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Abstract

After presenting our variant of constructive dependent type theory (CDTT), we develop the language necessary to postulate Voevodsky's univalence axiom (Univ), which formally encodes the identification of equivalent objects in any categorical model of CDTT + Univ. Afterwards, we describe the original construction of a model of CDTT + Univ in the (Quillen) model category of simplicial sets, due mainly to Voevodsky.

Contents

Overview

Our ultimate goal is to construct a certain model category in which every theorem of $\mathrm{CDTT} + \mathrm{Univ}$ is true. As we shall see, this model category interprets a dependent type as a fibration. The univalence axiom is so named because in any model category satisfying it, the canonical fibration over a chosen universe of types U is univalent, i.e., every fibration with small enough fibers is an essentially unique pullback of it. In the language of ∞-category theory, this means that a univalent fibration is a classifier for the class of all such fibrations.

It need *not* be the case, though, that $\mathrm{CDTT} + \mathrm{Univ}$ is modeled by any model category with object classifiers. In the categorical semantics of CDTT, the syntactic substitution of a term t for a variable occurring in a dependent type B is interpreted as a pullback of the fibration interpreting B along the morphism interpreting t. But substitution is strictly functorial, whereas pullback is merely functorial up to isomorphism. Thus, any model of $\mathrm{CDTT} + \mathrm{Univ}$ needs at least enough structure to make pullbacks in it along certain fibrations strictly associative. Our chosen model category will have such structure, as well as enough structure to model the strict behavior of the type constructors of CDTT. Finally, we must find an object classifier in our model category that is strictly "closed" under these type constructors.

1 Martin-Löf dependent type theory (MLDTT)

In this section and Section 2.1, we present a particular variant of (intensional) Martin-Löf dependent type theory, another name for constructive dependent type theory, in honor of the Swedish logician Per Martin-Löf. In the type theory literature, there are many other variants of the same theory. These differ from ours only in which logical/type constructors they include. The more one includes, the more expressive it is. Usually, other variants include at least our constructors for the unit type, the dependent product, and the dependent sum.

1.1 Syntax

A *Martin Löf dependent type theory* is a system of natural deduction whose object- and meta-languages are defined as follows. (See Section A for a review of deductive systems.)

Object language

Our presentation of the object language is inspired by [1].

First of all, we are given a countably infinite set of *variables*

$$\mathcal{V} := \{v_0, v_1, v_2, \ldots\}$$

along with the auxiliary symbols ':', ',', '(', and '('. Note that $\mathcal{V}$ inherits the well-ordering $\leq$ of $\mathbb{N}$.

We want to build our language out of variables and auxiliary symbols

Definition 1.1.1.

1. An *arity* is a tuple of the form

$$((n_1, \beta_1), (n_2, \beta_2), \ldots, (n_k, \beta_k), \beta)$$

where $k \in \mathbb{N}$ and $\beta, \beta_i \in \{0, 1\}$ for each $1 \le i \le k$. Let Ar denote the set of all arities.

2. A *signature* is a pair $(\Sigma^{\text{sym}}, \alpha)$ consisting of a set Σ^{sym} of *logical symbols/constructors* and a function $\alpha : \Sigma^{\text{sym}} \to Ar$. The value

$$\alpha(s) = ((n_1, \beta_1), (n_2, \beta_2), \ldots, (n_k, \beta_k), \beta)$$

is called the *arity of s*. If s has arity 0, then it is called a *term-valued symbol*. If it has arity 1, then it is called a *type-valued symbol*.

Definition 1.1.2 (0- and 1-expressions). Let $(\Sigma^{\text{sym}}, \alpha)$ be a signature. By mutual recursion, define the set Σ_0^{sym} of 0-*expressions* and the set Σ_1^{sym} of 1-*expressions* so that

(i) every variable is a 0-expression and

(ii) if $s \in \Sigma^{\text{sym}}$ has arity $((n_1, \beta_1), \ldots, (n_k, \beta_k), \beta)$ and M_i is a β_i-expression and $x_1^i, x_2^i, \ldots, x_{n_i}^i$ is a list of pairwise distinct variables for each $i \in \{1, 2, \ldots, k\}$, then

$$s\left(x_1^1.x_2^1.\ldots.x_{n_1}^1.M_1, \ \ldots, \ x_1^k.x_2^k.\ldots.x_{n_k}^k.M_k\right)$$

is a β-expression.

If $k = 0$, then we write s instead of $s()$ and say that s is a *constant symbol*.

Terminology. Other names for a 0-expression and 1-expression are *term expression/constructor* and *type expression/constructor*, respectively.

Think of the arity
$$\alpha(s) = ((n_1, \beta_1), (n_2, \beta_2), \ldots, (n_k, \beta_k), \beta)$$
of s as specifying an operation that

(a) takes k expressions as inputs (the sort of each indicated by β_i) with n_i pairwise distinct variables bound in the i-th input and

(b) outputs a new expression whose sort is indicated by β.

Let us add one more kind of expression to the object language.

Definition 1.1.3 (Context). A *context* is a list of the form

$$x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$$

such that

- each x_i denotes a variable,

- each A_i denotes a 1-expression, and

- for any $i, j \ge 1$ with $i \ne j$, we have that x_i and x_j denote distinct variables, i.e., each variable in the list is a *fresh* variable.

We say that the context *declares* the variables $x_1, \ldots, x_n$.

The set of raw terms / expressions is taken to be $\Sigma_0^{\text{exp}} \cup \Sigma_1^{\text{exp}} \cup \mathcal{X}$ where $\mathcal{X}$ denotes the set of all contexts.

Thus, the object language of a particular dependent type theory is determined by its signature.

All but one logical symbol in *our specific* MLDTT will appear in at least one of the inference rules postulated in Section 1.3, Section 1.5, or Section 2.1. The other one (namely univ) will appear in the univalence axiom (Definition 2.4.2). The arity of each logical symbol will be evident.

Table 1: A fragment of our signature

Symbol	Arity
Π	((0,1), (1,1), 1)
λ	((0,1), (1,1), (1,0), 0)
Σ	((0,1), (1,1), 1)
0	(1)
1	(1)
2	(1)
$0_{\mathbf{2}}$	(0)
U	(1)
el	((0,0), 1)
Id	((0,1), (0,0), (0,0), 1)

Definition 1.1.4 (Free variable).

1. Let t be a raw term other than a context. If t is a variable, then we say that t is *free in itself*.

   Otherwise, by construction, t is of the form

   $$s\left(x_1^1.x_2^1.\ldots.x_{n_1}^1.M_1, \ \ldots, \ x_1^k.x_2^k.\ldots.x_{n_k}^k.M_k\right).$$

   If x is a variable occurring in t and $x \notin \left\{x_j^i \mid i \in \{1,\ldots,k\}, \ j \in \{1,\ldots,n_i\}\right\}$, then x is *free in* t. If $x = x_j^i$ for some i and j, then we say that x is *bound in* M_i.

2. For any context $\underbrace{x_1 : A_1, \ldots, x_n : A_n}_{\Gamma}$, a variable x is *free in* Γ if $x = x_i$ for some $1 \leq i \leq n$ or x is free in A_i for some i.

The set of free variables in an expression ρ is denoted by $\text{FV}(\rho)$.

Example 1.1.5.

1. If A and B are 1-expressions and x is a variable, then the raw term $\Pi_{x:A}B := \Pi(A, x.B)$ is a 1-expression with x bound in B.

   This is similar to a first-order formula such as $\forall x.\psi$, where x is bound in ψ.

2. If A and B are 1-expressions, t is a term, and x is a variable, then the raw term $\lambda(x : A).t := \lambda(A, x.A, x.t)$ is a 0-expression with x bound in A and t.

   This is similar to a first-order definition of a function such as $f(x) = x + yz$, where x is bound in $x + yz$.

Meta-language

By assumption, the meta-language contains a countably infinite set of *meta-variables*, which range over raw terms.

Notation.

- The symbols x_i, y_i, x, y, z, x', etc. will refer to arbitrary variables.

- The symbols Γ, Δ, Θ, Γ', Γ_i, etc. will refer to arbitrary contexts.

- The symbols a, b, c, d, e, f, g, h, m, t, s, t', t_i, s_i, τ, etc. will refer to arbitrary 0-expressions.

- The symbols A, B, C, A', B', C', A_i, etc. will refer to arbitrary 1-expressions.

For readability, we may write $A(x_1, \ldots, x_n)$ in the meta-language to indicate that the variables occurring in the 1-expression denoted by A include $x_1, \ldots, x_n$.

Any judgment will have one of six forms.

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1. (well-formed context) "Γ is a well-formed context," formally,

$$\mathtt{ctx}(\Gamma).$$

2. (equality of contexts) "Γ and Δ are judgmentally equal well-formed contexts," formally,

$$\Gamma \equiv \Delta \, \mathtt{ctx}.$$

3. (typehood) "A is a well-formed type in context Γ," formally,

$$\Gamma \vdash A \, \mathtt{type}.$$

4. (typing declaration) "a is a (well-formed) term of type A / inhabiting A in context Γ," formally,
$$\Gamma \vdash a : A.$$

5. (equality of types) "A and B are judgmentally equal well-formed types in context Γ," formally,
$$\Gamma \vdash A \equiv B \, \mathtt{type}.$$

6. (equality of terms) "a and b are judgmentally equal well-formed terms of type A in context Γ," formally,
$$\Gamma \vdash a \equiv b : A.$$

A *generic judgment* refers to any judgment with one of the last four forms. A generic judgment consists of an antecedent Γ and a consequent, e.g., $A \, \mathtt{type}$. We call such a judgment a *hypothetical judgment*, thinking of Γ as a list of hypotheses. For example, the theory of a category $\mathscr{C}$ has as an axiom $x : \mathsf{obj}, y : \mathsf{obj} \vdash \mathsf{hom}(x, y) \, \mathtt{type}$. Intuitively, this is intended to mean that $\mathsf{hom}(x, y)$ is a well-formed type whenever x and y are objects in $\mathscr{C}$.

Notation. The symbol $\mathcal{K}$ will denote the consequent of a generic judgment.

Note that the inference rules of a MLDTT induce six relations on the set of all raw terms, which determine the subset of raw terms that are *well-formed*.

Finally, we define a family of total operations on expressions and then define another such family in terms of it.

Definition 1.1.6 (Capture-free substitution). Let y be a variable and t be a term expression. Let ρ be any expression. Define the *(capture-free) substitution of t for (free occurrences of) y in ρ*, denoted by

$$\rho\left[t/y\right],$$

as follows.

1. Suppose that ρ is not a context. Then $\rho\left[t/x\right]$ is the finite string obtained recursively by

   (a) $x\left[t/y\right] = \begin{cases} t & x = y \\ x & x \neq y \end{cases}$,

   (b) $\kappa\left[t/y\right] = \kappa$ where κ denotes a constant symbol, and

   (c) $s\left(\underbrace{x_1^1.x_2^1.\ldots.x_{n_1}^1.M_1, \ \ldots, \ x_1^k.x_2^k.\ldots.x_{n_k}^k.M_k}_{k \geq 1}\right)\left[t/y\right] = s\left(N_1, \ldots, N_k\right)$

   where for each $i \in \{1, \ldots, k\}$,

   $$N_i = \begin{cases} \tilde{x}_1^i.\tilde{x}_2^i.\ldots.\tilde{x}_{n_i}^i.M_i\left[t/y\right] & y \notin \left\{x_1^i, \ldots, x_{n_i}^i\right\} \\ x_1^i.x_2^i.\ldots.x_{n_i}^i.M_i & \text{otherwise} \end{cases}$$

   such that for any $1 \leq j \leq n_i$, we choose $\tilde{x}_j^i$ to be the least variable $z \geq x_j^i$ with $z \notin \left\{\tilde{x}_1^i, \ldots, \tilde{x}_{j-1}^i\right\} \cup \mathrm{FV}(M_i) \cup \mathrm{FV}(t)$.

2. If ρ is a context, then $\rho\left[t/y\right]$ is the context obtained recursively by

   (a) $\epsilon\left[t/y\right] = \epsilon$, where ϵ denotes the empty list, and

   (b) $\left(\underbrace{x_1 : A_1, \ldots, x_k : A_k}_{k \geq 1}\right)\left[t/y\right] =$

   $$\begin{cases} (x_1 : A_1, \ldots, x_{k-1} : A_{k-1})\left[t/y\right], \tilde{x}_k : A_k\left[t/y\right] & x_i \neq y, \ i = 1, \ldots, k \\ \epsilon & \text{otherwise} \end{cases}$$

   where we choose $\tilde{x}_k$ to be the least variable $z \geq x_k$ such that $z \notin \left\{x_1, \ldots, x_{k-1}\right\} \cup \mathrm{FV}(t)$.

We extend this definition in the obvious way to a definition of the *substitution of t for x in $\mathcal{K}$*, denoted by $\mathcal{K}\left[t/x\right]$.

Remark 1.1.7. We have defined substitution so as to avoid obtaining a raw term in which variables intended to be free are *captured*, i.e., become bound. For example, in the language of arithmetic, naively substituting the variable y for x in the formula $\exists y(x+y=1)$ results in $\exists y(y+y=1)$, which is not logically equivalent to the original formula. Rather, one ought to first convert $\exists y(x+y=1)$ to something like $\exists z(x+z=1)$ and then naively substitute y for x in it.

Definition 1.1.8 (Simultaneous substitution). Let $x_1, \ldots, x_n$ be pairwise distinct variables and let $t_1, \ldots, t_n$ be term expressions such that

$$(\mathrm{FV}(t_1) \cup \cdots \cup \mathrm{FV}(t_n)) \cap \{x_1, \ldots, x_n\} = \emptyset.$$

Let ρ be any expression. Define the *simultaneous substitution of $t_1, \ldots, t_n$ for $x_1, \ldots, x_n$ in ρ*, denoted by

$$\rho\,[t_1, \ldots, t_n / x_1, \ldots, x_n]\,,$$

as the term

$$\rho\,[t_1/x_1]\,[t_2/x_2] \cdots [t_n/x_n]\,,$$

which is obtained by iterated substitution. We extend this definition in the obvious way to a definition of the *simultaneous substitution of $t_1, \ldots, t_n$ for $x_1, \ldots, x_n$ in $\mathcal{K}$*, denoted by $\mathcal{K}\,[t_1, \ldots, t_n / x_1, \ldots, x_n]$.

Example 1.1.9.

1. $(\lambda(x : A).y)\,[z/x] = \lambda(x : A).y$.

2. $(\lambda(v_1 : A).v_2)\,[v_0, v_3 / v_2, v_1] = \lambda(v_1 : A).v_0$.

1.2 Structural rules

We also require any MLDTT to include certain inference rules known as *structural rules*, which we now list.

First, we postulate four structural rules, which govern the formation and equality of well-formed contexts:

$$\frac{}{\mathtt{ctx}(\epsilon)} \quad \text{(where } \epsilon \text{ denotes the empty list)}$$

$$\frac{x_1 : A_1, \ldots, x_{n-1} : A_{n-1} \vdash A_n\,\mathtt{type}}{\mathtt{ctx}\,(x_1 : A_1, \ldots, x_n : A_n)} \quad \text{(when } x_n \notin \{x_1, \ldots, x_{n-1}\})$$

$$\frac{}{\epsilon \equiv \epsilon\,\mathtt{ctx}} \qquad \frac{\Gamma \equiv \Delta\,\mathtt{ctx} \qquad \Gamma \vdash A \equiv B\,\mathtt{type}}{\Gamma, x : A \equiv \Delta, y : B\,\mathtt{ctx}} \quad \text{(when } x \notin \mathrm{FV}(\Gamma) \text{ and } y \notin \mathrm{FV}(\Delta))$$

where

$$x_1 : A_1, \ldots, x_{n-1} : A_{n-1}$$

is, by convention, the empty context when $n = 1$. It follows that a context is well-formed exactly when it is either empty or an expression of the form

$$x_1 : A_1, x_2 : A_2, \ldots, x_n : A_n$$

such that

- if $2 \leq k \leq n$, then

$$x_1 : A_1, x_2 : A_2, \ldots, x_{k-1} : A_{k-1} \vdash A_k\,\mathtt{type}$$

  and

- if $n = 1$, then
$$\vdash A_1 \, \texttt{type} \, .$$

In this case, we say that A_1 is a *closed type*. Also, if a is a term of type A in the empty context, then we say that a is a *closed term*.

Note 1.2.1. It turns out that, in light of all of our inference rules to be presented, the following meta-theoretic property will be true of our system:

For any well-formed context
$$x_1 : A_1, \ldots, x_n : A_n,$$

we have that $\mathrm{FV}(A_i) \subset \{x_1, \ldots, x_{i-1}\}$. Moreover, if we can derive both $\texttt{ctx}(\Gamma)$ and $\Gamma \vdash a : A$, then any free variable in either a or A must be declared by Γ.

In addition, we postulate the rules

$$\frac{\texttt{ctx}(\Gamma)}{\Gamma \equiv \Gamma \, \texttt{ctx}} \qquad \frac{\Gamma \equiv \Delta \, \texttt{ctx}}{\Delta \equiv \Gamma \, \texttt{ctx}} \qquad \frac{\Gamma \equiv \Delta \, \texttt{ctx} \qquad \Delta \equiv \Theta \, \texttt{ctx}}{\Gamma \equiv \Theta \, \texttt{ctx}} \quad ,$$

which together assert that equality of contexts is an equivalence relation.

To be able to manipulate variables for convenience in our derivations, we postulate four more structural rules:

$$\text{V{\footnotesize BLE}} \, \frac{\texttt{ctx}(\Gamma, x : A, \Delta)}{\Gamma, x : A, \Delta \vdash x : A} \qquad\qquad \text{S{\footnotesize UBST}} \, \frac{\Gamma \vdash a : A \qquad \Gamma, x : A, \Delta \vdash \mathcal{K}}{\Gamma, \Delta[a/x] \vdash \mathcal{K}[a/x]}$$

$$\text{W{\footnotesize KG}} \, \frac{\Gamma \vdash A \, \texttt{type} \qquad \Gamma, \Delta \vdash \mathcal{K}}{\Gamma, x : A, \Delta \vdash \mathcal{K}} \quad (\text{when } x \text{ is not free in } \Gamma, \Delta)$$

$$\text{E{\footnotesize XCHANGE}} \, \frac{\Gamma, x : A, y : B, \Delta \vdash \mathcal{K}}{\Gamma, x : B, y : A, \Delta \vdash \mathcal{K}} \quad (\text{when } x \text{ is not free in } B)$$

$$\alpha\text{-{\footnotesize CONV-CTX}} \, \frac{\Gamma, x : A, \Delta \vdash \mathcal{K}}{\Gamma, x' : A, \Delta[x'/x] \vdash \mathcal{K}[x'/x]} \quad (\text{when } x' \text{ is not free in } \Gamma, x : A, \Delta) \quad .$$

(i) The *variable rule* V{\footnotesize BLE} asserts that each declared variable in a well-formed context is well-typed.

(ii) The *substitution rule* S{\footnotesize UBST} asserts that substituting a declared variable with a term of the same type preserves $\mathcal{K}$.

(iii) The *weakening rule* W{\footnotesize KG} asserts that expanding the context by a fresh variable of type A (known as *weakening by A*) preserves $\mathcal{K}$.

(iv) The *exchange rule* E{\footnotesize XCHANGE} asserts that certain permutations of the context preserve $\mathcal{K}$.

(v) The *context α-conversion* α-{\footnotesize CONV-CTX} rule asserts that renaming a declared variable as a fresh variable preserves $\mathcal{K}$.

Additionally, we postulate the structural rules

$$\frac{\Gamma \vdash a \equiv b : A \qquad \Gamma, x : A, \Delta \vdash B \, \mathtt{type}}{\Gamma, \Delta[a/x] \vdash B[a/x] \equiv B[b/x] \, \mathtt{type}} \qquad \frac{\Gamma \vdash a \equiv a' : A \qquad \Gamma, x : A, \Delta \vdash b : B}{\Gamma, \Delta[a/x] \vdash b[a/x] \equiv b[a'/x] : B[a/x]},$$

which assert certain congruence conditions for judgmental equality of terms.

Next, we postulate those structural rules governing judgmental equality of types. Specifically, we have the rules

$$\frac{\Gamma \vdash A \, \mathtt{type}}{\Gamma \vdash A \equiv A \, \mathtt{type}} \qquad \frac{\Gamma \vdash A \equiv B \, \mathtt{type}}{\Gamma \vdash B \equiv A \, \mathtt{type}} \qquad \frac{\Gamma \vdash A \equiv B \, \mathtt{type} \qquad \Gamma \vdash B \equiv C \, \mathtt{type}}{\Gamma \vdash A \equiv C \, \mathtt{type}},$$

which together assert that judgmental equality of types is an equivalence relation, along with the *variable conversion* rule

$$\frac{\Gamma \vdash A \equiv B \, \mathtt{type} \qquad \Gamma, x : A, \Delta \vdash \mathcal{K}}{\Gamma, x : B, \Delta \vdash \mathcal{K}}.$$

Finally, we postulate those structural rules governing judgmental equality of terms.

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash a \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv b : A}{\Gamma \vdash b \equiv a : A} \qquad \frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash b \equiv c : A}{\Gamma \vdash a \equiv c : A}$$

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash A \equiv B \, \mathtt{type}}{\Gamma \vdash a : B} \qquad \frac{\Gamma \vdash a \equiv b : A \qquad \Gamma \vdash A \equiv B \, \mathtt{type}}{\Gamma \vdash a \equiv b : B}.$$

Together, these assert that judgmental equality of terms is an equivalence relation respected by typing.

1.3   Logical rules

In addition to our structural rules, our MLDTT postulates certain inference rules known as *logical rules*. These allow us to define various types inductively. We can describe five main kinds of logical rules.

1. A *type formation* rule, which asserts those conditions under which we can use a type constructor to form a new type B.

2. A *term introduction* rule, which asserts those conditions under which we can use a a term constructor to form a term of type B.

   This term is called a *canonical term* of type B.

3. A *term elimination* rule (sometimes called an *induction principle*), which asserts that to define a "section over B" it is both necessary and sufficient to define it on the term constructors for B. That is, it is enough to define it on the canonical terms of type B.

4. A *computation rule*, whose conclusion is a judgmental equality allowing us to rewrite the result of applying term elimination to a term formed by term introduction.

5. A *congruence rule*, which asserts that a given logical constructor preserves judgmental equality in each of its arguments.

An *inductive type* is governed by at least a type formation rule and an induction principle. An inductive type is *non-degenerate* if it is governed by all five kinds of rule (among others).

Dependent products (Π-types)

$$\Pi\text{-}{\rm FORM}\ \frac{\Gamma \vdash A\,\mathtt{type} \qquad \Gamma, x : A \vdash B(x)\,\mathtt{type}}{\Gamma \vdash \Pi_{x:A}B(x)\,\mathtt{type}}$$

$$\Pi\text{-}{\rm INTRO}\ \frac{\Gamma, x : A \vdash B(x)\,\mathtt{type} \qquad \Gamma, x : A \vdash b(x) : B(x)}{\Gamma \vdash \lambda(x : A).b(x) : \Pi_{x:A}B(x)}$$

$$\Pi\text{-}{\rm ELIM}\ \frac{\Gamma \vdash f : \Pi_{x:A}B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}(f, a) : B[a/x]}$$

$$\Pi\text{-}{\rm COMP}\ \frac{\Gamma, x : A \vdash B(x)\,\mathtt{type} \qquad \Gamma, x : A \vdash b(x) : B(x) \qquad \Gamma \vdash a : A}{\Gamma \vdash \mathsf{app}\left(\lambda(x : A).b(x), a\right) \equiv b[a/x] : B[a/x]}$$

$$\Pi\text{-}{\rm CONG}(1)\ \frac{\Gamma \vdash A \equiv A'\,\mathtt{type} \qquad \Gamma, x : A \vdash B(x) \equiv B'(x)\,\mathtt{type}}{\Gamma \vdash \Pi_{x:A}B(x) \equiv \Pi_{x:A'}B'(x)\,\mathtt{type}}$$

$$\Pi\text{-}{\rm CONG}(2)\ \frac{\Gamma \vdash A \equiv A'\,\mathtt{type} \qquad \Gamma, x : A \vdash B(x) \equiv B'(x)\,\mathtt{type} \qquad \Gamma, x : A \vdash b(x) \equiv b'(x) : B(x)}{\Gamma \vdash \lambda(x : A).b(x) \equiv \lambda(x : A').b'(x) : \Pi_{x:A}B(x)}$$

$$\Pi\text{-}\eta\ \frac{\Gamma \vdash f : \Pi_{x:A}B(x)}{\Gamma \vdash f \equiv \lambda(x : A).\mathsf{app}(f, x) : \Pi_{x:A}B(x)}$$

A judgment $\Gamma \vdash A\,\mathtt{type}$ together with a judgment

$$\Gamma, x : A \vdash B(x)\,\mathtt{type}$$

is called a *type family B over A*. Informally, this corresponds to the fiber bundle $\pi : \coprod_{x \in A} B(x) \twoheadrightarrow A$. Moreover, we can think of an inhabitant f of the dependent product $\Pi_{x:A}B(x)$ as a set-theoretic function $f : A \to \bigcup_{x \in A} B(x)$ where $f(x) \in B(x)$ for each $x \in A$ (i.e., f is a choice function). We can also think of f as a section of the fiber bundle determined by $\Gamma, x : A \vdash B(x)\,\mathtt{type}$. In any case, we shall call such an f a *section of B over A* or a *dependent function on A*.

Notation.

- We may write $\lambda x.b(x)$ for the expression $\lambda(x : A).b(x)$.

- We may write $f(a)$ and $f(x)$ for the expressions $\mathsf{app}(f, a)$ and $\mathsf{app}(f, x)$, respectively.

Example 1.3.1 (Function types). Using the weakening rule, we get the derivation

$$\frac{\dfrac{\Gamma \vdash A\,\mathtt{type} \qquad \Gamma \vdash B\,\mathtt{type}}{\Gamma, x : A \vdash B\,\mathtt{type}}}{\Gamma \vdash \Pi_{x:A}B\,\mathtt{type}} \qquad .$$

In context Γ, the expression $\Pi_{x:A}B$ is called the *type of (non-dependent) functions from A to B*. Thus, a non-dependent function is a special case of a dependent one.

Notation. We may write $A \to B$ or B^A for the type of functions from A to B. To avoid ambiguity, we stipulate that the symbol $\to$ is right associative.

Example 1.3.2 (Swap function). We can apply the exchange rule together with the context α-conversion rule to obtain the derivation

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash A\,\mathtt{type} \qquad \Gamma \vdash B\,\mathtt{type} \qquad \Gamma, x:A, y:B \vdash C(x,y)\,\mathtt{type}}{\mathtt{ctx}(\Gamma, f:\Pi_{(x:A)}\Pi_{(y:B)}C(x,y), x:A, y:B)}}{\Gamma, f:\Pi_{(x:A)}\Pi_{(y:B)}C(x,y), x:A, y:B \vdash f(x)(y):C(x,y)}}{\Gamma, f:\Pi_{(x:A)}\Pi_{(y:B)}C(x,y), y:B, x:A \vdash f(y)(x):C(x,y)}}{\Gamma, f:\Pi_{(x:A)}\Pi_{(y:B)}C(x,y) \vdash \lambda y.\lambda x.f(y)(x):\Pi_{(y:B)}\Pi_{(x:A)}C(x,y)}}{\Gamma \vdash \lambda f.\lambda y.\lambda x.f(y)(x):\big(\Pi_{(x:A)}\Pi_{(y:B)}C(x,y)\big) \to \big(\Pi_{(y:B)}\Pi_{(x:A)}C(x,y)\big)} \quad .$$

Intuitively, this shows that we can switch the order of two independent arguments of a dependent function.

Note 1.3.3 (α-equivalence). Moreover, we shall postulate certain α-conversion rules when defining a new inductive type. In the case of Π-types, these are precisely

$$\frac{\Gamma, x:A \vdash B(x)\,\mathtt{type}}{\Gamma \vdash \Pi_{x:A}B(x) \equiv \Pi_{x':A}B[x'/x]\,\mathtt{type}} \quad \text{(when } x' \text{ is not free in } \Gamma, x:A\text{)}$$

$$\frac{\Gamma, x:A \vdash b(x):B(x)}{\Gamma \vdash \lambda x.b(x) \equiv \lambda x'.b[x'/x]:\Pi_{x:A}B(x)} \quad \text{(when } x' \text{ is not free in } \Gamma, x:A\text{)}$$

These assert that we can always rename bound variables in λ- or Π-expressions so long as we avoid variable capture. We shall tacitly assert similar rules for each subsequent logical constructor.

Definition 1.3.4.

1. Using the variable rule, we get the derivation

$$\frac{\dfrac{\mathtt{ctx}(\Gamma, x:A) \qquad \Gamma \vdash A\,\mathtt{type}}{\Gamma, x:A \vdash x:A}}{\Gamma \vdash \lambda x.x:A \to A} \quad .$$

We call $\lambda x.x$ the *identity map on A*, written as $\mathtt{idmap}_A$.

2. Using both the variable and weakening rules, we get the derivation

$$\frac{\dfrac{\dfrac{\mathtt{ctx}(\Gamma, y:B, x:A) \qquad \Gamma \vdash A\,\mathtt{type}}{\Gamma, y:B \vdash y:B}}{\Gamma, y:B, x:A \vdash y:B}}{\Gamma, y:B \vdash \lambda x.y:A \to B} \quad .$$

We call $\lambda x.y$ the *constant map at y*, denoted by $\mathtt{cons}_y$.

3. It is straightforward yet tedious to derive the rule

$$\frac{\begin{array}{c} \Gamma \vdash A\,\mathsf{type} \qquad \Gamma \vdash B\,\mathsf{type} \qquad \Gamma \vdash C\,\mathsf{type} \\ \mathsf{ctx}(\Gamma, g : C^B, f : B^A, x : A, y : B) \end{array}}{\Gamma \vdash \lambda g.\lambda f.\lambda x.g(f(x)) : C^B \to (B^A \to C^A)} \ .$$

We write $j \circ h$ for

$$\mathsf{app}\left(\mathsf{app}\left(\lambda g.\lambda f.\lambda x.g(f(x)), j\right), h\right),$$

called the *composition of j with h*.[1]

Theorem 1.3.5. *The rule*

$$\frac{\Gamma \vdash f : A \to B \qquad \Gamma \vdash g : B \to C \qquad \Gamma \vdash h : C \to D}{\Gamma \vdash (h \circ g) \circ f \equiv h \circ (g \circ f) : A \to D}$$

is derivable.[2] *Hence function composition is associative.*

Dependent sums (Σ-types)

Remark 1.3.6. From now on, we shall postulate *tacitly* a congruence rule CONG for each new logical constructor that we define. This rule will be like that found in our definition of Π-types.

$$\Sigma\text{-FORM}\ \frac{\Gamma \vdash A\,\mathsf{type} \qquad \Gamma, x : A \vdash B(x)\,\mathsf{type}}{\Gamma \vdash \Sigma_{x:A}B(x)\,\mathsf{type}}$$

$$\Sigma\text{-INTRO}\ \frac{\Gamma, x : A \vdash B(x)\,\mathsf{type} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : B[a/x]}{\Gamma \vdash \mathsf{pair}(a, b) : \Sigma_{x:A}B(x)}$$

$$\Sigma\text{-ELIM}\ \frac{\Gamma, z : \Sigma_{x:A}B(x) \vdash C(z)\,\mathsf{type} \qquad \Gamma, x : A, y : B(x) \vdash d(x, y) : C\left[\mathsf{pair}(x, y)/z\right] \qquad \Gamma \vdash p : \Sigma_{x:A}B(x)}{\Gamma \vdash \mathsf{split}(z.C, x.y.d, p) : C[p/z]}$$

$$\Sigma\text{-COMP}\ \frac{\begin{array}{c} \Gamma, z : \Sigma_{x:A}B(x) \vdash C(z)\,\mathsf{type} \\ \Gamma, x : A, y : B(x) \vdash d(x, y) : C\left[\mathsf{pair}(x, y)/z\right] \\ \Gamma \vdash a : A \qquad \Gamma \vdash b : B[a/x] \end{array}}{\Gamma \vdash \mathsf{split}(z.C, x.y.d, \mathsf{pair}(x, y)) \equiv d[a, b/x, y] : C\left[\mathsf{pair}(a, b)/z\right]}$$

Informally, we can think of a dependent sum $\Sigma_{x:A}B(x)$ as a set-theoretic disjoint union $\coprod_{x \in A} B(x)$.

Notation. We may write $\mathsf{pair}(x, y)$ as (x, y).

Note 1.3.7. Intuitively, Σ-ELIM asserts that to construct a dependent function out of $\Sigma_{x:A}B(x)$, it suffices to construct, for each canonical element (a, b), a term of type $C\left[(a, b)/z\right]$.

---

[1] [20, Definition 2.2.4] includes a full derivation.
[2] [20, Lemma 2.2.5].

Definition 1.3.8. Suppose that we have a type family B over A.

1. Define the *first projection map* $\text{pr}_1 : (\Sigma_{x:A} B(x)) \to A$ inductively by

$$\text{pr}_1(x, y) := x.$$

Notation. In the style of programming languages, the symbol $:=$ here means that pr_1 is syntactic sugar for $\text{split}(A, x.y.x, (x, y))$. In particular, pr_1 is *not* a symbol in our object language. We shall make use of sugaring throughout.

2. Define the *second projection map* $\text{pr}_2 : \Pi_{\left(p:\Sigma_{(x:A)} B(x)\right)} B\left(\text{pr}_1(p)\right)$ inductively by

$$\text{pr}_2(x, y) := y.$$

Example 1.3.9 (Product types). Using the weakening rule, we get the derivation

$$\frac{\dfrac{\Gamma \vdash A \,\text{type} \qquad \Gamma \vdash B \,\text{type}}{\Gamma, x : A \vdash B \,\text{type}}}{\Gamma \vdash \Sigma_{x:A} B \,\text{type}} \quad .$$

In context Γ, the expression $\Sigma_{x:A} B$ is called the *(cartesian) product of A and B*.

Notation. We may write $A \times B$ for the product of A and B.

Empty type (0)

$$\textbf{0}\text{-FORM} \; \frac{\text{ctx}(\Gamma)}{\Gamma \vdash \mathbf{0} \,\text{type}} \qquad\qquad \textbf{0}\text{-ELIM} \; \frac{\Gamma, x : \mathbf{0} \vdash C(x) \,\text{type} \qquad \Gamma \vdash a : \mathbf{0}}{\Gamma \vdash \text{ind}_{\mathbf{0}}(x.C, a) : C[a/x]}$$

Note that the empty type is a degenerate inductive type. In particular, if one can derive a typing declaration of the form $\Gamma \vdash t : \mathbf{0}$, then one can derive any typing declaration with context Γ. Thus, the empty type corresponds, informally, to the empty set in set theory for if one can prove $\exists x \, (x \in \emptyset)$, then one can prove any sentence in the language of set theory.

Unit type (1)

$$\textbf{1}\text{-FORM} \; \frac{\text{ctx}(\Gamma)}{\Gamma \vdash \mathbf{1} \,\text{type}} \qquad\qquad \textbf{1}\text{-INTRO} \; \frac{\text{ctx}(\Gamma)}{\Gamma \vdash \star : \mathbf{1}}$$

$$\textbf{1}\text{-ELIM} \; \frac{\Gamma, x : \mathbf{1} \vdash C(x) \,\text{type} \qquad \Gamma \vdash c : C\left[\star/x\right] \qquad \Gamma \vdash a : \mathbf{1}}{\Gamma \vdash \text{ind}_{\mathbf{1}}(x.C, c, a) : C[a/x]}$$

$$\textbf{1}\text{-COMP} \; \frac{\Gamma, x : \mathbf{1} \vdash C(x) \,\text{type} \qquad \Gamma \vdash c : C\left[\star/x\right]}{\Gamma \vdash \text{ind}_{\mathbf{1}}(x.C, c, \star) \equiv c : C\left[\star/x\right]}$$

Note that in any well-formed context Γ, we have that $\star$ is the unique term of type $\mathbf{1}$. Thus, the unit type corresponds, informally, to a singleton set in set theory.

Boolean type (2)

$$\textbf{2-}\textsc{form}\ \frac{\texttt{ctx}(\Gamma)}{\Gamma \vdash \textbf{2}\,\texttt{type}} \qquad \textbf{2-}\textsc{intro}(1)\ \frac{\texttt{ctx}(\Gamma)}{\Gamma \vdash 0_{\textbf{2}} : \textbf{2}} \qquad \textbf{2-}\textsc{intro}(2)\ \frac{\texttt{ctx}(\Gamma)}{\Gamma \vdash 1_{\textbf{2}} : \textbf{2}}$$

$$\textbf{2-}\textsc{elim}\ \frac{\Gamma, x : \textbf{2} \vdash C(x)\,\texttt{type} \qquad \Gamma \vdash c : C\,[0_{\textbf{2}}/x] \qquad \Gamma \vdash d : C\,[1_{\textbf{2}}/x] \qquad \Gamma \vdash a : \textbf{2}}{\Gamma \vdash \mathsf{ind}_{\textbf{2}}(x.C, c, d, a) : C\,[a/x]}$$

$$\textbf{2-}\textsc{comp}(1)\ \frac{\Gamma, x : \textbf{2} \vdash C(x)\,\texttt{type} \qquad \Gamma \vdash c : C\,[0_{\textbf{2}}/x]}{\Gamma \vdash \mathsf{ind}_{\textbf{2}}(x.C, c, 0_{\textbf{2}}) \equiv c : C\,[0_{\textbf{2}}/x]}$$

$$\textbf{2-}\textsc{comp}(2)\ \frac{\Gamma, x : \textbf{2} \vdash C(x)\,\texttt{type} \qquad \Gamma \vdash d : C\,[1_{\textbf{2}}/x]}{\Gamma \vdash \mathsf{ind}_{\textbf{2}}(x.C, d, 1_{\textbf{2}}) \equiv d : C\,[1_{\textbf{2}}/x]}$$

Informally, the Boolean type corresponds to the set of truth values $\{F, T\}$ in propositional logic.

1.4 Propositions as types

The following table describes the so-called Curry-Howard correspondence.

Table 2: Logical and set-theoretic interpretations of type theory

FOL (with bounded quantifiers)	Set theory	Type theory
Proposition	Set	Well-formed type
Proof	Element	Inhabitant
$\neg A$	A^c	$A \to \textbf{0}$
$A \wedge B$	$A \times B$	$A \times B$
$A \to B$	B^A	$A \to B$
$\forall_{x:A} B(x)$	$\prod_{x \in A} B(x)$	$\Pi_{x:A} B(x)$
$\exists_{x:A} B(x)$	$\coprod_{x \in A} B(x)$	$\Sigma_{x:A} B(x)$
$\top$	$\{0\}$	$\textbf{1}$
$\bot$	$\emptyset$	$\textbf{0}$

In particular, this correspondence between first-order logic and type theory, called the *propositions-as-types doctrine*, encodes a system of constructive logic inside our type theory. (It is constructive in the sense that proving a proposition P corresponds to constructing a term of type P.) For example, negation in our MLDTT corresponds to the principle of explosion in constructive logic, and Π-ELIM corresponds to modus ponens. In addition, the propositions-as-types doctrine automatically provides us with a type-theoretic notion of logical equivalence.

Definition 1.4.1 (Logical equivalence). Suppose that we have derived a rule of the form

$$\frac{J_1 \quad \dots \quad J_n}{\begin{array}{cc} \Gamma \vdash A\,\texttt{type} & \Gamma \vdash B\,\texttt{type} \\ \Gamma \vdash f : A \to B & \Gamma \vdash g : B \to A \end{array}}$$

where each J_i denotes a judgment. Then we say that the type expressions A and B are *logically equivalent*.

1.5 The universe (U)

With the following logical rules, we define a closed type called the *universe (type)* such that it is closed under all of our logical constructors.

$$\frac{}{\vdash \mathsf{U}\,\mathtt{type}} \qquad\qquad \frac{}{x : \mathsf{U} \vdash \mathsf{el}(x)\,\mathtt{type}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma, x : \mathsf{el}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \hat{\Pi}(a, x.b) : \mathsf{U}} \qquad \frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma, x : \mathsf{el}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \mathsf{el}(\hat{\Pi}(a, x.b)) \equiv \Pi_{x:\mathsf{el}(a)}\mathsf{el}(b(x))\,\mathtt{type}}$$

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma, x : \mathsf{el}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \hat{\Sigma}(a, x.b) : \mathsf{U}} \qquad \frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma, x : \mathsf{el}(a) \vdash b(x) : \mathsf{U}}{\Gamma \vdash \mathsf{el}(\hat{\Sigma}(a, x.b)) \equiv \Sigma_{x:\mathsf{el}(a)}\mathsf{el}(b(x))\,\mathtt{type}}$$

$$\frac{}{\vdash \hat{\mathbf{0}} : \mathsf{U}} \qquad\qquad \frac{}{\vdash \mathsf{el}(\hat{\mathbf{0}}) \equiv \mathbf{0}\,\mathtt{type}}$$

$$\frac{}{\vdash \hat{\mathbf{1}} : \mathsf{U}} \qquad\qquad \frac{}{\vdash \mathsf{el}(\hat{\mathbf{1}}) \equiv \mathbf{1}\,\mathtt{type}}$$

$$\frac{}{\vdash \hat{\mathbf{2}} : \mathsf{U}} \qquad\qquad \frac{}{\vdash \mathsf{el}(\hat{\mathbf{2}}) \equiv \mathbf{2}\,\mathtt{type}}$$

Definition 1.5.1. We say that a well-formed type A in context Γ is *small* if we can derive

$$\Gamma \vdash \hat{A} : \mathsf{U} \qquad \Gamma \vdash \mathsf{el}(\hat{A}) \equiv A\,\mathtt{type}$$

for some expression $\hat{A}$.

From a set-theoretic viewpoint, $\hat{A}$ is a lift of A under el.

Example 1.5.2. If A is a small type in context Γ and $B(x)$ is a small type in context $\Gamma, x : A$, then $\Pi_{x:A}B(x)$ is a small type in context Γ.

We can view el as an interpretation operator (in a semantic sense) so that each inhabitant X of U in context Γ is a name for the genuine type $\mathsf{el}(X)$.

Aside. We could not assert that every type has type U, in which case U inhabits U. For then we'd obtain an encoding of Russell's paradox known as Girard's paradox, so that our MLDTT would be inconsistent (i.e., we could construct a closed term of type $\mathbf{0}$). We could postulate a sequence of universes $(\mathsf{U}_i)_{i \in \mathbb{N}}$ governed by the rule schemata

$$\frac{\mathtt{ctx}(\Gamma)}{\Gamma \vdash \mathsf{U}_i : \mathsf{U}_{i+1}} \qquad\qquad \frac{\Gamma \vdash A : \mathsf{U}_i}{\Gamma \vdash A : \mathsf{U}_{i+1}}.$$

Such a sequence is called a *cumulative hierarchy.*[3] In this case, we would alter our MLDTT by removing judgments of typehood and expressing that A is a well-formed type via a typing declaration such as $\Gamma \vdash A : \mathsf{U}_i$. This approach is taken by [25].

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[3]Each stage of this hierarchy is called a *universe à la Russell*. The universe in our MLDTT is called a *universe à la Tarski*.

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2 Homotopy type theory

In this section, our goal is to develop enough classical homotopy theory *within* our MLDTT to motivate and state the univalence axiom. This form of homotopy theory is known as *synthetic homotopy theory.* This is precisely the area that both [25] and [20] cover. Synthetic homotopy theory has produced some new proofs of old theorems, such as the Freudenthal suspension theorem.

Throughout this section, we shall mention a new, informal interpretation of Martin-Löf dependent type theory in which each well-formed type represents (the homotopy type of) a topological space. This interpretation will be made precise by way of categorical semantics.

Remark 2.0.1. After defining identity types, we shall mainly use informal notation so that our presentation matches the ordinary style of mathematics. (The type theory literature normally does the same.) Still, all of the definitions, theorems, proofs, etc. internal to our system can be syntactically formalized.[4]

2.1 Identity types $\big(\mathsf{Id}_-(-,-)\big)$

So far, our sole concept of equality is judgmental equality. But this fails to capture our ordinary concept of equality, as in the formula $\forall x, y \in \mathbb{R}\,(x + y = y + x)$. Indeed, this is a first-order sentence and thus should correspond to a well-formed type à la Curry-Howard. By contrast, judgmental equality, as a relation on the set of all raw terms, determines a rewriting system, i.e., a set of rules for replacing one raw term with another.

To reason about mathematical equality in our MLDTT, we define our final inductive type, the *(propositional) identity type*, as follows.

$$\mathsf{Id}\text{-FORM}\ \frac{\Gamma \vdash A\,\mathsf{type}}{\Gamma, x : A, y : A \vdash \mathsf{Id}_A(x,y)\,\mathsf{type}} \qquad \mathsf{Id}\text{-INTRO}\ \frac{\Gamma \vdash A\,\mathsf{type}}{\Gamma, x : A \vdash \mathsf{refl}(A,x) : \mathsf{Id}_A(x,x)}$$

$$\mathsf{Id}\text{-ELIM}\ \frac{\Gamma, x : A, y : A, p : \mathsf{Id}_A(x,y) \vdash C(x,y,p)\,\mathsf{type} \quad \Gamma, z : A \vdash c(z) : C[z,z,\mathsf{refl}(A,z)/x,y,p]}{\Gamma, x : A, y : A, p : \mathsf{Id}_A(x,y) \vdash \mathsf{J}(x.y.p.C, z.c, x, y, p) : C(x,y,p)}$$

$$\mathsf{Id}\text{-COMP}\ \frac{\Gamma, x : A, y : A, p : \mathsf{Id}_A(x,y) \vdash C(x,y,p)\,\mathsf{type} \quad \Gamma, z : A \vdash c(z) : C[z,z,\mathsf{refl}(A,z)/x,y,p]}{\Gamma, x : A \vdash \mathsf{J}(x.y.p.C, z.c, x, x, \mathsf{refl}(A,x)) \equiv c[x/z] : C[x,x,\mathsf{refl}(A,x)/x,y,p]}$$

Notation. We shall write refl_a for $\mathsf{refl}(A,a)$ when its first argument can easily be inferred.

Remark 2.1.1. Our MLDTT is *intensional* in that it leaves out both of the following inference rules.

1. (*equality reflection rule* (ERR))

$$\frac{\Gamma \vdash p : \mathsf{Id}_A(x,y)}{\Gamma \vdash x \equiv y : A}\ \ .$$

---

[4]For example, there are ongoing implementations of [25] in the proof assistants Coq and Agda available on GitHub.

2. (*uniqueness of identity proofs* (UIP))

$$\frac{\texttt{ctx}(\Gamma)}{\Gamma \vdash \mathsf{uip} : \prod_{A:\mathsf{U}} \prod_{x,y:\mathsf{el}(A)} \prod_{p,q:\mathsf{Id}_{\mathsf{el}(A)}(x,y)} \mathsf{Id}_{\mathsf{Id}_{\mathsf{el}(A)}(x,y)}(p,q)} .$$

A Martin-Löf dependent type theory is *extensional* if it includes ERR. It is a *set-level* type theory if it includes UIP.[5] In an extensional type theory, propositional equality implies judgmental equality. (The converse is always true.) In a set-level type theory, for any two well-formed terms, there is at most one proof that they are propositionally equal. If ERR is assumed, then UIP is provable with Id-ELIM.

Now, consider the logical rule Id-ELIM. Intuitively, this asserts that if we know that

(a) for any $x, y : A$ and $p : \mathsf{Id}_A(x,y)$, we have a type $C(x,y,p)$ and

(b) whenever $x = y$ and $p = \mathsf{refl}(A,x)$, we have a term t of type $C(x,x,\mathsf{refl}(A,x))$,

then we can construct a certain term J of type $C(x',y',p')$ for *any* $x', y' : A$ and $p' : \mathsf{Id}_A(x,y)$.

Terminology. Another name for Id-ELIM is *path induction*.

Indeed, we can interpret a well-formed type A as a topological space and each inhabitant of $\mathsf{Id}_A(x,y)$ as a path in A from the point x to the point y. In this case, we think of the term refl_x as the constant path at the point x.

Moreover, under our propositions-as-types doctrine, we view a path in A from x to y as a proof of the proposition that $x = y$.

