

Abstract

These notes are based on Tony Pantev’s “Algebra II” lectures given at UPenn. Any mistake in what follows is my own.

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1 Injective and flat modules

1.1 Lecture 1

Proposition 1. An R -module M is injective if and only if we can fill any injectivity diagram of ideal type, i.e.,

$$\begin{array}{ccc} & & M \\ & \nearrow & \uparrow \\ 0 & \longrightarrow \mathfrak{a} & \longrightarrow R \end{array}$$

where \mathfrak{a} is an ideal in R .

Proof. The (\implies) direction is obvious.

(\impliedby) Let

$$\begin{array}{ccc} & & M \\ & \nearrow \varphi & \\ 0 & \longrightarrow X' & \longrightarrow X \end{array}$$

be an injectivity diagram of R -modules and define

$$S = \{(A, \xi) \mid X' \subset A \subset X, \xi : A \rightarrow M, \xi|_{X'} = \varphi\}.$$

By Zorn's lemma, there is some maximal element (N, ψ) of S . Suppose, for contradiction, that $X \neq N$. Pick any $x \in X \setminus N$. We have the ideal

$$\mathfrak{a} := \{a \in R : ax \in N\}$$

in R . Define the R -module morphism $\theta : \mathfrak{a} \rightarrow M$ by $a \mapsto \psi(ax)$. By hypothesis, we get the following commutative diagram.

$$\begin{array}{ccc} & & M \\ & \nearrow \theta & \uparrow \tilde{\theta} \\ 0 & \longrightarrow \mathfrak{a} & \longrightarrow R \end{array}$$

Define the R -submodule $\tilde{N} = \langle N, x \rangle$. We can write any $z \in \tilde{N}$ as $z = y + ax$ for some $y \in N$ and some $a \in R$. Define $\tilde{\psi} : \tilde{N} \rightarrow M$ by $y + ax \mapsto \psi(y) + \tilde{\theta}(a)$. To see that this is well-defined, let $y + ax = y' + a'x$. Then $(y - y') = (a' - a)x$, so that

$$\psi(y - y') = \psi((a' - a)x) = \tilde{\theta}(a' - a) = \tilde{\theta}(a') - \tilde{\theta}(a).$$

This implies that $\tilde{\psi}$ is a well-defined homomorphism. But then $(\tilde{N}, \tilde{\psi}) > (N, \psi)$, a contradiction. \square

Aside. The categorical dual P^{op} of this recognition principle for injectivity expresses a recognition principle for projectivity, namely that for any R -module M , ideal $I \subset R$, and homomorphism $\varphi : M \rightarrow R/I$, we can fill the diagram

$$\begin{array}{ccc} M & & \\ \downarrow & \searrow \varphi & \\ R & \twoheadrightarrow R/I & \longrightarrow 0 \end{array} \quad (*)$$

if and only if M is projective. This is equivalent to saying that M is projective if and only if the obvious group map $\text{Hom}_R(M, R) \rightarrow \text{Hom}_R(M, R/I)$ is surjective. But then P^{op} is precisely an affirmative answer to what is known as “Faith’s problem on R -projectivity,” which Trlifaj (2017) proved to be undecidable in $ZFC + GCH$. Therefore, both P^{op} and $\neg(P^{op})$ are consistent with $ZFC + GCH$.

Corollary 1.

1. If R is an integral domain, then any injective R -module M is divisible.
2. If R is a PID, then M is injective if and only if it is divisible.

Proof.

1. Given any $a \in R$, we want to show that the homomorphism $\text{mult}_a : M \rightarrow M$ given by $x \mapsto ax$ is surjective. The assumption that R is an integral domain entails that $\text{mult}_a : R \rightarrow R$ is injective. Note that $\mathfrak{a} := \text{mult}_a(R)$ is an ideal in R , giving the short exact sequence

$$0 \rightarrow \mathfrak{a} \rightarrow R \rightarrow R/\mathfrak{a} \rightarrow 0.$$

By assumption, $\text{Hom}_R(-, M)$ is exact, so that the sequence

$$0 \rightarrow \text{Hom}(R/\mathfrak{a}, M) \rightarrow \text{Hom}_R(R, M) \rightarrow \text{Hom}_R(\mathfrak{a}, M) \rightarrow 0$$

is exact. Since R and \mathfrak{a} are free R -modules of rank 1, it follows that $\text{Hom}_R(R, M) \cong M \cong \text{Hom}_R(\mathfrak{a}, M)$. This makes the sequence

$$0 \rightarrow \text{Hom}(R/\mathfrak{a}, M) \rightarrow M \xrightarrow{\text{mult}_a} M \rightarrow 0$$

exact. In particular mult_a is surjective.

2. (\Leftarrow) Suppose that M is divisible and R is a PID. We want to fill the injectivity diagram

$$\begin{array}{ccccc} & & & & M \\ & & & \nearrow \varphi & \uparrow \psi \\ 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & R \end{array}.$$

where \mathfrak{a} is an ideal in R . We have that $\mathfrak{a} = (a)$. Therefore, the short exact sequence

$$0 \rightarrow \mathfrak{a} \rightarrow R \rightarrow R/\mathfrak{a} \rightarrow 0$$

is isomorphic to $0 \rightarrow R \xrightarrow{\text{mult}_a} R \rightarrow R/\mathfrak{a} \rightarrow 0$. Since M is divisible, we know that $M \xrightarrow{\text{mult}_a} M \rightarrow 0$ is exact. Apply $\text{Hom}_R(-, M)$ to get the sequence

$$\text{Hom}_R(R, M) \xrightarrow{(-) \circ \text{mult}_a} \text{Hom}_R(R, M) \rightarrow 0,$$

which is isomorphic $M \xrightarrow{\text{mult}_a} M \rightarrow 0$. This shows that $\text{Hom}_R(R, M) \xrightarrow{(-) \circ \text{mult}_a} \text{Hom}_R(R, M)$ is surjective. It follows that φ can be lifted to some $\psi : R \rightarrow M$.

□

1.2 Lecture 2

Corollary 2. Any abelian group is injective if and only if it's divisible.

Corollary 3. If R is a PID and M is an injective R -module, then every quotient of M is injective.

Proof. This follows from the fact that any quotient of a divisible group is divisible.

□

Example 1.

1. \mathbb{Q}/\mathbb{Z} is injective.
2. S^1 is injective.
3. Any non-trivial finitely generated abelian group G is never injective.

Proof. It suffices to show that G is never divisible. There exists a maximal proper subgroup $H \leq G$. Then G/H is a simple abelian group, so that $G/H \cong C_p$ for some prime p . If G is divisible, then so must G/H . But C_p is not divisible, a contradiction. \square

Theorem 1. (Baer embedding) If R is a ring, then every module embeds into an injective module.

Corollary 4. For any R -module M , we can find an injective resolution

$$0 \rightarrow M \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_k \rightarrow \cdots$$

Proof. We want to invent a duality operation that will convert $R\text{-Mod}^{\text{op}}$ to $R^{\text{op}}\text{-Mod}$ and then use projective objects in $R^{\text{op}}\text{-Mod}$. If T is an abelian group, then the functor

$$\mathbf{Ab} \xrightarrow{\text{Hom}_{\mathbf{Ab}}(-, T)} \mathbf{Ab}^{\text{op}}$$

will reverse arrows. The choice of T that ends up working is precisely \mathbb{Q}/\mathbb{Z} .

Claim 1. Let $\text{Hom}_{\mathbf{Ab}}(-, \mathbb{Q}/\mathbb{Z}) := (-)^D$. Note that for any abelian group A , we have a canonical homomorphism $\epsilon_A : A \rightarrow A^{DD}$ given by $a \mapsto ([\varphi : A \rightarrow \mathbb{Q}/\mathbb{Z}] \mapsto \varphi(a))$. Then ϵ_A is injective.

Proof. We need to show that if $a \in A$ is nonzero, then we can find some homomorphism $f : A \rightarrow \mathbb{Q}/\mathbb{Z}$ such that $f(a) \neq 0$.

Case 1: Suppose that $|a| = n < \infty$. Then define the homomorphism $\varphi : (a) \rightarrow \mathbb{Q}/\mathbb{Z}$ by $a \mapsto [\frac{1}{n}]$. Since \mathbb{Q}/\mathbb{Z} is divisible in \mathbf{Ab} , it is also injective. Thus, we may find some map ψ such that

$$\begin{array}{ccc} & & \mathbb{Q}/\mathbb{Z} \\ & \nearrow \varphi & \uparrow \psi \\ 0 \longrightarrow (a) & \hookrightarrow & A \end{array}$$

commutes. This makes $\psi(a) \neq 0$, as required.

Case 2: If (a) has infinite order, then define $\varphi : (a) \rightarrow \mathbb{Q}/\mathbb{Z}$ by $a \mapsto \frac{1}{2}$ and apply a similar argument to Case 1. \square

Note 1. The duality functor $(-)^D$ extends to a functor $(-)^D : R^{\text{op}}\text{-Mod} \rightarrow R\text{-Mod}^{\text{op}}$ that is compatible with forgetting the module structure. Indeed, if M is a left module over R^{op} , then its module structure is given by a collection of maps $\{\text{mult}_a : M \rightarrow M \mid a \in R\}$. Note that $\text{mult}_a \circ \text{mult}_b = \text{mult}_{a \cdot R^{\text{op}} b} = \text{mult}_{b \cdot_R a}$. For each $a \in R$, define $\underline{\text{mult}}_a(\varphi) = \varphi \circ \text{mult}_a$. Then the abelian group M^D has an R -module structure given by $\underline{\text{mult}}_a : M^D \rightarrow M^D$, which clearly satisfies $\underline{\text{mult}}_{ab} = \underline{\text{mult}}_a \circ \underline{\text{mult}}_b$.

Lemma 1. If M is a projective R^{op} -module, then M^D is an injective R -module.

Proof. Suppose that M is a projective R^{op} -module and consider the injectivity diagram

$$\begin{array}{ccc} & & M^D \\ & \nearrow \varphi & \\ 0 \longrightarrow X' & \longrightarrow & X \end{array}$$

of R -modules. We want to lift $\varphi : X' \rightarrow M^D$ to a map $\psi : X \rightarrow M^D$. Apply $(-)^D$ to get the commutative diagram

$$\begin{array}{ccc} & & M^{DD} \\ & \nwarrow \varphi^D & \\ 0 \longleftarrow (X')^D & \longleftarrow & X^D \end{array}$$

where the bottom row is exact because \mathbb{Q}/\mathbb{Z} is injective.

Exercise 1. Show that $\epsilon_M : M \rightarrow M^{DD}$ is a map of R^{op} -modules.

We now have the following projectivity diagram of R^{op} -modules.

$$\begin{array}{ccccc} M & & & & \\ & \searrow \epsilon_M \circ \varphi^D & & & \\ X^D & \longrightarrow & (X')^D & \longrightarrow & 0 \end{array}$$

By assumption, we may fill this diagram with some map $\psi : M \rightarrow X^D$. This induces the map $\psi^D : X^{DD} \rightarrow M^D$. Note that $(\epsilon_M)^D \circ \varphi^{DD} = \psi^D \circ i^{DD}$ where $i : X' \hookrightarrow X$. But $i^{DD} \downarrow_{X'} = i$ and $\varphi^{DD} \downarrow_{X'} = \varphi$, so that

$$\psi^D \circ i = (\epsilon_M)^D \circ \varphi = \varphi$$

on X' . It follows that

$$\begin{array}{ccccc} & & M^D & & \\ & \nearrow \varphi & \uparrow \psi^D \circ \epsilon_X & & \\ 0 & \longrightarrow & X' & \longrightarrow & X \end{array}$$

commutes. □

There is some surjection $\bigoplus_{j \in J} R \rightarrow M^D$. Therefore, we have a sequence of embeddings

$$M \hookrightarrow M^{DD} = \text{Hom}_{\mathbb{Z}}(M^D, \mathbb{Q}/\mathbb{Z}) \hookrightarrow \text{Hom}_{\mathbb{Z}}\left(\bigoplus_{j \in J} R, \mathbb{Q}/\mathbb{Z}\right) = \underbrace{\left(\bigoplus_{j \in J} R\right)^D}_{\text{injective by Lemma 1}}.$$

□

Definition. Given two R -modules M and N , the *additive invariants* of M and N are the abelian groups

$$\text{Ext}_R^i(M, N) := H^i(\text{Hom}_R(P^\bullet, N))$$

indexed by \mathbb{N} where P^\bullet is a projective resolution of M .

