  induces the universal property of  $K_1$  that any homomorphism  $f : \mathrm{GL}(R) \rightarrow H$  with  $H$  abelian has  $f = g \circ \pi$  for some unique  $g : K_1(R) \rightarrow H$ .

**Proposition 2.2.** *Any ring map  $f : R \rightarrow S$  induces a natural map  $\mathrm{GL}(R) \rightarrow \mathrm{GL}(S)$ . Hence  $K_1$  is a functor  $\mathbf{Rng} \rightarrow \mathbf{Ab}$ .*

Thanks to Whitehead, we know that the commutator subgroup  $[\mathrm{GL}(R), \mathrm{GL}(R)]$  is equal to  $E(R) = \bigcup_n E_n(R)$ , the group of elementary matrices  $E_{i,j}(r)$  where  $r \in R$  and  $i \neq j$ . Thus,  $K_1(R)$  can be viewed as the “stabilized” group of automorphisms of the trivial projective module modulo trivial automorphisms.

**Example 2.3.** If  $F$  is a field, then  $K_1(F) = F^\times$ .

*Proof.* It is each to check that  $E_n(F) \cong \mathrm{SL}_n(F)$  for any  $n \in \mathbb{N}$ . Therefore,  $E(F) \cong \mathrm{SL}(F)$ .  $\square$

**Proposition 2.4.** *Suppose  $R$  is commutative. Consider the sequence  $R^\times \cong \mathrm{GL}(1, R) \rightarrow \mathrm{GL}(R) \rightarrow K_1(R)$ . This induces a natural split exact sequence.*

$$1 \longrightarrow SK_1(R) \hookrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1,$$

where  $SK_1(R)$  denotes  $\ker(\det)$ .

This means that  $K_1(R) \cong R^\times \times SK_1(R)$ .

**Example 2.5.** Suppose  $R$  is a Euclidean domain. Then  $SK_1(R) = 1$ , so that  $K_1(R) \cong R^\times$ .

**Lemma 2.6.** *Let  $D$  be a division ring. Then  $K_1(D) \cong \mathrm{GL}_n(D) / E_n(D)$  for any  $n \geq 3$ .*

*Proof.* Any invertible matrix over  $D$  is reducible (a la Gaussian elimination) to a diagonal matrix of the form  $(r, 1, \dots, 1)$ . Moreover,  $E_n(D) \trianglelefteq \mathrm{GL}_n(D)$  for each  $n$ . In particular, Dieudonné (1943) showed that  $\mathrm{GL}_n(D) / E_n(D) \cong D^\times / (D^\times)'$  for any  $n \neq 2$ .  $\square$

Now, suppose that  $R$  is Noetherian of dimension  $d$ , so that  $E_n(R) \trianglelefteq \mathrm{GL}_n(R)$  for any  $n \geq d + 2$ .

**Proposition 2.7 (Vaserstein).**  $K_1(R) \cong \mathrm{GL}_n(R) / E_n(R)$  for any  $n \geq d + 2$ .

Let  $D$  be a  $d$ -dimensional division algebra over the field  $F := Z(D)$ . We know that  $d = n^2$  for some integer  $n$ . By Zorn’s lemma, there is some maximal subfield  $E \subset D$  such that  $[E : F] = n$ . Then  $D \otimes_F E \cong M_n(E)$ , where  $M_n$  denotes the  $n$ -dimensional matrix ring over  $E$ . Any field with this property is called a *splitting field* for  $D$ .

Let  $E'$  be a splitting field for  $D$ . For any  $r \in \mathbb{N}$ , the inclusions  $D \hookrightarrow M_n(E')$  and  $M_r(D) \hookrightarrow M_{nr}(E')$  induce maps  $D^\times \subset \mathrm{GL}_n(E') \xrightarrow{\det} (E')^\times$  and  $\mathrm{GL}_r(D) \rightarrow \mathrm{GL}_{nr}(E') \xrightarrow{\det} (E')^\times$  whose images are contained in  $F^*$ . The induced maps are called the *reduced norms*  $N_{\mathrm{red}}$  for  $D$ .

