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Abstract

These notes are based on Tony Pantev's "Algebra II" lectures at UPenn. Any mistake in what follows is my own.

Contents

1 Injective and flat modules

1.1 Lecture 1

Proposition 1.1.1. *An R-module M is injective if and only if we can fill any injectivity diagram of ideal type, i.e.,*

$$\begin{tikzcd} & & M \\ 0 \arrow[r] & \mathfrak{a} \arrow[ur] \arrow[r] & R \arrow[u, dashed] \end{tikzcd}$$

where $\mathfrak{a}$ is an ideal in R.

Proof.

($\Longrightarrow$)
This is obvious.

($\Longleftarrow$)
Let

$$\begin{tikzcd} & & M \\ 0 \arrow[r] & X' \arrow[ur, "\varphi"] \arrow[r] & X \end{tikzcd}$$

be an injectivity diagram of R-modules and define

$$S = \{(A, \xi) \mid X' \subset A \subset X,\ \xi : A \to M,\ \xi \restriction_{X'} = \varphi\}.$$

By Zorn's lemma, there is some maximal element (N, ψ) of S. Suppose, toward a contradiction, that $X \neq N$. Pick any $x \in X \setminus N$. We have the ideal

$$\mathfrak{a} \coloneqq \{a \in R : ax \in N\}$$

in R. Define the R-module morphism $\theta : \mathfrak{a} \to M$ by $a \mapsto \psi(ax)$. By hypothesis, we get the following commutative diagram.

$$\begin{tikzcd} & & M \\ 0 \arrow[r] & \mathfrak{a} \arrow[ur, "\theta"] \arrow[r, hook] & R \arrow[u, dashed, "\tilde{\theta}"'] \end{tikzcd}$$

Define the R-submodule $\widetilde{N} = \langle N, x \rangle$. We can write any $z \in \widetilde{N}$ as $z = y + ax$ for some $y \in N$ and some $a \in R$. Define $\tilde{\psi} : \widetilde{N} \to M$ by $y + ax \mapsto \psi(y) + \tilde{\theta}(a)$. To see that this is well-defined, let $y + ax = y' + a'x$. Then $(y - y') = (a' - a)\,x$, so that

$$\psi(y - y') = \psi((a' - a)\,x) = \tilde{\theta}\,(a' - a) = \tilde{\theta}(a') - \tilde{\theta}(a).$$

This implies that $\tilde{\psi}$ is a well-defined homomorphism. But then $\left(\widetilde{N}, \tilde{\psi}\right) > (N, \psi)$, a contradiction. $\square$

Aside. The categorical dual P^{op} of this recognition principle for injectivity expresses a recognition principle for projectivity, namely that for any R-module M, ideal $I \subset R$, and homomorphism $\varphi : M \to R/I$, we can fill the diagram

$$\begin{tikzcd} M \arrow[d, dashed] \arrow[dr, "\varphi"] & & \\ R \arrow[r, two heads] & R/I \arrow[r] & 0 \end{tikzcd} \tag{$*$}$$

if and only if M is projective. This is equivalent to saying that M is projective if and only if the natural group map $\mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R\left(M, R/I\right)$ is surjective. But then P^{op} is precisely an affirmative answer to what is known as "Faith's problem on R-projectivity," which Trlifaj (2017) proved to be undecidable in $\mathsf{ZFC} + \mathsf{GCH}$. Therefore, both P^{op} and $\neg(P^{op})$ are consistent with $\mathsf{ZFC} + \mathsf{GCH}$.

Corollary 1.1.2.

1. *If R is an integral domain, then any injective R-module M is divisible.*

2. *If R is a PID, then M is injective if and only if it is divisible.*

Proof.

1. Given any $a \in R$, we want to show that the homomorphism $\mathrm{mult}_a : M \to M$ given by $x \mapsto ax$ is surjective. The assumption that R is an integral domain entails that $\mathrm{mult}_a : R \to R$ is injective. Note that $\mathfrak{a} := \mathrm{mult}_a(R)$ is an ideal in R, giving the short exact sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0.$$

By assumption, $\mathrm{Hom}_R(-, M)$ is exact, so that the sequence

$$0 \to \mathrm{Hom}\left(R/\mathfrak{a}, M\right) \to \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(\mathfrak{a}, M) \to 0$$

is exact. Since R and $\mathfrak{a}$ are free R-modules of rank 1, it follows that $\mathrm{Hom}_R(R, M) \cong M \cong \mathrm{Hom}_R(\mathfrak{a}, M)$. This means that the sequence

$$0 \to \mathrm{Hom}\left(R/\mathfrak{a}, M\right) \to M \xrightarrow{\mathrm{mult}_a} M \to 0$$

exact. In particular mult_a is surjective.

2. ($\Longleftarrow$) Suppose that M is divisible and R is a PID. We want to fill the injectivity diagram

$$\begin{tikzcd} & & M \\ 0 \arrow[r] & \mathfrak{a} \arrow[ur, "\varphi"] \arrow[r] & R \arrow[u, dashed, "\psi"'] \end{tikzcd} .$$

where $\mathfrak{a}$ is an ideal in R. We have that $\mathfrak{a} = (a)$. Therefore, the short exact sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$$

is isomorphic to $0 \to R \xrightarrow{\text{mult}_a} R \to {}^{R}\!/\!_{\mathfrak{a}} \to 0$. Since M is divisible, we know that $M \xrightarrow{\text{mult}_a} M \to 0$ is exact. Apply $\text{Hom}_R(-, M)$ to get the sequence

$$\text{Hom}_R(R, M) \xrightarrow{(-)\circ\text{mult}_a} \text{Hom}_R(R, M) \to 0,$$

which is isomorphic to $M \xrightarrow{\text{mult}_a} M \to 0$. This shows that $(-) \circ \text{mult}_a$ is surjective. It follows that φ can be lifted to some $\psi : R \to M$.

□

1.2 Lecture 2

Corollary 1.2.1. *Any abelian group is injective if and only if it's divisible.*

Corollary 1.2.2. *If R is a PID and M is an injective R-module, then every quotient of M is injective.*

Proof. This follows from the fact that any quotient of a divisible group is divisible. □

Example 1.2.3.

1. ${}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$ is injective.

2. S^1 is injective.

3. Any non-trivial finitely generated abelian group G is never injective.

   Proof. It suffices to show that G is never divisible. There exists a maximal proper subgroup $H \leq G$. Then ${}^{G}\!/\!_{H}$ is a simple abelian group, so that ${}^{G}\!/\!_{H} \cong C_p$ for some prime p. If G is divisible, then so must ${}^{G}\!/\!_{H}$. But C_p is not divisible, a contradiction. □

Theorem 1.2.4 (Baer embedding). *If R is a ring, then every module embeds into an injective module.*

Corollary 1.2.5. *For any R-module M, we can find an injective resolution*

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_k \to \cdots.$$

Proof. We want to invent a duality operation that will convert $R{-}\mathbf{Mod}^{\text{op}}$ to $R^{\text{op}}{-}\mathbf{Mod}$ and then use projective objects in $R^{\text{op}}{-}\mathbf{Mod}$. If T is an abelian group, then the functor

$$\mathbf{Ab} \underset{\text{Hom}_{\mathbf{Ab}}(-,T)}{\longrightarrow} \mathbf{Ab}^{\text{op}}$$

will reverse arrows. The choice of T that ends up working is precisely ${}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$.

Claim. *Let $\text{Hom}_{\mathbf{Ab}}(-, {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}) := (-)^D$. Note that for any abelian group A, we have a canonical homomorphism $\epsilon_A : A \to A^{DD}$ given by $a \mapsto \left(\left[\varphi : A \to {}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}\right] \to \varphi(a)\right)$. Then ϵ_A is injective.*

Proof. We need to show that if $a \in A$ is nonzero, then we can find some homomorphism $f : A \to \mathbb{Q}/\mathbb{Z}$ such that $f(a) \neq 0$.

Case 1: Suppose that $|(a)| = n < \infty$. Then define the homomorphism $\varphi : (a) \to \mathbb{Q}/\mathbb{Z}$ by $a \mapsto [\frac{1}{n}]$. Since $\mathbb{Q}/\mathbb{Z}$ is divisible in **Ab**, it is also injective. Thus, we may find some map ψ such that

$$\begin{tikzcd} & & \mathbb{Q}/\mathbb{Z} \\ 0 \arrow[r] & (a) \arrow[ur, "\varphi"] \arrow[r, hook] & A \arrow[u, dashed, "\psi"'] \end{tikzcd}$$

commutes. This means that $\psi(a) \neq 0$, as required.

Case 2: If (a) has infinite order, then define $\varphi : (a) \to \mathbb{Q}/\mathbb{Z}$ by $a \mapsto \frac{1}{2}$ and apply a similar argument to Case 1. □

The duality functor $(-)^D$ extends to a functor $(-)^D : R^{\mathrm{op}}{-}\mathbf{Mod} \to R{-}\mathbf{Mod}^{\mathrm{op}}$ that is compatible with forgetting the module structure. Indeed, if M is a left module over R^{op}, then its module structure is given by a collection of maps $\{\mathrm{mult}_a : M \to M \mid a \in R\}$. Note that

$$\mathrm{mult}_a \circ \mathrm{mult}_b = \mathrm{mult}_{a \cdot_{R^{\mathrm{op}}} b} = \mathrm{mult}_{b \cdot_R a}\,.$$

For each $a \in R$, let $\underline{\mathrm{mult}}_a(\varphi) = \varphi \circ \mathrm{mult}_a$. Then the abelian group M^D has an R-module structure given by $\underline{\mathrm{mult}}_a : M^D \to M^D$, which clearly satisfies

$$\underline{\mathrm{mult}}_{ab} = \underline{\mathrm{mult}}_a \circ \underline{\mathrm{mult}}_b.$$

Lemma 1.2.6. *If M is a projective R^{op}-module, then M^D is an injective R-module.*

Proof. Suppose that M is a projective R^{op}-module and consider the injectivity diagram

$$\begin{tikzcd} & & M^D \\ 0 \arrow[r] & X' \arrow[ur, "\varphi"] \arrow[r] & X \end{tikzcd}$$

of R-modules. We want to lift $\varphi : X' \to M^D$ to a map $\psi : X \to M^D$. Apply $(-)^D$ to get a commutative diagram

$$\begin{tikzcd} & & M^{DD} \arrow[dl, "\varphi^D"'] \\ 0 & (X')^D \arrow[l] & X^D \arrow[l] \end{tikzcd}$$

where the bottom row is exact because $\mathbb{Q}/\mathbb{Z}$ is injective.

Exercise 1.2.7. *Show that $\epsilon_M : M \to M^{DD}$ is a map of R^{op}-modules.*

We now have the following projectivity diagram of R^{op}-modules.

$$\begin{tikzcd} M \arrow[dr, "\epsilon_M \circ \varphi^D"] & & \\ X^D \arrow[r] & (X')^D \arrow[r] & 0 \end{tikzcd}$$

By assumption, we may fill this diagram with some map $\psi : M \to X^D$. This induces the map $\psi^D : X^{DD} \to M^D$. Note that $(\epsilon_M)^D \circ \varphi^{DD} = \psi^D \circ i^{DD}$ where $i : X' \hookrightarrow X$. But $i^{DD} \restriction_{X'} = i$ and $\varphi^{DD} \restriction_{X'} = \varphi$, so that

$$\psi^D \circ i = (\epsilon_M)^D \circ \varphi = \varphi$$

on X'. It follows that

$$\begin{tikzcd} & & M^D \\ 0 \arrow[r] & X' \arrow[r] \arrow[ur, "\varphi"] & X \arrow[u, "\psi^D \circ \epsilon_X"'] \end{tikzcd}$$

commutes. □

There is some surjection $\bigoplus_{j\in J} R \to M^D$. Therefore, we have a sequence of embeddings

$$M \hookrightarrow M^{DD} = \mathrm{Hom}_{\mathbb{Z}}\left(M^D, \mathbb{Q}/\mathbb{Z}\right) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}\left(\bigoplus_{j\in J} R, \mathbb{Q}/\mathbb{Z}\right) = \underbrace{\left(\bigoplus_{j\in J} R\right)^D}_{\text{injective by Lemma 1.2.6}}.$$

□

Definition 1.2.8. Given two R-modules M and N, the *additive invariants of M and N* are the abelian groups

$$\mathrm{Ext}^i_R(M,N) := H^i(\mathrm{Hom}_R(P^\bullet, N))$$

indexed by $\mathbb{N}$ where $P^\bullet$ is a chosen projective resolution of M.

Proposition 1.2.9.

1. $\mathrm{Ext}^i_R(M,N)$ *is independent of our choice of projective resolution.*

   Proof. This follows from the fact that any two projective resolutions are chain homotopic. □

2. $\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(M, I_\bullet))$ *for any injective resolution $I_\bullet$ of N.*

Lemma 1.2.10.

1. $\mathrm{Ext}^0_R(M,N) = \mathrm{Hom}_R(M,N)$
2. $\mathrm{Ext}^1_R(M,N) =$ (*the group of isomorphism classes of extensions of N by M in $R-$***Mod**).

Proof. Let

$$\cdots \xrightarrow{\partial_1} P^1 \xrightarrow{\partial_0} P^0 \xrightarrow{\epsilon} M \to 0$$

be a projective resolution and let

$$(\xi) : 0 \to N \xrightarrow{f} T \xrightarrow{g} M \to 0$$

be a short exact sequence of R-modules. Note that $\mathrm{Hom}_R(P^k, -)$ is exact for each $k \geq 0$. Therefore, the sequence

$$0 \to \mathrm{Hom}_R(P^k, N) \xrightarrow{f_k} \mathrm{Hom}_R(P^k, T) \xrightarrow{g_k} \mathrm{Hom}_R(P^k, M) \to 0$$

is exact where $f_k \coloneqq f \circ (-)$ and $g_k \coloneqq g \circ (-)$. Letting $d_i \coloneqq (-) \circ \partial_i$, we get *short exact sequences of complexes* constituting the columns of

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}_R(P^0, N) \arrow[r, "f_0"] \arrow[d, "d_0"] & \operatorname{Hom}_R(P^0, T) \arrow[r, "g_0"] \arrow[d, "d_0"] & \operatorname{Hom}_R(P^0, M) \arrow[r] \arrow[d, "d_0"] & 0 \\
0 \arrow[r] & \operatorname{Hom}_R(P^1, N) \arrow[r, "f_1"] \arrow[d, "d_1"] & \operatorname{Hom}_R(P^1, T) \arrow[r, "g_1"] \arrow[d, "d_1"] & \operatorname{Hom}_R(P^1, M) \arrow[r] \arrow[d, "d_1"] & 0 \\
& \vdots & \vdots & \vdots &
\end{tikzcd}$$

By definition, $\operatorname{Ext}^i_R(M,N) = \overbrace{\ker d_i / \operatorname{im} d_{i-1}}^{\text{for the first column}}$. Since $\operatorname{Hom}_R(-,M)$ is left-exact and $P^1 \xrightarrow{\partial_1} P^0 \xrightarrow{\epsilon} M$ is exact, we also have the exact sequence

$$0 \to \operatorname{Hom}_R(M,M) \xrightarrow{(-)\circ\epsilon} \operatorname{Hom}_R(P^0,M) \xrightarrow{d_0} \operatorname{Hom}_R(P^1,M).$$

Let $\psi \in \operatorname{Hom}_R(P^0,M)$ satisfy $d_0(\psi) = 0$. Then $\psi = \varphi \circ \epsilon$ for some unique map $\varphi : M \to M$. Since g_0 is surjective, there exists $\alpha \in \operatorname{Hom}_R(P^0,T)$ such that $g_0(\alpha) = \psi = \varphi \circ \epsilon$. This implies that

$$g_1(d_0(\alpha)) = d_0(g_0(\alpha)) = d_0(\psi) = 0.$$

It follows that $d_0(\alpha) \in \ker g_1 = \operatorname{im} f_1$, so that $d_0(\alpha) = f_1(\beta)$ for some $\beta : P^1 \to N$. Since $f_2(d_1(\beta)) = d_1(f_1(\beta)) = d_1(d_0(\alpha)) = 0$, the fact that f_2 is injective means that $d_1(\beta) = 0$. Hence $\beta \in \ker d_1$, and $[\beta] \in \operatorname{Ext}^1_R(M,N)$

Exercise 1.2.11. *Show that $\psi \mapsto [\beta]$ is well-defined, i.e., that $[\beta]$ is independent of α.*

This defines a map of abelian groups $\delta_\xi : \operatorname{Hom}_R(M,M) \to \operatorname{Ext}^1_R(M,N)$ given by $\varphi \mapsto [\beta]$. Now, define the homomorphism

$$e : \operatorname{Ext}_R(M,N) \to \operatorname{Ext}^1_R(M,N), \quad (\xi) \mapsto \delta_\xi(\mathrm{id}_M).$$

Apply $\operatorname{Hom}_R(M,-)$ to (ξ) to get the exact sequence

$$0 \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,T) \to \operatorname{Hom}_R(M,M\ ).$$

Claim. *We can extend this sequence to a long exact sequence of abelian groups*

$$0 \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,T) \to \operatorname{Hom}_R(M,M) \xrightarrow{\delta_\xi} \operatorname{Ext}^1_R(M,N) \to \operatorname{Ext}^1_R(M,T) \to \operatorname{Ext}^1_R(M,M).$$

Exercise 1.2.12. *Show that if (ξ) is split, then $\delta_\xi(\mathrm{id}_M) = 0$.*

How is it that e is injective?

This implies that e is injective. We need to show that it is surjective as well. Suppose that $\gamma \in \operatorname{Ext}^1_R(M,N)$ and let $I_\bullet$ be an injective resolution of N. Apply $\operatorname{Hom}_R(M,-)$ to $I_\bullet$ to get

$$0 \to \operatorname{Hom}_R(M,N) \xrightarrow{\nu} \operatorname{Hom}_R(M,I_0) \xrightarrow{d_0} \operatorname{Hom}_R(M,I_1) \xrightarrow{d_1} \cdots$$

(where we have abused the notation d_i). By Proposition 1.2.9(2), we have that $\gamma = [f]$ for some $f \in \ker d_1$. Note that $f : M \to \ker \partial_1 = \operatorname{im} \partial_0$, giving

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & I_0 \arrow[r, "\partial_0"] & \operatorname{im} \partial_0 \arrow[r] & 0 \\ & & & M \arrow[u, "f"] & \end{tikzcd}$$

where the top row is exact. Take the pullback of ∂_0 and f to obtain T such that

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & I_0 \arrow[r, "\partial_0"] & \operatorname{im} \partial_0 \arrow[r] & 0 \\ 0 \arrow[r] & N \arrow[r] \arrow[u, equal] & T \arrow[r] \arrow[u] & M \arrow[r] \arrow[u, "f"] & 0 \end{tikzcd}.$$

Exercise 1.2.13.

1. *Show that the map $\rho : \operatorname{Ext}^1_R(M, N) \to \operatorname{Ext}_R(M, N)$ given by $\gamma \mapsto \xi$ is independent of our choice of f.*
2. *Show that ρ is the inverse of e.*

□

1.3 Lecture 3

Let N be a right R-module and M an R-module. Recall that $N \otimes_R M \in \operatorname{ob}(\mathbf{Ab})$ is precisely the object in $\mathbb{Z}-\mathbf{Mod}$ representing the functor $B_{M,N} : \mathbf{Ab} \to \mathbf{Ab}$ given by

$$A \mapsto \{f : M \times N \to A \mid f(ax, y) = f(x, ay)\}.$$

Moreover, recall that N is *flat* if the functor

$$N \otimes_R (-) : R-\mathbf{Mod} \to \mathbf{Ab}$$

is exact.

Definition 1.3.1. Let N be a right R-module and M an R-module. Let $x_1, \ldots, x_n \in M$.

(1) A *relation of the x_i's with coefficients in R* is a list of scalars $a_1, \ldots, a_n \in R$ such that

$$\sum_{i=1}^n a_i x_i = 0.$$

(2) A *relation of the x_i's with coefficients in N* is a list of elements $y_1, \ldots, y_n \in N$ such that

$$\sum_{i=1}^n y_i \otimes x_i = 0.$$

Since $R \otimes_R M \cong M$, we see that (1) is a special case of (2).

Let

$$\begin{aligned} a_1 &:= (a_{11}, \ldots, a_{1n}) \\ a_2 &:= (a_{21}, \ldots, a_{2n}) \\ &\vdots \\ a_m &:= (a_{21}, \ldots, a_{mn}). \end{aligned}$$

be relations of $x_1, \ldots, x_n$ with coefficients in R. Let $(z_1, \ldots, z_m) \in N^m$. If A denotes the matrix (a_{ij}), then $y = A^t z \in N^n$ is a relation with coefficients in N.

Definition 1.3.2. A relation y with coefficients in N *follows from R-relations* if y is of the form $A^t z$ for some z and some matrix A of relations in R.

Lemma 1.3.3. *A right R-module N is flat if and only if for any R-module M and any $x_1, \ldots, x_n \in M$, every N-relation among the x_i follows from R-relations.*

Proof.

($\Longrightarrow$)

We have a module homomorphism $\varphi : R^n \to M$ given by $(a_1, \ldots, a_n) \mapsto \sum_{i=1}^n a_i x_i$. Then

$$\underbrace{\ker \varphi}_{K} = \{(r_1, \ldots, r_n) \in R^n \mid (r_1, \ldots, r_n) \text{ is a relation of the } x_i\text{'s in } R\}.$$

We have an exact sequence

$$0 \to K \xrightarrow{i} R^n \xrightarrow{\varphi} M.$$

If N is flat, then $N \otimes_R (-)$ is exact, so that

$$0 \to N \otimes_R K \xrightarrow{\tilde{i}} N^n \xrightarrow{\tilde{\varphi}} N \otimes_R M$$

is exact. Thus, $\ker \tilde{\varphi} = (N\text{-relations}) = N \otimes_R K$.

($\Longleftarrow$)

Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence of R-modules. Since $N \otimes_R (-)$ is right exact, it suffices to show that

$$N \times_R M' \xrightarrow{\mathrm{id}_N \otimes f} N \otimes_R M$$

is injective. Let $z \in \ker \mathrm{id}_N \otimes f$. Then $z = \sum_{i=1}^n y_i \otimes z_i$. We know that

$$\sum_{i=1}^n y_i \otimes f(z_i) = \mathrm{id}_N \otimes f(z) = 0,$$

and thus $(y_1, \ldots, y_n)$ is an N -relation among the $f(z_i) \in M$. This shows that there exist $\left(a_i^j\right) \in R$ where $i = 1, \ldots, n$ and $j = 1, \ldots, m$ and elements $v_1, \ldots, v_m \in N$ such that $y_i = \sum_{i=1}^m v_j a_i^j$. Therefore, $\sum_{i=1}^n a_i^j f(z_i) = 0$ for each j. But

$$0 = \sum_{i=1}^n a_i^j f(z_i) = f\left(\sum_{i=1}^n a_i^j z_i\right).$$

As f is injective, it follows that $\sum_{i=1}^n a_i^j z_i = 0$ for each j. Finally, we compute

$$\begin{aligned}\sum_{i=1}^n y_i \otimes z_i &= \sum_{i=1}^n \sum_{j=1}^m (v_j a_i^j) \otimes z_i \\ &= \sum_{j=1}^n v_j \otimes \left(\sum_{i=1}^n a_i^j z_i \right) \\ &= \sum_{j=1}^n (v_j \otimes 0) = 0.\end{aligned}$$

□

Corollary 1.3.4.

1. *Any free module is flat.*
2. *Any colimit of flat modules is flat.*
3. *Any direct summand of a free module is flat, so that any projective module is flat.*
4. *Any colimit of projective modules is flat.*

2 Localization

2.1 Lecture 4

Let R be a commutative ring. Given $x \in R$, when can we make x multiplicatively invertible, perhaps in a new ring? This is a question of representability. We have a functor $\Phi_x : \mathbf{CommRing} \to \mathbf{Set}$ given by

$$B \mapsto \left\{ \varphi : R \to B \mid \varphi(x) \in B^\times \right\} \subset \mathrm{Hom}_{\mathbf{CommRing}}(R, B).$$

We are asking whether or not Φ_x is representable. That is, we want to find some pair (R_x, h) where R_x is a commutative ring and $h : R \to R_x$ is a morphism such that $h(x) \in (R_x)^\times$ and if $\varphi : R \to B$ with $\varphi(x) \in B^x$, then $\underline{\varphi} \circ h = \varphi$ for some map $\underline{\varphi} : R_x \to B$.

In general, we can consider a set S of nonzero elements and ask for a universal way of making them invertible. But if we make S invertible, then we shall also make the *multiplicative closure* $\mathrm{cl}(S)$ *of* S invertible.

Definition 2.1.1. Any $S \subset R$ is called *multiplicatively closed* if $0 \notin S$, $1 \in S$, and $x, y \in S \implies xy \in S$.

Given a multiplicatively closed subset $S \subset R$, we want to find a universal way of inverting every element of S. Equivalently, find a ring representing Φ_S. Equivalently, we want to find a pair $\left(S^{-1}R, h\right)$ where $h : R \to S^{-1}R$ such that $h(S) \subset (S^{-1}R)^\times$ and any $\varphi : R \to B$ with $\varphi(S) \subset B^\times$ has $\varphi = \underline{\varphi} \circ h$ for some unique map $\underline{\varphi} : R \to B$. We call the pair $\left(S^{-1}R, h\right)$ the *localization of* R *along* S.

Formally adjoin to R fractions with numerator in R and denominator in S. Consider the set $(R \times S, \sim)$ where $(a,s) \sim (b,t)$ if $u(at - bs) = 0$ for some $u \in S$. Set $S^{-1}R \coloneqq {R \times S}/{\sim}$. Let $\frac{a}{s} \coloneqq [(a,s)]$. Define

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

and

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Then $S^{-1}R$ becomes a ring with unity $\frac{1}{1}$. Also, we see that $h : R \to S^{-1}R$ given by $a \mapsto \frac{a}{1}$ is a ring homomorphism. Given a map $\varphi : R \to B$ such that $\operatorname{im} \varphi \subset B^\times$, we have a well-defined map of rings $\underline{\varphi} : S^{-1}R \to B$ given by $\frac{a}{s} \mapsto \varphi(a)\varphi(s)^{-1}$, which satisfies $\underline{\varphi} \circ h = \varphi$.

Example 2.1.2. Here are some natural choices for S.

(a) $\{1, x, x^2, \ldots\}$ with x not nilpotent.

(b) $R^\times$.

(c) $\{r \in R : r \text{ is not a zero divisor}\}$.

If A is an integral domain and we take any multiplicatively closed subset $S \subset A$, then $\operatorname{Frac}(A) \coloneqq (A \setminus \{0\})^{-1} A$ is a field and $h : A \to (A \setminus \{0\})^{-1} A$ is injective. For now, let S denote the set of non zero-divisors. If $\frac{a}{b} \in \operatorname{Frac}(A)$ is nonzero, then $\frac{a}{b} \neq \frac{0}{1}$, i.e., $a \cdot 1$ is not a zero divisor, so that $a \neq 0$ and thus $\frac{b}{a} \in \operatorname{Frac}(A)$. This shows that $\operatorname{Frac}(A)$ is a field. Moreover, if $a \in A$ satisfies $h(a) = \frac{a}{1} = 0 \in \operatorname{Frac}(A)$, then $\frac{a}{1} = \frac{0}{1} \implies a \cdot 1$ is a zero divisor. Hence $a = 0$, and h is injective.

If S is generic, then $S^{-1}A \subset \operatorname{Frac}(A)$ since $S^{-1}A$ equals the subring generated by $A \cong h(A)$ and $S^{-1} = \left\{\frac{1}{s} \mid s \in S\right\}$. In this case, $\left(S^{-1}A, h\right)$ represents the functor $\Phi : \mathbf{Field} \to \mathbf{Set}$ given by $k \mapsto \{\varphi : A \to k \mid \varphi \text{ is injective}\}$. This means that for any ring map $\varphi : A \to B$ with $\varphi(S) \subset B^\times$, there is some unique map ψ such that $\psi \circ h = \varphi$.

2.2 Lecture 5

Example 2.2.1.

1. If $S = \{1, x, x^2, \ldots\}$ with x not nilpotent, then $S^{-1}A = A_f \equiv \left\{\frac{a}{f^n} : n \geq 0,\ a \in A\right\}$.
2. If $S \subset A^\times$, then $h : A \to S^{-1}A$ is an isomorphism.
3. If A is any ring and $S \subset A$ denotes the set of all non-zero divisors, then $\operatorname{Frac}(A) = S^{-1}A$ is called the *fraction ring of* A. If A is an integral domain, then $\operatorname{Frac}(A)$ is a field (called the *field of fractions of* A) and $H : A \to \operatorname{Frac}(A)$ is injective. In this case, $(\operatorname{Frac}(A), h)$ represents the functor $F_A : \mathbf{Field} \to \mathbf{Set}$ given by $K \mapsto \{\varphi : A \to K \mid \varphi \text{ monomorphism}\}$.

Let A be a commutative ring and $S \subset A$ be multiplicatively closed. Let M be an A-module. Define the equivalence relation $(M \times S, \sim)$ where $(m,s) \sim (n,t)$ if $u(tm - sn) = 0$ for some $u \in S$.

Define the A-module $S^{-1}M = {}^{M \times S}\!/\!_{\sim}$ where $\frac{m}{s} + \frac{n}{t} := \frac{tm+sn}{st}$. Define the module homomorphism $h_M : M \to S^{-1}M$ by $m \mapsto [(m,1)]$. Let $\frac{m}{s}$ denote the equivalence class $[(m,s)]$.

Moreover, $S^{-1}M$ is naturally a module over $S^{-1}A$ via the action $\frac{a}{s} \cdot \frac{m}{t} := \frac{a \cdot m}{st}$. This makes h_M a module over $h : A \to S^{-1}A$ in that for any $a \in A$ and $m \in M$, we have that $h_M(a \cdot m) = h(a) \cdot h_M(m)$.

We see that $S^{-1}(-)$ is a functor which maps each homomorphism $\varphi : M \to N$ to $S^{-1}\varphi : S^{-1}M \to S^{-1}N$ given by $\frac{m}{s} \mapsto \frac{\varphi(m)}{s}$. It's easy to verify that $S^{-1}(-)$ is left adjoint to the pullback functor $h^\bullet$.

If $f : A \to B$ is a map of commutative rings, then there are natural functors $f^\bullet : B-\mathbf{Mod} \to A-\mathbf{Mod}$ and $f_\bullet : A-\mathbf{Mod} \to B-\mathbf{Mod}$, called the *pullback* and *pushforward*, respectively.

On the one hand, the pullback functor is already familiar to us. On the other hand, the pushforward acts on objects by

$$f_\bullet(M) \equiv B \otimes_A M$$

where B is viewed as an A-module via f along with the action $b \cdot (c \otimes m) \equiv (bc) \otimes m$. It acts on morphisms by $(\varphi : M \to N) \mapsto (\mathrm{id}_B \otimes \varphi : f_\bullet(M) \to f_\bullet(N))$.

Exercise 2.2.2. $(f_\bullet, f^\bullet)$ *is an adjoint pair.*

Corollary 2.2.3. $S^{-1}(-) \cong h_\bullet$.

Naively, we could have tried to define fractions in A by $(a,s) \sim_n (b,t)$ if $(at - bs = 0)$. But this is not in general an equivalence relation, for it is not transitive. Indeed, set $A = {}^{\mathbb{C}[x,y]}\!/\!_{(xy)}$ and $S = \{1, x, x^2, \ldots\}$. Consider the localization A_x. Note that $(y,1) \not\sim_n (0,1)$ but that $(y,1) \sim_n (0,x)$ and $(0,x) \sim_n (0,1)$.