Proposition 2.

1. $\text{Ext}_R^i(M, N)$ is independent of the choice of projective resolution. This follows from the fact that any two projective resolutions are chain homotopic.
2. $\text{Ext}_R^i(M, N) = H^i(\text{Hom}_R(M, I_\bullet))$ for any injective resolution I_\bullet of N .

Lemma 2.

1. $\text{Ext}_R^0(M, N) = \text{Hom}_R(M, N)$
2. $\text{Ext}_R^1(M, N) = (\text{the group of isomorphism classes of extensions of } N \text{ by } M \text{ in } R\text{-}\mathbf{Mod})$.

Proof. Let

$$\dots \xrightarrow{\partial_1} P^1 \xrightarrow{\partial_0} P^0 \xrightarrow{\epsilon} M \rightarrow 0$$

be a projective resolution and let

$$(\xi) : 0 \rightarrow N \xrightarrow{f} T \xrightarrow{g} M \rightarrow 0$$

be a short exact sequence of R -modules. Note that $\text{Hom}_R(P^k, -)$ is exact for each $k \geq 0$. Therefore, the sequence

$$0 \rightarrow \text{Hom}_R(P^k, N) \xrightarrow{f_k} \text{Hom}_R(P^k, T) \xrightarrow{g_k} \text{Hom}_R(P^k, M) \rightarrow 0$$

is exact where $f_k := f \circ (-)$ and $g_k := g \circ (-)$. Letting $d_i := (-) \circ \partial_i$, we get *short exact sequences of complexes* constituting the columns of

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Hom}_R(P^0, N) & \xrightarrow{f_0} & \text{Hom}_R(P^0, T) & \xrightarrow{g_0} & \text{Hom}_R(P^0, M) \longrightarrow 0 \\ & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 \\ 0 & \longrightarrow & \text{Hom}_R(P^1, N) & \xrightarrow{f_1} & \text{Hom}_R(P^1, T) & \xrightarrow{g_1} & \text{Hom}_R(P^1, M) \longrightarrow 0 \\ & & \downarrow d_1 & & \downarrow d_1 & & \downarrow d_1 \\ & & \vdots & & \vdots & & \vdots \end{array}$$

By definition, $\text{Ext}_R^i(M, N) = \underbrace{\ker d_i / \text{im } d_{i-1}}_{\text{for the first column}}$. Since $\text{Hom}_R(-, M)$ is left-exact and $P^1 \xrightarrow{\partial_1} P^0 \xrightarrow{\epsilon} M$ is exact, we also have the exact sequence

$$0 \rightarrow \text{Hom}_R(M, M) \xrightarrow{(-) \circ \epsilon} \text{Hom}_R(P^0, M) \xrightarrow{d_0} \text{Hom}_R(P^1, M).$$

Let $\psi \in \text{Hom}_R(P^0, M)$ satisfy $d_0(\psi) = 0$. Then $\psi = \varphi \circ \epsilon$ for some unique $\varphi : M \rightarrow M$. Since g_0 is surjective, there exists $\alpha \in \text{Hom}_R(P^0, T)$ such that $g_0(\alpha) = \psi = \varphi \circ \epsilon$. This implies that

$$g_1(d_0(\alpha)) = d_0(g_0(\alpha)) = d_0(\psi) = 0.$$

It follows that $d_0(\alpha) \in \ker g_1 = \text{im } f_1$, so that $d_0(\alpha) = f_1(\beta)$ for some $\beta : P^1 \rightarrow N$. Since $f_2(d_1(\beta)) = d_1(f_1(\beta)) = d_1(d_0(\alpha)) = 0$, the fact that f_2 is injective makes $d_1(\beta) = 0$. Hence $\beta \in \ker d_1$, and $[\beta] \in \text{Ext}_R^1(M, N)$

Exercise 2. Show that $\psi \mapsto [\beta]$ is well-defined, i.e., that $[\beta]$ is independent of α .

This defines a map of abelian groups $\delta_\xi : \text{Hom}_R(M, M) \rightarrow \text{Ext}_R^1(M, N)$ given by $\varphi \mapsto [\beta]$. Now, define the homomorphism

$$e : \text{Ext}_R(M, N) \rightarrow \text{Ext}_R^1(M, N), \quad (\xi) \mapsto \delta_\xi(\text{id}_M).$$

Claim 2. Apply $\text{Hom}_R(M, -)$ to (ξ) to get the exact sequence

$$0 \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, T) \rightarrow \text{Hom}_R(M, M).$$

We can extend this sequence to a long exact sequence of abelian groups

$$0 \rightarrow \text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M, T) \rightarrow \text{Hom}_R(M, M) \xrightarrow{\delta_\xi} \text{Ext}_R^1(M, N) \rightarrow \text{Ext}_R^1(M, T) \rightarrow \text{Ext}_R^1(M, M).$$

Exercise 3. Show that if (ξ) is split, then $\delta_\xi(\text{id}_M) = 0$.

This implies that e is injective. [[How?]] We need to show that it is surjective as well. Suppose that $\gamma \in \text{Ext}_R^1(M, N)$ and let I_\bullet be an injective resolution of N . Apply $\text{Hom}_R(M, -)$ to I_\bullet to get

$$0 \rightarrow \text{Hom}_R(M, N) \xrightarrow{\nu} \text{Hom}_R(M, I_0) \xrightarrow{d_0} \text{Hom}_R(M, I_1) \xrightarrow{d_1} \dots$$

(where we have abused the notation d_i). By Proposition 2(2), we have that $\gamma = [f]$ for some $f \in \ker d_1$. Note that $f : M \rightarrow \ker \partial_1 = \text{im } \partial_0$, giving

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & I_0 & \xrightarrow{\partial_0} & \text{im } \partial_0 \longrightarrow 0 \\ & & & & & \uparrow f & \\ & & & & & M & \end{array}$$

where the top row is exact. Take the pullback of ∂_0 and f to obtain T such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & I_0 & \xrightarrow{\partial_0} & \text{im } \partial_0 \longrightarrow 0 \\ & & \parallel & & \uparrow & \uparrow f & \\ 0 & \longrightarrow & N & \longrightarrow & T & \longrightarrow & M \longrightarrow 0 \end{array}$$

Exercise 4. Show that the map $\rho : \text{Ext}_R^1(M, N) \rightarrow \text{Ext}_R(M, N)$ given by $\gamma \mapsto \xi$ is independent of the choice of f . Also show that ρ is inverse to e .

□

1.3 Lecture 3

Remark 1. Let N be a right R -module and M an R -module. Then $N \otimes_R M \in \text{ob}(\mathbf{Ab})$ represents the functor $B_{M,N} : \mathbf{Ab} \rightarrow \mathbf{Ab}$ given by $A \mapsto \{f : M \times N \rightarrow A \mid f(ax, y) = f(x, ay)\}$.

Definition. Let N be a right R -module and M an R -module. Let $x_1, \dots, x_n \in M$.

1. A *relation of the x_i 's with coefficients in R* is a list of scalars $a_1, \dots, a_n \in R$ such that $\sum_{i=1}^n a_i x_i = 0$.
2. A *relation of the x_i 's with coefficients in N* is a list of elements $y_1, \dots, y_n \in N$ such that

$$\sum_{i=1}^n y_i \otimes x_i = 0.$$

Since $R \otimes_R M \cong M$, the last definition is a special case of this one.

Remark 2. Let

$$\begin{aligned} a_1 &:= (a_{11}, \dots, a_{1n}) \\ a_2 &:= (a_{21}, \dots, a_{2n}) \\ &\vdots \\ a_m &:= (a_{m1}, \dots, a_{mn}). \end{aligned}$$

be relations of x_1, \dots, x_n with coefficients in R . Let $(z_1, \dots, z_m) \in N^m$. If A denotes the matrix (a_{ij}) , then $y = A^t z \in N^n$ is a relation with coefficients in N .

Definition. A relation y with coefficients in N *follows from R -relations* if y is of the form $A^t z$ for some z and some A matrix of relations in R .

Lemma 3. A right R -module N is flat if and only if for any R -module M and any $x_1, \dots, x_n \in M$, every N -relation among the x_i follows from R -relations.

Proof.

(\implies) We have a module homomorphism $\varphi : R^n \rightarrow M$ given by $(a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_i x_i$. Then $K := \ker \varphi = \{(r_1, \dots, r_n) \in R^n \mid (r_1, \dots, r_n) \text{ is a relation of the } x_i\text{'s in } R\}$. We have an exact sequence

$$0 \rightarrow K \xrightarrow{i} R^n \xrightarrow{\varphi} M.$$

If N is flat, then $N \otimes_R (-)$ is exact, so that

$$0 \rightarrow N \otimes_R K \xrightarrow{\tilde{i}} N^n \xrightarrow{\tilde{\varphi}} N \otimes_R M$$

is exact. Thus, $\ker \tilde{\varphi} = (N\text{-relations}) = N \otimes_R K$.

(\impliedby) Let $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ be a short exact sequence of R -modules. Since $N \otimes_R (-)$ is right exact, it suffices to show that $N \times_R M' \xrightarrow{\text{id}_N \otimes f} N \otimes_R M$ is injective. Let $z \in \ker \text{id}_N \otimes f$. Then $z = \sum_{i=1}^n y_i \otimes z_i$. We know that $\sum_{i=1}^n y_i \otimes f(z_i) = \text{id}_N \otimes f(z) = 0$. Hence (y_1, \dots, y_n) is an N -relation among the $f(z_i) \in M$. This shows that there exist $(a_i^j) \in R$ where $i = 1, \dots, n$ and $j = 1, \dots, m$ and elements $v_1, \dots, v_m \in N$ such that $y_i = \sum_{j=1}^m v_j a_i^j$. Therefore, $\sum_{i=1}^n a_i^j f(z_i) = 0$ for each j . But $0 = \sum_{i=1}^n a_i^j f(z_i) = f(\sum_{i=1}^n a_i^j z_i)$. As f is injective, it follows that $\sum_{i=1}^n a_i^j z_i = 0$ for each j . We compute

$$\sum_{i=1}^n y_i \otimes z_i = \sum_{i=1}^n \sum_{j=1}^m (v_j a_i^j) \otimes z_i = \sum_{j=1}^m v_j \otimes \left(\sum_{i=1}^n a_i^j z_i \right) = \sum_{j=1}^m (v_j \otimes 0) = 0.$$

□

Corollary 5.

1. Any free module is flat.
2. Any colimit of flat modules is flat.
3. Any direct summand of a free module is flat, so that any projective module is flat.
4. Any colimit of projective modules is flat.

2 Localization

2.1 Lecture 4

Remark 3. Let R be a commutative ring. Given $x \in R$, when can we make x multiplicatively invertible, perhaps in a new ring? This is a question of representability. We have a functor $\Phi_x : \mathbf{CommRing} \rightarrow \mathbf{Set}$ given by $B \mapsto \{\varphi : R \rightarrow B \mid \varphi(x) \in B^\times\} \subset \text{Hom}_{\mathbf{CommRing}}(R, B)$. We are asking whether or not Φ_x is representable. That is, we want to find some pair (R_x, h) where R_x is a commutative ring and $h : R \rightarrow R_x$ is a morphism such that $h(x) \in (R_x)^\times$ and if $\varphi : R \rightarrow B$ with $\varphi(x) \in B^\times$, then $\varphi = \underline{\varphi} \circ h$ for some map $\underline{\varphi} : R_x \rightarrow B$.