**Example 2.8.** If  $D = \mathbb{H}$ , then  $N_{\text{red}}$  is the square of the usual norm. It induces an isomorphism  $K_1(\mathbb{H}) \cong \mathbb{R}_+^\times$ .

Let  $R$  be a commutative Banach algebra over  $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$  (i.e., a Banach space equipped with a commutative bilinear multiplication map  $m : R \times R \rightarrow R$  such that  $\|m(a, b)\| \leq \|a\| \cdot \|b\|$ ). Recall that both  $\text{GL}_n(R)$  and  $\text{SL}_n(R)$  are topological groups as subspaces of  $\mathbb{R}^{n^2}$ .

**Proposition 2.9.** *We have that  $E_n(R)$  is the path component of the identity matrix  $I_n$  for any  $n \geq 2$ .*

**Corollary 2.10.** *We may identify  $SK_1(R)$  with the set  $\pi_0 \text{SL}(R)$ .*

*Proof.* Note that  $E(R) \leq \text{SL}(R)$ . By the third isomorphism theorem, we get

$$\text{GL}(R)/_{E(R)} / \text{SL}(R)/_{E(R)} \cong \text{GL}(R)/_{\text{SL}(R)}.$$

Thus, we get the short exact sequence

$$1 \longrightarrow \text{SL}(R)/_{E(R)} \longrightarrow \text{GL}(R)/_{E(R)} \cong K_1(R) \longrightarrow \text{GL}(R)/_{\text{SL}(R)} \cong R^\times \longrightarrow 1$$

By Proposition 2.9, we know that  $\text{SL}(R)/_{E(R)} \cong \pi_0 \text{SL}(R)$ , yielding a short exact sequence.

$$1 \longrightarrow \pi_0 \text{SL}(R) \longrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1.$$

□

**Example 2.11.** If  $X$  is compact, then

$$\begin{aligned} SK_1(\mathbb{R}^X) &\leftrightarrow [X, \text{SL}(\mathbb{R})] \cong [X, \text{SO}] \\ SK_1(\mathbb{C}^X) &\leftrightarrow [X, \text{SL}(\mathbb{C})] \cong [X, \text{SU}]. \end{aligned}$$

In particular,  $SK_1(\mathbb{R}^{S^1}) \leftrightarrow \pi_1 \text{SO} \cong C_2$ .

Let  $P$  be a finitely generated projective  $R$ -module. Any choice of isomorphism  $P \oplus Q \cong R^n$  induces a group map

$$\text{Aut}(P) \rightarrow \text{Aut}(P) \oplus \text{Aut}(Q) \cong \text{Aut}(R^n) \cong \text{GL}(n, R).$$

The group map  $\text{Aut}(P) \rightarrow \text{GL}(R)$  is independent of our choice of isomorphism up to inner automorphism of  $\text{GL}(R)$ . Therefore, there is a well-defined homomorphism  $\Phi : \text{Aut}(R) \rightarrow K_1(R)$ .

**Lemma 2.12.** *Suppose that  $R$  is commutative and  $T$  is an  $R$ -algebra. Then  $K_1(T)$  has a natural module structure over  $K_0(R)$ .*

*Proof.* For any  $P \in \mathbf{P}(R)$  and  $m \in \mathbb{N}$ , consider the homomorphism  $\Phi : \text{Aut}(P \otimes T^m) \rightarrow K_1(R \otimes T)$ . For any  $\beta \in \text{GL}_m(T)$ , let

$$[P] \cdot \beta = \Phi(1_P \otimes \beta).$$

This action factors through  $K_0(R)$  and  $K_1(T)$ , inducing an operation  $K_0(R) \times K_1(T) \rightarrow K_1(R \otimes T)$ . Now, since  $T$  is an  $R$ -algebra, there is a ring map  $R \otimes T \rightarrow T$ . The induced composite  $K_0(R) \times K_1(T) \rightarrow K_1(R \otimes T) \rightarrow K_1(T)$  is the desired module structure. □

As it turns out,  $K_1(R)$  is completely determined by the category  $\mathbf{P}(R)$ . This means that  $K_1$  is invariant under Morita equivalence, just as  $K_0$  is.

