

## Abstract

This is an introduction to universal properties in category theory. The main sources for this talk are the following.

- nLab.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 4.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4.

## 1 Universal arrows

**Definition 1.1.** An object  $X$  of  $\mathcal{C}$  is *initial* if for each  $Y \in \text{ob } \mathcal{C}$ , there is a unique morphism  $f : X \rightarrow Y$ . Moreover, we say that  $X$  is *terminal* if for each  $Z \in \text{ob } \mathcal{C}$ , there is a unique morphism  $g : Z \rightarrow X$ . Either condition is called a *universal property* of  $X$ .

Any property  $P$  of  $\mathcal{C}$  has a dual property  $P^{\text{op}}$  of  $\mathcal{C}^{\text{op}}$  obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing  $P$ . Then  $P$  is true of  $\mathcal{C}$  iff  $P^{\text{op}}$  is true of  $\mathcal{C}^{\text{op}}$ .

**Example 1.2.** Being initial and being terminal are dual properties.

**Lemma 1.3.** *Any two initial objects of  $\mathcal{C}$  are canonically isomorphic. The same holds for any two terminal objects of  $\mathcal{C}$ .*

*Proof.* Let  $X$  and  $X'$  be two initial objects. Compose the two unique morphisms  $X \rightarrow X'$  and  $X' \rightarrow X$  to get an isomorphism between  $X$  and  $X'$ . Apply duality to this argument for the case of terminal objects.  $\square$

We can think of a universal property as follows. Let  $F : \mathcal{D} \rightarrow \mathcal{C}$  be a functor and  $X \in \text{ob } \mathcal{C}$ . A *universal arrow from  $X$  to  $F$*  is an ordered pair  $(Y, f)$  with  $Y \in \text{ob } \mathcal{D}$  and  $f : X \rightarrow F(Y)$  a morphism of  $\mathcal{C}$  with the property that for any  $X' \in \text{ob } \mathcal{D}$  and morphism  $f' : X \rightarrow F(X')$  of  $\mathcal{C}$ , there exists a unique morphism  $\hat{f} : Y \rightarrow X'$  of  $\mathcal{D}$  such that  $F(\hat{f}) \circ f = f'$ .

$$\begin{array}{ccc} X & \xrightarrow{f} & F(Y) \\ & \searrow f' & \downarrow F(\hat{f}) \\ & & F(X') \end{array}$$

Dually, a *universal arrow from  $F$  to  $X$*  is an ordered pair  $(Y, f)$  with  $Y \in \text{ob } \mathcal{D}$  and  $f : F(Y) \rightarrow X$  of  $\mathcal{C}$  with the property that for any  $X' \in \text{ob } \mathcal{D}$  and morphism  $f' : F(X') \rightarrow X$ , there exists a unique morphism

$\hat{f} : X' \rightarrow Y$  such that  $f' = f \circ F(\hat{f})$ .

$$\begin{array}{ccc} F(X') & \xrightarrow{F(\hat{f})} & F(Y) \\ & \searrow f' & \downarrow f \\ & & X \end{array}$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

**Definition 1.4.**

1. Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $Y \in \text{ob } \mathcal{D}$ . The *slice* or *left fiber category*, denoted by  $(F/Y)$  or  $(F \downarrow Y)$ , has as objects pairs  $(X, f)$  where  $f : F(X) \rightarrow Y$  and as morphisms from  $f : F(X) \rightarrow Y$  to  $f' : F(X') \rightarrow Y$  morphisms  $g : X \rightarrow X'$  such that  $f = f' \circ F(g)$ .
2. The *coslice* or *right fiber category*, denoted by  $(Y/F)$  or  $(Y \downarrow F)$ , has as objects pairs  $(X, f)$  where  $f : Y \rightarrow F(X)$  and as morphisms from  $f : Y \rightarrow F(X)$  to  $f' : Y \rightarrow F(X')$  morphisms  $g : X \rightarrow X'$  such that  $f' = F(g) \circ f$ .

If  $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$  is opposite to the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $Y \in \text{ob } \mathcal{D}$ , then  $(Y/F)^{\text{op}} = F^{\text{op}}/Y$ . Thus, the left and right fiber categories are dual in the sense that  $P(Y, F)$  is true of any right fiber category  $Y/F$  iff  $P^{\text{op}}(Y, F)$  is true of any left fiber category  $F/Y$ .

