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MATH 571
Exercises
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1. Let I be a countably infinite set. Let $\mathbb{D} := \langle I, E \rangle$ be a structure where E is an equivalence relation for which there is exactly one equivalence class of size k for each $k \in \mathbb{Z}_{\geq 1}$.

(1) Show that the set Λ of (first-order) sentences expressing that E is an equivalence relation with exactly one equivalence class of size k for each $k \in \mathbb{Z}_{\geq 1}$ axiomatizes $\mathbb{D}$, i.e., $\mathsf{Th}(\mathbb{D}) = \mathsf{Cn}(\Lambda)$ where

$$\mathsf{Cn}(\Lambda) := \{\varphi \in \mathsf{FO}_{\mathbb{D}} \mid \Lambda \models \varphi\}.$$

(2) Show that for every (first-order) formula $\theta(y, \overline{w})$ and every $\bar{a} \in I$, the set

$$\theta[\mathbb{D}, \bar{a}] := \{x \in \mathrm{dom}(\mathbb{D}) \mid \mathbb{D} \models \theta[x, \bar{a}]\}$$

is either finite or cofinite.

(1) It suffices to prove that Λ is complete. For, in this case, any two models of Λ must be elementarily equivalent.

Claim 1. *Let $\mathbb{E}$ be any model of Λ of size $\kappa \geq \omega$. There exists an elementary extension $\mathbb{E}_\kappa \succeq \mathbb{E}$ of size κ such that $\mathbb{E}_\kappa$ has exactly κ equivalence classes each of size κ.*

Proof. Let λ denote the cardinality of the set of all equivalence classes in $\mathrm{dom}(\mathbb{E})$. Note that $\lambda \leq \kappa$. For every $\alpha, \beta \in \kappa$, adjoin to the language of $\mathbb{E}$ a new constant symbol $c(\alpha, \beta)$. Consider the theory

$$\Delta := \Lambda \cup \{Ec(x,y)c(x,z) \mid x, y, z \in \kappa\} \cup \{\neg Ec(x,0)c(y,0) \mid x, y \in \kappa,\ x \neq y\}.$$

Any finite subset F of Δ is satisfiable by a suitable expansion $\mathbb{E}_F$ of $\mathbb{E}$. Then there exists an ultrafilter on the family of finite subsets of Δ such that the ultraproduct

$$\prod_{\substack{F \subset \Delta \\ \text{finite}}} \mathbb{E}_F / \mathcal{U}$$

satisfies Δ. Moreover, its reduct $\mathbb{A}$ to the language of $\mathbb{E}$ is an elementary extension of $\mathbb{E}$. By the downward Löwenheim-Skolem theorem, there exists a structure $\mathbb{E}_0$ of size κ such that $\mathbb{A} \succeq \mathbb{E}_0 \succeq \mathbb{E}$.

Now, repeat our preceding construction ω times to get an increasing chain

$$\mathbb{E} \preceq \mathbb{E}_0 \preceq \mathbb{E}_1 \preceq \mathbb{E}_2 \preceq \cdots$$

of structures such that each $\mathrm{dom}(\mathbb{E}_i)$ has cardinality κ. Note that $\mathbb{E}_\kappa$ is an elementary extension of $\mathbb{E}$. Further, the universe of $\mathbb{E}_\kappa := \bigcup_{i \in \omega} \mathbb{E}_i$ also has cardinality κ, so that $\mathbb{E}_\kappa$ has exactly κ equivalence classes. Finally, for any $x \in \mathbb{E}_\kappa$, x belongs to some $\mathbb{E}_n$. Hence the equivalence class $[x]$ has size κ in $\mathbb{E}_{n+1}$ and thus in $\mathbb{E}_\kappa$. It follows that every equivalence class in $\mathbb{E}_\kappa$ has size κ. $\square$