In general, we can consider a set S of nonzero elements and ask for a universal way of making them invertible. But if we make S invertible, then we will also make

$$\underbrace{\text{cl}(S)}_{\text{multiplicative closure}}$$

invertible.

Definition. Any $S \subset R$ is called *multiplicatively closed* if $0 \notin S$, $1 \in S$, and $x, y \in S \implies xy \in S$.

Given a multiplicatively closed subset $S \subset R$, find a universal way of inverting every element of S . Equivalently, find a ring representing Φ_S . Equivalently, find a pair $(S^{-1}R, h)$ where $h : R \rightarrow S^{-1}R$ such that $h(S) \subset (S^{-1}R)^\times$ and any $\varphi : R \rightarrow B$ with $\varphi(S) \subset B^\times$ has $\varphi = \underline{\varphi} \circ h$ for some map $\underline{\varphi} : S^{-1}R \rightarrow B$. We call the pair $(S^{-1}R, h)$ the *localization of R along S* .

Formally adjoin to R fractions with numerator in R and denominator in S . Consider the set $(R \times S, \sim)$ where $(a, s) \sim (b, t)$ if $u(at - bs) = 0$ for some $u \in S$. Set $S^{-1}R := R \times S / \sim$. Let $\frac{a}{s} := [(a, s)]$. Define

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st}$$

and

$$\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}.$$

Then $S^{-1}R$ becomes a ring with unity $\frac{1}{1}$. Also, we see that $h : R \rightarrow S^{-1}R$ given by $a \mapsto \frac{a}{1}$ is a ring homomorphism. Given $\varphi : R \rightarrow B$ such that $\text{im } \varphi \subset B^\times$, we have a well-defined map of rings $\underline{\varphi} : S^{-1}R \rightarrow B$ given by $\frac{a}{s} \mapsto \varphi(a)\varphi(s)^{-1}$, which satisfies $\underline{\varphi} \circ h = \varphi$.

Example 2. Here are some natural choices for S .

- (a) $S = \{1, x, x^2, \dots\}$ with x not nilpotent.
- (b) $S = R^\times$.
- (c) $S = \{r \in R : r \text{ is not a zero divisor}\}.$

Note 2. If A is an integral domain and we take any multiplicatively closed subset $S \subset A$, then $\text{Frac}(A) := (A \setminus \{0\})^{-1}A$ is a field and $h : A \rightarrow (A \setminus \{0\})^{-1}A$ is injective. For now, let S denote the set of non zero-divisors. If $\frac{a}{b} \in \text{Frac}(A)$ is nonzero, then $\frac{a}{b} \neq \frac{0}{1}$, i.e., $a \cdot 1$ is not a zero divisor, so that $a \neq 0$ and thus $\frac{b}{a} \in \text{Frac}(A)$. This shows that $\text{Frac}(A)$ is a field. Moreover, if $a \in A$ has $h(a) = \frac{a}{1} = 0 \in \text{Frac}(A)$, then $\frac{a}{1} = \frac{0}{1} \implies a \cdot 1$ is a zero divisor. Hence $a = 0$, and h is injective.

If S is generic, then $S^{-1}A \subset \text{Frac}(A)$ since $S^{-1}A$ equals the subring generated by $A \cong h(A)$ and $S^{-1} = \{\frac{1}{s} \mid s \in S\}$. In this case, $(S^{-1}A, h)$ represents the functor $\Phi : \mathbf{Field} \rightarrow \mathbf{Set}$ given by $k \mapsto \{\varphi : A \rightarrow k \mid \varphi \text{ is injective}\}$. This means that for any ring map $\varphi : A \rightarrow B$ with $\varphi(S) \subset B^\times$, there is some unique map ψ such that $\psi \circ h = \varphi$.

2.2 Lecture 5

Example 3.

1. If $S = \{1, x, x^2, \dots\}$ with x not nilpotent, then $S^{-1}A = \{\frac{a}{f^n} : n \geq 0, a \in A\} := A_f$.
2. If $S \subset A^\times$, then $h : A \rightarrow S^{-1}A$ is an isomorphism.
3. If A is any ring and $S \subset A$ denotes the set of all non-zero divisors, then $\text{Frac}(A) = S^{-1}A$ is called the *fraction ring of A* . If A is an integral domain, then $\text{Frac}(A)$ is a field and $H : A \rightarrow \text{Frac}(A)$ is injective. In this cases, $(\text{Frac}(A), h)$ represents the functor $F_A : \mathbf{Field} \rightarrow \mathbf{Set}$ given by $K \mapsto \{\varphi : A \rightarrow K \mid \varphi \text{ monomorphism}\}$.

Remark 4. Let A be a commutative ring and $S \subset A$ be multiplicatively closed. Let M be an A -module. Define the equivalence relation $(M \times S, \sim)$ where $(m, s) \sim (n, t)$ if $u(tm - sn) = 0$ for some $u \in S$. Define the A -module $S^{-1}M = M \times S / \sim$ where $\frac{m}{s} + \frac{n}{t} := \frac{tm + sn}{st}$. Define the module homomorphism $h_M : M \rightarrow S^{-1}M$ by $m \mapsto [(m, 1)]$. Let $\frac{m}{s} := [(m, s)]$.

Moreover, $S^{-1}M$ is naturally a module over $S^{-1}A$ via the action $\frac{a}{s} \cdot \frac{m}{t} := \frac{a \cdot m}{st}$. This makes h_M a module over $h : A \rightarrow S^{-1}A$ in that for any $a \in A$ and $m \in M$, we have that $h_M(a \cdot m) = h(a) \cdot h_M(m)$.

We see that $S^{-1}(-)$ is a functor which maps each homomorphism $\varphi : M \rightarrow N$ to $S^{-1}\varphi : S^{-1}M \rightarrow S^{-1}N$ given by $\frac{m}{s} \mapsto \frac{\varphi(m)}{s}$. It's easy to verify that $S^{-1}(-)$ is left adjoint to the pullback functor h^\bullet .

Definition. If $f : A \rightarrow B$ is a map of **commutative** rings, then there are natural functors $f^\bullet : B\text{-}\mathbf{Mod} \rightarrow A\text{-}\mathbf{Mod}$ and $f_\bullet : A\text{-}\mathbf{Mod} \rightarrow B\text{-}\mathbf{Mod}$, called the *pullback* and *pushforward*, respectively. The pullback is defined in the obvious way. Further, $f_\bullet : M \mapsto B \otimes_A M$ where B is viewed as an A -module via f and $b \cdot (c \otimes m) := (bc) \otimes m$ and $(\varphi : M \rightarrow N) \mapsto (\text{id}_B \otimes \varphi : f_\bullet(M) \rightarrow f_\bullet(N))$.

Exercise 5. (f_\bullet, f^\bullet) is an adjoint pair.

Corollary 6. $S^{-1}(-) \cong h_\bullet$.

Example 4. Naively, we could have tried to define fractions in A by $(a, s) \sim_n (b, t)$ if $(at - bs = 0)$. But this is not in general an equivalence relation, for it is not transitive. Indeed, set $A = \mathbb{C}[x, y]_{(xy)}$ and $S = \{1, x, x^2, \dots\}$. Consider the localization A_x . Note that $(y, 1) \not\sim_n (0, 1)$ but that $(y, 1) \sim_n (0, x)$ and $(0, x) \sim_n (0, 1)$.

Note 3. We have that $A_x = \mathbb{C}[x, x^{-1}]$, which is a field, and that $h : A \rightarrow A_x$ is given by $\frac{[f(x, y)]}{[p(x) + yq(y)]} \mapsto p(x)$,

which is non-injective.

Proposition 3.

1. If $h : A \rightarrow S^{-1}A$, then $\ker h = \{a \in A : \exists s \in S. sa = 0\}$.
2. $S^{-1}A$ is flat as an A -module.

Corollary 7. $S^{-1}(-)$ is an exact functor.

Proof. Let $M \xrightarrow{f} T \xrightarrow{g} N$ be an exact sequence of A -modules. We want to show that

$$S^{-1}M \xrightarrow{S^{-1}f} S^{-1}T \xrightarrow{S^{-1}g} S^{-1}N$$

is exact as well. Let $\frac{x}{s} \in S^{-1}T$ with $(S^{-1}g)(\frac{x}{s}) = 0$. This implies that $\frac{g(x)}{s} = \frac{0}{1}$, so that $ug(x) = 0$ for some $u \in S$. But since g is a morphism, we know that $0 = ug(x) = g(ux)$. This means that $f(y) = ux$ for some $y \in M$. Then $\frac{y}{us} \in S^{-1}M$ such that $(S^{-1}f)(\frac{y}{us}) = \frac{f(y)}{us} = \frac{ux}{us} = \frac{x}{s}$. \square

3. Suppose that $f \in A$ is not nilpotent. We can compute A_f explicitly as follows. There is a natural map $A_f[x] \rightarrow A_f$ given by $x \mapsto \frac{1}{f}$. This induces $A_f[x]_{(x - \frac{1}{f})} \xrightarrow{\cong} A_f$. We also have a map $A[x] \rightarrow A_f[x]$ from the map h on the coefficients. Define the map $\alpha : A[x] \rightarrow A_f$ by $a \mapsto h(a) = \frac{a}{1}$ and $x \mapsto \frac{1}{f}$. We must compute $\ker \alpha$ as an ideal in $A[x]$. This is surjective since any element in A_f is of the form $\frac{a}{f^n}$ for some $a \in A$ and $n \in \mathbb{N}$, so that $ax^n \mapsto \frac{a}{f^n}$.

2.3 Lecture 6

Claim 3. $\ker \alpha = (fx - 1)$.

Proof. Note that $xf - 1 \in \ker \alpha$. Also, note that $\exists n \geq 0. f^n g(\frac{1}{f}) = 0 \iff \alpha(g) = 0 \iff g(\frac{1}{f}) = 0$ in A_f . Without loss of generality, we may assume that $n \geq \deg g$. Thus, $f^n g(x)$ is a polynomial of fx with coefficients in A , so that there is some $G(y) \in A[y]$ such that $G(fx) = f^n g(x)$. Then $g \in \ker \alpha \iff \exists G(y) \in A[y]. G(fx) = f^n g(x) \wedge G(1) = 0$. But then $G(y) = (y - 1)h(y)$ where $h(y) \in A[y]$. This implies that $g(x) \in \ker \alpha \iff \exists n \geq 0. f^n g(x) \in (xf - 1)$. But $f, fx - 1 \in A[x]$ are relatively prime since $1 = fx + (fx - 1)(-1)$. Hence $1^n = (fx + (fx - 1)(-1))^n = f^n x^n + (fx - 1)s(x)$ for some $s(x) \in A[x]$. Multiply by $g(x)$ to get

$$g(x) = \underbrace{f^n g(x)}_{\in (xf-1)} + \underbrace{(fx - 1)s(x)g(x)}_{\in (xf-1)}.$$

Therefore, $g(x) \in (xf - 1)$, and $(xf - 1) = \ker \alpha$. \square

Remark 5. Suppose that $\varphi : A \rightarrow B$ is a map of commutative rings. Then we can transport the ideals along φ as follows.

Definition.

1. Given an ideal $\mathfrak{a} \trianglelefteq A$, the *extension of \mathfrak{a} along φ* is the ideal $\mathfrak{a}^e \trianglelefteq B$ that is generated by $\varphi(\mathfrak{a})$, i.e., $\mathfrak{a}^e = \varphi(\mathfrak{a}) \cdot B$.
2. Given an ideal $\mathfrak{b} \trianglelefteq B$, the *contraction of \mathfrak{b} along φ* is defined as the ideal $\mathfrak{b}^c = \varphi^{-1}(\mathfrak{b})$.