**Proposition 1.5.** *Let  $F : \mathcal{D} \rightarrow \mathcal{C}$  be a functor and  $x \in \text{ob } \mathcal{C}$ . Then  $u : x \rightarrow Fr$  is a universal arrow from  $x$  to  $F$  iff it is initial object of the coslice  $(x \downarrow F)$ . Dually,  $u' : Fr' \rightarrow x$  is a universal arrow from  $F$  to  $x$  iff it is a terminal object of the same category.*

*Proof.* Suppose that  $u$  is universal and  $f : x \rightarrow Fy$  is another object of  $(x \downarrow F)$ . Then there is some unique  $\hat{f} : r \rightarrow y$  such that  $F(\hat{f}) \circ u = f$ . Thus  $F(\hat{f})$  is a unique morphism of the coslice.

Conversely, suppose that  $u$  is initial. Then for any object  $f : x \rightarrow Fy$  of  $(x \downarrow F)$ , there is some unique arrow  $Sg : Fr \rightarrow Fy$  such that  $Sg \circ u = f$ . Hence setting  $\hat{f} = g$  makes  $u$  a universal arrow.  $\square$

**Corollary 1.6.** *Any two universal arrows from  $x$  to  $F$  can be canonically identified by Lemma 1.3.*

## 2 (Co)limits

**Definition 2.1.** A *zero object* of  $\mathcal{C}$  is an object that is both initial and terminal.

**Example 2.2.** The unique initial object of **Set** is  $\emptyset$ , and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in  $\text{iso}(\mathbf{Set})$ .

Given  $X \in \text{ob } \mathcal{C}$ , the *undercategory*  $X/\mathcal{C}$  has as objects morphisms in  $\mathcal{C}$  of the form  $i : X \rightarrow Y$  where  $X$  is fixed. Given  $i : X \rightarrow Y$  and  $i' : X \rightarrow Y'$  in  $\text{ob } X/\mathcal{C}$ , define the set of morphisms from  $i$  to  $i'$  as the morphisms  $f : Y \rightarrow Y'$  where

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow i' & \downarrow f \\ & & Y' \end{array}$$

commutes. (We call  $i$  the *structure morphism*.) Composition and identity carry over exactly from  $\mathcal{C}$ .

Likewise, given  $x \in \text{ob } \mathcal{C}$ , the *overcategory*  $\mathcal{C}/X$  has as objects morphisms in  $\mathcal{C}$  of the form  $i : Y \rightarrow X$  where  $X$  is fixed. Given  $i : Y \rightarrow X$  and  $i' : Y' \rightarrow X$  in  $\text{ob } \mathcal{C}/X$ , define the set of morphisms from  $i$  to  $i'$  as the morphisms  $f : Y \rightarrow Y'$  where

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & \searrow i & \downarrow i' \\ & & X \end{array}$$

commutes. Composition and identity carry over exactly from  $\mathcal{C}$ .

**Remark 2.3.** If  $X \in \text{ob } \mathcal{C}$ , then  $(X/\mathcal{C})^{\text{op}} = \mathcal{C}^{\text{op}}/X$ . Thus, the under- and overcategories are dual in the sense that  $P(X, \mathcal{C})$  is true of any undercategory  $X/\mathcal{C}$  iff  $P^{\text{op}}(X, \mathcal{C})$  is true of any overcategory  $\mathcal{C}/X$ .

**Lemma 2.4.** *For any  $X \in \mathcal{C}$ , the identity morphism on  $X$  is an initial object  $X/\mathcal{C}$ . Dually, it is a terminal object in  $\mathcal{C}/X$ .*

*Proof.* Any  $i : X \rightarrow Y$  is itself the unique morphism from  $\text{Id}_X$  to  $i$ . □

**Lemma 2.5.** *Let  $X$  be an initial object of  $\mathcal{C}$ . The identity morphism on  $X$  is a zero object  $\mathcal{C}/X$ . Dually, if  $Y \in \text{ob } \mathcal{C}$  is terminal, then  $\text{Id}_Y$  is a zero object in  $Y/\mathcal{C}$ .*

*Proof.* We already know that  $\text{Id}_X$  is terminal. If  $p : Y \rightarrow X$  is an object in  $\mathcal{C}/X$ , then there is a unique morphism  $f : X \rightarrow Y$ . Then  $f \circ p$  must equal  $\text{Id}_X$ . □

**Example 2.6.** Let  $(X, x)$  be a pointed set with  $X = \{x\}$ . Let  $\mathbf{Set}_*$  denotes the category of pointed sets with base point preserving functions. Since  $\mathbf{Set}_* \cong X/\mathbf{Set}$ , it follows that  $X$  is a zero object in  $\mathbf{Set}_*$ .