Suppose, toward a contradiction, that there is a sentence φ in the language of $\mathbb{D}$ such that neither φ nor $\neg\varphi$ belongs to $\mathsf{Cn}(\Lambda)$. Then there are models $\mathbb{A}^1$ and $\mathbb{A}^2$ of Λ a such that $\mathbb{A}^1 \models \neg\varphi$ and $\mathbb{A}^2 \models \varphi$. By the Löwenheim-Skolem theorem, we may assume that both of these have size $\kappa \geq \omega$. By Claim 1, we thus have two structures $\mathbb{A}^1_\kappa$ and $\mathbb{A}^2_\kappa$ such that $\mathbb{A}^1_\kappa \models \neg\varphi$ and $\mathbb{A}^2_\kappa \models \varphi$. But it's easy to see that $\mathbb{A}^1_\kappa$ and $\mathbb{A}^2_\kappa$ must be isomorphic, which yields a contradiction.

(2) Suppose, toward a contradiction, that there exist a formula $\theta(y, w_1, \ldots, w_n)$ and an element $\bar{a} \in I$ such that $\theta[\mathbb{D}, \bar{a}]$ is both infinite and coinfinite. Adjoin to the language of $\mathbb{D}$ new constant symbols $\bar{e} := (e_1, \ldots, e_n)$, c, and d. For each $k \in \mathbb{Z}_{\geq 1}$, let $\lambda_k(x)$ denote the formula expressing that the equivalence class of x has cardinality $> k$. Now, consider the theory

$$\begin{aligned} \Gamma := \Lambda \cup \{\lambda_k(c) \mid k \geq 1\} &\cup \{\lambda_k(d) \mid k \geq 1\} \\ &\cup \{\neg E e_i c \mid 1 \leq i \leq n\} \\ &\cup \{\neg E e_i d \mid 1 \leq i \leq n\} \\ &\cup \{\theta(c, \bar{e}), \neg\theta(d, \bar{e})\} \end{aligned}$$

in our new language.

Let F be any finite subset of Γ. Since both $\theta[\mathbb{D}, \bar{a}]$ and $\neg\theta[\mathbb{D}, \bar{a}]$ are infinite, we can find an expansion of $\mathbb{D}$ that satisfies F by interpreting $\bar{e}$ as $\bar{a}$ and both c and d as members of large enough equivalence classes. By the compactness theorem, it follows that there is some model $\mathbb{C}$ of Γ, which must be infinite. Let $\mathbb{C}'$ denote the reduct of $\mathbb{C}$ to the language of $\mathbb{D}$. Thanks to the Löwenheim-Skolem theorem, we may assume that $\mathrm{dom}(\mathbb{C}')$ is countable. Thus, the equivalence classes $\left[c^{\mathbb{C}}\right]$ and $\left[d^{\mathbb{C}}\right]$ are countable. Note that $e_i^{\mathbb{C}} \notin \left[c^{\mathbb{C}}\right] \cup \left[d^{\mathbb{C}}\right]$ for each $1 \leq i \leq n$. Therefore, there is an automorphism of $\mathbb{C}'$ sending $c^{\mathbb{C}}$ to $d^{\mathbb{C}}$ and fixing each $e_i^{\mathbb{C}}$. But this contradicts the fact that $\mathbb{C}' \models \theta\left[c^{\mathbb{C}}, \bar{e}^{\mathbb{C}}\right] \wedge \neg\theta\left[d^{\mathbb{C}}, \bar{e}^{\mathbb{C}}\right]$.

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2. Show that a $\mathcal{L}$-structure $\mathbb{A}$ is finite if and only if for any $\mathcal{L}$-structure $\mathbb{B}$,

$$\mathbb{A} \equiv \mathbb{B} \iff \mathbb{A} \cong \mathbb{B}.$$

($\Longrightarrow$)

It is always true that any two isomorphic structures are elementarily equivalent. Thus, it remains to show that $\mathbb{A} \equiv \mathbb{B} \implies \mathbb{A} \cong \mathbb{B}$.