With the following two rules, we postulate that the universe is closed under the logical constructor Id.

$$\frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma \vdash b,c : \mathsf{el}(a)}{\Gamma \vdash \hat{\mathsf{Id}}_a(b,c) : \mathsf{U}} \qquad\qquad \frac{\Gamma \vdash a : \mathsf{U} \qquad \Gamma \vdash b,c : \mathsf{el}(a)}{\Gamma \vdash \mathsf{el}(\hat{\mathsf{Id}}_a(b,c)) \equiv \mathsf{Id}_{\mathsf{el}(a)}(b,c) \,\texttt{type}}$$

Notation. We may write $x \rightsquigarrow_A y$ for $\mathsf{Id}_A(x,y)$, omitting the subscript A when it can be easily inferred.

The type $x \rightsquigarrow_A y$ can be viewed as the path space of A, which consists of the set of paths in A equipped with the compact-open topology.

Table 3: Homotopy interpretation of type theory

Type theory	Homotopy theory
Type	Space
Inhabitant	Point
Identity type	Path space
Type family	Fibration
Dependent sum	Total space

---

[5]It is easy to prove that UIP is incompatible with the univalence axiom.

In our propositions-as-types doctrine, we intuitively interpret *any* type as a certain logical proposition. Using identity types, we can define an alternative notion of proposition within our MLDTT as follows.

Definition 2.1.2. A type A is an *h-proposition* (or a *mere proposition*) if there is some term of type

$$\texttt{is\_prop}(A) \coloneqq \prod_{x,y:A} x \rightsquigarrow y.$$

Not every well-formed type A is necessarily an h-proposition within our MLDTT. It is one exactly when any two of its inhabitants are propositionally equal. According to our propositions-as-types doctrine, this means that any two of its proofs are the same. In this case, A is true exactly when it is inhabited by a single term and false exactly when its inhabitation leads to a contradiction.

As a result, the constructive logic encoded in our MLDTT now has the property that a specific proof of a proposition P carries no mathematical data other than the fact that P is true. Likewise, in propositional logic, a proposition is interpreted as nothing more than a truth value.

Aside. Suppose that we treated as propositions exactly those types A such that $\texttt{is\_prop}(A)$ is inhabited. Further, suppose that we postulated the rule

$$\frac{\Gamma \vdash A \,\texttt{type}}{\mathsf{dn}_A : \texttt{is\_prop}(A) \to (\neg\neg A \to A)} \,,$$

called the *law of double negation.*[6] Then the logic encoded in our MLDTT would become classical. Thus, we could recover classical mathematical reasoning, if desired, at the expense of constructiveness.

2.2 Basic properties of (type-theoretic) paths

Lemma 2.2.1 (Path inversion). *Let A be a type and let x and y be inhabitants of A (in context Γ). Then there is some function*

$$\texttt{inv} : (x \rightsquigarrow y) \to (y \rightsquigarrow x)$$

such that $\texttt{inv}(\mathsf{refl}_z) \equiv \mathsf{refl}_z$ for each $z : A$. For any $p : x \rightsquigarrow y$, let $p^{-1} \coloneqq \texttt{inv}(p)$.

Proof. By path induction, it suffices to construct a term

$$\texttt{inv}(\mathsf{refl}_x) : x \rightsquigarrow x.$$

Take $\texttt{inv}(\mathsf{refl}_x)$ to be refl_x. $\qquad\square$

Lemma 2.2.2 (Path concatenation). *Let A be a type and $x, y : A$. Then there is some function*

$$\texttt{concat} : (x \rightsquigarrow y) \to (y \rightsquigarrow z) \to (x \rightsquigarrow z)$$

such that $\texttt{concat}(\mathsf{refl}_x, q) \equiv q$ for any $q : x \rightsquigarrow z$. Let $p * q \coloneqq \texttt{concat}(p, q)$.

Proof. Again, it suffices to construct, for any $z : A$, a term

$$\texttt{concat}(\mathsf{refl}_x) : (x \rightsquigarrow z) \to (x \rightsquigarrow z).$$

---

[6]As it turns out, this is compatible with the univalence axiom, but it would not be if we omitted the antecedent "$\texttt{is\_prop}(A)$." See [25, Theorem 3.2.2].

Take $\texttt{concat}(\mathsf{refl}_x)$ to be $\texttt{idmap}_{x \rightsquigarrow z}$. The fact that $\texttt{concat}(\mathsf{refl}_x, q) \equiv q$ follows automatically from Id-COMP. $\square$

It follows that propositional equality is like a set-theoretic equivalence relation.

Lemma 2.2.3. *Let A be a type. Let $x, y : A$ and $p : x \rightsquigarrow y$.*

(1) Let $z, w : A$. Let $q : y \rightsquigarrow z$ and $r : z \rightsquigarrow w$. Then there is some path

$$\texttt{assoc}(p, q, r) : (p * q) * r \rightsquigarrow p * (q * r).$$

(2) There exist certain paths

$$\texttt{l\_unit}(p) : \mathsf{refl}_x * p \rightsquigarrow p$$
$$\texttt{r\_unit}(p) : p * \mathsf{refl}_x \rightsquigarrow p.$$

(3) There exist certain paths

$$\texttt{l\_inv}(p) : p^{-1} * p \rightsquigarrow \mathsf{refl}_y$$
$$\texttt{r\_inv}(p) : p * p^{-1} \rightsquigarrow \mathsf{refl}_x.$$

Proof. Let us just prove (1) for both (2) and (3) will follow similarly. By path induction, it suffices to construct a term of type

$$(\mathsf{refl}_x * q) * r \rightsquigarrow \mathsf{refl}_x * (q * r)$$

where $q : x \rightsquigarrow z$. Note that $(\mathsf{refl}_x * q) * r \equiv q * r$ and $\mathsf{refl}_x * (q * r) \equiv q * r$ due to Lemma 2.2.2. Thus, by some of our structural rules, we can just choose refl_{q*r}. $\square$

Corollary 2.2.4. *Every type has the structure of a fundamental groupoid of a topological space.*

Our next result shows that any non-dependent function is continuous in a certain sense.

Lemma 2.2.5 (Functoriality). *Any non-dependent function $f : A \to B$ preserves paths, i.e., for any path $p : x \rightsquigarrow_A y$, there is some path $\texttt{ap}_{f,p} : f(x) \rightsquigarrow_B f(y)$.*

Proof. It suffices to construct a term $\texttt{ap}_{f,p}(\mathsf{refl}_x) : f(x) \rightsquigarrow_B f(x)$. Choose $\mathsf{refl}_{f(x)}$. $\square$

Lemma 2.2.6 (Transport). *Let P be a type family over A. Let $p : x \rightsquigarrow_A y$. Then there is some function*

$$\texttt{transport}(p) : P(x) \to P(y)$$

such that $\texttt{transport}(\mathsf{refl}_x)(u) \equiv u$ for any $u : P(x)$. Let $p \cdot u := \texttt{transport}(p)(u)$.

Proof. It suffices to construct a term $\texttt{transport}(\mathsf{refl}_x) : P(x) \to P(x)$. Choose $\texttt{idmap}_{P(x)}$. $\square$

Lemma 2.2.7. *Let $f : \prod_{x:A} P(x)$ and $p : x \rightsquigarrow_A y$. Then there is some path*

$$\texttt{apd}_f(p) : p \cdot f(x) \rightsquigarrow_{P(y)} f(y).$$

Proof. It suffices to construct a term $\texttt{apd}_f(\mathsf{refl}_x) : f(x) \rightsquigarrow_{P(x)} f(x)$. Choose $\mathsf{refl}_{f(x)}$. $\square$

As fiber bundles in topology possess the homotopy lifting property, our next lemma leads us to interpret the dependent sum $\sum_{x:A} P(x)$ as the total space of a fiber bundle over A.

Lemma 2.2.8 (Path lifting). *Let P be a type family over A. Suppose that $p : x \rightsquigarrow_A y$ and that $u : P(x)$. Then there is some path*

$$p_\Sigma(u) : (x, u) \rightsquigarrow_{\Sigma_{x:A} P(x)} (y, p \cdot u).$$

Proof. It suffices to construct a term $(\mathsf{refl}_x)_\Sigma (u) : (x, u) \rightsquigarrow (x, u)$. Choose $\mathsf{refl}_{(x,u)}$. $\qquad\square$

Lemma 2.2.9. *Suppose that $x, x' : A$, $y : B(x)$, and $y' : B(x')$. Suppose that we have both a path from (x, y) to (x', y') and a path from y' to z where $z : B(x')$. Then there is some term of type*

$$\mathsf{Id}_{\Sigma_{x:A} B(x)} \left((x, y), (x', z) \right).$$

Proof. Another easy application of path induction. $\qquad\square$

As one may expect, if two dependent functions of the same type are propositionally equal, then they are pointwise propositionally equal.

Lemma 2.2.10. *Let $f, g : \prod_{x:A} P(x)$ where P is a type family over A. Let $\alpha : f \rightsquigarrow g$. Then α induces a path $\alpha(x) : f(x) \rightsquigarrow_{P(x)} g(x)$ for each $x : A$. Therefore, we have a function*

$$\mathtt{hApply}_{A,P} : \prod_{f,g:\Pi_{x:A} P(x)} (f \rightsquigarrow g) \to \left(\prod_{x:A} f(x) \rightsquigarrow g(x) \right).$$

Proof. It suffices to construct a term

$$\mathtt{hApply}_{A,P}(f, f)(\mathsf{refl}_f) : \prod_{x:A} f(x) \rightsquigarrow f(x).$$

Choose $\lambda x.\mathsf{refl}_{f(x)}$. $\qquad\square$

Definition 2.2.11 (Homotopy). Let $f, g : \prod_{x:A} P(x)$ where P is a type family over A. A *homotopy from f to g* is a term H of type

$$f \approx g := \prod_{x:A} f(x) \rightsquigarrow_{P(x)} g(x).$$

If this type is inhabited, then we say that f and g are *homotopic*, written as $f \sim g$.

As it turns out, if our MLDTT assumes the univalence axiom, then any model of it must satisfy so-called functional extensionality, which ensures that every type-theoretic homotopy induces a continuous *choice* of paths $p_x : f(x) \rightsquigarrow g(x)$, just as a homotopy in the classical sense.

The following notion corresponds to homotopy equivalence in topology, i.e., an isomorphism in the homotopy category of **Top**.

Definition 2.2.12 (Isomorphism). Let $f : A \to B$. We say that f is a *(homotopy) isomorphism* if there is some $g : B \to A$ such that $f \circ g \sim \mathtt{idmap}_B$ and $g \circ f \sim \mathtt{idmap}_A$. To be precise,

$$\mathtt{iso}(f) := \sum_{g:B \to A} (\mathtt{idmap}_B \approx f \circ g) \times (\mathtt{idmap}_A \approx g \circ f).$$

We call such a g a *homotopy inverse of f*.

Definition 2.2.13. Let B be a type family over A. Define *homotopy concatenation* as the function

$$\texttt{htpy\_concat} : \prod_{f,g,h:\Pi_{(x:A)} B(x)} (f \approx g) \to (g \approx h) \to (f \approx h)$$

where $\texttt{htpy\_concat}(H, K) \coloneqq \lambda x.H(x) * K(x)$. We write $H \bullet K$ for $\texttt{htpy\_concat}(H, K)$.

Lemma 2.2.14. *Suppose that $f, g : A \to B$ and $f', g' : B \to C$. Suppose that we have inhabitants H and H' of $f \approx g$ and $H' : f' \approx g'$, respectively. Then $f' \circ f \sim g' \circ g$.*

Proof. The term $\lambda a.g'(H(a)) \bullet H'(f(a))$ has type $f' \circ f \approx g' \circ g$. $\qquad\square$

Definition 2.2.15 (Contractible type). Given a type A, we say that A is *contractible* if the type

$$\texttt{is\_contr}(A) \coloneqq \sum_{a:A} \prod_{x:A} x \rightsquigarrow a$$

is inhabited. For any inhabitant (c, C) of $\texttt{is\_contr}(A)$, we call c a *center of contraction* for A and C a *contraction* of A.

To preserve our topological intuition, we should interpret this as saying that A is contractible exactly when we can construct a term $a : A$ and a homotopy from $\lambda x.x$ to $\lambda x.a$.

Example 2.2.16. The unit type is contractible.

Proof. Define $f : \prod_{x:\mathbf{1}} x \rightsquigarrow \star$ inductively by $f(\star) \coloneqq \mathsf{refl}_\star$. Then $(\star, f)$ has type $\texttt{is\_contr}(\mathbf{1})$. $\qquad\square$

The following result reveals that a type where each inhabitant is potentially non-contractible can collectively form a contractible type.

Lemma 2.2.17. *For any type A and any $a : A$, the type $\sum_{x:A} x \rightsquigarrow a$ is contractible.*

Proof. We claim that (a, refl_a) inhabits $\texttt{is\_contr}(\sum_{x:A} x \rightsquigarrow a)$. We must show that there is some path from (x, p) to (a, refl_a) for any $(x, p) : \sum_{x:A} x \rightsquigarrow a$. By the path lifting lemma along with Lemma 2.2.9, it suffices to construct a term $q : a \rightsquigarrow x$ such that $q \cdot \mathsf{refl}_a$ and p are propositionally equal. By path induction, it is easy to show that $p \cdot \mathsf{refl}_a$ and p are propositionally equal. Hence take q to be p. $\qquad\square$

The following notion corresponds to the fiber of a point under a continuous map between spaces.

Definition 2.2.18. Let $f : A \to B$ and $b : B$. The *homotopy fiber of b* is the type

$$\texttt{hFiber}(f, b) \coloneqq \sum_{x:A} f(x) \rightsquigarrow b.$$

We can now extend our notion of contractibility to functions.

Definition 2.2.19. We say that a function $f : A \to B$ is *contractible* if $\texttt{hFiber}(f, b)$ is contractible for each $b : B$, i.e., there is some term of type

$$\texttt{is\_contr\_map}(f) \coloneqq \prod_{y:B} \texttt{is\_contr}(\texttt{hFiber}(f, y)).$$

2.3 Type-theoretic equivalence

Definition 2.3.1. Consider a function $f : A \to B$.

1. Let
$$\texttt{retr}(f) \coloneqq \sum_{g:B\to A} g \circ f \approx \texttt{idmap}_A .$$

   For any term $(g, G) : \texttt{retr}(f)$, we call g a *retraction of f*.

2. Let
$$\texttt{sec}(f) \coloneqq \sum_{h:B\to A} f \circ h \approx \texttt{idmap}_B .$$

   For any term $(h, H) : \texttt{sec}(f)$, we call h a *section of f*.

3. We say that f is an *equivalence from A to B* if we have functions $g : B \to A$ and $h : B \to A$ such that $g \circ f \sim \texttt{idmap}_A$ and $f \circ h \sim \texttt{idmap}_B$, i.e., there is some term of type
$$\texttt{is\_equiv}(f) \coloneqq \texttt{retr}(f) \times \texttt{sec}(f).$$

   Then the type of equivalences from A to B is precisely
$$A \simeq B \coloneqq \sum_{f:A\to B} \texttt{is\_equiv}(f).$$

   If this is inhabited, then we say that A and B are *equivalent*.

Remark 2.3.2. We have defined logical equivalence and equivalence between types differently.

Example 2.3.3. For any type A, the map $\texttt{idmap}_A$ is clearly an equivalence.

Corollary 2.3.4. *Let P be a type family over A and $p : x \rightsquigarrow_A y$. Then $\texttt{transport}(p)$ is an equivalence from $P(x)$ to $P(y)$.*

Proof. An easy application of path induction. $\square$

Lemma 2.3.5. *Suppose that P and Q are h-propositions and that we have terms $f : P \to Q$ and $g : Q \to P$. Then P and Q are equivalent.*

Proof. It is easy to see that f and g are homotopy inverses of each other. $\square$

We now proceed to establish two new ways of logically characterizing $\texttt{is\_equiv}(f)$.

Lemma 2.3.6. *A function $f : A \to B$ is an equivalence if and only if it is an isomorphism, i.e., $\texttt{iso}(f)$ and $\texttt{is\_equiv}(f)$ are logically equivalent types.*

Proof. The ($\Longleftarrow$) direction is obvious. Conversely, suppose that f is an equivalence. Then we have a term $(g, G) : \texttt{retr}(f)$ and a term $(h, H) : \texttt{sec}(f)$. For any $y : B$, we can apply Theorem 1.3.5 to get a chain of paths

$$h(y) \xrightarrow{\;G(h(y))^{-1}\;} (g \circ f)(h(y)) \rightsquigarrow g((f \circ h)(y)) \xrightarrow{\;\texttt{apd}_g(H(y))\;} g(y) \ .$$

This shows that we can construct a homotopy $K : h \approx g$. It follows that $h \circ f \sim \texttt{idmap}_A$, so that h is a homotopy inverse of f. $\square$

Corollary 2.3.7. *Any two contractible types are equivalent. In particular, every contractible type is equivalent to the unit type.*

Corollary 2.3.8. *If the types A and B are contractible and $f : A \to B$, then f is an equivalence.*

Proof. It suffices to show that f is an isomorphism. By assumption, we have inhabitants (c, C) and (c', C') of $\texttt{is\_contr}(A)$ and $\texttt{is\_contr}(B)$, respectively. It is easy to check that $f \circ \texttt{cons}_c \sim \texttt{idmap}_B$ and $\texttt{cons}_c \circ f \sim \texttt{idmap}_A$. $\qquad\square$

Theorem 2.3.9. *A function $f : A \to B$ is an equivalence if and only if it is contractible, i.e., $\texttt{is\_equiv}(f)$ and $\texttt{is\_contr\_map}(f)$ are logically* equivalent types.

Proof. For the ($\Longrightarrow$) direction (which is much more difficult), see [20, Theorem 6.3.3].

Conversely, suppose that f is contractible. Then for each $y : B$, we get a term

$$(h(y), H(y)) : \texttt{is\_contr}(\texttt{hFiber}(t, y)).$$

Therefore, we have a term

$$\lambda y. (h(y), H(y)) : \prod_{y:B} \texttt{hFiber}(f, y).$$

From this we can construct a function $h : B \to A$ and a homotopy

$$H : \prod_{y:B} f(h(y)) \rightsquigarrow_B y.$$

Now, we also can construct a term of type $\prod_{x:A} h(f(x)) \rightsquigarrow_A x$. Indeed, for each $x : A$, we have a path

$$p : f(h(f(x))) \rightsquigarrow_B f(x),$$

so that $f(h(f(x)))$ inhabits $\texttt{hFiber}(f, f(x))$. Since $\texttt{hFiber}(f, f(x))$ is contractible, we get a path

$$q : (h(f(x)), p) \rightsquigarrow \left(x, \mathsf{refl}_{f(x)}\right).$$

Then $\texttt{pr}_1(h(f(x)), p) \rightsquigarrow \texttt{pr}_1((x, \mathsf{refl}_{f(x)}))$ is inhabited due to Lemma 2.2.5. Hence $h(f(x)) \rightsquigarrow x$ is inhabited as well. It follows that h is both a section and a retraction of f. In particular, f is an equivalence. $\qquad\square$

Definition 2.3.10. Let P and Q be dependent families over A. A function $\gamma : \prod_{x:A} P(x) \to Q(x)$ is a *fiberwise equivalence from P to Q* if each $\gamma(x)$ is an equivalence from $P(x)$ to $Q(x)$.

Theorem 2.3.11 (Voevodsky). *Let P and Q be type families over A. Consider a term*

$$\tau : \prod_{x:A} P(x) \to Q(x)$$

with the property that

$$\sigma_\tau := \lambda w. (\texttt{pr}_1(w), \tau(\texttt{pr}_1(w))(\texttt{pr}_2(w))) : \sum_{x:A} P(x) \to \sum_{x:A} Q(x)$$

is an equivalence. Then τ is a fiberwise equivalence.[7]

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[7][19, Theorem 2.4.19].

Functional extensionality

So far, we have intuitively regarded dependent functions as set-theoretic functions, i.e., ordinary mathematical functions. At this point, however, our deductive system is too weak to prove that any two dependent functions f and g that are pointwise propositionally equal are themselves propositionally equal. In this way, they are more like algorithms than ordinary functions. Indeed, two different algorithms may have the same output on each input. To avoid this issue, we may consider so-called functional extensionality principles.

Definition 2.3.12.

1. The *weak functional extensionality principle* (WFE) asserts that for any type family P over A, we have a term

$$\left(\prod_{x:A} \texttt{is\_contr}(P(x))\right) \to \texttt{is\_contr}\left(\prod_{x:A} P(x)\right).^{[8]}$$

2. The *functional extensionality principle* (FE) asserts that for any type family P over A, there is some term of type

$$\prod_{f,g:\Pi_{x:A}P(x)} \texttt{is\_equiv}(\texttt{hApply}(f,g)).$$

In particular, FE asserts that for any two dependent functions $f, g : \prod_{x:A} P(x)$, if f are g are homotopic, then they are propositionally equal. From a set-theoretic perspective, this corresponds to the fact that if two functions $f, g : X \to Y$ agree at each point in X, then $f = g$. From a topological perspective, it corresponds to the fact that a homotopy between f and g induces a path between f and g in the mapping space $M(X, Y)$. In this case, our type-theoretic notion of homotopy agrees with the classical notion.

Example 2.3.13. To see that functional extensionality is useful, let P be a mere proposition (as in Definition 2.1.2). We want to show that $\neg P$ is also a mere proposition, i.e., that $\prod_{x,y:\neg P} x \rightsquigarrow y$ is inhabited. To this end, assume FE and let $x, y : P \to \mathbf{0}$. We must construct a term of type $x \rightsquigarrow y$. Note that $x(z) \rightsquigarrow y(z)$ is inhabited for any $z : P$ by virtue of $\mathbf{0}$-ELIM. Thus, we get a term of type $\Pi_{z:P} x(z) \rightsquigarrow y(z)$. By FE, it follows that $x \rightsquigarrow y$ is inhabited, as desired.

It turns out that, despite our terminology, WFE is logically at least as strong as FE. Before proving this, we state a few intermediate results.

Whereas the axiom of choice is not provable in ZF, its type-theoretic formulation à la propositions-as-types is easily derivable in our MLDTT.

Theorem 2.3.14 (Axiom of choice). *Let P be a type family over A. Also, for each $x : A$, let $C(x)$ be a type family over $P(x)$. Then we have a term*

$$\texttt{ac} : \left(\prod_{x:A}\sum_{y:P(x)} C(x,y)\right) \to \left(\sum_{s:\Pi_{x:A}P(x)}\prod_{x:A} C\left(x,s(x)\right)\right).^{[9]}$$

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[8]This corresponds to the fact that any product of contractible spaces is contractible in classical homotopy theory.

[9]This is called the axiom of choice because it is a direct translation of the set-theoretic axiom of choice under the Curry-Howard correspondence. Despite this formal similarity, the set-theoretic version is much stronger in FOL than the type-theoretic version is in our deductive system. In fact, a suitably strong type-theoretic version is not derivable in our MLDTT.

Proof. Take `ac` to be the term $\lambda h.\,(\lambda x.\,\mathtt{pr}_1(h(x)), \lambda x.\,\mathtt{pr}_2(h(x)))$. $\qquad\square$

Under the propositions-as-types doctrine, the following result states that we can always switch the order of bounded quantifiers in a proposition of the form $\forall x\exists y C(x,y)$.

Theorem 2.3.15. *Assume* WFE. *Then* `ac` *is an equivalence.*[10]

Corollary 2.3.16. *Assume* WFE. *Let P be a type family over A and $f : \prod_{x:A} P(x)$. Then*

$$\sum_{g:\Pi_{x:A} P(x)} g \approx f$$

is contractible.

Proof. By Lemma 2.2.17, we know that the type $\sum_{y:P(x)} y \rightsquigarrow f(x)$ is contractible for each $x : A$. By WFE, it follows that

$$\prod_{x:A}\sum_{y:P(x)} y \rightsquigarrow f(x)$$

is also contractible. It is easy to see that any type that is equivalent (hence isomorphic) to a contractible type is contractible. Thus, it follows from Theorem 2.3.15 that $\sum_{g:\Pi_{x:A} P(x)} g \approx f$ is contractible. $\qquad\square$

Theorem 2.3.17. *If* WFE *is derivable in our MLDTT, then so is* FE.

Proof. Assume WFE. Let P be a type family over A and $f : \prod_{x:A} P(x)$. We want to show that

$$\prod_{g:\Pi_{x:A} P(x)} \mathtt{is\_equiv}\,(\mathtt{hApply}(f,g))$$

is inhabited.

Consider the function

$$\lambda g.\,\mathtt{hApply}_{A,P}(f,g) : \prod_{x:A} P(x) \to \left((f \rightsquigarrow g) \to \left(\prod_{x:A} f(x) \rightsquigarrow g(x) \right) \right).$$

By Theorem 2.3.11, it suffices to show that

$$\sigma_{\lambda g.\,\mathtt{hApply}_{A,P}(f,g)} : \left(\sum_{g:\prod_{x:A} P(x)} f \rightsquigarrow g \right) \to \sum_{g:\prod_{x:A} P(x)} f \approx g$$

is an equivalence. Note that Lemma 2.2.17 and Corollary 2.3.16 imply that the "domain" type and the "codomain" type here are contractible, respectively. Therefore, $\sigma_{\lambda g.\,\mathtt{hApply}_{A,P}(f,g)}$ is an equivalence by Corollary 2.3.8. $\qquad\square$

The derivability of WFE in our system is useful for proving many inferences about propositional equality of functions. The next two results are small examples of this.

Example 2.3.18. Assume WFE. For any type family P over $\mathbf{0}$, we have that $\prod_{x:\mathbf{0}} P(x)$ is contractible.

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[10][19, Lemma 2.5.6]. This relies on Π-η.

Proof. By induction, there is some dependent function $f : \prod_{x:\mathbf{0}} P(x)$. For any $g : \prod_{x:\mathbf{0}} P(x)$, another use of induction shows that $f \sim g$. By FE, it follows that $f \rightsquigarrow g$ is inhabited. Therefore, $\prod_{x:\mathbf{0}} P(x)$ is contractible, with center of contraction f. $\qquad\square$

Example 2.3.19. Assume WFE. Let B be a type family over A. Suppose that for each $x : A$, the type $B(x)$ is an h-proposition. Then $\prod_{x:A} B(x)$ is an h-proposition.

Proof. Let $f, g : \prod_{x:A} B(x)$. For any $x : A$, we see that $f(x)$ and $g(x)$ are propositionally equal. By FE, it follows that f and g are propositionally equal. $\qquad\square$

Corollary 2.3.20. *Assume* WFE. *For any function* $f : A \to B$, *the type*

$$\mathtt{is\_equiv}(f) \simeq \mathtt{is\_contr\_map}(f)$$

is inhabited.

Proof. By Lemma 2.3.5 along with Theorem 2.3.9, it suffices to show that both $\mathtt{is\_equiv}(f)$ and $\mathtt{is\_contr\_map}(f)$ are h-propositions. The fact that the former is an h-proposition is precisely [25, Theorem 4.3.2]. Moreover, [25, Lemma 3.1..4] states that $\mathtt{is\_contr}(E)$ is an h-proposition for any type E. By Example 2.3.19, it follows that $\mathtt{is\_contr\_map}(f)$ is an h-proposition, as desired. $\qquad\square$

Remark 2.3.21. By contrast, even if WFE holds, it is *not* the case that for any function $f : A \to B$, $\mathtt{iso}(f) \simeq \mathtt{is\_equiv}(f)$. Indeed, $\mathtt{iso}(f)$ need not be an h-proposition [25, p. 77].

2.4 Univalence

Lemma 2.4.1. *Let B be a type family over A. For any $x, y : A$, there is some function*

$$\mathtt{idtoequiv}_{x:A;B(x)}(x, y) : (x \rightsquigarrow y) \to (B(x) \simeq B(y)).$$

Proof. By path induction, it suffices to construct a term $\mathtt{idtoequiv}_{x,y}(\mathsf{refl}_x) : B(x) \simeq B(x)$. We can choose $\mathtt{idmap}_{B(x)}$ thanks to Example 2.3.3. $\qquad\square$

Definition 2.4.2 (Univalence axiom (Univ)). For any $A, B : \mathsf{U}$, the function

$$\mathtt{idtoequiv}_{x:\mathsf{U};\mathsf{el}(x)}(A, B) : (A \rightsquigarrow_\mathsf{U} B) \to (\mathsf{el}(A) \simeq \mathsf{el}(B))$$

is an equivalence. Formally, we postulate the logical rule

$$\vdash \mathsf{univ} : \prod_{x,y:\mathsf{U}} \mathtt{is\_equiv}\left(\mathtt{idtoequiv}_{x:\mathsf{U};\mathsf{el}(x)}(x, y)\right).$$

In general, we say that a type family B over A is *univalent* if the function $\mathtt{idtoequiv}_{x:A;B(x)}(a, b)$ is an equivalence for any $a, b : A$. Therefore, the univalence axiom states that the type family $x : \mathsf{U} \vdash \mathsf{el}(x) \, \mathsf{type}$ is univalent. In particular, there is some homotopy inverse

$$\mathtt{idtoequiv}(A, B)^{-1} : (\mathsf{el}(A) \simeq \mathsf{el}(B)) \to (A \rightsquigarrow_\mathsf{U} B).$$

Informally, this means that whenever two types are equivalent, they are propositionally equal.

Theorem 2.4.3. *Let A and B be types. Then* Univ *implies that there is some term of type*

$$\prod_{f,g:A\to B} (f \approx g) \to (f \rightsquigarrow g) .^{[11]}$$

Corollary 2.4.4 (Voevodsky). Univ $\implies$ WFE $\implies$ FE.

Proof. In light of Theorem 2.3.17, it just remains to prove that if Univ is derivable, then so is WFE .

Assume Univ. Suppose that P is a type family over A and that we have a term of type

$$\prod_{x:A} \mathtt{is\_contr}(P(x)).$$

Since U is closed under all logical constructors, we may assume that A as well as each $P(x)$ is a small type. We must show that $\prod_{x:A} P(x)$ is contractible. Define $F : A \to \mathsf{U}$ as the constant map $\lambda x.\hat{\mathbf{1}}$. Since both $P(x)$ and $\mathbf{1}$ are contractible for any $x : A$, we have that $P(x) \simeq \mathbf{1}$ is inhabited. By Univ, it follows that $\widehat{P(x)} \rightsquigarrow_{\mathsf{U}} \hat{\mathbf{1}}$. By path induction, we see that the type

$$\left(\hat{\prod}_{x:A} \widehat{P(x)} \right) \rightsquigarrow_{\mathsf{U}} \left(\hat{\prod}_{x:A} \overbrace{\widehat{F(x)}}^{\hat{\mathbf{1}}} \right)$$

is inhabited, so that

$$\prod_{x:A} P(x) \simeq \prod_{x:A} \mathsf{el}(F(x))$$

is also inhabited.

Hence it suffices to show that the righthand type is contractible. But any function $f : A \to \mathbf{1}$ is homotopic to $\lambda x.\star$. Theorem 2.4.3 thus shows that f and $\lambda x.\star$ are, in fact, propositionally equal. It follows that $\prod_{x:A} F(x)$ is contractible, with center of contraction $\lambda x.\star$. $\qquad\square$

For a concrete application of Univ to algebra within Martin-Löf dependent type theory, see Section B.

3 Categorical semantics

Any variant of CDTT is purely a formal language. To give its well-formed expressions meanings, we interpret them as certain mathematical objects. Specifically, we define an interpretation of them as certain structures within a suitable category, thereby providing the CDTT with a *categorical semantics*.

This section develops those notions from categorical semantics which Section 5 will rely on. First of all, it is worth summarizing our set-theoretic foundations for category theory.

Definition 3.0.1. A *Grothendieck universe* is a transitive set U such that

(i) $\mathbb{N} \in U$,

(ii) $x \in U \implies \mathcal{P}(x) \in U$, and

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[11] [19, Lemma 2.7.6].

(iii) for any $I \in U$ and function $u : I \to U$, $\bigcup_{i \in I} u(i) \in U$.

By a combination of (ii) and (iii), any subset of an element of U belongs to U. As a result, U is also closed under intersections, unions, and cartesian products.

Example 3.0.2. For any (strongly) inaccessible cardinal κ, the κ-th stage V_κ of the rank hierarchy is a Grothendieck universe.

In fact, any Grothendieck universe U satisfies ZFC. Thus, by Gödel's incompleteness theorems, it is impossible to prove the existence of a Grothendieck universe in ZFC. This leads us to the first-order *axiom of universes*:

For every set s, there exists a Grothendieck universe U such that $s \in U$.

This guarantees that every class definable from U-small sets for some Grothendieck universe U is U'-small for some larger universe U'.

Conversely, it is provable in ZFC that any Grothendieck universe U has the form V_κ for some inaccessible cardinal κ.[12] By Example 3.0.2, it follows that there exists a Grothendieck universe if and only if there exists an inaccessible cardinal. This means that the axiom of universes is equivalent to the large cardinal axiom that there exist arbitrarily large inaccessible cardinals.

For us, category theory will be formulated in the extension of ZFC by this axiom (which, of course, has *not* been proven inconsistent). Moreover, the category **Set** will consist of all U-small sets for a sufficiently large Grothendieck universe U. In particular, this means that if $\mathscr{C}$ is a locally small category, then we can pass to a universe U' larger than U via the axiom of universes so that $\mathscr{C}$ itself is a U'-small category. For this reason, we may regard, for example, **sSet** as a small category.

Remark 3.0.3. Most of our results, however, do hold in ZFC.

Remark 3.0.4. Throughout this section, all categories are assumed to be locally small.

3.1 The syntactic category

To begin with, we build a category directly out of our MLDTT. In Lemma 4.2.8, we shall see this category determines the canonical semantics of CDTT.

Notation. Let $\mathbb{T}$ denote our MLDTT (without Univ).

Definition 3.1.1 (Context morphism). Let Γ and $\Delta := x_1 : A_1, \ldots, x_n : A_n$ be well-formed contexts in $\mathbb{T}$. Further, let $t_1, \ldots, t_n$ denote 0-expressions.

- Recalling Definition 1.1.8, suppose that the following n judgments are derivable in $\mathbb{T}$:

$$\Gamma \vdash t_1 : A_1 \,\texttt{type}$$
$$\Gamma \vdash t_2 : A_2[t_1/x_1] \,\texttt{type}$$
$$\vdots$$
$$\Gamma \vdash t_n : A_n[t_1/x_1][t_2/x_2] \cdots [t_{n-1}/x_{n-1}] \,\texttt{type} \,.$$

Then the tuple $f := (t_1, \ldots, t_n)$ is a *context morphism from Γ to Δ*, written as $f : \Gamma \to \Delta$.

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[12]See [27].

- Let $f \coloneqq (t_1, \ldots, t_n)$ and $g \coloneqq (s_1, \ldots, s_n)$ be context morphisms from Γ to Δ. Suppose that the following n judgments are derivable in $\mathbb{T}$:

$$\Gamma \vdash t_1 \equiv s_1 : A_1$$
$$\Gamma \vdash t_2 \equiv s_2 : A_2[t_1/x_1]$$
$$\vdots$$
$$\Gamma \vdash t_n \equiv s_n : A_n[t_1/x_1][t_2/x_2] \cdots [t_{n-1}/x_{n-1}].$$

Then f and g are *judgmentally equal context morphisms from Γ to Δ*, written as $f \equiv g : \Gamma \to \Delta$.

Notation. $\mathcal{K}[f/\Delta] \coloneqq \mathcal{K}[t_1/x_1][t_2/x_2] \cdots [t_n/x_n]$.

Example 3.1.2 (Display map). Suppose that $\mathtt{ctx}(\Gamma, x : A)$ is derivable in $\mathbb{T}$ where $\Gamma \coloneqq x_1 : A_1, \ldots, x_n : A_n$. Then each of

$$\Gamma, x : A \vdash x_1 : A_1 \ \mathtt{type}$$
$$\Gamma, x : A \vdash x_2 : A_2 \ \mathtt{type}$$
$$\vdots$$
$$\Gamma, x : A \vdash x_n : A_n \ \mathtt{type}$$

is derivable in $\mathbb{T}$ due to the structural rule VBLE. Thus, $p_A \coloneqq (x_1, \ldots, x_n)$ is a context morphism from $\Gamma, x : A$ to Γ, called the *display map of A*.

From a syntactic perspective, this represents a type family A over the types appearing in Γ. From a topological perspective, it represents a fiber bundle over Γ.

Definition 3.1.3 (Syntactic category). Define the *syntactic category $\mathscr{C}(\mathbb{T})$ of $\mathbb{T}$* as follows.

- Let $\mathrm{Ob}\,\mathscr{C}(\mathbb{T})$ be the quotient of the set of all well-formed contexts in $\mathbb{T}$ by the equivalence relation $\sim$ where

$$\Gamma \sim \Delta \iff \Gamma \equiv \Delta \ \mathtt{ctx} \text{ is derivable in } \mathbb{T}.$$

- For any $[\Gamma], [\Delta] \in \mathrm{Ob}\,\mathscr{C}(\mathbb{T})$, let $\mathscr{C}(\mathbb{T})(\Gamma, \Delta)$ be the quotient of the set of all context morphisms from Γ to Δ by the equivalence relation $\sim'$ where

$$[f] \sim' [g] \iff f \equiv g : \Gamma \to \Delta \text{ is derivable in } \mathbb{T}.$$

This is well-defined because of the derived rule

$$\frac{\Gamma_1 \equiv \Gamma_2 \qquad \Gamma_3 \equiv \Gamma_4 \qquad f : \Gamma_1 \to \Gamma_3}{f : \Gamma_2 \to \Gamma_4}.$$

- For any two (equivalence classes of) context morphisms $f : \Gamma \to \Delta$ and $g \coloneqq (s_1, \ldots, s_n) : \Delta \to \Theta$, let

$$g \circ f = (s_1\,[f/\Delta], \ldots, s_n\,[f/\Delta]).$$

It is straightforward yet tedious to verify that the operation $\circ$ is well-defined and associative and that for any well-formed context $\Gamma \coloneqq x_1 : A_1, \ldots, x_n : A_n$, the morphism $(x_1, \ldots, x_n)$ is well-defined

and constitutes the identity morphism id_Γ for (the equivalence class of) Γ. It follows that $\mathscr{C}(\mathbb{T})$ is, indeed, a category.

Note 3.1.4.

(1) Let $\Gamma \coloneqq x_1 : A_1, \ldots, x_n : A_n$. We have a one-to-one correspondence between typing declarations $\Gamma \vdash a : A$ and sections of the display map p_A given by mapping $\Gamma \vdash a : A$ to $(x_1, \ldots, x_n, a)$.

(2) The set $\mathrm{Ob}\,\mathscr{C}(\mathbb{T})$ is $\mathbb{N}$-graded in that there is a natural bijection $\mathrm{Ob}\,\mathscr{C}(\mathbb{T}) \cong \coprod_{n \in \mathbb{N}} C_n$ where C_n denotes the quotient of the set of all well-formed contexts in $\mathbb{T}$ of length n by $\sim$. Thus, $\mathrm{Ob}\,\mathscr{C}(\mathbb{T})$ may be viewed as a rooted tree with the following properties.

    (a) Its root is precisely the empty context.

    (b) Its n-th level is precisely C_n.

    (c) For any node $\Gamma, x : A$ of degree $n \geq 1$, its parent is precisely Γ.

In particular, the empty context is the terminal object of $\mathscr{C}(\mathbb{T})$ as well as the unique object of degree zero.

Let $f \coloneqq (t_1, \ldots, t_n) : \Delta \to \Gamma$ be a morphism in $\mathscr{C}(\mathbb{T})$. Suppose that both $\mathtt{ctx}\,(\Delta, y : A\,[f/\Gamma])$ and $\mathtt{ctx}\,(\Gamma, x : A)$ are derivable in $\mathbb{T}$. Note that

$$q(f, A) \coloneqq (t_1, \ldots, t_n, y)$$

is a morphism from $\Delta, y : A\,[f/\Gamma]$ to $\Gamma, x : A$ because $\Delta, y : A\,[f/\Gamma] \vdash y : A\,[f/\Gamma]$ is derivable in $\mathbb{T}$ by the structural rule VBLE.
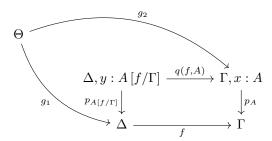
Lemma 3.1.5. *The commutative square*

$$
\begin{array}{ccc}
\Delta, y : A\,[f/\Gamma] & \xrightarrow{\;q(f,A)\;} & \Gamma, x : A \\
{\scriptstyle p_{A[f/\Gamma]}}\big\downarrow & & \big\downarrow{\scriptstyle p_A} \\
\Delta & \xrightarrow{\;\;f\;\;} & \Gamma
\end{array}
\tag{1}
$$

is a pullback in $\mathscr{C}(\mathbb{T})$.

Proof. Suppose that

commutes in $\mathscr{C}(\mathbb{T})$. We must show that there exists a unique morphism $g : \Theta \to \Delta, y : A\,[f/\Gamma]$ that fits into this diagram. Since $p_A \circ g_2 = f \circ g_1$, we see that $g_2 = (f \circ g_1, \tau)$ for some term

expression τ such that $\Theta \vdash \tau : A\,[f \circ g_1/\Gamma]$ is derivable in $\mathbb{T}$. But $A\,[f \circ g_1/\Gamma] = A\,[f/\Gamma]\,[g_1/\Delta]$, so that $g \coloneqq (g_1, \tau)$ is a morphism $\Theta \to \Delta, y : A[f/\Gamma]$. This satisfies

$$p_{A[f/\Gamma]} \circ g = g_1$$

$$q(f, A) \circ g = (f \circ g_1, \tau) \qquad\qquad (*)$$
$$= g_2,$$

and thus g fits into our diagram. To prove that g is unique, let $\tilde{g}$ be any other such morphism $\Theta \to \Delta, y : A\,[f/\Gamma]$. As $p_{A[f/\Gamma]} \circ \tilde{g} = g_1$, we have that $\tilde{g}$ is of the form $(g_1, \tilde{\tau})$ for some $\tilde{\tau}$ such that $\theta \vdash \tilde{\tau} : A[g_1/\Delta]$ is derivable in $\mathbb{T}$. Moreover, using $(*)$, we have that

$$(f \circ g_1, \tau) = g_2$$
$$= q(f, A) \circ \hat{g}$$
$$= (f, y) \circ (g_1, \tilde{\tau})$$
$$= (f \circ g_1, \tilde{\tau})\,.$$

This implies that $\Theta \vdash \tau \equiv \tilde{\tau} : \underbrace{A[f \circ g_1/\Gamma]}_{A[f/\Gamma][g_1/\Delta]}$ is derivable in $\mathbb{T}$. This means that

$$\tilde{g} = (g_1, \tilde{\tau}) = (g_1, \tau) = g,$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2 Contextual categories

There are at least three reasons for looking at contextual categories for our categorical semantics. First, the class of objects in a contextual category has a tree-like structures, just as the set of well-formed contexts of our MLDTT. Second, contextual categories are suitable for interpreting type equality judgments of our MLDTT as their objects can be compared for a kind of equality, not just for isomorphism. Third, they possess a class of distinguished pullbacks, which must be strictly functorial and must commute strictly with logical constructors such as dependent products, thereby resembling syntactic substitution.

Overall, the notion of a contextual category is designed to abstract the key structure from Definition 3.1.3.

Definition 3.2.1 (Contextual category). A *contextual category* $\mathscr{C}$ is a category such that

   (i) $\mathscr{C}$ comes equipped with a terminal object 1,

  (ii) $\mathrm{Ob}\,\mathscr{C}$ is $\mathbb{N}$-graded, i.e., is of the form $\coprod_{n \in \mathbb{N}} \mathrm{Ob}_n\,\mathscr{C}$,

 (iii) $\mathrm{Ob}_0(\mathscr{C}) = \{1\}$,

 (iv) for each n, $\mathscr{C}$ comes equipped with a (class) function $\mathrm{ft}_n : \mathrm{Ob}_{n+1}\,\mathscr{C} \to \mathrm{Ob}_n\,\mathscr{C}$,

  (v) for each $X \in \mathrm{Ob}_{n+1}\,\mathscr{C}$, $\mathscr{C}$ comes equipped with a morphism $p_X : X \to \mathrm{ft}_n(X)$ (called the *display map of X*),

(vi) for each $X \in \mathrm{Ob}_{n+1} \mathscr{C}$ and map $f : Y \to \mathrm{ft}_n(X)$, $\mathscr{C}$ comes equipped with an object f^*X and a morphism $q(f, X)$ such that $\mathrm{ft}\,(f^*X) = Y$ and

$$
\begin{array}{ccc}
f^*X & \xrightarrow{\;q(f,X)\;} & X \\
{\scriptstyle p_{f^*X}}\downarrow & & \downarrow{\scriptstyle p_X} \\
Y & \xrightarrow[\;f\;]{} & \mathrm{ft}_n(X)
\end{array}
$$

is a pullback square (called the *canonical pullback* of X along f), and

(vii) every canonical pullback is strictly functorial in the sense that

$$
\mathrm{id}^*_{\mathrm{ft}_n(X)} X = X
$$
$$
q\,(\mathrm{id}_{\mathrm{ft}_n X}, X) = \mathrm{id}_X
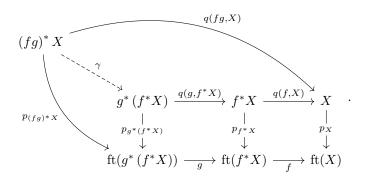$$
$$
(fg)^* X = g^*\,(f^*X)
$$
$$
q\,(fg, X) = q\,(f, X)\,q\,(g, f^*X)
$$

for any $X \in \mathrm{Ob}_{n+1} \mathscr{C}$ and morphisms $f : Y \to \mathrm{ft}_n(X)$ and $g : Z \to Y$.

Remark 3.2.2. To motivate condition (vii), let $\Gamma \vdash A\,\mathtt{type}$ be derivable in $\mathbb{T}$ and let $f : \Delta \to \Gamma$ and $g : \Theta \to \Delta$ be context morphisms in $\mathbb{T}$. On the one hand, it is easy to check that

$$
A\,[f/\Gamma]\,[g/\Delta] = A\,[fg/\Gamma]
$$

as strings. On the other hand, consider the commutative diagram

If (vii) is omitted, then the induced map γ, though an isomorphism in $\mathscr{C}$, need not be the identity morphism. Therefore, without *strict* functoriality, interpreting typehood judgments of the form $\Delta \vdash A\,[f/\Gamma]$ as canonical pullbacks in $\mathscr{C}$ may be unsound. Yet, we want a contextual category to carry a structure similar to that of the syntactic category of $\mathbb{T}$, in which $\Delta \vdash A\,[f/\Gamma]$ is correctly interpreted as the canonical pullback (1).

Example 3.2.3. The category **Set** of sets carries the data of a contextual category $\mathscr{D}$. Indeed, define the grading of $\mathrm{Ob}\,\mathscr{D}$ by mutual recursion with a function $\delta : \mathrm{Ob}\,\mathscr{D} \to \mathbf{Set}$ as follows.

- $\mathrm{Ob}_0\,\mathscr{D} \equiv \{\emptyset\}$, and $\delta(\emptyset) \equiv \{\emptyset\}$.

- The class $\mathrm{Ob}_{n+1}\,\mathscr{D}$ consists of pairs (X, A) where $X \in \mathrm{Ob}_n\,\mathscr{D}$ and $A : \delta(X) \to \mathbf{Set}$, and $\delta(X, A) \equiv \{(x, a) \mid x \in \delta(X), a \in A(x)\}$.

Let $\mathrm{Hom}_{\mathscr{D}}(X, Y)$ consist of all set-theoretic functions $\delta(X) \to \delta(Y)$. The remaining data of a contextual category are given as follows. For any $(X, A) \in \mathrm{Ob}_{n+1} \mathscr{D}$ and $f \in \mathrm{Hom}_{\mathscr{D}}(Y, X)$,

- $\mathrm{ft}(X, A) \equiv X$.

- $p_{(X,A)}(x, a) \equiv x$.