**Theorem 2.13.** *if  $R$  and  $S$  are Morita equivalent, then  $K_1(R) \cong K_1(S)$ .*

For an application of  $K_1$  to manifold theory, let  $\pi$  be a finitely generated group. Define the *Whitehead group*  $\text{Wh}(\pi)$  of  $\pi$  as the cokernel of the map  $\pi \times \{\pm 1\} \rightarrow K_1(\mathbb{Z}\pi)$  given by  $(g, \pm 1) \mapsto [\pm g]$ .

**Definition 2.14.** Suppose that  $W$ ,  $M$ , and  $N$  are compact manifolds (possibly smooth or piecewise-linear). Suppose that  $M$  and  $N$  are without boundary. Let  $\dim(M) = \dim(N) = n$  and  $\dim(W) = n + 1$ .

1. We say that  $W$  is a *cobordism of  $M$  and  $N$*  if  $\partial W \cong M \amalg N$ .
2. We say that  $W$  is an  *$h$ -cobordism of  $M$  and  $N$*  if it is a cobordism of  $M$  and  $N$  and the inclusion maps  $i_M : M \hookrightarrow \partial W$  and  $i_N : N \hookrightarrow \partial W$  are homotopy equivalences.

Let  $R$  be a ring. A *based chain complex over  $R$*  is a bounded chain complex

$$\cdots \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots$$

of finitely generated free  $R$ -modules together with a choice  $B_n$  of basis (ordered in a predetermined way) for each  $F_n$ . The *Euler characteristic of  $(F_*, d_n)$*  is the finite sum

$$\chi(F_*) \equiv \sum_n (-1)^n |B_n|.$$

If  $F_*$  is acyclic, then it is contractible, so that there is some map  $h : F_* \rightarrow F_{*+1}$  such that  $dh + hd = \text{Id}_{F_*}$ . In this case, one can check that

$$d + h : \bigoplus_n F_{2n} \rightarrow \bigoplus_n F_{2n+1}.$$

is an isomorphism of free  $R$ -modules. If  $\chi(F_*) = 0$ , then this yields an element  $\underbrace{\rho(F_*) := [d + h]}_{\text{Reidemeister torsion}}$  of

$K_1(R)_{/\{\pm 1\}}$ , which is independent of our choice of null-homotopy  $h$ .

Suppose that  $f : X_* \rightarrow Y_*$  is a quasi-isomorphism of based chain complexes over  $R$ . Then  $\text{cone}(f)$  is an acyclic based chain complex over  $R$ . Further, if  $\chi(X_*) = \chi(Y_*)$ , then  $\chi(\text{cone}(f)) = 0$ , in which case we may define the *torsion of  $f$*  as the element  $\rho(\text{cone}(f))$  of  $K_1(R)_{/\{\pm 1\}}$ .

Now, suppose that  $f : X \rightarrow Y$  is a homotopy equivalence of finite connected CW complexes. Since these are locally contractible, they admit respective universal covering spaces  $\tilde{X}$  and  $\tilde{Y}$ . If  $f$  is a cellular map, then it induces a map

$$\lambda_f : C_*(\tilde{X}; \mathbb{Z}) \rightarrow C_*(\tilde{Y}; \mathbb{Z})$$

of cellular chain complexes, which must be a quasi-isomorphism since  $f$  is assumed to be a homotopy equivalence. Note that  $C_*(\tilde{X}; \mathbb{Z})$  and  $C_*(\tilde{Y}; \mathbb{Z})$  may be viewed as based chain complexes over  $\mathbb{Z}\pi_1(Y)$ . In this case, the *Whitehead torsion  $\tau(f)$*  of  $f$  is the image of the torsion of  $\lambda_f$  under the natural projection  $K_1(\mathbb{Z}\pi_1(Y))_{/\{\pm 1\}} \rightarrow \text{Wh}(\mathbb{Z}\pi_1(Y))$ .