Suppose that A is a commutative ring and that $S \subset A$ is multiplicatively closed. Recall the map $h : A \rightarrow S^{-1}A$. We want to study $(-)^e$ and $(-)^c$ along h .

Proposition 4.

1. If $\mathfrak{a} \trianglelefteq A$, then $\mathfrak{a}^e = \{\frac{a}{s} \mid a \in \mathfrak{a}, s \in S\}$.

Proof. By definition, $\mathfrak{a}^e = h(\mathfrak{a}) \cdot S^{-1}A = \{\sum_i \frac{b_i}{t_i} \frac{a_i}{s_i} \mid a_i \in \mathfrak{a}, b_i \in A, s_i, t_i \in S\}$. Since $a_i b_i \in \mathfrak{a}$ and $s_i t_i \in S$, our proof is complete. \square

2. If $\mathfrak{a} \trianglelefteq A$, then $\mathfrak{a}^e = (1) \iff \mathfrak{a} \cap S \neq \emptyset$.

Proof. Note that $(S^{-1}A)^\times$ consists of every fraction $\frac{a}{s}$ for which we can find some fraction $\frac{b}{t}$ such that $\frac{a}{s} \frac{b}{t} = 1$. Therefore, we must have some element $u \in S$ such that $u(ab - st) = 0 \iff \exists \beta \in A. \beta a \in S$. Thus, $(S^{-1}A)^\times = \{\frac{a}{s} \mid \exists \beta \in A. \beta a \in S\}$. But then $\mathfrak{a}^e = (1) \iff \mathfrak{a}^e$ contains some unit $\iff \exists \frac{a}{s} \in \mathfrak{a}^e. \exists \beta \in A. \beta \cdot a \in S \iff \beta \cdot a \in S \cap \mathfrak{a}$. \square

3. Suppose that $I \trianglelefteq S^{-1}A$ is an ideal. Then we can form $I^{ce} \trianglelefteq S^{-1}A$. By definition, $I \supset I^{ce}$. In fact, $I = I^{ce}$.

Proof. If $\frac{a}{s} \in I$, then $a \in h^{-1}(I)$ because $h^{-1}(I) = \{r \in A \mid \frac{r}{1} \in I\}$. But $\frac{a}{1} = s \cdot \frac{a}{s}$ where $s \in S^{-1}A$ and $\frac{a}{s} \in I$, so that $a \in I$. This implies that $\frac{a}{s} \in I^{ce}$ for each $s \in S$, and thus $I \subset I^{ce}$. \square

4. If $\mathfrak{a} \trianglelefteq A$, then $\mathfrak{a}^{ec} = \{r \in A \mid \exists s \in S. sr \in \mathfrak{a}\}$.

Proof. Suppose that $a \in \mathfrak{a}^{ec}$. Then $\frac{a}{1} = h(a) \in \mathfrak{a}^e$, so that $\exists b \in \mathfrak{a}. \exists s \in S. \frac{a}{1} = \frac{b}{s}$. This implies that $\exists u \in S$ such that $u(sa - b) = 0$. Hence $(us)a = b$, and $\mathfrak{a}^{ec} \subset \{r \in A \mid \exists s \in S. sr \in \mathfrak{a}\}$. If $r \in A$ is such that $rs \in \mathfrak{a}$ for some $s \in S$, then $\frac{r}{1} = \frac{rs}{s} \in \mathfrak{a}^e$ and thus $r \in \mathfrak{a}^{ec}$. \square

5. We have that $\mathfrak{a} \trianglelefteq A$ is contracted (i.e., $\mathfrak{a} = I^c$ for some $I \trianglelefteq S^{-1}A$) if and only if $\mathfrak{a} = \mathfrak{a}^{ec}$ if and only if $[s] \in A/\mathfrak{a}$ is not a zero divisor for any $s \in S$.

6. The map $(-)^e$ induces a bijection $(-)^e : \{\mathfrak{a} \trianglelefteq A \mid \mathfrak{a} \text{ is a contraction of some ideal}\} \rightarrow \{I \mid I \trianglelefteq S^{-1}A\}$ that preserves inclusions of ideals.
7. Suppose that M is an A -module.

Definition. A submodule $N \subset M$ is *S-saturated* if $N = \{x \in M \mid \exists s \in S. sx \in N\}$. (If $M = A$ and $N = \mathfrak{a}$, then N is *S-saturated* if and only if $\mathfrak{a} = \mathfrak{a}^{ec}$.)

The localization on modules induces an inclusion-preserving bijection

$$S^{-1}(-) : \{N \subset M \mid N \text{ is } S\text{-saturated}\} \rightarrow \{M \mid M \subset S^{-1}M\}.$$

Exercise 6. An ideal $\mathfrak{b} \trianglelefteq A$ is prime if and only if $A \setminus \mathfrak{b}$ is multiplicatively closed.

2.4 Lecture 7

Proposition 5.

1. If $\mathfrak{p} \trianglelefteq A$ is prime and $S \subset A$ is multiplicatively closed, then $\mathfrak{p}^e \trianglelefteq S^{-1}A$ is prime if and only if $S \cap \mathfrak{p} = \emptyset$.

Proof. The forward direction is obvious. Conversely, suppose that $S \cap \mathfrak{p} = \emptyset$. Then $\mathfrak{p}^{ec} = \mathfrak{p}$. Indeed, $\mathfrak{p}^{ec} = \{a \in A \mid \exists s \in S. sa \in \mathfrak{p}\}$. But if $sa \in \mathfrak{p}$, then either $s \in \mathfrak{p}$ or $a \in \mathfrak{p}$. Since $S \cap \mathfrak{p} = \emptyset$, we see that $s \notin \mathfrak{p} \implies a \in \mathfrak{p}$. Suppose that $x \cdot y \in \mathfrak{p}^e$. Then $x = \frac{a}{s}$ for some $a \in A$ and $s \in S$, and $y = \frac{b}{t}$ for some $b \in A$ and $t \in B$. Then $\frac{ab}{st} \in \mathfrak{p}^e$, so that $\frac{ab}{t} \in \mathfrak{p}^e$ since \mathfrak{p}^e is an ideal. Hence $ab \in \mathfrak{p}$, which is prime by assumption. Say that $a \in \mathfrak{p}$. Then $\frac{a}{s} \in \mathfrak{p}^e$. \square

Corollary 8. If $S \subset A$ is multiplicatively closed, then we get a bijection

$$\{\mathfrak{p} \in \text{Spec}(A) \mid \mathfrak{p} \cap S = \emptyset\} \xrightarrow{(-)^e} \text{Spec}(S^{-1}A)$$

since \mathfrak{p}^c is prime in A with $\mathfrak{p}^c \cap S = \emptyset$ whenever \mathfrak{p} is prime in $S^{-1}A$.

2. Recall the property P that an ideal \mathfrak{a} in A is prime if and only if $A \setminus \mathfrak{a}$ is multiplicatively closed. We claim that \mathfrak{a} is prime if and only if there is some multiplicatively closed $S \subset A$ such that $S \cap \mathfrak{a} = \emptyset$ and \mathfrak{a} is maximal among all ideals satisfying P.

Proof. If \mathfrak{a} is prime, then $S = A \setminus \mathfrak{a}$ is multiplicatively closed and \mathfrak{a} is maximal. Conversely, let $a, b \in A$ such that $a, b \notin \mathfrak{a}$. We must show that $ab \notin \mathfrak{a}$. Consider $\mathfrak{a} + (a) \supsetneq \mathfrak{a}$ and $\mathfrak{a} + (b) \supsetneq \mathfrak{a}$. But we are given S such that $\mathfrak{a} \cap S = \emptyset$. Hence there are $s \in S \cap (\mathfrak{a} + (a))$ and $t \in S \cap (\mathfrak{a} + (b))$. Then $s = \alpha + x \cdot a$ and $t = \beta + y \cdot b$ where $\alpha, \beta \in \mathfrak{a}$ and $x, y \in A$. We compute

$$st = \alpha\beta + \alpha yb + \beta xa + xyab,$$

where $st \in S$ and $\alpha\beta, \alpha yb, \beta xa \in \mathfrak{a}$. If we assume that $ab \in \mathfrak{a}$, then $st \in S \cap \mathfrak{a}$, a contradiction. \square

3. If $S \subset A$ is multiplicatively closed, then there is some prime ideal \mathfrak{b} such that $\mathfrak{b} = A \setminus S$.

Proof. Apply Zorn's lemma. \square

Definition. Let A be a ring. We call A a *local ring* if any of the following equivalent conditions holds.

- (a) A has a unique maximal ideal \mathfrak{m} .
- (b) $A \setminus A^\times$ is an ideal.
- (c) If \mathfrak{m} is maximal and $x \in \mathfrak{m}$, then $1 + x \in A^\times$.

If A is a ring and \mathfrak{p} a prime ideal, we will denote the localization $(A - \mathfrak{p})^{-1}A$ by $A_{\mathfrak{p}}$.

Proposition 6. If \mathfrak{p} is prime, then $A_{\mathfrak{p}}$ is a local ring with unique maximal ideal \mathfrak{p}^e .

Proof. Let $S = A \setminus \mathfrak{p}$. Then $A_{\mathfrak{p}} = S^{-1}A$. Suppose that $I \leq A_{\mathfrak{p}} = S^{-1}A$ such that $I \neq (1)$. But any ideal in $S^{-1}A$ is of the form $I = \mathfrak{a}^e$ for some ideal \mathfrak{a} in A . Since $(1) \neq I = \mathfrak{a}^e$, it follows that $\mathfrak{a} \cap S = \emptyset$. Therefore, $\mathfrak{a} = A \setminus S = \mathfrak{p}$, so that $I = \mathfrak{a}^e \subset \mathfrak{p}^e$. Hence every nontrivial ideal in $A_{\mathfrak{p}}$ is contained in \mathfrak{p}^e , implying that \mathfrak{p}^e is the unique maximal ideal. \square

Corollary 9. In particular, the map (prime ideal of $A \mid A \subset \mathfrak{p}$) $\xrightarrow{(-)^e} \text{Spec}(A_{\mathfrak{p}})$ is a bijection that preserves inclusions of ideals.

Definition. If A is a commutative ring, then for every $\mathfrak{p} \leq A$ prime, we define the *height of \mathfrak{p}* as

$$\text{ht}(\mathfrak{p}) = \sup\{k \mid \mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \cdots \supsetneq \mathfrak{p}_k, \mathfrak{p}_i \leq A\}.$$

Note that $\text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{p}^e \text{ in } A_{\mathfrak{p}})$.

Definition. The *Krull dimension* of A is

$$\dim A = \sup\{\text{ht}(\mathfrak{m}) \mid \text{maximal } \mathfrak{m} \leq A\}.$$

Note 4. $\dim A_{\mathfrak{p}} = \text{ht}(\mathfrak{p})$, and $\dim A = \sup\{\dim A_{\mathfrak{m}} \mid \text{maximal } \mathfrak{m} \leq A\}$.

Example 5.

1. If k is a field, then $\dim k = 0$. (The converse is also true.)
2. If A is a PID, then $\dim A = 1$. For example, \mathbb{Z} , $\mathbb{Q}[x]$, and $\mathbb{Z}[i]$ have dimension 1.

Exercise 7.

1. Show that $\mathbb{Z}[-\sqrt{5}]$ is not a PID but has dimension 1.
2. Show that $\dim \mathbb{C}[x_1, \dots, x_n] = n$.

3 Basic algebraic geometry

Remark 6. Any information about A , any prime ideal in A , any localization in A , and their relations can be packaged in a geometrical entity: specifically, a topological space along with a distinguished class of maps.

Let $X := \text{Spec}(A)$, the set of all prime ideals in A . For any $f \in A$, define the *principal open subset associated with f* as

$$X_f = \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}.$$

Such subsets satisfy

- (a) $X_f \cap X_g = X_{fg}$.
- (b) $X_{f^n} = X_f$, $X_f = X \iff f \notin \mathfrak{p} \forall \mathfrak{p} \text{ prime} \iff f \in A^\times$.
- (c) $X_f = \emptyset \iff f \in \bigcap_{\mathfrak{p} \subset X} \mathfrak{p}$.