- $f^*(X, A) \equiv (Y, A \circ f)$, and $q(f, (X, A))(x, a) \equiv (f(x), a)$.
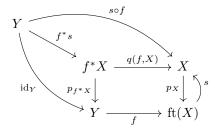
Let $g : \delta(Y) \to \delta(X)$ be any set-theoretic function. Define $A : \delta(X) \to \mathbf{Set}$ by $A(x) = g^{-1}(x)$. It is easy to check that the function $\psi : \delta(Y) \cong \delta(X, A)$ given by $y \mapsto (g(y), y)$ is bijective and that $p_{(X,A)} \circ \psi = g$. Thus, every map in $\mathscr{D}$ is isomorphic to a canonical projection. In an arbitrary contextual category, then, we may think of canonical projections as resembling set maps.

Note 3.2.4 (Binary products). Let $\mathscr{C}$ be a contextual category. For any object X in $\mathscr{C}$, we can form the pullback square

$$
\begin{array}{ccc}
p_X^* X & \xrightarrow{\ q(p_X, X)\ } & X \\
{\scriptstyle p_{p_X^*}} \downarrow & & \downarrow {\scriptstyle p_X} \\
X & \xrightarrow[\ p_X\ ]{} & \mathrm{ft}(X)
\end{array} \ .
$$

Then $p_X^* X$ is the product $X \times X$ in $\mathscr{C}$ with projection map $p_{p_X^*}$.

Note 3.2.5 (Pullback section). Suppose that s is a section of p_X. Then, by the universal property of pullback squares, there exists a unique morphism $f^* s : Y \to f^* X$ such that
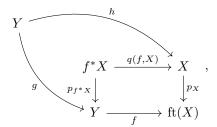

commutes. This means that $f^* s$ is a section of $p_{f^* X}$.

Notation.

1. For any $X \in \mathrm{Ob}_n \mathscr{C}$, we may write (X, A) for any object $Y \in \mathrm{Ob}_{n+1} \mathscr{C}$ such that $\mathrm{ft}(Y) = X$. Also, we may write (X, A, B) for any object $Y \in \mathrm{Ob}_{n+2} \mathscr{C}$ such that $\mathrm{ft}(Y) = (X, A)$.

2. For any such X and any map $f : Y \to X$ in $\mathscr{C}$, we may write $(Y, f^* A)$ for the canonical pullback $f^*(X, A)$.

3. For any such X and f, we may write p_A for the display map $p_{(X,A)} : (X, A) \to X$ and $q(f, A)$ for the map $q(f, X, A)$.

4. For any such X, we may write $p_{A,B}$ for the composite of display maps $p_A \circ p_B : (X, A, B) \to X$.

5. For any such X, we may write s_A for any section of p_A.

6. For any commutative diagram of the form

$$
\begin{array}{c}
Y \xrightarrow{\makebox[3cm]{h}} \\
\end{array}
$$

,

we may write $\langle f^*X, g, h\rangle$ for the unique morphism $Y \to f^*X$ fitting into it.

Using the setting of contextual categories, let us begin formally defining a notion of truth of a judgment in $\mathbb{T}$. Let $\mathscr{C}$ be a contextual category. We want $\mathscr{C}$ to carry a structure for each logical constructor in $\mathbb{T}$ as well as a structure for the universe type in $\mathbb{T}$. Defining such a structure on $\mathscr{C}$ amounts to

(a) translating the main logical rules (except congruence rules) governing the given constructor or universe into the language of contextual categories and

(b) stipulating a so-called stability condition so that canonical pullbacks commute strictly with the given constructor or universe.

In the interest of space, we describe here the structures for just Π, $\mathbf{0}$, Id, and U. For the other logical constructors, see [14, Appendix B].

A Π-*type structure on* $\mathscr{C}$ consists of the following data:

(i) for each $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathscr{C}$, an object $(\Gamma, \Pi(A,B)) \in \mathrm{Ob}_{n+1}\,\mathscr{C}$;

(ii) for each such (Γ, A, B) and each section b of p_B, a section $\lambda(b)$ of $p_{\Pi(A,B)}$;

(iii) for each such (Γ, A, B), each section k of $p_{\Pi(A,B)}$, and each section a of p_A, a section $\mathsf{app}(k,a)$ of the composite $p_{A,B}$ such that $p_B \circ \mathsf{app}(k,a) = a$,

(iv) $\mathsf{app}(\lambda(b), a) = b \circ a$, and

(v) for each map $f : \Gamma' \to \Gamma$ in $\mathscr{C}$, we have that

$$
\begin{aligned}
f^*(\Gamma, \Pi(A,B)) &= (\Gamma', \Pi(f^*A, f^*B)) \\
f^*\lambda(b) &= \lambda(f^*b) \\
f^*\mathsf{app}(k,a) &= \mathsf{app}(f^*k, f^*a).
\end{aligned}
$$

Remark 3.2.6. We have left out the data for Π-η for simplicity. Nevertheless, they are present in every model of $\mathbb{T}$ studied in Section 5.

A $\mathbf{0}$-*type structure on* $\mathscr{C}$ consists of the following data:

(i) for each $\Gamma \in \mathrm{Ob}\,\mathscr{C}$, an object $(\Gamma, \mathbf{0}_\Gamma)$;

(ii) for each object $(\Gamma, \mathbf{0}_\Gamma, A)$, a section $\mathsf{ind}_{\mathbf{0}}(A)$ of p_A such that

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(iii) for each $f : \Gamma' \to \Gamma$, we have that

$$f^*(\Gamma, \mathbf{0}_\Gamma) = (\Gamma', \mathbf{0}_{\Gamma'})$$
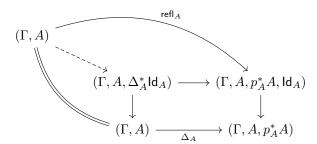$$f^*(\mathsf{ind}_\mathbf{0}(A)) = \mathsf{ind}_\mathbf{0}(f^*A).$$

A Id-*type structure on* $\mathscr{C}$ consists of the following data:

(i) for any $(\Gamma, A) \in \mathrm{Ob}_{n+1}\,\mathscr{C}$ and sections a and b of p_A, an object $(\Gamma, A, p_A^*A, \mathsf{Id}_A)$;

(ii) for each such (Γ, A), a morphism $\mathsf{refl}_A : (\Gamma, A) \to (\Gamma, A, p_A^*A, \mathsf{Id}_A)$ such that $p_{\mathsf{Id}_A} \circ \mathsf{refl}_A = \Delta_A$;

(iii) for each $(\Gamma, A, p_A^*A, \mathsf{Id}_A, B) \in \mathrm{Ob}_{n+4}\,\mathscr{C}$ and map $d : (\Gamma, A) \to (\Gamma, A, p_A^*A, \mathsf{Id}_A, B)$ satisfying $p_B \circ d = \mathsf{refl}_A$, a section $\mathsf{J}_{B,d}$ of p_B such that $\mathsf{J}_{B,d} \circ \mathsf{refl}_A = d$ and

(iv) for each $f : \Gamma' \to \Gamma$, we have that

$$f^*(\Gamma, A, p_A^*A, \mathsf{Id}_A) = \left(\Gamma', f^*A, (p_{f^*A})^* (f^*A), \mathsf{Id}_{f^*A}\right)$$
$$f^*\mathsf{refl}_A = \mathsf{refl}_{f^*A}$$
$$f^*\mathsf{J}_{B,d} = \mathsf{J}_{f^*B, f^*d}.$$

Note 3.2.7.

1. Consider the commutative diagram

where Δ_A stands for the diagonal morphism $\Delta_{(\Gamma, A)}$. Note that the dashed arrow

$$\left\langle (\Gamma, A, \Delta_A^*\mathsf{Id}_A), \mathrm{id}_{(\Gamma, A)}, \mathsf{refl}_A \right\rangle$$

is a section of $p_{\Delta_A^*\mathsf{Id}_A}$.

2. Consider the pullback square

Then for any section s of $p_{\mathsf{refl}_A^*B}$, we have that $p_B \circ \underbrace{q(\mathsf{refl}_A, B) \circ s}_{d} = \mathsf{refl}_A$. Thus, d yields a distinguished section $\mathsf{J}_{B,d}$.

A U-*type structure on* $\mathscr{C}$ is a distinguished object $(\mathsf{U}, \mathsf{el}) \in \mathrm{Ob}_2\,\mathscr{C}$ that is closed under each type structure on $\mathscr{C}$. For example, it is closed under Π-types in the sense that

(i) for any two maps $a : \Gamma \to \mathsf{U}$ and $b : (\Gamma, a^*\mathsf{el}) \to \mathsf{U}$, $\mathscr{C}$ comes equipped with a morphism $\hat{\Pi}(a, b) : \Gamma \to \mathsf{U}$ such that

(ii) $\left(\Gamma, \hat{\Pi}(a, b)^*\mathsf{el}\right) = (\Gamma, \Pi(a^*\mathsf{el}, b^*\mathsf{el}))$ and

(iii) for each $f : \Gamma' \to \Gamma$, we have that

$$\hat{\Pi}(a, b) \circ f = \hat{\Pi}\left(a \circ f, b \circ q(f, a^*\mathsf{el})\right).$$

Notation. Consider the diagrams

$$
\begin{array}{ccc}
(\Gamma, !_\Gamma^*\mathsf{U}) & \longrightarrow & \mathsf{U} \\
{\scriptstyle s}\uparrow\downarrow & & \downarrow \\
\Gamma & \xrightarrow{\ !_\Gamma\ } & 1
\end{array}
\qquad\qquad
\begin{array}{ccc}
(\Gamma, (q(!_\Gamma, \mathsf{U}) \circ s)^*\mathsf{el}, !^*\mathsf{U}) & \longrightarrow & \mathsf{U} \\
{\scriptstyle \tilde{s}}\uparrow\downarrow & & \downarrow \\
(\Gamma, (q(!_\Gamma, \mathsf{U}) \circ s)^*\mathsf{el}) & \xrightarrow{\ !\ } & 1
\end{array}
$$

where both s and $\tilde{s}$ denote sections. We shall refer to the composite maps $q \circ s : \Gamma \to \mathsf{U}$ and $q \circ \tilde{s} : (\Gamma, (q(!_\Gamma, \mathsf{U}) \circ s)^*\mathsf{el}) \to \mathsf{U}$ as $v_{s,\Gamma}$ and $v_{\tilde{s},s,\Gamma}$, respectively.

Definition 3.2.8 ($\mathbb{T}$-structure). A $\mathbb{T}$-*structure* is a contextual category equipped with a structure for each logical constructor in $\mathbb{T}$ and a structure for the universe type in $\mathbb{T}$.

Let $\mathscr{C}$ be a $\mathbb{T}$-structure. We proceed to define a partial function $[\![-]\!]$ (called an *interpretation function*) on the class of all judgments in $\mathbb{T}$ such that $[\![-]\!]$ has values of the forms

$$
\begin{aligned}
[\![\mathtt{ctx}(\Gamma)]\!] &= X, & X &\in \mathrm{Ob}\,\mathscr{C} \\
[\![\Gamma \vdash C\,\mathtt{type}]\!] &= p_A, & (X, A) &\in \mathrm{Ob}\,\mathscr{C} \\
[\![\Gamma \vdash c : C]\!] &= s, & p_A \circ s &= \mathrm{id}_X .
\end{aligned}
$$

Intuitively, this means that

- objects in $\mathscr{C}$ represent well-formed contexts in $\mathbb{T}$,

- display maps in $\mathscr{C}$ represent well-formed types, and

- sections of display maps represent inhabitants of well-formed types.

Our motivation for interpreting typing declarations as sections of display maps is precisely part (1) of Note 3.1.4.

Specifically, $[\![-]\!]$ is defined, in part, via mutual recursion as follows.[13]

<center>Well-formed contexts</center>

$$\frac{}{[\![\mathtt{ctx}(\epsilon)]\!] = 1}$$

$$\frac{[\![\mathtt{ctx}(\Gamma)]\!] = X \qquad [\![\Gamma \vdash C\,\mathtt{type}]\!] = p_A}{[\![\mathtt{ctx}(\Gamma, x : C)]\!] = (X, A)} \quad (\text{when } x \notin \mathrm{FV}(\Gamma))$$

---

[13] Our definition is an adaptation and extension of [16, Section 6.4].

Typing declarations

$$\frac{\begin{array}{cc}[\![\mathtt{ctx}(\Gamma)]\!] = X & [\![\Gamma \vdash C\,\mathtt{type}]\!] = p_A\end{array} \quad [\![\Gamma, x : C \vdash C'\,\mathtt{type}]\!] = p_{(p_A)^*A}}{[\![\Gamma, x : C \vdash x : C']\!] = \big\langle (X, A, p_A^*A)\,, \mathrm{id}_{(X,A)}, \mathrm{id}_{(X,A)}\big\rangle}$$

$$\frac{[\![\mathtt{ctx}(\Gamma)]\!] = X \quad [\![\Gamma \vdash C_1\,\mathtt{type}]\!] = p_A \quad [\![\Gamma, x : C_1 \vdash C_2\,\mathtt{type}]\!] = p_{(X,A,B)} \quad [\![\Gamma, x : C_1, x' : C_2 \vdash C_3\,\mathtt{type}]\!] = p_{(p_{B,A})^*A}}{[\![\Gamma, x : C_1, x' : C_2 \vdash x : C_3]\!] = \big\langle (X, A, B, (p_{A,B})^* A)\,, \mathrm{id}_{(X,A,B)}, p_B\big\rangle}$$

Dependent products

$$\frac{[\![\Gamma \vdash C\,\mathtt{type}]\!] = p_{(X,A)} \quad [\![\Gamma, x : C \vdash C'(x)\,\mathtt{type}]\!] = p_{(X,A,B)}}{[\![\Gamma \vdash \Pi_{x:C}C'(x)\,\mathtt{type}]\!] = p_{\Pi(A,B)}}$$

$$\frac{[\![\Gamma \vdash C\,\mathtt{type}]\!] = p_{(X,A)} \quad [\![\Gamma, x : C \vdash C'(x)\,\mathtt{type}]\!] = p_{(X,A,B)} \quad [\![\Gamma, x : C \vdash c'(x) : C'(x)]\!] = s_B}{[\![\Gamma \vdash \lambda(x : C).c'(x) : \Pi_{x:C}C'(x)]\!] = \lambda(s_B)}$$

$$\frac{[\![\Gamma \vdash C\,\mathtt{type}]\!] = p_{(X,A)} \quad [\![\Gamma, x : C \vdash C'(x)\,\mathtt{type}]\!] = p_{(X,A,B)} \quad [\![\Gamma \vdash k : \Pi_{x:C}C'(x)]\!] = s_{\Pi(A,B)} \quad [\![\Gamma \vdash a : C]\!] = s_A}{[\![\Gamma \vdash \mathsf{app}(k,a) : C'[a/x]]\!] = \big\langle (X, s_A^*B)\,, \mathrm{id}_X, \mathsf{app}(s_{\Pi(A,B)}, s_A)\big\rangle}$$

Zero type

$$\frac{[\![\mathtt{ctx}(\Gamma)]\!] = X}{[\![\Gamma \vdash \mathbf{0}\,\mathtt{type}]\!] = p_{\mathbf{0}_X}}$$

$$\frac{[\![\Gamma, x : \mathbf{0} \vdash C(x)\,\mathtt{type}]\!] = p_{(X,\mathbf{0}_X,A)} \quad [\![\Gamma \vdash a : \mathbf{0}]\!] = s_{\mathbf{0}_X}}{[\![\Gamma \vdash \mathsf{ind}_{\mathbf{0}}(x.C, a) : C[a/x]]\!] = \big\langle (X, s_{\mathbf{0}_X}^* A)\,, \mathrm{id}_X, \mathsf{ind}_{\mathbf{0}}(A) \circ s_{\mathbf{0}_X}\big\rangle}$$

$$\frac{[\![\Gamma \vdash C\,\texttt{type}]\!] = p_{(X,A)}}{[\![\Gamma, x:C, y:C \vdash \mathsf{Id}_C(x,y)\,\texttt{type}]\!] = p_{\left(X,A,p_A^*A,\mathsf{Id}_A\right)}}$$

$$\frac{[\![\Gamma \vdash C\,\texttt{type}]\!] = p_{(X,A)}}{[\![\Gamma, x:C \vdash \mathsf{refl}(C,x) : \mathsf{Id}_C(x,x)]\!] = \left\langle (X,A,\Delta_A^*\mathsf{Id}_A)\,, \mathrm{id}_{(X,A)}, \mathsf{refl}_A \right\rangle}$$

$$\frac{[\![\Gamma, x:C, y:C, p:\mathsf{Id}_C(x,y) \vdash C'(x,y,p)\,\texttt{type}]\!] = p_{\left(X,A,p_A^*A,\mathsf{Id}_A,B\right)} \qquad [\![\Gamma, z:C \vdash c:C'[z,z,\mathsf{refl}(C,z)/x,y,p]]\!] = s_{\mathsf{refl}_A^*B}}{[\![\Gamma, x:A, y:A, p:\mathsf{Id}_C(x,y) \vdash \mathsf{J}(z.c,x,y,p):C'(x,y,p)]\!] = \mathsf{J}_{B,q(\mathsf{refl}_A,B)\circ s_{\mathsf{refl}_A^*B}}}$$

$$\overline{[\![\vdash \mathsf{U}\,\texttt{type}]\!] = \mathop{!}: \mathsf{U} \to 1}$$

$$\overline{[\![x:\mathsf{U} \vdash \mathsf{el}(x)\,\texttt{type}]\!] = p_{\mathsf{el}}}$$

$$\frac{[\![\Gamma \vdash a:\mathsf{U}]\!] = s_{!_X^*\mathsf{U}} \qquad [\![\Gamma, x:\mathsf{el}(a) \vdash b(x):\mathsf{U}]\!] = s_{!_{v_{s_{!_X^*\mathsf{U}},X}^*\mathsf{el}}^*\mathsf{U}}}{[\![\Gamma \vdash \hat{\Pi}(a,x.b):\mathsf{U}]\!] = \left\langle (X,!_X^*\mathsf{U})\,, \mathrm{id}_X, \hat{\Pi}(v,v) \right\rangle}$$

$$\vdots$$

Due to its hideous form, we ought to describe the map $S \coloneqq s_{!_{v_{s_{!_X^*\mathsf{U}},X}^*\mathsf{el}}^*\mathsf{U}}$ explicitly:

The reason that we forego a separate group of semantic rules for typehood is that these correspond to the formation rules for $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$, and U, i.e., our four non-dependent types. In the interest of space, we have omitted any rule defining $[\![-]\!]$ for Σ-types, the unit type, or the Boolean type.

Notation. If φ is a judgment in $\mathbb{T}$, then the notation $[\![\varphi]\!]_\checkmark$ means that $[\![-]\!]$ *or an extension of it* is defined at φ.

Definition 3.2.9 (Model of type theory). Let $\mathscr{C}$ be a $\mathbb{T}$-structure and let φ be a judgment in $\mathbb{T}$. We say that $\mathscr{C}$ *satisfies* φ or φ *is true in* $\mathscr{C}$, written as $\mathscr{C} \models \varphi$, according to the following conditions.

- $\mathscr{C} \models \mathtt{ctx}(\Gamma)$ if and only if $[\![\mathtt{ctx}(\Gamma)]\!]_\checkmark$.

- $\mathscr{C} \models \Gamma \vdash C\,\texttt{type}$ if and only if $[\![\Gamma \vdash C\,\texttt{type}]\!]_\checkmark$.

- $\mathscr{C} \models \Gamma \vdash c : C$ if and only if $[\![\Gamma \vdash c : C]\!]_{\checkmark}$.

- $\mathscr{C} \models \Gamma \equiv \Gamma' \, \mathtt{ctx}$ if and only if $[\![\mathtt{ctx}(\Gamma)]\!] = [\![\mathtt{ctx}(\Gamma')]\!]$.

- $\mathscr{C} \models \Gamma \vdash C \equiv C' \, \mathtt{type}$ if and only if $[\![\Gamma \vdash C \, \mathtt{type}]\!] = [\![\Gamma \vdash C' \, \mathtt{type}]\!]$.

- $\mathscr{C} \models \Gamma \vdash t \equiv t' : C$ if and only if $[\![\Gamma \vdash t : C]\!] = [\![\Gamma \vdash t' : C]\!]$

where $[\![\varphi]\!] = [\![\varphi']\!]$ means that $[\![\varphi]\!]$ and $[\![\varphi']\!]$ are defined and equal. We say that a $\mathbb{T}$-structure $\mathscr{C}$ *models* $\mathbb{T}$ if every theorem of $\mathbb{T}$ is satisfied by $\mathscr{C}$.

Example 3.2.10 (Tautological model). The syntactic category $\mathscr{C}(\mathbb{T})$ is clearly a $\mathbb{T}$-structure. By its design, for any judgment φ in $\mathbb{T}$, $\mathscr{C}(\mathbb{T})$ satisfies φ if and only if φ is a theorem of $\mathbb{T}$.

Theorem 3.2.11 (Completeness). *For any judgment φ in $\mathbb{T}$, if φ is satisfied by every $\mathbb{T}$-structure, then φ is a theorem of $\mathbb{T}$.*

Proof. If φ is satisfied by every $\mathbb{T}$-structure, then it is satisfied in particular by $\mathscr{C}(\mathbb{T})$, which only satisfies theorems of $\mathbb{T}$. $\qquad\square$

Definition 3.2.12. Let $\mathscr{C}$ and $\mathscr{D}$ be contextual categories. A *contextual functor $F : \mathscr{C} \to \mathscr{D}$* is a functor $\mathscr{C} \to \mathscr{D}$ that preserves the structure of a contextual category on the nose, i.e.,

(i) if Γ is the parent of Γ', then $F(\Gamma)$ is the parent of $F(\Gamma')$,

(ii) $F(1_{\mathscr{C}}) = 1_{\mathscr{D}}$,

(iii) $F(p_{\Gamma}) = p_{F(\Gamma)}$ for any $\Gamma \in \mathrm{Ob}_{n+1} \mathscr{C}$,

(iv) $F(f^*\Gamma) = F(f)^*(F(\Gamma))$ for any $f : \Gamma' \to \Gamma$, and

(v) $F(q(f, \Gamma)) = q(F(f), F(\Gamma))$.

Likewise, if both $\mathscr{C}$ and $\mathscr{D}$ are $\mathbb{T}$-structures, then a contextual functor $F : \mathscr{C} \to \mathscr{D}$ is a *logical contextual functor* if it preserves all of the logical structures on the nose.

For each $\mathbb{T}$-structure $\mathscr{C}$, the function $[\![-]\!]$ induces a functor

$$[\![-]\!]^{\mathscr{C}} : \mathscr{C}(\mathbb{T}) \to \mathscr{C}$$
$$x_1 : C_1, \ldots, x_n : C_n \;\mapsto\; (1, X_1, \ldots, X_n), \; [\![x_1 : C_1, \ldots x_{i-1} : C_{i-1} \vdash C_i]\!] = p_{X_i}$$
$$\Gamma \xrightarrow{(t_1, \ldots, t_n)} x_1 : C_1, \ldots, x_n : C_n \;\mapsto\; [\![\Gamma]\!]^{\mathscr{C}} \xrightarrow{\tau} (1, X_1, \ldots, X_n),$$

where the morphism τ is defined as follows.

Conjecture 3.2.13 (Initiality of syntax). *If $\mathscr{C}$ is a $\mathbb{T}$-structure, then $[\![-]\!]^{\mathscr{C}}$ is a logical contextual functor $\mathscr{C}(\mathbb{T}) \to \mathscr{C}$. Moreover, it is unique.*

Informally, this means that every $\mathbb{T}$-structure correctly interprets the syntax of $\mathbb{T}$.

We shall assume that Conjecture 3.2.13 is true. Most, but not all, type theorists accept that it is a straightforward adaptation of the Correctness Theorem in [24], which proves the conjecture for a type theory smaller than ours (called the Calculus of Constructions).[14] If we do not assume Conjecture 3.2.13, then the soundness of the semantics of our MLDTT is left unverified.[15] In this case, proving that a contextual category (such as **sSet**) has suitable logical structure is insufficient to prove that it correctly interprets the syntax.

We are now in position to state a crucial consequence of Conjecture 3.2.13 (whose counterpart in classical FOL, by contrast, holds easily).

Theorem 3.2.14 (Soundness). *For any judgment φ of $\mathbb{T}$, if φ is a theorem of $\mathbb{T}$, then φ is satisfied by every $\mathbb{T}$-structure. Hence every $\mathbb{T}$-structure is a model for $\mathbb{T}$.*

We say that $\mathbb{T}$ is *consistent* if no judgment of the form $\vdash a : \mathbf{0}$ is derivable in $\mathbb{T}$. Otherwise, we say that $\mathbb{T}$ is *inconsistent*.

Corollary 3.2.15 (Consistency). *$\mathbb{T}$ is consistent if and only if it has a model in which the display map $p_{\mathbf{0}_1}$ has no section.*

Proof. For the forward direction, simply observe that if $\mathbb{T}$ is consistent, then $\mathscr{C}(\mathbb{T})$ is a model of it and contains no section of $p_{\mathbf{0}_1}$. The backward direction follows easily from Theorem 3.2.14 (just as it does for classical FOL). $\square$

Before moving on, let us formulate the univalence axiom in any sufficiently structured contextual category.

Let $\mathscr{C}$ be a $\mathbb{T}$-structure and let Γ be an object in $\mathscr{C}$. Since

$$\prod_{x,y:\mathsf{U}} \mathtt{is\_equiv}\left(\mathtt{idtoequiv}_{x:\mathsf{U};\mathsf{el}(x)}(x,y)\right)$$

is a closed type in $\mathbb{T}$, we can find, by virtue of our existing semantics, a certain object $(1, \Pi(A_u, B_u)) \in \mathrm{Ob}_1 \mathscr{C}$ such that

$$\left[\!\!\left[\vdash \prod_{x,y:\mathsf{U}} \mathtt{is\_equiv}\left(\mathtt{idtoequiv}_{x:\mathsf{U};\mathsf{el}(x)}(x,y)\right) \mathtt{type} \right]\!\!\right] = p_{\Pi(A_u,B_u)}.$$

Note that $\mathscr{C}$ satisfies a typing declaration of the form

$$\vdash \tau : \prod_{x,y:\mathsf{U}} \mathtt{is\_equiv}\left(\mathtt{idtoequiv}_{x:\mathsf{U};\mathsf{el}(x)}(x,y)\right)$$

if and only if $[\![-]\!]$ maps this judgment to a section of $p_{\Pi(A_u,B_u)}$. Therefore, we make the following definition.

---

[14]Currently, there is a communal project called the *Initiality Project*, hosted on nLab, that aims to rigorously establish the initiality of $\mathscr{C}(\mathbb{T})$.

[15]Note, however, that [11, Section 3.5] briefly sketches a direct proof of soundness.

Definition 3.2.16. We say that $\mathscr{C}$ *satisfies the univalence axiom* if it comes equipped with a section univ of $p_{\Pi(A_u, B_u)}$.

3.3 Universe categories

Definition 3.3.1. Let $\mathscr{C}$ be any category. A *universe in* $\mathscr{C}$ is an object U in $\mathscr{C}$ equipped with a morphism $p : \tilde{U} \to U$ and, for each map $f : X \to U$, a distinguished pullback square

$$
\begin{array}{ccc}
(X; f) & \xrightarrow{Q(f)} & \tilde{U} \\
P_{X,f} \downarrow & \lrcorner & \downarrow p \\
X & \xrightarrow{\quad f \quad} & U
\end{array}
\qquad (*)
$$

Intuitively, U corresponds to the universe type U, and the morphism p corresponds to the type family $\Gamma, x : \mathsf{U} \vdash \mathsf{el}(x) \, \mathsf{type}$ over U. Further, any map $\alpha : Y \to X$ isomorphic to $P_{X,f}$ in the over category $\mathscr{C}/X$ corresponds to a judgement of the form $\Gamma \vdash \mathsf{el}(a) \, \mathsf{type}$, i.e., a well-formed type.

Notation.

1. For any maps $f_1 : X \to U$ and $f_2 : (X; f_1) \to U$, write $(X; f_1, f_2)$ for the object $((X; f_1); f_2)$ and P_{X,f_1,f_2} for the map $P_{(X;f_1),f_2}$.

2. $(X; \;) := X$.

Definition 3.3.2.

1. A *universe category* is a triple $(\mathscr{C}, U, 1)$ where $\mathscr{C}$ is a category, U is a universe in $\mathscr{C}$, and 1 is a terminal object in $\mathscr{C}$.

2. A *functor of universe categories* from $(\mathscr{C}, U, 1)$ to $(\mathscr{C}', U', 1')$ is a triple $(\Phi, \varphi, \tilde{\varphi})$ where $\Phi : \mathscr{C} \to \mathscr{C}'$ is a functor and $\varphi : \Phi(U) \to U'$ and $\tilde{\varphi} : \Phi(\tilde{U}) \to \tilde{U}'$ are morphisms such that

   (a) Φ maps distinguished pullback squares in $\mathscr{C}$ to pullbacks squares in $\mathscr{C}'$,

   (b) Φ maps 1 to a terminal object in $\mathscr{C}$, and

   (c) the diagram

   $$
   \begin{array}{ccc}
   \Phi(\tilde{U}) & \xrightarrow{\tilde{\varphi}} & \tilde{U}' \\
   \Phi(p) \downarrow & & \downarrow p' \\
   \Phi(U) & \xrightarrow{\varphi} & U'
   \end{array}
   $$

   is a pullback square in $\mathscr{C}'$.

The notions of a universe category and a functor of universe categories are due to Voevodsky and can be found in [26].

Remark 3.3.3. Definition 3.3.2 makes the class of all universe categories into a *precategory*, i.e., a category except that each composition operation is partial rather than total. For simplicity, we shall refer to this precategory as a category.

Now, consider any universe category $(\mathscr{C}, U, 1)$. Define the contextual category $\mathscr{C}_U$ as follows.[16]

- $\mathrm{Ob}_n \, \mathscr{C}_U \equiv \{(f_1, \ldots, f_n) \in (\mathrm{Mor}(\mathscr{C}))^n \mid f_i : (1; f_1, \ldots, f_{i-1}) \to U, \ 1 \le i \le n\}$.

- $\mathrm{Hom}_{\mathscr{C}_U}\big((f_1, \ldots, f_n), (g_1, \ldots, g_m)\big) \equiv \mathrm{Hom}_{\mathscr{C}}\big((1; f_1, \ldots, f_n), (1; g_1, \ldots, g_m)\big)$.

- $1_{\mathscr{C}_U} \equiv ()$, the empty sequence.

- $\mathrm{ft}_n(f_1, \ldots, f_{n+1}) \equiv (f_1, \ldots, f_n)$.

- Take $P_{(1; f_1, \ldots, f_n), f_{n+1}}$ (as in $(*)$) to be the display map $p_{(f_1, \ldots, f_{n+1})}$.

- For any object $(f_1, \ldots, f_{n+1})$ and map $\alpha : (g_1, \ldots, g_m) \to (f_1, \ldots, f_n)$ in $\mathscr{C}_U$, the canonical pullback of $(f_1, \ldots, f_{n+1})$ along α is given by $(g_1, \ldots, g_m, f_{n_1} \circ \alpha)$, i.e., the canonical pullback square is precisely the lefthand square in

$$
\begin{array}{c}
\overset{\displaystyle Q(f_{n+1} \circ \alpha)}{} \\
(1; g_1, \ldots, g_m, f_{n+1} \circ \alpha) \dashrightarrow (1; f_1, \ldots, f_{n+1}) \xrightarrow[Q(f_{n+1})]{} \tilde{U} \\
p_{(1; g_1, \ldots, g_m, f_{n+1} \circ \alpha)} \Big\downarrow \quad \lrcorner \quad p_{(1; f_1, \ldots, f_{n+1})} \Big\downarrow \quad \lrcorner \quad \Big\downarrow p \\
(1; g_1, \ldots, g_m) \xrightarrow[\alpha]{} (1; f_1, \ldots, f_n) \xrightarrow[f_{n+1}]{} U
\end{array} \qquad (\star)
$$

Theorem 3.3.4. *There exists a certain functor $\mathcal{CC}$ from the category of universe categories to the category of contextual categories such that $\mathcal{CC}(\mathscr{C}, U, 1) = \mathscr{C}_U$.*[17]

Suppose that U is a universe in $\mathscr{C}$ when equipped with either of two choices C and C' of distinguished pullback squares. Further, suppose that both 1 and $1'$ are terminal objects in $\mathscr{C}$. These conditions yield two universe categories UC and UC'.

Corollary 3.3.5. *$\mathcal{CC}(UC)$ and $\mathcal{CC}(UC')$ are isomorphic as contextual categories.*

Proof. The triple $(\mathrm{id}_{\mathscr{C}}, \mathrm{id}_U, \mathrm{id}_{\tilde{U}})$ is a functor of universe categories $UC \to UC'$. Clearly, it is its own inverse and thus is an isomorphism. This implies that $\mathcal{CC}\big(\mathrm{id}_{\mathscr{C}}, \mathrm{id}_U, \mathrm{id}_{\tilde{U}}\big)$ is an isomorphism, thereby completing our proof. $\qquad \square$

This means that, up to canonical isomorphism, $\mathscr{C}_U$ is independent of our choice of distinguished pullback squares and terminal object in $\mathscr{C}$.

3.4 Logical structure on a universe category

Let $(\mathscr{C}, U, 1)$ be a universe category with morphism $p : \tilde{U} \to U$. As we did on contextual categories, we would like to endow U with logical structures based on the logical rules in $\mathbb{T}$. Moreover, we would like these structures to induce corresponding logical structures on the contextual category $\mathscr{C}_U$, thereby making $\mathscr{C}_U$ into a $\mathbb{T}$-structure. (We can think of the logical structure on $\mathscr{C}_U$ as internal to that on $\mathscr{C}$.) This section is devoted to describing certain structures on U that achieve these goals.

---

[16] [14, Definition 1.3.2].
[17] [26, Construction 4.7].

Remark 3.4.1. Throughout this section, we shall assume that $\mathscr{C}$ is a locally cartesian closed category (LCCC). See Section C for a review of this kind of category. Additionally, we shall assume that $\mathscr{C}$ has both an initial object 0 and the coproduct $1 \coprod 1$.

For convenience, Table 4 summarizes our results of this section.

Table 4: Logical structure on U

CDTT	LCCC
Dependent product	Dependent product
Dependent sum	Dependent sum
Empty type	Initial object
Unit type	Terminal object
Boolean type	$1 \coprod 1$
Identity type	Path space object
Universe type	Internal universe

Dependent products

Consider a distinguished pullback $P_{X,A}$ in $\mathscr{C}$, i.e., a well-formed type A. Consider also a distinguished pullback $P_{X,A,B}$, i.e., a type family B over A. Intuitively, to form the dependent product $\prod_{x:A} B(x)$ in $\mathscr{C}$, we need a map $\hat{\Pi}(A, B) : X \to U$ along with an isomorphism $P_{X,\hat{\Pi}(A,B)} \cong \Pi_{P_{X,A}} P_{X,A,B}$ such that $\left(X; \hat{\Pi}(A, B) \right)$ is stable under pullback.

To this end, we endow $\mathscr{C}$ with such a map $\hat{\Pi}(A, B)$ under the assumption that X equals

$$\Pi(U) \coloneqq \Sigma_{!_U} \Pi_p \big(\pi_2^U : U \times \tilde{U} \to \tilde{U} \big)$$

and (A, B) equals a special pair (A_g, B_g) of maps determined by $\Pi(U)$, called the *generic* pair. As it turns out, this is enough to produce in $\mathscr{C}_U$ *any* dependent product $\prod_{x:A} B(x)$ as well as ensure stability.

Note 3.4.2 (Functoriality of $\Pi(-)$). Let $\mathrm{Pb}(\mathscr{C})$ denote the category with morphisms in $\mathscr{C}$ as objects and squares of the form

$$
\begin{array}{ccc}
X_1 & \dashrightarrow & Y_1 \\
{\scriptstyle x}\downarrow & \lrcorner & \downarrow{\scriptstyle y} \\
X_2 & \dashrightarrow & Y_2
\end{array}
$$

as morphisms $x \to y$. (This is a wide subcategory of the arrow category $\mathrm{Arr}(\mathscr{C})$ of $\mathscr{C}$.) Now, let U' be another universe in $\mathscr{C}$ and let (f, g) be a map $p' \to p$ in $\mathrm{Pb}(\mathscr{C})$:

$$
\begin{array}{ccc}
\tilde{U}' & \xrightarrow{\ f\ } & \tilde{U} \\
{\scriptstyle p'}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
U' & \xrightarrow[\ g\ ]{} & U
\end{array}
\ . \tag{$\star$}
$$

We can define another map $\pi_2^{U'} \to \pi_2^U$ in $\mathrm{Arr}(\mathscr{C})$ by the commutative square

$$
\begin{array}{ccc}
U' \times \tilde{U}' & \xrightarrow{(g \circ \pi_1, f \circ \pi_2)} & U \times \tilde{U} \\
{\scriptstyle \pi_2} \downarrow & & \downarrow {\scriptstyle \pi_2} \\
\tilde{U}' & \xrightarrow{\quad f \quad} & \tilde{U}
\end{array} \quad .
$$

This induces a unique map $H(f, g) : \pi_2^{U'} \to f^* \pi_2^U$ in the over category $\mathscr{C}/\tilde{U}'$ thanks to the universal property of pullback squares. Applying the functor $\Pi_{p'}$ to H yields a new map

$$
\Pi_{p'}\left(\pi_2^{U'}\right) \to \Pi_{p'}\left(f^* \pi_2^U\right)
$$

in $\mathscr{C}/U'$. In light of Proposition C.0.16, there exists an isomorphism

$$
\Pi_{p'}\left(f^* \pi_2^U\right) \cong g^* \Pi_p\left(\pi_2^U\right)
$$

natural in π_2^U. From this, we get yet another map $\Pi_{p'}\left(\pi_2^{U'}\right) \to g^* \Pi_p\left(\pi_2^U\right)$ in $\mathscr{C}/U'$ and thus a map

$$
\Pi_{p'}\left(\pi_2^{U'}\right) \to \Pi_p\left(\pi_2^U\right)
$$

in $\mathrm{Arr}(\mathscr{C})$. This is our desired map $\Pi(g) : \Pi(U') \to \Pi(U)$ under the identification $\mathscr{C}/1 \cong \mathscr{C}$.

Now, let us return to describing the map $\hat{\Pi}(A_g, B_g) : \Pi(U) \to U$. The map $\Pi_p \pi_2$ can be viewed as a projection map $\mathtt{pr} : \Pi(U) \to U$ in $\mathscr{C}$. Take the pullback $\Pi(U) \times_U \tilde{U}$ to be A_g and let α_g denote the corresponding projection map $A_g \to \Pi(U)$. Note that $\alpha_g = p^*\left(\Pi_p \pi_2\right)$. Consider now the counit ϵ of the adjunction $\Pi_p \dashv p^*$ and define B_g and β_g so that

$$
\begin{array}{ccc}
B_g & \dashrightarrow & \tilde{U} \\
{\scriptstyle \beta_g} \downarrow \quad {\scriptstyle \lrcorner} & & \downarrow {\scriptstyle p} \\
A_g & \xrightarrow{\pi_1 \circ \epsilon_{\pi_2}} & U
\end{array} \quad .
$$

At last, we define a Π-*structure on* U as a map $\bar{\Pi} : \Pi(U) \to U$ together with an isomorphism $\bar{\Pi}^* p \cong \Pi_{\alpha_g} \beta_g$.

Empty type

Recall that we can think of the empty type as the empty set, i.e., the initial object in **Set**. This leads us to define a **0**-*structure on* U as a map $\bar{0} : 1 \to U$ together with an isomorphism $\bar{0}^* \tilde{U} \cong 0$.

Identity types

Let $\mathscr{D}$ be a contextual category. For simplicity, we shall refer to an object (Γ, A) in $\mathscr{D}$ by simply A and to the product $(\Gamma, A) \times (\Gamma, A)$ in $\mathscr{D}$ by $A \times A$. Recall the notion of an Id-type structure on $\mathscr{D}$

(p. 34). Note that condition (ii) exhibits a section

$$\begin{array}{ccc}
\Delta_A^* \mathsf{Id}_A & \longrightarrow & \mathsf{Id}_A \\
s_{\Delta_A^* \mathsf{Id}_A} \Big\uparrow \Big\downarrow & & \Big\downarrow p_{\mathsf{Id}_A} \\
A & \xrightarrow{\ \Delta_A\ } & A \times A
\end{array}$$

by way of refl_A. This is equivalent to exhibiting a lift

$$\begin{array}{ccc}
 & & \mathsf{Id}_A \\
 & \nearrow & \Big\downarrow p_{\mathsf{Id}_A} \\
\mathsf{refl}_A & & \\
A & \xrightarrow{\ \Delta_A\ } & A \times A
\end{array} \tag{$\dagger$}$$

Further, condition (iii) exhibits a diagonal fill-in

$$\begin{array}{ccc}
A & \xrightarrow{\ d\ } & B \\
\mathsf{refl}_A \Big\downarrow & \overset{\mathsf{J}_{B,d}}{\nearrow} & \Big\downarrow p_B \\
\mathsf{Id}_A & =\!=\!= & \mathsf{Id}_A
\end{array}$$

This implies that refl_A has the left lifting property against *any* display map p_C in $\mathscr{D}$. Indeed, for any map $f : \mathsf{Id}_A \to \mathrm{ft}(C)$ and commutative square

$$\begin{array}{ccc}
A & \xrightarrow{\ g\ } & C \\
\mathsf{refl}_A \Big\downarrow & & \Big\downarrow p_C \\
\mathsf{Id}_A & \xrightarrow{\ f\ } & \mathrm{ft}(C)
\end{array} ,$$

we have a lift

$$\begin{array}{ccccc}
 & & \overset{g}{\frown} & & \\
A & \cdots\!\cdots\!\rightarrow & f^*C & \longrightarrow & C \\
\mathsf{refl}_A \Big\downarrow & \nearrow & \Big\downarrow p_{f^*C} & & \Big\downarrow p_C \\
\mathsf{Id}_A & =\!=\!= & \mathsf{Id}_A & \xrightarrow{\ f\ } & \mathrm{ft}(C)
\end{array} .$$

In light of ($\dagger$), we see that Δ_A factors as a map $A \to \mathsf{Id}_A$ having the left lifting property against all display maps followed by a display map $\mathsf{Id}_A \to A \times A$.

Remark 3.4.3. In the language of model categories (Section 4.2), if we view a display map as a fibration and a map having the left lifting property against all fibrations as a trivial cofibration, then Id_A is precisely a *path space object* of A.

With this in mind, an Id-*structure on* U consists of maps $\overline{\mathsf{Id}} : \tilde{U} \times_U \tilde{U} \to U$ and $\zeta : \tilde{U} \to \overline{\mathsf{Id}}^* \tilde{U}$ such that

$$\begin{array}{ccc}
\tilde{U} & \xrightarrow{\ \zeta\ } & \overline{\mathsf{Id}}^* \tilde{U} \\
 & \Delta_p \searrow & \Big\downarrow \overline{\mathsf{Id}}^* p \\
 & & \tilde{U} \times_U \tilde{U}
\end{array}$$

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commutes and ζ is *stably orthogonal* to $p \times U$ over U, i.e., for any map $X \to U$, every lifting problem

$$\begin{array}{ccc}
X \times_U \tilde{U} & \xrightarrow{\ a\ } & \tilde{U} \times_U U \\
{\scriptstyle X \times \zeta}\downarrow & & \downarrow{\scriptstyle p \times U} \\
X \times_U \bar{\mathsf{Id}}^* \tilde{U} & \xrightarrow[\ b\ ]{} & \underbrace{U \times_U U} \\
& & {\scriptstyle U}
\end{array}$$

between $X \times \zeta$ and $p \times U$ comes equipped with a solution $D(a, b)$ such that for any map $f : Y \to X$ over U,

$$D\left(a, b\right) \circ \left(f \times \bar{\mathsf{Id}}^* U\right) = D\left(a \circ \left(f \times \tilde{U}\right), b \circ \left(f \times \bar{\mathsf{Id}}^* U\right)\right).$$

This structure can be thought of as the "generic" identity type just as a Π-structure is seen as the generic dependent product. This means that it produces in $\mathscr{C}_U$ *every* identity type thanks to the universal properties of a LCCC.

Universe type

A U-structure on U roughly amounts to a universe U_0 in $\mathscr{C}$ that is "nested" in U. To be precise, a *nested universe in U* is a pair (u_0, ι) where u_0 is a map $1 \to U$ and ι is a map $U_0 \coloneqq u_0^* \tilde{U} \to U$.

Terminology. U_0 is called an *internal universe*, and U a *meta-universe*.

Note that U_0 is, indeed, a universe in $\mathscr{C}$ when equipped with the morphism

$$\underbrace{\iota^* p}_{p_0} : \underbrace{\iota^* \tilde{U}}_{\tilde{U}_0} \to U_0$$

and, for each map $f : X \to U_0$, the lefthand square in the commutative diagram

$$\begin{array}{ccccc}
& & \overset{Q(\iota \circ f)}{\overgroup{}} & & \\
(X; \iota \circ f) & \dashrightarrow{\ \ } & \tilde{U}_0 & \xrightarrow{\ \ } & \tilde{U} \\
{\scriptstyle p_{X, \iota \circ f}}\downarrow & {\scriptstyle Q(f)} & \downarrow{\scriptstyle p_0} & \lrcorner & \downarrow{\scriptstyle p} \\
X & \xrightarrow[\ f\ ]{} & U_0 & \xrightarrow[\ \iota\ ]{} & U
\end{array}$$

as the distinguished pullback square. We say that U_0 is *closed under Π-types in U* if it has a Π-structure $\bar{\Pi}_0 : \Pi(U_0) \to U_0$ such that

$$\begin{array}{ccc}
\Pi(U_0) & \xrightarrow{\ \Pi(\iota)\ } & \Pi(U) \\
{\scriptstyle \bar{\Pi}_0}\downarrow & & \downarrow{\scriptstyle \bar{\Pi}} \\
U_0 & \xrightarrow[\ \iota\ ]{} & U
\end{array}$$

commutes (with $\Pi(i)$ as in Note 3.4.2). We say that U_0 is *closed under Σ-types, etc.* under similar circumstances.

Again, in the interest of space, we have omitted the definitions of Σ-*structure*, **1**-*structure*, and **2**-*structure*.

Induced logical structure on $\mathscr{C}_U$

The following result will ensure that the induced Π- and Σ-type structures on $\mathscr{C}_U$ are stable under substitution.
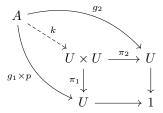
Lemma 3.4.4. *Let $B \xrightarrow{f_2} A \xrightarrow{f_1} \Gamma$ be a composite of morphisms in $\mathscr{C}$. Let $g_1 : \Gamma \to U$, $g_2 : A \to U$, $h_1 : A \to \tilde{U}$, and $h_2 : B \to \tilde{U}$ be morphisms in $\mathscr{C}$ such that*

$$
\begin{array}{ccc}
A & \xrightarrow{h_1} & \tilde{U} \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle p} \\
\Gamma & \xrightarrow[g_1]{} & U
\end{array}
\qquad\qquad
\begin{array}{ccc}
B & \xrightarrow{h_2} & \tilde{U} \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle p} \\
A & \xrightarrow[g_2]{} & U
\end{array}
$$

are pullback squares. Then there exists a unique map (A, B) such that

$$
\begin{array}{ccc}
B & \longrightarrow & B_g \\
{\scriptstyle f_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta_g} \\
A & \longrightarrow & A_g \\
{\scriptstyle f_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha_g} \\
\Gamma & \xrightarrow[(A,B)]{} & \Pi(U)
\end{array}
\qquad (*)
$$

in $\mathscr{C}$.

Proof. We have a unique mediating map

by the universal property of pullback squares. Note that k is a map $g_1 \times p \to \pi_1$ in $\mathscr{C}/U$. Thus, we can take its exponential transpose $\tilde{k} : g_1 \to (\pi_1)^p$. Since

$$
(\pi_1)^p \simeq \Pi_p p^* \pi_1 = \Pi_p \pi_2^U
$$

by Proposition C.0.15, we see that $\tilde{k}$ is a map $g_1 \to \Pi_p \pi_2^U$, i.e., a map $(A, B) : \Gamma \to \Pi(U)$ over U. It is easy to check that this map satisfies $(*)$. The fact that it is unique is clear from the way in which it is constructed. $\square$

Of course, this universal property fails to imply that the induced Id-type structure is stable. For this, we use instead the requirement that ζ be stably orthogonal to $p \times U$.