Given a morphism  $\alpha : X \rightarrow Z$  in  $\mathcal{C}$ , define the *under-and-overcategory*  $(X/\mathcal{C}/Z)_\alpha$  as having triples  $(Y, i, p)$  as objects where  $i : X \rightarrow Y$  and  $p : Y \rightarrow Z$  are morphisms in  $\mathcal{C}$  such that  $p \circ i = \alpha$ . Define the set of morphisms from  $(Y, i, p)$  to  $(Y', i', p')$  as the set of morphisms  $f : Y \rightarrow Y'$  such that

$$\begin{array}{ccc} X & \xrightarrow{i'} & Y' \\ i \downarrow & \begin{array}{c} \text{blue } \nearrow \\ \text{red } \searrow \end{array} & \downarrow p' \\ Y & \xrightarrow{p} & Z \end{array}$$

$\alpha$

commutes. If  $\alpha = \text{Id}_X$ , then we call  $(X/\mathcal{C}/X)_{\text{Id}_X}$  the category of *retractive* objects over  $X$ , with each triple  $(Y, i, p)$  being a retraction of  $Y$  onto  $X$ .

**Example 2.7.** If  $F : \mathcal{C} \rightarrow \mathcal{C}$  is the identity functor, then the undercategory  $Y/\mathcal{C}$  equals the right fiber category  $Y/F$ , and the overcategory  $\mathcal{C}/Y$  equals the left fiber category  $F/Y$ .

**Definition 2.8.** Let  $\mathcal{J}$  be a category. A *diagram of shape  $\mathcal{J}$  in  $\mathcal{C}$*  is a functor  $F : \mathcal{J} \rightarrow \mathcal{C}$ .

**Definition 2.9.** Given a functor  $F : \mathcal{J} \rightarrow \mathcal{C}$  and  $X \in \text{ob } \mathcal{C}$ , a *cone over  $F$*  consists of an *apex*  $X \in \text{ob } \mathcal{C}$  and *legs*  $f_j : X \rightarrow F(j)$  for each  $j \in \text{ob } \mathcal{J}$  such that for any  $\alpha : j \rightarrow j'$ , the triangle

$$\begin{array}{ccc} X & \xrightarrow{f_j} & F(j) \\ & \searrow f_{j'} & \downarrow F\alpha \\ & & F(j') \end{array}$$

commutes.

This is simply a natural transformation  $\Delta_{\mathcal{J}} X \Rightarrow F$  where  $\Delta_{\mathcal{J}} X$  denotes the constant functor on  $\mathcal{J}$  at  $X$ . If  $\mathcal{J}$  is small, then  $\Delta_{\mathcal{J}}$  is just a functor from  $\mathcal{C}$  to  $\mathbf{Fun}(\mathcal{J}, \mathcal{C})$ .

**Definition 2.10.** The *category of cones over  $F$*  is the right fiber category  $X/F$ . The *category of cones under  $F$*  is the left fiber category  $F/X$ .

**Definition 2.11.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories and  $g : Y \rightarrow Z$  a morphism in  $\mathcal{D}$ . Let  $\Delta_{\mathcal{C}} g : \Delta_{\mathcal{C}} Y \Rightarrow \Delta_{\mathcal{C}} Z$  be the natural transformation with components  $X \mapsto g$ .

1. A *colimit*  $\text{colim}_{\mathcal{C}} F$  of the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an object  $Y$  of  $\mathcal{D}$  together with a natural transformation  $i : F \Rightarrow \Delta_{\mathcal{C}} Y$  such that for any  $Z \in \text{ob } \mathcal{D}$  and natural transformation  $j : F \Rightarrow \Delta_{\mathcal{C}} Z$ , there is a unique morphism  $g : Y \rightarrow Z$  such that  $j = \Delta_{\mathcal{C}} g \circ i$ .
2. We say that  $\mathcal{D}$  *admits/has  $\mathcal{C}$ -shaped colimits* if each functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  has a colimit.
3. We say that  $\mathcal{D}$  is *cocomplete* if each functor  $G : \mathcal{C} \rightarrow \mathcal{D}$  with  $\mathcal{C}$  small has a colimit.