Definition. The minimal topology on X generated by $\{X_f\}_{f \in A}$ is called the *Zariski topology on X* .

Note 5. The subset $U \subset X$ is open if and only if there is some $T \subset A$ such that $U = \bigcup_{f \in T} X_f$. Also, $Y \subset X$ is closed if $Y = \bigcap_{f \in T} (X \setminus X_f)$ for some $T \subset A$. Hence $Y \subset X$ is closed if there is some $T \subset A$ such that

$$Y = \{\mathfrak{p} \mid \mathfrak{p} \supset \langle T \rangle\}.$$

In particular, for any ideal $\mathfrak{a} \leq A$, we can define a Zariski-closed subset $V(\mathfrak{a}) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supset \mathfrak{a}\}$. (Note that replacing \mathfrak{a} with a set $S \subset A$ determines an equivalent topology.) Every closed subset is of this form.

Exercise 8. Write arbitrary intersections of closed sets, finite unions of closed sets, X , and \emptyset in this form.

Remark 7. Any $f \in A$ can be viewed as a function on X in two ways. First, view f as a mapping $X \rightarrow \prod_{\mathfrak{p} \in X} A_{\mathfrak{p}}$ given by $\mathfrak{p} \mapsto \frac{f}{1} \in A_{\mathfrak{p}}$. Then for any \mathfrak{p} , the value of f on \mathfrak{p} is in $A_{\mathfrak{p}}$. Second, view f as a mapping $X \rightarrow \prod_{\mathfrak{p} \in X} k_{\mathfrak{p}}$ given by $f \mapsto \frac{f}{1} + \mathfrak{p}^e \in k_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}^e$. We call $k_{\mathfrak{p}}$ the *residue field* of $A_{\mathfrak{p}}$.

Example 6. If k is a field and $A = k[x_1, \dots, x_n]$ such that for any \mathfrak{m} , $k_{\mathfrak{m}} = k$. Then $f \in A$ induces a function (prime ideals in A) $\rightarrow k$ given by $(x_1 - a_1, \dots, x_n - a_n) \mapsto f(a_1, \dots, a_n)$.

Lemma 4. X is quasi-compact, meaning that for any Zariski-open $U \subset X$ and any open cover $\{U_{\alpha}\}$ of X , there is some finite subcover and $U = \bigcup_{\alpha} U_{\alpha}$.

Note 6. X is not Hausdorff in general.

Exercise 9. Let $A = \mathbb{C}[x]$. Show that $X = \text{Spec}(A)$ is not Hausdorff.

3.1 Lecture 8

Note 7.

1. We have that $V(S) = V(\mathfrak{a})$ when $\mathfrak{a} = \langle S \rangle$.
2. The Zariski topology is generated by the collection of principal open subsets on X , i.e., subsets of the form $X_f = \{\mathfrak{b} \in X \mid f \notin \mathfrak{b}\}$ where $f \in A$. The elements in the ring A may be viewed as kinds of functions on X . View $f \in A$ as a function $X \rightarrow \prod_{\mathfrak{b} \in X} A_{\mathfrak{b}} \rightarrow \prod_{\mathfrak{b} \in X} A_{\mathfrak{b}}/\mathfrak{b}^c$ defined by $\mathfrak{b} \mapsto \frac{f}{1} \in A_{\mathfrak{b}}$.

Remark 8. If k is a field and $A = k[x_1, \dots, x_n]$, then $V(a_1, \dots, a_n) \in A^n$. We get a maximal ideal $\langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle$. Thus, if $f(x) \in A$ and we restrict this function, then we get the evaluation of f on points $a \in A^n$.

$$\begin{array}{ccc} a \xrightarrow{\epsilon} \langle x_1 - a_1, x_2 - a_2, \dots, x_n - a_n \rangle X & \longrightarrow & \prod_{\mathfrak{b}} k_{\mathfrak{b}} \\ \uparrow & & \uparrow \\ A^n & \longrightarrow & \prod_{\mathfrak{a}} k \cong A^n \times k \end{array}$$

This geometric way of packaging information about A and all of its prime ideals is compatible with all natural rings homomorphisms between the $A_{\mathfrak{p}}$'s. If we have a principal open, then for every $f \in A$, we get a ring A_f , provided that f is not nilpotent, and a functor (poset of principal open sets in X)^{op} $\rightarrow \mathbf{CommRing}$ given by $X_f \mapsto A_f$. This extends to another functor (opens in X)^{op} $\rightarrow \mathbf{CommRing}$. Given $f \in A$, viewing f as a kind of function on X thus induces a compatible system of elements of all rings $A(U)$ where $U \in X$ is open.

Lemma 5. X is quasi-compact, meaning that any open cover $\{U_{\alpha}\}$ of X admits some finite subcover.

Proof. Let $X = \bigcup_{\alpha} U_{\alpha}$. The principal opens generate the Zariski topology, so that for any α , we can find a cover $U_{\alpha} = \bigcup_{\beta} X_{f_{\alpha}^{\beta}}$ where $f_{\alpha}^{\beta} \in A$. Then $X = \bigcup_{\alpha, \beta} X_{f_{\alpha}^{\beta}}$, so that $\emptyset = \bigcap_{\alpha, \beta} \underbrace{(X - X_{f_{\alpha}^{\beta}})}_{V(f_{\alpha}^{\beta})}$. But $\emptyset \cap_{\alpha, \beta} V(f_{\alpha}^{\beta}) =$

$V(\{f_{\alpha}^{\beta}\}_{\alpha, \beta})$. Hence $\langle (f_{\alpha}^{\beta})_{\alpha, \beta} \rangle$ is not contained in any prime ideal, so that $\langle (f_{\alpha}^{\beta})_{\alpha, \beta} \rangle = A$, hence $1 \in \langle (f_{\alpha}^{\beta})_{\alpha, \beta} \rangle$. We can find a collection of elements $\{a_{\alpha}^{\beta}\}_{\alpha, \beta}$ where $a_{\alpha}^{\beta} \in A$ such that $1 = \sum_{\alpha, \beta} a_{\alpha}^{\beta} f_{\alpha}^{\beta}$ and at most finitely many a_{α}^{β} are nonzero. Thus, there is sequence $(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)$ with $\langle f_{\alpha_1}^{\beta_1}, \dots, f_{\alpha_k}^{\beta_k} \rangle = \langle 1 \rangle = A$. Hence $V(f_{\alpha_1}^{\beta_1}) \cap \dots \cap V(f_{\alpha_k}^{\beta_k}) = \emptyset$, and $X = X_{f_{\alpha_1}^{\beta_1}} \cup \dots \cup X_{f_{\alpha_k}^{\beta_k}}$. But $X_{f_{\alpha_i}^{\beta_i}} \in U_{\alpha_i}$ for each $i = 1, \dots, k$. Therefore, $X = U_{\alpha_1} \cup \dots \cup U_{\alpha_k}$. \square

Example 7. Suppose that X is a compact and Hausdorff space. Let $A = C(X)$, the ring of complex-valued continuous functions on X . Consider $\text{Spec}(A)$ equipped with the Zariski topology and its subset $\text{Max}(A) := \{\mathfrak{a} \in C(X) \mid \mathfrak{a} \text{ maximal}\}$ equipped with subspace topology.

Claim 4. The natural map $X\text{Max}^\varphi(A)$ given by $x \mapsto \{f \in C(X) \mid f(x) = 0\}$ is a homomorphism.

Proof. Let $\mathfrak{a}_x := \ker(\text{ev}_x : A \rightarrow \mathbb{C})$. By Urysohn's lemma, for any two distinct points $x, y \in X$, there is some $f \in A$ such that $f(x) = 0$ and $f(y) = 1$. But $f \in \mathfrak{a}_x$ and $f \notin \mathfrak{a}_y$, making $\mathfrak{a}_x \neq \mathfrak{a}_y$. Now, suppose $\mathfrak{a} \in \text{Max}(A)$ and $\mathfrak{a} \neq \mathfrak{a}_x$ for any $x \in X$. This means that for any $x \in X$, there is some $f_x \in \mathfrak{a}$ such that $f_x(x) \neq 0$. Let $U_x \subset U$ be an open neighborhood of $x \in X$ such that $f_x|_{U_x} \neq 0$. Then $X = \bigcup_{x \in X} U_x$, so that there is some finite subcover U_{x_1}, \dots, U_{x_k} of X . Let $f = \sum_{i=1}^k |f_{x_i}|^2$, which does not vanish at any point of X . Note that $f = \sum_{i=1}^k f_{x_i} \cdot \bar{f}_{x_i}$, so that $f \in \mathfrak{a}$. But f is nowhere vanishing, so that $\frac{1}{f}$ is a well defined continuous function on X . Thus, $\frac{1}{f} \in A$, and $1 \in \mathfrak{a}$, contrary to the fact that \mathfrak{a} is maximal.

Exercise 10. Check that φ is continuous, hence a homeomorphism. □

Definition. Let A be a commutative ring and M and A -module. Then M defines a subset of $X := \text{Spec}(A)$, namely

$$\text{supp}(M) = \{\mathfrak{b} \in X \mid M_{\mathfrak{b}} \neq 0\},$$

called the *support* of M .

Proposition 7.

1. $\text{supp}(M) \subset V(\text{ann}(M))$, where $\text{ann}(M) = \{a \in A \mid a \cdot m = 0 \text{ for each } m \in M\}$.

Proof. Let $\mathfrak{b} \in \text{supp}(M)$. Then $M_{\mathfrak{b}} \neq (0)$. We need to show that $\text{ann}(M) \subset \mathfrak{b}$. Suppose that there is some $a \in \text{ann}(M)$ with $a \notin \mathfrak{b}$. Let $x \in M_{\mathfrak{b}}$. Then $x = \frac{m}{s}$ where $m \in M$ and $s \notin \mathfrak{b}$. We compute $\frac{a}{1} \cdot \frac{m}{s} = \frac{am}{s} = 0$ in $M_{\mathfrak{b}}$. Since $a \notin \mathfrak{b}$, it follows that $\frac{a}{1}$ is invertible in $A_{\mathfrak{b}}$, i.e., $\frac{1}{a} \in A_{\mathfrak{b}}$. Hence $\frac{m}{s} = \frac{1}{a} \left(\frac{a}{1} \frac{m}{s} \right) = 0$ in $M_{\mathfrak{b}}$, so that $M_{\mathfrak{b}} = (0)$, a contradiction. □

2. If M is finitely generated, then $\text{supp}(M) \supset V(\text{ann}(M))$.

Proof. Let $\mathfrak{b} \in V(\text{ann}(M))$ and $\mathfrak{b} \supset \text{ann}(M)$. We want to show that $M_{\mathfrak{b}} \neq (0)$. Suppose to the contrary. Then for any $m \in M_{\mathfrak{b}}$ we have that $\frac{m}{1} = 0$ in $M_{\mathfrak{b}}$. This shows that there exists $s \notin \mathfrak{b}$ such that $s \cdot m = 0$ in M . But M is finitely generated. Let $m_1, \dots, m_k \in M$ be generators of $M \setminus A$. Then there are $s_1, \dots, s_k \in A \setminus \mathfrak{b}$ such that $s_i m_i = 0$ in M for each i . Let $s = s_1 \cdots s_k \in A \setminus \mathfrak{b}$. Then for any $m \in M$, we have that $s \cdot m = 0$. Hence $s \in A \setminus \mathfrak{b}$, and $s \in \text{ann}(M)$, a contradiction. □

3.2 Lecture 9

Proposition 8. $M = (0) \iff \text{supp}(M) = \emptyset \iff \text{supp}(M) \cap \text{Max}(A) = \emptyset$.