Theorem 3.4.5. *Any given logical structure on U induces a corresponding logical structure on $\mathscr{C}_U$.[18]*

Proof. Other than the data for Id-ELIM (i.e., condition (iii) and part of condition (iv) for a Id-type structure (p. 34)), each corresponding structure flows from the "generic" structure on U in a

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[18] [14, Theorem 1.4.15].

predictable way. To understand our method for verifying this, it is enough to consider just the data for Π-FORM, i.e., condition (i) and part of condition (v) for a Π-type structure (p. 33).

Suppose that U has a Π-structure $\bar{\Pi} : \Pi(U) \to U$. Let $(\Gamma, A, B) \in \mathrm{Ob}_{n+2}\,\mathscr{C}_U$. This is precisely a pair of pullback squares

$$
\begin{array}{ccc}
(1; \Gamma, A) & \xrightarrow{Q(A)} & \tilde{U} \\
{\scriptstyle P_{\Gamma, A}}\downarrow & & \downarrow{\scriptstyle p} \\
(1; \Gamma) & \xrightarrow{A} & U
\end{array}
\qquad
\begin{array}{ccc}
(1; \Gamma, A, B) & \xrightarrow{Q(B)} & \tilde{U} \\
{\scriptstyle P_{\Gamma, A, B}}\downarrow & & \downarrow{\scriptstyle p} \\
(1; \Gamma, A) & \xrightarrow{B} & U
\end{array}
$$

in $\mathscr{C}$, which yields another diagram

$$
\begin{array}{ccc}
(1; \Gamma, A, B) & \longrightarrow & B_g \\
{\scriptstyle P_{\Gamma, A, B}}\downarrow & & \downarrow{\scriptstyle \beta_g} \\
(1; \Gamma, A) & \longrightarrow & A_g \\
{\scriptstyle P_{\Gamma, A}}\downarrow & & \downarrow{\scriptstyle \alpha_g} \\
(1; \Gamma) & \xrightarrow[(A,B)]{} & \Pi(U)
\end{array}
$$

consisting of pullback squares via Lemma 3.4.4. Take the composite $\bar{\Pi} \circ (A, B)$ to be $\Pi(A, B)$ in $\mathscr{C}_U$. It remains to check that $\Pi(A, B)$ is stable under substitution in $\mathscr{C}_U$. Let $f : (1; \Gamma') \to (1; \Gamma)$ be a map in $\mathscr{C}$. On the one hand, we have that

$$
f^*(\Gamma, \Pi(A, B)) = \left(\Gamma', \bar{\Pi} \circ (A, B) \circ f \right).
$$

On the other hand,

$$
(\Gamma', \Pi(f^*A, f^*B)) = \left(\Gamma', \bar{\Pi} \circ (f^*A, f^*B) \right)
$$
$$
(f^*A, f^*B) = (A, B) \circ f
$$

by the uniqueness of (f^*A, f^*B). Hence $\Pi(A, B)$ is stable. $\qquad\square$

Remark 3.4.6. It is easy to show that, under this logical structure, $\mathscr{C}_U$ also satisfies Π-η.

In conclusion, we have a method for building a model of $\mathbb{T}$ in a given LCCC: finding a "logically structured" universe in it. This is precisely the method we employ in Section 5.

3.5 Presheaf universes

At this time, it is worth outlining a modest extension of the map $U \mapsto \mathscr{C}_U$ due to [3]. This requires a certain concept from algebraic geometry. Suppose that $\mathscr{C}$ is any locally small category with a terminal object 1. At this point, we may pass to a larger Grothendieck universe than our current one so that $\mathscr{C}$ is small (see p. 27).

Notation. Let $\mathcal{Y} : \mathscr{C} \to \widehat{\mathscr{C}}$ denote the Yoneda embedding, as in Lemma C.0.7.

Definition 3.5.1 (Grothendieck). Let $\widetilde{\mathcal{U}}$ and $\mathcal{U}$ be objects in $\widehat{\mathscr{C}}$. A natural transformation

$$
\rho : \widetilde{\mathcal{U}} \to \mathcal{U}
$$

is *representable* if for any object C in $\mathscr{C}$ and any map $T : \mathcal{Y}_C \to \mathcal{U}$, ρ comes equipped with a distinguished pullback square

$$
\begin{array}{ccc}
\mathcal{Y}_{C.T} & \xrightarrow{\ q_T\ } & \widetilde{\mathcal{U}} \\
{\scriptstyle \mathcal{Y}(p_T)}\downarrow & \lrcorner & \downarrow{\scriptstyle \rho} \\
\mathcal{Y}_C & \xrightarrow{\ T\ } & \mathcal{U}
\end{array}
$$

in $\widehat{\mathscr{C}}$, which means that each fiber of ρ is chosen to be a representable object. In this case, we say that $\mathcal{U}$ is a *universe in $\widehat{\mathscr{C}}$*.

Since the functor $\mathrm{Hom}_{\mathscr{C}}(C, -) : \mathscr{C} \to \mathbf{Set}$ is limit preserving for each $C \in \mathrm{Ob}\,\mathscr{C}$, so is the Yoneda embedding. Therefore, we can apply $\mathcal{Y}$ to Definition 3.3.1 to get a special case of Definition 3.5.1. As h is fully faithful, this means that any universe in $\mathscr{C}$ may be viewed as a universe in $\widehat{\mathscr{C}}$.

As it turns out, a representable natural transformation ρ over $\mathscr{C}$ makes $\mathscr{C}$ into a *category with families*, equivalently a *category with attributes* [16, Definition 6.3.3]. This is the same as a contextual category with the $\mathbb{N}$-grading of $\mathrm{Ob}\,\mathscr{C}$ replaced by a chosen class $\mathrm{Ty}_{\mathscr{C}}(X)$ of *semantic types* as well as a chosen *total object* $X \ltimes A$ *of A* for each $X \in \mathrm{Ob}\,\mathscr{C}$ and $A \in \mathrm{Ty}_{\mathscr{C}}(X)$.

Remark 3.5.2. We assume that Conjecture 3.2.13 also holds for categories with attributes.

In particular, a canonical pullback square looks like

$$
\begin{array}{ccc}
Y \ltimes f^*A & \xrightarrow{\ q(f,A)\ } & X \ltimes A \\
{\scriptstyle p_{f^*A}}\downarrow & & \downarrow{\scriptstyle p_A} \\
Y & \xrightarrow{\ f\ } & X
\end{array}\ .
$$

If we regard $C.T \in \mathrm{Ob}\,\mathscr{C}$ as the total object of T, then any such pullback square exists in $\mathscr{C}$. Indeed, consider the cospan

$$
\begin{array}{c}
C.T \\
\downarrow{\scriptstyle p_T} \\
B \xrightarrow{\ f\ } C
\end{array}
$$

in $\mathscr{C}$. We have two pullbacks of ρ fitting into a commutative diagram

$$
\begin{array}{ccccc}
 & & \overset{q_{T \circ \mathcal{Y}(f)}}{\frown} & & \\
\mathcal{Y}_{B.T \circ \mathcal{Y}(f)} & \dashrightarrow{g} & \mathcal{Y}_{C.T} & \xrightarrow{\ q_T\ } & \widetilde{\mathcal{U}} \\
{\scriptstyle \mathcal{Y}(p_{T \circ \mathcal{Y}(f)})}\downarrow & & \downarrow{\scriptstyle \mathcal{Y}(p_T)} & & \downarrow{\scriptstyle \rho} \\
\mathcal{Y}_B & \xrightarrow{\ \mathcal{Y}(f)\ } & \mathcal{Y}_C & \xrightarrow{\ T\ } & \mathcal{U}
\end{array}\ .
$$

As both the total rectangle and the righthand square are pullbacks, so is the lefthand square. Further, the induced map g has the form $\mathcal{Y}(q)$ for some unique map $q : B.T \circ \mathcal{Y}(f) \to C.T$ in $\mathscr{C}$ since the Yoneda embedding is fully faithful. It also reflects all limits for the same reason. Therefore, we can take

$$
\begin{array}{ccc}
B.T \circ \mathcal{Y}(f) & \xrightarrow{\ q\ } & C.T \\
{\scriptstyle p_{T \circ \mathcal{Y}(f)}}\downarrow & & \downarrow{\scriptstyle p_T} \\
B & \xrightarrow{\ f\ } & C
\end{array}
$$

as a canonical pullback square in $\mathscr{C}$.

Moreover, [3] defines dependent products, dependent sums, and identity types on $\mathcal{U}$ just as in Section 3.4 and proves a corresponding result to Theorem 3.4.5. It is straightforward to extend that result to the empty type, the unit type, and boolean types on $\mathcal{U}$ again with similar definitions to those found in Section 3.4. In conclusion, if $\mathcal{U}$ is logically structured enough, then $\mathscr{C}$ models our type theory $\mathbb{T}$ excluding the universe type U.

Let $\mathbb{D}$ be a subclass of $\mathrm{Mor}(\mathscr{C})$. We want to specify conditions on $\mathbb{D}$ guaranteeing the existence of a universe $\mathcal{U}$ in $\widehat{\mathscr{C}}$ with sufficient logical structure.

Definition 3.5.3. We say that $\mathbb{D}$ is *stable* if

(a) for any $f \in \mathbb{D}$, the pullback of f along any map in $\mathscr{C}$ exists and

(b) for any pullback square

$$
\begin{array}{ccc}
X_4 & \longrightarrow & X_1 \\
{\scriptstyle e}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
X_3 & \longrightarrow & X_2
\end{array}
$$

in $\mathscr{C}$, we have that $g \in \mathbb{D} \implies e \in \mathbb{D}$.

Define the presheaves $\mathbb{D}_1$ and $\mathbb{D}_0$ on $\mathscr{C}$ as follows, where $C/\mathscr{C}$ denotes the under category.

$$\mathbb{D}_1(C) \equiv \{(a,d) \in C/\mathscr{C} \times \mathbb{D} \mid \mathrm{cod}(a) = \mathrm{dom}(d)\}$$
$$\mathbb{D}_0(C) \equiv \{(a,d) \in C/\mathscr{C} \times \mathbb{D} \mid \mathrm{cod}(a) = \mathrm{cod}(d)\}$$

$$\mathbb{D}_1(s : D \to C)(a,d) \equiv (a \circ s, d)$$
$$\mathbb{D}_0(s : D \to C)(a,d) \equiv (a \circ s, d)$$

Further, define the natural transformation $\zeta(\mathbb{D}) : \mathbb{D}_1 \to \mathbb{D}_0$ componentwise by

$$\zeta(\mathbb{D})_C(a,d) \equiv (d \circ a, d).$$

Lemma 3.5.4. *Suppose that $\mathbb{D}$ is stable. Then the natural transformation $\zeta(\mathbb{D})$ is representable.*

Proof sketch. Let $C \in \mathrm{Ob}\,\mathscr{C}$ and consider any map $T : h_C \to \mathcal{U}$. We must exhibit a pullback square in $\widehat{\mathscr{C}}$ of the form

$$
\begin{array}{ccc}
\mathcal{Y}_{C.T} & \xrightarrow{\ q_T\ } & \mathbb{D}_1 \\
{\scriptstyle \mathcal{Y}(p_T)}\downarrow & \lrcorner & \downarrow{\scriptstyle \zeta(\mathbb{D})} \\
\mathcal{Y}_C & \xrightarrow[\ T\ ]{} & \mathbb{D}_0
\end{array}
\qquad (\diamond)
$$

Note that T is an element of $\mathbb{D}_0(C)$ via the Yoneda lemma, so that it is a cospan of the form

$$
\begin{array}{ccc}
 & & B \\
 & & \downarrow{\scriptstyle d_T} \\
C & \xrightarrow[\ a_T\ ]{} & A
\end{array}
$$

where $d_T \in \mathbb{D}$. Take the pullback square

$$
\begin{array}{ccc}
a_T^*(B) & \xrightarrow{\tilde{q}_T} & B \\
{\scriptstyle p_T}\big\downarrow & & \big\downarrow{\scriptstyle d_T} \\
C & \xrightarrow[a_T]{} & A
\end{array}
$$

in $\mathscr{C}$. Let $C.T = a_T^*(B)$ and let q_T be the map corresponding to the pair $(\tilde{q}_T, d_T) \in \mathbb{D}_1(C.T)$ under the Yoneda lemma. A straightforward yet tedious argument, omitted here, confirms that $(\diamond)$ is a pullback square. $\qquad\square$

For any $X \in \mathscr{C}$, let $\mathbb{D}(X)$ denote the full subcategory of $\mathscr{C}/X$ consisting of all maps in $\mathbb{D}$ with codomain X.

Definition 3.5.5. We say that $\mathbb{D}$ is *closed* if

    (a) it is stable,

    (b) it is closed under composition,

    (c) every map of the form $C \to 1$ belongs to $\mathbb{D}$,

    (d) for any map $a : D \to C$ in $\mathscr{C}$, the base change functor $a^* : \mathbb{D}(C) \to \mathbb{D}(D)$ has a right adjoint, and

    (e) the inclusion functor $\mathbb{D}(C) \hookrightarrow \mathscr{C}/C$ preserves exponentials.

Definition 3.5.6. We say that $\mathbb{D}$ is *factorizing* if every map $a : C \to D$ in $\mathscr{C}$ factors as $a = d \circ f$ where $d \in \mathbb{D}$ and f has the left lifting property against all maps in $\mathbb{D}$.

Theorem 3.5.7. *Suppose that $\mathbb{D}$ is both closed and factorizing. Then the universe $\mathbb{D}_0$ in $\widehat{\mathscr{C}}$ carries enough logical structure that $\mathscr{C}$ (as a category with families) models $\mathbb{T}$ excluding U.*[19]

4 Homotopy theory

This section develops those notions from classical homotopy theory which Section 5 will rely on.

4.1 Simplicial sets

Here, we gather a number of standard concepts and properties about the category **sSet** of simplicial sets, i.e., the functor category $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ where Δ denotes the category of all nonempty finite ordinals with order-preserving functions as morphisms, known as the *simplex category*.

Recall that any simplicial set X admits a nice combinatorial description. Specifically, for any $n \in \mathbb{N}$ and integer $0 \le i \le n + 1$, consider the i-th coface morphism $\delta_i^n : [n] \to [n+1]$ defined by

$$
\delta_i^n(m) = \begin{cases} m & m < i \\ m+1 & m \ge i \end{cases}.
$$

---

[19][3, Theorem 32].

Also, for any integer $0 \le i \le n$, consider the i-th codegeneracy morphism $\sigma_i^n : [n+1] \to [n]$ defined by

$$\sigma_i^n(m) = \begin{cases} m & m \le i \\ m - 1 & m > i \end{cases}.$$

The following properties of $\mathrm{Mor}(\Delta)$ are easy to verify yet quite useful.

Lemma 4.1.1.

(1) *Any morphism $[n] \to [m]$ in Δ factors uniquely as the composite of an epimorphism $[n] \to [p]$ (i.e., an order-preserving surjection) and a monomorphism $[p] \to [m]$ (i.e., an order-preserving injection).*

(2) *Any epimorphism $\epsilon : [n] \to [p]$ in Δ factors uniquely as $\epsilon = \sigma_{j_1} \cdots \sigma_{j_t}$ where $t \equiv n - p$ and $0 \le j_1 < \cdots < j_t < n$.*

(3) *Any monomorphism $\mu : [p] \to [m]$ factors uniquely as $\mu = \delta_{i_r} \cdots \delta_{i_1}$ where $r \equiv m - p$ and $0 \le i_1 < \cdots < i_r \le m$.*

Now, let us form the i-th face operator $d_i := X(\delta_i^n) : X_{n+1} \to X_n$ and i-th degeneracy operator $s_i := X(\sigma_i^n) : X_n \to X_{n+1}$ in X.

Terminology. The *boundary* of an n-simplex $x \in X_n$ is the tuple $\partial x := (d_0 x, \ldots, d_n x)$.

For any n-simplex x in X, we can view $d_i(x)$ as the $(n-1)$-simplex, or face, in X missing the i-th vertex of x. Moreover, we can view $s_i(x)$ as the $(n+1)$-simplex having x as its i-th and $(i+1)$-th faces so that collapsing the edge between its i and $(i+1)$-th vertices yields x. For example, the map s_1 acts on 1-simplices by

Terminology. A simplex of the form $s_i(x)$ is called *degenerate*.

Since any epimorphism other than an identity morphism in Δ factors as a composite of degeneracy operators, we may say equivalently that x is degenerate if there exist an epimorphism $s : [n] \to [m]$ with $m < n$ and an m-simplex $y \in X_m$ such that $x = X(s)(y)$.

Lemma 4.1.2 (Simplicial identities).

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{for } i < j \\ d_i s_j = s_{j-1} d_i & \text{for } i < j \\ d_i s_j = \mathrm{id}_{X_n} & \text{for } j \le i \le j+1 \\ d_i s_j = s_j d_{i-1} & \text{for } j+1 < i \\ s_i s_j = s_{j+1} s_i & \text{for } i \le j \end{cases}.$$

Proof. It is easy to verify that the *cosimplicial identities*, those dual to the simplicial identities, hold in Δ, e.g.,

$$\delta_j \delta_i = \delta_i \delta_{j-1}, \quad i < j.$$

Applying now the contravariant functor X on Δ to the cosimplicial identities yields the simplicial ones. $\qquad\square$

Conversely, a family of set maps

$$\{d_i^n : X_{n+1} \to X_n \mid 0 \leq i \leq n+1, \ n \in \mathbb{N}\} \cup \{s_j^n : X_n \to X_{n+1} \mid 0 \leq j \leq n, \ n \in \mathbb{N}\}$$

satisfying the simplicial identities completely determines a simplicial set X.[20] In other words, a simplicial set amounts to an $\mathbb{N}$-graded set (X_n) equipped with such a family of maps.

Lemma 4.1.3 (Eilenberg-Zilber). *For any $x \in X_n$, there exists a unique pair (s, y) where $s : [n] \to [m]$ is an epimorphism and y is a non-degenerate m-simplex in X satisfying $x = X(s)(y)$.*

Proof. To see that such a pair exists, we have two cases to consider.

- If x is non-degenerate, then simply take the pair $\left(\text{id}_{[n]}, x\right)$.

- If x is degenerate, then by definition we can find an epimorphism $t : [n] \to [m]$ with $m < n$ and an m-simplex $w \in X_m$ such that $x = X(t)(w)$. In this case, if w is non-degenerate, then we are done. Otherwise, we can find another pair (r, z) witnessing the degeneracy of w, so that $r : [m] \to [p]$ for some $p < m$. But this process must terminate in finitely many steps, resulting in our desired pair since any composite of epimorphisms is epic.

To see that (s, y) is unique, suppose that (s', y') is another such pair.

Claim. *Any epimorphism in Δ is a split epimorphism.*

Proof. Note that any surjective map in Δ has a set-theoretic section,[21] which is easily seen to be order-preserving. But the epimorphisms in Δ are precisely the order-preserving surjections, which completes our proof. $\qquad\square$

Thus, we may choose sections σ and σ' of s and s' in Δ, respectively. This implies that

$$y = X(\sigma)(x) = X(\sigma)X(s')(y') = X(s'\sigma)(y').$$

Since y' is non-degenerate, $s'\sigma$ must be an automorphism. But the identity map is the only automorphism of a well-ordered set. Hence $y' = y$, and any section of s is a section of s'. This means that $s' = s$, so that $(s, y) = (s', y')$. $\qquad\square$

Before moving on, let us record a definition that will appear in Section 4.2.

Definition 4.1.4. A simplicial set is *finite* if it has only finitely many non-degenerate simplices.

Next, for any integer $n \geq 1$ and any simplicial set X, consider the discrete diagram D valued in the over category $\mathbf{sSet}/X$ consisting of all simplicial subsets $Y \hookrightarrow X$ that contain every non-degenerate simplex in X of degree $< n$. The limit $\text{sk}_n(X) \hookrightarrow X$ of D, which exists because $\mathbf{sSet}$ is complete, is called the *intersection* of the Y. We can view the *n-skeleton* $\text{sk}_n(X)$ *of X* as the smallest simplicial subset of X that contains every non-degenerate simplex in X of degree $< n$. Note that $\text{sk}_n(-)$ determines a functor $\mathbf{sSet} \to \mathbf{sSet}$.

Consider the Yoneda embedding $\mathcal{Y} : \Delta \to \widehat{\Delta}$ (Lemma C.0.7). For any $n \in \mathbb{N}$, we call the simplicial set

$$\Delta[n] := \mathcal{Y}[n] = \Delta(-, [n])$$

the *standard (combinatorial) n-simplex.*

---

[20][21, Lemma 6.2.8].
[21]This is provable in ZF as it follows from the finite version of the axiom of choice.

Remark 4.1.5. Notice that

$$\Delta[n]_k \cong \{(x_0, \dots, x_k) \mid 0 \leq x_i \leq x_j \leq n, \ i \leq j\}$$

for any $k \in \mathbb{N}$. Thus, we can view $\Delta[n]$ as the simplicial complex whose k-simplices are precisely the nonempty subsets of $\{0, 1, \dots, n\}$ endowed with their natural orders.

Now, the i-th face operator $d_i : \Delta[n]_{k+1} \to \Delta[n]_k$ is given by

$$\left([k+1] \xrightarrow{f} [n] \right) \mapsto \left([k] \xrightarrow{\delta_i} [k+1] \xrightarrow{f} [n] \right).$$

Additionally, the i-th degeneracy operator $s_i : \Delta[n]_k \to \Delta[n]_{k+1}$ is given by

$$\left([k] \xrightarrow{f} [n] \right) \mapsto \left([k+1] \xrightarrow{\sigma_i} [k] \xrightarrow{f} [n] \right).$$

Using Lemma 4.1.1, we see that the non-degenerate k-simplices in $\Delta[n]$ are precisely the monomorphisms belonging to $\Delta([k],[n])$. In particular, the unique non-degenerate n-simplex in $\Delta[n]$ is precisely $\mathrm{id}_{[n]}$. Thus, for any simplicial set X, there is a natural one-to-one correspondence

$$x \in X_n \longleftrightarrow \Delta[n] \xrightarrow{x} X$$

where the map x sends the unique non-degenerate n-simplex in $\Delta[n]$ to the element x.

Suppose that $n \in \mathbb{Z}_{\geq 1}$. For each $i \in \{0, 1, \dots, n\}$, The *i-th face* of $\Delta[n]$ is the simplicial subset (i.e., subfunctor) $\partial^i \Delta[n]$ of $\Delta[n]$ with

$$\partial^i \Delta[n]_k \equiv \mathrm{im}(\Delta(\delta_i^{n-1})_k) \subset \Delta[n]_k$$

for each $k \in \mathbb{N}$. The simplicial subset

$$\partial \Delta[n] \coloneqq \bigcup_{i=0}^{n} \partial^i \Delta[n]$$

of $\Delta[n]$, computed pointwise in **Set**, is called the *simplicial $(n-1)$-sphere* or *boundary of $\Delta[n]$*.

Remark 4.1.6. The k-simplices of $\partial \Delta[n]$ are precisely the non-surjective morphisms $[k] \to [n]$ in Δ. Thus, in light of Remark 4.1.5, we can view $\partial \Delta[n]$ as the simplicial complex whose k-simplices are precisely the nonempty *proper* subsets of $\{0, 1, \dots, n\}$ endowed with their natural orders.

Geometric realization

Let $\mathscr{D}$ be any cocomplete, locally small category and let $\mathscr{C}$ be any small category. Suppose that F is a covariant functor $\mathscr{C} \to \mathscr{D}$. For each $X \in \mathrm{Ob}\, \mathscr{D}$, consider the presheaf

$$R_F(X) \equiv \mathrm{Hom}_{\mathscr{D}}(F(-), X) : \mathscr{C}^{\mathrm{op}} \to \textbf{Set}.$$

Also, for any map $f : X \to Y$ in $\mathscr{D}$, define the natural transformation $R_F f : R_F(X) \to R_F(Y)$ componentwise by

$$R_F(X)_c \to R_F(Y)_c, \quad \varphi \mapsto f \circ \varphi.,$$

thereby yielding a functor $R_F : \mathscr{D} \to [\mathscr{C}^{\mathrm{op}}, \textbf{Set}]$.

Example 4.1.7. We can view each nonempty finite ordinal as an order category. This means that Δ is precisely the full subcategory of **Cat** on all nonempty finite ordinals. Consider the full inclusion $\iota : \Delta \hookrightarrow \mathbf{Cat}$. Then for each small category $\mathscr{E}$, $N\mathscr{E} \coloneqq R_\iota(\mathscr{E})$ is a simplicial set (called the *nerve of $\mathscr{E}$*) with $N\mathscr{E}_n = \mathrm{Hom}_{\mathbf{Cat}}([n], \mathscr{E})$ for each $n \in \mathbb{N}$, i.e., the set of all sequences of n composable morphisms in $\mathscr{E}$. The i-th face operator $d_i : N\mathscr{E}_{n+1} \to N\mathscr{E}_n$ is given by

$$x_0 \to x_1 \to \cdots \to x_{n+1} \to x_{n+2} \mapsto \begin{cases} x_1 \to x_2 \to \cdots \to x_{n+1} \to x_{n+2} & i = 0 \\ x_0 \to x_1 \to \cdots \to x_n \to x_{n+1} & i = n+1 \\ x_0 \to \cdots \to x_i \to x_{i+2} \to \cdots \to x_{n+2} & \text{otherwise} \end{cases},$$

and the j-th degeneracy operator $s_j : N\mathscr{E}_n \to N\mathscr{E}_{n+1}$ is given by

$$(x_0 \to x_1 \to \cdots \to x_n \to x_{n+1}) \mapsto \left(x_0 \to \cdots \to x_j \xrightarrow{\mathrm{id}_{x_j}} x_j \to x_{j+1} \to \cdots \to x_{n+1} \right)$$

for each $j = 0, 1, \ldots, n$.

In general, R_F has a left adjoint, which we now begin constructing. To this end, consider the following generalization of Definition C.0.2.

Definition 4.1.8. Let $\mathscr{A}$ be a closed monoidal category and $\mathscr{B}$ a category enriched over $\mathscr{A}$. Let $b \in \mathrm{Ob}\,\mathscr{B}$ and $S \in \mathrm{Ob}\,\mathscr{A}$. The *copower of b by S* is an object $S \odot b$ in $\mathscr{B}$ together with a natural isomorphism

$$\mathrm{Hom}_{\mathscr{B}}(S \odot b, y) \cong [S, \mathrm{Hom}_{\mathscr{B}}(b, y)]$$

in $y \in \mathrm{Ob}\,\mathscr{B}$.

For example, since $\mathscr{D}$ is enriched over **Set** and has all coproducts by assumption, the copower of $x \in \mathrm{Ob}\,\mathscr{D}$ by a set S is precisely the coproduct $\coprod_{s \in S} x$ of $|S|$ many copies of x along with the natural isomorphism

$$\mathrm{Hom}_{\mathscr{D}}\left(\coprod_{s \in S} x, y \right) \cong \prod_{s \in S} \mathrm{Hom}_{\mathscr{D}}(x, y) \cong \mathrm{Hom}_{\mathbf{Set}}(S, \mathrm{Hom}_{\mathscr{D}}(x, y))$$

witnessing the fact that $\mathrm{Hom}_{\mathscr{D}}(-, y) : \mathscr{D}^{\mathrm{op}} \to \mathbf{Set}$ is a continuous functor for each $y \in \mathrm{Ob}\,\mathscr{D}$.

Definition 4.1.9. Suppose that $\mathscr{A}$ is a small category and $\mathscr{B}$ is cocomplete. Let $G : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathscr{B}$ be a functor. The *coend of G* is the coequalizer of the diagram

$$\coprod_{f : a' \to a} G(a, a') \overset{G(a,f)}{\underset{G(f,a')}{\rightrightarrows}} \coprod_{a \in \mathrm{Ob}\,\mathscr{A}} G(a, a) \dashrightarrow \int^{a : \mathscr{A}} G(a, a) \qquad (\bullet)$$

in $\mathscr{B}$.

Here is an equivalent description of the coend of G. A *cowedge $k : G \to w$ for G* is an object w in $\mathscr{B}$ together with a family of morphisms $\{ k_a : G(a, a) \to w \mid a \in \mathrm{Ob}\,\mathscr{A} \}$ such that for each map $f : a' \to a$ in $\mathscr{A}$, the square

$$\begin{array}{ccc} w & \xleftarrow{\ k_{a'}\ } & G(a', a') \\ {\scriptstyle k_a} \uparrow & & \uparrow {\scriptstyle G(f, a')} \\ G(a, a) & \xleftarrow[G(a, f)]{} & G(a, a') \end{array}$$

commutes. The coend $\int^{a:\mathscr{A}} G(a,a)$ of G is defined as a universal cowedge $\left\{ \tilde{k}_a : G(a,a) \to w \mid a \in \mathrm{Ob}\,\mathscr{A} \right\}$ for G in the sense that there exists a unique map $\int^{a:\mathscr{A}} G(a,a) \to w$ in $\mathscr{B}$ such that any diagram of the form

$$
\begin{array}{ccc}
\int^{a:\mathscr{A}} G(a,a) & & \\
 & w \xleftarrow{\;k_{a'}\;} G(a',a') & \\
 & k_a \uparrow \qquad \qquad \uparrow G(f,a') & \\
 & G(a,a) \xleftarrow{\;G(a,f)\;} G(a,a') &
\end{array}
$$

commutes. We have a category $\mathbf{Cwd}(G)$ with cowedges for G as objects and maps $u : w \to w'$ in $\mathscr{B}$ for which the triangle

$$
\begin{array}{ccc}
w & \xrightarrow{\;u\;} & w' \\
k_a \uparrow & \nearrow_{k'_a} & \\
G(a,a) & &
\end{array}
$$

commutes for every $a \in \mathrm{Ob}\,\mathscr{A}$ as morphisms $w \to w'$. Note that the coend $\int^{a:\mathscr{A}} G(a,a)$ is precisely the initial object of $\mathbf{Cwd}(G)$. Thus, it may be viewed as either a colimit in $\mathscr{B}$ or a colimit in $\mathbf{Cwd}(G)$.

Definition 4.1.10. The *Yoneda extension of F* is the functor

$$
\widetilde{F} : [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathscr{D}, \quad X \mapsto \int^{c:\mathscr{C}} X(c) \odot F(c).
$$

We call $|X| \coloneqq \widetilde{F}(X)$ the *geometric realization* of X with respect to F.

If F denotes the functor $\Delta \to \mathbf{Top}$ mapping $[n]$ to the standard topological n-simplex Δ^n, then we recover the familiar definition of $|(X_n)|$ as a quotient space

$$
\coprod_{m \geq 0} (X_m \times \Delta^m)\big/_{\sim}
$$

where each set X_m is endowed with the discrete topology.

Proposition 4.1.11. *For any simplicial set X, $|X|$ is a CW-complex.*

Intuitive proof. For each integer $m \geq 0$, consider the subset $Y_m \subset X_m$ of all non-degenerate m-simplices. Then $|X|$ is obtained by gluing together countably many disjoint unions $Y_m \times \Delta^m \cong \coprod_{y \in Y_m} \Delta^m$ of topological simplices Δ^m along their individual boundaries $\partial \Delta^m$. Therefore, $|X|$ carries the structure of a CW-complex. $\qquad \square$

This means that the functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ takes values in the full subcategory $k\mathbf{Top}$ of *k-spaces*, i.e., quotient spaces of disjoint unions of compact Hausdorff spaces.

The subcategory $k\mathbf{Top} \subset \mathbf{Top}$ has a coreflection $k : \mathbf{Top} \to k\mathbf{Top}$, known as *$k$-ification*. For any topological space U, the k-space $k(U)$ is given by the set U topologized so that a subset $A \subset U$ is closed if and only if $A \cap H$ is closed in U for any compact Hausdorff subspace of U. Thus, $k(U)$ has a finer topology than U.

Since right adjoints preserve limits and **Top** is complete, we have that $k\mathbf{Top}$ is complete. Indeed, let $D : J \to k\mathbf{Top}$ be a functor. Then

$$k\left(\lim_J (\iota \circ D)_j\right) \cong \lim_J (k \circ \iota \circ D)_j$$

where $\iota : k\mathbf{Top} \hookrightarrow \mathbf{Top}$ denotes inclusion.

Proposition 4.1.12. *Consider any adjoint pair $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ of functors. If F is fully faithful, then the unit $\eta : \mathrm{id}_\mathscr{C} \to G \circ F$ is an isomorphism.*

Proof. Let the natural isomorphism $\varphi : \mathrm{Hom}_\mathscr{D}(F(-), -) \xrightarrow{\cong} \mathrm{Hom}_\mathscr{C}(-, G(-))$ witness our adjunction. Suppose that F is fully faithful. Then we have a composite of isomorphisms

$$\mathrm{Hom}_\mathscr{C}(x, y) \xrightarrow[\cong]{F(-)} \mathrm{Hom}_\mathscr{D}(Fx, Fy) \xrightarrow[\cong]{\varphi_{x,Fy}} \mathrm{Hom}_\mathscr{D}(x, GFy)$$

for any $x, y \in \mathrm{Ob}\,\mathscr{C}$. For any map $f : x \to y$ in $\mathscr{C}$, we have that

$$\begin{aligned}
\varphi_{x,Fy}(F(f)) &= G(F(f)) \circ \eta_x && \text{(unit identity)} \\
&= \eta_y \circ f. && \text{(naturality of } \eta\text{)}
\end{aligned}$$

Thus, our composite isomorphism is given by $f \mapsto \eta_y \circ f$, i.e., $\mathrm{Hom}_\mathscr{C}(x, \eta_y)$. As the Yoneda embedding reflects isomorphisms, we see that $\eta_y : y \xrightarrow{\cong} GFy$ is an isomorphism, i.e., η is an isomorphism. $\square$

As a consequence, $k \circ \iota$ is naturally isomorphic to the identity functor because. Hence $\lim_J D_j$ is given by $k\left(\lim_J (\iota \circ D)_j\right)$. Note that the limit of a diagram in $k\mathbf{Top}$ is obtained by k-ifying the limit of the same diagram in **Top**.

Theorem 4.1.13.

(1) *The functor $|-| : \mathbf{sSet} \to k\mathbf{Top}$ preserves finite products.*[22]

(2) *The functor $|-| : \mathbf{sSet} \to k\mathbf{Top}$ preserves equalizers, hence all finite limits.*[23]

Let us now return to our general setting.

Lemma 4.1.14. $\left(\widetilde{F}, R_F\right)$ *is an adjoint pair.*[24]

Proof. Let $x \in \mathrm{Ob}\,\mathscr{C}$ and consider the Yoneda embedding $\mathcal{Y} : \mathscr{C} \to \widehat{\mathscr{C}}$. For each $c \in \mathrm{Ob}\,\mathscr{C}$, define the map $k_c : \coprod_{s \in \mathcal{Y}_x(c)} F(c) \to F(x)$ in $\mathscr{D}$ by the copairing $(F(s) : F(c) \to F(x))_{s \in \mathcal{Y}_x(c)}$. For any map $f : c' \to c$ in $\mathscr{C}$ and any $s \in \mathcal{Y}_x(c)$, we have that

$$F(s) \circ F(f) = F(s \circ f) = F(\mathcal{Y}_x(f)(s)) = F(\mathcal{Y}_x(f)(s)) \circ \mathrm{id}_{F(c')}.$$

---

[22] [12, Lemma 3.1.8].
[23] [12, Lemma 3.2.4].
[24] Cf. [18, Section 4].

This implies that the square

$$
\begin{array}{ccc}
F(x) & \xleftarrow{\quad k_{c'} \quad} & \coprod_{s\in\mathcal{Y}_x(c')} F(c') \\
{\scriptstyle k_c}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathcal{Y}_x(f)\odot F(c')} \\
\coprod_{s\in\mathcal{Y}_x(c)} F(c) & \xleftarrow[\mathcal{Y}_x(c)\odot F(f)]{} & \coprod_{s\in\mathcal{Y}_x(c)} F(c')
\end{array}
$$

commutes, so that $\{k_c\}$ is a cowedge to $F(x)$. Let $\left\{k'_c : \coprod_{s\in\mathcal{Y}_x(c)} F(c) \to w\right\}$ be another cowedge. Consider the composite $k'_x \circ i_{\mathrm{id}_x} : F(x) \to w$, where i_{id_x} denotes inclusion. For any map $s : c \to x$ in $\mathscr{C}$, the fact that

$$
\begin{array}{ccc}
w & \xleftarrow{\quad k'_c \quad} & \coprod_{t\in\mathcal{Y}_x(c)} F(c) \\
{\scriptstyle k'_x}\Big\uparrow & & \Big\uparrow{\scriptstyle \mathcal{Y}_x(s)\odot F(c)} \\
\coprod_{t\in\mathcal{Y}_x(x)} F(x) & \xleftarrow[\mathcal{Y}_x(x)\odot F(s)]{} & \coprod_{t\in\mathcal{Y}_x(x)} F(c)
\end{array}
$$

commutes yields

$$
\begin{aligned}
k'_x \circ i_{\mathrm{id}_x} \circ k_c \circ i_s &= k'_x \circ i_{\mathrm{id}_x} \circ F(s) \\
&= k'_c \circ (\mathcal{Y}_x(s) \odot F(c)) \circ i_{\mathrm{id}_x} \\
&= k'_c \circ i_{\mathcal{Y}_x(s)(\mathrm{id}_x)} \circ \mathrm{id}_{F(c)} \\
&= k'_c \circ i_s.
\end{aligned}
$$

It follows that $F(x)$ is a universal cowedge. By uniqueness of colimits, we have a natural isomorphism

$$
F(x) \cong \int^{c:\mathscr{C}} \mathcal{Y}_x(c) \odot F(c) = \widetilde{F}(\mathcal{Y}_x)
$$

in x. Thanks to this as well as the Yoneda lemma, we have a sequence of isomorphisms

$$
\mathrm{Hom}_{\widehat{\mathscr{C}}}(\mathcal{Y}_x, R_F(d)) \cong R_F(d)_x = \mathrm{Hom}_{\mathscr{D}}(F(x), d) \cong \mathrm{Hom}_{\mathscr{D}}(\widetilde{F}(\mathcal{Y}_x), d) \qquad (\blacklozenge)
$$

natural in both x and d. By Theorem C.0.8, every presheaf $X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ is naturally isomorphic to a small colimit of representable presheaves. Further, colimits commute with colimits, and thus $\widetilde{F}$ commutes with all colimits. Since the hom-functor of any locally small category is continuous in its first variable, we can conclude that ($\blacklozenge$) holds with $\mathcal{Y}_x$ replaced by any presheaf X. $\qquad\square$

Kan fibrations

Let $n \in \mathbb{Z}_{\geq 1}$. For each integer $0 \leq k \leq n$, the simplicial subset

$$
\Lambda^k[n] := \bigcup_{i\in\{0,\ldots,k-1,k+1,\ldots,n\}} \partial^i \Delta[n]
$$

of $\Delta[n]$, computed pointwise in $\mathbf{Set}$, is called the *(simplicial) (n,k)-horn*. For any simplicial set X, an *(n,k)-horn in X* is a simplicial map $\Lambda^k[n] \to X$.

Terminology. If $0 < k < n$, then such a map is called an *inner horn* in X. Otherwise, it is called an
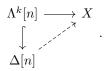
outer horn in X.

Example 4.1.15. The geometric realization of the inner horn in $\Delta[2]$ looks like

$$
\begin{array}{ccc}
 & 1 & \\
\nearrow & & \searrow \\
0 & & 2
\end{array}
\ ,
$$

whereas the geometric realizations of the two outer horns in $\Delta[2]$ look like

$$
\begin{array}{ccc}
 & 1 & \\
\nearrow & & \\
0 \ \rule{2em}{0.4pt}\ 2
\end{array}
\qquad
\begin{array}{ccc}
 & 1 & \\
 & & \searrow \\
0 \ \rule{2em}{0.4pt}\ 2
\end{array}
$$

We say that X is a *Kan complex* if every horn in X has a filler, i.e., can be extended to $\Delta[n]$ along the inclusion map:

$$
\begin{array}{ccc}
\Lambda^k[n] & \longrightarrow & X \\
\downarrow & \nearrow & \\
\Delta[n] & &
\end{array}
\ .
$$

Intuitively, by the Yoneda lemma, such an extension picks out a unique n-simplex in X all of whose faces but one are determined by the given horn in X.

Definition 4.1.16 (Kan fibration). A map $p \colon X \to Y$ of simplicial sets is a *Kan fibration* if any commutative square of the form

$$
\begin{array}{ccc}
\Lambda^k[n] & \longrightarrow & X \\
\downarrow & & \downarrow p \\
\Delta[n] & \longrightarrow & Y
\end{array}
$$

admits a lift

$$
\begin{array}{ccc}
\Lambda^k[n] & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow p \\
\Delta[n] & \longrightarrow & Y
\end{array}
\ .
$$

This means that X is a Kan complex if and only if the unique map $X \to 1$ from X to the terminal object $\Delta[0]$ is a Kan fibration.

Now, let $\mathscr{C}$ be any cocomplete category. We say that a subclass of $\mathrm{Mor}(\mathscr{C})$ is *saturated* if it

(i) contains all isomorphisms in $\mathscr{C}$,

(ii) is closed under pushouts,

(iii) is closed under retracts (see (†) below), and

(iv) is closed under transfinite compositions (see Definition 4.2.13 below).

Terminology. The smallest saturated class containing a given class K of morphisms in $\mathscr{C}$ is called the *saturated class generated by K*.

Definition 4.1.17 (Anodyne extension). A map of simplicial sets is an *anodyne extension* if it belongs to the saturated class generated by the set

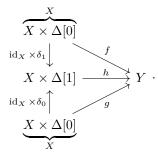$$\left\{ \Lambda^k[n] \hookrightarrow \Delta[n] \mid n \geq 1,\ 0 \leq k \leq n \right\}$$

of horn inclusions.

Notation. $\mathbb{A}$ will denote the class of all anodyne extensions.

Thanks to Remark 4.2.15 below, we see that a simplicial map is a Kan fibration if and only if it has the right lifting property against $\mathbb{A}$.

We want to look at a certain class of Kan fibrations that will play a key role in our interpretation of CDTT + Univ in **sSet**. For this, we must first gather some standard concepts of simplicial homotopy theory.

Definition 4.1.18 (Simplicial homotopy). Let $f, g : X \to Y$ be maps of simplicial sets. A *(simplicial) homotopy* $f \xrightarrow{\simeq} g$ *from* f *to* g is a map $h : X \times \Delta[1] \to Y$ of simplicial sets fitting into a commutative diagram

Notation.

- $h_0 := h \circ (\mathrm{id}_X \times \delta_1)$.

- $h_1 := h \circ (\mathrm{id}_X \times \delta_0)$.

The cospan $\Delta[0] \xrightarrow{0:=\delta_1} \Delta[1] \xleftarrow{1:=\delta_0} \Delta[0]$ is the standard *interval object* in **sSet**, analogous to the standard interval object

$$\{0\} \longhookrightarrow [0,1] \longleftarrow\hspace{-1em}\hookleftarrow \{1\}$$

in **Top**. From this perspective, any simplicial homotopy $h : f \xrightarrow{\simeq} g$ satisfies $h(x,0) = f(x)$ and $h(x,1) = g(x)$ for all simplices x in X.

Simplicial homotopy $\simeq$ is *not* an equivalence relation in general. For example, let $n \in \mathbb{Z}_{\geq 1}$ and consider the simplicial maps $\iota_0, \iota_1 : \Delta[0] \to \Delta[n]$ induced by the monomorphisms $0 \mapsto 0$ and $0 \mapsto 1$, respectively, in Δ. In light of Remark 4.1.5, it is easy to see, on the one hand, that the map sending, say, $(0, 0, 1, 1)$ to itself determines a homotopy from ι_0 to ι_1. On the other hand, there is no homotopy from ι_1 to ι_0, because $0 \leq 1$. This shows that $\simeq$ is not symmetric in general.

Lemma 4.1.19. *Simplicial homotopy is an equivalence relation on the class of all simplicial maps $X \to Y$ with Y a Kan complex.*

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Proof. For the moment, assume that $X = \Delta[0]$. Then, by the Yoneda lemma, there is a homotopy $f \xrightarrow{\simeq} g$ if and only if there is a 1-simplex $v \in Y_1$ such that $\partial v = (g, f)$. Thus, the equation $\partial s_0 f = (f, f)$ witnesses the fact that $\simeq$ is reflexive.

Next, to see that $\simeq$ is symmetric, let $\partial v_2 = (g, f)$. Let $v_1 = s_0 f$. Then $d_1 v_1 = d_1 v_2$. From this, we get a $(2, 0)$-horn (v_1, v_2) in Y where v_i acts on the i-th face of $\Lambda^0[2]$ for each $i = 1, 2$. As Y is a Kan complex by hypothesis, this has a filler

$$\Lambda^0[2] \xrightarrow{(v_1, v_2)} Y$$
$$\downarrow \qquad \nearrow \theta$$
$$\Delta[2]$$

By the simplicial identities, we have that

$$\begin{aligned} \partial (d_0 \theta) &= (d_0 d_0 \theta, d_1 d_0 \theta) \\ &= (d_0 d_1 \theta, d_0 d_2 \theta) \\ &= (f, g) \,, \end{aligned}$$

so that there is a homotopy from g to f.

Finally, to see that $\simeq$ is transitive, let $\partial v_2 = (g, f)$ and $\partial v_0 = (j, g)$. This means that $d_1 v_0 = d_0 v_2$, thereby yielding a $(2, 1)$-horn (v_0, v_2) in Y. As Y is a Kan complex, this has a filler:

$$\Lambda^1[2] \xrightarrow{(v_0, v_2)} Y$$
$$\downarrow \qquad \nearrow \theta'$$
$$\Delta[2]$$

We have that

$$\begin{aligned} \partial (d_1 \theta') &= (d_0 d_1 \theta', d_1 d_1 \theta') \\ &= (d_0 d_0 \theta', d_1 d_2 \theta') \\ &= (j, f) \,, \end{aligned}$$

so that f is homotopic to j.

Now, assume that X is arbitrary. Since **sSet** is cartesian closed by Lemma C.0.10, any simplicial map $f : X \times \Delta[0] \cong X \to Y$ naturally corresponds to a map $\tilde{f} : \Delta[0] \to Y^X$. Likewise, any simplicial homotopy $h : X \times \Delta[1] \to Y$ naturally corresponds to a map $\tilde{h} : \Delta[0] \times \Delta[1] \cong \Delta[1] \to Y^X$. Therefore, any homotopy $h : f \xrightarrow{\simeq} g$ naturally corresponds to a homotopy $\tilde{h} : \tilde{f} \xrightarrow{\simeq} \tilde{g}$. In this case, we have shown that $\simeq$ is an equivalence relation, and thus our proof is done. $\qquad\square$

It follows at once that for any $n \in \mathbb{Z}_{\geq 1}$, $\Delta[n]$ is *not* a Kan complex.

For any simplicial set X, the set $\pi_0(X)$ of *connected components* of X is precisely the coequalizer in the diagram

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \dashrightarrow \pi_0(X).$$

Explicitly, $\pi_0(X)$ equals the set of all connected components of the undirected graph

$$(X_0, E_{X_0}), \qquad E_{X_0} \equiv \{\{d_0(x), d_1(x)\} \mid x \in X_1\}.$$

By viewing a 1-simplex in X as a homotopy, we thus have $\pi_0(X)$ as the quotient of the set of all vertices in X by the equivalence relation $\simeq$.

Remark 4.1.20. We can generalize this notion a bit. Let $\mathscr{C}$ be a category enriched over the cartesian monoidal category **sSet**. Then the *category $\pi_0(\mathscr{C})$ of components of $\mathscr{C}$* is given by
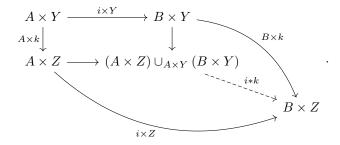
$$\operatorname{Ob}\pi_0(\mathscr{C}) \equiv \operatorname{Ob}\mathscr{C}$$
$$\operatorname{Hom}_{\pi_0(\mathscr{C})}(a,b) \equiv \pi_0(\operatorname{Hom}_{\mathscr{C}}(a,b)).$$

It follows easily from the universal property of coequalizers that any **sSet**-enriched functor $F : \mathscr{C} \to \mathscr{D}$ induces a functor $\pi_0(F) : \pi_0(\mathscr{C}) \to \pi_0(\mathscr{D})$.