Note 2.2.4. We have that $A_x = \mathbb{C}[x, x^{-1}]$, which is a field, and that $h : A \to A_x$ is given by

$$\underbrace{[f(x,y)]}_{[p(x)+yq(y)]} \mapsto p(x),$$

which is non-injective.

Proposition 2.2.5.

1. *If* $h : A \to S^{-1}A$*, then* $\ker h = \{a \in A : (\exists s \in S)\,(sa = 0)\}$.

2. $S^{-1}A$ *is flat as an* A*-module.*

Corollary 2.2.6. $S^{-1}(-)$ *is an exact functor.*

Proof. Let $M \xrightarrow{f} T \xrightarrow{g} N$ be an exact sequence of A-modules. We want to show that

$$S^{-1}M \xrightarrow{S^{-1}f} S^{-1}T \xrightarrow{S^{-1}g} S^{-1}N$$

is exact as well. Let $\frac{x}{s} \in S^{-1}T$ with $\left(S^{-1}g\right)\left(\frac{x}{s}\right) = 0$. This implies that $\frac{g(x)}{s} = \frac{0}{1}$, so that $ug(x) = 0$ for some $u \in S$. But since g is a morphism, we know that $0 = ug(x) = g(ux)$. This means that $f(y) = ux$ for some $y \in M$. Then $\frac{y}{us} \in S^{-1}M$ such that $\left(S^{-1}f\right)\left(\frac{y}{us}\right) = \frac{f(y)}{us} = \frac{ux}{us} = \frac{x}{s}$. $\square$

Suppose that $f \in A$ is not nilpotent. We can compute A_f explicitly as follows. There is a natural map $A_f[x] \to A_f$ given by $x \mapsto \frac{1}{f}$. This induces an isomorphism

$$A_f[x] \Big/ \left(x - \frac{1}{f}\right) \xrightarrow{\cong} A_f.$$

We also have a map $A[x] \to A_f[x]$ from the map h on the coefficients. Define the map $\alpha : A[x] \to A_f$ by $a \mapsto h(a) = \frac{a}{1}$ and $x \mapsto \frac{1}{f}$. We must compute $\ker \alpha$ as an ideal in $A[x]$. This is surjective since any element in A_f is of the form $\frac{a}{f^n}$ for some $a \in A$ and $n \in \mathbb{N}$, so that $ax^n \mapsto \frac{a}{f^n}$.

2.3 Lecture 6

Claim. $\ker \alpha = (fx - 1)$.

Proof. Note that $xf - 1 \in \ker \alpha$. Also, note that

$$\exists n \geq 0 \text{ s.t. } f^n g\left(\frac{1}{f}\right) = 0 \iff \alpha(g) = 0 \iff g\left(\frac{1}{f}\right) = 0 \text{ in } A_f.$$

Without loss of generality, we may assume that $n \geq \deg g$. Thus, $f^n g(x)$ is a polynomial of fx with coefficients in A, so that there is some $G(y) \in A[y]$ such that $G(fx) = f^n g(x)$. Then

$$g \in \ker \alpha \iff \exists G(y) \in A[y] \text{ s.t. } G(fx) = f^n g(x) \wedge G(1) = 0.$$

But then $G(y) = (y - 1)\, h(y)$ where $h(y) \in A[y]$. This implies that

$$g(x) \in \ker \alpha \iff \exists n \geq 0 \text{ s.t. } f^n g(x) \in (xf - 1).$$

But $f, fx-1 \in A[x]$ are relatively prime since $1 = fx + (fx - 1)(-1)$. Hence $1^n = (fx + (fx - 1)(-1))^n = f^n x^n + (fx - 1)s(x)$ for some $s(x) \in A[x]$. Multiply by $g(x)$ to get

$$g(x) = \underbrace{f^n g(x)}_{\substack{\cap \\ (xf-1)}} + \underbrace{(fx - 1)s(x)g(x)}_{\substack{\cap \\ (xf-1)}}.$$

Therefore, $g(x) \in (xf - 1)$, and $(xf - 1) = \ker \alpha$. $\square$

Suppose that $\varphi : A \to B$ is a map of commutative rings. Then we can transport the ideals along φ as follows.

Definition 2.3.1.

1. Given an ideal $\mathfrak{a} \trianglelefteq A$, the *extension of* $\mathfrak{a}$ *along* φ is the ideal $\mathfrak{a}^e \trianglelefteq B$ that is generated by $\varphi(\mathfrak{a})$, i.e., $\mathfrak{a}^e = \varphi(\mathfrak{a}) \cdot B$.
2. Given an ideal $\mathfrak{b} \trianglelefteq B$, the *contraction of* $\mathfrak{b}$ *along* φ is defined as the ideal $\mathfrak{b}^c = \varphi^{-1}(\mathfrak{b})$.

Suppose that A is a commutative ring and that $S \subset A$ is multiplicatively closed. Recall the localization morphism $h : A \to S^{-1}A$. We want to study $(-)^e$ and $(-)^c$ along h.

Proposition 2.3.2.

1. *If $\mathfrak{a} \trianglelefteq A$, then $\mathfrak{a}^e = \left\{ \frac{a}{s} \mid a \in \mathfrak{a},\ s \in S \right\}$.*

   Proof. By definition, $\mathfrak{a}^e = h(\mathfrak{a}) \cdot S^{-1}A = \left\{ \sum_i \frac{b_i}{t_i} \frac{a_i}{s_i} \mid a_i \in \mathfrak{a},\ b_i \in A,\ s_i, t_i \in S \right\}$. Since $a_i b_i \in \mathfrak{a}$ and $s_i t_i \in S$, our proof is complete. $\square$

2. *If $\mathfrak{a} \trianglelefteq A$, then $\mathfrak{a}^e = (1) \iff \mathfrak{a} \cap S \neq \emptyset$.*

   Proof. Note that $\left(S^{-1}A\right)^\times$ consists of every fraction $\frac{a}{s}$ for which we can find some fraction $\frac{b}{t}$ such that $\frac{a}{s}\frac{b}{t} = 1$. Therefore, we must have some element $u \in S$ such that $u(ab - st) = 0 \iff \exists \beta \in A$ s.t. $\beta a \in S$. Thus, $\left(S^{-1}A\right)^\times = \left\{ \frac{a}{s} \mid \exists \beta \in A \text{ s.t. } \beta a \in S \right\}$. But then

$$\begin{aligned} \mathfrak{a}^e = (1) &\iff \mathfrak{a}^e \text{ contains some unit} \\ &\iff \left(\exists \frac{a}{s} \in \mathfrak{a}^e\right)(\exists \beta \in A)(\beta \cdot a \in S) \\ &\iff \beta \cdot a \in S \cap \mathfrak{a}. \end{aligned}$$

   $\square$

Suppose that $I \trianglelefteq S^{-1}A$ is an ideal . Then we can form $I^{ce} \trianglelefteq S^{-1}A$. By definition, $I \supset I^{ce}$.

Proposition 2.3.3.

1. *In fact, $I = I^{ce}$.*

   Proof. If $\frac{a}{s} \in I$, then $a \in h^{-1}(I)$ because $h^{-1}(I) = \left\{ r \in A \mid \frac{r}{1} \in I \right\}$. But $\frac{a}{1} = s \cdot \frac{a}{s}$ where $s \in S^{-1}A$ and $\frac{a}{s} \in I$, so that $a \in I$. This implies that $\frac{a}{s} \in I^{ce}$ for each $s \in S$, and thus $I \subset I^{ce}$. $\square$

2. *If $\mathfrak{a} \trianglelefteq A$, then $\mathfrak{a}^{ec} = \{ r \in A \mid \exists s \in S \text{ s.t. } sr \in \mathfrak{a} \}$.*

   Proof. Suppose that $a \in \mathfrak{a}^{ec}$. Then $\frac{a}{1} = h(a) \in \mathfrak{a}^e$, so that $(\exists b \in \mathfrak{a})(\exists s \in S)\left(\frac{a}{1} = \frac{b}{s}\right)$. This implies that $\exists u \in S$ such that $u(sa - b) = 0$. Hence $(us)\, a = b$, and $\mathfrak{a}^{ec} \subset \{ r \in A \mid \exists s \in S \text{ s.t. } sr \in \mathfrak{a} \}$. If $r \in A$ satisfies $rs \in \mathfrak{a}$ for some $s \in S$, then $\frac{r}{1} = \frac{rs}{s} \in \mathfrak{a}^e$ and thus $r \in \mathfrak{a}^{ec}$. $\square$

3. *$\mathfrak{a} \trianglelefteq A$ is contracted (i.e., $\mathfrak{a} = I^c$ for some $I \trianglelefteq S^{-1}A$) if and only if $\mathfrak{a} = \mathfrak{a}^{ec}$ if and only if $[s] \in A/\mathfrak{a}$ is not a zero divisor for any $s \in S$.*

4. *The map $(-)^e$ induces a bijection*

$$(-)^e : \{ \mathfrak{a} \trianglelefteq A \mid \mathfrak{a} \text{ is a contraction of some ideal} \} \to \left\{ I \mid I \trianglelefteq S^{-1}A \right\}$$

   that preserves inclusions of ideals.

Suppose that M is an A-module.

Definition 2.3.4. A submodule $N \subset M$ is *S-saturated* if $N = \{x \in M \mid (\exists s \in S)\,(sx \in N)\}$.

If $M = A$ and $N = \mathfrak{a}$, then N is S-saturated if and only if $\mathfrak{a} = \mathfrak{a}^{ec}$. The localization on modules induces an inclusion-preserving bijection

$$S^{-1}(-) : \{N \subset M \mid N \text{ is } S\text{-saturated}\} \to \left\{M \mid M \subset S^{-1}M\right\}.$$

Definition 2.3.5. Let $\mathfrak{a}$ be an ideal in A.

1. We say that $\mathfrak{a}$ is a *maximal ideal* if it is properly contained in A and is maximal in the set of all properly contained ideals in A partially ordered by inclusion.

2. We say that $\mathfrak{a}$ is a *prime ideal* if $xy \in \mathfrak{a} \implies x \in \mathfrak{a} \vee y \in \mathfrak{a}$.

Exercise 2.3.6. *An ideal $\mathfrak{b} \trianglelefteq A$ is prime if and only if $A \setminus \mathfrak{b}$ is multiplicatively closed.*

2.4 Lecture 7

Proposition 2.4.1. *If $\mathfrak{p} \trianglelefteq A$ is prime and $S \subset A$ is multiplicatively closed, then $\mathfrak{p}^e \trianglelefteq S^{-1}A$ is prime if and only if $S \cap \mathfrak{p} = \emptyset$.*

Proof. The forward direction is obvious. Conversely, suppose that $S \cap \mathfrak{p} = \emptyset$. Then $\mathfrak{p}^{ec} = \mathfrak{p}$. Indeed, $\mathfrak{p}^{ec} = \{a \in A \mid \exists s \in S \text{ s.t. } sa \in \mathfrak{p}\}$. But if $sa \in \mathfrak{p}$, then either $s \in \mathfrak{p}$ or $a \in \mathfrak{p}$. Since $S \cap \mathfrak{p} = \emptyset$, we see that $s \notin \mathfrak{p} \implies a \in \mathfrak{p}$. Suppose that $x \cdot y \in \mathfrak{p}^e$. Then $x = \frac{a}{s}$ for some $a \in A$ and $s \in S$, and $y = \frac{b}{t}$ for some $b \in A$ and $t \in B$. Then $\frac{ab}{st} \in \mathfrak{p}^e$, so that $\frac{ab}{t} \in \mathfrak{p}^e$ since $\mathfrak{p}^e$ is an ideal. Hence $ab \in \mathfrak{p}$, which is prime by assumption. Say that $a \in \mathfrak{p}$. Then $\frac{a}{s} \in \mathfrak{p}^e$. $\square$

Corollary 2.4.2. *If $S \subset A$ is multiplicatively closed, then we get a bijection*

$$\{\mathfrak{p} \in \mathsf{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset\} \xrightarrow{(-)^e} \mathsf{Spec}(S^{-1}A).$$

Proof. This is because $\mathfrak{p}^c$ is prime in A with $\mathfrak{p}^c \cap S = \emptyset$ whenever $\mathfrak{p}$ is prime in $S^{-1}A$. $\square$

Now, recall the property P that an ideal $\mathfrak{a}$ in A is prime if and only if $A \setminus \mathfrak{a}$ is multiplicatively closed.

Proposition 2.4.3. *$\mathfrak{a}$ is prime if and only if there is some multiplicatively closed $S \subset \mathfrak{a}$ such that $S \cap A = \emptyset$ and $\mathfrak{a}$ is maximal among all ideals satisfying* P.

Proof. If $\mathfrak{a}$ is prime, then $S = A \setminus \mathfrak{a}$ is multiplicatively closed and $\mathfrak{a}$ is maximal. Conversely, let $a, b \in A$ such that $a, b \notin \mathfrak{a}$. We must show that $ab \notin \mathfrak{a}$. Consider $\mathfrak{a} + (a) \supsetneq \mathfrak{a}$ and $\mathfrak{a} + (b) \supsetneq \mathfrak{a}$. But we are given S such that $\mathfrak{a} \cap S = \emptyset$. Hence there are $s \in S \cap (\mathfrak{a} + (a))$ and $t \in S \cap (\mathfrak{a} + (b))$. Then $s = \alpha + x \cdot a$ and $t = \beta + y \cdot b$ where $\alpha, \beta \in \mathfrak{a}$ and $x, y \in A$. We compute

$$st = \alpha\beta + \alpha yb + \beta xa + xyab,$$

where $st \in S$ and $\alpha\beta, \alpha yb, \beta xa \in \mathfrak{a}$. If we assume that $ab \in \mathfrak{a}$, then $st \in S \cap \mathfrak{a}$, a contradiction. $\square$

Note 2.4.4. If $S \subset A$ is multiplicatively closed, then by Zorn's lemma there is some prime ideal $\mathfrak{b}$ such that $\mathfrak{b} = A \setminus S$.

Definition 2.4.5. Let A be a ring. We call A a *local ring* if any of the following equivalent conditions holds.

(a) A has a unique maximal ideal $\mathfrak{m}$.

(b) $A \setminus A^{\times}$ is an ideal.

(c) If $\mathfrak{m}$ is maximal and $x \in \mathfrak{m}$, then $1 + x \in A^{\times}$.

If A is a ring and $\mathfrak{p}$ a prime ideal, we shall denote the localization $(A \setminus \mathfrak{p})^{-1} A$ by $A_{\mathfrak{p}}$.

Proposition 2.4.6. *If $\mathfrak{p}$ is prime, then $A_{\mathfrak{p}}$ is a local ring with unique maximal ideal $\mathfrak{p}^e$.*

Proof. Let $S = A \setminus \mathfrak{p}$. Then $A_{\mathfrak{p}} = S^{-1}A$. Suppose that $I \trianglelefteq A_{\mathfrak{p}} = S^{-1}A$ such that $I \neq (1)$. But any ideal in $S^{-1}A$ is of the form $I = \mathfrak{a}^e$ for some ideal $\mathfrak{a}$ in A. Since $(1) \neq I = \mathfrak{a}^e$, it follows that $\mathfrak{a} \cap S = \emptyset$. Therefore, $\mathfrak{a} = A \setminus S = \mathfrak{p}$, so that $I = \mathfrak{a}^e \subset \mathfrak{p}^e$. Hence every nontrivial ideal in $A_{\mathfrak{p}}$ is contained in $\mathfrak{p}^e$, implying that $\mathfrak{p}^e$ is the unique maximal ideal. $\square$

Corollary 2.4.7. *In particular, the map*

$$(\textit{prime ideal of } A \mid A \subset \mathfrak{p}) \xrightarrow{(-)^e} \mathsf{Spec}(A_{\mathfrak{p}})$$

is a bijection that preserves inclusions of ideals.

Definition 2.4.8. Let A be a commutative ring, For every $\mathfrak{p} \trianglelefteq A$ prime, the *height of* $\mathfrak{p}$ is

$$\mathsf{ht}(\mathfrak{p}) \equiv \sup\{k \mid \mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \cdots \supsetneq \mathfrak{p}_k,\ \mathfrak{p}_i \trianglelefteq A\}.$$

Note that $\mathsf{ht}(\mathfrak{p}) = \mathsf{ht}(\mathfrak{p}^e \text{ in } A_{\mathfrak{p}})$.

Definition 2.4.9. The *Krull dimension* of A is

$$\dim A \equiv \sup\left\{\mathsf{ht}(\mathfrak{m}) \mid \text{maximal } \mathfrak{m} \ntrianglelefteq A\right\}.$$

Note that $\dim A_{\mathfrak{p}} = \mathsf{ht}(\mathfrak{p})$ and that $\dim A = \sup\left\{\dim A_{\mathfrak{m}} \mid \text{maximal } \mathfrak{m} \ntrianglelefteq A\right\}$.

Example 2.4.10.

1. If k is a field, then $\dim k = 0$. (The converse is also true.)

2. If A is a PID, then $\dim A = 1$. For example, $\mathbb{Z}$, $\mathbb{Q}[x]$, and $\mathbb{Z}[i]$ have dimension 1.

Exercise 2.4.11.

1. *Show that $\mathbb{Z}\left[-\sqrt{5}\right]$ is not a PID but has dimension* 1.

2. *Sow that* $\dim \mathbb{C}[x_1, \ldots, x_n] = n$.

3 Basic algebraic geometry

Any information about a commutative ring A, a prime ideal in A, a localization in A, and the relations between them can be packaged into a geometrical object, specifically, a topological space along with a distinguished class of maps.

Let $X = \mathsf{Spec}(A)$, the set of all prime ideals in A, or *spectrum of* A. For any $f \in A$, define the *principal open subset associated with* f as

$$X_f \equiv \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\} .$$

Such subsets satisfy

(a) $X_f \cap X_g = X_{fg}$.

(b) $X_{f^n} = X_f$, $X_f = X \iff f \notin \mathfrak{p}\ \forall \mathfrak{p}$ prime $\iff f \in A^\times$.

(c) $X_f = \emptyset \iff f \in \bigcap_{\mathfrak{p} \subset X} \mathfrak{p}$.

Definition 3.0.1. The minimal topology on X generated by $\{X_f\}_{f \in A}$ is called the *Zariski topology* on X.

The subset $U \subset X$ is open if and only if there is some $T \subset A$ such that $U = \bigcup_{f \in T} X_f$. Also, $Y \subset X$ is closed if $Y = \bigcap_{f \in T}(X \setminus X_f)$ for some $T \subset A$. Hence $Y \subset X$ is closed if there is some $T \subset A$ such that

$$Y = \{\mathfrak{p} \mid \mathfrak{p} \supset \langle T \rangle\} .$$

In particular, for any ideal $\mathfrak{a} \trianglelefteq A$, we can define a Zariski-closed subset $V(\mathfrak{a}) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supset \mathfrak{a}\}$. (Note that replacing $\mathfrak{a}$ with a set $S \subset A$ determines an equivalent topology.) Every closed subset is of this form.

Exercise 3.0.2. *Write arbitrary intersections of closed sets, finite unions of closed sets, X, and $\emptyset$ in this form.*

Any $f \in A$ can be viewed as a function on X in two ways. First, view f as a mapping $X \to \coprod_{\mathfrak{p} \in X} A_\mathfrak{p}$ given by $\mathfrak{p} \mapsto \frac{f}{1} \in A_\mathfrak{p}$. Then for any $\mathfrak{p}$, the value of f on $\mathfrak{p}$ is in $A_\mathfrak{p}$. Second, view f as a mapping $X \to \coprod_{\mathfrak{p} \in X} k_\mathfrak{p}$ given by

$$f \mapsto \frac{f}{1} + \mathfrak{p}^e \in k_\mathfrak{p} := {}^{A_\mathfrak{p}}\!/_{\mathfrak{p}^e}.$$

We call $k_\mathfrak{p}$ the *residue field of* $A_\mathfrak{p}$.

Example 3.0.3. Suppose that k is a field and $A = k[x_1, \ldots, x_n]$ such that for any $\mathfrak{m}$, $k_\mathfrak{m} = k$. Then $f \in A$ induces a function (prime ideals in A) $\to k$ given by $(x_1 - a_1, \ldots, x_n - a_n) \mapsto f(a_1, \ldots, a_n)$.

Lemma 3.0.4. *The space X is quasi-compact, which means that for any Zariski-open $U \subset X$, any open cover $\{U_\alpha\}$ of X admits a finite subcover of U.*

The space X, however, is *not* Hausdorff in general.

Exercise 3.0.5. *Let $A = \mathbb{C}[x]$ and $X = \mathsf{Spec}(A)$. Show that X is not Hausdorff.*

3.1 Lecture 8

Note 3.1.1.

1. We have that $V(S) = V(\mathfrak{a})$ whenever $\mathfrak{a} = \langle S \rangle$.

2. The Zariski topology is generated by the collection of principal open subsets on X, i.e., subsets of the form $X_f \coloneqq \{\mathfrak{b} \in X \mid f \notin \mathfrak{b}\}$ where $f \in A$. The elements in the ring A may be viewed as kinds of functions on X. View $f \in A$ as a function $X \to \coprod_{\mathfrak{b}\in X} A_{\mathfrak{b}} \to \coprod_{\mathfrak{b}\in X} {A+\mathfrak{b}}/{\mathfrak{b}^c}$ defined by $\mathfrak{b} \mapsto \frac{f}{1} \in A_{\mathfrak{b}}$.

If k is a field and $A = k[x_1, \ldots, x_n]$, then $V(a_1, \ldots, a_n) \in A^n$. We get a maximal ideal

$$\langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n \rangle .$$

Thus, if $f(x) \in A$ and we restrict this function, then we get the evaluation of f on points $a \in A^n$.

$$\begin{tikzcd} a \arrow[r, "\in"] & \langle x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n \rangle\, X \arrow[r] & \coprod_{\mathfrak{b}} k_{\mathfrak{b}} \\ & A^n \arrow[u, hook] \arrow[r] & \coprod_a k \cong A^n \times k \arrow[u, hook] \end{tikzcd}$$

This geometric way of packaging information about A and all of its prime ideals is compatible with all natural rings homomorphisms between the A_p's. If we have a principal open, then for every $f \in A$, we get a ring A_f, provided that f is not nilpotent, and a functor (poset of principal open sets in $X)^{\text{op}} \to$ **CommRing** given by $X_f \mapsto A_f$. This extends to another functor (opens in $X)^{\text{op}} \to$ **CommRing**. Given $f \in A$, viewing f as a kind of function on X thus induces a compatible system of elements of all rings $A(U)$ where $U \in X$ is open.

Lemma 3.1.2. *The space X is quasi-compact, which means that any open cover $\{U_\alpha\}$ of X admits a finite subcover.*

Proof. Let $X = \bigcup_\alpha U_\alpha$. The principal opens generate the Zariski topology, so that for any α, we can find a cover $U_\alpha = \bigcup_\beta X_{f_\alpha^\beta}$ where $f_\alpha^\beta \in A$. Then $X = \bigcup_{\alpha,\beta} X_{f_\alpha^\beta}$, so that

$$\emptyset = \bigcap_{\alpha,\beta} \underbrace{(X - X_{f_\alpha^\beta})}_{V(f_\alpha^\beta)}.$$

But $\emptyset = \bigcap_{\alpha,\beta} V(f_\alpha^\beta) = V(\{f_\alpha^\beta\}_{\alpha,\beta})$. Hence $\left\langle (f_\alpha^\beta)_{\alpha,\beta} \right\rangle$ is not contained in a ny prime ideal, so that $\left\langle (f_\alpha^\beta)_{\alpha,\beta} \right\rangle = A$, hence $1 \in \left\langle (f_\alpha^\beta)_{\alpha,\beta} \right\rangle$. We can find a collection of elements $\left\{ a_\alpha^\beta \right\}_{\alpha,\beta}$ where $a_\alpha^\beta \in A$ such that $1 = \sum_{\alpha,\beta} a_\alpha^\beta f_\alpha^\beta$ and at most finitely many a_α^β are nonzero. Thus, there is sequence $(\alpha_1, \beta_1), \ldots, (\alpha_{,k}, \beta_k)$ with

$$\left\langle f_{\alpha_1}^{\beta_1}, \ldots, f_{\alpha_k}^{\beta_k} \right\rangle = \langle 1 \rangle = A.$$

Hence $V(f_{\alpha_1}^{\beta_1}) \cap \cdots \cap V(f_{\alpha_k}^{\beta_k}) = \emptyset$, and $X = X_{f_{\alpha_1}^{\beta_1}} \cup \cdots \cup X_{f_{\alpha_k}^{\beta_k}}$. But $X_{f_{\alpha_i}^{\beta_i}} \in U_{\alpha_i}$ for each $i = 1, \ldots, k$. Therefore, $X = U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$. $\square$

Example 3.1.3. Suppose that X is a compact Hausdorff space. Let $A = C(X)$, the ring of complex-valued continuous functions on X. Consider $\mathsf{Spec}(A)$ equipped with the Zariski topology and its subset $\mathrm{Max}(A) := \{\mathfrak{a} \in C(X) \mid \mathfrak{a} \text{ maximal}\}$ equipped with the subspace topology.

Claim. *The natural map $X \xrightarrow{\varphi} \mathrm{Max}(A)$ given by $x \mapsto \{f \in C(X) \mid f(x) = 0\}$ is a homomorphism.*

Proof. Let $\mathfrak{a}_x = \ker(\mathrm{ev}_x : A \to \mathbb{C})$. By Urysohn's lemma, for any two distinct points $x, y \in X$, there is some $f \in A$ such that $f(x) = 0$ and $f(y) = 1$. But $f \in \mathfrak{a}_x$ and $f \notin \mathfrak{a}_y$, making $\mathfrak{a}_x \neq \mathfrak{a}_y$. Now, suppose $\mathfrak{a} \in \mathrm{Max}(A)$ and $\mathfrak{a} \neq \mathfrak{a}_x$ for any $x \in X$. This means that for any $x \in X$, there is some $f_x \in \mathfrak{a}$ such that $f_x(x) \neq 0$. Let $U_x \subset U$ be an open neighborhood of $x \in X$ such that $f_x \restriction_{U_x} \neq 0$. Then $X = \bigcup_{x \in X} U_x$, so that there is some finite subcover $U_{x_1}, \ldots, U_{x_k}$ of X. Let $f = \sum_{i=1}^k |f_{x_i}|^2$, which does not vanish at any point of X. Note that $f = \sum_{i=1}^k f_{x_i} \cdot \bar{f}_{x_i}$, so that $f \in \mathfrak{a}$. But f is nowhere vanishing, so that $\frac{1}{f}$ is a well defined continuous function on X. Thus, $\frac{1}{f} \in A$, and $1 \in \mathfrak{a}$, contrary to the fact that $\mathfrak{a}$ is maximal.

Exercise 3.1.4. *Check that φ is continuous, hence a homeomorphism.*

□

Let A be a commutative ring and M an A-module. Then M defines a subset of $X := \mathsf{Spec}(A)$, namely

$$\mathrm{supp}(M) \equiv \{\mathfrak{b} \in X \mid M_{\mathfrak{b}} \neq 0\},$$

called the *support of M*.

Proposition 3.1.5.

1. $\mathrm{supp}(M) \subset V(\mathrm{ann}(M))$ *where* $\mathrm{ann}(M) \equiv \{a \in A \mid a \cdot m = 0$ *for each* $m \in M\}$.

   Proof. Let $\mathfrak{b} \in \mathrm{supp}(M)$. Then $M_{\mathfrak{b}} \neq (0)$. We need to show that $\mathrm{ann}(M) \subset \mathfrak{b}$. Suppose that there is some $a \in \mathrm{ann}(M)$ with $a \notin \mathfrak{b}$. Let $x \in M_{\mathfrak{b}}$. Then $x = \frac{m}{s}$ where $m \in M$ and $s \notin \mathfrak{b}$. We compute $\frac{a}{1} \cdot \frac{m}{s} = \frac{am}{s} = 0$ in $M_{\mathfrak{b}}$. Since $a \notin \mathfrak{b}$, it follows that $\frac{a}{1}$ is invertible in $A_{\mathfrak{b}}$, i.e, $\frac{1}{a} \in A_{\mathfrak{b}}$. Hence $\frac{m}{s} = \frac{1}{a}\left(\frac{a}{1}\frac{m}{s}\right) = 0$ in $M_{\mathfrak{b}}$, so that $M_{\mathfrak{b}} = (0)$, a contradiction. □

2. *If M is finitely generated, then* $\mathrm{supp}(M) \supset V(\mathrm{ann}(M))$.

   Proof. Let $\mathfrak{b} \in V(\mathrm{ann}(M))$ and $\mathfrak{b} \supset \mathrm{ann}(M)$. We want to show that $M_{\mathfrak{b}} \neq (0)$. Suppose to the contrary. Then for any $m \in M$¡ we have that $\frac{m}{1} = 0$ in $M_{\mathfrak{b}}$. This shows that there exists $s \notin \mathfrak{b}$ such that $s \cdot m = 0$ in M. But M is finitely generated. Let $m_1, \ldots, m_k \in M$ be generators of $M \setminus A$. Then there are $s_1, \ldots, s_k \in A \setminus \mathfrak{b}$ such that $s_i m_i = 0$ in M for each i. Let $s = s_1 \cdots s_k \in A \setminus \mathfrak{b}$. Then for any $m \in M$, we have that $s \cdot m = 0$. Hence $s \in A \setminus \mathfrak{b}$, and $s \in \mathrm{ann}(M)$, a contradiction. □

3.2 Lecture 9

Proposition 3.2.1. $M = (0) \iff \mathrm{supp}(M) = \emptyset \iff \mathrm{supp}(M) \cap \mathrm{Max}(A) = \emptyset$.

Proof. It's clear that $M = \emptyset \implies \mathrm{supp}(M) = (0) \implies \mathrm{supp}(M) \cap \mathrm{Max}(A) = \emptyset$. Hence it suffices to show that

$$\mathrm{supp}(M) \cap \mathrm{Max}(A) = \emptyset \implies M = (0)\,.$$

On the one hand, if M is finitely generated, then $\mathrm{supp}(M) = V(\mathrm{ann}(M))$, so that $\mathrm{supp}(M)$ must contain any maximal idea that contains $\mathrm{ann}(M) \trianglelefteq A$. Thus, the assumption that $\mathrm{supp}(M) \cap \mathrm{Max}(A) = \emptyset$ implies that $\mathrm{ann}(A)$ is not contained in any maximal ideal, meaning that $\mathrm{ann}(M) = A$. This means that $M = (0)$.

On the other hand, if M is arbitrary, then $M = \mathrm{colim}_\alpha N_\alpha$ with each $N_\alpha \subset M$ finitely generated. But then $M_\mathfrak{a} = \mathrm{colim}_\alpha (N_\alpha)_\mathfrak{a}$ because localization is exact. Since each $N_\alpha = (0)$, it follows that $\mathrm{colim}_\alpha N_\alpha = 0$ as well. ◻

Corollary 3.2.2. *If we have a sequence of modules*

$$\eta : M \xrightarrow{f} T \xrightarrow{g} N,$$

then η is exact at $T \iff \eta_\mathfrak{p}$ is exact at $T_\mathfrak{p}$ for each $\mathfrak{p} \in \mathsf{Spec}(A) \iff \eta_\mathfrak{a}$ is exact at $T_\mathfrak{a}$ for each $\mathfrak{a} \in \mathrm{Max}(A)$¿

Proof. All of the forward directions are clear. Conversely, if $\eta_\mathfrak{a}$ is for every $\mathfrak{a}$, then $M_\mathfrak{a} \xrightarrow{f_\mathfrak{a}} T_\mathfrak{a} \xrightarrow{g_\mathfrak{a}} N_\mathfrak{a}$ is exact. If $H = {}^{\ker g}\!/_{\mathrm{im}\, f}$, then $H_\mathfrak{a} = {}^{\ker g_\mathfrak{a}}\!/_{\mathrm{im}\, f_\mathfrak{a}} = 0$. Thus, $\mathrm{supp}(H) \cap \mathrm{Max}(A) = \emptyset$, so that $H = 0$. ◻

Definition 3.2.3. Suppose that Π is a property of A-modules or of morphisms of A-modules. We say that Π *holds locally for A* if $\Pi_\mathfrak{a}$ holds for every $\mathfrak{a} \in \mathsf{Spec}(A)$.