First, assume that $\mathcal{L}$ is finite. Consider the *atomic diagram* of $\mathbb{A}$, i.e., the set

$$\mathtt{D}(\mathbb{A}) := \{\varphi \mid \underline{\mathbb{A}} \models \varphi,\ \varphi \text{ is either atomic or the negation of an atomic formula}\}$$

where $\underline{\mathbb{A}}$ denotes the expansion of $\mathbb{A}$ obtained by adjoining a constant symbol c_a for each $a \in \mathrm{dom}(\mathbb{A})$. Since both $\mathcal{L}$ and $\mathrm{dom}(\mathbb{A})$ are finite, we can encode $\mathtt{D}(\mathbb{A})$ with a single sentence ψ. Therefore, the sentence

$$\psi_{\mathbb{A}} := \forall x \left(\bigvee_{a \in \mathrm{dom}(\mathbb{A})} x = c_a \right) \wedge \psi$$

has the property that $\underline{\mathbb{B}} \models \psi_{\mathbb{A}} \implies \mathbb{B} \cong \mathbb{A}$ for any other $\mathcal{L}$-structure $\mathbb{B}$. Now, if $\mathbb{A} \equiv \mathbb{B}$, then clearly both $\underline{\mathbb{A}}$ and $\underline{\mathbb{B}}$ satisfy $\psi_{\mathbb{A}}$, so that $\mathbb{B} \cong \mathbb{A}$.

Next, let $\mathcal{L}$ be arbitrary and let $\mathbb{A} \equiv \mathbb{B}$. Suppose, toward a contradiction, that $\mathbb{A} \ncong \mathbb{B}$. Then for any bijection $f : \mathrm{dom}(\mathbb{A}) \to \mathrm{dom}(\mathbb{B})$, there is some finite sublanguage $\mathcal{L}_f$ of $\mathcal{L}$ such that f is *not* an isomorphism $\mathbb{A}^{\mathcal{L}_f} \to \mathbb{B}^{\mathcal{L}_f}$ of reducts to $\mathcal{L}_f$. Consider the language

$$\mathcal{L}' := \bigcup_{\substack{f : \mathrm{dom}(\mathbb{A}) \to \mathrm{dom}(\mathbb{B}) \\ \text{bijection}}} \mathcal{L}_f,$$

which is finite as the finite union of finite sets. Thanks to our preceding discussion, we obtain an isomorphism $g : \mathbb{A}^{\mathcal{L}'} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}'}$. But $\mathcal{L}_g \subset \mathcal{L}'$ by our construction of $\mathcal{L}'$, and thus g induces an isomorphism $\mathbb{A}^{\mathcal{L}_g} \xrightarrow{\cong} \mathbb{B}^{\mathcal{L}_g}$, contrary to our choice of $\mathcal{L}_g$.

($\Longleftarrow$)

Suppose that $\mathbb{A}$ is infinite. We must find a structure $\mathbb{B}$ such that $\mathbb{A} \equiv \mathbb{B}$ but $\mathbb{A} \ncong \mathbb{B}$. But this follows at once from the the Löwenheim-Skolem theorem, which implies that $\mathsf{Th}(\mathbb{A})$ has a model of any infinite size.

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3. Let $\mathbb{N}^* = \langle \omega, < \rangle$. Show that for any infinite cardinal κ, $\mathsf{Th}(\mathbb{N}^*)$ is not κ-categorical.

Expand the language of $\mathbb{N}^*$ by adjoining countably many constants $\{c_i\}_{i \in \mathbb{Z}}$. Consider the theory

$$T \coloneqq \mathsf{Th}(\mathbb{N}^*) \cup \{c_i > c_{i+1} \mid i \in \mathbb{Z}\}\,. \tag{$\star$}$$

in our new language. Any finite subset of T is satisfied by an expansion of $\mathbb{N}^*$ suitably interpreting the c_i since $\mathbb{N}^*$ has descending chains of all finite lengths. By the compactness theorem, it follows that there is some model $\mathbb{A}$ of T, which must be infinite. If $|\mathbb{A}| > \aleph_0$, then apply the Löwenheim-Skolem theorem to get a model $\mathbb{B}$ of T such that $|\mathbb{B}| = \aleph_0$. Let

$$\mathbb{A}' = \begin{cases} \mathbb{B} & |\mathbb{A}| > \aleph_0 \\ \mathbb{A} & |\mathbb{A}| = \aleph_0 \end{cases}.$$