Moving on, for any two simplicial maps $i : A \to B$ and $k : Y \to Z$, consider the *pushout product*

$$i * k : (A \times Z) \cup_{A \times Y} (B \times Y) \to B \times Z,$$

i.e., the unique map fitting into a commutative diagram

Notice that $B \times Z \cong Z \times B$ and $(A \times Z) \cup_{A \times Y} (B \times Y) \cong (Y \times B) \cup_{Y \times A} (Z \times A)$. Moreover, it is easy to check that if both i and k are monic (i.e., levelwise injections), then so is $i * k$.

Theorem 4.1.21 (Gabriel-Zisman). *If i and k are monic and i is anodyne, then $i * k$ is anodyne.*[25]

Proof. Let $k : Y \to Z$ be any simplicial map. Let $\mathcal{D}$ denote the class of all monomorphisms $i : A \to B$ such that $i * k$ is an anodyne extension. We must show that $\mathbb{A} \subset \mathcal{D}$.

Fact. *Let $\mathcal{C}$ denote the saturated class generated by*

$$\left\{ \iota_e * m : (\{e\} \times Z') \cup (\Delta[1] \times Y') \to \Delta[1] \times Z' \mid m : Y' \xrightarrow{\text{monic}} Z', \ e = 0, 1 \right\}$$

where ι_e denotes the horn inclusion $\{e\} \hookrightarrow \Delta[1]$ for each $e = 0, 1$. Then $\mathbb{A} = \mathcal{C}$.[26]

Therefore, it suffices to show that $\mathcal{C} \subset \mathcal{D}$.

---

[25][13, Theorem 3.2.2].
[26][13, Theorem 3.2.3].

We have that $\mathcal{D}$ is saturated. For example, to see that $\mathcal{D}$ is closed under pushouts, let $i \in \mathcal{D}$ and consider the pushout square

$$
\begin{array}{ccc}
A & \xrightarrow{\ f\ } & C \\
{\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle f_*i} \\
B & \xrightarrow{\ f'\ } & D
\end{array},
\tag{1}
$$

where f_*i is monic since pushouts preserve monomorphisms in **Set**. This square induces another pushout square

$$
\begin{array}{ccc}
A \times Z & \xrightarrow{\ f\times Z\ } & C \times Z \\
{\scriptstyle i\times Z}\downarrow & \ulcorner & \downarrow{\scriptstyle f_*i\times Z} \\
B \times Z & \xrightarrow{\ f'\times Z\ } & D \times Z
\end{array}.
\tag{2}
$$

Let $p : U \to V$ be a Kan fibration. We must show any lifting problem of the form

$$
\begin{array}{ccc}
(C \times Z) \cup (D \times Y) & \longrightarrow & U \\
{\scriptstyle f_*i*k}\downarrow & & \downarrow{\scriptstyle p} \\
D \times Z & \longrightarrow & V
\end{array}
\tag{3}
$$

has a solution. To this end, note that the commutative diagram

$$
\begin{array}{ccccc}
(A \times Z) \cup (B \times Y) & \longrightarrow & (C \times Z) \cup (D \times Y) & \longrightarrow & U \\
{\scriptstyle i*k}\downarrow & & {\scriptstyle f_*i*k}\big\downarrow & & \downarrow{\scriptstyle p} \\
B \times Z & \longrightarrow & D \times Z & \longrightarrow & V
\end{array}
$$

induced by (2) admits a lift $B \times Z \to U$. By the universal property of (2), we obtain a unique mediating map $D \times Z \to U$, which is a solution to (3).

Thus, it suffices to show that for each monomorphism $m : Y' \to Z'$ and each integer $0 \le e \le 1$, the map

$$
\iota_e * m : (\{e\} \times Z') \cup (\Delta[1] \times Y') \to \Delta[1] \times Z'
$$

belongs to $\mathcal{D}$. It is easy to see that the maps

$$
(\iota_e * m) * k : (((\{e\} \times Z') \cup (\Delta[1] \times Y')) \times Z) \cup ((\Delta[1] \times Z') \times Y) \longrightarrow (\Delta[1] \times Z') \times Z
$$
$$
\iota_e * (m * k) : (\{e\} \times Z' \times Z) \cup (\Delta[1] \times (Y' \times Z \cup Z' \times Y)) \longrightarrow \Delta[1] \times (Z' \times Z)
$$

are isomorphic in $\mathrm{Arr}(\mathbf{sSet})$. But $\iota_e * (m * k)$ belongs to $\mathcal{C}$ and thus is anodyne. This implies that $(\iota_e * m) * k$ is also anodyne since $\mathbb{A}$ is closed under retracts. Hence the monomorphism $\iota_e * m$ belongs to $\mathcal{D}$, as desired. $\qquad\square$

Corollary 4.1.22 (Covering homotopy extension property). *Let $p : U \to V$ be a Kan fibration and $k : Y \to Z$ be a monomorphism. Any commutative diagram of the form*

$$
\begin{array}{ccc}
(Y \times \Delta[1]) \cup (Z \times \{e\}) & \longrightarrow & U \\
{\scriptstyle k*\iota_e}\downarrow & & \downarrow{\scriptstyle p} \\
Z \times \Delta[1] & \longrightarrow & V
\end{array}
$$

has a diagonal fill-in for each $e = 0, 1$.

Suppose that L is a simplicial subset of X and that $f \upharpoonright_L = g \upharpoonright_L$. We say that a homotopy $h : f \xRightarrow{\simeq} g$ is a *simplicial homotopy from f to g relative L* if the square

$$
\begin{array}{ccc}
L \times \Delta[1] & \xrightarrow{\pi_1} & L \\
{\scriptstyle \iota \times \mathrm{id}_{\Delta[1]}} \downarrow & & \downarrow {\scriptstyle f\upharpoonright_L = g\upharpoonright_L} \\
X \times \Delta[1] & \xrightarrow{h} & Y
\end{array}
$$

commutes. The relation $\simeq \mathrm{rel}\, L$ is also an equivalence relation when Y is a Kan complex.

Suppose that $p : X \to Y$ is a Kan fibration. We say that p is *minimal* if for any commutative diagram of the form

$$
\begin{array}{ccc}
\partial\Delta[n] \times \Delta[1] & \xrightarrow{\pi_1} & \partial\Delta[n] \\
\downarrow & & \downarrow \\
\Delta[n] \times \Delta[1] & \xrightarrow{h} & X \\
{\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle p} \\
\Delta[n] & \longrightarrow & Y
\end{array}
\qquad , \qquad (*)
$$

the diagram

$$
\Delta[n] \underset{\left(\mathrm{id}_{\Delta[n]}, \delta_1\right)}{\overset{\left(\mathrm{id}_{\Delta[n]}, \delta_0\right)}{\rightrightarrows}} \Delta[n] \times \Delta[1] \xrightarrow{h} X
$$

commutes. The bottom square of $(*)$ exhibits h as a *fiberwise* homotopy. Note that p is minimal precisely when

- h is a homotopy relative boundary,

- $p \circ h_0 = p \circ h_1$ (i.e., h_0 and h_1 are in the same fiber of p), and

- whenever h is a fiberwise homotopy relative $\partial\Delta[n]$, we have that $h_0 = h_1$.

In general, we say that two n-simplices $e, e' : \Delta[n] \to X$ of X are *p-fiberwise homotopic relative boundary*, written as $e \simeq_p e' \, \mathrm{rel}\, \partial\Delta[n]$, if there is a diagram of the form $(*)$ such that $h_0 = e$ and $h_1 = e'$. Then p is a minimal fibration if and only if $e \simeq_p e' \implies e = e'$.

Proposition 4.1.23. $\simeq_p \mathrm{rel}\, \partial\Delta[n]$ *is an equivalence relation.*

Definition 4.1.24 (Deformation retraction). Let $i : A \to B$ be a map of simplicial sets and let $r : B \to A$ be a retraction of i.

1. We say that r is a *deformation retraction of i* if there is a homotopy $h : B \times \Delta[1] \to B$ from id_B to $i \circ r$.

2. We say that r is a *strong deformation retraction of i* if it is a deformation retraction of i and the homotopy h is stationary on A in the sense that

$$A \times \Delta[1] \xrightarrow{i \times \mathrm{id}_{\Delta[1]}} B \times \Delta[1]$$
$$\pi_1 \downarrow \qquad\qquad\qquad \downarrow h$$
$$A \xrightarrow{\quad i \quad} B$$

commutes.

Theorem 4.1.25. *Let $p : X \to Y$ be any Kan fibration. There exists a commutative diagram*

$$Z \xrightarrow{\; j \;} X \xrightarrow{\; g \;} Z$$

$(\bullet)$

such that g is a fiberwise strong deformation retraction of j and p' is a minimal fibration.[27]

Lemma 4.1.26 (Quillen). *The map g in $(\bullet)$ has the right lifting property against the inclusion $i_n : \partial\Delta[n] \hookrightarrow \Delta[n]$ for every $n \in \mathbb{N}$.[28]*

Proof. Since g is a fiberwise strong deformation retraction of j by hypothesis, we have a homotopy $h : X \times \Delta[1] \to X$ fitting into commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{\;g\;} & Z \\ \delta_1 \downarrow & & \downarrow j \\ X \times \Delta[1] & \xrightarrow{\;h\;} & X \\ \delta_0 \uparrow & \nearrow_{\mathrm{id}_X} & \\ X & & \end{array} \qquad \begin{array}{ccc} X \times \Delta[1] & \xrightarrow{\;h\;} & X \\ \pi_1 \downarrow & & \downarrow p \\ X & \xrightarrow{\;p\;} & Y \end{array} .$$

Now, consider any lifting problem

$$\begin{array}{ccc} \partial\Delta[n] & \xrightarrow{\;a\;} & X \\ i_n \downarrow & & \downarrow g \\ \Delta[n] & \xrightarrow{\;b\;} & Z \end{array}$$

between i_n and g with $n \geq 1$. It is easy to check that these two squares along with $(\bullet)$ yield additional commutative diagrams

$$\begin{array}{ccccccc} \partial\Delta[n] & \xrightarrow{\;a\;} & X & \xrightarrow{\;g\;} & Z & \xrightarrow{\;j\;} & X \\ i_n \downarrow & & & & {}^j \nearrow & & \downarrow p \\ \Delta[n] & & \xrightarrow{\;\;\;\;b\;\;\;\;} & & Z & \xrightarrow{\;p'\;} & Y \end{array}$$

$$\begin{array}{ccccccc} \partial\Delta[n] \times \Delta[1] & \xrightarrow{a \times \mathrm{id}_{\Delta[1]}} & X \times \Delta[1] & \xrightarrow{\;h\;} & X \\ i_n \times \mathrm{id}_{\Delta[1]} \downarrow & & & & \downarrow p \\ \Delta[n] \times \Delta[1] & \xrightarrow{\qquad \pi_1 \qquad} & \Delta[n] & \xrightarrow{\;p' \circ b\;} & Y \end{array}$$

$$\Downarrow$$

[27][13, Theorem 3.3.3].
[28][8, Lemma 10.11].

$$\big(\partial\Delta[n]\times\Delta[1]\big)\cup\big(\Delta[n]\times\{0\}\big)\xrightarrow{\big(h\circ(a\times\mathrm{id}_{\Delta[1]}),\,j\circ b\big)}X$$

Since p is a Kan fibration, our last diagram admits a diagonal fill-in $H:\Delta[n]\times\Delta[1]\to X$ by Corollary 4.1.22. Consider the n-simplex v in X given by

$$\Delta[n]\times\Delta[0]\xrightarrow{\delta_0}\Delta[n]\times\Delta[1]\xrightarrow{H}X\ .$$

Then the triangle

$$\partial\Delta[n]\xrightarrow{a}X$$
$$i_n\downarrow\qquad\nearrow v$$
$$\Delta[n]$$

commutes since $h_1=\mathrm{id}_X$. Therefore, the composite $t\coloneqq g\circ h\circ\big(v\times\mathrm{id}_{\Delta[1]}\big):\Delta[n]\times\Delta[1]\to Z$ is a homotopy $\underbrace{g\circ j\circ g\circ v}_{g\circ v}\xrightarrow{\simeq}g\circ v$ such that $t\restriction_{\partial\Delta[n]\times\Delta[1]}=g\circ h\circ\big(a\times\mathrm{id}_{\Delta[1]}\big)$. Note that $g\circ H$ and t together determine a map $\langle g\circ H,t\rangle:\Delta[n]\times\Lambda^0[2]\to Z$, with

$$\langle g\circ H,t\rangle\restriction_{\Delta[n]\times\partial^1\Delta[2]}=g\circ H$$
$$\langle g\circ H,t\rangle\restriction_{\Delta[n]\times\partial^2\Delta[2]}=t.$$

We now have a commutative diagram

$$\big(\partial\Delta[n]\times\Delta[2]\big)\cup\big(\Delta[n]\times\Lambda^0[2]\big)\xrightarrow{(g\circ h\circ(a\times\sigma_1),\,\langle g\circ H,t\rangle)}Z$$

which admits a diagonal fill-in $H':\Delta[n]\times\Delta[2]\to Z$ thanks to Corollary 4.1.22. This yields another commutative diagram

$$\partial\Delta[n]\times\Delta[1]\xrightarrow{\pi_1}\partial\Delta[n]$$
$$\Delta[n]\times\Delta[1]\xrightarrow{H'\circ\big(\mathrm{id}_{\Delta[n]}\times\delta_0\big)}Z$$
$$\pi_1\downarrow\qquad\qquad\downarrow p'$$
$$\Delta[n]\xrightarrow{p'\circ b}Y$$

This shows that $g\circ v\simeq_{p'}b$. As p' is minimal, it follows that $g\circ v=b$, so that v is a solution to our lifting problem. This completes our proof. $\qquad\square$

Weak homotopy equivalences

Recall from Definition 4.1.10 the geometric realization functor $|{-}|:\mathbf{sSet}\to k\mathbf{Top}$. We say that a simplicial map $f:X\to Y$ is a *weak homotopy equivalence* if $|f|:|X|\to|Y|$ is a homotopy

equivalence of topological spaces.

Remark 4.1.27. By the Whitehead theorem, f is a weak homotopy equivalence in **sSet** if and only if $|f|$ is a weak homotopy equivalence in $k\mathbf{Top}$.

Definition 4.1.28. A simplicial set X is *contractible* if the unique map $X \to 1$ is a weak homotopy equivalence.

Example 4.1.29. Any horn $\Lambda^k[n]$ is contractible.

Suppose that $p : Y \to X$ is a map of simplicial sets. Let F be a simplicial set such that for each vertex $x_0 \in X_0$, the fiber

$$
\begin{array}{ccc}
Y \times_X \Delta[0] & \longrightarrow & \Delta[0] \\
\downarrow & & \downarrow{\scriptstyle x_0} \\
Y & \xrightarrow{\;\;p\;\;} & X
\end{array}
$$

of p over x_0 is isomorphic to F. In this case, we say that p is a *fiber bundle with standard fiber F*.

Theorem 4.1.30. *If p is a minimal fibration with two pullback squares*

$$
\begin{array}{ccc}
f_1^*Y & \longrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle p} \\
A & \xrightarrow{\;\;f_1\;\;} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
f_2^*Y & \longrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle p} \\
A & \xrightarrow{\;\;f_2\;\;} & X
\end{array}
$$

and a homotopy $f_0 \xrightarrow{\simeq} f_1$, then there is a commutative triangle of the form

$$
\begin{array}{ccc}
f_1^*Y & \xrightarrow{\;\cong\;} & f_2^*Y \\
& \searrow & \downarrow \\
& & A
\end{array}
\;.
$$

Proof reference. See [8, Corollary 10.7]. $\qquad\square$

Corollary 4.1.31. *If p is a minimal fibration and X is connected (i.e., $\pi_0(X) = 1$), then it is a fiber bundle.*

Proof. Suppose that p is a minimal fibration and that X is connected. Consider any two vertices $v_1, v_2 : \Delta[0] \to X$ in X. Since X is connected, there is a 1-simplex z in X whose boundary ∂z equals (v_2, v_1). Then z is precisely a homotopy $v_1 \xrightarrow{\simeq} v_2$. By Theorem 4.1.30, the fiber of p over v_1 is thus isomorphic to that over v_2. $\qquad\square$

Theorem 4.1.32. *If p is a minimal fibration and X is contractible, then p is trivializable, i.e., isomorphic to the trivial bundle*

$$
\begin{array}{c}
F \times X \\
\downarrow{\scriptstyle \pi_2} \\
X
\end{array}
$$

over X with fiber F.[29]

---

[29][4, Corollary III.5.6].

Lemma 4.1.33. *If p is merely a Kan fibration, then the trivial bundle $F \times X \xrightarrow{\pi_2} X$ is a Kan fibration.*

Proof. Note that the unique map $F \to \Delta[0]$ is precisely the pullback

$$
\begin{array}{ccc}
F & \longrightarrow & Y \\
\downarrow & \ulcorner & \downarrow{\scriptstyle p} \\
\Delta[0] & \xrightarrow[x_0]{} & X
\end{array}
\; .
$$

Thus, F is a Kan complex by Lemma 4.2.6 below. Moreover, π_2 is precisely the pullback

$$
\begin{array}{ccc}
F \times X & \xrightarrow{\pi_1} & F \\
{\scriptstyle \pi_2}\downarrow & \ulcorner & \downarrow \\
X & \longrightarrow & \Delta[0]
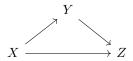\end{array}
\; .
$$

Hence π_2 is a Kan fibration again by Lemma 4.2.6. $\qquad\square$

4.2 Quillen model categories

This section begins to develop the theory of model categories, a generalized setting for homotopy theory (whether simplicial or topological) originally developed by Quillen. This will provide the background for Section 4.3, which examines the classical model structure on **sSet**. This, in turn, will play a key role in Section 5.

Our treatment of model categories is mainly based on [9] and [12].

Definition 4.2.1. We say that a category $\mathscr{C}$ is a *category with weak equivalences* if it is equipped with a subclass W of $\mathrm{Mor}(\mathscr{C})$ consisting of *weak equivalences* such that every isomorphism in $\mathscr{C}$ belongs to W and W satisfies *two-out-of-three*, i.e., for any commutative triangle

$$
\begin{array}{ccc}
 & Y & \\
\nearrow & & \searrow \\
X & \longrightarrow & Z
\end{array}
$$

in $\mathscr{C}$, if two of these three morphisms are in W, then so is the third.

Definition 4.2.2 (Model category). Let $\mathscr{C}$ be a category with all small limits and colimits.

1. A *weak factorization system (WFS) on $\mathscr{C}$* is a pair $(\mathcal{L}, \mathcal{R})$ of subclasses of $\mathrm{Mor}(\mathscr{C})$ such that

    (i) any morphism in $\mathscr{C}$ factors as a morphism in $\mathcal{L}$ followed by a morphism in $\mathcal{R}$,

    (ii) $\mathcal{L}$ consists of all $\mathcal{R}$-*projective* morphisms in $\mathscr{C}$, i.e., those with the left lifting property against every morphism in $\mathcal{R}$, and

    (iii) $\mathcal{R}$ consists of all $\mathcal{L}$-*injective* morphisms in $\mathscr{C}$, i.e., those with the right lifting property against every morphism in $\mathcal{L}$.

2. A category $(\mathscr{C}, W)$ with weak equivalences is a *model category* if $\mathscr{C}$ is equipped with two additional subclasses Fib and Cof of $\mathrm{Mor}(\mathscr{C})$ consisting of *fibrations* and *cofibrations*, respectively,

such that the pairs

$$
\begin{pmatrix}
\overbrace{W \cap Cof}^{trivial\ cofibrations} & , Fib
\end{pmatrix}
$$

$$
\begin{pmatrix}
Cof, & \underbrace{W \cap Fib}_{trivial\ fibrations}
\end{pmatrix}
$$

are both weak factorization systems on $\mathscr{C}$.

In this case, the triple (Fib, Cof, W) is called a *model structure on $\mathscr{C}$*.

Remark 4.2.3. For any WFS $(\mathcal{L}, \mathcal{R})$ on $\mathscr{C}$, both $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms in $\mathscr{C}$. Hence Fib, Cof, and W contain all isomorphisms in $\mathscr{C}$. Thus, the condition of Definition 4.2.1 that W must contain all isomorphisms in $\mathscr{C}$ is superfluous for Definition 4.2.2.

Terminology. Let $\mathscr{C}$ be a model category and let $X \in \mathrm{Ob}\,\mathscr{C}$.

1. We call X *cofibrant* if the unique map $0 \to X$ is a cofibration.

2. We call X *fibrant* if the unique map $X \to 1$ is a fibration.

Definition 4.2.4 (Properness). Let $(\mathscr{C}, Fib, Cof, W)$ be a model category.

1. We say that $\mathscr{C}$ is *right proper* if for any pullback square

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
{\scriptstyle w'}\downarrow & \lrcorner & \downarrow{\scriptstyle w} \\
Y' & \underset{f}{\longrightarrow} & Y
\end{array}
$$

   where $f \in Fib$ and $w \in W$, we have $w' \in W$.

2. We say that $\mathscr{C}$ is *left proper* if for any pushout square

$$
\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & X' \\
{\scriptstyle w}\downarrow & \ulcorner & \downarrow{\scriptstyle w'} \\
Y & \longrightarrow & Y'
\end{array}
$$

   where $f \in Cof$ and $w \in W$, we have $w' \in W$.

We say that $\mathscr{C}$ is *proper* if it is both right proper and left proper.

Lemma 4.2.5. *Suppose that both $\mathscr{C}$ and $\mathscr{D}$ are model categories. Let $(F : \mathscr{C} \to \mathscr{D}, G : \mathscr{D} \to \mathscr{C})$ be an adjoint pair of functors.*

(a) *F preserves cofibrations if and only if G preserves trivial fibrations.*

(b) *Dually, G preserves fibrations if and only if F preserves trivial cofibrations.*

Proof sketch. Suppose that F preserves cofibrations. Let $f : A \to B$ be a cofibration in $\mathscr{C}$ and $g : C \to D$ be a trivial fibration in $\mathscr{D}$. Any lifting problem between f and $G(g)$ induces a lifting problem between $F(f)$ and g by adjunction.

$$
\begin{array}{ccc}
A \longrightarrow G(C) & \Longrightarrow & F(A) \longrightarrow C \\
\Big\downarrow f \qquad \Big\downarrow G(g) & & F(f)\Big\downarrow \qquad \Big\downarrow g \\
B \longrightarrow G(D) & \Longrightarrow & F(B) \longrightarrow D
\end{array}
$$

Since g is a trivial fibration, there is a solution $\hat{g} : F(B) \to C$ to our righthand lifting problem. The conjugate of $\hat{g}$ under our adjunction is a solution to our lefthand lifting problem. This proves that $G(g)$ has the right lifting property against every cofibration in $\mathscr{C}$, i.e., that $G(g)$ is a trivial fibration in $\mathscr{C}$.

A similar argument proves that if G preserves trivial fibrations, then F preserves cofibrations. $\qquad\square$

Lemma 4.2.6. *Let $(\mathcal{L}, \mathcal{R})$ be a WFS on $\mathscr{C}$. Then $\mathcal{L}$ is closed under pushouts. Dually, $\mathcal{R}$ is closed under pullbacks.*

Proof sketch. Consider any pushout square

$$
\begin{array}{ccc}
A & \xrightarrow{\;y\;} & C \\
x\Big\downarrow & & \Big\downarrow y_* x \\
B & \longrightarrow & B \cup_A C
\end{array}
$$

where $x \in \mathcal{L}$. Let

$$
\begin{array}{ccc}
C & \xrightarrow{\;t\;} & X \\
y_* x\Big\downarrow & & \Big\downarrow f \\
B \cup_A C & \longrightarrow & Y
\end{array}
$$

be a commutative diagram with $f \in \mathcal{R}$. We must find a lift $B \cup_A C \to X$. To this end, note that the commutative diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{\;y\;} & C & \xrightarrow{\;t\;} & X \\
x\Big\downarrow & & & & \Big\downarrow f \\
B & \longrightarrow & B \cup_A C & \longrightarrow & Y
\end{array}
$$

admits a lift $s : B \to X$ because $x \in \mathcal{L}$ and $f \in \mathcal{R}$. By the universal property of pushout squares, the pair (s, t) induces a unique mediating map $B \cup_A C \to X$, which is a solution to our lifting problem between $y_* x$ and f. $\qquad\square$

Lemma 4.2.7 (Ken Brown). *Suppose that $(\mathscr{C}, Fib, Cof, W)$ is a model category and $(\mathscr{D}, W)$ is a category with weak equivalences. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor sending any trivial fibration of fibrant objects to a weak equivalence. Then F sends any weak equivalence of fibrant objects to a weak equivalence.*

Proof. Let $f : A \to B$ be a weak equivalence of fibrant objects in $\mathscr{C}$. The pullback square

$$
\begin{array}{ccc}
A \times B & \xrightarrow{\ \pi_2\ } & B \\
{\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow \\
A & \longrightarrow & 1
\end{array}
$$

exhibits the projections π_1 and π_2 as fibrations in $\mathscr{C}$ by Lemma 4.2.6. Now, factor the map $h :=$ $(\mathrm{id}_A, f) : A \to A \times B$ as

$$
A \xrightarrow[p_1]{} C \xrightarrow[p_2]{} A \times B
$$

(with h labeled over the arc from A to $A \times B$)

where $p_1 \in W \cap Cof$ and $p_2 \in Fib$. Both $\pi_1 \circ p_2$ and $\pi_2 \circ p_2$ are weak equivalences by two-out-of-three. They are also fibrations since Fib is closed under composition. For the same reason, C is fibrant. Therefore, both $\pi_1 \circ p_2$ and $\pi_2 \circ p_2$ are trivial fibrations of fibrant objects. By hypothesis, it follows that both $F(\pi_1 \circ p_2)$ and $F(\pi_2 \circ p_2)$ are weak equivalences in $\mathscr{D}$. As

$$
F(\pi_1 \circ p_2 \circ p_1) = F(\mathrm{id}_A) = \mathrm{id}_{F(A)}
$$

is a weak equivalence in $\mathscr{D}$ as well, we have that $F(p_1)$ is a weak equivalence by two-out-of-three. Applying two-out-of-three yet again shows that

$$
F(\pi_2 \circ p_2 \circ p_1) = F(f)
$$

is a weak equivalence, as desired. $\qquad\square$

Let I denote the *interval category* $\{0 \to 1\}$. The functor category $\mathscr{C}^I := [I, \mathscr{C}]$ is isomorphic to the arrow category $\mathrm{Arr}(\mathscr{C})$ of $\mathscr{C}$.

Lemma 4.2.8. *Let $(\mathcal{L}, \mathcal{R})$ be a WFS on $\mathscr{C}$. Then $\mathcal{L}$ is closed under coproducts in $\mathscr{C}^I$. Dually, $\mathcal{R}$ is closed under products in $\mathscr{C}^I$.*

Proof. Let $\{f_s : A_s \to B_s\}_{s \in S}$ be any set of elements of $\mathcal{L}$. As colimits in $\mathscr{C}^I$ are computed pointwise, their coproduct is precisely the map

$$
\coprod_{s \in S} A_s \xrightarrow{\ (f_s)_{s \in S}\ } \coprod_{s \in S} B_s
$$

induced by the universal property of coproducts. Let

$$
\begin{array}{ccc}
\coprod_{s \in S} A_s & \longrightarrow & X \\
{\scriptstyle (f_s)_{s \in S}}\big\downarrow & & \big\downarrow {\scriptstyle p} \\
\coprod_{s \in S} B_s & \longrightarrow & Y
\end{array}
$$

be any lifting problem such that $p \in \mathcal{R}$. By the universal property of coproducts, this naturally corresponds to the set

$$
\left\{
\begin{array}{ccc}
A_s & \longrightarrow & X \\
{\scriptstyle f_s}\big\downarrow & & \big\downarrow {\scriptstyle p} \\
B_s & \longrightarrow & Y
\end{array}
\ \Big|\ s \in S
\right\}
$$

of lifting problems, each of which has a solution $\ell_s : B_s \to X$ by hypothesis. Again, by the universal property of coproducts, the induced map $(\ell_s)_{s \in S}$ is a solution to our original lifting problem. Hence $(f_s)_{s \in S}$ belongs to $\mathcal{L}$. $\qquad\square$

Consider a commutative diagram of the form

$$
\begin{array}{ccccc}
& \xrightarrow{\mathrm{id}_A} & & & \\
A & \longrightarrow & C & \longrightarrow & A \\
f \downarrow & & \downarrow g & & \downarrow f \\
B & \longrightarrow & D & \longrightarrow & B \\
& & \xrightarrow{\mathrm{id}_B} & &
\end{array}
\tag{$\dagger$}
$$

in a category $\mathscr{C}$. In this situation, we say that f is a *retract of g*. This corresponds to a retraction (i.e., left-inverse) in the arrow category $\mathscr{C}^I$.

Notation. Let J be a subclass of $\mathrm{Mor}(\mathscr{C})$.

- $\mathrm{ret}(J)$ will denote the class of all retracts of elements in J.

- $\mathrm{rlp}(J)$ will denote the class of all maps in $\mathscr{C}$ with the right lifting property against every element of J.

- $\mathrm{llp}(J)$ will denote the class of all maps in $\mathscr{C}$ with the left lifting property against every element of J.

Lemma 4.2.9 (Retract argument). *Consider any composite $h \equiv f \circ g$ of maps in $\mathscr{C}$.*

(a) If h has the left lifting property against f, then h is a retract of g.

(b) Dually, if h has the right lifting property against g, then h is a retract of f.

Proof. Suppose that h has the left lifting property against f. We have a lifting problem of the form

$$
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
h \downarrow & & \downarrow f \\
B & = & B,
\end{array}
$$

which has a solution $t : B \to C$ by hypothesis. As a result, we get a commutative diagram

$$
\begin{array}{ccccc}
A & = & A & = & A \\
h \downarrow & & \downarrow g & & \downarrow h \\
B & \xrightarrow{t} & C & \xrightarrow{f} & B \\
& & \xrightarrow{\mathrm{id}_B} & &
\end{array}
$$

which means that h is a retract of g. $\qquad\square$

Lemma 4.2.10. *Let $(\mathcal{L}, \mathcal{R})$ be a WFS on $\mathscr{C}$. Then both $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.*

Proof. For simplicity, let us just show that $\mathcal{L}$ is closed under retracts. To this end, let f and g be as in (†) and suppose that $g \in \mathcal{L}$. We must show that any commutative square of the form

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
f \downarrow & & \downarrow j \\
B & \longrightarrow & Y
\end{array}
$$

where $j \in \mathcal{R}$ has a lift $B \to X$. The commutative diagram

$$
\begin{array}{ccccc}
C & \longrightarrow & A & \longrightarrow & X \\
\downarrow g & & \downarrow f & & \downarrow j \\
D & \longrightarrow & B & \longrightarrow & Y
\end{array}
$$

must have a lift $t : D \to X$ since $g \in \mathcal{L}$ by assumption. Therefore, the composite

$$
A \xrightarrow{f} B \longrightarrow D \xrightarrow{t} X
$$

is a lift for the expanded commutative diagram

$$
\begin{array}{ccccccc}
A & \longrightarrow & C & \longrightarrow & A & \longrightarrow & X \\
f \downarrow & & \downarrow g & & \downarrow f & & \downarrow j \\
B & \longrightarrow & D & \longrightarrow & B & \longrightarrow & Y
\end{array} .
$$

It follows easily that the same composite is a lift for our original square. This means that $f \in \mathcal{L}$, as desired. $\qquad\square$

Corollary 4.2.11. *Let $(\mathscr{C}, Fib, Cof, W)$ be a model category. The classes Fib, Cof, $W \cap Fib$, and $W \cap Cof$ are closed under retracts.*

Corollary 4.2.12. *The class W is closed under retracts.*[30]

Proof. Let f and g be as in (†) and suppose that $g \in W$. We must show that $f \in W$.

Step 1: Assume that f is a fibration. We can factor g as a cofibration $g_1 : C \to D'$ followed by a trivial fibration $g_2 : D' \to D$. As W satisfies two-out-of-three, g_1 is actually a trivial cofibration. Thus, the commutative diagram

$$
\begin{array}{ccc}
C & \longrightarrow & A \\
g_1 \downarrow & & \downarrow f \\
D' \xrightarrow{g_2} D & \longrightarrow & B
\end{array}
$$

---

[30][17, Lemma 2.4].

admits a lift $\ell : D' \to A$, which fits into the extended commutative diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & C & \longrightarrow & A \\
\| & & \downarrow{\scriptstyle g_1} & & \| \\
A & \longrightarrow & D' & \xrightarrow{\ell} & A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle f} \\
B & \longrightarrow & D & \longrightarrow & B
\end{array}
\;\cdot
$$

This implies that the composite $A \longrightarrow D' \xrightarrow{\ell} A$ equals the identity map id_A, so that f is a retract of g_2. By Corollary 4.2.11, f is a trivial fibration, hence a weak equivalence.

Step 2: Let f be arbitrary. We can factor f as a trivial cofibration $f_1 : A \to B'$ followed by a fibration $f_2 : B' \to B$. By the universal property of pushout squares, we obtain a unique map d such that the diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & C & \longrightarrow & A \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle f_1} \\
B' & \xrightarrow{c_2} & E & {\scriptstyle g} & B' \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle f_2} \\
B & \longrightarrow & D & \longrightarrow & B
\end{array}
\tag{1}
$$

commutes. Likewise, there is a unique map $y : E \to B'$ such that the diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & C & \longrightarrow & A \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle c_1} & & \downarrow{\scriptstyle f_1} \\
B' & \xrightarrow{c_2} & E & \xdashrightarrow{y} & B'
\end{array}
\tag{2}
$$

$$\underset{\mathrm{id}_{B'}}{\curvearrowright}$$

commutes. But $f_2 \circ y \circ c_1$ equals the composite $C \longrightarrow A \xrightarrow{f_1} B' \xrightarrow{f_2} B$, and $f_2 \circ y \circ c_2$ equals the composite $B' \xrightarrow{f_2} B \longrightarrow D \longrightarrow B$. Hence $f_2 \circ y$ is a mediating map for our pullback square. Similarly, one can check that the composite $E \xrightarrow{d} D \longrightarrow B$ is a mediating map. But such a map is unique by the universal property of pushout squares. Therefore, y fits into (1), so that f_1 is a retract of d.

Note that c_1 is a weak equivalence thanks to Lemma 4.2.6. Hence d is one as well by the two-out-of-three property. By Step 1, we see that f_2 is also a weak equivalence. Thus, the composite $f = f_2 \circ f_1$ is a weak equivalence by the two-out-of-three property.

$$\square$$

It will be useful to extend our ordinary notion of composition to the transfinite case. Let $\mathscr{C}$ be a cocomplete, locally small category. Let $(\alpha, \in)$ be any ordinal viewed as an order category and let J be any subclass of $\mathrm{Mor}(\mathscr{C})$.

Definition 4.2.13. An α-*sequence of maps in* J is an α-shaped diagram $F : \alpha \to \mathscr{C}$ such that
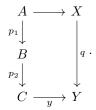
(i) F sends the successor morphism $\beta \xrightarrow{\in} \beta + 1$ to a map in J for each $\beta + 1 \in \alpha$ and

(ii) for any limit ordinal $\gamma \in \alpha$, F_γ together with the family of induced maps $\{F_\beta \to F_\gamma \mid \beta \in \gamma\}$ is the colimiting cocone under $F_\bullet$ restricted to the full-subdiagram $\{\beta \mid \beta \in \gamma\}$; in short, we have an isomorphism
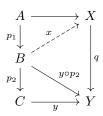
$$\operatorname*{colim}_{\beta \in \gamma} F_\beta \xrightarrow{\cong} F_\gamma.$$

The *(transfinite) composition* of such a sequence is the induced map $F_0 \to F_\alpha := \operatorname{colim}_{\beta \in \alpha} F_\beta$.

Note 4.2.14. Let $(\mathcal{L}, \mathcal{R})$ be a WFS on $\mathscr{C}$. Suppose that $p_1, p_2 \in \mathcal{L}$ and $q \in \mathcal{R}$. Consider the lifting problem

We have a lift

because $p_1 \in \mathcal{L}$. Since $p_2 \in \mathcal{L}$, this yields a solution $C \to X$ to the lifting problem (x, y). This is a solution to our original lifting problem. This proves that $\mathcal{L}$ (dually, $\mathcal{R}$) is closed under finite composition. By the universal property of colimits, it follows easily that $\mathcal{L}$ is closed under transfinite compositions as well.

Remark 4.2.15. Note 4.2.14, Lemma 4.2.10, and Lemma 4.2.6 together show that for any class M of maps in $\mathscr{C}$, the class of all maps in $\mathscr{C}$ with the left lifting property against M is saturated.

Proposition 4.2.16. *Consider a set $\{g_s\}_{s \in S}$ of objects in the arrow category $\operatorname{Arr}(\mathscr{C})$. The coproduct $\coprod_{s \in S} g_s$ arises as a transfinite composite of pushouts of the g_s.[31]*

Definition 4.2.17. The class $\operatorname{cell}(J)$ of *relative J-cell complexes* consists of all maps in $\mathscr{C}$ arising as transfinite composites of pushouts of elements of J.

Example 4.2.18. Let $\mathscr{C} = \mathbf{Top}$, which is cocomplete because it has all coproducts and coequalizers, and let J consist of all Hurewicz cofibrations, including all inclusions of subcomplexes into CW-complexes. Let $X_{-1} = \emptyset$ and consider any CW-complex $Y := \bigcup_{n \geq -1} X_n$ with attaching maps $\{\varphi_\alpha : S^{n-1} \to X_{n-1} \mid \alpha \in A_n\}$. Then we have a pushout square

[31] [9, Proposition 10.2.7].

for each $n \geq 0$, and Y is precisely the colimit of the induced diagram

$$X_{-1} \hookrightarrow X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots .$$

This means that the map $\emptyset \hookrightarrow Y$ is a relative J-complex. In this case, we call Y simply a *J-cell complex*, thereby recovering our usual notion of a cell complex.

Suppose that κ is a cardinal and $F : \kappa \to \mathscr{C}$ is a functor. The colimit of such a functor is called a *κ-sequential colimit*. Further, if F satisfies condition (i) of Definition 4.2.13, then its colimit is called a *κ-sequential colimit relative to J*.

More generally, if κ is regular and A is a poset such that any subset $B \subset A$ with cardinality $< \kappa$ has an upper bound in A, then we call A a *κ-directed set* and the colimit of any A-shaped diagram D a *κ-directed colimit*. In this case, if every morphism in D belongs to J, then we call such a colimit a *κ-directed colimit relative to J*.

Definition 4.2.19.

1. An object X in $\mathscr{C}$ is *κ-compact relative to J* if for any regular cardinal $\lambda \geq \kappa$, the covariant functor $\mathrm{Hom}_{\mathscr{C}}(X, -)$ preserves λ-directed colimits relative to J, specifically, the set map

$$\operatorname*{colim}_{\beta \in \lambda} \mathrm{Hom}_{\mathscr{C}}(X, F_\beta) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{C}}\left(X, \operatorname*{colim}_{\beta \in \lambda} F_\beta\right)$$

$$[f_\beta : X \to F_\beta] \mapsto \left(X \xrightarrow{f_\beta} F_\beta \to \operatorname*{colim}_{\beta \in \lambda} F_\beta\right)$$

   is an isomorphism.

   We say that X is *small relative to J* if it is κ-compact relative to J for some cardinal κ.

   We say that X is *small* if it is small relative to $\mathrm{Mor}(\mathscr{C})$.

2. We say that $\mathscr{C}$ is *locally presentable* if there is a regular cardinal λ along with a set S of λ-compact objects of $\mathscr{C}$ such that every object in $\mathscr{C}$ arises as a λ-directed colimit of a diagram with objects in S.

Note that if X is κ-compact relative to J, then any map $X \to \mathrm{colim}_{\beta \in \lambda} F_\beta$ factors through one of the maps i_β

$$
\begin{array}{ccc}
X & \longrightarrow & \mathrm{colim}_{\beta \in \lambda} F_\beta \\
\vdots & \nearrow & \\
\downarrow & \hspace{-1em} i_\beta & \\
F_\beta & &
\end{array}
$$

of the colimiting cocone so that any other such map $X \to F_{\beta'}$ with $\beta < \beta'$ is precisely the composite $X \to F_\beta \to F_{\beta'}$.

Lemma 4.2.20. *Let κ be any infinite regular cardinal. Then any finite simplicial set (Definition 4.1.4) is κ-compact.*[32]

Proof. Let K be a finite simplicial set and let $D : \kappa \to \mathbf{sSet}$ be a κ-sequence. We must show that the canonical set map

$$\operatorname*{colim}_{\beta \in \kappa} \mathrm{Hom}_{\mathbf{sSet}}(K, D_\beta) \to \mathrm{Hom}_{\mathbf{sSet}}\left(K, \operatorname*{colim}_{\beta \in \kappa} D_\beta\right)$$

---

[32][12, Lemma 3.1.2].

is bijective.

To see that it is injective, suppose that we have equal simplicial maps

$$K \xrightarrow{f_1} D_\beta \longrightarrow \operatorname*{colim}_{\beta \in \kappa} D_\beta$$

$$K \xrightarrow{f_2} D_\beta \longrightarrow \operatorname*{colim}_{\beta \in \kappa} D_\beta.$$

Since K is finite, there is some ordinal $\beta < \alpha < \kappa$ such that

$$D(\beta \to \alpha) \circ f_1 \restriction_{\Delta K_{\mathrm{nondeg}}} = D(\beta \to \alpha) \circ f_2 \restriction_{\Delta K_{\mathrm{nondeg}}}$$

where $\Delta K_{\mathrm{nondeg}}$ denotes the category of non-degenerate simplices $\Delta[n] \to K$ in K. By Lemma 4.1.3, it follows easily that

$$D(\beta \to \alpha) \circ f_1 \restriction_{\Delta K_{\mathrm{nondeg}}} = D(\beta \to \alpha) \circ f_2 \restriction_{\Delta K_{\mathrm{nondeg}}},$$

so that $[f_1] = [f_2]$ in $\operatorname*{colim}_{\beta \in \kappa} \operatorname{Hom}_{\mathbf{sSet}}(K, D_\beta)$, as desired.

To see that our canonical function is surjective, consider any simplicial map $g : K \to \operatorname*{colim}_{\beta \in \kappa} D_\beta$. For every integer $n \geq 0$ and every non-degenerate n-simplex x in K, there exists an ordinal $\alpha_x \in \kappa$ along with an n-simplex y_x in D_{α_x} such that $g_n(x) = (i_{\alpha_x})_n (y_x)$. Since K is finite, we may take

$$\mu := \max\{\alpha_z \mid z \text{ is a non-degenerate simplex in } K\}$$

to form a levelwise *set* map $h : \Delta K_{\mathrm{nondeg}} \to D_\mu$ such that $i_\mu \circ h = g \restriction_{\Delta K_{\mathrm{nondeg}}}$. Using Lemma 4.1.3, we can extend h to a levelwise set map $h' : K \to D_\mu$ that both commutes with all degeneracy operators and satisfies $i_\mu \circ h' = g$.

Now, let x be a non-degenerate n-simplex x in K. For any integer $0 \leq i \leq n + 1$, we have that

$$i_\mu \left(h'_{n-1} d_i x \right) = g_{n-1} d_i x = d_i g_n x = d_i i_\mu h'_n x = i_\mu \left(d_i h'_n x \right)$$

Hence $[d_i h'_n x] = \left[h'_{n-1} d_i x \right]$ in the quotient set $\operatorname*{colim}_{\beta \in \kappa} (D_\beta)_n$. This means that

$$D(\mu \to \alpha(x,i))_n (d_i h'_n x) = D(\mu \to \alpha(x,i))_n (h'_{n-1} d_i x)$$

for some ordinal $\mu < \alpha(x,i) < \kappa$. As K is finite, we can take

$$\mu' := \max \left\{ \alpha(x,i) \mid x \text{ is a non-degnerate simplex in } K, \ 0 \leq i \leq n+1 \right\}$$

to form a levelwise set map $h'' : K \to X_{\mu'}$ such that

(a) $i_{\mu'} \circ h'' = g$,

(b) h'' commutes with all degeneracy operators, and

(c) $h'' \restriction_{\Delta K_{\mathrm{nondeg}}}$ commutes with all face operators.

By applying Lemma 4.1.3 to (c), we see that h'' commutes with all face operators as well. Thanks to the simplicial identities, this means that h'' is a simplicial map. By condition (a), this completes our proof.

$\square$

Lemma 4.2.21. *Let κ be a regular cardinal and A be a κ-directed set. Let $\mathscr{D}$ be a small category. Let $\mathscr{B}$ be a cocomplete, locally small category and $\mathscr{E}$ be a finite category.*

(a) Any representable functor $\mathrm{Hom}_{\mathscr{D}}(-, C) : \mathscr{D}^{\mathrm{op}} \to \mathbf{Set}$ is κ-compact.

(b) Let $F : \mathscr{E} \to \mathscr{B}$ be a diagram where $F(e)$ is a κ-compact object for each $e \in \mathrm{Ob}\,\mathscr{E}$. Then $\mathrm{colim}_e F_e$ is also a κ-compact object.

Proof.

(a) Let $D : A \to [\mathscr{D}^{\mathrm{op}}, \mathbf{Set}]$ be any diagram. Using the Yoneda lemma twice, we obtain a chain of natural isomorphsisms

$$\mathrm{Hom}_{\widehat{\mathscr{D}}}\left(\mathrm{Hom}_{\mathscr{D}}(-, X), \mathrm{colim}_a D_a\right) \cong \left(\mathrm{colim}_a D_a\right)(X)$$
$$\cong \mathrm{colim}_a D_a(X)$$
$$\cong \mathrm{colim}_a \mathrm{Hom}_{\widehat{\mathscr{D}}}\left(\mathrm{Hom}_{\mathscr{D}}(-, X), D_a\right).$$

(b) Let $D' : A \to \mathscr{B}$ be any diagram. It is well-known that κ-directed colimits commute with finite limits in $\mathbf{Set}$. Since the bifunctor $\mathrm{Hom}_{\mathscr{B}}(-, -)$ is continuous in each variable, it follows that

$$\mathrm{Hom}_{\mathscr{B}}\left(\mathrm{colim}_e F_e, \mathrm{colim}_a D'_a\right) \cong \lim_e \mathrm{Hom}_{\mathscr{B}}\left(F_e, \mathrm{colim}_a D'_a\right)$$
$$\cong \lim_e \mathrm{colim}_a \mathrm{Hom}_{\mathscr{B}}(F_e, D'_a)$$
$$\cong \mathrm{colim}_a \lim_e \mathrm{Hom}_{\mathscr{B}}(F_e, D'_a)$$
$$\cong \mathrm{colim}_a \mathrm{Hom}_{\mathscr{B}}\left(\mathrm{colim}_e F_e, D'_a\right).$$

$\square$

Corollary 4.2.22. *If $\mathscr{D}$ is a small category, then the presheaf category $[\mathscr{D}^{\mathrm{op}}, \mathbf{Set}]$ is locally presentable.*

Proof sketch. Thanks to Theorem C.0.8, it suffices to show that any small colimit of representable presheaves $\mathscr{D}^{\mathrm{op}} \to \mathbf{Set}$ arises as an $\aleph_0$-directed colimit of a diagram with objects in a set S of $\aleph_0$-compact objects of $[\mathscr{D}^{\mathrm{op}}, \mathbf{Set}]$. To this end, take S to be the full subcategory $\widehat{\mathscr{D}}_{\mathrm{fp}}$ of $[\mathscr{D}^{\mathrm{op}}, \mathbf{Set}]$ on the class of all $\aleph_0$-compact objects.

Let $F : \mathscr{I} \to [\mathscr{D}^{\mathrm{op}}, \mathbf{Set}]$ be any small diagram of representable presheaves. Consider the poset $(\mathcal{F}, \subset)$ of all finite full subcategories of $\mathscr{I}$. Note that $\mathcal{F}$ is an $\aleph_0$-directed set. It can be shown that

$$\mathrm{colim} F = \mathrm{colim}_{T \in \mathcal{F}} \mathrm{colim}_{t \in T} F_t.$$

By Lemma 4.2.21, $\mathrm{colim}_t F_t$ belongs to $\widehat{\mathscr{D}}_{\mathrm{fp}}$ for every $T \in \mathcal{F}$.