Example 3.2.4.

1. $M = (0)$ holds locally if and only if it holds globally.
2. $M \to T \to N$ is exact locally if and only if it's exact globally.

Lemma 3.2.5. *TFAE.*

(a) M is flat over A.

(b) M is locally flat over A.

(c) $M_\mathfrak{a}$ is flat over $A_\mathfrak{a}$ for every $\mathfrak{a} \in \mathrm{Max}(A)$.

(d) $M_\mathfrak{a}$ is flat over A for every $\mathfrak{a} \in \mathrm{Max}(A)$.

Proof. The fact that (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) is obvious. To see that (c) $\implies$ (a), suppose that M is an A-module such that $M_{\mathfrak{a}}$ is flat as an $A_{\mathfrak{a}}$-module. Suppose that $0 \to X \to Y$ is an exact sequence of A-modules. Let $K = \ker(X \otimes_A M \to Y \otimes_A M)$. We want to show that $K = 0$.

Localizing $0 \to K \to X \otimes_A M \to Y \otimes_A M$ along $\mathfrak{a}$ gives an exact sequence $0 \to K_{\mathfrak{a}} \to X_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}} \to Y_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}}$, where we have used the fact that $(X \otimes_A M)_{\mathfrak{a}} = X \otimes_A M_{\mathfrak{a}} = X_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}}$. But $M_{\mathfrak{a}}$ is flat over $A_{\mathfrak{a}}$. Hence if we tensor the exact sequence $0 \to X_{\mathfrak{a}} \to Y_{\mathfrak{a}}$ with $M_{\mathfrak{a}}$ over $A_{\mathfrak{a}}$, then it will remain exact. This implies that $\ker(X_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}} \to Y_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}}) = 0$, so that $K_{\mathfrak{a}} = 0$ for each $\mathfrak{a}$. It follows that $\operatorname{supp}(K) = \emptyset$, which implies that $K = (0)$. $\square$

Definition 3.2.6. If A is commutative ring, then the *Jacobson radical of A* is the ideal

$$\operatorname{Jac}(A) \equiv \bigcap_{\mathfrak{a} \in \operatorname{Max}(A)} \mathfrak{a}.$$

Lemma 3.2.7 (Nakayama). *If A is a commutative ring and M is a finitely generated A-module with $\operatorname{Jac}(A) \cdot M = M$, then $M = (0)$.*

Proof. Let M be finitely generated over A. Choose some finite set of generators $m_1, \ldots, m_t$ of M of minimal cardinality. If $M \neq (0)$, then $t > 0$. Then $m_t \in M = \operatorname{Jac}(A) \cdot M$. Thus there are $a_1, \ldots, a_t \in \operatorname{Jac}(A)$ such that $m_t = \sum_{i=1}^{t} a_i m_i$. Then

$$(1 - a_t)\, m_t = \sum_{i=1}^{t-1} a_i m_i.$$

But $a_t \in \operatorname{Jac}(A)$, meaning that m_t belongs to every maximal ideal. Then $1 - a_t$ cannot be in any maximal ideal. Hence $1 - a_t$ is a unit in A. Let $u \in A$ such that $u(1 - a_t) = 1$. Then $m_t = \sum_{i=1}^{t-1} a_i u m_i$. This contradicts the fact that t is minimal. $\square$

Corollary 3.2.8 (Classical Nakayama). *Suppose A is a local ring with maximal ideal $\mathfrak{a}_A$. Let M be a finitely generated A-module such that $\mathfrak{a}_A M = M$. Then $M = (0)$.*

Proposition 3.2.9.

1. *If A is a commutative ring, then the functor $(-) \otimes {}^{A}\!/_{\operatorname{Jac}(A)} : A\text{–}\mathbf{Mod}^{\mathbf{fg}} \to {}^{A}\!/_{\operatorname{Jac}(A)} - \mathbf{Mod}^{\mathbf{fg}}$ is faithful.*

2. *If M is a finitely generated A-module and $m_1, \ldots, m_t \in M$ are such that their images $\bar{m}_1, \ldots, \bar{m}_t \in {}^{M}\!/_{\operatorname{Jac}(A) \cdot M}$ generate the module ${}^{M}\!/_{\operatorname{Jac}(A) \cdot M}$, then they generate M.*

Proof. If $N = \langle m_1, \ldots, m_t \rangle \subset M$, then $\overline{M/N} = (0)$ since $\overline{M/N} = \overline{M}/\overline{N}$. But then $M \setminus N = 0$ thanks to Lemma 3.2.7. $\square$

Proposition 3.2.10. *If A is a local ring and t is the minimal number of generators of a finitely generated A-module M, then every generating set for M contains a generating set of t elements.*

Proof. Let $m_1, \dots, m_k$ be a generating set for M. Then $\bar{m}_1, \dots, \bar{m}_k$ generate ${}^{M}\!/_{\mathfrak{a}_A M}$ = (finite dimensional vector space over $k_A = {}^{A}\!/_{\mathfrak{a}_A}$). This must have dimension t since every spacing subset in ${}^{M}\!/_{\mathfrak{a}_A M}$ lifts to a spanning subset of M. Choose a linearly independent subset in $\{\bar{m}_1, \dots, \bar{m}_k\}$ and lift this to M. □

Theorem 3.2.11. *Let A be a local ring and M an A-module. Assume that one of the following conditions holds.*

(a) A is Noetherian with M finitely generated.

(b) M is finitely presentable.

Then M is free $\iff$ M is projective $\iff$ M is flat.

Proof. We only need to show that if M is flat, then M is free. Suppose that M is flat and finitely presentable. We want to show that M is free. Let $0 \to K \to A^t \to M \to 0$ be a finite presentation where K is finitely generated. Since M being flat implies that $(-) \otimes k_A$ is exact, we have that

$$\eta : 0 \to K \otimes_A k_A \to k_A^t \to M \otimes_A k_A \to 0$$

is exact. Indeed, if $0 \to N' \to N \to N'' \to 0$ is a short exact sequence of A-modules and N^n is flat, then for every A-module, the sequence

$$0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$$

is exact. To see this, choose a presentation $0 \to K \to F \to M \to 0$, where F is free. Then we get a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & N'' \arrow[d] \\
 & K \otimes N' \arrow[r] \arrow[d, "\delta_1"] & K \otimes N \arrow[r] \arrow[d, "\delta_2"] & K \otimes N'' \arrow[d, "\delta_3"] \\
0 \arrow[r] & F \otimes N' \arrow[r] \arrow[d] & F \otimes N \arrow[r] \arrow[d] & F \otimes N'' \arrow[d] \\
 & M \otimes N' \arrow[r, "\theta"] \arrow[d] & M \otimes N \arrow[r] \arrow[d] & M \otimes N'' \arrow[d] \\
 & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd} .$$

Apply the snake lemma (Lemma 11.2.6 below) to the first two rows.

Returning to η, note that k_A^t and $M \otimes_A k_A$ are t-dimensional vector spaces over k_A. Hence $K \otimes_A k_A = 0$. But K is a finitely generated A module. Therefore, Lemma 3.2.7 implies that $K = 0$. □

4 Algebraic extensions

4.1 Lecture 10

Definition 4.1.1. Suppose that $A \subset B$ where A and B are commutative rings.

1. We say that $u \in B$ is *algebraic over* A if there is some $f(x) \in A[x]$ such that $f(u) = 0$ in B and $f \neq 0$. We say that u is *transcendental over* A if it is not algebraic over A.

2. In general, we say that a collection of elements $u_1, \ldots, u_k \in B$ are *algebraically independent over* A if there is some $f(x_1, \ldots, x_k) \in A[x_1, \ldots, x_k]$ such that $f \neq 0$ and $f(u_1, \ldots, u_k) = 0$ in B. We say that $u_1, \ldots, u_k \in B$ are *independent transcendentals over* A if they are not algebraically independent over A.

3. We say that $B \supset A$ is *algebraic* if each $u \in B$ is algebraic over A.

Our goal is to understand any algebraic extension of a ring. If A and B are domains, then we have a Cartesian diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & \mathrm{Frac}(A) \arrow[d, hook] \\ B \arrow[r, hook] & \mathrm{Frac}(B) \end{tikzcd}.$$

We have that B is an algebraic extension of A if and only if $\mathrm{Frac}(B)$ is an algebraic extension of $\mathrm{Frac}(A)$. This motivates the study of algebraic extensions of fields.

Definition 4.1.2. If $L \supset K$ is a field extension, we say that L is a *finite extension* if L is finite dimensional as a vector space over K. We call $[L : K] := \dim_K L$ the *degree of the extension*.

Remark 4.1.3. Finite field extensions arise naturally from polynomials.

Definition 4.1.4. If K is a field, then $f(x) \in K[x]$ is called *irreducible* if $\deg f > 0$ and f cannot be written as $f = gh$ where $g, h \in K[x]$ not units.

Theorem 4.1.5. *If $h(x) \in K[x]$ is irreducible, then the ring ${}^{K[x]}\!/_{(h)}$ is a field and the inclusion $K \subset {}^{K[x]}\!/_{(h)}$ is a finite field extension of degree* $\deg h$.

Proof. Recall that $K[x]$ is a Euclidean domain, in particular, a PID.

Lemma 4.1.6. *Let A be a PID and $u \in A$ be nonzero. TFAE.*

(a) ${}^{A}\!/_{(u)}$ is a field.

(b) (u) is prime.

(c) u is simple.

Proof. The fact that (b) and (c) are equivalent is obvious.

Suppose that u is not simple, so that $u = vw$ with $v, w \in A$ not units. Then in ${}^{A}\!/\!_{(u)}$ we have two elements $[v]$ and $[w]$ such that $[v] \cdot [w] = [u] = [0]$. But both $[v]$ and $[w]$ are nonzero since A has cancellations as a PID. Thus, ${}^{A}\!/\!_{(u)}$ is not a field.

Conversely, if $u \in A$ is simple, then for any $x \in A \setminus (u)$ we have that $(x, u) = (1)$ since x and u are coprime. This means that we can find $a, b \in A$ such that $ax + bu = 1$. Then $[x] \cdot [a] = [1]$. Hence $[x]$ is a unit, so that ${}^{A}\!/\!_{(u)}$ is a field. $\square$

From this our theorem follows immediately. $\square$

Note 4.1.7.

1. If $h(x) \in K[x]$ is irreducible and $L = {}^{K[x]}\!/\!_{(h)}$, then $h(x)$ has a natural root in L, namely, $t + (h)$. Moreover, every element in L can be written in the form $g(\alpha)$ for some $g(x) \in K[x]$.

2. If $B \supset A$ is a ring extension and $\alpha_1, \ldots, \alpha_k \in B$, we get an intermediate ring $A \subset A[\alpha_1, \ldots, \alpha_k] \subset B$ where $A[\alpha_1, \ldots, \alpha_k]$ is the image of the evaluation map $\mathtt{ev}_\alpha : f(x_1, \ldots, x_k) \mapsto f(\alpha_1, \ldots, \alpha_k)$. Thus, if K is a field and $h(x) \in K[x]$ is irreducible and $\alpha = t{+}{+}(h)$, then $L := {}^{K[x]}\!/\!_{(h)} = K[\alpha]$. Observe that α is algebraic over K, meaning that L is generated by a single algebraic element α.

Definition 4.1.8. We say that field extension $L \supset K$ is *simple* if it is isomorphic to ${}^{K[x]}\!/\!_{(h)}$ for some irreducible h.

Example 4.1.9.

1. $\mathbb{C} = \mathbb{R}[i] = {}^{\mathbb{R}[x]}\!/\!_{(x^2+1)}$.

2. If K is any field and $a \in K$ is not a square, then $x^2 - a$ is irreducible and we get a simple field extension $K\left[\sqrt{a}\right] := {}^{K[t]}\!/\!_{(t^2-a)}$.

Let $L \supset K$ be any field extension and $u \in L$ be algebraic over K. Consider

$$\operatorname{ann}(u) \equiv \{g(x) \in K[x] \mid g(u) = 0\},$$

which is an ideal in $K[x]$. Since $K[x]$ is a PID, we see that this ideal is generated by a single element $s(x)$. If we require that $s(x)$ be monic, then it is uniquely determined. We call this the *minimal polynomial of* u, denoted by $\min_u(x)$.

Lemma 4.1.10. *If $L \supset K$ is a field extension and $u \in L$ is algebraic over K, then $\min_u(x)$ is irreducible and $K[u]$ is isomorphic to the simple field extension ${}^{K[t]}\!/\!_{\min_u(x)}$.*

Proof. If $\min_u(x) = f(x)g(x)$, then $0 = \min_u(u) = f(u)g(u)$, so that either $f(u) = 0$ or $g(u) = 0$. But $f, g \mid \min_u$, so that $\deg f, \deg g \leq \deg \min_u$. By the minimality of $\min_u$, this implies that $\deg f = \deg \min_u$ or $\deg g = \deg \min_u$. Then either $\deg f = 0$ or $\deg g = 0$. $\square$

Theorem 4.1.11. *Let $L \supset K$ be a field extension and $u \in L$.*

(a) u is algebraic over K if and only if $K[u]$ is a finite dimensional vector space over K.

(b) If u is algebraic, then $[K(u) : K] = \deg \min_u$.

Proof. We have proven (b) in Lemma 4.1.10. For (a), suppose that the ring $K[u]$ is finite dimensional as a vector space over K. Then there exist nonnegative integers $k_1, \dots, k_s$ such that $k(u) = \operatorname{span}_K(u^{k_1}, \dots, u^{k_2})$. Thus, if $m > \max(k_1, \dots, k_s)$, then u^m is a K-linear combination of $u^{k_1}, \dots, u^{k_s}$. Write $u^m = a_1 u^{k_1} + \cdots + a_s u^{k_s}$. Then $f(x) = x^m - \sum_{i=1}^{s} a_i x^{k_i}$ satisfies $f(u) = 0$. Conversely, if $u \in L$ is algebraic over K, then there is some $n > 0$ such that $u^n = \operatorname{span}_K(1, u, \dots, u^{n-1})$. Then $u^m \in \operatorname{span}_K(1, u, \dots, u^{n-1})$ for any m. This implies that $K[u]$ is finite dimensional over$K$. $\square$

Corollary 4.1.12. *If $L \supset K$ is a finite field extension, then L is algebraic over K.*

4.2 Lecture 11

Definition 4.2.1. A finite field extension of $\mathbb{Q}$ is called a *number field.*

Fix a prime $p > 0$. Let $\epsilon_p = e^{\frac{2\pi i}{p}} \in \mathbb{C}$. Then $\mathbb{Q}(\epsilon_p) \supset \mathbb{Q}$ is a finite extension because ϵ_p is annihilated by the polynomial $x^p - 1$. It is called the *p-th cyclotomic field.* Note that $x^p - 1$ is not minimal since we can factor out $(x-1)$. We claim that $\frac{x^p-1}{x-1}$ is the minimal polynomial, so that $[\mathbb{Q}(\epsilon_p) : \mathbb{Q}] = p - 1$. This will hold if we can prove that $\frac{x^p-1}{x-1}$ is irreducible in $\mathbb{Q}[x]$.

Lemma 4.2.2 (Gauss). *If $f(x) \in \mathbb{Z}[x]$ is irreducible, then it is irreducible in $\mathbb{Q}[x]$.*

Proof. Note that if $p(x) \in \mathbb{Q}[x]$, then there exists $N \in \mathbb{Z}_{>0}$ such that $Np(x) \in \mathbb{Z}[x]$ and the coefficients of Np are pairwise coprime. Suppose that $f(x) \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$. Suppose, towards a contradiction, that there are $g(x), h(x) \in \mathbb{Q}[x]$ non-units such that $f(x) = g(x)h(x)$. Then $g(x)$ and $h(x)$ are $\mathbb{Q}$-proportional to some $\tilde{g}(x)$ and $\tilde{h}(x)$, respectively, over $\mathbb{Z}$ with each having pairwise coprime coefficients. Thus, $f(x) = \lambda \tilde{g}(x)\tilde{h}(x)$ for some $\lambda \in \mathbb{Q}^\times$. Let $\lambda = \frac{a}{b}$ with $(a, b) = 1$. If $b \neq \pm 1$, then there is some $p > 0$ where $p \mid b$ and $pf = a\tilde{g}\tilde{h}$. We have that $bf, a\tilde{g}, \tilde{h} \in \mathbb{Z}[x]$. We can reduce $\mod p$ to get $[bf]_p = [a]_p\,[\tilde{g}]_p[\tilde{h}]_p$ But $f(x) \in \mathbb{Z}[x]$, so that $[bf]_p = [b]_p\,[f]_p = 0$.Hence $[a]_p\,[\tilde{g}]_p[\tilde{h}]_p = 0$ in $(\mathbb{Z}/p)\,[x]$, so that one of $[a]_p$, $[\tilde{g}]_p$, and $\left[\tilde{h}_p\right]$ must be 0. But $(a, b) = 1$, so that $[a]_p \neq 0$. Since each of $\tilde{g}$ and $\tilde{h}$ has coprime coefficients, we have that $[\tilde{g}]_p \neq 0$ and $\left[\tilde{h}\right]_p \neq 0$, a contradiction. $\square$

Thus, it suffices to show that $\frac{x^p-1}{x-1}$ is irreducible in $\mathbb{Z}[x]$. Let $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$. Then $f(x)\,(x-1) = x^p - 1$. By the binomial formula, we see that $[(x-1)^p]_p = [x^p - 1]_p$. Thus, $[f]_p[x-1]_p = \left[(x-1)^p\right]_p$, so that $[f]_p[(x-1)]_p = \left([x-1]_p\right)^p$ and $[f]_p = [(x-1)]^{p-1}$ If $f = gh$ for some non-units g and h, then $[g]_p[h]_p = ([(x-1)]_p)^{p-1}$, which implies that $[g]_p = [(x-1)^r]_p$ and $[h]_p = [(x-1)^s]_p$ for some r and s. Thus, $[g(1)]_p = [g]_p(1) = 0 = [h]_p(1) = [h(1)]_p$, meaning that $p \mid g(1)$ and $p \mid h(1)$. Since $f = gh$, it follows that $p^2 \mid f(1) = p$, a contradiction.

Theorem 4.2.3. *Suppose that $M \supset L \supset K$ is a chain of finite field extensions. Then $M \supset K$ is also finite with $[M : K] = [M : L]\,[L : K]$.*

Proof. Let $e_1, \dots, e_n$ be a basis of L over K and $f_1, \dots, f_m$ be a basis of M over K. Then $\{e_i \cdot f_j\}_{i,j}$ forms a basis of M over K. ☐

Note 4.2.4. Suppose that $L \supset K$ is a field extension with $u_1, \dots, u_n \in L$. We get a ring $K[u_1, \dots, u_n] = \operatorname{im} \mathsf{ev}_u$, which is a domain since it's contained in L. Let $K(u_1, \dots, u_n) \coloneqq \operatorname{Frac}(K[u_1, \dots, u_n])$. Then we fave that $K \subset K[u_1, \dots, u_n] \subset K(u_1, \dots, u_n) \subset L$. Note that if $u \in L$ is algebraic over K, then $K \subset K[u] = K(u) \subset L$.

Theorem 4.2.5. *Suppose that $L \supset K$ is a field extension and let $u_1, \dots, u_n \in L$ be algebraic over K. Then $\dim_K K(u_1, \dots, u_n) < \infty$. In particular, $K(u_1, \dots, u_n) \supset K$ is an algebraic extension.*

Proof. Note that

$$\begin{aligned} &K \subset K(u_1) \subset K(u_1, u_2) \subset \cdots \subset K(u_1, \dots, u_n) \\ &K(u_1, \dots, u_k) = K(u_1, \dots, u_{k-1})(u_k). \end{aligned}$$

Since each u_k is algebraic over K, we see that u_k is algebraic over any field containing K. Thus, u_k is algebraic over $K(u_1, \dots, u_{k-1})$. Hence $\dim_{K(u_1,\dots,u_{k-1})} K(u_1, \dots, u_k) < \infty$. By Theorem 4.2.3, $\dim_K K(u_1, \dots, u_n) < \infty$. ☐

Definition 4.2.6.

1. A field K is *algebraically closed* if for every $L \supset K$ and every $u \in L$ algebraic over K, we have that $u \in K$.

2. We say that $K \subset L$ is *algebraically closed in* L if any $u \in L$ that is algebraic over K belongs to K.

Theorem 4.2.7. *If $L \supset K$ is a field extension, then*

$$\overline{K} \coloneqq \{u \in L \mid u \textit{ is algebraic over } K\}$$

is a field that is algebraically closed in L.

Proof. Let $u, v \in \overline{K}$. Then both are algebraic over K. If $K \subset K(u, v) \subset L$, then Theorem 4.2.5 shows that $K(u, v) \supset K$ is an algebraic extension. Since $K(u, v) \subset \overline{K}$, it follows that $\overline{K}$ is a field. ☐

Suppose $u \in L$ is algebraic over $\overline{K}$. Then we can find $f(x) = \sum_{i=1}^n a_i x^i \in \overline{K}[x]$ such that $\deg f > 0$ and $f(u) = 0$. Hence $f \in K(a_1, \dots, a_n)[x]$, so that u is algebraic over $K(a_1, \dots, a_n)$. Hence $K(a_1, \dots, a_n, u)$ is finite dimensional over $K(a_1, \dots, a_n)$. But $a_1, \dots, a_n \in K$ are algebraic over K, so that $K(a_1, \dots, a_n)$ is algebraic over K. This means that u is algebraic over K.

Let h is an irreducible polynomial over K. Write $\widetilde{K} = {}^{K[x]}\!\big/_{(h)}$ and let α denote the marked root $x + (h)$ of h viewed as a polynomial in $\widetilde{K}[t]$.

Lemma 4.2.8 (Main lemma of Galois theory). *For any $\varphi : K \to F$ field homomorphism, the natural map*
$$\left\{\psi : \widetilde{K} \to F \mid \psi\restriction_K = \varphi\right\} \to (\textit{distinct roots of } h^\varphi \in F[x])$$
given by $\psi \mapsto \psi(\alpha)$ is a bijection, where h^φ denotes the polynomial obtained by applying φ to the coefficients of h.

Proof. Let $\psi : \widetilde{K} \to F$ be a homomorphism with $\psi\restriction_K = \varphi$. Then
$$\begin{aligned}h^\varphi(\psi(\alpha)) &= \varphi(a_n)\psi(\alpha)^n + \varphi(a_{n-1})\psi(\alpha)^{n-1} + \cdots + \varphi(a_1)\psi(\alpha) + \varphi(a_0)\\ &= \psi(a_n)\psi(\alpha)^n + \psi(a_{n-1})\psi(\alpha)^{n-1} + \cdots + \psi(a_1)\psi(\alpha) + \psi(a_0) = \psi(h(\alpha))\\ &= 0.\end{aligned}$$

Now, let $\xi \in F$ be a root of h^φ. Define a homomorphism $K[x] \to F$ by $f(x) \mapsto f^\varphi(\xi)$. Then $h(x) \mapsto h^\varphi(\xi) = 0$. Thus, this homomorphism descends to a homomorphism $\psi : K[x]/(h) \to F$ such that $\psi(\alpha) = \xi$. This implies that the assignment $\psi \mapsto \psi(\alpha)$ is surjective.

Finally, suppose that $\tilde{\varphi} : \widetilde{K} \to F$ is any homormophism such that $\tilde{\varphi}\restriction_K = \varphi$. Then $\tilde{\varphi}(\alpha)$ is a root of h^φ. Let $\psi_{\tilde{\varphi}(\alpha)} : \widetilde{K} \to F$ be the extension that we constructed. Then $\tilde{\varphi}\restriction_K = \varphi$, and $\psi_{\tilde{\varphi}(\alpha)\restriction_K} = \varphi$. Also, we have that $\tilde{\varphi}(\alpha) = \xi$ and $\psi_{\tilde{\varphi}(\alpha)}(\alpha) = \xi$. This shows that $\tilde{\varphi}\restriction_{K(\alpha)} = \psi_{\tilde{\varphi}(\alpha)}\restriction_{K(\alpha)}$. But $K(\alpha) = \widetilde{K}$. ☐

5 Splitting fields

5.1 Lecture 12

Definition 5.1.1. If K is a field and $f(x) \in K[x]$, then a field extension $L \supset K$ is a *splitting field for* f if

(a) $f(x) = a\prod_{i=1}^n (x - c_i)$ with $a, c_i \in L$ and

(b) $L = K(c_1, \ldots, c_n)$.

Theorem 5.1.2. *For every $f(x) \in K[x]$, a splitting field for f exists and is unique up to an isomorphism over K.*

Proof. Consider the tower of fields $K = K_0 \subset K_1 \subset K_2 \subset \cdots$ where $K_i = K_{i-1}[\alpha_i]$ and α_i is a root of an irreducible factor f_i of f over K_{i-1} with $\deg f_i > 0$. The degree of f is fixed, but the number of irreducible factors of f strictly increases after each step. Hence this sequence of fields will stabilize at some K_s, which is thus a splitting field for f.

To prove uniqueness, suppose that $L \supset K$ is another splitting field for f. We have $\varphi_0 : K_0 = K \hookrightarrow L$. By Lemma 4.2.8, we can extend φ_0 to a homomorphism $\varphi_1 : K_1 \to L$ provided that $f_1^{\varphi_0}$ has a root in L. But by assumption, f^{φ_0} has each of its roots in L. Since $f_1 \mid f$, it follows that $f_1^{\varphi_0}$ has each of its roots in L as well. This implies that $\varphi_1 : K_1 \to L$ will extend to a map provided that $\varphi_2 : K_2 \to L$ $f_2^{\varphi_1} = f_2^{\varphi_0}$ has some root in L. But this holds since $f_2 \mid f$. Continuing in this way, we get $\varphi_s : K_s \to L$ such that $f_1^{\varphi_{s-1}}$ has all of its roots in L. Thus, $f^{\varphi_s} = f^{\varphi_0}$ has all of its roots in L. But $\varphi_s \restriction_K = \varphi_0$, so that φ_s is injective. But $L = K(\text{all roots of f})$. By construction, all roots of f belong to $\operatorname{im}\varphi_s$. Also, $K \subset \operatorname{im}\varphi_s$. Hence φ_s is surjective and thus an isomorphism. $\square$

Exercise 5.1.3. *Describe all splitting fields of polynomials of degree* 2.

Example 5.1.4. Suppose that K is a field of characteristic $\neq 2$. Let $f(x) = x^3+a_2x^2+a_1x+a_0 \in K[x]$. Let L be a splitting field for f. What can L be? This depends on the splitting behavior of f over K

(A) Suppose that f has all of its roots in K. Then $L = K$, and $[L : K] = 1$.

(B) Suppose that f has exactly one root in K Then $f(x) = (x - \alpha)\, g(x)$ with $\alpha \in K$ and $g(x)$ a quadratic irreducible in $K[x]$. Consider $L = {K[x]}/{(g)}$. Then $[L : K] = 2$, and g has a root in L. This implies that g has all of its roots in L. Hence L is the splitting field for f.

(C) Suppose that f has no roots in K. Then f is irreducible in $K[x]$. Let $K_1 = {K[x]}/{(f)}$, which is a simple extension of degree 3. Note that f has a root α_1 in K_1. Thus, $K_1 = K[\alpha_1]$. Consider $f(x) = (x - \alpha_1)\, g(x)$ with $g \in K_1[x]$ and $\deg g = 2$. There are two sub-cases to consider.

   (a) Suppose that g has two roots in K_1. Then $L = K_1$, so that $[L : K] = 3$.

   (b) Suppose that f is irreducible in K_1. Then $L = K_2 = {K_1[x]}/{(g)}$, so that $[L : K] = 6$.

We conclude that if L is the splitting field for f, then $[L : K] \in \{1, 2, 3, 5\}$.

How can we compute $[L : K]$ from the coefficients of f? We have that $f(x) = (x - \alpha_1)\,(x - \alpha_2)\,(x - \alpha_3)$ in $L[x]$. Look at $\operatorname{Discr}(f) \coloneqq (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2 \in L$. This is a symmetric function in $\alpha_1, \alpha_2, \alpha_3$. Hence it is expressible in terms of a_2, a_1, a_0. Note that

$$\operatorname{Discr}(f) = a_1^2a_2^2 - 4a_2^2a_0 - 4a_1^3 + 18a_0a_1a_2 - 27a_0^2.$$

Proposition 5.1.5. *Suppose that f has no roots in K. Then $[L : K] = 3 \iff \operatorname{Discr} f \in K^2$.*

Proof. We know that f is irreducible over K. Hence $K_1 = {K[x]}/{(f)}$ is an extension of degree 3 in which f has a root α_1. Note that $\operatorname{Discr}(f) \notin K^2 \iff \operatorname{Discr}(f) \notin K_1^2$. The ($\Longleftarrow$) direction is obvious. For the reverse direction, suppose, towards a contradiction, that $\operatorname{Discr}(f) \notin K^2$ but $\operatorname{Discr}(f) \in K_1^2$. This means that $\left[K\left[\sqrt{\operatorname{Discr}(f)}\right] : K\right] = 2$ and $K \subset K\left[\sqrt{\operatorname{Discr}(f)}\right] \subset K_1$. Thus, $3 = [K_1 : K] = \left[K\left[\sqrt{\operatorname{Discr}(f)}\right] : K\right] \cdot \left[K_1 : K\left[\sqrt{\operatorname{Discr}(f)}\right]\right] = 2 \cdot 1$, a contradiction.

Now, $\operatorname{Discr}(f) \in K_1^2 \iff (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \in K_1$. This implies that $\alpha_2 - \alpha_3 \in K_1$. Indeed, $f \in K_1[x]$ satisfies $f(x) = (x - \alpha_1)\, g(x)$, and $\alpha_2, \alpha_3 \in L$ are roots of g. Therefore, we have in L that $g(\alpha_1) = (\alpha_1 - \alpha_2)\,(\alpha_1 - \alpha_3) \in L$. But $g \in K_1[x]$ and $\alpha_1 \in K_1$, so that $g(\alpha_1) \in K_1$. It follows that $\alpha_2 - \alpha_3 \in K_1$. Hence $\operatorname{Discr}(f) \in K_1^2 \iff \alpha_2 - \alpha_1 \in K_1$. But $-\alpha_2 - \alpha_3$ is a coefficient of g in K_1. Therefore, $\alpha_2, \alpha_3 \in K_1$. $\square$

Note 5.1.6.

1. Suppose that K is a finite field. Then $\mathsf{char}\, K > 0$.

2. Suppose that K is any field such that $\mathsf{char}\, K = p > 0$. Then the natural map $\varphi : K \to K$ given by $x \mapsto x^p$ respects addition due to the binomial theorem. Hence it is a field homomorphism, called the *Frobenius morphism*. If K is finite, then this map is an automorphism. In general, $\operatorname{im} \varphi = K^p \subset K$ is a subfield.