If  $\mathcal{C}$  is small, then a colimit of  $F : \mathcal{C} \rightarrow \mathcal{D}$  is just an initial object in the right fiber category  $F/\Delta_{\mathcal{C}}$ , which has as objects pairs  $(Z, j : F \rightarrow \Delta Z)$  and as morphisms from  $(Y, i)$  to  $(Z, j)$  the morphisms  $g : Y \rightarrow Z$  in  $\mathcal{D}$  such that  $\Delta g \circ i = j$ .

*Remark 2.12.* There is a natural bijection  $\mathcal{D}(Y, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta Z)$  iff  $Y = \text{colim}_{\mathcal{C}} F$ .

**Proposition 2.13.** *Any two colimits are canonically isomorphic.*

*Proof.* When  $\mathcal{C}$  is small, this is immediate from Lemma 1.3. But note that the proof of Lemma 1.3 does *not* require that  $\mathcal{C}$  be locally small (a property which Rognes stipulates of any category).  $\square$

Assume that  $\mathcal{D}$  has  $\mathcal{C}$ -shaped colimits and that  $\mathcal{C}$  is small. Then a (possibly global) choice function  $\text{colim}_{\mathcal{C}} : \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$  given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor  $\Delta_{\mathcal{C}} : \mathcal{D} \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$ . Indeed, for any functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , there is a bijection  $\mathcal{D}(\text{colim}_{\mathcal{C}} F, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta_{\mathcal{C}} Z)$ .

**Definition 2.14.** A *limit* of the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is the colimit of  $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ .

Explicitly, a limit for  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an object  $Z$  of  $\mathcal{D}$  along with a natural transformation  $p : \Delta_{\mathcal{C}} Z \Rightarrow F$  such that for any  $Y \in \text{ob } \mathcal{D}$  and natural transformation  $q : \Delta_{\mathcal{C}} Y \Rightarrow F$ , there is a unique morphism  $g : Y \rightarrow Z$  such that  $q = p \circ \Delta_{\mathcal{C}} g$ .

*Remark 2.15.* The colimit of a functor  $F$  is the limit of  $F^{\text{op}}$ . Hence *limit* and *colimit* are dual properties, and our results so far for colimits can be dualized for limits.

**Example 2.16.** If  $\mathcal{C}$  is the empty category, then the empty functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  satisfies  $F/\Delta_{\mathcal{C}} \cong \mathcal{D}$ , so that the colimit is an initial object of  $\mathcal{D}$ .

**Definition 2.17.** Let  $\mathcal{J}$  be a discrete small category. Consider a diagram  $\{A_i\}_{i \in \text{ob } \mathcal{J}}$  of shape  $\mathcal{J}$ .

1. The limit of this diagram is called the *product*  $\prod_i A_i$ , equipped with projections  $\pi_i : \prod_i A_i \rightarrow A_i$  such that for every  $f_i : U \rightarrow A_i$  there is some unique map  $f := (f_i) : U \rightarrow \prod_i A_i$  satisfying  $\pi_i \circ f = f_i$ .

2. Dually, the colimit of the diagram is called the *coproduct*  $\coprod_i A_i$ , equipped with inclusions  $u_i : A_i \rightarrow \coprod_i A_i$  such that for any  $f_i : A_i \rightarrow Y$ , there is some unique map  $f := (f_i) : \coprod_i A_i \rightarrow Y$  satisfying  $f_i = f \circ u_i$ .

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Let  $\mathcal{J}$  be the category  $\bullet \rightrightarrows \bullet$ . Then a diagram of shape  $\mathcal{J}$  looks like  $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$ . A cone over this with apex  $C$  and legs  $f_1 : C \rightarrow A$  and  $f_2 : C \rightarrow B$  satisfies  $f f_1 = f_2 = g f_1$ .

**Definition 2.18.** If such an object  $C$  together with  $f_1$  is the limit of the diagram, then we say it is the *equalizer* of  $f$  and  $g$ . Dually, the colimit is called the *coequalizer* of  $f$  and  $g$ .

**Example 2.19.** The equalizer in **Set** of  $f, g : X \rightarrow Y$  is the subset  $X' := \{x \in X : f(x) = g(x)\}$  together with the inclusion function  $X' \hookrightarrow X$ . The coequalizer of  $(f, g)$  is  $Y/\sim$  together with the quotient map on  $B$  where  $\sim$  is the smallest equivalence relation under which  $f(x) \sim g(x)$  for every  $x$ .