It remains to show that $\widehat{\mathscr{D}}_{\mathrm{fp}}$ is small. Since $\mathscr{D}$ is small by hypothesis, there are only small many representable presheaves $\mathscr{D}^{\mathrm{op}} \to \mathbf{Set}$. Therefore, there are only small many finite colimits of representable presheaves. Let $X : \mathscr{D}^{\mathrm{op}} \to \mathbf{Set}$ be any $\aleph_0$-compact presheaf. It can be shown that X

arises as an $\aleph_0$-directed colimit of finite colimits of representable presheaves: $X \xrightarrow{\cong} \text{colim}_{a \in A} G_a$. As X is $\aleph_0$-compact, this yields a commutative triangle

$$X \xrightarrow{\cong} \text{colim}_a G_a$$
$$\downarrow \qquad \nearrow$$
$$G_a$$

for some $a \in A$. Here, the map $X \to G_a$ must be monic, so that X is a *subpresheaf* of G_a. This shows that all $\aleph_0$-compact presheaves are subpresheaves of finite colimits of representable presheaves. This completes our proof. $\qquad\square$

Lemma 4.2.23 (Quillen's small object argument). *Suppose that J permits the small object argument, i.e., for any $f \in J$, the object $\text{dom}(f)$ is small relative to $\text{cell}(J)$. Then any map in $\mathscr{C}$ can be factored as a map in $\text{cell}(J)$ followed by a map in $\text{rlp}(J)$.*[33]

Proof. Let $z : X_0 \to Y$ be a map in $\mathscr{C}$. Consider the set

$$L_0 := \left\{ \begin{matrix} A_f \longrightarrow X_0 \\ f\downarrow \qquad \quad \downarrow z \\ B_f \longrightarrow Y \end{matrix} \mid f \in J \right\}$$

of lifting problems in $\mathscr{C}$. The universal property of coproducts yields a map $\coprod_{f \in J} f \to z$ in the arrow category $\text{Arr}(\mathscr{C})$. In particular, we have universal morphisms $\coprod_{f \in J} A_f \to X_0$ and $\coprod_{f \in J} B_f \to Y$. Form the pushout square

$$\begin{matrix} \coprod_{f \in J} A_f \longrightarrow X_0 \\ (f)_{f \in J}\downarrow \qquad \qquad \downarrow y_1 \\ \coprod_{f \in J} B_f \longrightarrow X_1 \end{matrix} \qquad (A)$$

together with the unique mediating map

$$\begin{matrix} & X_0 & \\ & \downarrow y_1 \quad \searrow^{p_0 := z} & \\ \coprod_{f \in J} B_f \longrightarrow & X_1 & \\ & \searrow_{p_1} \quad \downarrow & \\ & & Y \end{matrix} \qquad (B)$$

Note that $y_1 \in \text{cell}(J)$ by Proposition 4.2.16. In general, for any successor ordinal $\beta \equiv \alpha + 1$, suppose that we have constructed a triple

$$(y_\beta : X_\alpha \to X_\beta, p_\alpha : X_\alpha \to Y, p_\beta : X_\beta \to Y)$$

of maps fitting into a diagram like (B). Then repeat our construction of (y_1, p_0, p_1) with p_0 replaced by p_β to obtain a new triple $(y_{\beta+1}, p_\beta, p_{\beta+1})$ of maps fitting into (B). Now, choose the least regular

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[33][9, Proposition 10.5.16].

cardinal κ such that for any $f \in J$, $\mathrm{dom}(f)$ is κ-compact relative to cell(J). For any limit ordinal γ, suppose that we have constructed a cocone under a γ-sequence

$$X_0 \xrightarrow{y_1} X_1 \longrightarrow \cdots \longrightarrow X_\alpha \xrightarrow{y_{\alpha+1}} X_{\alpha+1} \longrightarrow \cdots$$

of maps in cell(J). Take the transfinite composition $y : X_0 \to X_\gamma$ of this sequence. By transfinite induction, we now have defined a κ-sequence

$$X_0 \xrightarrow{y_1} X_1 \longrightarrow \cdots \longrightarrow X_\gamma \xrightarrow{y_{\gamma+1}} X_{\gamma+1} \longrightarrow \cdots \tag{C}$$

of maps in cell(J). The universal property of colimits yields a map $p_\gamma : X_\gamma \to Y$ fitting into the cocone

$$X_0 \xrightarrow{y_1} X_1 \longrightarrow \cdots \longrightarrow X_\gamma \xrightarrow{y_{\gamma+1}} X_{\gamma+1} \longrightarrow \cdots \quad .$$

Now, take the transfinite composition $x : X_0 \to X_\kappa$ of (C). We see that the diagram

$$\begin{array}{c} X_0 \\ {\scriptstyle y_1}\downarrow \quad \searrow^{\,x} \\ X_1 \longrightarrow X_\kappa \xrightarrow{p_\kappa} Y \end{array}$$

commutes where p_κ is induced by the universal property of colimits. Therefore, $z = p_1 \circ y_1 = p_\kappa \circ x$.

It remains to show that $p_\kappa \in \mathrm{rlp}(J)$. To this end, let

$$\begin{array}{ccc} U & \longrightarrow & X_\kappa \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle p_\kappa} \\ V & \longrightarrow & Y \end{array}$$

be a commutative square with $h \in J$. We must exhibit a lift $V \to X_\kappa$. Since U is κ-compact, the map $U \to X_\kappa$ factors as $U \to X_\epsilon \to X_\kappa$ for some $\epsilon \in \kappa$. In light of (A), we get a commutative diagram

$$\begin{array}{ccc} U & \longrightarrow X_\epsilon & \longrightarrow X_\kappa \\ & {\scriptstyle y_{\epsilon+1}}\downarrow \quad \nearrow & \downarrow{\scriptstyle p_\kappa} \\ {\scriptstyle h}\; & X_{\epsilon+1} & Y \\ & \nearrow & \\ V & & \end{array}$$

and thus our desired lift. $\qquad\square$

Definition 4.2.24 (Cofibrantly generated). A model category $(\mathscr{C}, Fib, Cof, W)$ is *cofibrantly generated* if it comes equipped with a pair (J, K) of sets of maps in $\mathscr{C}$ such that

   (i) $\mathrm{ret}(\mathrm{cell}(J)) = Cof$,

   (ii) $\mathrm{ret}(\mathrm{cell}(K)) = W \cap Cof$, and

   (iii) both J and K permit the small object argument.

In this case, we call elements of J *generating cofibrations* and elements of K *generating trivial cofibrations*.

Lemma 4.2.25. *Let $(\mathscr{C}, J, K)$ be a cofibrantly generated model category. Then*

   (a) $\mathrm{ret}(\mathrm{cell}(J)) = \mathrm{llp}(\mathrm{rlp}(J))$, *and*

   (b) $\mathrm{ret}(\mathrm{cell}(K)) = \mathrm{llp}(\mathrm{rlp}(K))$.

Proof. For simplicity, let us just prove (a). Since $J \subset \mathrm{llp}(\mathrm{rlp}(J))$, the fact that

$$\mathrm{ret}(\mathrm{cell}(J)) \subset \mathrm{llp}(\mathrm{rlp}(J))$$

follows from Lemma 4.2.10 along with Lemma 4.2.8 and Note 4.2.14.

For the reverse inclusion, let $f \in \mathrm{llp}(\mathrm{rlp}(J))$. By Lemma 4.2.23, we can factor f as

$$A \xrightarrow{\ f_1\ } B \xrightarrow{\ f_2\ } C$$

such that $f_1 \in \mathrm{cell}(J)$ and $f_2 \in \mathrm{rlp}(J)$. Hence f has the right lifting property against f_2. By Lemma 4.2.9, we have that f is a retract of f_1. It follows that $f \in \mathrm{ret}(\mathrm{cell}(J))$. $\square$

Lemma 4.2.25 immediately implies that $Cof = \mathrm{rlp}(K)$ and $W \cap Fib = \mathrm{rlp}(J)$.

Our next notion will serve as a noteworthy generalization of **sSet** equipped with its classical model structure.

Definition 4.2.26. Let $(\mathscr{C}, Fib, Cof, W)$ be a model category

   1. We say that $\mathscr{C}$ is a *combinatorial* model category if it is locally presentable as a category and cofibrantly generated as a model category.

   2. We say that $\mathscr{C}$ is a *type-theoretic* model category if it is locally cartesian closed as a category and proper as a model category and Cof consists of all monomorphisms in $\mathscr{C}$.

4.3   Classical model structure on sSet

The *classical model structure* on **sSet** is due to Quillen and consists of

   - Kan fibrations as fibrations,

   - monomorphisms as cofibrations, and

   - weak homotopy equivalences as weak equivalences.

Notation. We shall write $\mathbf{sSet}_{\texttt{Quillen}}$ for the category **sSet** equipped with this model structure $(Fib_{\texttt{SS}}, Cof_{\texttt{SS}}, W_{\texttt{SS}})$.

Note 4.3.1. Every object of $\mathbf{sSet}_{\mathtt{Quillen}}$ is cofibrant because the empty map $\emptyset \to X$ is trivially levelwise injective for any simplicial set X.

In particular, $\Delta[n]$ is cofibrant. Therefore, for any trivial fibration $p : X \to Y$ of simplicial sets, we can find a diagonal fill-in of the form

$$
\begin{array}{ccc}
\emptyset & \longrightarrow & X \\
\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\
\Delta[n] & \longrightarrow & Y
\end{array} \ .
$$

In terms of the Yoneda lemma, this means that the preimage $p_n^{-1}(y)$ is nonempty for any n-simplex y in Y. In other words, every trivial fibration is levelwise surjective.

In the interest of space, we shall only *partially* verify that $\mathbf{sSet}_{\mathtt{Quillen}}$ is, indeed, a model category as well as cofibrantly generated and proper.

Recall the class $\mathbb{A}$ of anodyne extensions (Definition 4.1.17) and let J denote the saturated class generated by the set
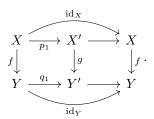
$$
B := \{ \partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0 \}
$$

of canonical inclusions, i.e., $J = \mathrm{ret}(\mathrm{cell}(B))$.

Lemma 4.3.2. *A map in $\mathbf{sSet}_{\mathtt{Quillen}}$ is a cofibration if and only if it belongs to J.*

Proof.

($\Longleftarrow$) It is clear that every element of B is a levelwise injection, i.e., a monomorphism. One can readily check that the class of all levelwise injections is closed under pushouts, transfinite compositions, and retracts. For example, suppose that a simplicial map $g : X' \to Y'$ is a levelwise injection and that $f : X \to Y$ is a retract of g. Then we have a commutative diagram of the form

$$
\begin{array}{ccccc}
 & & \overset{\mathrm{id}_X}{\overbrace{\hspace{3cm}}} & & \\
X & \overset{p_1}{\longrightarrow} & X' & \longrightarrow & X \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\
Y & \underset{q_1}{\longrightarrow} & Y' & \longrightarrow & Y \\
 & & \underset{\mathrm{id}_Y}{\underbrace{\hspace{3cm}}} & &
\end{array} \ .
$$

Recall that a set map is injective if and only if it has a left inverse. Thus, $g \circ p_1 = q_1 \circ f$ is levelwise injective as the composite of two levelwise injective maps. Hence f must be levelwise injective, as desired.

It follows that any element of J belongs to Cof_{SS}.

($\Longrightarrow$) Suppose that $f_0 : X_0 \to Y$ is a monomorphism. By induction, let us construct an ω-sequence $X_\bullet$ of pushouts of coproducts of maps in B along with a sequence $(f_n : X_n \to Y)_{n \geq 0}$ of monomorphisms such that $f_n \upharpoonright_{\mathrm{sk}_{n-1}(X_n)}$ is an isomorphism $\mathrm{sk}_{n-1}(X_n) \overset{\cong}{\longrightarrow} \mathrm{sk}_{n-1}(Y)$ for each $n \geq 1$. Let $n \in \mathbb{Z}_{\geq 1}$ and suppose that we have constructed such an X_n and $f_n : X_n \to Y$. Let S_n denote the category of all n-simplices $\Delta[n] \to Y$ in Y that are *not* in $\mathrm{im}(f_n)$. Note that all objects of S_n are non-degenerate.

Claim. *For each $s \in \mathrm{Ob}\, S_n$, the map $s\restriction_{\partial\Delta[n]}\colon \partial\Delta[n] \to Y$ factors as*

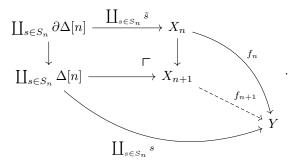$$\partial\Delta[n] \xrightarrow{\ \tilde{s}\ } X_n \xrightarrow{\ f_n\ } Y$$

for some unique map $\tilde{s}$.

Proof. Notice that the non-degenerate k-simplices in $\partial\Delta[n]$ are precisely the non-identity monomorphisms in Δ of the form $[k] \to [n]$. This means that $\partial\Delta[n] \cong \mathrm{sk}_{n-1}(\partial\Delta[n])$. Moreover, $f_n \restriction_{\mathrm{sk}_{n-1}(X_n)}$ is an isomorphism $\mathrm{sk}_{n-1}(X_n) \xrightarrow{\cong} \mathrm{sk}_{n-1}(Y)$ by assumption. This induces an isomorphism $s' \colon \partial\Delta[n] \xrightarrow{\cong} \mathrm{sk}_{n-1}(X_n)$. Now, take $\tilde{s}$ to be the composite

$$\partial\Delta[n] \xrightarrow{\ s'\ } \mathrm{sk}_{n-1}(X_n) \overset{i}{\hookrightarrow} X_n,$$

which must be unique since f_n is monic. $\qquad\square$

This provides us with a commutative diagram

Recall that the Yoneda lemma specifies a natural one-to-one correspondence

$$y \in Y_n \longleftrightarrow \Delta[n] \overset{y}{\to} Y$$

where the map y sends the unique non-degenerate n-simplex in $\Delta[n]$ to the element y. From this, we can see that $\coprod_{s \in S_n} s$ is monic. Since f_n is also monic and $\mathrm{im}(f) \cap \mathrm{im}(h) = \emptyset$, it follows that f_{n+1} is monic. Further, $f_{n+1} \restriction_{\mathrm{sk}_n(X_n)}$ is levelwise surjective by construction. Therefore, $f_{n+1} \restriction_{\mathrm{sk}_n(X_n)}$ is an isomorphism, completing our induction step.

The transfinite composition of $X_\bullet$ is precisely f_0. In light of Proposition 4.2.16, f_0 belongs to $\mathrm{cell}(B) \subset J$, as desired. $\qquad\square$

The following property of the geometric realization functor is a nontrivial consequence of all of Lemma 4.3.2, Theorem 4.1.25, and Lemma 4.1.26.

Theorem 4.3.3 (Quillen). *If $p : E \to X$ is a Kan fibration, then $|p| : |E| \to |X|$ is a (Serre) fibration of topological spaces.*[34]

---

[34][8, Theorem 10.10].

Lemma 4.3.4. *Any map $f : X \to Y$ in* $\mathbf{sSet}_{\texttt{Quillen}}$ *can be factored as*

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle a}\downarrow & \nearrow{\scriptstyle b} & \\
Z & &
\end{array}
$$

where $a \in \mathbb{A}$ and $b \in Fib_{\mathsf{SS}}$.[35]

Proof. The follows directly from Lemma 4.2.23 applied to the class

$$
\left\{ \Lambda^k[n] \hookrightarrow \Delta[n] \mid n \geq 1,\ 0 \leq k \leq n \right\}
$$

together with the fact that $\Lambda^k[n]$ is small by Lemma 4.2.20. $\qquad\square$

Corollary 4.3.5. *Any map in* $\mathbf{sSet}_{\texttt{Quillen}}$ *with the left lifting property against Fib_{SS} is anodyne.*

Proof. Suppose that $f : X \to Y$ has the left lifting property against Fib. By Lemma 4.3.4, we can factor f as an anodyne extension a followed by a Kan fibration b. Then we have a diagonal fill-in of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\ a\ } & Z \\
{\scriptstyle f}\downarrow & \nearrow{\scriptstyle t} & \downarrow{\scriptstyle b} \\
Y & =\!\!=\!\!= & Y
\end{array} \ .
$$

Thus, the diagram

$$
\begin{array}{ccccc}
X & =\!\!=\!\!= & X & =\!\!=\!\!= & X \\
{\scriptstyle f}\downarrow & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{\ t\ } & Z & \xrightarrow{\ b\ } & Y
\end{array}
$$

commutes, which exhibits f as a retract of b. Since $\mathbb{A}$ is saturated, it follows that $f \in \mathbb{A}$. $\qquad\square$

The converse of Corollary 4.3.5 is clear, and thus $\mathbb{A}$ equals the class of all Fib-projective morphisms.

Note 4.3.6. Thanks to Lemma 4.3.2, by applying the same argument for Lemma 4.3.4 to the generating set

$$
\{ \partial\Delta[n] \hookrightarrow \Delta[n] \mid n \geq 0 \}
$$

instead of $\left\{ \Lambda^k[n] \hookrightarrow \Delta[n] \mid n \geq 1,\ 0 \leq k \leq n \right\}$, we have that any map $f : X \to Y$ in $\mathbf{sSet}_{\texttt{Quillen}}$ can be factored as a cofibration followed by a J-injective morphism.

Theorem 4.3.7.

(1) $\underbrace{J\text{–inj}}_{\substack{J\text{-injective} \\ morphisms}} = W_{\mathsf{SS}} \cap Fib_{\mathsf{SS}}.$

(2) $\mathbb{A} = W_{\mathsf{SS}} \cap Cof_{\mathsf{SS}}.$

---

[35][13, Theorem 3.1.1].

Proof.

(1) See [13, Proposition 3.4.1], which is based on a different yet equivalent definition of W_{SS}.

(2) For the inclusion $\mathbb{A} \subset W_{\mathsf{SS}} \cap Cof_{\mathsf{SS}}$, see [12, Proposition 3.2.3]. For the reverse inclusion, suppose that $f : X \to Y$ is a trivial cofibration. Apply Lemma 4.3.4 to factor f as

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
 & a \searrow & \uparrow b \\
 & & Z
\end{array}
$$

where $a \in \mathbb{A}$ and $b \in Fib_{\mathsf{SS}}$. By Lemma 4.3.8 below, we have that b is a weak equivalence because both a and f are weak equivalences. Thanks to part (1), it follows that $b \in J-\mathsf{inj}$. Hence there is a lift of the form

$$
\begin{array}{ccc}
X & \xrightarrow{\ a\ } & Z \\
f \downarrow & \overset{k}{\nearrow} & \downarrow b \\
Y & =\!=\!= & Y
\end{array} \ .
$$

Then the diagram

$$
\begin{array}{ccccc}
X & =\!=\!= & X & =\!=\!= & X \\
f \downarrow & & \downarrow a & & \downarrow f \\
Y & \xrightarrow{\ k\ } & Z & \xrightarrow{\ b\ } & Y
\end{array}
$$
$$
\mathrm{id}_Y
$$

commutes, so that f is a retract of a and thus is anodyne.

$\square$

In light of Theorem 4.3.7, we now can see that $(W_{\mathsf{SS}} \cap Cof_{\mathsf{SS}}, Fib_{\mathsf{SS}})$ is a WFS on **sSet**, as required. Likewise, $(Cof_{\mathsf{SS}}, W_{\mathsf{SS}} \cap Fib_{\mathsf{SS}})$ is a WFS on **sSet**.

Lemma 4.3.8. W_{SS} *satisfies two-out-of-three.*

Proof. Let $W_{k\mathbf{Top}}$ denote the class of all weak homotopy equivalences of k-spaces.

Claim. $W_{k\mathbf{Top}}$ *satisfies two-out-of-three.*

Proof. Consider any commutative triangle

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
 & h \searrow & \downarrow g \\
 & & Z
\end{array}
$$

in $k\mathbf{Top}$. The following two facts are obvious.

- If both f and g belong to $W_{k\mathbf{Top}}$, then so does h.

- If both g and h belong to $W_{k\mathbf{Top}}$, then so does s.

Finally, suppose that both f and h belong to $W_{k\mathbf{Top}}$. In particular, $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection. Therefore, for any $y \in Y$, there is some $x \in X$ along with a path p from y to $f(x)$. Let $n \in \mathbb{Z}_{\geq 1}$. When $n = 1$, the mapping $[\gamma] \mapsto [\bar{p} * \gamma * p]$ defines an isomorphism $\hat{p} : \pi_n(Y, y) \xrightarrow{\cong} \pi_n(Y, f(x))$ where $*$ denotes concatenation. In this case, we likewise have an isomorphism $\hat{\hat{p}} : \pi_n(Z, g(f(x))) \xrightarrow{\cong} \pi_n(Z, g(y))$ given by $[\gamma] \mapsto [(g \circ p) * \gamma * (g \circ \bar{p})]$. If $n > 1$, then we can define $\hat{p}$ by sending any map $s : (I^n, \partial I^n) \to (Y, y)$ to a new map $s_p : (I^n, \partial I^n) \to (Y, f(x))$ given as follows. Shrink the n-cube I^n to $C := \left[\frac{1}{3}, \frac{2}{3}\right]^n$ and draw a radial segment ℓ_t from each point t on ∂I^n to $\left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right) \in C$. Re-parameterize s so that it has domain C and then extend it to a map s_p on I^n that equals p on each segment ℓ_t.

When $n = 1$, it is straightforward to check that the composite

$$\pi_n(Y, y) \xrightarrow{\hat{p}} \pi_n(Y, f(x)) \xrightarrow{\pi_n(g, f(x))} \pi_n(Z, g(f(x))) \xrightarrow{\hat{\hat{p}}} \pi_n(Z, g(y))$$

is precisely $\pi_n(g, y)$. This remain true even if $n > 1$. By functoriality of $\pi_n(-, -)$, the map $\pi_n(g, f(x))$ equals the composite $\pi_n(h, x) \circ \pi_n(f^{-1}, f(x))$, which is an isomorphism since both f and h are weak homotopy equivalences. Therefore, $\pi_n(g, y)$ is an isomorphism.

It is clear that $\pi_0(g)$ is a bijection, and thus $g \in W_{k\mathbf{Top}}$. $\qquad\square$

As $|-| : \mathbf{sSet} \to k\mathbf{Top}$ both preserves and reflects weak homotopy equivalences, it follows that $W_{\mathbf{SS}}$ also satisfies two-out-of-three. $\qquad\square$

We have established that $\mathbf{sSet}_{\mathtt{Quillen}}$ is, in fact, a model category.

Theorem 4.3.9. $\mathbf{sSet}_{\mathtt{Quillen}}$ *is cofibrantly generated.*

Proof. $\mathbf{sSet}_{\mathtt{Quillen}}$ satisfies conditions (i) and (ii) of Definition 4.2.24 thanks to Lemma 4.3.2 and Theorem 4.3.7(2), respectively. It satisfies condition (iii) because every simplicial set is small by an argument similar to that given for Lemma 4.2.20. $\qquad\square$

Corollary 4.3.10. $\mathbf{sSet}_{\mathtt{Quillen}}$ *is a combinatorial model category.*

Proof. This is an immediate consequence of Theorem 4.3.9 together with Corollary 4.2.22. $\qquad\square$

Finally, we want to show that $\mathbf{sSet}_{\mathtt{Quillen}}$ is right proper. For this, the following lemma is useful.

Lemma 4.3.11. *The pullback of a weak homotopy equivalence of k-spaces along a Serre fibration of k-spaces is again a weak homotopy equivalence.*

Proof. Consider any pullback square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle w'}\downarrow & \lrcorner & \downarrow{\scriptstyle w} \\ Y' & \xrightarrow{f} & Y \end{array}$$

in $k\mathbf{Top}$ where f is a Serre fibration and w is weak homotopy equivalence. We must show that w' is also a weak homotopy equivalence.

As a pullback of f, f' is a fibration with fibers vertically isomorphic to those of f. This yields a commutative diagram of the form

$$
\begin{array}{ccccc}
F & \lhook\joinrel\longrightarrow & X' & \xrightarrow{\ f'\ } & X \\
{\scriptstyle\cong}\downarrow & & {\scriptstyle w'}\downarrow & & \downarrow{\scriptstyle w} \\
\widetilde{F} & \lhook\joinrel\longrightarrow & Y' & \xrightarrow[\ f\ ]{} & Y
\end{array}
\qquad (*)
$$

for any two corresponding fibers F and $\widetilde{F}$. By applying the long exact sequence in homotopy for f' and for f to the top and bottom row of $(*)$, respectively, we get a commutative diagram

$$
\begin{array}{ccccc}
\pi_n(F,\cdot) & \longrightarrow & \pi_n(X',\cdot) & \xrightarrow{\pi_n(f',\cdot)} & \pi_n(X,\cdot) \\
{\scriptstyle\cong}\downarrow & & {\scriptstyle \pi_n(w',\cdot)}\downarrow & & \downarrow{\scriptstyle \pi_n(w,\cdot)} \\
\pi_n(\widetilde{F},\cdot) & \longrightarrow & \pi_n(Y',\cdot) & \xrightarrow[\pi_n(f,\cdot)]{} & \pi_n(Y,\cdot)
\end{array}
$$

for every integer $n \geq 0$. But $\pi_n(w,\cdot)$ is an isomorphism because w is a weak homotopy equivalence. The short split five lemma now implies that $\pi_n(w',\cdot)$ is an isomorphism, as required. $\qquad\square$

Theorem 4.3.12. $\mathbf{sSet}_{\mathtt{Quillen}}$ *is right proper.*

Proof. Consider any pullback square

$$
\begin{array}{ccc}
X' & \longrightarrow & X \\
{\scriptstyle w'}\downarrow \ \ \lrcorner & & \downarrow{\scriptstyle w} \\
Y' & \xrightarrow[\ f\ ]{} & Y
\end{array}
$$

in $\mathbf{sSet}$ where $f \in Fib_{\mathtt{SS}}$ and $w \in W_{\mathtt{SS}}$. We must show that $w' \in W_{\mathtt{SS}}$. To this end, let us gather a few properties of $|-| : \mathbf{sSet} \to k\mathbf{Top}$ established thus far.

- $|-|$ preserves pullbacks by Theorem 4.1.13(2).

- $|-|$ sends any Kan fibration to a Serre fibration by Theorem 4.3.3.

- $|-|$ both preserves and reflects all weak homotopy equivalences.

As a result, we have a pullback square

$$
\begin{array}{ccc}
|X'| & \longrightarrow & |X| \\
{\scriptstyle |w'|}\downarrow \ \ \lrcorner & & \downarrow{\scriptstyle |w|} \\
|Y'| & \xrightarrow[\ |f|\ ]{} & |Y|
\end{array}
$$

in $k\mathbf{Top}$ such that $|f|$ is a Serre fibration and $|w|$ is a weak homotopy equivalence. By Lemma 4.3.11, $|w'|$ is also a weak homotopy equivalence. Thus, $w' \in W_{\mathtt{SS}}$. $\qquad\square$

For any model category $(\mathscr{C}, Fib, Cof, W)$ and $x \in \mathrm{Ob}\,\mathscr{C}$, the over category $\mathscr{C}/x$ inherits a model structure from $\mathscr{C}$ where a morphism

$$
\begin{array}{ccc}
y & \xrightarrow{\ \zeta\ } & z \\
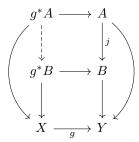& \searrow \quad & \downarrow \\
& & x
\end{array}
$$

in $\mathscr{C}/x$ is

- a fibration if and only if $\zeta \in Fib$,

- a cofibration if and only if $\zeta \in Cof$, and

- a weak equivalence if and only if $\zeta \in W$.

Note that an object of $\mathscr{C}/x$ is fibrant if and only if it belongs to Fib.

Corollary 4.3.13. *Let $g : X \to Y$ be a simplicial map.*

(1) The base change functor $g^* : \mathbf{sSet}_{\mathtt{Quillen}}/Y \to \mathbf{sSet}_{\mathtt{Quillen}}/X$ preserves cofibrations.

(2) The dependent product $\Pi_g : \mathbf{sSet}_{\mathtt{Quillen}}/X \to \mathbf{sSet}_{\mathtt{Quillen}}/Y$ preserves trivial fibrations in $\mathbf{sSet}_{\mathtt{Quillen}}$.

(3) If g is a Kan fibration, then g^* preserves weak equivalences (hence trivial cofibrations).

(4) If g is a Kan fibration, then Π_g preserves Kan fibrations.

Proof.

(1) Let $\begin{smallmatrix} A & \overset{j}{\to} & B \\ & \searrow & \downarrow \\ & & Y \end{smallmatrix}$ be a cofibration in $\mathbf{sSet}_{\mathtt{Quillen}}/Y$, so that j is a monomorphism of simplicial

sets. We have a commutative diagram

$$
\begin{array}{ccc}
g^*A & \longrightarrow & A \\
\vdots & & \downarrow{\scriptstyle j} \\
g^*B & \longrightarrow & B \\
\downarrow & & \downarrow \\
X & \underset{g}{\longrightarrow} & Y
\end{array}
$$

obtained by the universal property of pullback squares. By definition, g^* sends j to this dotted arrow g^*j. Since the upper square must be a pullback and monomorphisms are stable under pullback, it follows that g^*j is a monomorphism. Hence it is a cofibration in $\mathbf{sSet}_{\mathtt{Quillen}}/X$.

Note 4.3.14. The preceding argument with "monomorphism" replaced by "trivial fibration" shows that g^* preserves trivial fibrations as well.

(2) Let $f : A \to X$ be a trivial fibration of simplicial sets. We can view this as a trivial fibration

$$
\begin{array}{ccc}
A & \overset{f}{\longrightarrow} & X \\
& {\scriptstyle f}\searrow & \| \\
& & X
\end{array}
$$

in $\mathbf{sSet}_{\mathtt{Quillen}}/X$. We must show that the *object* $\Pi_g f$ of $\mathbf{sSet}_{\mathtt{Quillen}}/Y$ is a trivial fibration in $\mathbf{sSet}_{\mathtt{Quillen}}$. By part (1) together with Lemma 4.2.5(a), we deduce that the map $\Pi_g f : \Pi_g f \to \Pi_g \mathrm{id}_X$ is a trivial fibration. Since $\Pi_g(-)$ is right adjoint, it preserves the terminal object id_X. Hence the map $\Pi_g f$ coincides with the object $\Pi_g f$, which is thus a trivial fibration in $\mathbf{sSet}_{\mathtt{Quillen}}$.

(3) Suppose that g is a Kan fibration. Let $\begin{smallmatrix} A & \xrightarrow{w} & B \\ & \searrow & \downarrow \\ & & Y \end{smallmatrix}$ be a weak equivalence in $\mathbf{sSet}_{\mathtt{Quillen}}/Y$, so

that w is a weak equivalence of simplicial sets. We again have a pasting

$$
\begin{array}{ccc}
g^*A & \longrightarrow & A \\
\downarrow{\scriptstyle g^*w} & & \downarrow{\scriptstyle w} \\
g^*B & \xrightarrow{\ g'\ } & B \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ g\ } & Y
\end{array}
$$

of pullback squares. Lemma 4.2.6 implies that g' is a Kan fibration, and thus g^*w is a weak equivalence because $\mathbf{sSet}_{\mathtt{Quillen}}$ is right proper.

(4) This follows from a nearly identical argument to (2).

$\square$

Remark 4.3.15. It can be shown that any model category where all objects are cofibrant is left proper. Hence $\mathbf{sSet}_{\mathtt{Quillen}}$ is also left proper and thus proper.

Global model structure on $[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]$

Let $\mathscr{C}$ be a small category. Corollary 4.2.22 and Theorem 4.3.9 together imply that $\mathbf{sSet}_{\mathtt{Quillen}}$ is combinatorial. As it turns out, this ensures that the category $[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]$ of simplicial presheaves over $\mathscr{C}$ inherits a model structure from $\mathbf{sSet}_{\mathtt{Quillen}}$ in at least two ways:

(a) The *projective model structure* $[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathtt{proj}}$ consists of

  - levelwise weak equivalences as weak equivalences,
  - levelwise fibrations as fibrations, and
  - W-projective morphisms as cofibrations.

(b) The *injective model structure* $[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathtt{inj}}$ consists of

  - levelwise weak equivalences as weak equivalences,
  - levelwise cofibrations as cofibrations, and
  - W-injective morphisms as fibrations.

Each of (a) and (b) is called a *global* model structure on $[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]$. It is known that $[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathtt{inj}}$ is both proper and cofibrantly generated, just as $\mathbf{sSet}_{\mathtt{Quillen}}$. Therefore, it is also a combinatorial model category.

4.4 ∞-Categories

This section sets forth a generalization of **sSet** that, in some sense, is the right setting for modeling CDTT, as we shall see in Section 5.2.

Definition 4.4.1. An ∞-*category* is a category enriched over the cartesian monoidal category **sSet**.

Example 4.4.2.

1. **sSet** is an ∞-category. Indeed, since **sSet** is cartesian closed, we have a sequence of natural isomorphisms

$$Z^Y \cong \mathrm{Hom}_{\mathbf{sSet}}\left(1, Z^Y\right)$$
$$\cong \mathrm{Hom}_{\mathbf{sSet}}\left(1 \times Y, Z\right)$$
$$\cong \mathrm{Hom}_{\mathbf{sSet}}\left(Y, Z\right)$$

   for any simplicial sets Y and Z. This shows that every hom-set for **sSet** is itself a simplicial set, as desired.

2. $[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]$ is an ∞-category for any small category $\mathscr{C}$. Indeed, this is also cartesian closed as it is isomorphic to the presheaf category $[(\mathscr{C} \times \Delta)^{\mathrm{op}}, \mathbf{Set}]$. We thus have another sequence of natural isomorphisms

$$Z^Y \cong \mathrm{Hom}_{[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]}\left(1, Z^Y\right)$$
$$\cong \mathrm{Hom}_{[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]}\left(1 \times Y, Z\right)$$
$$\cong \mathrm{Hom}_{[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]}\left(Y, Z\right)$$

   for any simplicial presheaves Y and Z over $\mathscr{C}$. Note that any functor of the form $G : (\mathscr{C} \times \Delta)^{\mathrm{op}} \to \mathbf{Set}$ restricts to the simplicial set given by $[n] \mapsto G(*, [n])$. Hence Z^Y and thus $\mathrm{Hom}_{[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}]}\left(Y, Z\right)$ may be regarded as simplicial sets, as desired.

Next, suppose that $F : K \to \mathscr{C}$ and $w : K \to \mathscr{V}$ are small diagrams enriched over a closed symmetric monoidal category $\mathscr{V}$. The *weighted limit* $\lim^w F$ *of F with weight w* is the object, if it exists, of $\mathscr{C}$ that represents the $\mathscr{V}$-valued presheaf

$$\mathrm{Hom}_{[K, \mathscr{V}]}\left(w, \mathrm{Hom}_{\mathscr{C}}(x, F(-))\right) : \mathscr{C}^{\mathrm{op}} \to \mathscr{V}$$

naturally in $x \in \mathrm{Ob}\,\mathscr{C}$.

Let $\mathscr{V} = \mathbf{sSet}$, so that $\mathscr{C}$ is an ∞-category. Define the weight w by

$$k \mapsto N(K/k)$$

where $N(-)$ denotes the nerve of a small category. In this case, we call $\lim^w F$ the *homotopy limit of F*, denoted by $\mathrm{holim}_K F$.

Definition 4.4.3. Let $\mathscr{C}$ be an ∞-category with all finite homotopy limits. We say that $\mathscr{C}$ is a *locally cartesian closed (LCC) ∞-category* if for every map $f : x \to y$ in $\mathscr{C}$, the homotopy pullback functor $f^* : \mathscr{C}/y \to \mathscr{C}/x$ has a right adjoint Π_f.

We also have a variant of *locally presentable* for ∞-categories, but it requires a lot of machinery to state.

Definition 4.4.4. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor of ∞-categories (i.e., a **sSet**-enriched functor). We say that F is a *Dwyer-Kan equivalence* if

(i) the functor $\pi_0(F) : \pi_0(\mathscr{C}) \to \pi_0(\mathscr{D})$ (Remark 4.1.20) is essentially surjective and

(ii) for all $x, y \in \mathrm{Ob}\,\mathscr{C}$, the map $F_{x,y} : \mathrm{Hom}_{\mathscr{C}}(x, y) \to \mathrm{Hom}_{\mathscr{D}}(F(x), F(y))$ is a weak equivalence in **sSet**$_{\texttt{Quillen}}$.

In this case, we write $\mathscr{C} \simeq_{\mathrm{DK}} \mathscr{D}$.

We now want to define a way of forming an ∞-category out of a given category with weak equivalences $(\mathscr{C}, W)$. To this end, consider a directed graph $G := (V, E)$. The *free category FG of G* has vertices of G as objects and lists of the form

$$(a_n, f_n, a_{n-1}, \ldots, a_1, f_1, a_0)$$
$$n \in \mathbb{N}, \ a_i \in V, \ a_0 \equiv a, \ a_n \equiv b$$
$$f_i \text{ is an edge from } a_{i-1} \text{ to } a_i \text{ for all } 0 < i \leq n$$

as morphisms $a \to b$. Here, composition is given by

$$(a_n, f_n, a_{n-1}, \ldots, a_1, f_1, a_0) \circ (b_m, g_m, a_{m-1}, \ldots, b_1, g_1, b_0)$$
$$\|\|$$
$$(a_n, f_n, a_{n-1}, \ldots, a_1, f_1, a_0 = b_m, g_m, a_{m-1}, \ldots, b_1, g_1, b_0).$$

The identity map id_a is precisely the empty path $\emptyset$ from a to itself.

A *reflexive quiver* is a quiver equipped with an edge i_v from v to itself for each vertex v. Every small category $\mathscr{E}$ may be regarded as a reflexive quiver $(\mathrm{Ob}\,\mathscr{E}, \mathrm{Mor}(\mathscr{E}))$. This provides us with two functors

$$\varepsilon_{\mathscr{E}} : F\mathscr{E} \longrightarrow \mathscr{E}, \qquad v \mapsto v$$
$$\delta_{\mathscr{E}} : F\mathscr{E} \longrightarrow FF\mathscr{E}, \quad v \mapsto v.$$

It can be shown that the induced triple $\mathbb{F}(\mathscr{E}) := (F(-), \epsilon, \delta)$ is a comonad in **Cat**. This induces a simplicial category $\mathbb{F}_\bullet\mathscr{E} : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ defined on objects by

$$\mathbb{F}(\mathscr{E})_n = F^{n+1}\mathscr{E}$$

with face $d_i : \mathbb{F}_k(\mathscr{E}) \to \mathbb{F}_{k-1}(\mathscr{E})$ and degeneracy $s_i : \mathbb{F}_k(\mathscr{E}) \to \mathbb{F}_{k+1}(\mathscr{E})$ operators given by

$$F^{k+1}(\mathscr{E}) \xrightarrow{F^i \epsilon_{F^{k-i}}} F^k(\mathscr{E})$$

,

$$F^{k+1}(\mathscr{E}) \xrightarrow{F^i \delta_{F^{k-i}}} F^{k+2}(\mathscr{E})$$

respectively.

Let $U : \mathbf{Cat} \to \mathbf{Set}$ denote the functor given by $\mathscr{B} \mapsto \mathrm{Ob}(\mathscr{B})$. Then the composite functor $U \circ \mathbb{F}_\bullet \mathscr{E}$ is the constant simplicial set at $\mathrm{Ob}\,\mathscr{E}$.

Moving to our next piece of machinery, suppose that $(\mathscr{E}, W_{\mathscr{E}})$ is a category with weak equivalences. For any arrow f in $\mathscr{E}$, let $\bar{f}$ denote the reverse of f. Let

$$W_{\mathscr{E}}^{\mathrm{op}}(x, y) = \left\{ \bar{f} \mid f : x \to y, \ f \in W_{\mathscr{E}} \right\}.$$

Consider the directed graph $\mathcal{G}$ with objects of $\mathscr{E}$ as vertices, elements of $\mathrm{Hom}_{\mathscr{E}}(x, y)$ as edges $x \to y$, and elements of $W_{\mathscr{E}}^{\mathrm{op}}(x, y)$ as edges $y \to x$. Now, define $\sim$ as the smallest equivalence relation on the set $\mathrm{Mor}(F\mathcal{G})$ such that

- for any $x \in \mathrm{Ob}\,\mathscr{E}$, $(x, \mathrm{id}_x, x) \sim (x, \emptyset, x)$,

- for any maps $f : x \to y$ and $g : y \to z$ in $\mathscr{E}$, $(z, g, y, f, x) \sim (z, g \circ f, x)$, and

- for any map $f : x \to y$ in $W_{\mathscr{E}}$,

$$\left(x, \bar{f}, y, f, x \right) \sim (x, \mathrm{id}_x, x)$$
$$\left(y, f, x, \bar{f}, y \right) \sim (y, \mathrm{id}_y, y).$$

The quotient category $\mathscr{E}\left[W_{\mathscr{E}}^{-1} \right] := {}^{F\mathcal{G}}\!\big/\!_{\sim}$ is called the *localization of $\mathscr{E}$ by $W_{\mathscr{E}}$*.

Definition 4.4.5. If $\mathscr{C}$ is small, then the *(standard) simplicial localization of $\mathscr{C}$* is the simplicial category

$$\mathbb{F}_\bullet \mathscr{C} \left[\mathbb{F}_\bullet W^{-1} \right] : \Delta^{\mathrm{op}} \to \mathbf{Cat},$$

where W may be treated as a subcategory of $\mathscr{C}$.[36]

The composite $U \circ \mathbb{F}_\bullet \mathscr{C} \left[\mathbb{F}_\bullet W^{-1} \right]$ is again a constant simplicial set.

Proposition 4.4.6. *Let $T : \Delta^{\mathrm{op}} \to \mathbf{Cat}$ be a functor so that $U \circ T$ is a constant simplicial set at, say, $\mathcal{S}$. For any $x, y \in \mathcal{S}$ and any $n \in \mathbb{N}$, let*

$$T(x, y)_n = \left\{ \tau \in \mathrm{Mor}(T_n) \mid \tau : x \to y \right\}.$$

(a) The family $\{T(x, y)_n\}_{n \geq 0}$ has the structure of a simplicial set $T(x, y)$.

(b) Levelwise composition of T induces a composition operation $T(x, y) \times T(y, z) \to T(x, z)$.

(c) The category $\langle T \rangle$ with $\mathrm{Ob}\,\langle T \rangle \equiv \mathcal{S}$ and $\mathrm{Hom}_{\langle T \rangle}(x, y) \equiv T(x, y)$ is enriched over $\mathbf{sSet}$.

It follows at once that

$$L_W \mathscr{C} := \left\langle \mathbb{F}_\bullet \mathscr{C} \left[\mathbb{F}_\bullet W^{-1} \right] \right\rangle$$

is an ∞-category.

Definition 4.4.7. We say that an ∞-category $\mathscr{D}$ is *locally presentable* if it has a *presentation* by a combinatorial simplicial model category $\mathscr{A}$ in the sense that $\mathscr{D} \simeq_{\mathrm{DK}} L_W \mathscr{A}$.

---

[36][6, Section 4.1].

Here, by "simplicial model category" we mean a model category $\mathscr{C}$ enriched over **sSet** such that for every fibration $p : X \to Y$ and cofibration $i : A \to B$ in $\mathscr{C}$, the unique mediating map

$$
\begin{array}{ccc}
& \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(i,\operatorname{id}_X)} & \\
\operatorname{Hom}_{\mathscr{C}}(B,X) & & \\
& \operatorname{Hom}_{\mathscr{C}}(B,Y) \times_{\operatorname{Hom}_{\mathscr{C}}(A,Y)} \operatorname{Hom}_{\mathscr{C}}(A,X) \longrightarrow \operatorname{Hom}_{\mathscr{C}}(A,X) & \\
& \downarrow & \downarrow{\scriptstyle \operatorname{Hom}_{\mathscr{C}}(\operatorname{id}_A,p)} \\
\operatorname{Hom}_{\mathscr{C}}(\operatorname{id}_B,p) & \operatorname{Hom}_{\mathscr{C}}(B,Y) \xrightarrow{\operatorname{Hom}_{\mathscr{C}}(i,\operatorname{id}_Y)} \operatorname{Hom}_{\mathscr{C}}(A,Y) &
\end{array}
$$

with dashed map labeled $i^* \times p_*$

is a Kan fibration. Moreover, this map must be a weak equivalence whenever p or i is one.

Finally, let us define a certain kind of locally presentable ∞-category with good structure for modeling not only CDTT but also Univ. (We shall make this feature precise at the end of Section 5.4.) For this, we need a few auxiliary concepts.

Definition 4.4.8. Let $\mathscr{C}$ be an ∞-category with all homotopy pullbacks and all homotopy colimits of shape D. We say that a D-shaped homotopy colimit $\operatorname{hocolim}_{d \in D} F(d)$ in $\mathscr{C}$ is *universal* if for any homotopy pullback square of the form

$$
\begin{array}{ccc}
Y \times_Z (\operatorname{hocolim}_d F(d)) & \longrightarrow & \operatorname{hocolim}_d F(d) \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Z
\end{array}
$$

in $\mathscr{C}$, we have that $\operatorname{hocolim}_d (F(d) \times_Z Y) \cong Y \times_Z (\operatorname{hocolim}_d F(d))$.

Definition 4.4.9. Let $\mathscr{C}$ be an ∞-category. Let J be a class of morphisms in $\mathscr{C}$ closed under homotopy pullbacks. We say that a map $\widehat{J\mathtt{Type}} \to J\mathtt{Type}$ is a *J-classifier* if for any map $X \to B$ in J, there exists a unique homotopy pullback square of the form

$$
\begin{array}{ccc}
X & \longrightarrow & \widehat{J\mathtt{Type}} \\
\downarrow & \lrcorner & \downarrow \\
B & \xrightarrow{\ulcorner X \urcorner} & J\mathtt{Type}
\end{array} .
$$

Let κ be a cardinal. We say that a map $X \to Y$ in $\mathscr{C}$ is *relatively κ-compact* if for any κ-compact object Y' of $\mathscr{C}$ and any homotopy pullback square

$$
\begin{array}{ccc}
Z \times_Y X & \longrightarrow & X \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Y
\end{array}
$$

in $\mathscr{C}$, the object $X \times_Y Z$ is also κ-compact.

Terminology.

- If J denotes the class of all morphisms in $\mathscr{C}$, then a J-classifier is called an *object classifier*.

- If J denotes the class of all monomorphisms in $\mathscr{C}$, then a J-classifier is called a *subobject classifier*.

- If κ is a regular cardinal and J denotes the class of all relatively κ-compact morphisms in $\mathscr{C}$, then a J-classifier is called a *κ-compact-object classifier*.

Example 4.4.10. To gain a bit of intuition about J-classifiers, let $\mathscr{C} = \mathbf{Set}$. Then the function

$$T : 1 \to \underbrace{\{F, T\}}_{1 \coprod 1}$$

picking out the truth value T is a subobject classifier in $\mathscr{C}$. Indeed, for any inclusion function $S \hookrightarrow B$, define $\ulcorner S \urcorner : B \to \{F, T\}$ as the function

$$\chi_S(b) \equiv \begin{cases} T & b \in S \\ F & b \notin S \end{cases}.$$

Definition 4.4.11 (Rezk). We say that an ∞-category $\mathscr{C}$ is a *Grothendieck ∞-topos* if

(i) it is locally presentable,

(ii) has all universal colimits, and

(iii) has a κ-compact-object classifier for all sufficiently large regular cardinals κ.[37]

5   A simplicial model of HoTT

This section is devoted to examining [14, Sections 2 and 3], which constructs a certain model of $\mathrm{CDTT} + \mathrm{Univ}$ in the category $\mathbf{sSet}$ of simplicial sets. In Section 5.1 and Section 5.2, we choose particular universes in our chosen class of presheaf categories and show that they carry all of the logical structure found in our MLDTT without Univ, respectively. Next, turning out attention to the univalence axiom, we define in Section 5.3 a simplicial notion of univalence for these models that is logically equivalent to our type-theoretic notion of univalence. Finally, in Section 5.4, we prove that our chosen universes are univalent in the simplicial sense and then state a remarkable generalization of this result.

5.1   Fibrant universes of "small" fibrations

This section recounts [14, Sections 2.1 and 2.2], which defines a class of Kan complexes serving as universes in $\mathbf{sSet}$ both in the sense of Definition 3.3.1 and in the sense of *internal universe* (p. 45).