3. If K has characteristic p, then the natural map $\mathbb{F}_p \to K$ given by $[n] \mapsto \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}$ is a field extension. Therefore, if K is finite, then $K \supset \mathbb{F}_p$ is a finite field extension. In this case, if K has degree n, then $K \cong \mathbb{F}_p^{\oplus n}$ is a vector space over $\mathbb{F}_p$. Hence $|K| = |\mathbb{F}_p|^n = p^n$.

6 Finite fields

Theorem 6.0.1. *For every prime p and integer $n > 0$, there is some finite field K consisting of p^n elements that is unique up to an isomorphism over $\mathbb{F}_p$.*

Proof. We first prove uniqueness. If F is a finite field with $q := p^n$ elements, then $|F^\times| = q - 1$. It follows that for any $a \in F^\times$, $a^{q-1} = 1$. But then for any $a \in F$, $a^q = a$, so that each element of F is a root of $x^q - x \in \mathbb{F}_p[x]$. Then $\prod_{a \in F}(x - a) \mid x^q - x$ in $F[x]$. This implies that $x^q - x = \prod_{a \in F}(x - a)$ in $F[x]$. This means that F is a splitting field for $x^q - x$ over $\mathbb{F}_p$, which must be unique up to isomorphism.

To prove existence, consider F the splitting field for $x^q - x$ over $\mathbb{F}_p$. We want to show that $|F| = q$.

Note 6.0.2. If A is any commutative ring, then $A[x]$ has a natural derivation. There exists a unique map $\frac{d}{dx} : A[x] \to A[x]$ such that $\frac{d}{dx}(a) = 0$ for any $a \in A$, $\frac{d}{dx}(x) = 1$, and $\frac{d}{dx}$ satisfies the Leibniz rule, i.e., $\frac{d}{dx}(fg) = \frac{df}{dx} g + f \frac{dg}{dx}$. Note that $\frac{d}{dx}$ is given by $\frac{d}{dx}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0) = n a_n + (n-1)a^{n-1}x^{n-2} + \cdots + 0$. Then $\frac{d}{dx}$ is an A-module homomorphism. If $A \subset B$ is a subring, then we get compatible derivations $\frac{d}{dx} \hookrightarrow A[x] \subset B[x] \hookleftarrow \frac{d}{dx}$.

Returning to our proof, consider $f(x) = x^q - x$. Then since $F \supset \mathbb{F}_p$ is the splitting field for $f(x)$, it follows that $f(x) = \prod_{i=1}^q (x - c_i)$ where $c_i \in F$. How many distinct roots does $f(x)$ have in F? If $f(x)$ has a repeated root, then we can write $f(x) = (x - c)^2 g(x)$ in $F[x]$. This implies that $\frac{df}{dx}(x) = 2(x - c)g + (x - c)^2 \frac{dg}{dx}$ will also have c has a root. But $\frac{df}{dx} = qx^{q-1} - 1 = -1$ in $\mathbb{F}_p[x] \subset F[x]$. But in this case $\frac{df}{dx}$ has no roots. Thus, $f(x)$ has no repeated roots in F, so that $|F| \geq q$.

Now consider $R_f := \{c \in F \mid f(c) = 0\}$. Note that $\mathbb{F}_p \subset R_f \subset F$ and that $R_f = \{c \in F \mid \varphi^n(c) = c\}$ where φ denotes the Frobenius map. But since φ is a field automorphism of F, so is φ^n. Hence the fixed points of φ^n form a subfield. This means that R_f is a subfield, hence a splitting field for f. Thus, $R_f \cong F$. $\square$

6.1 Lecture 13

We write $\mathbb{F}_q$ for the splitting field for $x^q - x \in \mathbb{F}_p[x]$.

Proposition 6.1.1. *The group $\mathbb{F}_q^\times$ is a cyclic group of order $q-1$.*

Proof. By the structure theorem for finite abelian groups, we get

$$\begin{aligned}\mathbb{F}_q^\times \cong \mathbb{Z}/p_1^{m_{11}} \times \mathbb{Z}/p_1^{m_{12}} \times \cdots \times \mathbb{Z}/p_1^{m_{1k}} \times \mathbb{Z}/p_2^{m_{21}} \times \mathbb{Z}/p_2^{m_{22}} \times \cdots \\ \times \mathbb{Z}/p_2^{m_{2k}} \times \cdots \times \mathbb{Z}/p_s^{m_{s1}} \times \mathbb{Z}/p_s^{m_{s2}} \times \cdots \times \mathbb{Z}/p_s^{m_{sk}}.\end{aligned}$$

Let $\alpha_i = p_1^{m_{1i}} p_2^{m_{2i}} \cdots p_s^{m_{si}}$ for each $i = 1, \ldots, k$. Hence $\left|\mathbb{F}_q^\times\right| \mid d_1 d_2 \cdots d_k$ where $d_1 \mid d_2 \mid \cdots \mid d_k$. Hence every element in $\mathbb{F}_q^\times$ has order dividing d_k. For any $a \in \mathbb{F}_q$, $a^{d_k+1} = a$, so that $|\mathbb{F}_q| = \deg x^{d_k+1} - x = d_k + 1$. Then $q \leq d_k + 1$, so that $q - 1 \leq d_k$. Since $d_k \mid q-1$, we have that $d_k = q-1$, and thus $d_1 = d_2 = \cdots = d_{k-1} = 1$. Hence

$$\mathbb{F}_q^\times \cong \mathbb{Z}/p_1^{m_{1k}} \times \mathbb{Z}/p_2^{m_{2k}} \times \cdots \times \mathbb{Z}/p_s^{m_{sk}}.$$

Since the $p_i^{m_{ik}}$ are pairwise coprime, it follows that $\mathbb{F}_q^\times \cong \mathbb{Z}/p_1^{m_{1k}} p_2^{m_{2k}} \cdots p_s^{m_{sk}}$. ☐

Corollary 6.1.2. $\mathbb{F}_q = \mathbb{F}_p(\sigma)$.

Proof. Since $\mathbb{F}_q^\times$ is cyclic , we know that $\mathbb{F}_q^\times = \langle \sigma \rangle$. ☐

Proposition 6.1.3. $\operatorname{Aut}(\mathbb{F}_q)$ *is a cyclic group of order n. In fact,* $\operatorname{Aut}(\mathbb{F}_q) \cong \langle \varphi \rangle$ *where φ denotes the Frobenius map.*

Proof. We have arranged it so that $\mathbb{F}_q$ is unique up to isomorphism over $\mathbb{F}_p$, so that each $\psi \in \operatorname{Aut}(\mathbb{F}_q)$ restricts to the identity on $\mathbb{F}_p \subset \mathbb{F}_q$. This implies that $\operatorname{Aut}(\mathbb{F}_q) \cong \operatorname{Aut}(\mathbb{F}_q/\mathbb{F}_p)$, which is the subgroup of all $\psi : \mathbb{F}_q \xrightarrow{\cong} \mathbb{F}_q$ such that $\psi \restriction_{\mathbb{F}_p} = \mathrm{id}_{\mathbb{F}_p}$.

Lemma 6.1.4. *Let $L \supset K$ be a finite field extension of degree n. Then we have that $|\operatorname{Aut}(L/K)| \leq n$.*

Proof. Since $[L : K] = n < \infty$, we can construct L as a tower $K = K_0 \subset K_1 \subset \cdots \subset K_s = L$ where $K_{i+1} = K_i[\alpha_i]$ and α_i is a root of an irreducible $f_i(x) \in K_i[x]$. Consider $\varphi_0 : K \hookleftarrow L$ the natural inclusion. Applying Lemma 4.2.8, we see that φ_0 extends to $\varphi_1 : K_1 \to L$ in finitely many ways such that the number of such φ_1's equals the number of distinct roots of $f_0^{\varphi_0}$ in L. This quantity is $\leq \deg f_0 = [K_1 : K_0]$ Each φ_1 extends to a map $\varphi_2 : K_2 \to L$ in at most $\deg f_1 = [K_2 : K_1]$ ways. Therefore, φ_0 will extend to a map $\varphi_s : L \to L$ in $[K_1 : K_0]\,[K_2 : K_1] \cdots [K_s : K_{s-1}]$ many ways. It follows that

$$|\operatorname{Aut}(L/K)| \leq \prod_{i=0}^{s-1} [K_{i+1} : K_i] = [L : K] = n.$$

☐

Corollary 6.1.5. *If $f(x) \in K[x]$ and L is a splitting field for f and f has distinct roots in L, then* $|\operatorname{Aut}(L/K)| = [L : K]$.

We have that $|\operatorname{Aut}(\mathbb{F}_q/\mathbb{F}_p)| = [\mathbb{F}_q : \mathbb{F}_p] = n$. If $\varphi \in \operatorname{Aut}(\mathbb{F}_q/\mathbb{F}_p)$, then $\varphi^n = \mathrm{id}$. Thus, it suffices to show that $\varphi^m \neq \mathrm{id}$ for any $m < n$. Suppose that m has $\varphi^m = \mathrm{id}$. Then $\varphi^m(a) = a$ for every $a \in \mathbb{F}_q$. Therefore, $q^{p^m} = a$ for each $a \in \mathbb{F}_q$, so that $p^n = q = |\mathbb{F}_q| \leq p^m$. Then $m \geq n$.

This completes our main proof. ☐

Proposition 6.1.6. *There is a bijection* (*subfields of* $\mathbb{F}_q$) $\cong_{\mathbf{Set}}$ (*subgroups of* $\operatorname{Aut}(\mathbb{F}_q)$).

Proof. Let $F \subset \mathbb{F}_q$ be a subfield, so that $\mathbb{F}_p \subset F \subset \mathbb{F}_q$. We have that $|F| \mid |\mathbb{F}_q| = p^n$, so that $|F| \leq p^d$ for some $d \leq n$. Since $\mathbb{F}_q \supset F$, we have that $\mathbb{F}_q$ is a vector space over F. If $[\mathbb{F}_q : F] = r$, then $\mathbb{F}_q \cong F^{\oplus r}$ as F-vector spaces. Note that

$$p^n = |\mathbb{F}_q| = |F|^r = \left(p^d\right)^r = p^{dr},$$

which implies that $d \mid n$.

Since F is a finite field, it follows that $F^\times$ is cyclic of order $p^d - 1$. Hence any $a \in F \subset \mathbb{F}_q$ satisfies $a^{p^d} = a$. But if $d \mid n$, then $x^q - x = x^{p^n} - x = \left(x^{p^d} - x\right) g(x)$ because $p^n - 1 = p^{dr} - 1 = \left(p^d\right)^r - 1 = \left(p^d - 1\right) m$ so that $x^{q-1} - 1 = \left(x^{p^d-1} - 1\right) g(x)$. But $\mathbb{F}_q$ is the splitting field for $x^q - x$, and all roots of this are distinct. Thus, there are exactly p^d roots of $x^q - x$ that are the distinct roots of $x^{p^d} - x$. Therefore,

$$F = \mathbb{F}_{p^d} = \left(\text{subfield of } \mathbb{F}_q \text{ that is the splitting field for } x^{p^d} - x\right) = \left(\text{fixed subfield of } \varphi^d\right).$$

Hence F is the fixed point subgroup of $\left\langle \varphi^d \right\rangle \trianglelefteq \operatorname{Aut}(\mathbb{F}_q)$.

Let $\psi \in \operatorname{Aut}(\mathbb{F}_q)$ with $\psi \notin \left\langle \varphi^d \right\rangle$. Then $\psi = \varphi^e$ for some $e \geq 0$ such that $d \nmid e$. If ξ generates $F^\times$ and $\xi^{p^e} = \psi(\xi) = \xi$, then $p^d - 1 \mid p^e - 1$ since $|F^\times| = p^d - 1$. But this is impossible, which implies that $\psi \restriction_F \neq \mathrm{id}_F$. Therefore, $\left\langle \varphi^d \right\rangle = \operatorname{Aut}(\mathbb{F}_q/F)$, and we have a bijection

$$\begin{aligned} (\text{subfields of } \mathbb{F}_q) &\cong_{\mathbf{Set}} (\text{subgroups of } \operatorname{Aut}(\mathbb{F}_q)) \\ F &\mapsto \operatorname{Aut}(\mathbb{F}_q/F) \\ \mathbb{F}_q^G &\mapsfrom G. \end{aligned}$$

□

6.2 Lecture 14

Proposition 6.2.1.

1. *Let $\mathbb{F}_q^\times = \langle \theta \rangle$. Then $\mathbb{F}_q = \mathbb{F}_p(\theta)$, meaning that θ is a primitive element for the extension $\mathbb{F}_q \supset \mathbb{F}_p$. Further, if h denotes the minimal polynomial of θ over $\mathbb{F}_p$, then $\mathbb{F}_q$ is the splitting field for h.*

   Proof. Every nonzero element of $\mathbb{F}_q$ is a power of θ. Hence $\mathbb{F}_q = \mathbb{F}_p(\theta)$. Now, note that $\deg h = n$ because $[\mathbb{F}_q : \mathbb{F}_p] = n$. Write $h(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where each $a_i \in \mathbb{F}_p$. If we view h over $\mathbb{F}_q$, then $\varphi(a_i) = a_i$ due to Fermat's little theorem. Hence $\varphi(h(x)) = h(\varphi(x))$ for any $x \in \mathbb{F}_q$, meaning that $\varphi(c)$ is a root of h whenever c is a root. Thus, we get n roots of h.

   $$\theta,\ \theta^p,\ \theta^{p^2}, \ldots, \theta^{p^{n-1}}$$

   If K is the splitting field for h, then $\mathbb{F}_p \subset K \subset \mathbb{F}_q$. But $[K : \mathbb{F}_p] = n = [\mathbb{F}_q : \mathbb{F}_p]$, so that $K = \mathbb{F}_q$. □

2. *Let $m \geq 0$ be any integer and $q = p^n$. Then there is some irreducible polynomial over $\mathbb{F}_q$ of degree m*

Proof. Let $\mathbb{F}_{p^{mn}}^{\times} = \langle\theta\rangle$. Then the minimal polynomial $p(x)$ of θ over $\mathbb{F}_q$ has degree m, and $p(x)$ is irreducible since it is minimal. $\square$

7 Cyclotomic fields

Let $q = p^n$ and $d > 0$ be any integer. Among the finitely many polynomials over $\mathbb{F}_q$ of degree d, how many of these are irreducible? We have just shown that at least one is irreducible.

Define the *Möbius function* $\mu : \mathbb{Z}_{>0} \to \{-1, 0, 1\}$ by

$$n \mapsto \begin{cases} -1 & n = 1 \\ (-1)^k & n = p_1 \cdots p_k \text{ where the } p_i \text{ are pairwise distinct} \\ 0 & n \text{ is divisible by a square} \end{cases}.$$

Proposition 7.0.1.

(i) $\mu(k) \neq 0$ *for some* k.

(ii) $\mu(nm) = \mu(n)\mu(m)$ *when* $(n, m) = 1$.

(iii) $\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}.$

Proof. Let $n > 0$ be an integer and write $p_1^{k_1} \cdots p_k^{r_k}$ where the prime p_i are pairwise distinct. Let $n_0 = p_1 \cdots p_k$. Then $\sum_{d|n} \mu(d) = \sum_{d|n_0} \mu(d)$. If $d \mid n_0$, then $d = p_{i_1} \cdots p_{i_s}$, so that $\mu(d) = (-1)^s$. By the binomial theorem, it follows that

$$\begin{aligned} \sum_{d|n_0} \mu(d) &= \sum_{s=0}^{k} \binom{k}{s} (-1)^2 \\ &= (1-1)^k \\ &= \begin{cases} 1 & k = 0 \\ 0 & k > 0 \end{cases}. \end{aligned}$$

Therefore, $\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}.$ $\square$

Corollary 7.0.2. *For any $m, d \in \mathbb{Z}_{>0}$ such that $d \mid m$, we have that*

$$\sum_{d|n|m} \mu\left(\frac{m}{n}\right) = \begin{cases} 1 & d = m \\ 0 & d \neq m \end{cases}.$$

Remark 7.0.3. Proposition 7.0.1 completely characterizes the Möbius function.

Lemma 7.0.4 (Möbius inversion formula). *Let A be an abelian group and $f, g : \mathbb{Z}_{>0} \to A$ be functions such that $f(n) = \sum_{d|n} g(d)$ for every n. Then*

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d). \tag{†}$$

If A is written multiplicatively, then this becomes

$$g(n) = \prod_{d|n} f(d)^{\mu\left(\frac{n}{d}\right)}.$$

Proof. We compute

$$\begin{aligned}
\sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{k|d} g(k) \\
&= \sum_{d|n} \sum_{k|d} \mu\left(\frac{n}{d}\right) g(k) \\
&= \sum_{k|n} g(k) \sum_{d:\ k|d|n} \mu\left(\frac{n}{d}\right) \\
&= \sum_{k|n} g(k) \delta(k, n) \\
&= g(n).
\end{aligned}$$

□

Definition 7.0.5. Define the *Euler (totient) function* $\varphi : \mathbb{Z}_{>0} \to \mathbb{Z}$ by

$$\varphi(n) = \#\left\{m \in \mathbb{Z}_{>0} : m \leq n,\ (m, n) = 1\right\}.$$

If $n \in \mathbb{Z}_{>0}$. then $n = \sum_{d|n} \varphi(d)$. Therefore, if $f : \mathbb{Z}_{>0} \to \mathbb{Z}$ is given by $f(n) = n$ and $g \coloneqq \varphi$, then we can apply (†) to get

$$\begin{aligned}
\varphi(n) = g(n) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) \\
&= \sum_{d|n} \mu\left(\frac{n}{d}\right) d \\
&= \sum_{m|n} \mu(m) \frac{n}{m} = \left(\sum_{m|n} \frac{\mu(m)}{m}\right) n.
\end{aligned}$$

Lemma 7.0.6. *If $n = p_1^{r_1} \cdots p_k^{r_k}$, then $\varphi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$.*

Proof. Let $n_0 = p_1 \cdots p_k$. Then

$$\begin{aligned}\sum_{m|n} \frac{\mu(m)}{m} = \sum_{m|n_0} \frac{\mu(m)}{m} \\ &= \underbrace{1}_{m=1} - \sum_{i=1}^{k} \frac{1}{p_i} \\ &+ \sum_{i<j} \frac{1}{p_i p_j} + \cdots + (-1)^s \sum_{i_1 < \cdots < i_s} \frac{1}{p_{i_1} \cdots p_{i_s}} \\ &+ \cdots + (-1)^k \frac{1}{p_1 \cdots p_k} \\ &= \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right).\end{aligned}$$

□

Let $p(x) = x^n - 1 \in \mathbb{Q}[x]$ with $n > 0$. Let Γ_n be the splitting field for $p(x)$. We know that $\Gamma_n = \mathbb{Q}(\zeta_n)$, where ζ_n denotes a primitive n-th root of unity in $\mathbb{C}$. Let the set Prim_n consist of all the primitive n-th roots of unity.

We have that $\mu_n = \left\{1, \zeta_n, \zeta_n^2, \ldots, \zeta_n^{n-1}\right\} = \coprod_{d|n} \mathrm{Prim}_d$. Define the *d-th cyclotomic polynomial* as

$$\Phi_d(x) = \prod_{\alpha \in \mathrm{Prim}_d} (x - \alpha).$$

For example,

$$\begin{aligned}\Phi_1 &= x - 1 \\ \Phi_2 &= x + 1 \\ \Phi_3 &= x^2 + x + 1 \\ \Phi_4 &= x^2 + 1 \\ &\vdots \\ \Phi_p &= x^{p-1} + x^{p-2} + \cdots + x + 1 \text{ with } p \text{ prime.}\end{aligned}$$

Note that

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Consider the functions $\Phi_{(-)} : \mathbb{Z}_{>0} \to \mathbb{C}(x)^\times$ and $f : \mathbb{Z}_{>0} \to \mathbb{C}(x)^\times$ where $f(n) = x^n - 1$. We can apply (†) to get

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu\left(\frac{n}{d}\right)},$$

which is a rational function over $\mathbb{Z}$. We can write $\Phi_n(x) = \frac{a(x)}{b(x)}$ with $b(x)$ monic. Write $\Phi_n(x) = x^m + p_{m-1} X^{m-1} + \cdots + p_0$ and set $M = \mathrm{lcm}\{c_i \mid p_i = \frac{t_i}{c_i},\ i = 1, \ldots, m-1\}$. Let $P_i = M p_i$ for each $i = 1, \ldots, m-1$ and $P_m = M$. Since $M\Phi_n(x) b(x) = M a(x) \in \mathbb{Z}[x]$, we see that M divides each coefficient of $M\Phi_n(x)b(x)$. Suppose, towards a contradiction, that $M > 1$. Then there exists a prime divisor p of M. By our choice of M, there exists a maximal $0 \le i_0 \le m$ such that $p \nmid P_{i_0}$. If $\deg b(x) = s$, then the coefficient of X^{m+s} in $M\Phi_n(x)b(x)$ has the form $M + p \cdot t$ for some $t \in \mathbb{Z}$. But this is not divisible by p and thus not divisible by M, a contradiction. Thus, $M = 1$, so that $\Phi_n(x) \in \mathbb{Z}[x]$.

Moreover, since $\deg \Phi_n = \varphi(n)$, it follows that $[\Gamma_n : \mathbb{Q}] = \varphi(n)$.

7.1 Lecture 15

Let $q = p^n$. Let $\psi_d(q) = \#\{p(x) \text{ irreducible over } \mathbb{F}_q \mid \deg p(x) = d\}$. If $f(x) \in \mathbb{F}_q[x]$ is irreducible, then $F = {}^{\mathbb{F}_q[x]}\!/\!_{(f)}$ is a finite field. Thus, $\#F = g^d = p^{nd}$, so that F is the splitting field for $x^{p^{nd}} - x$ over $\mathbb{F}_p$. Also, F is just the set of roots of $x^{p^{nd}} - x$. By construction, the polynomial $f(x) \in \mathbb{F}_q[x]$ has a root over F, and $x^{p^{nd}} - x \in \mathbb{F}_q[x]$ has a root in F.

Since $f(x)$ is irreducible in $\mathbb{F}_q[x]$, we see that $\left(f, x^{p^{nd}} - x\right) \in \{1, f\}$ in $\mathbb{F}_q[x]$. But if $\left(f, x^{p^{nd}} - x\right) = 1$, then $1 = a(x)f(x) + b(x)(x^{p^{nd}} - x)$. with $a, b \in \mathbb{F}_q[x]$. If we write this as an equation in $F[x]$, then evaluating on $\alpha \in F$ a common root of $f(x)$ and $x^{p^{nd}} - x$ will give us a contradiction. Hence $f(x) \mid x^{p^{nd}} - x$ in $\mathbb{F}_q[x]$. Since all roots of $x^{p^{nd}} - x$ are pairwise distinct, we see that any irreducible monic polynomial of degree d over $\mathbb{F}_q$ appears exactly once in the decomposition of $x^{p^{nd}} - x$ into irreducibles. Note that if $m = dr$, then

$$x^{q^d} - x \mid \underbrace{x^{q^m} - x}_{\text{distinct roots}},$$

and thus every irreducible monic polynomial over $\mathbb{F}_q$ of degree dividing m appears exactly once in the irreducible decomposition of $x^{q^m} - x$.

For each $d \ge 1$, let $f_{d,1}, f_{d,2}, \dots, f_{d,\psi_d(q)}$ be irreducible monic polynomials over $\mathbb{F}_q$ of degree d. Then for any $m \ge 1$, we get

$$x^{q^m - x} - x = \prod_{d \mid m} \prod_{k=1}^{\psi_d(q)} f_{d,k}(x),$$

so that $q^m = \sum_{d \mid m} d\psi(q)$. Then

$$\psi_d(q) = \frac{1}{m} \sum_{d \mid m} \mu\left(\frac{m}{d}\right) q^d.$$

Example 7.1.1.

$$\psi_2(2) = \frac{1}{2}(2^2 - 2) = 1.$$
$$\psi_3(2) = \frac{1}{2}(2^3 - 2) = 2.$$

Remark 7.1.2. A randomly chosen polynomial over $\mathbb{F}_q$ of degree d will be irreducible with probability $\approx \frac{1}{d}$. Given a polynomial $p(x)$ over $\mathbb{F}_q$ of degree d, there is no known algorithm with complexity polynomial in d that decides whether $p(x)$ is irreducible.

8 Galois theory

Definition 8.0.1. If $L \supset K$ is a field extension, then the *Galois group* is

$$\mathrm{Gal}(L/K) \equiv \{\varphi \in \mathrm{Aut}(L) : \varphi \restriction_K = \mathrm{id}_K\}.$$

Theorem 8.0.2. *Let $L \supset K$ be a field extension of degree $n < \infty$. Let $G \leq \mathrm{Gal}(L/K)$.*

(a) $L^G = K \iff |G| = n$.

(b) If $L^G = K$ and $K \subset P \subset Q \subset L$ is a chain of field extensions, then every homomorphism $\varphi : P \to L$ over K extends to a homomorphism $Q \to L$ in exactly $[Q : P]$ many ways.

Proof.

(a) For the ($\Longleftarrow$) direction, note that if $G \leq \mathrm{Aut}(L)$, then tautologically $G \leq \mathrm{Gal}(L/L^G)$. Hence $|G| \leq \left|\mathrm{Gal}(L/L^G)\right| = \left[L : L^G\right]$. If $G \leq \mathrm{Gal}(L/K)$, then $L \subset L^G \subset L$, so that $\left[L : L^G\right] \leq [L : K] = n$. This means that $|G| \leq n$.

Conversely, let $L^G = K$. Take $\alpha \in L$ and let $\mathrm{Orb}_G(\alpha) = \{\lambda_1, \ldots, \lambda_m\} \subset L$. Consider

$$f(x) = \prod_{i=1}^{m}(x - \lambda_i) \in L[x].$$

But the coefficients are symmetric polynomials in λ_i, and any $g \in G$ permutes the λ_i. In this case, g permutes the coefficients of $f(x)$. Hence $f(x) \in L^G[x] = K[x]$. By construction, α is a root of $f(x)$, the minimal polynomial of α. We can decompose $f(x)$ into linear factors in L.

Apply part (b) to $P = K$ and $Q = L$. In this case, (b) implies that if $L^G = K$, then $|\mathrm{Gal}(L/K)| = [L : K] = n$. Thus, we must show that $G = \mathrm{Gal}(L/K)$.

Let $\varphi \in \mathrm{Gal}(L/K)$. Recall that $f(x)$ is the minimal polynomial of α over K. Note that $\varphi(x)$ is a root of $f^\varphi(x)$. Indeed, since $\varphi_K = \mathrm{id}_K$, we have that $f^\varphi(x) = f(x)$. Hence $\varphi(\alpha) \in \mathrm{Orb}_G(\alpha)$, so that there exists $g \in G$ such that $\varphi(\alpha) = g(\alpha)$. If L is a finite field, then we can take α to be the generator of $L^\times$, in which case $\varphi(\alpha) = g(\alpha) \implies \varphi(\alpha^k) = g(\alpha^k)$ for each $k \implies \varphi = g$. If L is infinite, then K is infinite and for any $g \in G$, we consider $L_g = \{a \in L \mid \varphi(a) = g(a)\} \subset L$. By definition, $L_g = L^{g^{-1} \circ \varphi}$ is a subfield in L. This contains K because $g^{-1} \circ \varphi \in \mathrm{Gal}(L/K)$. Therefore, $K \subset L_g \subset L$ is a field extension, meaning that L_g is a K-vector subspace in L.

We have shown that $L = \bigcup_{g \in G} L_g$.

Claim. *If K is an infinite field and V is a finite-dimensional K-vector space and $V_1, V_2, \ldots, V_g \subset V$ are subspaces, then $V = \bigcup_{i=1}^{g} V_i \implies V = V_k$ for some k.*

Proof. Suppose that each $V_i \subsetneq V$ and that $V = \bigcup_{i=1}^{g} V_i$. Then there exists a linear map $f_i : V \to K$ such that $f_i \restriction_{V_i} = 0$ and $f_i \neq 0$. Then $f : V \to K$ given by $f = \prod_{i=1}^{k} f_i$ is the function associates with a nonzero polynomial in $V = K^n$ of degree s. But f is the zero function since $V = \bigcup_{i=1}^{k} V_i$, a contradiction. $\square$

(b) Suppose that $K \subset P \subset L$. Let $K \subset P \subset Q \subset L$ where $Q = P(\alpha) = P[\alpha]$ and α is a root of some irreducible $h(x) \in P[x]$. Let $f(x) = \prod_{i=1}^{m}(x - \lambda_i)$ where $\mathrm{Orb}_G(\alpha) = \{\lambda_1, \dots, \lambda_m\} \subset L$. Then $f(x), g(x) \in P[x]$ have a common root, and $h(x)$ is irreducible. Hence $h \mid f$ in $P[x]$.

Let $\varphi : P \to L$ be any field homomorphism over K. Then $h^\varphi \mid \underbrace{f^\varphi}_{f}$ in $L[x]$. But f decomposes into distinct linear factors in $L[x]$. Hence h^φ equals a product of a subcollection of these factors. It follows that h^φ has $\deg h^\varphi = \deg h$ distinct roots in L. By Lemma 4.2.8, since $Q \cong P[x]/(h)$, we see that φ extends in exactly $\deg h - [Q : P]$ many ways.

This proves our result for simple field extensions. Since every finite extension is a tower of simple extensions, we are done by induction on the length of the tower.

$\square$

Definition 8.0.3. A finite field extension $L \supset K$ is a *Galois extension* if $|\mathrm{Gal}(L/K)| = [L : K]$.

Corollary 8.0.4. *If $L \supset K$ is a Galois extension, then $K \subset P \subset L \implies L \supset P$ is Galois as well.*

Proof. Take $Q = L$ and apply (b) then (a). $\square$

Definition 8.0.5. If K is a field and $f(x) \in K[x]$, then we say that f is *separable over K* if f has no repeated roots in any finite extension of K. Equivalently, f has no repeated roots in its splitting field.

8.1 Lecture 16

Proposition 8.1.1. *A polynomial $f(x) \in K[x]$ is separable over K if and only if $(f, f') = 1$.*

Proof. If $f, g \in K[x]$, then $(f, g) \in K[x]$. Suppose there exsits $L \supset K$ such that f has a multiple root in L. Then there exists an irreducible polynomial $h(x) \in L[x]$ such that $h^2 \mid f$. This implies that $f = h^2 q$, so that $f' = 2hh'q + h^2 q' = h(2h'q + hq')$. Hence $h \mid f'$ in $L[x]$. Then $h \mid (f, f')$ in $L[x]$, making $(f, f') \neq 1$.