Note that $\mathbb{A}' \models T$. Since the property of being a linearly ordered set is expressible by a first-order sentence, we see that $\mathbb{A}'$ is linearly ordered by $<$. Further, we see that $\mathbb{A}'$ has an infinite descending chain, which means that $\mathbb{A}'$ is not well-ordered by $<$. But $(\omega, <)$ is a well-ordered set, and thus the reduct of $\mathbb{A}'$ to the language of $\mathbb{N}^*$ is not isomorphic to $\mathbb{N}^*$. It does, however, satisfy $\mathsf{Th}(\mathbb{N}^*)$. This shows that $\mathsf{Th}(\mathbb{N}^*)$ is not $\aleph_0$-categorical.

Unfortunately, it's unclear that this method can be adapted to show that $\mathsf{Th}(\mathbb{N}^*)$ is not κ-categorical when κ is uncountable. In this case, we instead shall employ two binary operations on the class of all linear orderings. Let L_1 and L_2 be linearly ordered sets.

- $L_1 \cdot L_2$ refers to $L_1 \times L_2$ equipped with the lexicographic order.
- $L_1 + L_2$ refers to L_1 with its ordering followed by L_2 with its ordering.

Now, consider the following linearly ordered structures:

$$\begin{gathered} \mathbb{N}^* + (\mathbb{Z} \cdot \kappa) \\ \mathbb{N}^* + (\mathbb{Z} \cdot (\mathbb{Q} + \kappa)), \end{gathered}$$

both of which have cardinality κ. These orderings possess minimal elements and are *discrete* in the sense that both structures satisfy the sentences

$$\begin{gathered} \forall x \exists y \, (x < y \wedge \neg \exists z \, (x < z \wedge z < y)) \\ \forall x \, (\exists w \, (w < x) \rightarrow \exists y \, (y < x \wedge \neg \exists z \, (y < z \wedge z < x)))\,. \end{gathered} \tag{1}$$

Note that, on the one hand, $\mathbb{N}^* + (\mathbb{Z} \cdot \kappa)$ cannot possess an descending chain of length ω^2, for otherwise κ, which is well-ordered, would possess an infinite descending chain. On the other hand, $\mathbb{N}^* + (\mathbb{Z} \cdot (\mathbb{Q} + \kappa))$ does possess such a chain since ω^* (the order type of $\mathbb{Z}_{<0}$) can be embedded in $\mathbb{Q}$. Therefore,

$$\mathbb{N}^* + (\mathbb{Z} \cdot \kappa) \not\cong \mathbb{N}^* + (\mathbb{Z} \cdot (\mathbb{Q} + \kappa)).$$

Definition 1. Suppose that $\mathcal{L}$ is a finite language without function symbols. Let $\mathbb{D}$ and $\mathbb{E}$ be two $\mathcal{L}$-structures. Let $n \in \omega$. The *Ehrenfeucht-Fraïssé game* $\mathrm{EF}_n(\mathbb{D}, \mathbb{E})$ *of length* n *on* $\mathbb{D}$ *and* $\mathbb{E}$ is a game of perfect information played as follows.

(a) There are exactly two players, the *spoiler* and the *duplicator*.

(b) There are exactly n rounds.

(c) The spoiler begins round $k \le n$ by picking an element of either $\mathrm{dom}(\mathbb{D})$ or $\mathrm{dom}(\mathbb{E})$. Next, the duplicator picks an element of the other domain.

(d) This yields two sequences $(d_1, \ldots, d_n)$ and $(e_1, \ldots, e_n)$ such that $d_i \in \operatorname{dom}(\mathbb{D})$ and $e_i \in \operatorname{dom}(\mathbb{E})$ for each $i = 1, \ldots, n$. If the mapping $d_i \mapsto e_i$ defines an isomorphism of finite substructures, then we say that the duplicator has won $\mathrm{EF}_n(\mathbb{D}, \mathbb{E})$. Otherwise, we say that the spoiler has won.