Proof. It's clear that $M = \emptyset \implies \text{supp}(M) = (0) \implies \text{supp}(M) \cap \text{Max}(A) = \emptyset$. Hence it suffices to show that $\text{supp}(M) \cap \text{Max}(A) = \emptyset \implies M = (0)$. If M is finitely generated, then $\text{supp}(M) = V(\text{ann}(M))$, so that $\text{supp}(M)$ must contain any maximal idea that contains $\text{ann}(M) \trianglelefteq A$. Thus, the assumption that $\text{supp}(M) \cap \text{Max}(A) = \emptyset$ implies that $\text{ann}(A)$ is not contained in any maximal ideal, meaning that $\text{ann}(M) = A$. This makes $M = (0)$.

If M is arbitrary, then $M = \text{colim}_{\alpha} N_{\alpha}$ with each $N_{\alpha} \subset M$ finitely generated. But then $M_{\mathfrak{a}} = \text{colim}_{\alpha} (N_{\alpha})_{\mathfrak{a}}$ because localization is exact. Since each $N_{\alpha} = (0)$, it follows that $\text{colim}_{\alpha} N_{\alpha} = (0)$. □

Corollary 10. If we have a sequence of modules $\eta : M \xrightarrow{f} T \xrightarrow{g} N$, then η is exact at $T \iff \eta_{\mathfrak{p}}$ is exact at $T_{\mathfrak{p}}$ for each $\mathfrak{p} \in \text{Spec}(A) \iff \eta_{\mathfrak{a}}$ is exact at $T_{\mathfrak{a}}$ for each $\mathfrak{a} \in \text{Max}(A)$.

Proof. The forward directions are clear. Conversely, if $\eta_{\mathfrak{a}}$ is for every \mathfrak{a} , then $M_{\mathfrak{a}} \xrightarrow{f_{\mathfrak{a}}} T_{\mathfrak{a}} \xrightarrow{g_{\mathfrak{a}}} N_{\mathfrak{a}}$ is exact. If $H = \ker g /_{\text{im } f}$, then $H_{\mathfrak{a}} = \ker g_{\mathfrak{a}} /_{\text{im } f_{\mathfrak{a}}} = 0$. Thus, $\text{supp}(H) \cap \text{Max}(A) = \emptyset$, so that $H = 0$. □

Definition. Suppose that Π is a property of A -modules or of morphisms of A -modules. We say that Π *holds locally for A* if $\Pi_{\mathfrak{a}}$ holds for every $\mathfrak{a} \in \text{Spec}(A)$.

Example 8.

1. $M = (0)$ holds locally if and only if it holds globally.
2. $M \rightarrow T \rightarrow N$ is exact locally if and only if it's exact globally.
3. TFAE.
 - (a) M is flat over A .
 - (b) M is locally flat over A .
 - (c) $M_{\mathfrak{a}}$ is flat over $A_{\mathfrak{a}}$ for every $\mathfrak{a} \in \text{Max}(A)$.
 - (d) $M_{\mathfrak{a}}$ is flat over A for every $\mathfrak{a} \in \text{Max}(A)$.

Proof. The fact that (a) \implies (b) \implies (c) \implies (d) is obvious. To see that (c) \implies (a), suppose that M is an A -module such that $M_{\mathfrak{a}}$ is flat as an $A_{\mathfrak{a}}$ -module. Suppose that $0 \rightarrow X \rightarrow Y$ is an exact sequence of A -modules. Let $K = \ker(X \otimes_A M \rightarrow Y \otimes_A M)$. We want to show that $K = 0$.

Localizing $0 \rightarrow K \rightarrow X \otimes_A M \rightarrow Y \otimes_A M$ along \mathfrak{a} gives an exact sequence $0 \rightarrow K_{\mathfrak{a}} \rightarrow X_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}} \rightarrow Y_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}}$, where we have used the fact that $(X \otimes_A M)_{\mathfrak{a}} = X_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}}$. But $M_{\mathfrak{a}}$ is flat over $A_{\mathfrak{a}}$. Hence if we tensor the exact sequence $0 \rightarrow X_{\mathfrak{a}} \rightarrow Y_{\mathfrak{a}}$ with $M_{\mathfrak{a}}$ over $A_{\mathfrak{a}}$, then it will remain exact. This implies that $\ker(X_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}} \rightarrow Y_{\mathfrak{a}} \otimes_{A_{\mathfrak{a}}} M_{\mathfrak{a}}) = 0$, so that $K_{\mathfrak{a}} = 0$ for each \mathfrak{a} . It follows that $\text{supp}(K) = \emptyset$, which implies that $K = (0)$. \square

Definition. If A is commutative ring, then the *Jacobson radical of A* is the ideal

$$\text{Jac}(A) \equiv \bigcap_{\mathfrak{a} \in \text{Max}(A)} \mathfrak{a}.$$

Lemma 6. (Nakayama) If A is a commutative ring and M is a finitely generated A -module with $\text{Jac}(A) \cdot M = M$, then $M = (0)$.

Proof. Let M be finitely generated over A . Choose some finite set of generators m_1, \dots, m_t of M of minimal cardinality. If $M \neq (0)$, then $t > 0$. Then $m_t \in M = \text{Jac}(A) \cdot M$. Thus there are $a_1, \dots, a_t \in \text{Jac}(A)$ such that $m_t = \sum_{i=1}^t a_i m_i$. Then $(1 - a_t)m_t = \sum_{i=1}^{t-1} a_i m_i$. But $a_t \in \text{Jac}(A)$, meaning that m_t belongs to every maximal ideal. Then $1 - a_t$ cannot be in any maximal ideal. Hence $1 - a_t$ is a unit in A . Let $u \in A$ such that $u(1 - a_t) = 1$. Then $m_t = \sum_{i=1}^{t-1} a_i u m_i$. This contradicts that t is minimal. \square

Corollary 11. (Classical Nakayama) Suppose A is a local ring with maximal ideal \mathfrak{a}_A . Let M be a finitely generated A -module such that $\mathfrak{a}_A M = M$. Then $M = (0)$.

Proposition 9.

1. If A is a commutative ring, then the functor $(-) \otimes^A_{/\text{Jac}(A)} : A - \mathbf{Mod}^{\text{fg}} \rightarrow A_{/\text{Jac}(A)} - \mathbf{Mod}^{\text{fg}}$ is faithful.
2. If M is a finitely generated A -module and $m_1, \dots, m_t \in M$ such that their images $\bar{m}_1, \dots, \bar{m}_t \in M_{/\text{Jac}(A)} \cdot M$ generate the module $M_{/\text{Jac}(A)} \cdot M$, then they generate M .

Proof. If $N = \langle m_1, \dots, m_t \rangle \subset M$, then $\overline{M/N} = (0)$ since $\overline{M/N} = \overline{M}/\overline{N}$. But then $M \setminus N = 0$ by Nakayama. \square

Proposition 10. If A is a local ring and t is the minimal number of generators of the finitely generated A -module M , then every generating set for M contains a generating set of t -elements.

Proof. Let m_1, \dots, m_k be a generating set for M . Then $\bar{m}_1, \dots, \bar{m}_k$ generate $M/\mathfrak{a}_A M = (\text{finite dimensional vector space over } k_A = A/\mathfrak{a}_A)$. This must have dimension t since every spanning subset in $M/\mathfrak{a}_A M$ lifts to a spanning subset of M . Choose a linearly independent subset in $\{\bar{m}_1, \dots, \bar{m}_k\}$ and lift this to M . \square

Theorem 2. Let A be a local ring and M an A -module. Assume that one of the following conditions holds.

- (a) A is Noetherian with M finitely generated.
- (b) M is finitely presentable.

Then M is free $\iff M$ is projective $\iff M$ is flat.

Proof. We only need to show that if M is flat, then M is free. Suppose that M is flat and finitely presentable. We want to show that M is free. Let $0 \rightarrow K \rightarrow A^t \rightarrow M \rightarrow 0$ be a finite presentation where K is finitely generated. Since M being flat implies that $(-) \otimes k_A$ is exact, we have that $\eta : 0 \rightarrow K \otimes_A k_A \rightarrow k_A^t \rightarrow M \otimes_A k_A \rightarrow 0$ is exact. Indeed, if $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ is a short exact sequence of A -modules and N^n is flat, then for every A -module, we see that $0 \rightarrow N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0$ is exact. To see this, choose a presentation $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$, where F is free. Then we get the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \longrightarrow & N'' & & \\
\downarrow & & \downarrow & & \downarrow & & \\
K \otimes N' & \longrightarrow & K \otimes N & \longrightarrow & K \otimes N'' & & \\
\downarrow \delta_1 & & \downarrow \delta_2 & & \downarrow \delta_3 & & \\
0 \longrightarrow & F \otimes N' & \longrightarrow & F \otimes N & \longrightarrow & F \otimes N'' & \cdot \\
\downarrow & & \downarrow & & \downarrow & & \\
M \otimes N' & \xrightarrow{\theta} & M \otimes N & \longrightarrow & M \otimes N'' & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & &
\end{array}$$

Apply the snake lemma to the first two rows. Returning to η , note that k_A^t and $M \otimes_A k_A$ are t -dimensional vector spaces over k_A . Hence $K \otimes_A k_A = 0$. But K is a finitely generated A module. Therefore, Nakayama implies that $K = 0$. \square

4 Algebraic extensions

4.1 Lecture 10

Definition. Suppose that $A \subset B$ where A and B are commutative rings.

1. We say that $u \in B$ is *algebraic over A* if there is some $f(x) \in A[x]$ such that $f(u) = 0$ in B and $f \neq 0$. We say that u is *transcendental over A* if it is not algebraic over A .
2. In general, we say that a collection of elements $u_1, \dots, u_k \in B$ are *algebraically independent over A* if there is some $f(x_1, \dots, x_k) \in A[x_1, \dots, x_k]$ such that $f \neq 0$ and $f(u_1, \dots, u_k) = 0$ in B . We say that $u_1, \dots, u_k \in B$ are *independent transcendentals over A* if they are not algebraically independent over A .
3. We say that $B \supset A$ is *algebraic* if each $u \in B$ is algebraic over A .

Note 8. Our goal is to understand any algebraic extension of a ring. If A and B are domains, then we have a Cartesian diagram

$$\begin{array}{ccc}
A & \hookrightarrow & \text{Frac}(A) \\
\downarrow & & \downarrow \\
B & \hookrightarrow & \text{Frac}(B)
\end{array}
\cdot$$

We have that B is an algebraic extension of A if and only if $\text{Frac}(B)$ is an algebraic extension of $\text{Frac}(A)$. This motivates the study of algebraic extensions of fields.

Definition. If $L \supset K$ is a field extension, we say that L is a *finite extension* if L is finite dimensional as a vector space over K . We call $[L : K] := \dim_K L$ the *degree of the extension*.

Remark 9. Finite field extensions arise naturally from polynomials.

Definition. If K is a field, then $f(x) \in K[x]$ is called *irreducible* if $\deg f > 0$ and f cannot be written as $f = gh$ where $g, h \in K[x]$ not units.

Theorem 3. If $h(x) \in K[x]$ is irreducible, then the ring $K[x]_{/(h)}$ is a field and the inclusion $K \subset K[x]_{/(h)}$ is a finite field extension of degree $\deg h$.

Proof. Recall that $K[x]$ is a Euclidean domain, in particular, a PID.

Lemma 7. Let A be a PID and $u \in A$ be nonzero. Then the following are equivalent.

- (a) $A_{/(u)}$ is a field.
- (b) (u) is prime.
- (c) u is simple.

Proof. The fact that (b) and (c) are equivalent is obvious.

Suppose that u is not simple, so that $u = vw$ with $v, w \in A$ not units. Then in $A_{/(u)}$ we have two elements $[v]$ and $[w]$ such that $[v] \cdot [w] = [u] = [0]$. But both $[v]$ and $[w]$ are nonzero since A has cancellations as a PID. Thus, $A_{/(u)}$ is not a field.