Conversely, suppose that $(f, f') \neq 1$. Then there exists h irreducible in $K[x]$ such that $h \mid f'$ and $h \mid f$ in $K[x]$. We can write $f = hg$, so that $f' = h'g + hg'$. Either $h \mid g$ or $h' = 0$. In the former case, we have that $h \mid g \implies h^2 \mid f \implies f$ has a double root in $L = {}^{K[x]}\!/_{(h)}$. In the latter case, we see that $\mathsf{char}\, K = p > 0$ and $h(x) = a_0 + a_1 x^p + a_2 x^{2p} + \cdots + a_s x^{sp}$ with $a_s \neq 0$. Let $L \supset K$ be a finite field extension such that for any $i = 0, \dots, s$, we have b_i such that $b_i^p = q_i$. Then viewing $h(x) \in L[x]$, we get $h(x) = \big(b_0 + b_1 x + b_2 x^2 + \cdots + b_s x^s\big)^p$ since $b_s \neq 0$. Hence if $\varphi(x)$ is an irreducible factor of $b_0 + b_1 x + \cdots + b_s x^s$ in $L[x]$, then if $F = {}^{L[x]}\!/_{(\varphi)}$, then φ has a root in F and h will have a root with multiplicity p in F. In this case, f has a root with multiplicity p in F. $\square$

Corollary 8.1.2. *If K has $\mathsf{char}\, K = 0$, then every irreducible $f(x) \in K[x]$ is separable.*

Proof. If $\mathsf{char}\, K = 0$, then $f \neq 0$ and f is irreducible. Since $\deg f > 0$, it follows that $f' \neq 0$. Hence (f, f') is a polynomial of degree ≥ 0. Since f is irreducible, this means that $(f, f') \in \{1, f\}$. But $\deg f' < \deg f$, so that $(f, f') = 1$. $\square$

Corollary 8.1.3. *If $f(x) \in K[x]$ is irreducible and $\mathsf{char}\, K \nmid \deg f$, then f is separable.*

Corollary 8.1.4. *Every irreducible polynomial f over a finite field F is separable.*

Proof. If f is irreducible and $f' \neq 0$, then apply a similar argument to the proof of Corollary 8.0.4. Suppose $f' = 0$. Then $f(x) = a_0 + a_1 x^p + \cdots + a_s x^{sp}$ with $p = \mathsf{char}\, F$. But as F is finite, we know that the Frobenius map φ is an automorphism. Thus, any element in F has a p-th root in F. Hence there exists $b_i \in F$ such that $b_i^p = a_i$. This shows that $f(x) = (b_0 + b_1 x + \cdots + b_s x^s)^p$, which contradicts that f is irreducible over F. □

Example 8.1.5. There are irreducible polynomials over fields of characteristic > 0 that are not separable. For example, let $K = \mathbb{F}_p(t)$ and $f(x) = x^p - t$. This is irreducible in $K[x]$ but not separable over K.

Indeed, if $L \supset K$ is such that f has a root α in L, then f splits in $L[x]$. We can write $f(x) = (x-\alpha)^p$. But if $0 < k < p$, then $\alpha^k \notin K$. This shows that f is irreducible in $F[x]$ but has a root of multiplicity p.

Theorem 8.1.6. *If $f(x) \in K[x]$ and every irreducible factor of f is separable over K, then the splitting field L of f is Galois over K.*

Proof. We constructed L as a tower

$$K = K_0 \subset K_1 \subset \cdots \subset K_s = L$$

where $K_{i+1} = K_i(\alpha_{i+1})$ and α_{i+1} is a root of some irreducible factor $f_{i+1}(x)$ of $f(x) \in K_i[x]$. Since f_{i+1} is irreducible in $K_i[x]$ and $f_{i+1} \mid f$ in $K_i[x]$, it follows that f_{i+1} must divide one of the irreducible factors of $f(x)$ in $K[x]$. But these are separable, which implies that f_{i+1} is separable for each i. By Lemma 4.2.8, a field homomorphism $\varphi : K \to L$ extends to an isomorphism $\varphi : L \to L$ in

$$(\#\text{ of distinct roots in } f_1) \cdot (\#\text{ of distinct roots in } f_2) \cdots (\#\text{ of distinct roots in } f_{s-1})$$

many ways. Note that

$$\begin{aligned}(\#\text{ of distinct roots in } f_1) \cdot (\#\text{ of distinct roots in } f_2) \cdots (\#\text{ of distinct roots in } f_{s-1})\\ = \deg f_1 \cdot \deg f_2 \cdots \deg f_{s-1}.\end{aligned}$$

Hence

$$|\mathrm{Gal}(L/K)| = \deg f_1 \cdot \deg f_2 \cdots \deg f_{s-1} = [K_1 : K_0]\,[K_2 : K_1] \cdots [K_s : K_{s-1}] = [L : K]\,.$$

□

If $f(x) \in K[x]$ and $L \supset K$ is the splitting field for f, then let $\alpha_1, \ldots, \alpha_m$ denote the distinct roots of f in L. We have that $L = K(\alpha_1, \ldots, \alpha_m)$ and any $\varphi \in \mathrm{Gal}(L/K)$ sends $\{\alpha_1, \ldots, \alpha_m\}$ to itself. This gives us a homomorphism $\mathrm{Gal}(L/K) \to S_m$ that is injective by Lemma 4.2.8. Therefore, $\mathrm{Gal}(L/K) \subset S_m$.

Example 8.1.7.

1. Let K be a field and let $f(x) \in K[x]$ be irreducible of degree 2. Let L denote the splitting field for $f(x)$. Then $K\left[\sqrt{D}\right]$ where $D = \mathrm{Discr}(f) \in K$. In this case, $[L : K] = 2$, and $\mathrm{Gal}(L/K) \subset S_2$ since $D \neq 0$. Thus, f must have distinct roots in L. Note that $\mathrm{Gal}(L/K) \neq \{\mathrm{id}\}$, since these roots are not in K. This shows that $\mathrm{Gal}(L/K) = \langle \sigma \rangle = S_2$ where $\sigma : L \to L$ is given by $a + b\sqrt{D} \mapsto a - b\sqrt{D}$.

2. Let $q = p^n$. Consider the extension $\mathbb{F}_q \supset \mathbb{F}_p$. Then $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p) = \langle \varphi \rangle \cong \mathbb{Z}/n$.

3. Recall that the cyclotomic field $\Gamma_n \supset \mathbb{Q}$ is the splitting field for

$$x^n - 1 = (x-1)\left(x^{n-1} + x^{n-2} + \cdots + x + 1\right).$$

This polynomial has irreducible factors each of which is separable. Thus, $\Gamma_n \supset \mathbb{Q}$ is a Galois extension such that $\mathrm{Gal}(\Gamma_n/\mathbb{Q}) \subset S_{\mu_n}$. Since any $g \in \mathrm{Gal}(\Gamma_n/\mathbb{Q})$ respects multiplication in Γ_n, we see that $g \restriction_{\mu_n}: \mu_n \to \mu_n$ is a a group automorphism. It follows that $\mathrm{Gal}(\Gamma_n/\mathbb{Q}) \subset \mathrm{Aut}_{\mathbf{Grp}}(\mu_n) \cong (\mathbb{Z}/n)^\times$, which has order $\phi(n)$. We have shown that the minimal polynomial of a root of 1 over $\mathbb{Q}$ is precisely $\Phi_n(x)$, where $\deg \Phi_n(x) = \phi(n)$. Hence $[\Gamma_n : \mathbb{Q}] = \phi(n)$, so that $\mathrm{Gal}(\Gamma_n/\mathbb{Q}) = (\mathbb{Z}/n)^\times$.

4. Suppose that $\mathsf{char}\, K \notin \{2, 3\}$. Let $f(x) \in K[x]$ be irreducible and monic of degree 3. Let $D \in K$ denote the discriminant of f. Let $L \supset K$ be the splitting field for f, so that $L \supset K$ is Galois. Then

$$|\mathrm{Gal}(L/K)| = \begin{cases} 6 & D \notin K^2 \\ 3 & D \in K^2 \end{cases}.$$

But $\mathrm{Gal}(L/K) \subset S_3$. This shows that

$$\mathrm{Gal}(L/K) = \begin{cases} S_3 & D \notin K^2 \\ A_3 & D \in K^2 \end{cases}.$$

8.2 Lecture 17

Definition 8.2.1. Let k be a field and A be a finitely generated k-algebra. A collection $u_1, \ldots, u_n \in A$ is a *transcendence basis of A/k* if

(i) the u_i are independent transcendentals over k and

(ii) every $a \in A$ is algebraically dependent with $k[u_1, \ldots, u_n]$.

If A is a domain and $u_1, \ldots, u_n$ forms a transcendence basis of A/k, then they also form a transcendence basis of $\mathrm{Frac}(A)$ over k. Observe that $x \in \mathrm{Frac}(A)$ is algebraic over $k[u_1, \ldots, u_n]$ if and only if it is algebraic over $k(u_1, \ldots, u_n)$. Then

$$S \coloneqq \{x \in \mathrm{Frac}(A) \mid x \text{ is algebraic over } k[u_1, \ldots, u_n]\}$$

is a subfield. But $A \subset S \subset \mathrm{Frac}(A)$, so that, by the universal property, $S = \mathrm{Frac}(A)$. Hence $\mathrm{Frac}(A)$ is algebraic over $k(u_1, \ldots, u_n)$.

Let $A = k[u_1, \ldots, u_n]$ and suppose that $\{u_1, \ldots, u_d\}$ is a maximal subset of algebraically independent elements over k in $\{u_1, \ldots, u_n\}$. Then $u_1, \ldots, u_d$ form a transcendence basis of A/k. Indeed, K equals the algebraic closure of $k(u_1, \ldots, u_d)$ in $\mathrm{Frac}(A)$. Thus, $u_1, \ldots, u_n \in K$, so that $K = \mathrm{Frac}(A)$. It follows that $K \supset A$.

As a result, if A is a finitely generated algebra without zero divisions, then A has a transcendence basis over k. Indeed, choose any system of generators of A/k and then choose a maximal subset of algebraically independent elements.

Lemma 8.2.2. *Suppose that $\{u_1, \ldots, u_n\}$ is a transcendence basis of A/k and that v is transcendental over $k[u_1, \ldots, u_n]$. Then $\{v, u_2, u_3, \ldots, u_n\}$ is also a transcendence basis of A/k.*

Proof. Note that $v, u_2, \ldots, u_n$ are algebraically independent over k whereas $v, u_1, u_2, \ldots, u_n$ are algebraically dependent. A nontrivial algebraic relation among these will be given by a polynomial $p(x)$ over k such that $p(x)$ includes a monomial involving u_1 with a nonzero coefficient. Then $p(x)$ can be viewed as a nonzero polynomial in $(k[v, u_1, \ldots, u_n])[u_1]$ with deg ≥ 1 on u. We have that u_1 is algebraic over $k[v, u_1, \ldots, u_n]$. Thus, the algebraic closure of $k[v, u_2, \ldots, u_n]$ in $\mathrm{Frac}(A)$ contains u_1, hence contains A. It follows that the algebraic closure of $k[v, u_2, \ldots, u_n]$ equals $\mathrm{Frac}(A)$.

This shows that any transcendence basis of A/k has the same cardinality. Indeed, let $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ be transcendence bases of A/k. Then at least one of the v_i's must be transcendental over $k[u_2, \ldots, u_n]$. This is because if each v_i is algebraic over $k[u_2, \ldots, u_n]$, then $A \supset k[u_2, \ldots, u_n]$ will be algebraic, in which case $u_2, \ldots, v_m$ is also a basis, a contradiction.

Say that v_1 is transcendental over $k[u_2, \ldots, u_n]$. Then $A \supset k[v_1, u_2, \ldots, u_n]$ is algebraic .... One of $v_1, \ldots, v_m$ must be transcendental over $k[v_1, \ldots, u_2, \ldots, u_n]$. Hence $A \supset k[v_1, v_2, u_3, \ldots, u_n]$ is algebraic. If $m \leq n$, then $A \supset k[v_1, v_2, \ldots, v_m, u_{m+1}, \ldots, u_n]$ is algebraic and $v_1, v_2, \ldots, v_m, u_{m+1}, \ldots, u_n$ are dependent. This is a contradiction unless $n = m$. $\square$

If F is a field and $\widetilde{F} \supset F$ is a field extension, then we can measure how far $\widetilde{F}$ is from being an algebraic extension of F by its *transcendence degree over F*

$$\mathrm{trdeg}\left(\widetilde{F}/F\right) \equiv \mathrm{card}(\text{independent transcendentals we need to add to } F \text{ to generate } \widetilde{F}).$$

Corollary 8.2.3. trdeg *is an invariant of the extension $\widetilde{F}$.*

Example 8.2.4. Let k be a field and $a_1, a_2, \ldots, a_n$ be indeterminates. Let $K \coloneqq k(a_1, \ldots, a_n)$. Consider $f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots a_n \in K[x]$. Then $\mathrm{Gal}(L/K) = S_n$ where L denotes the splitting field for f.

Proof. Let $x_1, x_2, \ldots, x_n \in L$ denote the roots of f. Then $a_i = (-1)^i \sigma_i(x_1, \ldots, x_n)$ where σ_i denotes the i-th elementary symmetric function. Hence $L = K(x_1, \ldots, x_n) = k(x_1, \ldots, x_n)$. Consider the chain of field extensions $L \supset K \supset k$. Note that $L \supset K$ is an algebraic extension and that $K \supset k$ is a transcendental extension because K is obtained from adding n independent transcendentals to K.

Since $\operatorname{trdeg}(L/k) = \operatorname{trdeg}(K/k) = n$ and $L = K(x_1, \ldots, x_n)$, we see that $x_1, \ldots, x_n$ are algebraically independent over k. Therefore, there are pairwise distinct. This shows that $f(x) \in K[x]$ has distinct roots, so that $L \supset K$ is separable and thus a Galois extension. It follows that $\operatorname{Gal}(L/K) = S_n$ and

$$L^{S_n} = (k(x_1, \ldots, x_n))^{S_n} = K = k(\sigma_1, \ldots, \sigma_n).$$

□

Theorem 8.2.5 (Main theorem of Galois theory). *Let $L \supset K$ be a Galois extension. Then the mappings*

$$(K \subset P \subset L : P \text{ field}) \mapsto (G \leq \operatorname{Gal}(L/K))$$
$$L^G \leftarrow\!\shortmid G$$

are inverse to each other.

Furthermore, if $L \supset P \supset K$, then $P \supset K$ is a Galois extension of K if and only if $\operatorname{Gal}(L/P) \trianglelefteq \operatorname{Gal}(L/K)$.

Proof. Consider $K \subset P \subset L$ and $K \subset L^{\operatorname{Gal}(L/P)} \subset L$. Then $L^{\operatorname{Gal}(L/P)} \supset P$. From a theorem from two lectures ago, we have the following two results.

(a) $[L : P] = |\operatorname{Gal}(L/P)|$ for any $K \subset P \subset L$.

(b) $\left[L : L^G\right] = |G|$ for any $G \leq \operatorname{Gal}(L/K)$.

Therefore, $\left[K : L^{\operatorname{Gal}(L/P)}\right] \cdot [L : P] = [L : P]$, so that $\left[L^{\operatorname{Gal}(L/P)} : L\right] = 1$. Hence $L^{\operatorname{Gal}(L/P)} = L$. Similarly, $\operatorname{Gal}(L/L^G) \leq G$ satisfies $\left|\operatorname{Gal}(L/L^G)\right| = |G|$, so that $\operatorname{Gal}(L/L^G) = G$.

For the second part our theorem, note that any automorphism of P/K will extend to an automorphism of L/K. This shows that the map $\{\varphi \in \operatorname{Gal}(L/K) \mid \varphi(P) \subset P\} \to \operatorname{Gal}(P/K)$ given by $\varphi \mapsto \varphi \restriction_P$ is surjective. Then $P \supset K$ will be Galois if and only if the elements of $\{\varphi \in \operatorname{Gal}(L/K) \mid \varphi(P) \subset P\}$ induce $[P : K]$ distinct elements of $\operatorname{Gal}(P/K)$.

We compute

$$\begin{aligned} |\operatorname{Gal}(L/P)| &= [L : P] \\ [P : K] &= \frac{[L : K]}{[L : P]} = \frac{|\operatorname{Gal}(L/K)|}{|\operatorname{Gal}(L/K)|} \\ [P : K] &= [\operatorname{Gal}(L/K) : \operatorname{Gal}(L/P)]\,. \end{aligned}$$

Thus, $P \supset K$ is a Galois extension if and only if any element of $\operatorname{Gal}(L/K)$ leaves P invariant. But $P = L^{\operatorname{Gal}(L/P)}$, and $P = P^{\operatorname{Gal}(L/P)}$. Hence any $g \in \operatorname{Gal}(L/P)$ satisfies

$$g(P) = g(L^{\operatorname{Gal}(L/P)}) = L^{g \operatorname{Gal}(L/P) g^{-1}}.$$

It follows that $g(P) = P \iff g \operatorname{Gal}(L/P) g^{-1} = \operatorname{Gal}(L/P)$. □

8.3 Lecture 18

Example 8.3.1. Let K be a field with $\operatorname{char} K \notin \{2,3\}$. Let f be an irreducible, monic, cubic polynomial over K. Let L be the splitting field for f. Let $D \coloneqq \operatorname{Discr} f \in K \setminus K^2$. Then $\operatorname{Gal}(L/K) = S_3$. We get

$$L \supset L^{A_3} \supset K$$
$$\operatorname{Gal}(L/K) \trianglerighteq A_3 \trianglerighteq \{e\}\,.$$

It follows that $L^{A_3} \supset K$ is Galois with $\operatorname{Gal}(L^{A_3}/K) \cong \operatorname{Gal}(L/K)/A_3 \cong C_2$. In fact, $L^{A_3} \cong K\left[\sqrt{D}\right]$.

Now, let $p > 2$ be prime. Consider the cyclotomic field $\Gamma_p \supset \mathbb{Q}$. We have that

$$\operatorname{Gal}(\Gamma_p/\mathbb{Q}) \cong (\mathbb{Z}/p)^\times \cong \mathbb{Z}/(p-1).$$

Let $H \trianglelefteq \operatorname{Gal}(\Gamma_p/\mathbb{Q})$ be the unique subgroup of index 2. Then $\left[\Gamma_p^H : \mathbb{Q}\right] = 2$.

Let $\langle \varphi \rangle = \operatorname{Gal}(\Gamma_p/\mathbb{Q})$. Then $\varphi \restriction_{\mu_p} \colon \mu_p \to \mu_p$ is a group automorphism and uniquely determines φ, which in turn is uniquely determined by the image of ζ the positive p-th root of 1. Write $\varphi(\zeta) = 1\zeta^r$ for some $r \in \mathbb{Z}_{>0}$ so that $[r]_p \in \mathbb{Z}/p$ is a generator of $(\mathbb{Z}/p)^\times$.

Definition 8.3.2. Given $k \in \mathbb{Z}_{>0}$ and prime $p > 2$, define the *Legendre symbol*

$$\left(\frac{k}{p}\right) = \begin{cases} 1 & [k]_p \in \left((\mathbb{Z}/p)^\times\right)^2 \\ -1 & [k]_p \notin \left((\mathbb{Z}/p)^\times\right)^2 \end{cases}.$$

Consider $\alpha \in \Gamma_p$ given by

$$\alpha = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \zeta^{r^{k-1}} = \zeta - \zeta^r + \zeta^{r^2} - \cdots \zeta^{r^{p-2}}.$$

If $g \in \operatorname{Gal}(\Gamma_p/\mathbb{Q})$, then

$$g(\alpha) = \begin{cases} \alpha & g \in H \\ -\alpha & g \notin H \end{cases}.$$

Then $\alpha \in \Gamma_p^H$. Also, α^2 is fixed by every element of $\operatorname{Gal}(\Gamma_p/\mathbb{Q})$ and thus is rational. This implies that $\Gamma_p^H = \mathbb{Q}[\alpha]$.

Lemma 8.3.3. *$\alpha^2 = (-1)^{\frac{p-1}{2}} p$, so that*

$$\Gamma_p^H = \begin{cases} \mathbb{Q}\left[\sqrt{p}\right] & p \equiv 1 \mod 4 \\ \mathbb{Q}\left[\sqrt{-p}\right] & p \not\equiv 1 \mod 4 \end{cases}.$$

Proof. Let $L \supset K$ be a finite extension of fields. Then for any $u \in L$, we get a map $\operatorname{mult}_u : L \to L$, which is linear over K. Applying trace determines a K-linear map $L \to K$ given by $u \mapsto \operatorname{tr}(\operatorname{mult}_u)$. This induces a symmetric bilinear map $\langle \cdot, \cdot \rangle : L \otimes_K L \to K$ given by $u \otimes v \mapsto \operatorname{tr}(\operatorname{mult}_u \circ \operatorname{mult}_v)$. Note that if $u \neq 0$, then

$$\left\langle u, u^{-1} \right\rangle = \operatorname{tr}(\operatorname{mult}_{uu^{-1}}) = \operatorname{tr}(\operatorname{id}_L) = [L : K]$$

since $(\mathsf{char}\, K, [L : K]) = 1$. Now, the vector space Γ_p has a $\mathbb{Q}$-basis $\{1, \zeta, \zeta^2, \ldots, \zeta^{p-2}\}$. Hence mult_{ζ^2} is a cyclic operator, and $\mathrm{tr}(\mathrm{mult}_1) = p - 1$ and $\mathrm{tr}(\mathrm{mult}_{\zeta^k}) = -1$ for each $k = 1, \ldots, p-2$. It follows that

$$\langle \zeta^k, \zeta^l \rangle = \begin{cases} p-1 & k + l \equiv 0 \mod p \\ 1 & \text{otherwise} \end{cases}.$$

If $x = \sum_{i=0}^{p-1} x_i \zeta^i$ and $y = \sum_{i=0}^{p-1} y_i \zeta^i$ are two elements of Γ_p, then we can choose x_i and y_i such that $\sum x_i = 0$ and $\sum y_i = 0$. Thus,

$$\langle x, y \rangle = p(x_0 y_0 + \sum k = 1^{p-1} x_k y_{p-k}).$$

But $\alpha = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \zeta^{r^{k-1}}$, so that

$$\begin{aligned} \alpha^2 = \frac{1}{p-1} \langle \alpha, \alpha \rangle &= \frac{1}{p-1} \sum_{k=1}^{p-1} p \left(\frac{k}{p}\right) \left(\frac{-k}{p}\right) \\ &= \frac{p}{p-1} \sum_{k=1}^{p} \left(\frac{k}{p}\right) \left(\frac{-k}{p}\right) \\ &= p \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} p. \end{aligned}$$

□

Definition 8.3.4. Let $L \supset K$ be a field extension and $\alpha \in L$¿ We say that α *can be expressed in radicals over* K if it can be obtained from elements in K by applying $+$, $\cdot$, and $\sqrt[r]{\cdot}$, i.e., there exists a tower of subfields

$$K = K_0 \subset K_1 \subset \cdots \subset K_s \subset L$$

such that $K_{i+1} = K_i(\alpha_{i+1})$ where $\alpha_{i+1}^{r_{i+1}} \in K_i$ and $\alpha \in K_s$.

Proposition 8.3.5. *If $f(x) \in K[x]$ is irreducible, $L \supset K$ is an extension, and α is a root of $f(x)$, then α can be expressed in radicals if and only if any root of f can be expressed in radicals in the splitting field for f.*

Proof. If $L_1, L_2 \supset K$ are field extensions and $\alpha_1 \in L_1$ and $\alpha_2 \in L_2$ are roots of f, then by Lemma 4.2.8 there is a unique map $\varphi : K(\alpha_1) \xrightarrow{\cong} K(\alpha_2)$ such that $\varphi(\alpha_1) = \alpha_2$. Now transport all suitable expressions by φ or φ^{-1}. □

9 Solvability in radicals

Definition 9.0.1. We say that $f(x) \in K[x]$ is *solvable in radicals* if every root of f can be expressed in radicals over K.

This is equivalent to saying that L is a splitting field for f, then there is a tower of subfields $K = K_0 \subset K_1 \subset \cdots \subset K_s = L$ such that $K_{i+1} = K_i(\alpha_{i+1})$ where $\alpha_{i+1}^{r_i+1} \in K_i$.

Theorem 9.0.2. *If K is a field with characteristic 0, $f(x) \in K[x]$ is irreducible, and $L \supset K$ is the splitting field for f, then f is solvable in radicals over K if and only if $\mathrm{Gal}(L/K)$ is solvable.*

Note 9.0.3.

1. A generic polynomial equation over K of deg ≥ 5 will not be solvable in radicals, since Gal $\cong S_n$.

2. If $f(x) \in \mathbb{Q}[x]$ is irreducible of degree 5, then f will not be solvable in radicals as soon as $\mathrm{Gal}(L/\mathbb{Q}) \in \{S_5, A_5\}$. Suppose $f \in \mathbb{Q}[x]$ is such a polynomial and let $\alpha_1, \ldots, \alpha_5$ be the roots of f. Note that $\mathrm{Gal}(L/\mathbb{Q}) \subset S_5$. Since f is irreducible, it must be separable, which means that the α_i are pairwise distinct. Hence $5 \mid |\mathrm{Gal}(L/\mathbb{Q})|$. Therefore, $\mathrm{Gal}(L/\mathbb{Q})$ must contain an element of order 5, so that $\mathrm{Gal}(L/\mathbb{Q})$ contains a 5-cycle. If we can choose f so that $\mathrm{Gal}(L/\mathbb{Q})$ contains a transposition, then $\mathrm{Gal}(L/\mathbb{Q}) = S_5$.

   Choose f so that it has exactly three real roots. In this case, complex conjugation will belong to $\mathrm{Gal}(L/\mathbb{Q})$, so that $\mathrm{Gal}(L/\mathbb{Q}) = S_5$. Start with $x^5 - 16x = x(x-2)(x+2)(x^2+4)$, which has exact three real roots. To make this irreducible, shift its graph to obtain the polynomial $f(x) = x^5 - 16x + 2$.

9.1 Lecture 19

Theorem 9.1.1. *If K is a field with characteristic 0, $f(x) \in K[x]$ is irreducible, and $L \supset K$ is the splitting field for f, then f is solvable in radicals over K if and only if $\mathrm{Gal}(L/K)$ is solvable.*

Proof.

($\Longleftarrow$)

We have a series

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \cdots \trianglerighteq G^{(s)} \trianglerighteq \{e\},$$

which we can refine to get a normal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_r = \{e\}$$

such that $G_{i+1}/G_i \cong \mathbb{Z}/n_i$. Letting $K_i = L^{n_i}$, we have a tower

$$K = K_0 \subset K_1 \subset \cdots \subset K_r = L.$$

Let F be the cyclotomic field that contains all roots of 1 of order $n = n_1 n_2 \cdots n_r$. Consider the tower

$$KF \subset K_1 F \subset \cdots \subset K_r F = LF.$$

Then $K_{n_i} F \supset K_1 F$ is a cyclic extension of degree dividing n_i.

Lemma 9.1.2. *Let K be a field and let n have $\mathsf{char}\, K \nmid n$. Suppose that $K \supset \mu_n$.*

(a) For any $\alpha \in K$, the extension $K(\sqrt[n]{\alpha}) \supset K$ is cyclic of order dividing n.

(b) For every $\widetilde{K} \supset K$ Galois and cyclic of order n, there exists $\alpha \in K$ such that $\widetilde{K} \cong K(\sqrt[n]{\alpha})$.

Before proving this, note that (b) implies that $K_{i+1}F \supset K_iF$ for every i.

Proof.

(a) By definition, $K(\sqrt[n]{\alpha})$ contains some root of $x^n - \alpha$. But K contains μ_n, so that $K(\sqrt[n]{\alpha})$ contains every root of $x^n - \alpha$. Thus, $K(\sqrt[n]{\alpha})$ is the splitting field for $x^n - \alpha$. Hence $K(\sqrt[n]{\alpha}) \supset K$ is Galois. If $\sigma \in \mathrm{Gal}(K(\sqrt[n]{\alpha})/K)$, then $\sigma(\sqrt[n]{\alpha}) = \zeta_\sigma \sqrt[n]{\alpha}$ where ζ_σ is some n-th root of 1 depending on σ. Then we get a map

$$\zeta : \mathrm{Gal}(K(\sqrt[n]{\alpha})/K) \to \mu_n$$

given by $\sigma \mapsto \zeta_\sigma$. But since $K \supset \mu_n$, if $\sigma, \tau \in \mathrm{Gal}(K(\sqrt[n]{\alpha})/K)$, then

$$\sigma(\tau(\sqrt[n]{\alpha})) = \sigma(\zeta_\tau \sqrt[n]{\alpha}) = \sigma(\zeta_\tau)\sigma(\sqrt[n]{\alpha}).$$

As $\zeta_\tau \in \mu_n \subset K$, we see that $\sigma(\zeta_\tau) = \zeta_\tau$. This implies that

$$\sigma(\tau(\sqrt[n]{\alpha})) = \zeta_\tau\zeta_\sigma \sqrt[n]{\alpha}.$$

But $\sigma(\tau(\sqrt[n]{\alpha})) = \zeta_{\tau\sigma} \sqrt[n]{\alpha}$ as well, so that $\zeta_{\tau\sigma} = \zeta_\tau\zeta_\sigma$. This shows that ζ is a homomorphism.

Moreover, if $\sigma \in \ker\zeta$, i.e., $\zeta_\sigma = 1$, then $\sigma(\sqrt[n]{\alpha}) = 1 \cdot \sqrt[n]{\alpha} = \sqrt[n]{\alpha}$. Since any $\sigma \in \mathrm{Gal}(K(\sqrt[n]{\alpha})/K)$ preserving $\sqrt[n]{\alpha}$ must be the identity, it follows that ζ is injective. As a result, we get an embedding $\mathrm{Gal}(K(\sqrt[n]{\alpha})/K) \leq \mu_n$.

(b) Suppose that $\mathrm{Gal}\left(\widetilde{K}/K\right)$ is cyclic of order $d \mid n$. We want to show that there exists $\alpha \in K$ such that $\widetilde{K} \cong K(\sqrt[d]{\alpha})$.

Let $\alpha \in \widetilde{K}$ and $\xi \in \mu_d \subset \mu_n \subset K$. The *Lagrange resolvent* of (α, ξ) is the element

$$\ell(\alpha,\xi) = \alpha + \xi\sigma(\alpha) + \xi^2\sigma^2(\alpha) + \cdots + \xi^{d-1}\sigma^{d-1}(\alpha)$$

of $\widetilde{K}$ where $\sigma \in \mathrm{Gal}\left(\widetilde{K}/K\right)$ is a generator.

Note that $\sigma(\ell(\alpha,\xi)) = \xi^{-1}\ell(\alpha,\xi)$, so that $\sigma(\ell(\alpha,\xi)^2) = \xi^{-k}\ell(\alpha,\xi)^k$.

Suppose that ξ is a primitive d-th root of unity. We see that $\mathrm{id} + \xi\sigma + \xi^2\sigma^2 + \cdots + \xi^{d-1}\sigma^{d-1}$ is a linear combination of operators $L \to L$ viewed as a K-vector space. But in $\mathrm{End}_K\left(\widetilde{K}\right)$ the generators are linearly independent. Therefore, module this statement, we have that

$$\sum_{k=0}^{d-1} \xi^k\sigma^k \neq 0$$

in $\mathrm{End}_K\left(\widetilde{K}\right)$. Hence there exists $\alpha \in \widetilde{K}$ such that $\ell(\alpha,\xi) = \sum_{k=0}^{d-1}\xi^k\sigma^k(\alpha) \neq 0$. But for each $i = 0, \ldots, d-1$, we see that $\sigma^i(\ell(\alpha,\xi)) = \xi^{-i}\ell(\alpha,\xi)$. This implies that

$$\ell(\alpha,\xi),\ \sigma(\ell(\alpha,\xi)),\ \sigma^2(\ell(\alpha,\xi)),\ \ldots,\ \sigma^{d-1}(\ell(\alpha,\xi))$$

are pairwise distinct in $\widetilde{K}$. Therefore, $\ell(\alpha,\xi) \in \widetilde{K}$ does not belong to any proper subfield of $\widetilde{K}$. It follows that $\widetilde{K} = K(\ell(\alpha,\xi))$. But $\sigma(\ell(\alpha,\xi)^d) = \underbrace{\xi^{-1}}_{1}\ell(\alpha,\xi)^d = \ell(\alpha,\xi)^d$. Hence

$$\ell(\alpha,\xi)^d = \widetilde{K}^{\mathrm{Gal}\left(\widetilde{K}/K\right)} = K.$$

This proves our lemma modulo the statement that id, σ, σ^2, $\dots$, σ^{d-1} are linearly independent linear operators.