Theorem 2 (Fraïssé). *The duplicator has a winning strategy in* $\mathrm{EF}_n(\mathbb{D}, \mathbb{E})$ *for each* $n \in \omega$ *if and only if* $\mathbb{D} \equiv \mathbb{E}$.

Claim 2. *Suppose that* $(\mathbb{E}, <)$ *is a discrete linear ordering with a minimal element but no maximal element. Then* $\mathbb{E} \equiv \mathbb{N}^*$.

Proof sketch. Consider the Ehrenfeucht-Fraïssé game $\mathrm{EF}_n(\mathbb{E}, \mathbb{N}^*)$. The duplicator has a winning strategy in $\mathrm{EF}_n(\mathbb{E}, \mathbb{N}^*)$ by adhering to the following rules.

(i) If, in round m, the spoiler chooses an element of one of the structures that is connected to a previously chosen element or the minimal element by a path of successors of length $k < \infty$, then choose the corresponding element of the other structure in round m.

(ii) Otherwise, make sure that any chosen element of $\operatorname{dom}(\mathbb{N}^*)$ is always separated by at least $n+1$ elements from any previously chosen element of $\operatorname{dom}(\mathbb{N}^*)$ while preserving the required order of your choices.

In this case, choose first a natural number separated by more than 3^n elements from the greatest previously chosen element of $\operatorname{dom}(\mathbb{N}^*)$.

□

Thanks to Theorem 2, it follows that both $\mathbb{N}^* + (\mathbb{Z} \cdot \kappa)$ and $\mathbb{N}^* + (\mathbb{Z} \cdot (\mathbb{Q} + \kappa))$ are elementarily equivalent to $\mathbb{N}^*$ and thus models of $\mathsf{Th}(\mathbb{N}^*)$. Hence $\mathsf{Th}(\mathbb{N}^*)$ is not κ-categorical. ■

4. Show that any set definable over $\mathbb{N}^*$ is either finite or cofinite.

Remark. This shows that $\mathbb{N}^*$ is *o-minimal* in the sense that every definable set over $\mathbb{N}^*$ is a finite union of points and intervals in ω.

Note that any set definable over $\mathbb{N}^*$ is 0-definable because any natural number n is uniquely determined by the first-order property

$$\begin{cases} \text{“}n \text{ is less than any other element”} & n = 0 \\ \text{“there are exactly } n-1 \text{ elements between 0 and } n\text{”} & n > 1 \end{cases}.$$

Suppose, toward a contradiction, that there exist a formula $\theta(y)$ such that $\theta[\mathbb{N}^*]$ is both infinite and coinfinite. Consider, again, the theory $(\star)$. Let

$$T' = T \cup \{\theta(c_0), \neg\theta(c_1)\}.$$

Since both $\theta[\mathbb{N}^*]$ and $\neg\theta[\mathbb{N}^*]$ are infinite, we can find an expansion of $\mathbb{N}^*$ that satisfies any finite subset of T', By the compactness theorem together with the Löwenheim-Skolem theorem, we thus can find a countable model $\mathbb{D}$ of T' and take its reduct $\mathbb{C}$ to the language of $\mathbb{N}^*$. Note that $(\operatorname{dom}(\mathbb{C}), <)$ is a countable linear ordering with an infinite descending and ascending chain χ on which both $c_0^{\mathbb{D}}$ and $c_1^{\mathbb{D}}$ lie. Moreover, this ordering is discrete in the sense of (1). Therefore, we may assume that χ has the form

$$\cdots < x_{m-1} < x_m < x_{m+1} < \cdots$$

where x_{m+1} denotes the immediate successor of x_m. There is an automorphism of $\mathbb{C}$ mapping $c_0^{\mathbb{D}}$ to $c_1^{\mathbb{D}}$ by suitably shifting χ finitely many places to the left and fixing all elements outside χ. But this contradicts the fact that $\mathbb{C} \models \theta\left[c_0^{\mathbb{D}}\right] \wedge \neg\theta\left[c_1^{\mathbb{D}}\right]$. ■