Conversely, if $u \in A$ is simple, then for any $x \in A \setminus (u)$ we have that $(x, u) = (1)$ since x and u are coprime. This means that we can find $a, b \in A$ such that $ax + bu = 1$. Then $[x] \cdot [a] = [1]$. Hence $[x]$ is a unit, so that $A_{/(u)}$ is a field. \square

From this our theorem follows immediately. \square

Note 9. If $h(x) \in K[x]$ is irreducible and $L = K[x]_{/(h)}$, then $h(x)$ has a natural root in L , namely, $t + (h)$. Moreover, every element in L can be written in the form $g(x)$ for some $g(x) \in K[x]$.

Note 10. If $B \supset A$ is a ring extension and $\alpha_1, \dots, \alpha_k \in B$, we get an intermediate ring $A \subset A[\alpha_1, \dots, \alpha_k] \subset B$ where $A[\alpha_1, \dots, \alpha_k]$ is the image of the evaluation map $\text{ex}_\alpha : f(\overset{x}{}) = f(\alpha_1, \dots, \alpha_k)$. Thus, if K is a field and $h(x) \in K[x]$ is irreducible and $\alpha = t + (h)$, then $L := K[x]_{/(h)} = K[\alpha]$. Observe that α is algebraic over K , meaning that L is generated by a single algebraic element α .

Definition. We say that field extension $L \supset K$ is *simple* if it is isomorphic to $K[x]_{/(h)}$ for some irreducible h .

Example 9.

1. $\mathbb{C} = \mathbb{R}[i] = \mathbb{R}[x]_{/(x^2 + 1)}$
2. If K is any field and $a \in K$ is not a square, then $x^2 - a$ is irreducible and we get a simple field extension $K[\sqrt{a}] := K[t]_{/(t^2 - a)}$.

Definition. Let $L \supset K$ be any field extension and $u \in L$ be algebraic over K . Consider

$$\text{ann}(u) := \{g(x) \in K[x] \mid g(u) = 0\},$$

which is an ideal in $K[x]$. Since $K[x]$ is a PID, we see that this ideal is generated by a single element $s(x)$. If we require that $s(x)$ be monic, then it is uniquely determined. We call this the *minimal polynomial* of u , denoted by $\min_u(x)$.

Lemma 8. If $L \supset K$ is a field extension and $u \in L$ is algebraic over K , then $\min_u(x)$ is irreducible and $K[u]$ is isomorphic to the simple field extension $K[t]/\min_u(x)$.

Proof. If $\min_u(x) = f(x)g(x)$, then $0 = \min_u(u) = f(u)g(u)$, so that either $f(u) = 0$ or $g(u) = 0$. But $f, g \mid \min_u$, so that $\deg f, \deg g \leq \deg \min_u$. By the minimality of \min_u , this implies that $\deg f = \deg \min_u$ or $\deg g = \deg \min_u$. Then either $\deg f = 0$ or $\deg g = 0$. \square

Theorem 4. Let $L \supset K$ be a field extension and $u \in L$. Then

- (a) u is algebraic over K if and only if $K[u]$ is a finite dimensional vector space over K .
- (b) if u is algebraic, then $[K(u) : K] = \deg \min_u$.

Proof. We have proven (b) in the last lemma. For (a), suppose that the ring $K[u]$ is finite dimensional as a vector space over K . Then there exist nonnegative integers k_1, \dots, k_s such that $K[u] = \text{span}_K(u^{k_1}, \dots, u^{k_s})$. Thus, if $m > \max(k_1, \dots, k_s)$, then u^m is a K -linear combination of u^{k_1}, \dots, u^{k_s} . Write $u^m = a_1 u^{k_1} + \dots + a_s u^{k_s}$. Then $f(x) = x^m - \sum_{i=1}^s a_i x^{k_i}$ satisfies $f(u) = 0$. Conversely, if $u \in L$ is algebraic over K , then there is some $n > 0$ such that $u^n = \text{span}_K(1, u, \dots, u^{n-1})$. Then $u^m \in \text{span}_K(1, u, \dots, u^{n-1})$ for any m . This implies that $K[u]$ is finite dimensional over K . \square

Corollary 12. If $L \supset K$ is a finite field extension, then L is algebraic over K .

4.2 Lecture 11

Definition. A finite field extension of \mathbb{Q} is called a *number field*.

Example 10. Fix a prime $p > 0$. Let $\epsilon_p = e^{\frac{2\pi i}{p}} \in \mathbb{C}$. Then $\mathbb{Q}(\epsilon_p) \supset \mathbb{Q}$ is a finite extension because ϵ_p is annihilated by the polynomial $x^p - 1$. It is called the *p-th cyclotomic field*. Note that $x^p - 1$ is not minimal since we can factor out $(x - 1)$. We claim that $\frac{x^p - 1}{x - 1}$ is the minimal polynomial, so that $[\mathbb{Q}(\epsilon_p) : \mathbb{Q}] = p - 1$. This will hold if we can prove that $\frac{x^p - 1}{x - 1}$ is irreducible in $\mathbb{Q}[x]$.

Lemma 9. (Gauss) If $f(x) \in \mathbb{Z}[x]$ is irreducible, then it is irreducible in $\mathbb{Q}[x]$.

Proof. Note that if $p(x) \in \mathbb{Q}[x]$, then there exists $N \in \mathbb{Z}_{>0}$ such that $Np(x) \in \mathbb{Z}[x]$ and the coefficients of Np are pairwise coprime. Suppose that $f(x) \in \mathbb{Z}[x]$ is irreducible in $\mathbb{Z}[x]$. Suppose, towards a contradiction, that there are $g(x), h(x) \in \mathbb{Q}[x]$ non-units such that $f(x) = g(x)h(x)$. Then $g(x)$ and $h(x)$ are \mathbb{Q} -proportional to some $\tilde{g}(x)$ and $\tilde{h}(x)$, respectively, over \mathbb{Z} with each having pairwise coprime coefficients. Thus, $f(x) = \lambda \tilde{g}(x)\tilde{h}(x)$ for some $\lambda \in \mathbb{Q}^\times$. Let $\lambda = \frac{a}{b}$ with $(a, b) = 1$. If $b \neq \pm 1$, then there is some $p > 0$ where $p \mid b$ and $pf = a\tilde{g}\tilde{h}$. We have that $b\tilde{f}, a\tilde{g}, \tilde{h} \in \mathbb{Z}[x]$. We can reduce mod p to get $[b\tilde{f}]_p = [a]_p[\tilde{g}]_p[\tilde{h}]_p$. But $f(x) \in \mathbb{Z}[x]$, so that $[b\tilde{f}]_p = [b]_p[\tilde{f}]_p = 0$. Hence $[a]_p[\tilde{g}]_p[\tilde{h}]_p = 0$ in $(\mathbb{Z}/p)[x]$, so that one of $[a]_p$, $[\tilde{g}]_p$, and $[\tilde{h}]_p$ must be 0. But $(a, b) = 1$, so that $[a]_p \neq 0$. Since each of \tilde{g} and \tilde{h} has coprime coefficients, we have that $[\tilde{g}]_p \neq 0$ and $[\tilde{h}]_p \neq 0$, a contradiction. \square

Thus, it suffices to show that $\frac{x^p - 1}{x - 1}$ is irreducible in $\mathbb{Z}[x]$. Let $f(x) = x^{p-1} + x^{p-2} + \dots + x + 1$. Then $f(x)(x - 1) = x^p - 1$. By the binomial formula, we see that $[(x - 1)^p]_p = [x^p - 1]_p$. Thus, $[f]_p[x - 1]_p = [(x - 1)^p]_p$, so that $[f]_p[(x - 1)]_p = ([x - 1]_p)^p$ and $[f]_p = [(x - 1)]_p^{p-1}$. If $f = gh$ for some non-units g and h , then $[g]_p[h]_p = ([x - 1]_p)^{p-1}$, which implies that $[g]_p = [(x - 1)]_p^r$ and $[h]_p = [(x - 1)]_p^s$ for some r and s . Thus, $[g(1)]_p = [g]_p(1) = 0 = [h]_p(1) = [h(1)]_p$, meaning that $p \mid g(1)$ and $p \mid h(1)$. Since $f = gh$, it follows that $p^2 \mid f(1) = p$, a contradiction.

Theorem 5. Suppose that $M \supset L \supset K$ is a chain of finite field extensions. Then $M \supset K$ is also finite with $[M : K] = [M : L][L : K]$.

Proof. Let e_1, \dots, e_n be a basis of L over K and f_1, \dots, f_m be a basis of M over L . Then $\{e_i \cdot f_j\}_{i,j}$ forms a basis of M over K . \square

Note 11. Suppose $L \supset K$ is a field extension with $u_1, \dots, u_n \in L$. We get a ring $K[u_1, \dots, u_n] = \text{im } \text{ev}_u$, which is a domain since it's contained in L . Let $K(u_1, \dots, u_n) := \text{Frac}(K[u_1, \dots, u_n])$. Then we have that $K \subset K[u_1, \dots, u_n] \subset K(u_1, \dots, u_n) \subset L$. Note that if $u \in L$ is algebraic over K , then $K \subset K[u] = K(u) \subset L$.

Theorem 6. Suppose that $L \supset K$ is a field extension and let $u_1, \dots, u_n \in L$ be algebraic over K . Then $\dim_K K(u_1, \dots, u_n) < \infty$. In particular, $K(u_1, \dots, u_n) \supset K$ is an algebraic extension.

Proof. Note that $K \subset K(u_1) \subset K(u_1, u_2) \subset \dots \subset K(u_1, \dots, u_n)$ and that $K(u_1, \dots, u_k) = K(u_1, \dots, u_{k-1})(u_k)$. Since each u_k is algebraic over K , we see that u_k is algebraic over any field containing K . Thus, u_k is algebraic over $K(u_1, \dots, u_{k-1})$. Hence $\dim_{K(u_1, \dots, u_{k-1})} K(u_1, \dots, u_k) < \infty$. By our last theorem, $\dim_K K(u_1, \dots, u_n) < \infty$. \square

Definition.

1. A field K is *algebraically closed* if for every $L \supset K$ and every $u \in L$ algebraic over K , it follows that $u \in K$.
2. We say that $K \subset L$ is *algebraically closed in L* if any $u \in L$ that is algebraic over K belongs to K .

Theorem 7. If $L \supset K$ is a field extension, then $\bar{K} := \{u \in L \mid u \text{ is algebraic over } K\}$ is a field that is algebraically closed in L .

Proof. Let $u, v \in \bar{K}$. Then both are algebraic over K . If $K \subset K(u, v) \subset L$, then our previous theorem shows that $K(u, v) \supset K$ is an algebraic extension. Since $K(u, v) \subset \bar{K}$, it follows that \bar{K} is a field. \square

Note 12. Suppose $u \in L$ is algebraic over \bar{K} . Then we can find $f(x) = \sum_{i=1}^n a_i x^i \in \bar{K}[x]$ such that $\deg f > 0$ and $f(u) = 0$. Hence $f \in K(a_1, \dots, a_n)[x]$, so that u is algebraic over $K(a_1, \dots, a_n)$. Hence $K(a_1, \dots, a_n, u)$ is finite dimensional over $K(a_1, \dots, a_n)$. But $a_1, \dots, a_n \in K$ are algebraic over K , so that $K(a_1, \dots, a_n)$ is algebraic over K . This means that u is algebraic over K .

Let h is an irreducible polynomial over K . Write $\tilde{K} = K[x]_{(h)}$ and let α denote the marked root $x + (h)$ of h viewed as a polynomial in $\tilde{K}[t]$.

Lemma 10. (Main lemma of Galois theory) For any $\varphi : K \rightarrow F$ field homomorphism, the natural map

$$\{\psi : \tilde{K} \rightarrow F \mid \psi|_{K} = \varphi\} \rightarrow (\text{distinct roots of } h^\varphi \in F[x])$$

given by $\psi \mapsto \psi(\alpha)$ is a bijection, where h^φ denotes the polynomial obtained by applying φ to the coefficients of h .