**Example 2.20.** The equalizer in **Grp** is defined as in Example 2.19 except that the relation  $\sim$  becomes a certain minimal normal subgroup.

Now, let  $\mathcal{J}$  be the category  $\bullet \rightarrow \bullet \leftarrow \bullet$ . Then a diagram of this shape looks like  $B \xrightarrow{f} D \xleftarrow{g} A$ , and a cone over this diagram looks like

$$\begin{array}{ccc} C & \xrightarrow{j} & A \\ \downarrow i & \searrow \alpha & \downarrow g \\ B & \xrightarrow{f} & D \end{array}$$

**Definition 2.21.** If such an object  $C$  together with  $i$  and  $j$  is the limit of this diagram, then we call it the *pullback* of  $f$  and  $g$ , denoted by  $B \times_D A$ .

We can perform an analogous construction for  $\mathcal{J}^{\text{op}}$ . Then the colimit of the resulting diagram is called the *pushout*, denoted by  $B \cup_D A$ .

**Example 2.22.** The pullback in **Set** of  $f : X \rightarrow Z$  and  $g : Y \rightarrow Z$  is the subset  $\{(x, y) \in X \times Y : f(x) = g(y)\}$ , called the *fiber product* of  $X$  and  $Y$  over  $Z$ .

All coequalizers  $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B \xrightarrow{h} C$  can be obtained from taking binary coproducts and pushouts as follows.

$$\begin{array}{ccc} A \coprod A & \xrightarrow{(f,g)} & B \\ (\text{Id}_A, \text{Id}_A) \downarrow & & \downarrow h \\ A & \longrightarrow & C \end{array}$$

Therefore, any category with binary coproducts and pushouts has coequalizers.

Moreover, any colimit of a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \quad (*)$$

is precisely the coequalizer of

$$\coprod_n X_n \xrightleftharpoons[(u_{n+1} \circ f_n)]{\text{Id}} \coprod_n X_n.$$

Therefore, any category with coequalizers and small coproducts has colimits of diagrams like  $(*)$ . This fact can be generalized as follows.

**Theorem 2.23 (Freyd).**

- (i) If  $\mathcal{C}$  has equalizers and small (resp. finite) products, then it has small (resp. finite) limits.
- (ii) If  $\mathcal{C}$  has pullbacks and a terminal object, then it has finite limits.

*Proof.*

1. Let  $F : \mathcal{J} \rightarrow \mathcal{C}$  be any diagram with  $\mathcal{J}$  small. Consider the following two morphisms in  $\mathcal{C}$ :

$$\begin{aligned} f, g : \prod_{j \in \text{ob } \mathcal{J}} F_j &\rightarrow \prod_{\alpha : i \rightarrow j} F_j \\ \pi_{\alpha : i \rightarrow j} \circ f &\equiv \pi_j \\ \pi_{\alpha : i \rightarrow j} \circ g &\equiv F(\alpha) \circ \pi_i. \end{aligned}$$

Then  $\lim_{\mathcal{J}} F$  is precisely the equalizer of  $f$  and  $g$ .

2. Thanks to part (i), it suffices to show that  $\mathcal{C}$  has equalizers and finite products. By assumption, there is some terminal object  $1$ . Then any product  $A_1 \times A_2$  can be realized as the pullback of  $A_1 \rightarrow 1 \leftarrow A_2$ . By induction,  $\mathcal{C}$  has finite products. Moreover, for morphisms  $f, g : A \rightarrow B$ , note that any cone over the diagram

$$A \xrightarrow{(\text{Id}_A, g)} A \times B \xleftarrow{(\text{Id}_A, f)} A$$

yields morphisms  $h : A \rightarrow C$  and  $k : C \rightarrow A$  such that  $h = k$  and  $fk = gh$ . As a result, the pullback for this diagram is an equalizer of  $f$  and  $g$ , and thus our proof is complete.

□

**Corollary 2.24.** *Both **Set** and **Grp** are complete and cocomplete (or bicomplete).*

It turns out that adjoints interact nicely with (co)limits under mild conditions.