Note 9.1.3. The σ^i belong to $\mathrm{End}_K\left(\widetilde{K}\right)$ and commute with each other. They can be simultaneously diagonalized over $L \supset K$ the splitting field for $f(x) = \det(\sigma - x \cdot \mathrm{id})$. Writing a linear combination of the σ^i and evaluating it on a basis of eigenvectors will produce a homogenous linear system with a Vandermonde coefficient matrix. Then one needs to show that σ has distinct eigenvalues.

$\square$

$(\Longrightarrow)$
See Section 9.2. $\square$

9.2 Lecture 20

Definition 9.2.1. If G is a group and k a field, then a *k-character of G* is a group homomorphism $\chi : G \to \mathrm{GL}_1(k) = k^\times$.

Each k-character χ of G can be viewed as a function with values in k.

Lemma 9.2.2 (Dedekind). *If $\chi_1, \dots, \chi_s$ are pairwise distinct k-characters of G, then they are linearly independent in* $\mathrm{Fun}(G, k)$.

Proof. We induct on s. If $s = 1$, then χ_1 must be linearly independent since $\chi_1 \neq 0$. Suppose, inductively, that any collection $\sigma_1, \dots, \sigma_t$ of characters with $t \leq s$ is linearly independent. Suppose that $\chi_1, \dots, \chi_s$ are linearly dependent. Then there are $a_1, \dots, a_s \in k$ such that $a_1\chi_1 + \cdots + a_s\chi_s$ is the zero function. By our IH, each a_i must be nonzero, say, a_s. Let $b_i = -\frac{a_i}{a_s}$. Then

$$\sum_{i=1}^{s-1} b_i\chi_i = \chi_s.$$

If $g, h \in G$, then

$$\chi_s(h)\chi_s(g) = \sum_{i=1}^{s-1} b_i\chi_i(h)\chi(g),$$

in which case $\chi_s(g) = \sum_{i=1}^{s-1}(b_i\frac{\chi_i(h)}{\chi_s(h)})\chi_i(g)$. Fix $h \in G$, so that

$$\chi_s = \chi_s = \sum_{i=1}^{s-1}(b_i\frac{\chi_i(h)}{\chi_s(h)})\chi_i$$

and $\chi_s = \sum_{i=1}^{s-1} b_i\chi_i$. It follows that $0 = \sum_{i=1}^{s-1}(b_i\frac{\chi_i(h)}{\chi_s(h)} - b_i)\chi_i$. By our IH, we see that $b_i\frac{\chi_i(h)}{\chi_s(h)} - b_i = 0$. But $b_i \neq 0$ for any i. We have that $\chi_i(h) = \chi_s(h)$ for any $i = 1, \dots, s-1$. This proves that $\chi_i = \chi_s$ for any $i = 1, \dots, s-1$. This contradicts the assumption that the $\chi_1, \dots, \chi_s$ are pairwise distinct. $\square$

Definition 9.2.3. If $K_1, K_2 \subset L$, then the *composite of K_1 and K_2 in L* is the field

$$K_1K_2 = \bigcap \left\{ P \mid \underset{\text{subfield}}{P \subset L}, \ K_1, K_2 \subset P \right\}.$$

Let K_1 and K_2 be finite extensions of k, so that $K_1 = k(a_1, \ldots, a_s)$ and $K_2 = k(b_1, \ldots, b_t)$. Then the field $k(a_1, \ldots, a_s, b_1, \ldots, b_t)$ both contains K_1K_2 and is contained in some L such that $K_1, K_2 \subset L$. Hence

$$K_1K_2 = k(a_1, \ldots, a_s, b_1, \ldots, b_t).$$

Lemma 9.2.4. *Suppose that K and F are two finite field extensions of k. Then*

(a) If $K \supset k$ is Galois, then so is $KF \supset F$.

(b) $\mathrm{Gal}(KF/F) = \mathrm{Gal}(K/K \cap F)$.

Proof.

(a) If $K \supset k$ is Galois, then K is the splitting field of some separable polynomial $f(x) \in k[x]$. Thus, KF is the splitting field of $f(x)$ viewed over F. But if f is separable over k, then it is separable over F. Therefore, $KF \supset F$ is Galois.

(b) Consider the tower of extensions $k \subset K \subset KF$. The main theorem of Galois theory says that $\mathrm{Gal}(K/k) \trianglelefteq \mathrm{Gal}(KF/k)$ since $K \supset k$ is assumed to be Galois. Thus, if $\sigma \in \mathrm{Gal}(KF/k)$, then $\sigma(K) \subset K$. Indeed, $\sigma(K) = K$ as a subfield in KF if and only if $\sigma(K) = (KF)^{\mathrm{Gal}(KF/K)}$. Let $g \in \mathrm{Gal}(KF/K) \subset \mathrm{Gal}(KF/k)$. Then $g(x) = x$ for any $x \in K$.

Let $x \in K$. Consider $\sigma(x) \in KF$. We must show that $g(\sigma(x)) = \sigma(x)$ for any $g \in \mathrm{Gal}(KF/K)$, i.e., $\left(\sigma^{-1}g\sigma\right)(x) = x$ for any g. But since $\sigma^{-1}g\sigma \in \sigma^{-1}\mathrm{Gal}(KF/K)\sigma$, we see that $\sigma^{-1}g\sigma(x) = x$ for any $x \in K$. Hence we get a natural homomorphism $\rho : \mathrm{Gal}(KF/F) \to \mathrm{Gal}(K/k)$ given by $\sigma \mapsto \sigma \restriction_K$. Note that

$$\begin{aligned} \ker \rho &= \{\sigma \in \mathrm{Gal}(KF/F) \mid \sigma \restriction_K = \mathrm{id}_K\} \\ &= \{\sigma \in \mathrm{Gal}(KF/k) \mid \sigma \restriction_K = \mathrm{id}_K, \ \sigma \restriction_F = \mathrm{id}_F\}. \end{aligned}$$

But KF is generated by K and F, so that $\sigma = \mathrm{id}_{KF}$. This shows that $\ker \rho = \{\mathrm{id}_{KF}\}$. We see that $\mathrm{Gal}(KF/F) \subset \mathrm{Gal}(K/k)$.

Let $H \coloneqq \mathrm{im}\, \rho \subset \mathrm{Gal}(K/k)$ and consider its fixed subfield K^H. Note that (b) is equivalent to saying that $K^H - K \cap F$. We have that $K^H \supset K \cap F$ because $K^H = \{x \in K \mid (\forall \sigma \in \mathrm{Gal}(KF/F))\, (\sigma(x) = x)\}$. Moreover, if we view K^H as subfield of KF, then $k \subset K^H \subset KF$ and $k \subset F \subset KF$. Since $\mathrm{Gal}(KF/F)$ fixes K^H and F (pointwise), it follows that $\mathrm{Gal}(KF/F)$ fixes K^HF. Therefore, $K^HF \subset KF^{\mathrm{Gal}(KF/F)} = F$, so that $K^H \subset H$. This proves that $K^H \subset F \cap K$.

$\square$

Corollary 9.2.5. *If both K and F are Galois field extensions of k, then*

$$\begin{array}{ccc} & KF & \\ \diagup & & \diagdown \\ K & & F \\ \diagdown & & \diagup \\ & K \cap F & \\ & | & \\ & k & \end{array}$$

is a lattice of Galois field extensions.

Theorem 9.2.6. *If K is a field with characteristic 0, $f(x) \in K[x]$ is irreducible, and $L \supset K$ is the splitting field for f, then f is solvable in radicals over K if and only if $\mathrm{Gal}(L/K)$ is solvable.*

Proof.

($\Longleftarrow$)

This was proven in Section 9.1.

($\Longrightarrow$)

For any root α of f, we can find an extension $K_\alpha \supset K$ such that $\alpha \in L_\alpha \subset L$ and there exists a tower of radical extensions

$$K = K_0 \subset K_1 \subset \cdots \subset K_s = K_\alpha \subset L$$

with $K_{i+1} = K_i(\alpha_{i+1})$ and $\alpha_{i+1}^{n_{i+1}} \in L_i$.

Claim. *Without loss of generality, we may assume that K_α satisfies the following properties.*

- *$\alpha \in K_\alpha$.*
- *$K_\alpha \supset K$ is Galois.*
- *Each step of K_α (viewed as our tower of radical extensions) is Galois and cyclic.*

Proof. Since $K_\alpha \supset K$ is a finite extension, we can find a K-basis $e_1, \ldots, e_n$ of K_α. Let $f_i \in K[x]$ denote the minimal polynomial of e_i. Let S_i denote the splitting field of f_i. Then $S_i \supset K$ is a Galois extension and contains e_i. Note that the composite of the S_i contains each e_i. Let $L_\alpha = S_1 S_2 \cdots S_n$. Then $K \subset K_\alpha \subset L_\alpha$. (We call L_α the *Galois closure of* K_α.) Consider the tower $K = K_0 \subset K_1 \subset \cdots \subset K_s = K_\alpha$ where $K_{i+1} \supset K_i$ is a radical extension of degree n_i. If $\sigma \in \mathrm{Gal}(L_\alpha/K)$, then $K = \sigma K \subset \sigma K_1 \subset \cdots \subset \sigma K_\alpha$ is still a tower of radical extensions.

By taking the composites $K_1 \sigma K_1 \subset \cdots \subset K_1 \sigma K_s$ and $K_2 K_1 \sigma K_1 \cdots$, we get a composite of all $\{\sigma K \alpha\}_{\sigma \in \mathrm{Gal}(L_\alpha/K)}$, which will be a tower of radical extensions. But $K \subset \prod_\sigma \sigma K_\alpha \subset L_\alpha$, and L_α is generated by all σK_α. Hence $L_\alpha = \prod_\sigma \sigma K_\alpha = L$.

We still must prove that each step in our radical tower is Galois and cyclic. Let $n = n_1 n_2 \cdots n_k$. Let $F = K[\mu_n]$. If the tower $K = K_0 \subset K_1 \subset \cdots \subset K_t = L_\alpha$ has $K_i = K_{i-1}\left[\sqrt[n_i]{a_i}\right]$, then we can pass to composites

$$K \subset K_0 F \subset K_1 F \subset \cdots \subset K_t F = L_\alpha F.$$

We see that $LF \supset K$ is radical and Galois as the splitting field for $x^n - 1$ and that $K_iF \supset K_{i+1}F$ is radical of degree n_i and contains μ_{n_i}. Thus, $K_iF \supset K_{i+1}F$ is Galois and cyclic of degree dividing n_i by Lemma 8.3.3(a).

We have constructed an extension $LF \supset K$ such that

- $\alpha \in LF$,
- $LF \supset L$ is Galois, and
- LF is a tower of radical, cyclic, Galois extensions.

It follows that $\mathrm{Gal}(LF/K)$ is solvable. But $LF \supset L \supset K$ where $L \supset K$ is Galois. Hence $\sigma(L) \subset L$ for any $\sigma \in \mathrm{Gal}(LF/K)$, so that $\mathrm{Gal}(L/K) < \mathrm{Gal}(LF/K)$. This proves that $\mathrm{Gal}(L/K)$ is solvable. □

□

9.3 Lecture 21

Definition 9.3.1. Let K be a field and $f(x) \in K[x]$. We say that f is *solvable in quadratic radicals* if the splitting field L for f is a tower

$$K = K_0 \subset K_1 \subset \cdots \subset K_s = L$$

such that $K_i = K_{i-1}\left[\sqrt{a_i}\right]$ for some $a_i \in K_{i-1}$.

Theorem 9.3.2. *Let K be a field with $\mathsf{char}\, K \neq 2$ and $f(x) \in K[x]$ be irreducible. Then f is solvable in quadratic radicals if and only if $[L : K] = 2^n$ for some n where L denotes the splitting field for f.*

Proof.

($\Longrightarrow$)

We have that $L \supset K$ is a tower of quadratic extensions. Hence $[L : K] = 2^n$ for some n.

($\Longleftarrow$)

We have that $[L : K] = 2^n$ for some $n \geq 0$ and $\deg f = [K(\alpha) : K] \mid [L : K]$ where α is a root of $f(x)$. Thus, $[K(\alpha) : K]$ equals a power of 2, so that f is separable. This shows that $L \supset K$ is Galois and thus that $G \coloneqq \mathrm{Gal}(L/K)$ has order 2^n. It follows that there is some normal series

$$G = G^0 \trianglerighteq G^1 \trianglerighteq \cdots \trianglerighteq G^s = \{e\}$$

such that ${G^i}/{G^{i+1}} \cong \mathbb{Z}/2$. This induces a tower of field extensions

$$K = L^{G^0} \subset L^{G_1} \subset \cdots \subset L^{G_s} = L$$

such that $[L^{G^{i+1}} : L^{G_i}] = 2$. □

Note 9.3.3 (The construction problem). Given a unit measure and segments of lengths $a_1, \dots, a_k$, we want to construct a segment of length α using ruler and compass. Elementary geometry shows that such a construction is possible if and only if α can be expressed in quadratic radicals over $\mathbb{Q}(a_1, \dots, a_k)$. If α is transcendental over $\mathbb{Q}(a_1, \dots, a_k)$, then our construction is impossible.

Example 9.3.4. We see that π cannot be constructed over $\mathbb{Q}$, i.e., we cannot square the circle.

Moreover, if α is algebraic over $\mathbb{Q}(a_1, \dots, a_k)$, then α can be constructed by Theorem 9.3.2 if and only if the minimal polynomial of α has degree power of 2.

Example 9.3.5.

(a) Doubling the cube. Given a segment of length one, construct a segment of length $\sqrt[3]{2}$. Since the minimal polynomial of $\sqrt[3]{2}$ is $x^3 - 2$, such a construction is impossible.

(b) Trisecting an angle φ. Given a segment of length $\cos\varphi$, construct a segment of length $\cos\left(\frac{\varphi}{3}\right)$. The minimal polynomial of $\cos\left(\frac{\varphi}{3}\right)$ over $\mathbb{Q}(\cos\varphi)$ is $4x^3 - 3x - \cos\varphi$. In general, this is irreducible, in which case our construction is impossible.

(c) Constructing regular n-gons. Given a segment of length i, construct a segment of length $\cos\left(\frac{2\pi}{n}\right)$. This is possible if and only if $e^{\frac{2\pi i}{n}}$ is expressible in quadratic radicals over $\mathbb{Q}$. In turn, this happens if and only if

$$\underbrace{[\Gamma_n : \mathbb{Q}]}_{\varphi(n)} = 2^s.$$

For example, if p is prime, then we can construct a regular p-gon if and only if $1 + 2^k$ for some k. Currently, the largest known such p is 65,537.

10 Further applications of Galois theory

10.1 Lecture 22

To begin, note that the following statements are true.

- If $f(x) \in \mathbb{R}[x]$ has odd degree, then it has a real root.
- Every $\alpha \in \mathbb{C}$ has a square root in $\mathbb{C}$.

Now, suppose that $K \supsetneq \mathbb{R}$ is a finite field extension. If $[K : \mathbb{R}]$ is odd and $\alpha \in K \setminus \mathbb{R}$, then $K \supset \mathbb{R}(\alpha) \supset \mathbb{R}$, in which case $\deg f \mid [K : \mathbb{R}]$ where f denotes the minimal polynomial of α over $\mathbb{R}$. In this case, f has odd degree and thus has a root in $\mathbb{R}$, so that $\mathbb{R}(\alpha) = \mathbb{R}$, a contradiction. This proves that $[K : \mathbb{R}]$ is odd.

We want to prove the *fundamental theorem of algebra*: that any $f(x) \in \mathbb{C}[x]$ has a root in $\mathbb{C}$. Note that if c is a complex root of $f(x)\overline{f(x)} \in \mathbb{R}[x]$, then either c or $\bar{c}$ is a root of $f(x)$. Thus, it suffices to show that any polynomial over $\mathbb{R}$ has a root in $\mathbb{C}$.

To this end, let $g(x) \in \mathbb{R}[x]$ be non-constant and irreducible. Let L denote the splitting field for g. Then $[L : K] = |\mathrm{Gal}(L/\mathbb{R})|$ is even, so that there is some nontrivial 2-Sylow subgroup $H \leq \mathrm{Gal}(L/\mathbb{R})$. This means that the intermediate extension $L \supset L^H \supset \mathbb{R}$ has odd degree. But then $L^H = \mathbb{R}$. This means that $L \supset L^H$ is Galois, so that

$$[L : \mathbb{R}] = \left[L : L^H\right] = \left|\mathrm{Gal}(L/L^H)\right| = |H| = 2^n$$

for some n. By Theorem 9.3.2, it follows that $g(x)$ is solvable in quadratic radicals. Therefore, $L = \mathbb{C}$ since $[\mathbb{C} : \mathbb{R}] = 2$.

Theorem 10.1.1 (Primitive element theorem). *Suppose that $L \supset K$ is a finite field extension. This has a primitive element, i.e., $L = K(\theta)$ for some $\theta \in L_j$, if and only if there are at most finitely many intermediate fields $K \subset F \subset L$.*

Proof. If K is finite, then L is a finite group with cyclic multiplicative group $\langle\theta\rangle$. In this case, we have shown that $L = K(\theta)$.

($\Longleftarrow$)
For any $\alpha, \beta \in L$, consider the collection of intermediate fields

$$K \subset K(\alpha + c\beta) \subset L$$

where $c \in K$. Thus, $\exists c, c' \in K$ such that $E \coloneqq K(\alpha + c\beta) = K(\alpha + c'\beta)$. Hence $(c - c')\,\beta \in E$, and $c - c' \in K \backslash \{0\}$. Then $\beta \in E$, so that $\alpha \in E$. This shows that $E \supset K(\alpha, \beta)$. It's clear that $E \subset K(\alpha, \beta)$. Hence $E = K(\alpha, \beta)$. But $L \supset K$ is a finite extension, which implies that $L = K(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_1, \ldots, \alpha_n$. By indiction on n, we can find elements $c_2, \ldots, c_n \in K$ such that

$$K(\alpha_1, \ldots, \alpha_n) = K(\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \cdots + c_n\alpha_n).$$

($\Longrightarrow$)
We have that $L = K(\theta)$. Let $f(x) \in K[x]$ denote the minimal polynomial of θ over K. Let $K \subset F \subset L$ be an intermediate field extension. Let $g_F(x) \in F[x]$ denote the minimal polynomial over F. This proves that $g_F(x) \mid f(x)$ in $F[x]$. We get a map

$$(\text{intermediate field extensions } K \subset F \subset L) \to (\text{divisors of } f(x))$$

given by $F \mapsto g_F(x)$. Since there are at most finitely many divisors of $f(x)$, it suffices to check that this map is injective.

Suppose that $K \subset F \subset L$. Let $F_0 \subset F$ be the subfield obtained from K by adjoining the coefficients of $g_F(x)$. It is enough to show that $F_0 = F$. Note that $g_F(x)$ is irreducible in $F[x]$, so that $g_F(x)$ is irreducible in $F_0[x]$ Therefore, $g_F(x) \in F_0[x]$, which means that $g_F(x)$ is the minimal polynomial of θ over F_0. Then $[L : F_0] = \deg g_F = [L : F]$, so that $F_0 = F$. □

Corollary 10.1.2. *If $L \supset K$ is a (finite) separable extension, then L has a primitive element.*

Proof. It suffices to show that if $\alpha, \beta \in L$ are separable over K, then $K(\alpha, \beta) = K(\theta)$ for some θ. If K is finite, then we're done. Suppose that K is infinite. Let $\varphi_1, \dots, \varphi_n$ denote the distinct embeddings of $K(\alpha, \beta)$ in $\overline{K}$ over K. Consider

$$f(x) = \prod_{i \neq j} (\varphi_i(\alpha) + x\varphi_i(\beta) - \varphi_j(\alpha) - x\varphi_j(\beta)).$$

Since this is not the zero polynomial, there is some $c \in K$ such that $f(c) \neq 0$. It follows that the $\varphi_i(\alpha + c\beta)$ are pairwise distinct in $\overline{K}$. Then $[K(\alpha + c\beta) : K] \geq n$. But $[K(\alpha, \beta) : K] = n$, so that $K(\alpha, \beta) = K(\alpha + c\beta)$. □

Let K be a field and $f(x) \in K[x]$ be a monic separable polynomial. Let L denote the splitting field of f, so that $L \supset K$ is Galois. Let $G_f \coloneqq \mathrm{Gal}(L/K) \subset S_n$ where $n = \deg f$. Let $\mathsf{char}\, K \neq 2$.

Theorem 10.1.3. *$L^{G_f \cap A_n} = K(\Delta(f))$ where $\Delta(f) = \prod_{i<j} (\lambda_i - \lambda_j)$ and $\lambda_1, \dots, \lambda_n$ denote the distinct roots of f.*

Before proving this, note that $\Delta(f)$ is a square root of $\mathrm{Discr}(f) \in K$.

Proof. Consider $x_1, \dots, x_n$ purely transcendental elements over K. Let $K(x_1, \dots, x_n) \supset K$ be the corresponding extension. There is a group homomorphism $\Phi : S_n \to \mathrm{Gal}(K(x_1, \dots, x_n)/K)$ given by $\sigma \mapsto \Phi_\sigma$ where

$$\Phi_\sigma(f)(x_1, \dots, x_n) = f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

This is injective, and $K(x_1, \dots, x_n)^{S_n} = K(\sigma_1, \dots, \sigma_n)$ where $\sigma_1, \dots, \sigma_n \in K[x_1, \dots, x_n]$ are the alternating symmetric polynomials. Further, $\mathrm{Gal}(K(x_1, \dots, x_n)/K(\sigma_1, \dots, \sigma_n)) = S_n$. Let $\Delta_n = \prod_{i<j} (x_i - x_j) \in K(x_1, \dots, x_n)$. Then $\Phi_\sigma(\Delta_n) = \mathrm{sgn}(\sigma)\Delta_n$, and $\Delta_n \notin K(\sigma_1, \dots, \sigma_n)$.

Define $\mathrm{ev} : K(x_1, \dots, x_n) \to L$ by $x_i \mapsto \lambda_i$. Then $\mathrm{ev} \circ \Phi_\sigma = \sigma^{-1} \circ \mathrm{ev}$. Thus,

$$\mathrm{ev}(\Delta(f)) = \mathrm{ev}(\Phi_\sigma(\Delta_n)) = \sigma^{-1}(\Delta(f)).$$

This shows that the subgroup in G_f fixing $\Delta(f)$ is precisely $G_f \cap A_n$. □

Corollary 10.1.4. *If $\mathsf{char}\, K \neq 2$ and $f(x)$ is monic and separable over K, then $G_f \subset A_n$ if and only if $\mathrm{Discr}(f) \in K^2$.*

10.2 Lecture 23

Theorem 10.2.1. *Suppose that K is a field and $f(x) \in K[x]$ is separable. Then f is irreducible if and only if the Galois group G_f acts transitively on the set of roots of f.*

Proof.

($\Longrightarrow$)

For any two roots λ_i, λ_j of f, we have that $K(\lambda_I) \cong K(\lambda_j)$ as fields over K because both $\mathrm{ev}_{\lambda_i} : K[x] \to K(\lambda_i)$ and $\mathrm{ev}_{\lambda_j} : K[x] \to K(\lambda_j)$ induces isomorphisms with ${K[x]}/{(f)}$. By Lemma 4.2.8, we can extend this isomorphisms to an automorphism $\sigma : L \to L$ of the splitting field L for f. Thus, $\sigma \in \mathrm{Gal}(L/K)$ with $\sigma(\lambda_i) = \lambda_j$.

($\Longleftarrow$)

Let $\{\lambda_1, \ldots, \lambda_n\}$ denote the set of roots of f. Let $f(x) = g(x)h(x)$ where $\deg g \geq 1$ and g is irreducible. We must show that h is constant. Let λ be any root of g. Then there exists $\sigma_i \in G_f$ such that $\sigma_i(\lambda) = \lambda_i$ for each $i = 1, \ldots, n$. Note that

$$g(\lambda_i) = g(\sigma_i(\lambda)) = \sigma_i(g(\lambda)) = 0,$$

so that each λ_i is a root of g. Hence $f \mid g$, which implies that h is constant. $\square$

Theorem 10.2.2. *Suppose that p is prime and that $f(x) \in \mathbb{Q}[x]$ is monic and irreducible with* $\deg f = p$. *Suppose that f has exactly two non-real roots in $\mathbb{C}$. Then $G_f = S_p$.*

Proof. Let L be the splitting field for $f(x)$. Write $f(x) = \prod_{i=1}^{p}(x - x_i)$ with each $\lambda_i \in \mathbb{C}$. Then $\mathbb{Q}(\lambda_1, \ldots, \lambda_p) \subset \mathbb{C}$. We see that

$$\mathbb{Q} \subset \mathbb{Q}(\lambda_i) \subset \mathbb{Q}(\lambda_1, \ldots, \lambda_p) \subset \mathbb{C},$$

so that $[\mathbb{Q}(\lambda_i) : \mathbb{Q}] \mid [L : \mathbb{Q}]$. Since $p \mid [L : \mathbb{Q}] = |G_f| \subset S_p$, it follows from Sylow that G_p contains an element of order p, i.e., that G_f contains a p-cycle. Also, the element in G_f that switches the roots is the complex conjugate pair of a transposition. $\square$

Theorem 10.2.3 (Brouwer). *For any prime $p \geq 5$, there are infinitely many polynomials in $\mathbb{Q}[x]$ of degree p with Galois group S_p.*

Proof. Let k be an odd integer and let $0 \leq m, n_1 \leq n_2 < \cdots < n_{k-2}$ be even integers. Consider

$$g(x) = \left(x^2 + m\right)(x - n_1)(x - n_2) \cdots (x - n_{k-2}).$$

This polynomial has $\frac{k-3}{2}$ local maxima. Also, for each odd $h \in \mathbb{Z}$, $|g(h)| > 2$. Hence if c denotes a local maximum of g, then $g(c) > 2$. This shows that if $f(x) = g(x) - 2$, then there are

- $\frac{k-3}{2}$ positive local maxima in $[n_1, n_{k-2}]$ and
- $\frac{k-3}{2}$ negative local maxima in $[n_1, n_{k-2}]$.

It follows that $f(x)$ has k-3 real roots in $[n_1, n_{k-2}]$ with $f(n_{k-2}) = -2$ and $\lim_{x\to\infty} f(x) > 0$. Therefore, we have another real roots $> n_{k-2}$. Hence $f(x)$ has at least $k-2$ real roots. Let $\lambda, \ldots, \lambda_n \in \mathbb{C}$ denote the distinct roots of f. Then

$$\prod_{i=1}^{k}(x - \lambda_i) = f(x) = \left(x^2 + m\right)(x - n_1)(x - n_2) \cdots (x - n_{k-2}) - 2$$

, and $-\sum_{i=1}^{k}\lambda_i = -\sum_{i=1}^{k-2} n_i$. From this, we compute

$$\sum_{i<j}\lambda_i\lambda_j = m + \sum_{a<b} n_a n_b$$

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i^2 &= \left(\sum_{i=1}^{k}\lambda_i\right)^2 = \sum\sum_{i<j}\lambda_i\lambda_j \\ &= \left(\sum_{i=1}^{k-2} n_i\right)^2 - 2m - 2\left(\sum_{a<b} n_a n_b\right) \\ &= \sum_{i=1}^{k-2} n_i^2 - 2m.\end{aligned}$$

Choose $m \gg \sum n_i^2$ so that $\sum_{i=1}^{k}\lambda_i^2 < 0$. This implies that there exists a non-real root. Hence we must have exactly two real roots. Further, we can write $f(x) = x^k + a_1x^{k-1} + \cdots + a_{k-1}x + a_k$ with each $a_i \in 2\mathbb{Z}$. Since $a_k = f(0) = g(0) - 2$, we see that $2 \mid a_{k-1}$ but $4 \nmid a_{k-1}$. By Eisenstein's criterion, f must be irreducible. We thus get infinitely many f's such that $G_f = S_p$. □

11 Chain complexes and chain maps

The originators of homological algebra include Betti, Poincaré, and Riemann. The main goal of this subject is to extract invariants from topological spaces. Decompose X into contractible pieces (such as cells or simplices) to reduce X to combinatorial data. Specifically, reduce X to a collection of pieces of various dimensions where the boundary of a piece of dimension n is glued to a sub-collection of pieces of dimension $n-1$.

Emmy Noether introduced groups of chains $C_i(X)$, a free abelian group generated by the collection of i-dimensional pieces, equipped with boundary relations $\partial_i : C_i(X) \to C_{i-1}(X)$. From this, we obtain abelian groups $H_i(X) \equiv {}^{\ker\partial_i}\!/\!_{\operatorname{im}\partial_{i+1}}$, which are algebraic invariants of X.

Hilbert wanted to extract numerical invariants from a module. Specifically, if k is a field and $K \coloneqq k[x_1, \ldots, x_n]$, then he wanted to understand the complexity of a module over K (or, more generally, any graded module over k).

A typical graded module over R will be a module of the form MR/I where $I \trianglelefteq R$ is a homogeneous ideal. By the Hilbert basis theorem, $I \trianglelefteq R$ is generated by finitely many homogenous polynomials $f_1, f_2, \ldots, f_{r_0}$. Thus, we have surjective map $\psi : R^{\oplus r_o} \to I$ given by $(a_1, \ldots, a_{r_0}) \mapsto \sum a_i f_i$. But, there generators are not, in general, independent. Therefore, we consider the module of relations $Z_0(I) \equiv \ker\psi$ among the f_i. Note that $Z_0(I)$ is finitely generated. We can choose generators and get a map $\psi' : R^{\oplus r_1} \twoheadrightarrow Z_0(I)$. Then

$$R^{\oplus r_1} \to R^{\oplus r_0} \to I \to 0$$

is an exact sequence of graded R-modules. If $Z_1(I) \equiv \ker\psi'$ is not zero, then choose generators again to get a map $\psi'' : R^{\oplus r_2} \twoheadrightarrow Z_1(I)$. Continuing in this way, we get an exact sequence

$$\cdots \to R^{\oplus r_2} \to R^{\oplus r_1} \to R^{\oplus r_0} \to I \to 0.$$

The *length* of this sequence is defined to be $\max\{i \mid r_i \neq 0\}$. This is an invariant of I and of R/I.

Theorem 11.0.1 (Hilbert's syzygy theorem). *Hilbert's syzygy theorem states that $Z_{n-1}(I)$ is free, i.e., that there is an exact sequence of graded R-modules*

$$0 \to R^{\oplus r_n} \to R^{\oplus r_{n-1}} \to \cdots \to R^{\oplus r_0} \to I \to 0.$$

11.1 Lecture 24

Definition 11.1.1.

1. A *chain complex* (in $\mathbf{Ab}$) is a pair $(M_\bullet, \partial_\bullet)$ where $M_\bullet = \{M_i\}_{i\in\mathbb{Z}}$ is a set of abelian groups and $\partial_\bullet = \{\partial_i\}_{i\in\mathbb{Z}}$ is a set of morphisms in $\mathbf{Ab}$ such that the *i-th differential* $\partial_i : M_i \to M_{i-1}$ satisfies $\partial_{i-1} \circ \partial_i = 0$.