**Theorem 2.15 ( $s$ -cobordism).** *Suppose that  $W$ ,  $M$ , and  $N$  are compact manifolds and that  $W$  is an  $h$ -cobordism of  $M$  and  $N$ . If  $\dim(M) \geq 5$ , then  $(W, M, N) \cong (M \times [0, 1], M \times 0, M \times 1)$  iff  $\tau(i_M)$  vanishes.*

**Corollary 2.16 (Generalized Poincaré conjecture).** *Let  $M$  be an  $n$ -manifold that is homotopy equivalent to  $S^n$ . If  $n \geq 5$ , then  $M$  is homeomorphic to  $S^n$ .*

**Definition 2.17.** Let  $I$  be an ideal in  $R$ . Define  $\mathrm{GL}(I)$  as the kernel of the map  $\mathrm{GL}(R) \rightarrow \mathrm{GL}\left(\frac{R}{I}\right)$ . Moreover, define  $E(R, I)$  as the smallest normal subgroup of  $E(R)$  that contains  $E_{i,j}(r)$  for any  $r \in I$  and  $i \neq j$ .

**Proposition 2.18.**  $[\mathrm{GL}(I), \mathrm{GL}(I)] \subset E(R, I) \trianglelefteq \mathrm{GL}(I)$

**Definition 2.19.** The *relative group*  $K_1(R, I)$  is the the abelian group  $\mathrm{GL}(I)/_{E(R, I)}$ .

*Remark 2.20.* Swan has shown that a ring homomorphism  $f : R \rightarrow S$  mapping the ideal  $I$  isomorphically to the ideal  $J$  need *not* induce an isomorphism  $K_1(R, I) \rightarrow K_1(S, J)$ .

**Proposition 2.21.** *We have an exact sequence*

$$K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1\left(\frac{R}{I}\right) \longrightarrow K_0(I) \longrightarrow K_0(R) \longrightarrow K_0\left(\frac{R}{I}\right) .^1$$

### 3 $K_2$ for rings

**Definition 3.1.** Let  $n \geq 3$  and  $R$  be a ring. The *Steinberg group*  $\mathrm{St}_n(R)$  is the group generated by the symbols  $x_{ij}(r)$  with  $1 \leq i \neq j \leq n$  and  $r \in R$  that satisfy the following relations.

(i)

$$x_{ij}(r)x_{ij}(s) = x_{ij}(r + s)$$

(ii)

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & j \neq k, \quad i \neq l \\ x_{il}(rs) & j = k, \quad i \neq l \\ x_{kj}(-sr) & j \neq k, \quad i = l \end{cases}$$

We have a natural group surjection  $\phi_n : \mathrm{St}_n(R) \rightarrow E_n(R)$  given by  $x_{ij}(r) \mapsto E_{ij}(r)$ . Moreover, there is a group map  $\mathrm{St}_n(R) \hookrightarrow \mathrm{St}_{n+1}(R)$ . Since  $\mathrm{St}(R) := \mathrm{colim}_n \mathrm{St}_n(R)$  exists, the  $\phi_n$  form a group epimorphism  $\phi : \mathrm{St}(R) \rightarrow E(R)$ . Let

$$K_2(n, R) = \ker \phi_n$$

$$K_2(R) = \ker \phi.$$

Note that  $K_2(-)$  is a functor  $\mathbf{Rng} \rightarrow \mathbf{Ab}$ . Furthermore, we have an exact sequence

$$1 \longrightarrow K_2(R) \longrightarrow \mathrm{St}(R) \xrightarrow{\phi} \mathrm{GL}(R) \longrightarrow K_1(R) \longrightarrow 1 .$$

**Lemma 3.2.**  $K_2(R) \cong Z(\mathrm{St}(R))$ .

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<sup>1</sup>Section III.2.3 (Weibel).