For any such Kan complex U, we want to find a simplicial map $\widehat{U} \to U$ acting as a classifier for a specific class of Kan fibrations in the sense of Definition 4.4.9. To ensure that this class is closed under the categorical versions of our type-forming operations, we shall take the class of all *κ-small*

---

[37] A precise definition of "sufficiently large" is found in the proof of [15, Proposition 6.1.6.7].

well-ordered Kan fibrations. Moreover, since dependent types will be interpreted as Kan fibrations, $\widehat{U} \to U$ must be a Kan fibration to interpret the dependent type $x : \mathsf{U} \vdash \mathsf{el}(x)\,\mathtt{type}$.

Pick any regular cardinal κ.

Definition 5.1.1. Let $f : X \to Y$ be a map of simplicial sets.

1. We say that f is *well-ordered* if it is equipped with a well-ordering of $Y_x := f_n^{-1}(x)$ for each simplex $x \in X_n$.

2. We say that f is κ-*small* if $|Y_x| < \kappa$ for every simplex x.

Let $f : X \to Y$ and $g : Z \to Y$ be well-ordered simplicial maps. A *morphism* $f \to g$ is a simplicial map $h : X \to Z$ fitting into a commutative triangle

$$
\begin{array}{ccc}
X & \xrightarrow{\ h\ } & Z \\
 & \searrow{\scriptstyle f} & \downarrow{\scriptstyle g} \\
 & & Y
\end{array}
$$

such that $h_n : f_n^{-1}(y) \to g_n^{-1}(y)$ is order-preserving for every $n \in \mathbb{N}$ and $y \in Y_n$.

Note 5.1.2. Recall that for any two well-ordered sets x and y, there is exactly one isomorphism of the form $x \xrightarrow{\cong} y$. Thus, for any two well-ordered simplicial maps X and Y, there is exactly one isomorphism of the form $X \xrightarrow{\cong} Y$.

Now, the isomorphism class of any κ-small well-ordered map is a proper class. We can, however, apply Scott's trick to make this a set. In this case, for any simplicial set X, we have a definable class $\mathcal{W}_\kappa(X)$ consisting of all isomorphism classes of κ-small well-ordered maps $Y \to X$. In fact, $\mathcal{W}_\kappa(X)$ is a set. This gives rise to a presheaf

$$
\mathcal{W}_\kappa : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}
$$

that sends each simplicial map $f : B \to A$ to the pullback action $f^* : \mathcal{W}_\kappa(A) \to \mathcal{W}_\kappa(B)$ on equivalence classes.

Lemma 5.1.3. *For any functor $F : J \to \mathbf{sSet}$, $\mathcal{W}_\kappa(\mathrm{colim}_j F_j) \cong \lim_j(\mathcal{W}_\kappa(F_j))$.*

Proof. Applying $\mathcal{W}_\kappa(-)$ to the colimiting cocone $\{v_j : F_j \to \mathrm{colim}_j F_j \mid j \in \mathrm{Ob}\,J\}$ induces a cone over $\mathcal{W}_\kappa(F_\bullet)$ and thus a canonical map

$$
\psi : \mathcal{W}_\kappa\left(\mathop{\mathrm{colim}}_j F_j\right) \to \lim_j\left(\mathcal{W}_\kappa(F_j)\right)
$$

by the universal property of limits. We want to show that ψ is bijective.

To see that ψ is surjective, let $[f_j : Y_j \to F_j]_{j \in \mathrm{Ob}\,J}$ be a tuple of equivalence classes in $\lim_j(\mathcal{W}_\kappa(F_j))$. For each simplex $x \in \mathrm{colim}_j (F_j)_n$, choose an index j_x along with a simplex $\tilde{x} \in (F_{j_x})_n$ such that $(v_{j_x})_n (\tilde{x}) = x$. Define the fiber Y_x over x as the fiber $(Y_{j_x})_{\tilde{x}}$. For any other such j'_x and $\tilde{x}'$, there is

some map $j_x \to j'_x$ in J along with a pullback square

$$
\begin{array}{ccc}
Y_{j_x} & \dashrightarrow & Y_{j'_x} \\
{\scriptstyle f_{j_x}}\downarrow & \lrcorner & \downarrow{\scriptstyle f_{j'_x}} \\
F_{j_x} & \longrightarrow & F_{j'_x}
\end{array}
$$

By the pasting law for pullbacks, the total rectangle

$$
\begin{array}{ccccc}
(F_{j_x})_{\tilde{x}} & \longrightarrow & Y_{j_x} & \dashrightarrow & Y_{j'_x} \\
\downarrow & \lrcorner & {\scriptstyle f_{j_x}}\downarrow & \lrcorner & \downarrow{\scriptstyle f_{j'_x}} \\
\Delta[n] & \xrightarrow[\tilde{x}]{} & F_{j_x} & \longrightarrow & F_{j'_x}
\end{array}
$$

is a pullback. Since $\tilde{x} = \tilde{x}'$ in $\operatorname{colim}_j F_j$, the uniqueness of pullbacks yields an isomorphism

$$
(F_{j_x})_{\tilde{x}} \cong \left(F_{j'_x}\right)_{\tilde{x}'}
$$

of well-ordered sets. Note 5.1.2 now implies that the fiber Y_x is defined up to *canonical* isomorphism. Thus, we may patch the f_j together to form a κ-small well-ordered map $f : Y \to \operatorname{colim}_j F_j$ such that

$$
v_j^* f \cong f_j
$$

for each $j \in \operatorname{Ob} J$, as desired.

By similar reasoning, we can show that ψ is also injective. $\qquad\square$

Consider the opposite Yoneda embedding $\mathcal{Y}^{op} : \Delta^{\mathrm{op}} \to \mathbf{sSet}^{\mathrm{op}}$, from which we can form the simplicial set

$$
\mathrm{W}_\kappa := \mathcal{W}_\kappa \circ \mathcal{Y}^{op} : \Delta^{\mathrm{op}} \to \mathbf{Set} \,.
$$

Corollary 5.1.4. *The functor $\mathcal{W}_\kappa$ is represented by W_κ.*

Proof. For any $n \in \mathbb{N}$, the Yoneda lemma implies that

$$
\operatorname{Hom}_{\mathbf{sSet}}\left(\Delta[n], \mathrm{W}_\kappa\right) \cong \left(\mathrm{W}_\kappa\right)_n = \mathcal{W}_\kappa\left(\Delta[n]\right) \,.
$$

By Theorem C.0.8, every simplicial set is naturally isomorphic to a small colimit of standard simplices. Moreover, by Lemma 5.1.3, $\mathcal{W}_\kappa(-)$ is continuous, and the hom-functor of any locally small category is continuous in its first variable. It follows that $\operatorname{Hom}_{\mathbf{sSet}}(-, \mathrm{W}_\kappa)$ and $\mathcal{W}_\kappa(-)$ are isomorphic functors, as desired. $\qquad\square$

As a result, we have a natural isomorphism

$$
\Psi : \mathcal{W}_\kappa(-) \xrightarrow{\ \cong\ } \operatorname{Hom}_{\mathbf{sSet}}(-, \mathrm{W}_\kappa). \tag{Ψ}
$$

Notation.

1. For any simplicial set X and κ-small well-ordered map $f : Y \to X$, let $\ulcorner f \urcorner$ refer to the map $\Psi([f]) : X \to \mathrm{W}_\kappa$.

2. Let $\left[\Omega_\kappa : \widehat{\mathrm{W}}_\kappa \to \mathrm{W}_\kappa\right]$ denote the element $\Psi^{-1}\left(\mathrm{id}_{\mathrm{W}_\kappa}\right)$.

For every κ-small well-ordered map $f : Y \to X$, the map Ψ specifies a commutative square

$$\begin{array}{ccc} \mathcal{W}_\kappa(\mathrm{W}_\kappa) & \xrightarrow{\ \cong\ } & \mathrm{Hom}_{\mathbf{sSet}}(\mathrm{W}_\kappa, \mathrm{W}_\kappa) \\ {}^{\ulcorner f \urcorner *}- \downarrow & & \downarrow {}^{-\circ \ulcorner f \urcorner} \\ \mathcal{W}_\kappa(X) & \xrightarrow[\ \cong\ ]{} & \mathrm{Hom}_{\mathbf{sSet}}(X, \mathrm{W}_\kappa) \end{array}$$

of sets. By evaluating this square at $\Psi^{-1}(\mathrm{id}_{\mathrm{W}_\kappa})$, we see that there exists a unique pullback square of the form

$$\begin{array}{ccc} Y & \longrightarrow & \widehat{\mathrm{W}}_\kappa \\ f \downarrow & \lrcorner & \downarrow \Omega_\kappa \\ X & \xrightarrow[\ulcorner f \urcorner]{} & \mathrm{W}_\kappa \end{array} .$$

This means that Ω_κ is a classifier for the class of all κ-small well-ordered maps.

Note 5.1.5. By the axiom of choice, we can choose a well-ordering of each fiber of any κ-small simplicial map, thereby converting it into a well-ordered map. This map can be expressed as a pullback of Ω_κ. As a result, any κ-small simplicial map can be expressed as a pullback of Ω_κ. Such a pullback square, however, may *not* be unique, as our choice of well-orderings need not be unique.

We want to isolate the Kan fibrations found in $\mathcal{W}_\kappa(-)$. Formally, consider the subpresheaf

$$\mathcal{U}_\kappa \hookrightarrow \mathcal{W}_\kappa$$

such that $\mathcal{U}_\kappa(X)$ consists of all κ-small well-ordered *Kan fibrations* for each simplicial set X. Also, let

$$\mathrm{U}_\kappa = \mathcal{U}_\kappa \circ \mathcal{Y}^{\mathrm{op}} : \Delta^{\mathrm{op}} \to \mathbf{Set}$$

and consider the pullback square

$$\begin{array}{ccc} \widehat{\mathrm{U}}_\kappa & \longrightarrow & \widehat{\mathrm{W}}_\kappa \\ p_\kappa \downarrow & \lrcorner & \downarrow \Omega_\kappa \\ \mathrm{U}_\kappa & \longhookrightarrow & \mathrm{W}_\kappa \end{array} .$$

Proposition 5.1.6. *The map $p_\kappa : \widehat{\mathrm{U}}_\kappa \to \mathrm{U}_\kappa$ is a Kan fibration.*[38]

Lemma 5.1.7. *Let $f : Y \to X$ be a κ-small well-ordered map. Then f is a Kan fibration if and only if $\ulcorner f \urcorner : X \to \mathrm{W}_\kappa$ factors through the inclusion $\mathrm{U}_\kappa \hookrightarrow \mathrm{W}_\kappa$.*

Proof.

($\Longrightarrow$) Suppose that f is a Kan fibration. For any n-simplex $x : \Delta[n] \to X$ in X, the pullback

$$x^* f : \Delta[n] \times_X Y \to \Delta[n]$$

is also a Kan fibration by Lemma 4.2.6. Further, this map is κ-small and well-ordered because $\mathrm{im}(\Delta[n] \times_X Y \to Y)$ is exactly the fiber of f over $x \in X_n$. Now, by pasting the two pullback

---

[38][14, Lemma 2.1.10].

squares

$$\begin{CD}
\Delta[n] \times_X Y @>>> Y @>>> \widehat{W}_\kappa \\
@V{x^*f}VV @VV{f}V @VV{\Omega_\kappa}V \\
\Delta[n] @>>{x}> X @>>{\ulcorner f \urcorner}> W_\kappa
\end{CD} \quad ,$$

we see that $\ulcorner f \urcorner \circ x = \ulcorner x^*f \urcorner$ as the total rectangle must be a pullback as well. But $\ulcorner x^*f \urcorner \in (U_\kappa)_n$ because x^*f is a Kan fibration. Hence $\ulcorner f \urcorner$ sends each simplex in X to a simplex in U_κ and thus factors through $U_\kappa \hookrightarrow W_\kappa$.

($\Longleftarrow$) Suppose that $\ulcorner f \urcorner$ factors through $U_\kappa \hookrightarrow W_\kappa$. This yields a commutative diagram of the form

$$\begin{CD}
Y @>>> \widehat{U}_\kappa @>>> \widehat{W}_\kappa \\
@V{f}VV @VV{p_\kappa}V @VV{\Omega_\kappa}V \\
X @>>> U_\kappa @>>> W_\kappa
\end{CD} \quad .$$

Since the total rectangle is also a pullback, so is the lefthand square. By Proposition 5.1.6 together with Lemma 4.2.6, we thus have that $\ulcorner f \urcorner$ is a Kan fibration. $\qquad\square$

Corollary 5.1.8. *The functor $\mathcal{U}_\kappa$ is represented by U_κ.*

Proof. It follows directly from Lemma 5.1.7 that (Ψ) restricts to an isomorphism

$$\mathcal{U}_\kappa(-) \overset{\cong}{\longrightarrow} \mathrm{Hom}_{\mathbf{sSet}}(-, U_\kappa).$$

$\qquad\square$

Consider the pullback square

$$\begin{CD}
Y @>>> \widehat{W}_\kappa \\
@V{f}VV @VV{\Omega_\kappa}V \\
X @>>{\ulcorner f \urcorner}> W_\kappa
\end{CD} \quad .$$

from above. If f is a Kan fibration, then by Lemma 5.1.7 we obtain a pasting of pullback squares

$$\begin{CD}
Y @>>> \widehat{U}_\kappa @>>> \widehat{W}_\kappa \\
@V{f}VV @VV{p_\kappa}V @VV{\Omega_\kappa}V \\
X @>>> U_\kappa @>>> W_\kappa
\end{CD} \quad .$$

This shows that p_κ is a classifier for the class of all κ-small well-ordered Kan fibrations. Further, by the axiom of choice, any κ-small simplicial map can be expressed as a (not necessarily unique) pullback of p_κ.

In this way, U_κ is a universe in the sense of Definition 3.3.1. Any closed type in our model will be

a fibration of the form $\cdot \to 1$. Hence if we want U_κ (or a smaller copy thereof) to serve also as an internal universe, we must show that it is fibrant. For this, the following result due to Joyal will be useful.

Lemma 5.1.9. *Let $j : A \to B$ be a cofibration and $p : C \to A$ be a trivial fibration of simplicial sets.*

(a) There exists a pullback square of the form

$$
\begin{array}{ccc}
C & \longrightarrow & D \\
{\scriptstyle p}\downarrow & \raise2pt{\hbox{$\lrcorner$}} & \downarrow{\scriptstyle \tilde{p}} \\
A & \underset{j}{\longrightarrow} & B
\end{array}
$$

where $\tilde{p}$ is a trivial fibration.

(b) If p is κ-small, then $\tilde{p}$ can be made κ-small.

Proof.

(a) Let $\tilde{p} = \Pi_j p$. This a trivial fibration by Corollary 4.3.13(2). It remains to show that $j^* \Pi_j p \cong p$. To this end, recall the adjoint triple

$$
\Sigma_j(-) \dashv j^*(-) \dashv \Pi_j(-).
$$

This induces an adjunction

$$
j^* \circ \Sigma_j \dashv j^* \circ \Pi_j.
$$

Since adjoints are unique up to isomorphism and $\mathrm{id}_{\mathbf{sSet}/A}$ is right adjoint to itself, it suffices to exhibit a natural isomorphism $j^* \circ \Sigma_j \xrightarrow{\cong} \mathrm{id}_{\mathbf{sSet}/A}$. For any simplicial map $h : E \to A$, it is easy to check that the square

$$
\begin{array}{ccc}
E & =\!=\!= & E \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle j \circ h} \\
A & \underset{j}{\longrightarrow} & B
\end{array}
$$

is a pullback because j is monic. Hence $j^* \Sigma_j h \cong h$, from which we can define our desired isomorphism.

(b) Let $x : \Delta[n] \to B$ be any simplex in B. Note that

$$
(\Pi_j p)_x \cong \mathrm{Hom}_{\mathbf{sSet}/B}(x, \Pi_j p) \cong \mathrm{Hom}_{\mathbf{sSet}/A}(j^* x, p). \tag{$*$}
$$

Note that $j^* \Delta[n]$ is a simplicial subset of $\Delta[n]$ because monomorphisms are stable under pullback. Now, recall that the non-degenerate k-simplices in $\Delta[n]$ are precisely the monomorphisms belonging to $\Delta([k], [n])$. In particular, $\Delta[n]$ is finite (Definition 4.1.4), and thus so is $j^* \Delta[n]$. Moreover, Lemma 4.1.3 implies that any simplicial map $X \to Y$ is determined by its action on all non-degenerates simplices in X. We thus can find an embedding of $\mathrm{Hom}_{\mathbf{sSet}/A}(j^* x, p)$ into the finite product

$$
P := \prod_{\substack{z \text{ non-deg.} \\ \text{simplex in} \\ j^* \Delta[n]}} p^{-1}(j^* x(z))
$$

of fibers of p. Assuming that p is κ-small, we have that $|P| < \kappa$ by basic cardinal arithmetic. In this case, it follows that $\Pi_j p$ is κ-small by $(*)$.

$\square$

Theorem 5.1.10. *The simplicial set* U_κ *is a Kan complex.*

Proof. Let $n \in \mathbb{Z}_{\geq 1}$. For each integer $0 \leq k \leq n$, we must find a filler of the form

$$
\begin{array}{ccc}
\Lambda^k[n] & \xrightarrow{\ \ulcorner q \urcorner\ } & U_\kappa \\
\downarrow & \nearrow & \\
\Delta[n] & &
\end{array}
\ .
$$

Thanks to Corollary 5.1.8, the map $\ulcorner q \urcorner$ naturally corresponds to a κ-small well-ordered map $q : Z \to \Lambda^k[n]$. It suffices to find a pullback square of the form

$$
\begin{array}{ccc}
Z & \dashrightarrow & Z' \\
{\scriptstyle q}\downarrow & \ulcorner & \downarrow{\scriptstyle q'} \\
\Lambda^k[n] & \longhookrightarrow & \Delta[n]
\end{array}
$$

such that q' is a κ-small well-ordered Kan fibration and the map $Z \to Z'$ induces an order-preserving function $q^{-1}(x) \to (q')^{-1}(x)$ for any simplex x in $\Lambda^k[n]$. For, in this case, we can form a pasting of two pullback squares

$$
\begin{array}{ccccc}
Z & \longrightarrow & Z' & \longrightarrow & U_\kappa \\
{\scriptstyle q}\downarrow & \ulcorner & \downarrow{\scriptstyle q'} & \ulcorner & \downarrow{\scriptstyle p_\kappa} \\
\Lambda^k[n] & \longhookrightarrow & \Delta[n] & \xrightarrow[\ulcorner q' \urcorner]{} & \widehat{U}_\kappa
\end{array}
$$

where the lower composite must equal $\ulcorner q \urcorner$ because p_κ is a classifier. Hence $\ulcorner q' \urcorner$ would serve as our desired filler.

By Lemma 4.1.26, we can factor q as a trivial fibration $q_t : Z \to W$ followed by a minimal fibration $q_m : W \to \Lambda^k[n]$.

Claim. *Both q_t and q_m are κ-small.*

Proof. First, to see that q_t is κ-small, note that

$$
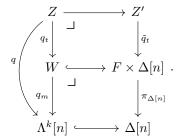q_t^{-1}(w) \subseteq (q_m \circ q_t)^{-1}(q_m(w)) = q^{-1}(q_m(w))
$$

for any simplex w in W. As q is κ-small, it follows that q_t is also κ-small.

Next, to see that q_m is κ-small, note that q_t is levelwise surjective as a trivial fibration. Therefore, for any simplex ℓ in $\Lambda^k[n]$, every element of $(q_m)^{-1}(\ell)$ has the form $q_t(z)$ for some simplex z in Z. Then q sends z to ℓ, which shows that the function $q^{-1}(\ell) \to (q_m)^{-1}(\ell)$ given by $x \mapsto q_t(x)$ is surjective. Hence q_m is κ-small. $\square$

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By Theorem 4.1.32, the map q_m is isomorphic to the trivial bundle $F \times \Lambda^k[n] \xrightarrow{\pi_2} \Lambda^k[n]$. This yields a pullback square

$$
\begin{array}{ccc}
W & \hookrightarrow & F \times \Delta[n] \\
{\scriptstyle q_m} \downarrow & \lrcorner & \downarrow {\scriptstyle \pi_{\Delta[n]}} \\
\Lambda^k[n] & \longrightarrow & \Delta[n]
\end{array} \; .
$$

Since q_m is κ-small, so is the trivial bundle $\pi_{\Delta[n]}$. Further, since q_t is κ-small, Lemma 5.1.9 provides us with a κ-small Kan fibration $\tilde{q}_t$ fitting into a commutative diagram

$$
\begin{array}{ccc}
Z & \longrightarrow & Z' \\
{\scriptstyle q_t} \downarrow & \lrcorner & \downarrow {\scriptstyle \tilde{q}_t} \\
W & \hookrightarrow & F \times \Delta[n] \\
{\scriptstyle q_m} \downarrow & \lrcorner & \downarrow {\scriptstyle \pi_{\Delta[n]}} \\
\Lambda^k[n] & \longrightarrow & \Delta[n]
\end{array} \; .
$$

Take q' to be $\pi_{\Delta[n]} \circ \tilde{q}_t$, which is a Kan fibration as the composite of two fibrations. To see that q' is κ-small, observe that for any simplex z in $\Delta[n]$, we have that

$$
(q')^{-1}(z) = \left(\pi_{\Delta[n]} \circ \tilde{q}_t\right)^{-1}(z) = \bigcup_{w \in \left(\pi_{\Delta[n]}\right)^{-1}(z)} (\tilde{q}_t)^{-1}(w).
$$

Since κ is regular, it follows that $\left|(q')^{-1}(z)\right| < \kappa$. Thus, q' is κ-small.

Finally, we must extend the well-ordering of q to a well-ordering of q'. This is possible because any well-founded binary relation R on a set Q can be extended to a well-ordering of Q. Indeed, consider the rank function $\mathrm{rank}_R : Q \to \alpha$, defined inductively by

$$
\mathrm{rank}_R(x) = \sup \{\mathrm{rank}_R(y) + 1 \mid yRx\},
$$

where α is an ordinal. By induction on rank_R together with the axiom of choice, we can extend R to a well-ordering of Q. $\qquad\square$

5.2   Modeling MLDTT without Univ

Notation. Continue to let $\mathbb{T}$ denote our MLDTT without Univ.

<div align="center">Assume now that κ is inaccessible.</div>

In this section, we verify that U_κ has sufficient logical structure to induce a model of $\mathbb{T}$ via Theorem 3.4.5. In the interest of space, we shall describe just the Π- and Id-structure on U_κ along with smaller copies of U_κ serving as nested universes in U_κ. See [14, Theorem 2.3.4] for sketches of the remaining cases.

Recall from page 43 that an Π-structure on U_κ is precisely a map

$$
\bar{\Pi} : \Pi\left(\mathrm{U}_\kappa\right) \to \mathrm{U}_\kappa
$$

together with an isomorphism $\bar{\Pi}^* p_\kappa \cong \Pi_{\alpha_g} \beta_g$. By Corollary 5.1.8, both α_g and β_g are κ-small Kan fibrations.

Proposition 5.2.1. *Let $h : X \to Y$ and $f : Y \to Z$ be κ-small Kan fibrations. Then the dependent product $\Pi_f h$ is a κ-small Kan fibration.*

Proof. Corollary 4.3.13(4) immediately implies that $\Pi_f h$ is a Kan fibration. To see that it is κ-small, let $n \in \mathbb{N}$ and notice from ($*$) that

$$(\Pi_f h)_x \cong \mathrm{Hom}_{\mathbf{sSet}/Y}(f^* x, h)$$

for any simplex $x : \Delta[n] \to Z$ in Z. The cardinality of $\mathrm{Hom}_{\mathbf{sSet}/Y}(f^* x, h)$ is at most the cardinality of the set S of all functions

$$f_n^{-1}(x) \to (f \circ h)_n^{-1}(x).$$

As κ is regular, both $f_n^{-1}(x)$ and $(f \circ h)_n^{-1}(x)$ have cardinality $< \kappa$. Hence

$$\left| (\Pi_f h)_x \right| \leq |S| < \kappa$$

because κ is inaccessible. $\qquad\qquad\square$

We have seen that any κ-small simplicial map can be expressed as a pullback of p_κ. Thus, by Proposition 5.2.1, the dependent product $\Pi_{\alpha_g} \beta_g$ can be expressed as a pullback $\bar{\Pi}^* p_\kappa$, as required.
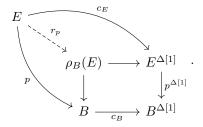
Next, we want to define an Id-structure on U_κ. For any Kan fibration $p : E \to B$, the *fibered path space object* of p is the pullback

$$
\begin{array}{ccc}
\rho_B(E) & \longrightarrow & E^{\Delta[1]} \\
\downarrow & \lrcorner & \downarrow{\scriptstyle p^{\Delta[1]}} \\
B & \xrightarrow{\ c_B\ } & B^{\Delta[1]}
\end{array}
$$

of the exponential object $E^{\Delta[1]}$ along the constant path map. We have a unique mediating map

We also have composites

$$s_p : \rho_B(E) \to E^{\Delta[1]} \xrightarrow{\ \mathrm{ev}_0\ } E$$
$$t_p : \rho_B(E) \to E^{\Delta[1]} \xrightarrow{\ \mathrm{ev}_1\ } E.$$

Proposition 5.2.2. *The diagonal map $\Delta_p : E \to E \times_B E$ factors as*

$$E \xrightarrow{\ r_p\ } \rho_B(E) \xrightarrow{\ (s_p, t_p)\ } E \times_B E$$

over B such that

- r_p *is a trivial cofibration,*

- (s_p, t_p) *is a Kan fibration, and*

- r_p *is stably orthogonal to* (s_p, t_p) *over* B.[39]

If $p : E \to B$ is κ-small, then so is (s_p, t_p) because r_p is monic. In this case, there is a pullback square of the form

$$
\begin{array}{ccc}
\rho_B(E) & \longrightarrow & E \\
{\scriptstyle (s_p, t_p)}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
E \times_B E & \xrightarrow[\overline{\mathsf{Id}}_p]{} & B
\end{array} \quad .
$$

Specifically, the Kan fibration p_κ is κ-small. Thus, we may take $\left(\overline{\mathsf{Id}}_{p_\kappa}, r_{p_\kappa}\right)$ as our Id-structure.

Finally, to define a U-structure on U_κ, suppose that $\lambda < \kappa$ is another inaccessible cardinal. Then U_λ is again a Kan complex. Also, the unique map $\mathrm{U}_\lambda \to 1$ is κ-small, i.e., the set

$$
(\mathrm{U}_\beta)_n = \mathcal{U}_\lambda(\Delta[n])
$$

has cardinality $< \kappa$ for each $n \in \mathbb{N}$. Indeed, each isomorphism class $[f : X \to \Delta[n]] \in \mathcal{U}_\lambda(\Delta[n])$ is determined by

(i) a family of isomorphism classes $\left[f^{-1}(z)\right]$ over $\Delta[n]$ indexed by the countable set of all simplices z in $\Delta[n]$ together with

(ii) all of the face and degeneracy operators between the total spaces of k-simplices ($k \in \mathbb{N}$) induced by the fibers of f.

Since every well-ordered set is isomorphic to exactly one ordinal, there are exactly λ many isomorphism classes of well-ordered sets of size $< \lambda$. Hence there are exactly $\aleph_0 \cdot \lambda = \lambda$ many possible families of isomorphism classes as in (i). Moreover, the domain and codomain of any face or degeneracy operator as in (ii) are of size $< \lambda$. Thus, there are exactly λ many ways of defining such an operator because λ is inaccessible. Also, there are countably many face and degeneracy operators to define in total. Therefore, there are exactly $\aleph_0 \cdot \lambda = \lambda$ many possible face and degeneracy operators as in (ii). We can conclude that

$$
|\mathcal{U}_\lambda(\Delta[n])| \leq \lambda \cdot \lambda = \lambda < \kappa.
$$

We thus have a unique pullback square of the form

$$
\begin{array}{ccc}
\mathrm{U}_\lambda & \longrightarrow & \widehat{\mathrm{U}}_\kappa \\
\downarrow & \lrcorner & \downarrow{\scriptstyle p_\kappa} \\
1 & \xrightarrow[u_\lambda]{} & \mathrm{U}_\kappa
\end{array}
$$

as well as an inclusion map

$$
\iota : \mathrm{U}_\lambda \to \mathrm{U}_\kappa, \quad [f] \mapsto [f].
$$

Finally, we take the pair (u_λ, ι) as our U-structure.

---

[39] [14, Proposition 2.3.3].

Generic structure for modeling CDTT

We can check that any LCCC $\mathscr{C}$ carries all data of a $\mathbb{T}$-structure (forgetting its contextual-categorical structure) aside from an Id-type structure (p. 34). It is possible, however, to *almost* correctly interpret *extensional* identity types (Remark 2.1.1) in $\mathscr{C}$. Indeed, suppose that we can derive

$$\Gamma \vdash A \,\texttt{type}$$
$$\Gamma \vdash a : A$$
$$\Gamma \vdash b : A$$

in $\mathbb{T}$. Then the well-formed identity type $\Gamma \vdash \mathsf{Id}_A(a,b) \,\texttt{type}$ is interpreted as the equalizer in the diagram

$$\llbracket \Gamma, z : \mathsf{Id}_A(a,b) \rrbracket \dashrightarrow^{p_{\mathsf{Id}_A(a,b)}} \llbracket \Gamma \rrbracket \underset{\llbracket b \rrbracket}{\overset{\llbracket a \rrbracket}{\rightrightarrows}} \llbracket \Gamma, x : A \rrbracket \,.$$

As a result, the canonical projection $p_{\mathsf{Id}_A(a,b)}$ must be monic. Further, we interpret the canonical term $\Gamma \vdash \mathsf{refl}_a : \mathsf{Id}_A(a,a)$ as the morphism $\llbracket \Gamma \rrbracket \to \llbracket \Gamma, z : \mathsf{Id}_A(a,b) \rrbracket$ induced by the universal property of equalizers, i.e., fitting into a commutative triangle

In general, our interpretation function $\llbracket - \rrbracket$ sends any well-formed term of type $\mathsf{Id}_A(a,b)$ to a section of $p_{\mathsf{Id}_A(a,b)}$. Since $p_{\mathsf{Id}_A(a,b)}$ is monic, it follows that $\mathscr{C}$ satisfies UIP. To see that it satisfies ERR, recall that any morphism that is both a monomorphism and a split epimorphism is an isomorphism. Therefore, $p_{\mathsf{Id}_A(a,b)}$ is an isomorphism, so that $\llbracket a \rrbracket = \llbracket b \rrbracket$, as desired.

The reason that such an interpretation is *almost* correct is that endowing $\mathscr{C}$ with a strictly functorial pullback operation may be impossible. One can, however, convert $\mathscr{C}$ into an equivalent category with attributes, which has such an operation [10]. In this sense, every LCCC admits a model of extensional CDTT.

Moreover, thanks to [7, Theorem 7.10], we know that every locally presentable LCC ∞-category $\mathscr{C}$ has a presentation by a type-theoretic model category $\underline{\mathscr{C}}$.[40] Consider the full subcategory $\underline{\mathscr{C}}_{\mathbf{f}}$ of $\underline{\mathscr{C}}$ on all fibrant objects.

Proposition 5.2.3. *The class Fib of all fibrations in $\underline{\mathscr{C}}_{\mathbf{f}}$ is both closed and factorizing. (See Section 3.5.)*

Proof. Note that Fib is factorizing because $\underline{\mathscr{C}}$ is a model category. For the same reason, Fib satisfies condition (b) of Definition 3.5.5. It also satisfies condition (c) because all objects of $\underline{\mathscr{C}}_{\mathbf{f}}$ are fibrant.

To see that Fib satisfies condition (d) of Definition 3.5.5, note that Fib is closed under pullbacks and that the pullback of a fibrant object in any model category is again fibrant. Thus, for every map $f : A \to B$ in $\underline{\mathscr{C}}_{\mathbf{f}}$, we have a base change functor $f^* : \underline{\mathscr{C}}_{\mathbf{f}}/B \to \underline{\mathscr{C}}_{\mathbf{f}}/A$ preserving all fibrations. Since $\underline{\mathscr{C}}$ is locally cartesian closed by definition of *type-theoretic model category*, it follows that f^* has a

---

[40] It is hoped that, eventually, we can drop the hypothesis that $\mathscr{C}$ is locally presentable.

right adjoint Π_f. We must show that Π_f preserves fibrations. But f^* preserves trivial cofibrations by Lemma 4.2.5.

It remains to show that Fib satisfies condition (e) of Definition 3.5.5. By Corollary 4.3.13(4) together with Proposition C.0.15 and the fact that fibrations are stable under pullback, we have that the exponential of two fibrations over $C \in \mathrm{Ob}\,\mathscr{C}_{\mathbf{f}}$ is again a fibration. It is easy to see that this remains the exponential in $Fib(C)$. Hence the inclusion functor $Fib(C) \hookrightarrow \mathscr{C}_{\mathbf{f}}/C$ preserves exponentials. $\square$

Assuming Conjecture 3.2.13, Theorem 3.5.7 directly implies that $\mathscr{C}_{\mathbf{f}}$ models $\mathbb{T}$ (without the universe type). In this sense, every locally presentable LCC ∞-category $\mathscr{C}$ can be presented by a model of intensional CDTT. In particular, $[\mathscr{D}, \mathbf{sSet}]$ has such a presentation for any small category $\mathscr{D}$.

5.3   The simplicial notion of univalence

At this point, let us turn to proving that the induced contextual category $\mathbf{sSet}_{\mathrm{U}_\kappa}$ (p. 41) satisfies the univalence axiom. To begin with, we define a simplicial notion of univalence that will be equivalent to our type-theoretic one.
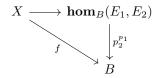
Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be Kan fibrations. The over category $\mathbf{sSet}/B$ is cartesian closed by Theorem C.0.17. Thus, we may form the exponential

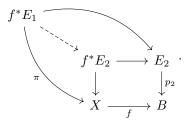$$p_2^{p_1} : \mathbf{hom}_B(E_1, E_2) \to B$$

of p_1 and p_2 in $\mathbf{sSet}/B$.

Note 5.3.1. The map $p_2^{p_1}$ is a Kan fibration.

By adjunction, any map

over B naturally corresponds to a map $f^*E_1 \to E_2$ over B. This, in turn, naturally corresponds to a map $f^*E_1 \to f^*E_2$ fitting into a commutative diagram

By the Yoneda lemma, it follows that an n-simplex x in $\mathbf{hom}_B(E_1, E_2)$ is precisely a pair of maps

$$(x : \Delta[n] \to B, s_x : x^*E_1 \to x^*E_2)\,.$$

Lemma 5.3.2. *Let $g : E_1 \to E_2$ be a weak equivalence in $\mathbf{sSet}/B$ and let $h : B' \to B$ be a simplicial map. Then the map $h^*g : h^*E_1 \to h^*E_2$ is a weak equivalence in $\mathbf{sSet}/B'$.*

Proof. By Note 4.3.14, the base change functor h^* preserves trivial fibrations. Therefore, Lemma 4.2.7 implies that h^* preserves weak equivalences of fibrant objects. In particular, h^*g is a weak equivalence. $\qquad\square$

Proposition 5.3.3. *Let $g : E_1 \to E_2$ be a map over B. If every connected component of B has a vertex $v : \Delta[0] \to B$ such that the induced map $v^*g : v^*E_1 \to v^*E_2$ of fibers is a weak equivalence, then g is also a weak equivalence.*[41]

Consider any map $f : [n] \to [m]$ in Δ along with induced map $\mathcal{Y}(f) : \Delta[n] \to \Delta[m]$. Then the function

$$\mathbf{hom}_B(E_1, E_2)(f) : \mathbf{hom}_B(E_1, E_2)_m \to \mathbf{hom}_B(E_1, E_2)_n$$

is given by

$$(x : \Delta[m] \to B, s_x : x^*E_1 \to x^*E_2) \;\mapsto\; \left(x \circ \mathcal{Y}(f), s_{x \circ \mathcal{Y}(f)} : (x \circ \mathcal{Y}(f))^* E_1 \to (x \circ \mathcal{Y}(f))^* E_2\right).$$

If s_x is a weak equivalence, then so is

$$s_{x \circ \mathcal{Y}(f)} : \mathcal{Y}(f)^* (x^*E_1) \to \mathcal{Y}(f)^* (x^*E_2).$$

by Lemma 5.3.2. Thus, we have a simplicial subset $\mathbf{eq}_B(E_1, E_2) \subset \mathbf{hom}_B(E_1, E_2)$ whose n-simplices are exactly pairs of maps

$$(x : \Delta[n] \to B, s_x : x^*E_1 \to x^*E_2)$$

$$s_x \text{ is a weak equivalence.}$$

Now, consider any map

$$(f : X \to B, s_f : f^*E_1 \to f^*E_2) : X \to \mathbf{hom}_B(E_1, E_2)$$

over B. This sends any $x \in X_n$ to the n-simplex $(f \circ x, x^*s_f)$ in $\mathbf{hom}_B(E_1, E_2)$. If x^*s_f is a weak equivalence for every simplex x in X, then s_f must be a weak equivalence by Proposition 5.3.3. Conversely, if s_f is a weak equivalence, then any such map x^*s_f must be a weak equivalence because the base change functor x^* preserves weak equivalences of fibrant objects. In conclusion, the map s_f is a weak equivalence if and only if (f, s_f) factors through $\mathbf{eq}_B(E_1, E_2) \hookrightarrow \mathbf{hom}_B(E_1, E_2)$. As a result, any map $X \to \mathbf{eq}_B(E_1, E_2)$ corresponds naturally to a pair of maps

$$(f : X \to B, s_f : f^*E_1 \to f^*E_2) \qquad\qquad (\bigstar)$$

$$s_f \text{ is a weak equivalence.}$$

Lemma 5.3.4. *The restriction $p_2^{p_1} : \mathbf{eq}_B(E_1, E_2) \to B$ is a Kan fibration.*

Proof. We must exhibit a lift of the form

$$\begin{array}{ccc} \Lambda^k[n] & \longrightarrow & \mathbf{eq}_B(E_1, E_2) \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p_2^{p_1}} \\ \Delta[n] & \xrightarrow{\quad x \quad} & B \end{array} \quad .$$

---

[41][14, Lemma 3.2.7].

Since $p_2^{p_1} : \mathbf{hom}_B(E_1, E_2) \to B$ is a Kan fibration, we can find a diagonal fill-in of the form

$$(x : \Delta[n] \to B, s_x : x^*E_1 \to x^*E_2) : \Delta[n] \to \mathbf{hom}_B(E_1, E_2).$$

In light of ($\bigstar$), it suffices to show that s_x is a weak equivalence. We already have a weak equivalence $s_{x \circ i} : i^*x^*E_1 \to i^*x^*E_2$ from our original square. Therefore, Proposition 5.3.3 implies that s_x is a weak equivalence because $\Delta[n]$ is connected. $\qquad\square$

We are now in position to formulate our simplicial notion of univalence. Let $p : E \to B$ be a Kan fibration and let

$$\mathbf{eq}(E) = \mathbf{eq}_{B \times B} \left(\pi_1^*E, \pi_2^*E \right).$$

As the product of two simplicial sets is computed levelwise, the n-simplices of $\mathbf{eq}(E)$ are precisely triples of the form

$$(b_1, b_2, s_{b_1, b_2})$$
$$b_1, b_2 \in B_n$$
$$s_{b_1, b_2} : b_1^*E \to b_2^*E.$$

From ($\bigstar$), we see that any map $X \to \mathbf{eq}(E)$ corresponds naturally to a triple of maps

$$(f_1 : X \to B, f_2 : X \to B, s_{f_1, f_2} : f_1^*E \to f_2^*E). \tag{I}$$

In particular, we have a map $\delta_E : B \to \mathbf{eq}(E)$ corresponding to the triple $(\mathrm{id}_B, \mathrm{id}_B, \mathrm{id}_E)$, with

$$(\delta_E)_n (b) = (b, b, \mathrm{id}_{b^*E}), \qquad b \in B_n. \tag{II}$$

Then δ_E has two retractions defined by the composites

$$\mathbf{eq}(E) \xrightarrow{\pi_2^*p^{\pi_1^*p}} B \times B \xrightarrow{\pi_i} B, \qquad i = 1, 2. \tag{III}$$

Therefore, δ_E is a split monomorphism.

Definition 5.3.5 (Simplicial univalence). A Kan fibration $p : E \to B$ is *univalent* if δ_E is a weak equivalence in $\mathbf{sSet}_{\mathtt{Quillen}}$.

Note that the diagram

$$
\begin{array}{ccc}
 & \Delta_B & \\
B \xrightarrow{\delta_E} \mathbf{eq}(E) & \xrightarrow{\pi_2^*p^{\pi_1^*p}} & B \times B
\end{array}
$$

commutes. Thus, since δ_E is monic and $\pi_2^*p^{\pi_1^*p}$ is a Kan fibration by Lemma 5.3.4, we have that δ_E is univalent if and only if $\mathbf{eq}(E)$ is a path space object of B.

Example 5.3.6. For any Kan complex X, the unique map $X \to 1$ is univalent if and only if the space of homotopy autoequivalences of X is contractible.

5.4   Proof of univalence

This section first unifies our type-theoretic and simplicial definitions of univalence and then shows that the Kan fibration $p_\kappa : \widehat{U}_\kappa \to U_\kappa$ is simplicially univalent. This implies that the contextual category $\mathbf{sSet}_{U_\kappa}$ satisfies Voevodsky's univalence axiom (Definition 2.4.2).

Remark 5.4.1. Recall that we interpret dependent types as pullbacks of p_κ. For convenience, we may write such pullbacks as dependent types *within* our MLDTT, thereby abusing notation.

Let $p_1 : E_1 \to B$ and $p_2 : E_2 \to B$ be pullbacks of p_κ. Suppose that B is a Kan complex (i.e., a closed type). Consider both the function type

$$[\mathbf{E_1}, \mathbf{E_2}] \coloneqq [\![x : B \vdash E_1 \to E_2 \ \mathtt{type}]\!]$$

and the type

$$\mathbf{E_1} \simeq \mathbf{E_2} \coloneqq [\![x : B \vdash E_1 \simeq E_2 \ \mathtt{type}]\!]$$

of equivalences from E_1 to E_2 interpreted as Kan fibrations over B.

Proposition 5.4.2.

(1) There is an isomorphism $[\mathbf{E_1}, \mathbf{E_2}] \xrightarrow{\cong} \mathbf{hom}_B(E_1, E_2)$ over B.[42]

(2) The induced map $\mathbf{E_1} \simeq \mathbf{E_2} \to \mathbf{hom}_B(E_1, E_2)$ factors through $\mathbf{eq}_B(E_1, E_2) \hookrightarrow \mathbf{hom}_B(E_1, E_2)$, and $\mathbf{E_1} \simeq \mathbf{E_2} \to \mathbf{eq}_B(E_1, E_2)$ is a trivial fibration.[43]

Theorem 5.4.3. *Suppose that $p : E \to B$ is a Kan fibration. Then p is simplicially univalent if and only if it is type-theoretically univalent in the sense that the Kan fibration*

$$\left[\!\left[x, y : B \vdash \mathtt{is\_equiv}\left(\mathtt{idtoequiv}_{x:B;E(x)}(x, y) \right) \ \mathtt{type} \right]\!\right]$$

over $B \times B$ has a section.

Proof sketch. Consider the map

$$w_E \coloneqq \left[\!\left[x, y : B, z : x \rightsquigarrow_B y \vdash \mathtt{idtoequiv}_{x:B;E(x)}(x, y)(z) : E(x) \simeq E(y) \right]\!\right].$$

It follows from [14, Lemma 3.3.2] that p is type-theoretically univalent if and only if

$$[\![x, y : B, z : x \rightsquigarrow_B y]\!] \xrightarrow{\quad w_E \quad} [\![x, y : B, f : \mathtt{is\_equiv}\left(E(x), E(y)\right)]\!]$$

$$\searrow \qquad\qquad \swarrow$$

$$B \times B$$

is a weak equivalence over $B \times B$. The same Lemma together with Proposition 5.4.2 provides us

---

[42][14, Corollary 3.3.3].
[43][14, Lemma 3.3.4].

with a commutative diagram

$$
\begin{array}{ccccc}
B & \xrightarrow{\;r_B\;} & \rho_1(B) & \xrightarrow{\;w_E\;} & \pi_1^*\mathbf{E} \simeq \pi_2^*(\mathbf{E}) \\
& & & & \downarrow \\
& & & & \mathbf{eq}_{B\times B}\left(\pi_1^*E, \pi_2^*E\right) \\
& (s_B,t_B) & & & \downarrow \\
& & & & \mathbf{hom}_{B\times B}\left(\pi_1^*E, \pi_2^*E\right) \\
& \Delta_B & & & \downarrow \\
& & & & B \times B
\end{array}
$$

Recall that the map r_B is precisely the interpretation $[\![x : B \vdash \mathsf{refl}(B,x) : x \rightsquigarrow_B x]\!]$ of reflexivity. By applying the inference rule Id-COMP to our construction of $\mathtt{idtoequiv}_{x:B;E(x)}(x,y)$ (Lemma 2.4.1), we deduce that the composite

$$
B \longrightarrow \mathbf{hom}_{B\times B}\left(\pi_1^*E, \pi_2^*E\right)
$$

is precisely the interpretation $[\![x : B \vdash \lambda(y : E(x)).y : E(x) \to E(x)]\!]$ of the identity map on $E(x)$. Therefore, the composite $B \longrightarrow \mathbf{eq}_{B\times B}\left(\pi_1^*E, \pi_2^*E\right)$ is precisely δ_E, defined by (II).

Both r_B and $\mathbf{E_1} \simeq \mathbf{E_2} \to \mathbf{eq}_B(E_1, E_2)$ are weak equivalences by Proposition 5.2.2 and Proposition 5.4.2(2), respectively. Hence the two-out-of-three property implies that δ_E is a weak equivalence if and only if w_E is one. $\qquad\square$
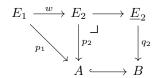
Theorem 5.4.4. *The Kan fibration $p_\kappa : \widehat{\mathrm{U}}_\kappa \to \mathrm{U}_\kappa$ is simplicially univalent.*

Proof. We must show that $\delta_{\widehat{\mathrm{U}}_\kappa}$ is a weak equivalence. Recall from (III) that the composite

$$
\tau := \pi_2 \circ \pi_2^* p_\kappa^{\;\pi_1^* p_\kappa}
$$

is a retraction of $\delta_{\widehat{\mathrm{U}}_\kappa}$. Therefore, by two-out-of-three, it suffices to show that τ is a weak equivalence. In fact, we shall show that it is a trivial fibration. To this end, consider any lifting problem of the form

$$
\begin{array}{ccc}
A & \longrightarrow & \mathbf{eq}\left(\widehat{\mathrm{U}}_\kappa\right) \\
j\downarrow & & \downarrow{\scriptstyle \delta_{\widehat{\mathrm{U}}_\kappa}} \\
B & \longrightarrow & \mathrm{U}_\kappa
\end{array}
$$

By Corollary 5.1.8 and our characterization (I) of maps into $\mathbf{eq}\left(\widehat{\mathrm{U}}_\kappa\right)$, this square naturally corresponds to a commutative diagram of the form

$$
\begin{array}{ccccc}
E_1 & \xrightarrow{\;w\;} & E_2 & \longrightarrow & \underline{E}_2 \\
& {\scriptstyle p_1}\searrow & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q_2} \\
& & A & \lhook\joinrel\longrightarrow & B
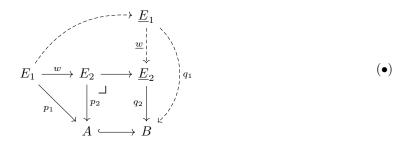\end{array}
$$

where

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- w is a weak equivalence and

- p_1, p_2, and q_2 are κ-small well-ordered Kan fibrations.

Our pullback square here comes from our lifting problem, which exhibits p_2 as the pullback of p_κ along the composite $A \hookrightarrow B \to U_\kappa$.

Now, a solution to our lifting problem naturally corresponds to a commutative diagram of the form

$$
(\bullet)
$$

where

- $\underline{w}$ is a weak equivalence,

- q_1 is a κ-small well-ordered Kan fibration, and

- the square

$$
\begin{array}{ccc}
E_1 & \dashrightarrow & \underline{E}_1 \\
{\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_1} \\
A & \longhookrightarrow & B
\end{array}
$$

is a pullback.