   We call $Z_n \equiv \ker \partial_n$ *the group of degree n cycles* and $B_n \equiv \operatorname{im} \partial_{n+1}$ *the group of degree n boundaries.* Finally, we call $H_n \equiv Z_n/B_n$ the *degree n homology group.*

2. A *((co)chain) complex* (in $\mathbf{Ab}$) is a pair $(M^\bullet, d^\bullet)$ where $M^\bullet = \left\{M^i\right\}_{i\in\mathbb{Z}}$ is a set of abelian groups and $d^\bullet = \left\{d^i\right\}_{i\in\mathbb{Z}}$ is a set of morphisms in $\mathbf{Ab}$ such that the *i-th differential* $d^i : M^i \to M^{i+1}$ satisfies $d^{i+1} \circ d^i = 0$.

   We call $Z^n \equiv \ker d^n$ *the group of degree n cocycles* and $B^n \equiv \operatorname{im} d^{n-1}$ *the group of degree n coboundaries.*

   Finally, we call $H^n \equiv Z^n/B^n$ the *degree n cohomology group.*

Definition 11.1.2. Let $\left(A^\bullet, d_A^i\right)$ and $\left(B^\bullet, d_B^i\right)$ be complexes. A *chain map* $f^\bullet : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ consists of group homomorphisms $f^i : A^i \to B^i$ for each $i \in \mathbb{Z}$ such that $d_B^i \circ f^i = f^{i+1} \circ d_A^i$.

Note 11.1.3.

1. Any chain map $f^\bullet : (A^\bullet, d_A^\bullet) \to (B^\bullet, d_B^\bullet)$ restricts term-wise to maps $f^i : Z^i(A^\bullet) \to Z^i(B^\bullet)$ and maps $f^i : B^i(A^\bullet) \to B^i(B^\bullet)$. Thus, it induces a map $f^* : H^i(A^\bullet) \to H^i(B^\bullet)$.

2. We have a natural isomorphism $\mathbf{Ch}(\mathbf{Ab}) \to \mathbf{CoCh}(\mathbf{Ab})$ given by $N_i \mapsto M^{-i}$ and $\partial_i \mapsto d^{-i}$.

Definition 11.1.4. We say that $(A^\bullet, d^\bullet)$ is *bounded above* if there is some N such that $A^n = 0$ for any $n \geq N$. We define *bounded below* similarly. We say that $(A^\bullet, d^\bullet)$ is *bounded* if it is both bounded above and bounded below.

As a result, we have the subcategories $\mathbf{CoCh}^-(\mathbf{Ab})$, $\mathbf{CoCh}^+(\mathbf{Ab})$, and $\mathbf{CoCh}^b(\mathbf{Ab})$, respectively.

If $C^\bullet = \bigoplus_{i\in\mathbb{Z}} C^i$ is a graded abelian group, then it induces a natural complex $(\underline{C}^\bullet, 0)$ where $\underline{C}^i \equiv C^i$. In particular, any abelian group may be viewed as a complex.

Conversely, given a complex $(M^\bullet, d^\bullet)$, we can form the graded abelian group $M^\bullet \equiv \bigoplus_{i\in\mathbb{Z}} M^i$ and package the differential d^i into a single group map $D : M^\bullet \to M^\bullet$ such that $D \restriction_{M^i} = d^i$ and $D^2 = 0$.

We can write D as the block diagonal matrix

$$\begin{bmatrix} 0 & & & & \\ d^i & 0 & & & \\ & d^{i+1} & 0 & & \\ & & d^{i+2} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

As a result, we obtain the *cochain functor* given by $(A^\bullet, d^\bullet) \to \bigoplus_{i\in\mathbb{Z}} A^i$ and $f^\bullet \mapsto (f^i)_{i\in\mathbb{Z}}$.

Definition 11.1.5. We say that $(A^\bullet, d^\bullet)$ is *acyclic* or *exact* if $H^\bullet(A^\bullet, d^\bullet) = 0$.

Theorem 11.1.6. *Let $K^\bullet$ be an exact complex of R-modules and $I^\bullet$ a bounded below complex of injective R-modules. Any chain map $f : K^\bullet \to I^\bullet$ is homotopic to zero.*

Proof. By hypothesis, there is some $r \in \mathbb{Z}$ such that $I^k = 0$ for any $k < r$. Then $f^k = 0$ for any $k < r$. Define $h^k : K^k \to I^{k-1}$ by $h^k = 0$ for each $k \leq r$. Then $f^k = 0 = d_I h^k + h^{k+1} d_K$ for any $k < r$. Let $s > r$ and assume, for induction, that, for each $k < s$, we have constructed a map $h^k : K^k \to I^{k-1}$ such that $f^{k-1} = d_I h^{k-1} + h^k d_K$. We must construct a map $h^s : K^s \to I^{s-1}$ such that $f^{s-1} = d_I h^{s-1} + h^s d_K$.

Let $g^{s-1} = f^{s-1} - d_I h^{s-1}$. Note that

$$\begin{aligned} g^{s-1} d_K &= \left(f^{s-1} - d_I h^{s-1}\right) d_K \\ &= f^{s-1} d_K - d_I h^{s-1} d_K \\ &= d_I f^{s-2} - d_I \left(f^{s-2} - d_I h^{s-2}\right) \\ &= 0. \end{aligned}$$

Therefore, g^{s-1} descends to a map $g^{s-1} : K^{s-1} / \operatorname{im} d_K \to I^{s-1}$. Since $K^\bullet$ is exact, we have

$$g^{s-1} : K^{s-1} / \ker d_K \to I^{s-1}.$$

Moreover, since I^{s-1} is injective, we can find some map $h^s : K^s \to I^{s-1}$ such that

$$\begin{tikzcd} I^{s-1} & & \\ K^{s-1}/\ker d_K \arrow[u, "g^{s-1}"] \arrow[r, "\cong"'] & \operatorname{im} d_K \arrow[r, hook] & K^s \arrow[llu, dashed, "h^s"'] \end{tikzcd}$$

commutes. Hence $h^s d_K = g^{s-1}$. It follows that

$$\begin{aligned} d_I h^{s-1} + h^s d_K &= d_I h^{s-1} + g^{s-1} \\ &= d_I h^{s-1} + f^{s-1} - d_I h^{s-1} \\ &= f^{s-1}, \end{aligned}$$

as desired $\square$

Definition 11.1.7. If A is an abelian group, then a *left resolution of* A is an exact complex $(C^\bullet, d^\bullet) \in \operatorname{ob}\mathbf{CoCh}^{\leq 0}(\mathbf{Ab})$ of the form

$$\cdots \to C^{i-1} \to C^i \to \cdots \to C^0 \to A \to 0.$$

Example 11.1.8. If $I \trianglelefteq k[x_1, \ldots, x_n]$ is a homogenous ideal, then Hilbert's syzygy theorem says that I has a left resolution of length $n+1$ with $n+1$ terms free finitely generated R-modules.

Let $a \in \mathbb{Z}$. Define the *shift functor*

$$-[a] : \mathbf{CoCh}(\mathbf{Ab}) \to \mathbf{CoCh}(\mathbf{Ab})$$

as follows. Let $(M^\bullet, d_M^\bullet)$ be a complex. Form the pair $\left(M^\bullet[a], d^\bullet_{M[a]}\right)$ where $(M^\bullet[a])^n \equiv M^{a+n}$ and $\left(d_{M[a]}\right)^n \equiv (-1)^a d_M^{a+n}$. If $f^\bullet$ is a chain map, then let $(f^\bullet[a])^n = f^{a+n}$.

Proposition 11.1.9. *The shift functor is an equivalence that preserves* $\mathbf{CoCh}^-(\mathbf{Ab})$, $\mathbf{CoCh}^+(\mathbf{Ab})$, *and* $\mathbf{CoCh}^b(\mathbf{Ab})$.

Definition 11.1.10. Let $f : M \to N$ be a chain map. Form $\operatorname{cone}(f)$ the *cone of* f as a new complex where $\operatorname{cone}(f)^\bullet \equiv N \oplus M[1]$ and $d^\bullet_{\operatorname{cone}(f)} \equiv \begin{bmatrix} d_N & f \\ 0 & d_{M[1]} \end{bmatrix}$.

We see that

$$\operatorname{cone}(f)^n = N^n \oplus M^{n+1}$$

and $d^n_{\operatorname{cone}(f)} : N^n \oplus M^{n+1} \to N^{n+1} \oplus M^{n+2}$ with

$$d^n_{\operatorname{cone}(f)} = \begin{bmatrix} d_N^n & f^{n+1} \\ 0 & -d_M^{n+1} \end{bmatrix}.$$

Exercise 11.1.11. *Show that* $d^{i+1}_{\operatorname{cone}(f)} \circ d^i_{\operatorname{cone}(f)} = 0$.

Definition 11.1.12.

1. A *double complex* is a triple $(A^{\bullet,\bullet}, d^\bullet, \delta^\bullet)$ where $A^{i,j} = \left\{A^{i,j}\right\}_{(i,j)\in\mathbb{Z}^2}$ and both $d : A^{\bullet,\bullet} \to A^{\bullet+1,\bullet}$ and $\delta : A^{\bullet,\bullet} \to A^{\bullet,\bullet+1}$ are homomorphisms such that $d\delta = \delta d$ and $d^2 = \delta^2 = 0$. As a commutative diagram, this has the form

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
\cdots & \xrightarrow{\delta} & A^{p,q} & \xrightarrow{\delta} & A^{p,q+1} & \xrightarrow{\delta} & A^{p,q+2} & \xrightarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
\cdots & \xrightarrow{\delta} & A^{p+1,q} & \xrightarrow{\delta} & A^{p+1,q+1} & \xrightarrow{\delta} & A^{p+1,q+2} & \xrightarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
\cdots & \xrightarrow{\delta} & A^{p+2,q} & \xrightarrow{\delta} & A^{p+2,q+1} & \xrightarrow{\delta} & A^{p+2,q+2} & \xrightarrow{\delta} & \cdots \\
 & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}.$$

2. The *total complex of* $(A^{\bullet,\bullet}, d^\bullet, \delta^\bullet)$ is the complex $\mathrm{Tot}(A)$ where $\mathrm{Tot}(A)^n \equiv \bigoplus_{p+q=n} A^{p,q}$ and $d_{\mathrm{Tot}(A)} \restriction_{A^{p,q}} \equiv d + (-1)^p \delta$.

Proposition 11.1.13. *Any chain map $f : M \to N$ induces a double complex*

$$\begin{array}{ccccccccc} M^{i-1,0} & \xrightarrow{d_M} & M^{i,0} & \xrightarrow{d_M} & M^{i+1,0} & \xrightarrow{d_M} & M^{i+2,0} & \xrightarrow{d_M} & \cdots \\ \downarrow f & & \downarrow f & & \downarrow f & & \downarrow f & & \\ N^{i-1,1} & \xrightarrow{d_N} & N^{i,1} & \xrightarrow{d_N} & N^{i+1,1} & \xrightarrow{d_N} & N^{i+2,1} & \xrightarrow{d_N} & \cdots \end{array}.$$

The total complex of this is precisely $\mathrm{cone}(f)$.

Let N and C be complexes. Suppose that $C \xrightarrow{\iota} N$ is a chain map where each $\iota^n : N^n \to C^n$ is injective. Let $s^n : C^n \to N^n$ be a group homomorphism such that $s^n \circ \iota^n = \mathrm{id}_{N^n}$. Then $M := \big(C/N, d_{C/N}\big)$ is a complex. Our choice of s^n produces a splitting $C^\bullet \cong N^\bullet \oplus M^\bullet[1]$ in the category of graded abelian groups. Thus, we have the map $d_C = \begin{bmatrix} d_N & f \\ 0 & d_{M[1]} \end{bmatrix}$ where $f : M \to N$ is a map of graded abelian groups.

Exercise 11.1.14. *Show that f is a chain map and $C \cong$* $\mathrm{cone}(f)$.

11.2 Lecture 25

Definition 11.2.1. Let $f, g : A^\bullet \to B^\bullet$ be two chain maps. A *homotopy between f and g* is a map of graded abelian groups $h : A^\bullet \to B^{\bullet-1}$ such the

$$d_B h + h d_A = f - g.$$

We say that f and g are *homotopy equivalent* (written as $f \sim g$) if there is a homotopy between them.

Proposition 11.2.2.

1. *Homotopy is an equivalence relation.*
2. *The class* $\mathrm{mor}^{\sim 0}\, \mathbf{CoCh}(\mathbf{Ab})$ *of all chain maps homotopic to* 0 *is a two-sided ideal in* $\mathrm{mor}\, \mathbf{CoCh}(\mathbf{Ab})$.
3. *If $f \simeq g : A^\bullet \to B^\bullet$, then $H^\bullet(f) = H^\bullet(g)$.*
4. *If $f \simeq g$ and c is a cocycle, then $f(c) - g(c) = d_B h(c)$, which is a coboundary.*

Notation. Let $\mathcal{C}(\mathbf{Ab})$ denote the category with complexes as objects and homotopy classes of chain maps as morphisms.

Note 11.2.3.

1. We have that $\mathrm{Hom}_{\mathcal{C}(\mathbf{Ab})}(A, B) = \mathrm{Hom}_{\mathbf{CoCh}(\mathbf{Ab})}(A, B) \big/ \mathrm{Hom}^{\sim 0}_{\mathbf{CoCh}(\mathbf{Ab})}(A, B)$.

2. $H^\bullet$ descends to a well-defined functor in the sense that the diagram

$$\begin{tikzcd} \mathbf{CoCh}(\mathbf{Ab}) \arrow[r] \arrow[d, "H^\bullet"'] & \mathcal{C}(\mathbf{Ab}) \arrow[dl, "H^\bullet"] \\ \mathbf{grAb} & \end{tikzcd}.$$

commutes.

Definition 11.2.4. A *short exact sequence of complex* is a sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of complexes such that each sequence

$$0 \to A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \to 0$$

is exact in **Ab**.

Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence of complexes. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A^{n-1} \arrow[r, "f^{n-1}"] \arrow[d, "d_A^{n-1}"'] & B^{n-1} \arrow[r, "g^{n-1}"] \arrow[d, "d_B^{n-1}"] & C^{n-1} \arrow[r] \arrow[d, "d_C^{n-1}"] & 0 \\ 0 \arrow[r] & A^n \arrow[r, "f^n"] \arrow[d, "d_A^n"'] & B^n \arrow[r, "g^n"] \arrow[d, "d_B^n"] & C^n \arrow[r] \arrow[d, "d_C^n"] & 0 \\ 0 \arrow[r] & A^{n+1} \arrow[r, "f^{n+1}"] & B^{n+1} \arrow[r, "g^{n+1}"] & C^{n+1} \arrow[r] & 0 \end{tikzcd}.$$

Define a collection of *edge homomorphisms* $\left\{\delta^n : H^n(C) \to H^{n+1}(A)\right\}_{n\in\mathbb{Z}}$ as follows. Let $c \in C^n$ with $d_C^n(c) = 0$. By exactness, there is some $b \in B^n$ such that $g^n(b) = c$. But then

$$d_B^n(b) \in \ker g^{n+1} = \operatorname{im} f^{n+1}.$$

Since f^{n+1} is injective, this means that there is a unique $a \in A^{n+1}$ such that $f^n(a) = d_B^n(b)$. Let $\delta^n([c]) = [a]$.

Exercise 11.2.5. *Check that δ^n is a homomorphism and that it is independent both of our choice of c and of our choice of b.*

Lemma 11.2.6 (Snake). *Any short exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

complexes induces a long exact sequence in cohomology

$$\begin{tikzcd} & \cdots \arrow[r] & H^{n-1}(C) \arrow[dll, "\delta^{n-1}"'] \\ H^n(A) \arrow[r, "f^*"'] & H^n(B) \arrow[r, "g^*"'] & H^n(C) \arrow[dll, "\delta^n"'] \\ H^{n+1}(A) \arrow[r, "f^*"'] & H^{n+1}(B) \arrow[r] & \cdots \end{tikzcd}.$$

Proof.

Exactness at $H^n(B)$: We have that $0_{H^n(C)} = H^n(0) = H^n(g \circ f) = H^n(g) \circ H^n(f)$. Hence $\operatorname{im} H^n(f) \subset \ker H^n(g)$.

For the reverse inclusion, let $[b] \in \ker H^n(g) \subset H^n(B)$. Then $g(b) \in C^n$ must be a coboundary, so that there is some $c \in C^{n-1}$ such that $g(b) = d_C c$. Choose a lift $b_1 \in B^{n-1}$ of c, meaning that $g(b_1) = c$. Then $b - d_B b_1 \in Z^n(B)$, and $[b] = [b - d_B b_1]$. But

$$g(b - d_B b_1) = g(b) - g(d_B b_1) = g(b) - d_C g(b_1) = g(b) - d_C c = 0.$$

Hence $b - d_B b_1 \in \ker g \subset B^n$. This implies that there exists a unique $a \in A^n$ such that $b - d_B b_1 = f(a)$. Also,

$$f(d_A a) = d_B(f(a)) = d_B(b - d_B b_1) = 0.$$

Since f is injective, we see that $d_A a = 0$, i.e., $a \in Z^n(A)$. Thus, $H^n(f)([a]) = [f(a)] = [b - d_B b_1] = [b]$. This proves that $[b] \in \operatorname{im} H^n(f)$.

Exactness at $H^n(C)$: Let $[b] \in H^n(B)$. Note that $\delta^n(H^n(g)([b])) = [a]$ where $a \in A^{n+1}$ denotes the unique element such that $f(a) = d_B b$. Since $d_B b = 0$ and f is injective, it follows that $a = 0$. Hence $\operatorname{im} H^n(g) \subset \ker \delta^n$.

Conversely, let $[c] \in \ker \delta^n$. Choose $b \in B^n$ such that $g(b) = c$ and then the unique $a \in A^{n+1}$ such that $f(a) = d_B b$. Thus, $\delta^n([c]) = [a] = 0$, so that $a \in B^{n+1}(A)$, i.e., $d_A a_1 = a$ for some $a_1 \in A^n$. Note that $g(b - f(a_1)) = g(b) - g(f(a_1)) = c - 0 = c$. Further,

$$\begin{aligned} d_B(b - f(a_1)) &= d_B(b) - d_B(f(a_1)) \\ &= f(a) - f(d_A a_1) \\ &= f(a) - f(a) \\ &= 0. \end{aligned}$$

This shows that $b - f(a_1)$ is a cocycle. Thus, $H^n(g)([b - f(a_1)]) = [g(b - f(a_1))] = [c]$, so that $[c] \in \operatorname{im} H^n(g)$.

Exactness at $H^{n+1}(A)$: Let $[c] \in H^n(C)$ and find $[a] = \delta^n([c])$, where

$$\begin{tikzcd} & b \arrow[r, "g"] \arrow[d] & c \\ a \arrow[r, "f"] & d_B b & \end{tikzcd}.$$

Then $H^{n+1}(f)([a]) = [f(a)] = [d_B b] = 0$. It follows that $\operatorname{im} \delta^n \subset \ker H^{n+1}(f)$.

Conversely, let $[a] \in \ker H^{n+1}(f)$, so that $H^{n+1}(f)([a]) = [f(a)] = 0$. This means that $f(a) = d_B b$ for some $b \in B^n$. Then $\delta^n([g(b)]) = [a]$. This shows that $\operatorname{im} \delta^n \supset \ker H^{n+1}(f)$. ∎

12 Additive categories

Definition 12.0.1.

1. A category $\mathscr{C}$ is *enhanced over* **Ab** if $\mathrm{Hom}_{\mathscr{C}}(a, b)$ is an abelian group for any $a, b \in \mathrm{ob}\,\mathscr{C}$ and the function
$$\mathrm{Hom}_{\mathscr{C}}(y, z) \times \mathrm{Hom}_{\mathscr{C}}(x, y) \xrightarrow{\circ} \mathrm{Hom}_{\mathscr{C}}(x, z)$$
is bilinear for any $x, y, z \in \mathrm{ob}\,\mathscr{C}$.

2. A category $\mathscr{C}$ is called *additive* if it is enhanced over **Ab** and has finite products.

Example 12.0.2. The following are additive categories.

1. **Ab**.

2. $R-$**Mod**.

Note 12.0.3.

1. Let $\mathscr{C}$ be category with finite products. The product of the empty diagram is the terminal object in $\mathscr{C}$ since it is the initial object in **Set**.

2. If $\mathscr{C}$ is additive and $*$ is the terminal object in $\mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(*, *)$ consists of a single element, which must equal the group identity element.

Exercise 12.0.4. *Verify the following statements.*

1. *If $\mathscr{C}$ is a additive, then its terminal object is also initial and thus is a zero object in $\mathscr{C}$.*

2. *A zero object $0_{\mathscr{C}}$ satisfies $\mathrm{Hom}_{\mathscr{C}}(x, 0_{\mathscr{C}}) = 0$ and $\mathrm{Hom}_{\mathscr{C}}(0_{\mathscr{C}}, x) = 0$ for any $x \in \mathrm{ob}\,\mathscr{C}$.*

3. *Any additive category has finite coproducts, and these are equal to finite products.*

12.1 Lecture 26

Definition 12.1.1. Let $\mathscr{C}$ be an additive category. Let $f : x \to y$ be a morphism in $\mathscr{C}$.

1. A *kernel (object) for f* is a pair (k, q) where $k \in \mathrm{ob}\,\mathscr{C}$ and $q : k \to x$ such that for any $z \in \mathrm{ob}\,\mathscr{C}$, the natural sequence
$$\mathrm{Hom}(z, k) \xrightarrow{q\circ -} \mathrm{Hom}(z, x) \xrightarrow{f\circ -} \mathrm{Hom}(z, y)$$
is exact.

2. A *cokernel (object) for f* is a pair (c, p) where $c \in \mathrm{ob}\,\mathscr{C}$ and $p : y \to c$ such that for any $z \in \mathrm{ob}\,\mathscr{C}$, the natural sequence
$$\mathrm{Hom}(c, z) \xrightarrow{-\circ p} \mathrm{Hom}(y, z) \xrightarrow{-\circ f} \mathrm{Hom}(x, z)$$
is exact.

Definition 12.1.2. We say that a category $\mathscr{A}$ is *abelian* if

1. $\mathscr{A}$ is additive and
2. for any morphism $f : x \to y$ in $\mathscr{A}$, there exists a sequence $k \xrightarrow{q} x \xrightarrow{a} i \xrightarrow{b} y \xrightarrow{p} c$ in $\mathscr{A}$ such that
   (a) (k, q) is a kernel for f,
   (b) (c, p) is a cokernel for f,
   (c) (c, a) is a cokernel for q, and
   (d) (i, b) is a kernel for p.

   We call i the *image of* f.

Definition 12.1.3. If $\mathscr{A}$ is a abelian, then a sequence $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathscr{A}$ is *exact* if $\operatorname{im} f = \ker g$.

Example 12.1.4.

1. **Ab**.
2. $R-$**Mod**.
3. $\mathbf{PreShAb}_X$ where X is a space.

Remark 12.1.5. Our notion of and results for cohomology for complexes of abelian groups hold for complexes of objects in an abelian category.

Theorem 12.1.6 (Freyd-Mitchell). *Every abelian category admits a fully faithful embedding into $R-$**Mod** for some ring R.*

Remark 12.1.7. It is not, in general, possible to complete an additive category $\mathscr{C}$ to an abelian one. Still, we can always add enough images to $\mathscr{C}$ to get cones of maps of complexes.

Let $\mathscr{C}$ be additive. A map $e : x \to x$ in $\mathscr{C}$ is an *idempotent* if $e^2 = e$. Let $\mathscr{C} = \mathbf{Vect}_k$. Then an idempotent map $e : x \to x$ is a projection map, i.e., $x = x_1 \oplus x_2$ such that $e = i_1 \circ p_1$.

If $\mathscr{C}$ is additive and $e : x \to x$ is idempotent in $\mathscr{C}$, then we say that e *has an image in* $\mathscr{C}$ if there exists a decomposition $x = x_1 \oplus x_2$ such that

$$e = \begin{bmatrix} \mathrm{id}_{x_1} & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to this decomposition. We say that x_1 is the *image of* e.

Let $e : x \to x$ be an idempotent. Then $\mathrm{id}_x e : x \to x$ is also an idempotent. Indeed,

$$(\mathrm{id}_x - e)^2 = \mathrm{id}_x^2 - \mathrm{id}_x e - e\mathrm{id}_x + e^2 = \mathrm{id}_x = e.$$

If $x = x_1 \oplus x_2$ has $e = \begin{bmatrix} \mathrm{id}_{x_1} & 0 \\ 0 & 0 \end{bmatrix}$, then $\mathrm{id}_x - e = \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{id}_{x_2} \end{bmatrix}$, so that $\mathrm{id}_x - e$ has x_2 as an image.

Definition 12.1.8. A category $\mathscr{C}$ is *idempotent complete* or *Karoubian* if $\mathscr{C}$ is additive and any idempotent in $\mathscr{C}$ has an image in $\mathscr{C}$.

Exercise 12.1.9. *Show that for any additive category $\mathscr{C}$, there exists a unique (up to unique isomorphism) category $\mathscr{C}^{\mathrm{Kor}}$ together with a functor $F : \mathscr{C} \to \mathscr{C}^{\mathrm{Kor}}$ such that*

1. *$\mathscr{C}^{\mathrm{Kor}}$ is idempotent complete,*
2. *F is fully faithful, and*
3. *every object in $\mathscr{C}^{\mathrm{Kor}}$ is an image of an idempotent in $\mathscr{C}$.*

Definition 12.1.10. A *graded additive category* is an additive category $\mathscr{C}$ such that for any $x, y \in \operatorname{ob}\mathscr{C}$, $\operatorname{Hom}(x,y)$ is a graded abelian group, i.e., $\operatorname{Hom}(x,y) \cong \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}^n(x,y)$ and $\operatorname{Hom}(x,y) \times \operatorname{Hom}(y,z) \xrightarrow{\circ} \operatorname{Hom}(x,z)$ has the form $\operatorname{Hom}^n(x,y) \times \operatorname{Hom}^m(y,z) \xrightarrow{\circ} \operatorname{Hom}^{n+m}(x,z)$ where $\circ$ is bilinear.

Definition 12.1.11. A graded additive category $\mathscr{C}$ is a *differential graded category* if for any $x, y \in \operatorname{ob}\mathscr{C}$, the graded group $\operatorname{Hom}(x,y)$ is equipped with with a homomorphism $d : \operatorname{Hom}(x,y) \to \operatorname{Hom}(x,y)$ such that

(a) $d : \operatorname{Hom}^n(x,y) \to \operatorname{Hom}^{n+1}(x,y)$,

(b) $d^2 = 0$, and

(c) d satisfies the *graded Leibniz rule*, i.e., if $f \in \operatorname{Hom}^n(x,y)$ and $g \in \operatorname{Hom}(a,x)$, then

$$d(f \circ g) = df \circ g + (-1)^n f \circ dg.$$

Proposition 12.1.12. *Let $\mathscr{C}$ be a category.*

1. *If $\mathscr{C}$ is additive, then for any $x \in \operatorname{ob}\mathscr{C}$, $\operatorname{Hom}(x,x)$ is a ring (in fact, a $\mathbb{Z}$-algebra).*
2. *If $\mathscr{C}$ is a graded additive category, then for any $x \in \operatorname{ob}\mathscr{C}$, $\operatorname{End}(x)$ is a graded ring.*
3. *If $\mathscr{C}$ is differential graded category, then for any $x \in \operatorname{ob}\mathscr{C}$, $\operatorname{End}(x)$ is a differential graded algebra.*

Definition 12.1.13. If $\mathscr{C}$ is a differential graded category, then the *homotopy category of $\mathscr{C}$* is the category $\operatorname{Ho}(\mathscr{C})$ (or $[\mathscr{C}]$) given by

$$\begin{aligned}
\operatorname{ob}\operatorname{Ho}(\mathscr{C}) &\equiv \operatorname{ob}\mathscr{C}\\
\operatorname{Hom}_{\operatorname{Ho}(\mathscr{C})}(x,y) &\equiv H^0(\operatorname{Hom}_{\mathscr{C}}(x,y), d)\\
&= \frac{\ker(\operatorname{Hom}^0_{\mathscr{C}}(x,y) \xrightarrow{d} \operatorname{Hom}^1_{\mathscr{C}}(x,y))}{\operatorname{im}(\operatorname{Hom}^{-1}_{\mathscr{C}}(x,y) \xrightarrow{d} \operatorname{Hom}^0_{\mathscr{C}}(x,y))}.
\end{aligned}$$

Let $\mathscr{B}$ be an additive category. Define the category $\mathbf{Compl}(\mathscr{B})$ of complexes in $\mathscr{B}$ by

$$\begin{aligned}\operatorname{ob}\mathbf{Compl}(\mathscr{B}) &= (\text{complexes of objects in } \mathscr{B})\\ \operatorname{mor}\mathbf{Compl}(\mathscr{B}) &= (\text{morphisms of complexes})\,.\end{aligned}$$

This is an additive category. We can also refine this definition by incorporating degree-shifting maps to get a differential graded category of complexes in $\mathscr{B}$. Define the category $\mathbf{Compl}^\bullet(\mathscr{B})$ by

$$\begin{aligned}\operatorname{ob}\mathbf{Compl}^\bullet(\mathscr{B}) &= (\text{complexes of objects in } \mathscr{B})\\ \operatorname{Hom}_{\mathbf{Compl}^\bullet(\mathscr{B})}(M,N) &= \bigoplus_{n\in\mathbb{Z}}\operatorname{Hom}^n(M,N)\end{aligned}$$

where

$$\operatorname{Hom}^n(M,N) \equiv \prod_{a\in\mathbb{Z}}\operatorname{Hom}_{\mathscr{B}}(M^a, N^{a+n}).$$

The composition is obtained component-wise from the composition in $\mathscr{B}$. Define $d : \operatorname{Hom}^n(M,N) \to \operatorname{Hom}^{n+1}(M,N)$ by

$$(f_a)_{a\in\mathbb{Z}} \to (d_N \circ f_a + (-1)^n f_{a+1}\circ d_M)_{a\in\mathbb{Z}}\,.$$

This makes $\mathbf{Compl}^\bullet(\mathscr{B})$ a differential graded category.

Let $M, N \in \operatorname{ob}\mathbf{Compl}^\bullet(\mathscr{B})$. Then

$$\begin{aligned}Z^0(\operatorname{Hom}_{\mathbf{Compl}^\bullet(\mathscr{B})}(M,N)) &= \ker(\operatorname{Hom}^0 \xrightarrow{d} \operatorname{Hom}^1)\\ &= \operatorname{Hom}_{\mathbf{Compl}(\mathscr{B})}(M,N)\\ B^0(\operatorname{Hom}_{\mathbf{Compl}^\bullet(\mathscr{B})}(M,N)) &= \operatorname{im}(\operatorname{Hom}^{-1} \xrightarrow{d} \operatorname{Hom}^0)\\ &= (\text{homotopies of 0-maps of complexes})\,.\end{aligned}$$

Also, we have that

$$H^0(\operatorname{Hom}_{\mathbf{Compl}^\bullet(\mathscr{B})}(M,N)) = {}^{(\text{maps of complexes})}\!\big/\!{}_{(\text{homotopies})}.$$

Example 12.1.14. $\operatorname{Ho}(\mathbf{Comp}^\bullet(\mathbf{Ab})) = \mathcal{C}(\mathbf{Ab})$, and $Z^0(\mathbf{Comp}^\bullet(\mathbf{Ab})) = \mathbf{CoCh}(\mathbf{Ab})$.