*Proof.* The fact that  $K_2(R) \supset Z(\text{St}(R))$  follows from the fact that  $Z(E(R))$  is trivial. The reverse containment is easy but more tedious to prove. See III.5.2.1 (Weibel).  $\square$

**Example 3.3.** A certain sort of Euclidean algorithm yields the following computations.

1.  $K_2(\mathbb{Z}) \cong C_2$
2.  $K_2(\mathbb{Z}[i]) = 1$
3.  $K_2(F) \cong K_2(F[t])$  when  $F$  is a field

**Theorem 3.4.** Suppose that  $R$  is Noetherian of dimension  $d$ . Then  $K_2(n, R) \cong K_2(R)$  for any  $n \geq d + 3$ .

**Theorem 3.5.** If  $R$  and  $S$  are Morita equivalent, then  $K_2(R) \cong K_2(S)$ .

**Example 3.6.** Let  $n \in \mathbb{Z}_{\geq 1}$ . Let  $R$  be any ring and let  $S = M_n(R)$ . These are Morita equivalent, so that

$$K_i(R) \cong K_i(M_n(R))$$

for each  $i = 0, 1, 2$ . Indeed, in one direction, define  $F : M \mapsto M^n$ . In the other direction, define  $G : M \mapsto e_{11}M$  where  $e_{11}$  denotes the matrix with 1 in position  $(1, 1)$  and 0 elsewhere. Define the natural isomorphism  $\text{Id}_{\text{Mod}_R} \Rightarrow G \circ F$  by the components  $f_M : M \rightarrow \{(m, 0, \dots, 0) : m \in M\}$ . Further, define the natural isomorphism  $\text{Id}_{\text{Mod}_S} \Rightarrow F \circ G$  by the components  $g_M : M \rightarrow (e_{11}M)^n$  given by  $m \mapsto (e_{11}m, \dots, e_{1n}m)$ . Hence  $\text{Mod}_R$  and  $\text{Mod}_S$  are equivalent, hence Morita equivalence as they are preadditive.

**Lemma 3.7.** Let  $R$  be a commutative Banach algebra. Then there is a surjection from  $K_2(R)$  onto  $\pi_1 \text{SL}(R)$ .<sup>2</sup>

**Example 3.8.** There is a surjection  $K_2(\mathbb{R}) \rightarrow \pi_1 \text{SL}(\mathbb{R}) \cong \pi_1 \text{SO} \cong C_2$ . Hence  $K_2(\mathbb{R})$  is nontrivial.

**Theorem 3.9 (Matsumoto 1969).** Let  $F$  be a field. Then  $K_2(F)$  is isomorphic to the free abelian group with system of generators  $\{a, b\}$  satisfying the following relations.

- (i)  $\{ac, b\} = \{a, b\}\{c, b\}$
- (ii)  $\{a, bd\} = \{a, b\}\{a, d\}$
- (iii)  $\{a, 1 - a\} = 1$  when  $a \neq 1 \neq 1 - a$ .

*Terminology.* The  $\{a, b\}$  are called *Steinberg symbols*.

Suppose that  $A, B \in E(F)$  commute. Write  $\phi(a) = A$  and  $\phi(b) = B$ . Then define

$$A \star B = [a, b] \in K_2(R).$$

If  $a, b \in F$ , then we can alternatively define the Steinberg symbol

$$\{a, b\} = \begin{bmatrix} r & & \\ & r^{-1} & \\ & & 1 \end{bmatrix} \star \begin{bmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{bmatrix}.$$

**Corollary 3.10.**  $K_2(\mathbb{F}_p^n) = 1$  for any prime  $p$  and any integer  $n \geq 1$ .

*Proof.* The proof is entirely computational. See III.6.1.1 (Weibel).  $\square$

**Proposition 3.11.** If  $F \supset \mathbb{Q}(t)$ , then  $|K_2(F)| = |F|$ .

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<sup>2</sup>III.5.9 (Weibel).