Let us define $\underline{w}$ as the pullback

$$
(\bigcirc)
$$

where η denotes the unit of the adjunction $j^* \dashv \Pi_j$. First, we want to show that $E_1 \cong j^*\underline{E}_1$ and that $j^*\underline{w} \cong w$ in $\mathbf{sSet}\,/A$ so that $(\bullet)$ commutes. To this end, recall from our proof of Lemma 5.1.9(a) that $j^*\Pi_j \cong \mathrm{id}_{\mathbf{sSet}\,/A}$. Thus, using the triangle identities for (j^*, Π_j), we can apply the functor $j^*(-)$ to $(\bigcirc)$ to get

$$
\begin{array}{ccc}
j^*\underline{E}_1 & \longrightarrow & E_1 \\
{\scriptstyle j^*\underline{w}}\downarrow & & \downarrow{\scriptstyle w} \\
\underbrace{j^*\underline{E}_2}_{E_2} & \xrightarrow{\mathrm{id}_{E_2}} & E_2
\end{array}.
$$

This square is a pullback because $j^*(-)$ preserves products as a right adjoint to Σ_j. Since the pullback of an isomorphism is again an isomorphism, it follows that $E_1 \cong j^*\underline{E}_1$ and $j^*\underline{w} \cong w$, as desired.

Next, we want to show that q_1 is κ-small and well-ordered. In light of our proof of Lemma 5.1.9(b), both $\Pi_j p_1$ and $\Pi_j p_1$ are κ-small. Since $q_2 = \Pi_j p_2 \circ \eta_{\underline{E}_2}$, it follows that $\eta_{\underline{E}_2}$ is also κ-small. Hence the map $\underline{E}_1 \to \Pi_j E_1$ is κ-small as the pullback of $\eta_{\underline{E}_2}$. As κ is regular, this implies that q_1 is κ-small as the composite of two κ-small maps. Moreover, by the axiom of choice, we may assume that q_1 is a well-ordered extension of p_1 as in Theorem 5.1.10.

It remains to show that q_1 is a Kan fibration and that $\underline{w}$ is a weak equivalence. As a weak equivalence, w can be factored as trivial cofibration w_c followed by a trivial fibration w_f. Further, the partial mapping $\mathrm{Arr}(\mathbf{sSet}\,/A) \to \mathrm{Arr}(\mathbf{sSet}\,/B)$ given by $w \mapsto \underline{w}$ respects composition, so that

$$\underline{w} = \underline{w_f} \circ \underline{w_c}.$$

Hence we may assume that w is either a trivial cofibration or a trivial fibration. First, suppose that w is a trivial fibration. Then $\Pi_j w$ is also a trivial fibration by Corollary 4.3.13(2). Hence so is $\underline{w}$ as the pullback of a trivial fibration, and now q_1 is a Kan fibration as the composite of two Kan fibrations.

Suppose, instead, that w is a trivial cofibration. In this case, $\Pi_j w$ must be a monic in $\mathbf{sSet}$ because $\Pi_j(-)$ preserves monomorphisms as a right adjoint. Hence $\underline{w}$ is also monic in light of ($\bigcirc$). Without loss of generality, we may assume that $\underline{w}$ is an inclusion map into $\underline{E}_2$.

Claim. *There exists a strong deformation retraction $H : E_2 \times \Delta[1] \to E_1$ of w.*

Proof. Since w is a trivial cofibration, we have a diagonal fill-in

$$
\begin{array}{ccc}
E_1 & \!\!=\!\!\!=\!\!\!= & E_1 \\
{\scriptstyle w}\big\downarrow & {\scriptstyle r}\nearrow & \big\downarrow{\scriptstyle p_1} \\
E_2 & \xrightarrow{\ p_2\ } & A
\end{array}\ .
$$

Thanks to Corollary 4.1.22, this yields another diagonal fill-in

$$
\begin{array}{ccc}
(E_2 \times \{0\}) \cup (E_1 \times \Delta[1]) \cup (E_2 \times \{1\}) & \xrightarrow{\ \mathrm{id}_{E_2}\,\cup\, w \,\cup\, w\circ r\ } & E_2 \\
\big\downarrow & \nearrow^{H} & \big\downarrow{\scriptstyle p_2} \\
E_2 \times \Delta[1] & \xrightarrow[\ p_2 \circ \pi_1\ ]{} & A
\end{array}\ .
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now, we may apply Corollary 4.1.22 to obtain a diagonal fill-in

$$
\begin{array}{ccc}
(E_2 \times \Delta[1]) \cup (\underline{E}_1 \cup \Delta[1]) \cup (\underline{E}_2 \times \{0\}) & \xrightarrow{\ H \cup \underline{w} \cup \mathrm{id}_{\underline{E}_2}\ } & \underline{E}_2 \\
\big\downarrow & \nearrow^{H'} & \big\downarrow{\scriptstyle q_2} \\
\underline{E}_2 \times \Delta[1] & \xrightarrow[\ q_2 \circ \pi_1\ ]{} & B
\end{array}\ .
$$

The induced map H_1' factors through $\underline{E}_1$ because H_1 factors through E_1 as a retraction of q. Therefore, H_1' is a strong deformation retraction of $\underline{w}$. This means that $|\underline{w}|$ is a homotopy equivalence, i.e., that $\underline{w}$ is a weak equivalence in $\mathbf{Top}$. Moreover, H_1' exhibits q_1 as a retract of q_2 in the sense of ($\dagger$), and thus q_1 is a Kan fibration by Lemma 4.2.10.

$\square$

Corollary 5.4.5. *For any inaccessible cardinal $\lambda < \kappa$, the contextual category* $\mathbf{sSet}_{U_\lambda}$ *models* $\mathrm{CDTT} + \mathrm{Univ}$.

Awodey's conjecture

In closing, it is worth mentioning some subsequent generalizations of [14]. To this end, let us introduce a new kind of category.

Definition 5.4.6 (Reedy category). A *Reedy category* is a category $\mathscr{R}$ equipped with two wide subcategories R_+ and R_- along with a *degree function* $d : \mathrm{Ob}\,\mathscr{R} \to \alpha$ where α is an ordinal such that

   (i) every morphism in $\mathscr{R}$ factors as a map in R_- followed by a map in R_+,

   (ii) every non-identity map $a \to b$ in R_+ satisfies $d(a) < d(b)$, and

   (iii) every non-identity map $a \to b$ in R_- satisfies $d(b) < d(a)$.

Example 5.4.7.

1. Any ordinal α is a Reedy category with $R_+ \equiv \alpha$, R_- the discrete category on $\mathrm{Ob}\,\alpha$, and $d \equiv \mathrm{id}_\alpha$.

2. By Lemma 4.1.1(1), the simplex category Δ is Reedy with R_+ the wide subcategory on all monomorphisms, R_- the subcategory on all epimorphisms, and d the inclusion map $\mathrm{Ob}\,\Delta \hookrightarrow \omega$.
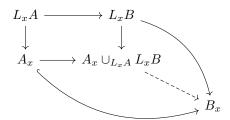
Also, note that the opposite of a Reedy category is a Reedy category with R_+ and R_- switched. In particular, Δ^{op} is a Reedy category.

Let $\mathscr{R}$ be a Reedy category and let G be a simplicial presheaf $\mathscr{R}^{\mathrm{op}} \to \mathbf{sSet}$ over $\mathscr{R}$. For any $r \in \mathrm{Ob}\,\mathscr{R}$, the *latching object of G over r* is the simplicial set

$$L_r G := \operatorname*{colim}_{f \in (\partial R_+/r)^{\mathrm{op}}} G_{\mathrm{dom}(f)}$$

where $\partial R_+/r$ denotes the full subcategory of the over category R_+/r on all objects except the identity map id_r. The universal property of colimits yields a unique map $L_r G \to G_r$.

Now, we say that $\mathscr{R}$ is an *elegant Reedy category* if for any monomorphism $A \hookrightarrow B$ in $[\mathscr{R}^{\mathrm{op}}, \mathbf{sSet}]$ and any $x \in \mathrm{Ob}\,\mathscr{R}$, the unique mediating map

is monic in $\mathbf{sSet}$. (This notion is due to [5].)

Example 5.4.8. The simplex category Δ is elegant Reedy.

Thanks to Proposition 5.2.3, we know that $[\mathscr{R}^{\mathrm{op}}, \mathbf{sSet}]$ can be presented by a type-theoretic model category for any small elegant Reedy category $\mathscr{R}$. Specifically, in [23], Michael Shulman modifies the construction of [14] to show that $\mathrm{CDTT} + \mathrm{Univ}$ can be interpreted in the model category $[\mathscr{R}^{\mathrm{op}}, \mathbf{sSet}]_{\mathtt{inj}}$. Since the trivial category $*$ is elegant Reedy and

$$[*^{\mathrm{op}}, \mathbf{sSet}] \cong [(* \times \Delta)^{\mathrm{op}}, \mathbf{Set}] \cong \mathbf{sSet},$$

it follows that [23] directly generalizes [14].

The general belief that compact-object classifiers model univalent universes leads to "Awodey's conjecture":

Every Grothendieck ∞-topos is presentable by a model category that models $\mathrm{CDTT} + \mathrm{Univ}$.

Assuming Conjecture 3.2.13, Shulman released a proof of Awodey's conjecture in April of 2019 [22]. Therefore, every theorem of homotopy type theory is true in any ∞-topos. In other words, a result in synthetic homotopy theory (including Univ) holds in any ∞-topos. In this sense, homotopy type theory serves as a formal language for reasoning within a number of general settings for algebraic topology at once.

A   Deductive systems

Definition A.0.1. A *deductive system* consists of the following data:

(a) a countable set $\mathcal{A}$ of symbols,

(b) a countable set $\mathcal{S}$ of (finite) strings over $\mathcal{A}$ called *expressions* or *raw terms*,

(c) a finite set $\mathcal{B}$ of positive integers,

(d) a finite set $\{\sigma_i\}_{i \in \mathcal{B}}$ where each σ_i is a subset of $\mathcal{S}^i$, and

(e) a finite set of ordered pairs called *inference rules*.

The set $\mathcal{S}$ is known as the *object language* of the deductive system (whereas the language of set theory is chosen as the metalanguage).

For any $n \in \mathcal{B}$, we say that an element of σ_n is a *judgment (of order n)*. By definition, every inference rule is a pair $(\{J_1, \ldots, J_n\}, J)$ where

- $n \in \mathbb{N}$ and,

- for all $1 \leq i \leq n$, J_i and J denote judgments.

This is represented graphically as

$$\text{Name} \; \frac{J_1 \quad \ldots \quad J_n}{J} \; .$$

We call J the *conclusion* and each J_i a *premise* of the rule. When $n = 0$, the inference rule is called an *axiom*.

The set of inference rules generates, via mutual recursion, a finite set $\{R_i\}_{i \in \mathcal{B}}$ of i-place relations on $\mathcal{S}$ as follows.

1. If J is the conclusion of an axiom and has order i, then $J \in R_i$.

2. If $(\{J_1, \ldots, J_n\}, J)$ is an inference rule, each J_i has order n_i, J has order k, and $J_i \in R_{n_i}$, then $J \in R_k$.

That is, $\{R_i\}$ is precisely the smallest family of relations closed under the inference rules.

Definition A.0.2. A *theorem* is an element of $\bigcup_{i \in \mathcal{B}} R_i$.

Definition A.0.3. A *derivation* (or a *derived rule*) is a finite rooted tree with the following properties.

- Each node is a judgment.

- For any non-leaf F with children $F_1, \ldots, F_k$, there is some inference rule $(\{F_1, \ldots, F_k\}, F)$.

Given a derivation, we say that the root is *derivable from* the leaves.

A theorem can be characterized recursively as either the conclusion of an axiom or a judgment derivable from a theorem.

Note A.0.4. Since each inference rule has only finitely many premises, each node of a proof tree has at most k children for some fixed $k \in \mathbb{N}$. Hence any proof tree can be viewed as a complete k-ary tree with nodes marked by inference rules. We can define the set of such trees inductively and thus perform so-called structural induction to prove properties about proof trees.

B Univalent group theory

In this appendix, we shall define some basic notions of group theory within our MLDTT. Our goal is to state, in a precise way, the fact that any two isomorphic groups are propositionally equal provided that Univ is true. In ordinary group theory, identifying two isomorphic groups is common practice but is technically an abuse of notation when they are not equal as sets. A virtue of homotopy type theory is that in it, such an abuse of notation becomes formally true.

We shall follow [20, Section 11.2] and adopt our informal notation from Section 2.

Definition B.0.1. We say that a (well-formed) type A is an *h-set* (or *set*) if there is some term of type
$$\texttt{is\_set}(A) \coloneqq \prod_{x,y:A} \texttt{is\_prop}(x \rightsquigarrow y).$$

Remark B.0.2. The axiom UIP asserts that every type is a set.

Now, we can encode the set-theoretic notion of a group in our MLDTT as follows.

Definition B.0.3. We say that a small type G is a *group* if there is some term of type

$$\texttt{is\_group}(G) \coloneqq \sum_{p:\texttt{is\_set}(G)} \sum_{e:G} \sum_{i:G\to G} \sum_{\mu:G\to(G\to G)}$$

$$\left(\prod_{(x,y,z:G)} \mu(\mu(x,y),z) \rightsquigarrow \mu(x,\mu(y,z)) \right) \times$$

$$\left(\prod_{(x:G)} \mu(e,x) \rightsquigarrow x \right) \times \left(\prod_{(x:G)} \mu(x,e) \rightsquigarrow x \right) \times$$

$$\left(\prod_{(x:G)} \mu(i:x,x) \rightsquigarrow e \right) \times \left(\prod_{(x:G)} \mu(x,i(x)) \rightsquigarrow x \right).$$

We may write xy for $\mu(x,y)$.

We require G to be a set so that it is invariant (up to propositional equality) under a different proof of, say, associativity of μ. Moreover, Voevodsky has proven that from Univ we can derive that $\texttt{is\_set}(X)$ is a mere proposition for any small type X. Hence the definition of G is invariant under a different proof of $\texttt{is\_set}(G)$.

Example B.0.4. We can turn the groupoid from Corollary 2.2.4 into a group just as we turn the fundamental groupoid into the fundamental group in classical topology.

Specifically, for any small type X and element $x : X$ such that $\texttt{is\_set}(x \rightsquigarrow x)$, define the *loop space of X at x* as the type $x \rightsquigarrow_X x$.

Definition B.0.5. If G and H are groups, then the type of *homomorphisms from G to H* is
$$\texttt{hom}(G,H) \coloneqq \sum_{f:G\to H} \prod_{x,y:G} f(xy) \rightsquigarrow_H f(x)f(y).$$

Example B.0.6. For any group G, the *identity homomorphism* is $\underline{\texttt{idmap}}_G \coloneqq (\texttt{idmap}_G, p)$ where $p(x,y) \coloneqq \texttt{refl}_{xy}$.

Definition B.0.7.

   1. Let $(h, p) : \mathtt{hom}(G, H)$ and $(k, q) : \mathtt{hom}(H, K)$. The *composition of k with h* is the term

$$k \underline{\circ} h := (k \circ h, p * q) : \mathtt{hom}(G, K)$$

   2. For any groups G and H, the type of *group isomorphisms from G to H* is

$$G \cong H := \sum_{h:\mathtt{hom}(G,H)} \sum_{k:\mathtt{hom}(H,G)} \left(k\underline{\circ}h \rightsquigarrow \underline{\mathtt{idmap}}_G \right) \times \left(h\underline{\circ}k \rightsquigarrow \underline{\mathtt{idmap}}_H \right).$$

Theorem B.0.8. *Assume* Univ*. Let $A, B : \mathsf{U}$ and suppose that $\mathsf{el}(A)$ and $\mathsf{el}(B)$ are groups. Define the function*

$$\mathtt{iso\_eq}_{A,B} : (A \rightsquigarrow_{\mathsf{U}} B) \to (\mathsf{el}(A) \cong \mathsf{el}(B))$$

inductively by

$$\mathtt{iso\_eq}_{A,B}(\mathsf{refl}_A) := \left(\underline{\mathtt{idmap}}_{\mathsf{el}(A)}, \left(\underline{\mathtt{idmap}}_{\mathsf{el}(A)}, \left(\mathsf{refl}_{\underline{\mathtt{idmap}}_{\mathsf{el}(A)}}, \mathsf{refl}_{\underline{\mathtt{idmap}}_{\mathsf{el}(A)}} \right) \right) \right).$$

Then $\mathtt{iso\_eq}_A$ is an equivalence.

C   Locally cartesian closed categories

Remark C.0.1. All categories in this section are assumed to be locally small.

Definition C.0.2. A symmetric monoidal category $(\mathscr{C}, \otimes)$ is *closed* if for every $X \in \mathrm{Ob}\,\mathscr{C}$, the functor $- \otimes X : \mathscr{C} \to \mathscr{C}$ has a right adjoint, denoted by

$$[X, -] : \mathscr{C} \to \mathscr{C}.$$

In this case, an object of the form $[X, Y]$ is called the *internal hom from X to Y*.

This generalizes the tensor-hom adjunction found in the category of R-modules.

Definition C.0.3 (Cartesian closed). A category $\mathscr{C}$ is *cartesian closed* if it has all finite products (hence a terminal object) and for any object X in $\mathscr{C}$, the functor $- \times X : \mathscr{C} \to \mathscr{C}$ has a right adjoint, denoted by

$$-^X : \mathscr{C} \to \mathscr{C}.$$

In this case, an object of the form Y^X is called the *exponential of Y by X*.

This means that $\mathscr{C}$ is cartesian closed exactly when it is closed as a cartesian monoidal category, with Y^X being the internal hom of X and Y.

Suppose that $\mathscr{C}$ is cartesian closed. For any object X of $\mathscr{C}$, there is some natural isomorphism $(\varphi_Z)_{Z \in \mathrm{Ob}(\mathscr{C}^{\mathrm{op}} \times \mathscr{C})}$ between the bifunctors

$$\mathrm{Hom}_{\mathscr{C}}\left(-, -^X \right) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}$$
$$\mathrm{Hom}_{\mathscr{C}}\left(- \times X, - \right) : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathbf{Set}.$$

For any $Y \in \mathrm{Ob}\,\mathscr{C}$, we call

$$\mathrm{ev}_{X,Y} := \varphi_{(Y^X,Y)}\left(\mathrm{id}_{Y^X}\right)$$

the *evaluation* morphism for Y^X. Note that $(\mathrm{ev}_{X,Y})_{Y \in \mathrm{Ob}\,\mathscr{C}}$ is precisely the counit of our chosen adjunction. This satisfies the following universal property.

Proposition C.0.4. *For any object Z and any morphism $f : Z \times X \to Y$, there is a unique morphism $\tilde{f} : Z \to Y^X$ such that $f = \mathrm{ev}_{X,Y} \circ \left(\tilde{f} \times \mathrm{id}_X\right)$.*

$$
\begin{array}{ccc}
Z & \qquad & Z \times X \\
\tilde{f}\downarrow & & \tilde{f}\times\mathrm{id}_X\downarrow \quad \searrow{\scriptstyle f} \\
Y^X & & Y^X \times X \xrightarrow[\mathrm{ev}_{X,Y}]{} Y
\end{array}
$$

Proof. It suffices to show that the bijection $\varphi_{(Z,Y)} : \mathrm{Hom}_{\mathscr{C}}(Z, Y^X) \to \mathrm{Hom}_{\mathscr{C}}(Z \times X, Y)$ is given by

$$\left(Z \xrightarrow{\psi} Y^X\right) \;\mapsto\; \left(Z \times X \xrightarrow{\psi \times \mathrm{id}_X} Y^X \times X \xrightarrow{\mathrm{ev}_{X,Y}} Y\right).$$

By naturality, we have that

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{C}}\left(Y^X, Y^X\right) & \xrightarrow{\mathrm{Hom}_{\mathscr{C}}\left(\psi^{\mathrm{op}},\,\mathrm{id}_{Y^X}\right)} & \mathrm{Hom}_{\mathscr{C}}\left(Z, Y^X\right) \\
\varphi_{(Y^X,Y)}\downarrow & & \downarrow\varphi_{(Z,Y)} \\
\mathrm{Hom}_{\mathscr{C}}\left(Y^X \times X, Y\right) & \xrightarrow[\mathrm{Hom}_{\mathscr{C}}\left(\psi^{\mathrm{op}}\times\mathrm{id}_X,\,\mathrm{id}_Y\right)]{} & \mathrm{Hom}_{\mathscr{C}}\left(Z \times X, Y\right)
\end{array}
\quad .
$$

In particular,

$$
\begin{aligned}
\mathrm{ev}_{X,Y} \circ (\psi \times \mathrm{id}_X) &= \mathrm{id}_Y \circ \varphi_{(Y^X,Y)}\left(\mathrm{id}_{Y^X}\right) \circ (\psi \times \mathrm{id}_X) \\
&= \varphi_{(Z,Y)}\left(\mathrm{id}_{Y^X} \circ \mathrm{id}_{Y^X} \circ \psi\right) \\
&= \varphi_{(Z,Y)}.
\end{aligned}
$$

$\square$

Terminology. The morphism $\tilde{f}$ is known as the *(exponential) transpose of f relative to φ*.

This is precisely the ordinary adjunct of f under φ.

Note C.0.5. Conversely, for any morphism $g : Z \to Y^X$, let

$$\bar{g} := \mathrm{ev}_{X,Y} \circ \left(g \times \mathsf{Id}_X\right).$$

By uniqueness of the transpose, we have that $\tilde{\bar{g}} = g$ and $\bar{\tilde{f}} = f$. Thus, we get a new adjunction $\mathrm{Hom}_{\mathscr{C}}\left(Z \times X, Y\right) \cong \mathrm{Hom}_{\mathscr{C}}\left(Z, Y^X\right)$ given by $f \mapsto \tilde{f}$ and $\bar{g} \leftarrow g$. We call such an adjunction *trans*.

Example C.0.6. We can think of the exponential Y^X as a generalized version of the set of functions from X to Y. In particular, both **Set** and **FinSet** are cartesian closed, where Y^X is taken to be $\{f \mid f : X \to Y\}$.

Let us review now the Yoneda lemma, through which we can establish a key class of additional examples of cartesian closed categories.

Notation. If $\mathscr{C}$ is a category, then $\widehat{\mathscr{C}}$ will denote the presheaf category $[\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$.

Lemma C.0.7 (Yoneda). *Let $\mathscr{C}$ be a category and let $\mathcal{Y} : \mathscr{C} \to \widehat{\mathscr{C}}$ denote the Yoneda embedding. In particular, for any $C \in \mathrm{Ob}\,\mathscr{C}$, $\mathcal{Y}_C := \mathcal{Y}(C)$ denotes the unique (set-valued) presheaf on $\mathscr{C}$ represented by C. If $F \in \mathrm{Ob}\,\widehat{\mathscr{C}}$, then the set map*

$$\varphi : \mathrm{Hom}_{\widehat{\mathscr{C}}}(\mathcal{Y}_C, F) \to F(C)$$
$$\varphi(f) = f_C\,(\mathrm{id}_C)$$

is a natural bijection in both C and F. In particular, $\mathcal{Y}$ is fully faithful.

For any $x \in F(C)$, the natural transformation $\varphi^{-1}(x)$ is given componentwise by

$$\left(\varphi^{-1}(x)\right)_D : \mathrm{Hom}_{\mathscr{C}}(D, C) \to F(D)$$
$$\left(\varphi^{-1}(x)\right)_D (g) = F(g)(x).$$

Therefore, from the natural one-to-one correspondence

$$x \in F(B) \longleftrightarrow \mathcal{Y}_B \xrightarrow{x} F,$$

we obtain another such correspondence

$$F(f)(x) \in F(A) \longleftrightarrow x \circ \mathcal{Y}(f)$$

for any map $f : A \to B$ in $\mathscr{C}$.

Turning now to a well-known consequence of the Yoneda lemma, consider any presheaf $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$. In order to prove our second corollary, define the *category $\int_{\mathscr{C}} F$ of elements of F* as follows. Its objects are precisely pairs (C, x) with $C \in \mathrm{Ob}\,\mathscr{C}$ and $x \in F(C)$, and its morphisms $(C, x) \to (C', x')$ are precisely morphisms $g : C \to C'$ in $\mathscr{C}$ such that

$$F(g)(x') = x.$$

Note that $\int_{\mathscr{C}} F$ is small whenever $\mathscr{C}$ is small.

Theorem C.0.8 (Density theorem). *Let $\mathscr{C}$ be a small category. For any $F \in \mathrm{Ob}\,\widehat{\mathscr{C}}$, there is a functor $G : J \to \mathscr{C}$ such that J is small and $\mathrm{colim}_{j \in J}\mathcal{Y}_{G_j} \cong F$.*[44]

Proof sketch. Take $\int_{\mathscr{C}} F$ as J. Consider the projection functor $\pi : \int_{\mathscr{C}} F \to \mathscr{C}$ given by $\pi(C, x) = C$ and $\pi(g) = g$ on objects and morphisms, respectively. For each element (C, x) of F, define $\alpha_{(C,x)} : \mathcal{Y}_{\pi(C,x)} \to F$ as the unique map $x : \mathcal{Y}_C \to F$ corresponding to x via the Yoneda lemma. This determines a cocone $\{\alpha_{(C,x)}\}$ under $\mathcal{Y} \circ \pi$. Further, it can be shown that this is colimiting. □

Lemma C.0.9. *Let $\mathscr{C}$ be any category and J be a small category. For any functors $A : J \to [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$ and $B : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$, there is a natural isomorphism*

$$\mathrm{colim}_j\,(A_j \times B) \cong \left(\mathrm{colim}_j A_j\right) \times B.$$

---

[44] [2, Proposition 8.10].

Proof. By definition of a colimit, we have a canonical cocone $\{\alpha_j : A_j \to \operatorname{colim}_j A_j\}_j$. Applying the functor $- \times B$ to this, we get another cocone

$$\left\{ \alpha_j \times \operatorname{id}_B : A_j \times B \to \left(\operatorname{colim}_j A_j \right) \times B \right\}_j .$$

By the universal property of colimits, this induces a unique morphism

$$\alpha : \operatorname{colim}_j (A_j \times B) \to \left(\operatorname{colim}_j A_j \right) \times B.$$

We want to show that α is a natural isomorphism. It suffices to show that each component α_C is a bijection. Since colimits in $\mathscr{C}$ are computed pointwise, we thus may assume wlog that $A_j, B \in \operatorname{Ob} \mathbf{Set}$ for any $j \in J$. Applying the fact that $\mathbf{Set}$ is cartesian closed, we obtain the following chain of bijections natural in $X \in \operatorname{Ob} \mathbf{Set}$:

$$
\begin{aligned}
\operatorname{Hom}_{\mathbf{Set}} \left(\operatorname{colim}_j (A_j \times B), X \right) &\cong \lim_j \operatorname{Hom}_{\mathbf{Set}} (A_j \times B, X) \\
&\cong \lim_j \operatorname{Hom}_{\mathbf{Set}} \left(A_j, X^B \right) \\
&\cong \operatorname{Hom}_{\mathbf{Set}} \left(\operatorname{colim}_j A_j, X^B \right) \\
&\cong \operatorname{Hom}_{\mathbf{Set}} \left(\left(\operatorname{colim}_j A_j \right) \times B, X \right) .
\end{aligned}
$$

Since the Yoneda embedding is fully faithful and thus reflects isomorphisms, it follows that $\operatorname{colim}_j (A_j \times B) \cong (\operatorname{colim}_j A_j) \times B$, as desired. $\qquad\square$

Lemma C.0.10. *If $\mathscr{C}$ is small, then $\widehat{\mathscr{C}}$ is cartesian closed.*

Proof. First of all, recall that $\widehat{\mathscr{C}}$ has all binary products where $P \times Q$ is computed pointwise for any $P, Q \in \operatorname{Ob} \widehat{\mathscr{C}}$, i.e., $(P \times Q)(C) = P(C) \times P(Q)$. Recall also that $\widehat{\mathscr{C}}$ has a terminal object, namely the constant functor at any singleton set. Hence it has all finite products.

It remains to show that $- \times P$ always has a right adjoint. For any presheaf Q on $\mathscr{C}$, define Q^P to be the presheaf $\operatorname{Hom}_{\widehat{\mathscr{C}}}(\mathcal{Y}_- \times P, Q)$ on $\mathscr{C}$. Given any $F \in \operatorname{Ob} \widehat{\mathscr{C}}$, apply Theorem C.0.8 to get a natural isomorphism $\operatorname{colim}_{j \in J} \mathcal{Y}_{G_j} \cong F$. Using the Yoneda lemma together with Lemma C.0.9, we thus get the following chain of bijections natural in (F, Q):

$$
\begin{aligned}
\operatorname{Hom}_{\widehat{\mathscr{C}}}(F \times P, Q) &\cong \operatorname{Hom}_{\widehat{\mathscr{C}}} \left(\left(\operatorname{colim}_{j \in J} \mathcal{Y}_{G_j} \right) \times P, Q \right) \\
&\cong \operatorname{Hom}_{\widehat{\mathscr{C}}} \left(\operatorname{colim}_{j \in J} \left(\mathcal{Y}_{G_j} \times P \right), Q \right) \\
&\cong \lim_j \operatorname{Hom}_{\widehat{\mathscr{C}}} \left(\mathcal{Y}_{G_j} \times P, Q \right) \\
&\cong \lim_j \operatorname{Hom}_{\widehat{\mathscr{C}}} \left(\mathcal{Y}_{G_j}, Q^P \right) \\
&\cong \operatorname{Hom}_{\widehat{\mathscr{C}}} \left(\operatorname{colim}_j \mathcal{Y}_{G_j}, Q^p \right) \\
&\cong \operatorname{Hom}_{\widehat{\mathscr{C}}} (F, Q^p) .
\end{aligned}
$$

Hence $\left(-\times P, -^{P}\right)$ is an adjoint pair, as required. $\qquad\square$

If $\mathscr{C}$ has all pullbacks, then any morphism $f : X \to Y$ in $\mathscr{C}$ induces a *base change* functor $f^{*} : \mathscr{C}/Y \to \mathscr{C}/X$ defined on objects and morphisms, respectively, by

$$(p : K \to Y) \mapsto \begin{pmatrix} X \times_Y K & \longrightarrow & K \\ {\scriptstyle f^{*}(p)} \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{pmatrix}$$

$$\begin{pmatrix} K & \xrightarrow{r} & K' \\ {\scriptstyle p} \searrow & & \swarrow {\scriptstyle q} \\ & Y & \end{pmatrix} \mapsto \begin{pmatrix} X \times_Y K & \xrightarrow{r^{*}} & X \times_Y K' \\ {\scriptstyle f^{*}(p)} \searrow & & \swarrow {\scriptstyle f^{*}(q)} \\ & X & \end{pmatrix}$$

where $f^{*}(r)$ is the unique morphism such that

$$f^{*}(p) \begin{pmatrix} \begin{array}{ccc} X \times_Y K & \longrightarrow & K \\ {\scriptstyle f^{*}(r)} \downarrow & & \downarrow {\scriptstyle r} \\ X \times_Y K' & \longrightarrow & K' \\ {\scriptstyle f^{*}(q)} \downarrow & & \downarrow {\scriptstyle q} \\ X & \xrightarrow{\ f\ } & Y \end{array} \end{pmatrix} p$$
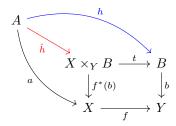
commutes.

Lemma C.0.11. *Let $\mathscr{C}$ be a category with pullbacks and $f : X \to Y$ be any morphism in $\mathscr{C}$. Define the dependent sum functor $\Sigma_f : \mathscr{C}/X \to \mathscr{C}/Y$ by post-composition with f. Then (Σ_f, f^{*}) is an adjoint pair.*

Proof. We must show that there is a natural isomorphism of bifunctors

$$\operatorname{Hom}_{\mathscr{C}/Y} (\Sigma_f(-), -) \to \operatorname{Hom}_{\mathscr{C}/X}(-, f^{*}(-)).$$

For each $(a, b) \in \operatorname{Ob}\left((\mathscr{C}/X)^{\mathrm{op}} \times (\mathscr{C}/Y)\right)$, define $\varphi_{(a,b)}$ by $(h : \Sigma_f(a) \to b) \mapsto \left(\hat{h} : a \to f^{*}(b)\right)$.

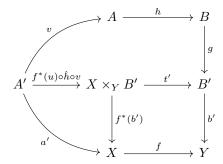The universal property of pullbacks implies that this is a bijection natural in (a, b). To see explicitly that it is natural, let $(a : A \to X, b : B \to Y)$ and $(a' : A' \to X, b' : B' \to Y)$ be objects of $(\mathscr{C}/X)^{\mathrm{op}} \times (\mathscr{C}/Y)$ and consider any morphism $(v, u) : (a, b) \to (a', b')$. We must prove that

$$\varphi_{(a',b')} (u \circ h \circ \Sigma_f(v)) = f^{*}(u) \circ \varphi_{(a,b)}(h) \circ v$$

for any $h : \Sigma_f(a) \to b$. Note that $\Sigma_f(v) = v$. By the universal property of pullbacks, it suffices to

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prove that

$$
\begin{array}{ccc}
 & A \xrightarrow{\ \ h\ \ } B \\
v \nearrow & & \downarrow g \\
A' \xrightarrow{f^*(u)\circ\hat{h}\circ v} X \times_Y B' \xrightarrow{\ t'\ } B' \\
 & \downarrow f^*(b') & \downarrow b' \\
a' \searrow & X \xrightarrow{\ \ f\ \ } Y
\end{array}
$$

commutes. We have that $h = t \circ \hat{h}$ by definition of $\hat{h}$. Further, by definition of $f^*(u)$, we have that $u \circ t = t' \circ f^*(u)$. Therefore,

$$
t' \circ f^*(u) \circ \hat{h} \circ v = u \circ t \circ \hat{h} \circ v = u \circ h \circ v,
$$

i.e., the top trapezoid commutes.
Next, note that

$$
\begin{aligned}
& f^*(b') \circ f^*(u) \circ \hat{h} \circ v \\
&= f^*(b) \circ \hat{h} \circ v \\
&= a \circ v \\
&= a'.
\end{aligned}
$$

Thus, the bottom left triangle commutes as well, so that the whole diagram commutes. $\qquad\square$

Example C.0.12. Any morphism $g : A \to X$ in **Set** corresponds uniquely to an X-indexed family of disjoint sets $\left(g^{-1}(x)\right)_{x \in X}$. Under this identification, the functor Σ_f is given by

$$
g \mapsto \left(\coprod_{x \in f^{-1}(y)} g^{-1}(x) \right)_{y \in Y}
$$

for any set function $f : X \to Y$.

We have established that f^* always has a left adjoint. If f^* always has a right adjoint and $\mathscr{C}$ has all finite limits (equivalently, $\mathscr{C}$ has a terminal object in addition to all pullbacks), then we obtain the following notion.

Definition C.0.13 (Locally cartesian closed category (LCCC)). A category $\mathscr{C}$ with finite limits is *locally cartesian closed (LCC)* if for each morphism $f : X \to Y$ in $\mathscr{C}$, the base change functor $f^* : \mathscr{C}/Y \to \mathscr{C}/X$ has a right adjoint, called the *dependent product* Π_f.

Example C.0.14. The category **Set** of sets is LCC. Indeed, for any set functions $f : X \to Y$ and $g : A \to X$, we have

$$
\Pi_f(g) = \left(\prod_{x \in f^{-1}(y)} g^{-1}(x) \right)_{y \in Y},
$$

which is precisely a Y-indexed family of sets of choice functions $f^{-1}(y) \to \bigcup_{x \in f^{-1}(y)} g^{-1}(x)$.

Proposition C.0.15. *Let* $f : X \to Y$ *be any map in a LCCC* $\mathscr{C}$*. There is a unique isomorphism*

$$(-)^f \cong \Pi_f \circ f^*$$

of right adjoint functors $\mathscr{C}/Y \to \mathscr{C}/Y$*.*

Proof. Note that the binary product functor $(-) \times f : \mathscr{C}/Y \to \mathscr{C}/Y$ is precisely the composite $\Sigma_f \circ f^*$. Hence both $\Pi_f \circ f^*$ and $(-)^f$ are right adjoint to $(-) \times f$. But right adjoints are unique up to unique adjunction-compatible isomorphism, and thus our proof is complete. $\square$

Proposition C.0.16 (Beck-Chevalley condition). *Suppose that*

$$
\begin{array}{ccc}
D & \xrightarrow{\ h\ } & C \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\
A & \xrightarrow[\ f\ ]{} & B
\end{array}
$$

is a pullback square in a LCCC $\mathscr{C}$*. Then there are isomorphisms*

$$\Sigma_k h^* \varphi \cong f^* \Sigma_g \varphi$$
$$\Pi_k h^* \varphi \cong f^* \Pi_g \varphi$$

natural in $\varphi \in \operatorname{Ob}\mathscr{C}/C$*.*

Proof. The first isomorphism follows immediately from the pasting law for pullbacks. From this, we obtain the second isomorphism. Indeed, we have a chain of isomorphisms

$$
\begin{aligned}
\operatorname{Hom}(\psi, f^* \Pi_g \varphi) &\cong \operatorname{Hom}(g^* \Sigma_f \psi, \varphi) \\
&\cong \operatorname{Hom}(\Sigma_h k^* \psi, \varphi) \\
&\cong \operatorname{Hom}(\psi, \Pi_k h^* \varphi)
\end{aligned}
$$

natural in $(\psi, \varphi) \in \operatorname{Ob}(\mathscr{C}/A \times \mathscr{C}/C)$. Our desired isomorphism now follows from the fact that the Yoneda embedding is conservative. $\square$

Notation. The terminal object of a category will always be denoted by 1.

Theorem C.0.17. *Suppose that* $\mathscr{C}$ *has a terminal object. Then* $\mathscr{C}$ *is LCC if and only if* $\mathscr{C}/X$ *is cartesian closed for every object* X *of* $\mathscr{C}$*.*

Proof.
($\Longrightarrow$) Let $X \in \operatorname{Ob}\mathscr{C}$. We must show that $\mathscr{C}/X$ is cartesian closed. Let $a : A \to X$ and $b : B \to X$ be morphisms in $\mathscr{C}$. Recall that $\mathscr{C}/X$ has all finite products, with binary products being pullbacks

$$
\begin{array}{ccc}
A \times_X B & \xrightarrow{\ \pi_2\ } & B \\
{\scriptstyle \pi_1}\downarrow & \searrow & \downarrow{\scriptstyle b} \\
A & \xrightarrow[\ a\ ]{} & X
\end{array}
$$

and the terminal object being $\mathrm{id}_X : X \to X$. To see that $- \times a$ has a right adjoint, note that it is the same as the composite functor

$$\mathscr{C}/X \xrightarrow{a^*} \mathscr{C}/A \xrightarrow{\Sigma_a} \mathscr{C}/X.$$

Since $\mathscr{C}$ is LCC, we have an adjoint triple $\Sigma_a \dashv a^* \dashv \Pi_a$. This yields an adjunction

$$\Sigma_a \circ a^* \dashv \Pi_a \circ a^*,$$

so that $- \times a$ has a right adjoint.

($\Longleftarrow$) Let us show that $\mathscr{C}$ has all finite limits. It suffices to show that $\mathscr{C}$ has a terminal object and all pullbacks. By assumption, it has the former. To see that is has the latter, note that each slice of $\mathscr{C}$ is cartesian closed and thus has all binary products, which are precisely pullback squares in $\mathscr{C}$. It follows that $\mathscr{C}$ has all pullbacks.

Next, let us show that Π_f exists for every morphism $f : X \to Y$ in $\mathscr{C}$. Define $\Pi_f : \mathscr{C}/X \to \mathscr{C}/Y$ on objects by mapping $a : A \to X$ to the pullback

$$
\begin{array}{ccc}
\Pi_f(a) & \longrightarrow & (f \circ a)^f \\
\downarrow & \lrcorner & \downarrow{\scriptstyle a^f} \\
\mathrm{id}_Y & \xrightarrow{\tilde{\tau}} & f^f
\end{array}
$$

in $\mathscr{C}/Y$ where $\tilde{\tau}$ denotes the transpose of the canonical isomorphism
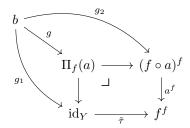
$$\tau : \mathrm{id}_Y \times f \xrightarrow{\cong} f$$

relative to *trans* (defined in Note C.0.5). We see that Π_f is functorial as the composite $U_f \circ \tilde{\tau}^* \circ -^f$ where U_f denotes a suitable forgetful functor that outputs just the object of the pullback.

It remains to exhibit a bijection

$$\mathrm{Hom}_{\mathscr{C}/Y}\left(b, \Pi_f(a)\right) \cong \mathrm{Hom}_{\mathscr{C}/X}\left(f^*(b), a\right)$$

natural in $(b : B \to Y, a : A \to Y) \in (\mathscr{C}/Y)^{\mathrm{op}} \times \mathscr{C}/X$. By the universal property of pullbacks, every $g \in \mathrm{Hom}_{\mathscr{C}/Y}(b, \Pi_f(a))$ naturally corresponds to a pair (g_1, g_2) of morphisms in $\mathscr{C}/Y$ such that

$$
\begin{array}{ccc}
b & \xrightarrow{\quad g_2 \quad} & \\
\downarrow{\scriptstyle g} & & \\
g_1 \searrow & \Pi_f(a) \longrightarrow (f \circ a)^f & \\
& \downarrow \quad \lrcorner \quad \downarrow{\scriptstyle a^f} & \\
& \mathrm{id}_Y \xrightarrow{\tilde{\tau}} f^f &
\end{array}
$$

commutes. But g_1 must equal b, so that g naturally corresponds to a morphism $g' : b \to (f \circ a)^f$ such that $a^f \circ g' = \widetilde{\mathrm{id}_Y} \circ b$. Furthermore, under the equivalence $(\mathscr{C}/X) \simeq (\mathscr{C}/Y)/f$, every $g \in$

$\mathrm{Hom}_{\mathscr{C}/X}(f^*(b), a)$ is precisely a morphism

$$b \times f \xrightarrow{\;g\;} f \circ a$$
$$\pi_X \searrow \quad \downarrow a$$
$$f$$

in $(\mathscr{C}/Y)/f$ where π_X is as in

$$B \times_Y X \xrightarrow{\;\pi_X\;} X$$
$$\pi_B \downarrow \qquad \downarrow f \quad .$$
$$B \xrightarrow{\;b\;} Y$$

Now, consider the product-exponential adjunction given by $trans$ in $\mathscr{C}/Y$. We want to show that $trans$ restricts to a bijection

$$\left\{ g' : b \to (f \circ a)^f \mid a^f \circ g' = \widetilde{\mathrm{id}_Y} \circ b \right\} \xrightarrow{\;\cong\;} \left\{ g : b \times f \to f \circ a \mid a \circ g = \pi_X \right\},$$

which must be natural. That is, we want to show that $a \circ trans_{b, f \circ a}(g') = \pi_X$. Since we have assumed that

$$b \xrightarrow{\;g'\;} (f \circ a)^f$$
$$b \downarrow \qquad \downarrow a^f$$
$$\mathrm{id}_Y \xrightarrow[\;\widetilde{\tau}\;]{} f^f$$

commutes, it follows from adjunction that

$$b \times f \xrightarrow{\;trans_{b, f \circ a}(g')\;} (f \circ a)$$
$$b \times \mathrm{id}_f \downarrow \qquad \qquad \downarrow a$$
$$\mathrm{id}_Y \times f \xrightarrow[\;trans_{\mathrm{id}_{Y,f}}(\widetilde{\tau})\;]{} f$$

does as well. But $trans_{\mathrm{id}_{Y,f}}(\widetilde{\tau}) = \widetilde{\widetilde{\tau}} = \tau$, and $\tau \circ (b \times \mathrm{id}_f)$ equals the projection $b \times f \xrightarrow{\;\pi_f\;} f$. Since π_f is precisely π_X, we have that $\pi_X = \tau \circ (b \times \mathrm{id}_f) = a \circ trans_{b, f \circ a}(g')$, as desired. $\qquad\square$

Corollary C.0.18. *Let $\mathscr{C}$ be LCC.*

1. *$\mathscr{C}$ is cartesian closed.*

2. *Every slice $\mathscr{C}/A$ of $\mathscr{C}$ is LCC.*

Proof.

1. Simply observe that $\mathscr{C} \cong \mathscr{C}/1$.

2. It is clear that $\mathscr{C}/X \simeq (\mathscr{C}/A)/a$ for every $X \in \mathrm{Ob}\,\mathscr{C}$ and every morphism $a : X \to A$ in $\mathscr{C}$. Hence every slice of $\mathscr{C}/A$ is cartesian closed, so that $\mathscr{C}/A$ is LCC.

$\square$

Corollary C.0.19. *If $\mathscr{C}$ is small, then $\widehat{\mathscr{C}}$ is LCC.*

Proof. Suppose that $\mathscr{C}$ is small. Then $\int_{\mathscr{C}} P$ is also small for any presheaf P on $\mathscr{C}$. By Lemma C.0.10, it suffices to show that $\widehat{\mathscr{C}}/P \simeq \widehat{\int_{\mathscr{C}} P}$. Let us construct such an equivalence.

Define the functor $\varphi : \widehat{\mathscr{C}}/P \to \widehat{\int_{\mathscr{C}} P}$ as follows. For each morphism $f : F \to P$ in $\widehat{\mathscr{C}}$, define the presheaf $\varphi(f)$ on $\int_{\mathscr{C}} P$ by mapping each object (C, x) to the fiber product

$$
\begin{array}{ccc}
\overbrace{F(C) \times_{P(C)} \{x\}}^{f_C^{-1}(x) \times \{x\}} & \longrightarrow & \{x\} \\
\downarrow & & \uparrow \\
F(C) & \xrightarrow[\quad f_C \quad]{} & P(C)
\end{array}
$$

and each morphism $a : (C, x) \to (C', x')$ to the set function given by $(y, x') \mapsto (F(a)(y), x)$. Also, for each morphism

$$
\begin{array}{ccc}
F & \xrightarrow{\ h\ } & G \\
& {\scriptstyle f}\searrow & \downarrow{\scriptstyle g} \\
& & P
\end{array}
$$

in $\widehat{\mathscr{C}}/P$, define the natural transformation $\varphi(h) : \varphi(f) \to \varphi(g)$ componentwise by

$$
\varphi(h)_{C,x} : f_C^{-1}(x) \times \{x\} \to g_C^{-1}(x) \times \{x\}
$$
$$
(y, x) \mapsto (h_C(y), x).
$$

For the reverse direction, define the functor $\pi : \widehat{\int_{\mathscr{C}} P} \to \widehat{\mathscr{C}}/P$ as follows. For each presheaf f on $\int_{\mathscr{C}} P$, first define the presheaf $R_f : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ on objects by $C \mapsto \coprod_{x \in P(C)} f(C, x)$ and on morphisms by

$$
(a : C \to C') \mapsto ((x, y) \mapsto (P(a)(x), f(a)(y))),
$$

where a is also a morphism $(C, P(a)(x)) \to (C', x)$ in $\int_{\mathscr{C}} P$. Now, define the natural transformation $\pi(f) : R_f \to P$ componentwise by $\pi(f)_C(x, y) = x$.

Also, for each morphism $K \coloneqq \left(k_{(C,x)}\right) : f \to g$ in $\widehat{\int_{\mathscr{C}} P}$, define the natural transformation $\pi(K) : R_f \to R_g$ componentwise by

$$\pi(K)_C(x,y) = \left(x, k_{(C,x)}(y)\right).$$

Clearly, $\pi(g) \circ \pi(K) = \pi(f)$, so that $\pi(K)$ is indeed a morphism in $\widehat{\mathscr{C}}/P$.

Finally, it is easy yet tedious to check that $\varphi \circ \pi \cong \operatorname{id}_{\widehat{\int_{\mathscr{C}} P}}$ and $\pi \circ \varphi \cong \operatorname{id}_{\widehat{\mathscr{C}}/P}$, and we won't do so here. $\qquad\square$

Example C.0.20.

1. **sSet**, the category of simplicial sets.

2. $[\mathscr{C}^{\mathrm{op}}, \mathbf{sSet}] \cong [(\mathscr{C} \times \varDelta)^{\mathrm{op}}, \mathbf{Set}]$, the category of *simplicial presheaves* over a small category $\mathscr{C}$.

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