**Proposition 2.25.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be functors such that  $(F, G)$  is an adjoint pair. Let  $\mathcal{E}$  be small category. If  $X : \mathcal{E} \rightarrow \mathcal{C}$  is a functor whose colimit exists, then*

$$\text{colim}_{\mathcal{E}}(F \circ X) = F(\text{colim}_{\mathcal{E}} X).$$

*Dually, if  $Y : \mathcal{E} \rightarrow \mathcal{D}$  is a functor whose limit exists, then*

$$\lim_{\mathcal{E}}(G \circ Y) = G(\lim_{\mathcal{E}} Y).$$

*Proof.* We have the following chain of bijections natural in  $Y \in \mathcal{D}$ :

$$\begin{aligned}
\mathcal{D}(F(\operatorname{colim}_{\mathcal{E}} X), Y) &\cong \mathcal{C}(\operatorname{colim}_{\mathcal{E}} X, G(Y)) \\
&\cong \lim_{\mathcal{E}} \mathcal{C}(X(-), G(Y)) \\
&\cong \lim_{\mathcal{E}} \mathcal{D}(F(X(-)), Y) \\
&\cong \mathbf{Fun}(\mathcal{E}, \mathcal{D})(F \circ X, \Delta Y).
\end{aligned}$$

The second bijection exists because both sets can be identified with the components of all natural transformations  $X \Rightarrow \Delta G(Y)$ .  $\square$

### 3 Fibrations

**Definition 3.1.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. The *fiber category*  $F^{-1}(Y)$  is the full subcategory of  $\mathcal{C}$  generated by the objects  $X$  with  $F(X) = Y$ .

**Definition 3.2.** Suppose  $\mathcal{C}$  has a terminal object  $1$ .

1. Given a morphism  $p : 1 \rightarrow Y$ , the *fiber* of  $f$  at  $p$  is the pullback  $f^{-1}(p)$  of  $1 \rightarrow Y \leftarrow X$ .
2. The *cofiber* of a morphism  $f : X \rightarrow Y$  is the pushout  $Y/X$  of  $1 \leftarrow X \rightarrow Y$ .

Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. For each  $Y \in \operatorname{ob} \mathcal{D}$ , there is a full and faithful functor  $F^{-1}(Y) \rightarrow F/Y$  given by  $X \mapsto (X, \operatorname{Id}_Y)$ .

**Definition 3.3.** We say that  $\mathcal{C}$  is a *precofibered category* over  $\mathcal{D}$  if  $F$  admits a left adjoint given by  $(Z, g : F(Z) \rightarrow Y) \mapsto g_*(Z)$ .

Likewise, there is a full and faithful functor  $F^{-1}(Y) \rightarrow Y/F$ . We say that  $\mathcal{C}$  is a *prefibered category* over  $\mathcal{D}$  if this functor admits a right adjoint given by  $(Z, g : Y \rightarrow F(Z)) \mapsto g_*(Z)$ .

**Definition 3.4.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor.

1. Let  $f : c' \rightarrow c$  be a morphism in  $\mathcal{C}$ . We say  $f$  is *cartesian* if for any morphism  $f' : c'' \rightarrow c$  in  $\mathcal{C}$  and any morphism  $g : F(c'') \rightarrow F(c')$  in  $\mathcal{D}$  such that  $Ff \circ g = Ff'$ , there exists a unique  $\phi : c'' \rightarrow c$  such that  $f' = f \circ \phi$  and  $F\phi = g$ .

In other words, any filler of

$$\begin{array}{ccc}
c'' & \xrightarrow{\exists!} & c' \\
& \searrow f' & \downarrow f \\
& & c
\end{array}$$

can be lifted to a filler in  $\mathcal{D}$ .

2. We say that  $F$  is a *fibration* if for any  $c \in \mathcal{C}$  and morphism  $f : d \rightarrow Fc$ , there is a cartesian  $\phi : c' \rightarrow c$  such that  $F\phi = f$ . Such an  $\phi$  is called a *cartesian lifting* of  $f$  to  $c$ .

**Example 3.5.** Let the category **Mod** consist of pairs  $(R, M)$  as objects where  $R$  is a ring and  $M$  is a left  $R$ -module and pairs  $(f, \tilde{f})$  as morphisms where  $f : R \rightarrow R'$  is a ring homomorphism and  $\tilde{f} : M \rightarrow M'$  is an  $R$ -linear map with  $M'$  viewed as an  $R$ -module via  $f$ . Then the forgetful functor  $U : \mathbf{Mod} \rightarrow \mathbf{Ring}$  is a fibration.