13 Triangulated categories

13.1 Lecture 27

Let $\mathscr{C}$ be a category. For any $x \in \operatorname{ob}\mathscr{C}$, define $x[n]$ as the object, if it exists, in $\mathscr{C}$ that represents the shift functor on morphisms $\operatorname{Hom}_{\mathscr{C}}(-,x)[n] : \mathscr{C}^{\mathrm{op}} \to \mathbf{Compl}(\mathbf{Ab})$. If $f : x \to y$ is a morphism in $\mathscr{C}$, then define the *cone* $\operatorname{cone}(f)$ *of* f to be the object, it it exists, in $\mathscr{C}$ that represents the functor $\mathscr{C}^{\mathrm{op}} \to \mathbf{Comp}(\mathbf{Ab})$ given by $z \mapsto \operatorname{cone}(\operatorname{Hom}_{\mathscr{C}}(z,x) \xrightarrow{f\circ -} \operatorname{Hom}_{\mathscr{C}}(z,y))$.

Definition 13.1.1. A category $\mathscr{C}$ is called *strongly pre-triangulated* if every object in $\mathscr{C}$ has shifts in $\mathscr{C}$ and every morphism in $\mathscr{C}$ has cones in $\mathscr{C}$. We call $\mathscr{C}$ *pre-triangulated* if every object in $\mathscr{C}$ has shifts in $\operatorname{Ho}(\mathscr{C})$ and every morphism in $\mathscr{C}$ has cones in $\operatorname{Ho}(\mathscr{C})$.

Note 13.1.2. Both the assignment $x \mapsto x[n]$ and the assignment $f \mapsto \operatorname{cone}(f)$ are functorial.

Definition 13.1.3. Given a differential graded category $\mathscr{C}$, we define $\mathrm{Ho}^\bullet(\mathscr{C})$ as the graded additive category such that

$$\begin{aligned} \operatorname{ob} \mathrm{Ho}^\bullet(\mathscr{C}) &= \operatorname{ob} \mathscr{C} \\ \mathrm{Hom}_{\mathrm{Ho}^\bullet(\mathscr{C})}(x, y) &= H^\bullet(\mathrm{Hom}_{\mathscr{C}}(x, y)). \end{aligned}$$

If $\mathscr{C}$ is strongly pre-triangulated, then $\mathrm{Ho}^\bullet(\mathscr{C})$ and $\mathrm{Ho}(\mathscr{C})$ contain the same information. Indeed, $\mathrm{Ho}(\mathscr{C})$ is precisely the degree zero piece of $\mathrm{Ho}^\bullet(\mathscr{C})$. Conversely, $if x, y \in \operatorname{ob} \mathscr{C}$, then

$$\mathrm{Hom}_{\mathrm{Ho}^\bullet(\mathscr{C})}(x, y) = \bigoplus_{a \in \mathbb{Z}} \mathrm{Hom}^a_{\mathrm{Ho}^\bullet(\mathscr{C})}(x, y)$$

where $\mathrm{Hom}^a_{\mathrm{Ho}^\bullet(\mathscr{C})}(x, y) = \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(x, y\,[a]) = \mathrm{Hom}_{\mathrm{Ho}(\mathscr{C})}(x, y)\,[a]$.

Notation. From now on, if $\mathscr{C}$ is strongly pre-triangulated, then we write $\mathrm{Ho}(\mathscr{C})$ for the graded homotopy category.

Definition 13.1.4. If $\mathscr{C}$ is strongly pre-triangulated, then a *triangle* $\triangle$ *in* $\mathrm{Ho}(\mathscr{C})$ is a sequence of degree zero maps $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} x[1]$. We represent this as

$$\begin{tikzcd} x \arrow[r, "u"] & y \arrow[d, "v"] \\ & z \arrow[ul, "w"] \end{tikzcd} .$$

Let $\mathscr{C}$ be strongly pre-triangulated. Given a triangle

$$\begin{tikzcd} x \arrow[r, "u"] & y \arrow[d, "v"] \\ & z \arrow[ul, "w"] \end{tikzcd} ,$$

we have a long sequence of maps

$$\begin{tikzcd} x[-1] \arrow[r, "{u[-1]}"] & y[-1] \arrow[r, "{v[-1]}"] & z[-1] \arrow[dll, "{w[-1]}"'] & \\ x \arrow[r, "u"] & y \arrow[r, "v"] & z \arrow[dll, "w"'] & \\ x[1] \arrow[r, "{u[1]}"'] & y[1] \arrow[r, "{v[1]}"'] & z[1] \arrow[r, "{w[1]}"'] & \cdots \end{tikzcd} .$$

in $\mathscr{C}$.

Definition 13.1.5. Let $\mathscr{C}$ be strongly pre-triangulated. We say that a triangle in $\mathrm{Ho}(\mathscr{C})$ is *exact* if it is isomorphic to the triangle

$$x \xrightarrow{\quad u \quad} y \xrightarrow{\text{“inclusion”}} \operatorname{cone}(u) \xrightarrow{\text{“projection''}} x[1] \ .$$

Definition 13.1.6. A graded additive category $\mathscr{D}$ is *triangulated* if $\mathscr{D}$ is equipped with a shift functor $[1] : \mathscr{D} \to \mathscr{D}$ and a collection of *distinguished triangles* such that the following axioms hold.

(0) Every triangle that is isomorphic to a distinguished triangle is distinguished.

(1) For any object x in $\mathscr{D}$, the triangle $x \xrightarrow{\mathrm{id}_x} x \to 0 \to x[1]$ is distinguished.

(2) (*rotation invariance*) The shift rotation of a triangle $\triangle$ is distinguished if and only if $\triangle$ is, i.e., the triangle $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} x[1]$ is distinguished if and only if the triangle

$$y \xrightarrow{v} z \xrightarrow{w} x[1] \xrightarrow{-u[1]} y[1]$$

is distinguished.

(3) Every morphism $u : x \to y$ can be included in a distinguished triangle $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} x[1]$, and every commutative square

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ x' \arrow[r, "u'"'] & y' \end{tikzcd}$$

can be completed to a commutative diagram of distinguished triangles, i.e.,

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "f"'] & y \arrow[r, "v"] \arrow[d, "g"] & z \arrow[r, "w"] \arrow[d, "h"'] & x[1] \arrow[d, "f[1]"] \\ x' \arrow[r, "u'"'] & y' \arrow[r, "v'"'] & z' \arrow[r, "w'"'] & x'[1] \end{tikzcd}.$$

(4) (*octahedron axiom*) Given any two distinguished triangles $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} x[1]$ and $y \xrightarrow{f} y' \xrightarrow{g} q \xrightarrow{h} y[1]$, we can complete them to a commutative diagram

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, equal] & y \arrow[r, "v"] \arrow[d, "f"] & z \arrow[r, "w"] \arrow[d, "a"] & x[1] \arrow[d, equal] \\ x \arrow[r] & y' \arrow[r] \arrow[d, "g"] & z' \arrow[r] \arrow[d, "b"] & x[1] \arrow[d, "u[1]"] \\ & q \arrow[r, equal] \arrow[d, "h"] & q \arrow[r] \arrow[d, "c"] & y[1] \\ & y[1] \arrow[r] & z[1] & \end{tikzcd}$$

where each new triangle is distinguished.

The octahedron axiom is the formal transplant of the second isomorphism theorem for $\mathbf{Comp}(\mathbf{Ab})$,[1] which states that given two complexes L and M, an inclusion $f : L \hookrightarrow M$, and a subcomplex N of L

[1] The second isomorphism theorem holds in some form for any abelian category.

and of M, we have that $M/L \cong (M/N)/(L/N)$, i.e., if

$$\begin{tikzcd} & N \arrow[d] & N \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & L \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L/N \arrow[r] \arrow[d] & M/N \arrow[r] \arrow[d] & (M/N)/(L/N) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

has exact rows and exact left two columns, then the third column is also exact.

Now, suppose that $\mathscr{C}$ is strongly pre-triangulated and let $\alpha : M \to N$ be a morphism in $\mathscr{C}$ such that α is injective (i.e., $\ker\alpha$ exists and is trivial) with $d\alpha = 0$ and α is split (i.e., there exists $\beta : N \to M$ with $p \circ \alpha = \mathrm{id}_M$). We call such an α a *split monomorphism in $\mathscr{C}$*.

Lemma 13.1.7.

(i) The map $\mathrm{cone}(\alpha) \to N/M$ *is a homotopy equivalence.*

(ii) Any morphism in $\mathscr{C}$ is homotopy equivalent to a split mono, i.e., given $f : M \to L$, we can construct a natural diagram

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[dr, "f"'] & N \arrow[d, "g"] \\ & L \end{tikzcd}$$

in $\mathscr{C}$ such that α is a split mono and g is an iso in $\mathrm{Ho}(\mathscr{C})$.

Partial proof. For (ii), take $N = L \oplus \mathrm{cone}(\mathrm{id}_M)$. ☐

Theorem 13.1.8. *If $\mathscr{C}$ is a strongly pre-triangulated differential graded category and $\mathscr{D} = \mathrm{Ho}(\mathscr{C})$, then $\mathscr{D}$ is triangulated with exact triangles as the distinguished triangles.*

Proof.

Verifying axioms (0) and (1) is trivial.

For axiom (2), if $x \to y \to z \to x[1]$ is a triangle, then we can use Lemma 13.1.7 to rewrite it as a homotopy equivalent triangle $M \to N \to L \to M[1]$ where $M \xrightarrow{\alpha} N$ is a split mono. In this case, we can check that $N \to L \to M[1] \to N[1]$ is exact by using the splitting.

For axiom (3), note that any $u : x \to y$ is included in $x \to y \to \mathrm{cone}(u) \to x[1]$. Moreover, if

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ x' \arrow[r, "u'"'] & y' \end{tikzcd}$$

is commutative in $\mathrm{Ho}(\mathscr{C})$ and we lift f, g, u, and u' to maps $\tilde{}$ in $\mathscr{C}$, then we get a diagram

$$\begin{tikzcd} x \arrow[r, "\tilde{u}"] \arrow[d, "\tilde{f}"'] & y \arrow[r] \arrow[d, "\tilde{g}"] & \mathrm{cone}(u) \arrow[r] \arrow[d, "M"'] & x[1] \arrow[d, "\tilde{f}[1]"] \\ x' \arrow[r, "\tilde{u'}"'] & y' \arrow[r] & \mathrm{cone}(u') \arrow[r] & x'[1] \end{tikzcd}$$

in $\mathscr{C}$ where $M \equiv \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}$, $\delta \in \mathrm{Hom}^{-1}(x, y')$, and $\tilde{g} \circ \tilde{u} - \underbrace{\tilde{u'} \circ \tilde{f}}_{d(\delta)} \sim 0$.

For axiom (4), given a distinguished triangle $M \to N \to L \to M[1]$, we apply Lemma 13.1.7 twice to get a homotopy equivalent distinguished triangle $M \to N' \to L'' \to M[1]$ where each map in this is a split mono. We are done after an application of the second isomorphism theorem. □

Remark 13.1.9. Such reasoning can be applied to complete any differential graded category to a triangulated one.

13.2 Lecture 28

Definition 13.2.1. If $\mathscr{A}$ and $\mathscr{B}$ are differential graded categories, then a *differential graded functor* $F : A \to B$ has the following properties.

(i) F is additive, i.e., $F : \mathrm{Hom}_{\mathscr{A}}(x, y) \to \mathrm{Hom}_{\mathscr{B}}(F(x), F(y))$ is a group homomorphism for any $x, y \in \mathrm{ob}\,\mathscr{A}$.

(ii) F respects differentials, i.e., if $x, y \in \mathrm{ob}\,\mathscr{A}$, then $F : \mathrm{Hom}_{\mathscr{A}}(x, y) \to \mathrm{Hom}_{\mathscr{B}}(F(x), F(y))$ is a map of complexes.

If $F, G : \mathscr{A} \to \mathscr{B}$ are two differential graded functors between differential graded categories, then define, for each $n \in \mathbb{Z}$, the group

$$\mathrm{Hom}^n(F, G) \equiv \{\varphi_x \mid \varphi_x : F(x) \to G(x) \text{ in } \mathrm{Hom}^n_{\mathscr{B}}(F(x), G(x)),\ x \in \mathrm{ob}\,\mathscr{A}\}.$$

A map $F \to G$ is defined as a natural transformation $F \to G$ such that each component φ_x : $F(x) \to G(x)$ belongs to $\mathrm{Hom}^n_{\mathscr{B}}(F(x), G(x))$. The differential on $\prod_{x \in \mathrm{ob}\,\mathscr{A}} \mathrm{Hom}^\bullet(F(x), G(x))$ induces a differential on

$$\mathrm{Hom}^\bullet(F, G) \equiv \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}^n(F, G).$$

This produces a complex of maps between F and G, and we get a differential graded category $\mathbf{dgFun}(\mathscr{A}, \mathscr{B})$.

Exercise 13.2.2. *Prove the following assertions.*

1. *If $F : \mathscr{A} \to \mathscr{B}$ is a differential graded functor, then $H^0(F) : H^0(\mathscr{A}) \to H^0(\mathscr{B})$ is an additive functor.*

2. *If $F, G : \mathscr{A} \to \mathscr{B}$ are differential graded functors, then there is an embedding $H^0(\operatorname{Hom}(F, G)) \subset \operatorname{Hom}(H^0(F), H^0(G))$.*

Definition 13.2.3. If $\mathscr{A}$ is a differential graded category, then a *left $\mathscr{A}$-module* is a differential graded functor $\mathscr{A} \to \mathbf{Compl}(\mathbf{Ab})$ and a *right $\mathscr{A}$-module* is a differential graded functor $\mathscr{A}^{\mathrm{op}} \to \mathbf{Compl}(\mathbf{Ab})$.

If $\mathscr{A}$ is a differential graded category with a single object $*$, then $\mathscr{A} \leftrightarrow R \coloneqq \operatorname{Hom}_{\mathscr{A}}(*, *)$, which is precisely the complex of abelian groups equipped with a multiplication-like operation $\cdot$ such that λ satisfies the graded Leibniz rule for $\cdot$.

Exercise 13.2.4. *Show that a module over $\mathscr{A}$ is precisely the data of a complex x of abelian groups together with a differential graded algebra homomorphism $R \to \operatorname{Hom}_{\mathbf{Compl}(\mathbf{Ab})}(x, x)$.*

Given a differential graded category $\mathscr{A}$, we have respective categories of left and right modules over $\mathscr{A}$ that are linear over a field k, namely

$$\begin{aligned} \mathscr{A}{-}\mathbf{dgmod}_k &\equiv \mathbf{dgFun}(\mathcal{A}, \mathbf{Compl}(k{-}\mathbf{Vect})) \\ \mathbf{dgmod}_k{-}\mathscr{A} &\equiv \mathbf{dgFun}(\mathcal{A}^{\mathrm{op}}, \mathbf{Compl}(k{-}\mathbf{Vect})). \end{aligned}$$

Exercise 13.2.5. *Show that the functors*

$$\begin{aligned} h^\bullet : \mathscr{A}^{\mathrm{op}} &\to \mathscr{A}{-}\mathbf{dgmod}_k \\ x &\mapsto h^\times \equiv \operatorname{Hom}_{\mathscr{A}}(x, -) \\ h_\bullet : \mathscr{A} &\to \mathscr{A}^{\mathrm{op}}{-}\mathbf{dgmod}_k \\ h_\times &\equiv \operatorname{Hom}_{\mathscr{A}^{\mathrm{op}}}(x, -) = \operatorname{Hom}_{\mathscr{A}}(-, x) \end{aligned}$$

are fully faithful differential graded functors.

Proposition 13.2.6.

1. *If $\mathscr{A}$ is a small differential graded category, then $H^0(\mathscr{A}^{\mathrm{op}}{-}\mathbf{dgmod}_k)$ is triangulated.*

2. *If $\mathscr{A}$ is a pre-triangulated differential graded category, then the fully faithful functor*

$$H^0(h_\bullet) : H^0(\mathscr{A}) \to H^0(\mathscr{A}^{\mathrm{op}}{-}\mathbf{dgmod}_k)$$

gives a triangulated structure on $H^0(\mathscr{A})$.

Definition 13.2.7. We say that an object F in $\mathscr{A}^{\mathrm{op}}{-}\mathbf{dgmod}_k$ is *compact* or *perfect* if $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{Compl}(k{-}\mathbf{Vect})$ commutes with arbitrary coproducts.

Note 13.2.8. $h^\times$ is compact for any $x \in \operatorname{ob} \mathscr{A}$.

Definition 13.2.9. We say that a k-linear differential graded category $\mathscr{A}$ is *triangulated* if every compact object in $\mathscr{A}^{\mathrm{op}}{-}\mathbf{dgmod}_k$ is representable.

Note 13.2.10. A triangulated differential graded category is automatically strongly pre-triangulated, and $H^0(\mathscr{A})$ is triangulated.

Exercise 13.2.11.

1. *Suppose that $\mathscr{D}$ is a triangulated additive category. Let $M \to N \to C \to M[1]$ be a distinguished triangle. Show that for every $L \in \operatorname{ob} \mathscr{D}$, the sequence*

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Hom}(L,M) \arrow[r] & \operatorname{Hom}(L,N) \arrow[r] & \operatorname{Hom}(L,C) \arrow[dll] & \\ & \operatorname{Hom}(L,M[1]) \arrow[r] & \operatorname{Hom}(L,N[1]) \arrow[r] & \operatorname{Hom}(L,C[1]) \arrow[r] & \cdots \end{tikzcd}$$

is a long exact sequence of abelian groups.

2. *Suppose that $\mathscr{D}$ is triangulated. Show that the sum $\triangle_1 \oplus \triangle_2$ of two triangles in $\mathscr{D}$ is distinguished if and only if both $\triangle_1$ and $\triangle_2$ are distinguished.*

Definition 13.2.12. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are triangulated additive categories, then a *triangulated* (or *exact*) *functor* $F : \mathscr{D}_1 \to \mathscr{D}_2$ is an additive functor such that

(i) F is equipped with an isomorphism $\sigma : F \circ [1] \to [1] \circ F$ and

(ii) F sends distinguished triangles to distinguished triangles.

A *morphism of two triangulated functors* (F, θ_F) *and* (G, θ_G) is a morphism $f : F \to G$ of additive functors such that f intertwines θ_F and θ_G. As a result, we have a category of triangulated functors $\mathscr{D}_1 \to \mathscr{D}_2$.

If $\mathscr{A}$ and $\mathscr{B}$ are differential graded categories and $F : \mathscr{A} \to \mathscr{B}$ is a differential graded functor, then we have a natural differential graded functor $\mathscr{A}^{\mathrm{op}}-\mathbf{dgmod}_k \xrightarrow{F} \mathscr{B}^{\mathrm{op}}-\mathbf{dgmod}_k$ so that $H^0(F)$ is triangulated.

Definition 13.2.13. If $\mathscr{D}$ is a triangulated category and $\mathscr{A}$ is an abelian category, then a *cohomological functor* is a functor $H : \mathscr{D} \to \mathscr{A}$ such that

(i) H is additive and

(ii) H sends distinguished $\triangle$'s in $\mathscr{D}$ into long exact sequences in $\mathscr{A}$.

Example 13.2.14.

1. If $\mathcal{C}(\mathbf{Ab})$ denotes the triangulated category of homotopy classes of complexes of abelian groups, then $H^\bullet : \mathcal{C}(\mathbf{Ab}) \to \mathbf{grAb}$ is a cohomological functor.

2. If $\mathscr{D}$ is a triangulated category and $L \in \operatorname{ob} \mathscr{D}$, then $h^L : \mathscr{D} \to \mathbf{Ab}$ given by $M \mapsto Z^0(\operatorname{Hom}_{\mathscr{B}}(L, M))$ is a cohomological functor.

13.3 Lecture 29

Let $\mathscr{D}$ be a triangulated category and $\mathscr{V} \subset \mathscr{D}$ a triangulated subcategory (i.e., the inclusion functor is triangulated). We wish to construct a quotient category $\mathscr{D}/\mathscr{V}$, i.e., a triangulated category $\mathscr{D}/\mathscr{V}$ together with a triangulated functor $q : \mathscr{D} \to \mathscr{D}/\mathscr{V}$ such that

- $q(x) = 0$ for any $x \in \mathrm{ob}\,\mathscr{V}$ and
- for any triangulated functor $f : \mathscr{D} \to \mathscr{D}'$ satisfying $x \in \mathrm{ob}\,\mathscr{V} \implies f(x) = 0$, we have $g \circ q = f$.

Note 13.3.1.

1. In the triangulated category of triangulated categories with exact functors, the triangle $\mathscr{V} \to \mathscr{D} \to \mathscr{D}/\mathscr{V} \to \mathscr{V}[1]$ is exact.
2. If $\mathscr{D}$ is triangulated and $u : x \to y$ is a morphism in $\mathscr{D}$, then there exists an object $\mathrm{cone}(u)$ in $\mathscr{D}$ that is unique up to a non-unique isomorphism. This is the third term in a distinguished $\triangle$ completing u.

Exercise 13.3.2. *Show that if* $u : x \to y$ *is a map in* $\mathscr{D}$*, then it is an isomorphism in* $\mathscr{D}$ *if and only if* $\mathrm{cone}(u) = 0$.

Definition 13.3.3. If $\mathscr{D}$ is a triangulated category and $\mathscr{V} \subset \mathscr{D}$ a triangulated subcategory, then a morphism $u : x \to y$ in $\mathscr{D}$ is a *$\mathscr{V}$-quasi-isomorphism* if $\mathrm{cone}(u) \in \mathrm{ob}(\mathscr{V})$.

Exercise 13.3.4. *Let* $\mathscr{V} \subset \mathscr{D}$ *be a pair of triangulated categories. Use the octahedron axiom to show that if* f *and* g *are compassable morphisms in* $\mathscr{D}$*, then every morphism in* $\{f, g, g \circ f\}$ *is a* $\mathscr{V}$*-quasi-isomorphism if and only if at least two morphisms in it are* $\mathscr{V}$*-quasi-isomorphisms.*

Remark 13.3.5. One may define $\mathscr{D}/\mathscr{V}$ as the localization of $\mathscr{D}$ in the set of all $\mathscr{V}$-quasi-isomorphisms. But doing so requires a lot of work.

Definition 13.3.6. Suppose that $\mathscr{I}$ is a small category. We say that $\mathscr{I}$ is a *directed category* if it satisfies the following properties.

(1) If $x_1, x_2 \in \mathrm{ob}\,\mathscr{I}$, then there is some diagram

$$\begin{tikzcd} x_1 \arrow[r] & x_3 \\ x_2 \arrow[ur] & \end{tikzcd}$$

of maps in $\mathscr{I}$.

(2) If

$$\begin{tikzcd} x_1 \arrow[r] \arrow[dr] & x_2 \\ & x_3 \end{tikzcd}$$

is a diagram of maps in $\mathscr{I}$, then there exist maps $x_2 \to x_4; x_3 \to x_4$ in $\mathscr{I}$ such that

$$\begin{tikzcd} x_1 \arrow[r] \arrow[dr] & x_2 \arrow[r] & x_4 \\ & x_3 \arrow[ur] & \end{tikzcd}$$

commutes in $\mathscr{I}$.

(3) For any two parallel maps $f, g : x \to y$, there exists a map $h : y \to z$ such that $h \circ f = h \circ g$.

Let $\mathscr{I}$ be small. There is a well-defined functor $\operatorname{colim} : \operatorname{Fun}(\mathscr{I}, \mathbf{Ab}) \to \mathbf{Ab}$, but this need *not* be exact even though both $\mathbf{Ab}$ and $\operatorname{Fun}(\mathscr{I}, \mathbf{Ab})$ are abelian.

Exercise 13.3.7. *Show, however, that if $\mathscr{I}$ is directed, then* colim *is an exact functor.*

Now, let $\mathscr{V} \subset \mathscr{D}$ be a pair of triangulated categories. Let $x \in \operatorname{ob}\mathscr{D}$ and let $\mathscr{Q}/x$ be the full subcategory of $\mathscr{D}/x$ consisting of morphisms $y \to x$ that are $\mathscr{V}$-quasi-isomorphisms. Similarly, let $x/\mathbb{Q}$ be the full subcategory of $x/\mathscr{D}$ consisting morphisms $x \to z$ that are $\mathscr{V}$-quasi-isomorphisms.

Exercise 13.3.8. *Prove the following assertions.*

1. *Both $x/\mathscr{Q}$ and $(\mathscr{Q}/x)^{\mathrm{op}}$ are directed categories.*
2. *Any map in $x/\mathscr{Q}$ or $\mathscr{Q}/x$ is automatically a $\mathscr{V}$-quasi-isomorphism.*

Definition 13.3.9. Define the *Verdier quotient of $\mathscr{D}$ by $\mathscr{V}$* as the category $\mathscr{D}/\mathscr{V}$ with $\operatorname{ob}\mathscr{D}/\mathscr{V} \equiv \operatorname{ob}\mathscr{D}$ and $\operatorname{Hom}_{\mathscr{D}/\mathscr{V}}(a, b) \equiv \operatorname{colim}_{a' \in (\mathscr{Q}/a)^{\mathrm{op}}} \operatorname{Hom}_{\mathscr{D}}(a', b)$.

There exists a canonical isomorphism

$$\operatorname*{colim}_{a' \in (\mathscr{Q}/a)^{\mathrm{op}}} \operatorname{Hom}_{\mathscr{D}}(a', b) \cong \operatorname*{colim}_{b' \in (b/\mathscr{Q})} \operatorname{Hom}_{\mathscr{D}}(a, b').$$

For this, we must check that given a top triangle

$$\begin{tikzcd} a' \arrow[r, "\text{q-iso}"] \arrow[dr] & a \arrow[d] \\ & b \end{tikzcd},$$

we can form a commutative double triangle

$$\begin{tikzcd} a' \arrow[r, "\text{q-iso}"] \arrow[dr] & a \arrow[d] \arrow[r] & b' \\ & b \arrow[ur, "\text{q-iso}"'] & \end{tikzcd}.$$

As a result, we get $q : \mathscr{D} \to \mathscr{D}/\mathscr{V}$.

Lemma 13.3.10. *If $x \in \operatorname{ob}\mathscr{D}$ has $q(x) = 0$ in $\mathscr{D}/\mathscr{V}$, then x is a direct summand of an object in $\mathscr{V}$.*

Proof. We have that $q(x) = 0 \iff$ there is some $y \in \mathscr{D}$ such that $\varphi : y \to x$ is a $\mathscr{V}$-quasi-isomorphism. In this case, $\underbrace{\operatorname{cone}(\varphi)}_{y[1] \oplus x} \in \mathscr{V}$. □

Definition 13.3.11. A triangulated subcategory $\mathscr{V} \subset \mathscr{D}$ is *thick* if any object in $\mathscr{D}$ that is isomorphic to a direct summand of an object in $\mathscr{V}$ is an object in $\mathscr{V}$.

Note 13.3.12. If $\mathscr{V}$ is a strict full thick triangulated subcategory of $\mathscr{D}$, then $q : \mathscr{D} \to \mathscr{D}/\mathscr{V}$ kills all and only objects in $\mathscr{V}$.

Definition 13.3.13. If $\mathscr{D}$ is triangulated and $\mathscr{U}, \mathscr{V} \subset \mathscr{D}$ are strict full triangulated subcategories, then $(\mathscr{U}, \mathscr{V})$ is an *admissible pair of subcategories* if

(a) $\mathrm{Hom}_{\mathscr{D}}(x, y) = 0$ for any $x \in \mathrm{ob}\, u$ and $y \in \mathrm{ob}\, \mathscr{V}$ and

(b) any object $z \in \mathrm{ob}\, \mathscr{D}$ fits in a distinguished triangle $x \to z \to y \to x[1]$ with $x \in \mathrm{ob}\, \mathscr{U}$ and $y \in \mathrm{ob}\, \mathscr{V}$.

Exercise 13.3.14. *Prove the following assertions.*

1. *The $\triangle$ in condition (b) is unique up to a unique isomorphism and is functorial in z.*

2. *The functor $\mathscr{D} \to \mathscr{U}$ given by $z \mapsto x(z)$ is triangulated and is right adjoint to $\mathscr{U} \hookrightarrow \mathscr{D}$.*

   Dually, the functor $\mathscr{D} \to \mathscr{V}$ given by $z \mapsto y(z)$ is triangulated and is left adjoint to $\mathscr{V} \hookrightarrow \mathscr{D}$.

3. *Each of $\mathscr{U}$ and $\mathscr{V}$ determines the other. Specifically,*

$$\mathscr{V} = \mathscr{U}^{\perp} \equiv \underbrace{\{y \in \mathrm{ob}\, \mathscr{D} \mid \mathrm{Hom}_{\mathscr{D}}(x, y) = 0,\ x \in \mathrm{ob}\, \mathscr{U}\}}_{\textit{full subcategory}}$$

$$\mathscr{U} = {}^{\perp}\mathscr{V} \equiv \underbrace{\{x \in \mathrm{ob}\, \mathscr{D} \mid \mathrm{Hom}_{\mathscr{D}}(x, y) = 0,\ y \in \mathrm{ob}\, \mathscr{V}\}}_{\textit{full subcategory}}.$$

   In particular, both $\mathscr{U}$ and $\mathscr{V}$ are thick subcategories.

4. *The natural compositions $\mathscr{U} \hookrightarrow \mathscr{D} \to \mathscr{D}/\mathscr{V}$ and $\mathscr{V} \hookrightarrow \mathscr{D} \to \mathscr{D}/\mathscr{U}$ are triangulated equivalences.*

Definition 13.3.15. An additive pair $(\mathscr{U}, \mathscr{V})$ is called a *semiorthogonal decomposition of $\mathscr{D}$ into $\mathscr{U}$ and $\mathscr{V}$*.

Proposition 13.3.16. *If $\mathscr{U} \subset \mathscr{D}$ is a strict full triangulated thick subcategory, then TFAE.*

1. *The inclusion $\mathscr{U} \hookrightarrow \mathscr{D}$ has a left adjoint.*

2. *The quotient $\mathscr{D} \to \mathscr{D}/\mathscr{U}$ has a right adjoint.*

3. *$\left(\mathscr{U}, \mathscr{U}^{\perp}\right)$ is admissible.*

Definition 13.3.17. If $\mathscr{A}$ is an abelian category, then the *derived category of $\mathscr{A}$* is the triangulated category

$$\mathcal{D}(\mathscr{A}) \equiv {}^{\mathcal{C}(\mathscr{A})}\!/\!_{\mathcal{C}(\mathscr{A})^{\mathrm{acyclic}}},$$

where $\mathcal{C}(\mathscr{A})^{\mathrm{acyclic}}$ is the full subcategory of $\mathcal{C}(\mathscr{A})$ on those complexes with zero cohomology.

To do computations in $\mathcal{D}(\mathscr{A})$, we must understand when it can be embedded in $\mathcal{C}(\mathscr{A})$ so that $\left(\mathcal{C}(\mathscr{A})^{\mathrm{acyclic}}, \mathcal{D}(\mathscr{A})\right)$ is an adjoint pair. This requires $\left(\mathcal{C}(\mathscr{A})^{\mathrm{acyclic}}\right)^{\perp}$ to be large.

Define ${}^{\perp}\mathcal{C}(\mathscr{A})^{\mathrm{acyclic}}$ as the category of *homotopically projective objects* in $\mathcal{C}(\mathscr{A})$ and $\left(\mathcal{C}(\mathscr{A})^{\mathrm{acyclic}}\right)^{\perp}$ as the category of *homotopically injective objects* in $\mathcal{C}(\mathscr{A})$.

Proposition 13.3.18.

1. *Every bounded-above complex of projectives is a homotopically projective object in $\mathcal{C}(\mathscr{A})$.*

2. *Any bounded-below complex of injectives is a homotopically injective object in $\mathcal{C}(\mathscr{A})$.*