Proof. Let $\psi : \tilde{K} \rightarrow F$ be a homomorphism with $\psi|_{K} = \varphi$. Then

$$\begin{aligned} h^\varphi(\psi(\alpha)) &= \varphi(a_n)\psi(\alpha)^n + \varphi(a_{n-1})\psi(\alpha)^{n-1} + \dots + \varphi(a_1)\psi(\alpha) + \varphi(a_0) \\ &= \psi(a_n)\psi(\alpha)^n + \psi(a_{n-1})\psi(\alpha)^{n-1} + \dots + \psi(a_1)\psi(\alpha) + \psi(a_0) = \psi(h(\alpha)) \\ &= 0. \end{aligned}$$

Now, let $\xi \in F$ be a root of h^φ . Define a homomorphism $K[x] \rightarrow F$ by $f(x) \mapsto f^\varphi(\xi)$. Then $h(x) \mapsto h^\varphi(\xi) = 0$. Thus, this homomorphism descends to a homomorphism $\psi : K[x]_{(h)} \rightarrow F$ such that $\psi(\alpha) = \xi$. This implies that the assignment $\psi \mapsto \psi(\alpha)$ is surjective.

Now, suppose that $\tilde{\varphi} : \tilde{K} \rightarrow F$ is any homomorphism such that $\tilde{\varphi}|_{K} = \varphi$. Then $\tilde{\varphi}(\alpha)$ is a root of h^φ . Let $\psi_{\tilde{\varphi}(\alpha)} : \tilde{K} \rightarrow F$ be the extension that we constructed. Then $\tilde{\varphi}|_{K} = \varphi$, and $\psi_{\tilde{\varphi}(\alpha)}|_{K} = \varphi$. Also, we have that $\tilde{\varphi}(\alpha) = \xi$ and $\psi_{\tilde{\varphi}(\alpha)}(\alpha) = \xi$. This shows that $\tilde{\varphi}|_{K(\alpha)} = \psi_{\tilde{\varphi}(\alpha)}|_{K(\alpha)}$. But $K(\alpha) = \tilde{K}$. \square

5 Splitting fields

5.1 Lecture 12

Definition. If K is a field and $f(x) \in K[x]$, then a field extension $L \supset K$ is a *splitting field* for f if

- (a) $f(x) = a \prod_{i=1}^n (x - c_i)$ with $a, c_i \in L$ and
- (b) $L = K(c_1, \dots, c_n)$.

Theorem 8. For every $f(x) \in K[x]$, a splitting field for f exists and is unique up to an isomorphism over K .

Proof. Consider the tower of fields $K = K_0 \subset K_1 \subset K_2 \subset \dots$ where $K_i = K_{i-1}[\alpha_i]$ and α_i is a root of an irreducible factor f_i of f over K_{i-1} with $\deg f_i > 0$. The degree of f is fixed, but the number of irreducible factors of f strictly increases after each step. Hence this sequence of fields will stabilize at some K_s , which is thus a splitting field for f .

To prove uniqueness, suppose that $L \supset K$ is another splitting field for f . We have $\varphi_0 : K_0 = K \hookrightarrow L$. By the main lemma of Galois theory, we can extend φ_0 to a homomorphism $\varphi_1 : K_1 \rightarrow L$ provided that $f_1^{\varphi_0}$ has a root in L . But by assumption, f^{φ_0} has each of its roots in L . Since $f_1 \mid f$, it follows that $f_1^{\varphi_0}$ has each of its roots in L as well. This implies that $\varphi_1 : K_1 \rightarrow L$ will extend to a map provided that $\varphi_2 : K_2 \rightarrow L$ $f_2^{\varphi_1} = f_2^{\varphi_0}$ has some root in L . But this holds since $f_2 \mid f$. Continuing in this way, we get $\varphi_s : K_s \rightarrow L$ such that $f_1^{\varphi_{s-1}}$ has all roots in L . Thus, $f^{\varphi_s} = f^{\varphi_0}$ has all roots in L . But $\varphi_s \upharpoonright_K = \varphi_0$, so that φ_s is injective. But $L = K(\text{all roots of } f)$. By construction, all roots of f belong to $\text{im } \varphi_s$. Also, $K \subset \text{im } \varphi_s$. Hence φ_s is surjective and thus an isomorphism. \square

Exercise 11. Describe all splitting fields of polynomials of degree 2.

Example 11. Suppose that K is a field of characteristic $\neq 2$. Let $f(x) = x^3 + a_2x^2 + a_1x + a_0 \in K[x]$. Let L be a splitting field for f . What can L be? This depends on the splitting behavior of f over K

1. Suppose that f has all roots in K . Then $L = K$, and $[L : K] = 1$.
2. Suppose that f has exactly one root in K . Then $f(x) = (x - \alpha)g(x)$ with $\alpha \in K$ and $g(x)$ a quadratic irreducible in $K[x]$. Consider $L = K[x]_{(g)}$. Then $[L : K] = 2$, and g has a root in L . This implies that g has all roots in L . Hence L is the splitting field for f .
3. Suppose that f has no roots in K . Then f is irreducible in $K[x]$. Let $K_1 = K[x]_{(f)}$, which is a simple extension of degree 3. Note that f has a root α_1 in K_1 . Thus, $K_1 = K[\alpha_1]$. Consider $f(x) = (x - \alpha_1)g(x)$ with $g \in K_1[x]$ and $\deg g = 2$. There are two sub-cases to consider.
 - (a) Suppose that g has two roots in K_1 . Then $L = K_1$, so that $[L : K] = 3$.
 - (b) Suppose that f is irreducible in K_1 . Then $L = K_2 = K_1[x]_{(g)}$, so that $[L : K] = 6$.

We conclude that if L is the splitting field for f , then $[L : K] \in \{1, 2, 3, 5\}$.

How can we compute $[L : K]$ from the coefficients of f ? We have that $f(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$ in $L[x]$. Look at $\text{Discr}(f) := (\alpha_1 - \alpha_2)^2(\alpha_1 - \alpha_3)^2(\alpha_2 - \alpha_3)^2 \in L$. This is a symmetric function in $\alpha_1, \alpha_2, \alpha_3$. Hence it is expressible in terms of a_2, a_1, a_0 . Note that

$$\text{Discr}(f) = a_1^2a_2^2 - 4a_2^2a_0 - 4a_1^3 + 18a_0a_1a_2 - 27a_0^2.$$

Proposition 11. Suppose that f has no roots in K . Then $[L : K] = 3 \iff \text{Discr } f \in K^2$.

Proof. We know that f is irreducible over K . Hence $K_1 = K[x]_{(f)}$ is an extension of degree 3 in which f has a root α_1 . Note that $\text{Discr}(f) \notin K^2 \iff \text{Discr}(f) \notin K_1^2$. The (\implies) direction is obvious. For the reverse direction, suppose, towards a contradiction, that $\text{Discr}(f) \notin K^2$ but $\text{Discr}(f) \in K_1^2$. This means that $[K[\sqrt{\text{Discr}(f)}] : K] = 2$ and $K \subset K[\sqrt{\text{Discr}(f)}] \subset K_1$. Thus, $3 = [K_1 : K] = [K[\sqrt{\text{Discr}(f)}] : K] \cdot [K_1 : K[\sqrt{\text{Discr}(f)}]] = 2 \cdot 1$, a contradiction.

Now, $\text{Discr}(f) \in K_1^2 \iff (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_2 - \alpha_3) \in K_1$. This implies that $\alpha_2 - \alpha_3 \in K_1$. Indeed, $f \in K_1[x]$ satisfies $f(x) = (x - \alpha_1)g(x)$, and $\alpha_2, \alpha_3 \in L$ are roots of g . Therefore, we have in L that $g(\alpha_1) = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \in L$. But $g \in K_1[x]$ and $\alpha_1 \in K_1$, so that $g(\alpha_1) \in K_1$. It follows that $\alpha_2 - \alpha_3 \in K_1$. Hence $\text{Discr}(f) \in K_1^2 \iff \alpha_2 - \alpha_1 \in K_1$. But $-\alpha_2 - \alpha_3$ is a coefficient of g in K_1 . Therefore, $\alpha_2, \alpha_3 \in K_1$. \square

Note 13.

1. Suppose that K is a finite field. Then $\text{char}K > 0$.
2. Suppose that K is any field such that $\text{char}K = p > 0$. Then the natural map $\varphi : K \rightarrow K$ given by $x \mapsto x^p$ respects addition due to the binomial theorem. Hence it is a field homomorphism, called the *Frobenius morphism*. If K is finite, then this map is an automorphism. In general, $\text{im } \varphi = K^p \subset K$ is a subfield.
3. If K has characteristic p , then the natural map $\mathbb{F}_p \rightarrow K$ given by $[n] \mapsto \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}$ is a field extension. Therefore, if K is finite, then $K \supset \mathbb{F}_p$ is a finite field extension. In this case, if K has degree n , then $K \cong \mathbb{F}_p^{\oplus n}$ is a vector space over \mathbb{F}_p . Hence $|K| = |\mathbb{F}_p|^n = p^n$.

6 Finite fields

Theorem 9. For every prime p and integer $n > 0$, there is some finite field K consisting of p^n elements that is unique up to an isomorphism over \mathbb{F}_p .

Proof. We first prove uniqueness. If F is a finite field with $q := p^n$ elements, then $|F^\times| = q - 1$. It follows that for any $a \in F^\times$, $a^{q-1} = 1$. But then for any $a \in F$, $a^q = a$, so that each element of F is a root of $x^q - x \in \mathbb{F}_p[x]$. Then $\prod_{a \in F} (x - a) \mid x^q - x$ in $F[x]$. This implies that $x^q - x = \prod_{a \in F} (x - a)$ in $F[x]$. This means that F is a splitting field for $x^q - x$ over \mathbb{F}_p , which must be unique up to isomorphism.

To prove existence, consider F the splitting field for $x^q - x$ over \mathbb{F}_p . We want to show that $|F| = q$.

Note 14. If A is any commutative ring, then $A[x]$ has a natural derivation. There exists a unique map $\frac{d}{dx} : A[x] \rightarrow A[x]$ such that $\frac{d}{dx}(a) = 0$ for any $a \in A$, $\frac{d}{dx}(x) = 1$, and $\frac{d}{dx}$ satisfies the Leibniz rule, i.e., $\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}$. Note that $\frac{d}{dx}$ is given by $\frac{d}{dx}(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0) = na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \cdots + 0$. Then $\frac{d}{dx}$ is an A -module homomorphism. If $A \subset B$ is a subring, then we get compatible derivations $\frac{d}{dx} \hookrightarrow A[x] \subset B[x] \leftarrow \frac{d}{dx}$.

Consider $f(x) = x^q - x$. Then since $F \supset \mathbb{F}_p$ is the splitting field for $f(x)$, it follows that $f(x) = \prod_{i=1}^q (x - c_i)$ where $c_i \in F$. How many distinct roots does $f(x)$ have in F ? If $f(x)$ has a repeated root, then we can write $f(x) = (x - c)^2 g(x)$ in $F[x]$. This implies that $\frac{df}{dx}(x) = 2(x - c)g + (x - c)^2 \frac{dg}{dx}$ will also have c as a root. But $\frac{df}{dx} = qx^{q-1} - 1 = -1$ in $\mathbb{F}_p[x] \subset F[x]$. But in this case $\frac{df}{dx}$ has no roots. Thus, $f(x)$ has no repeated roots in F , so that $|F| \geq q$.

Now consider $R_f := \{c \in F \mid f(c) = 0\}$. Note that $\mathbb{F}_p \subset R_f \subset F$ and that $R_f = \{c \in F \mid \varphi^n(c) = c\}$ where φ denotes Frobenius. But since φ is a field automorphism of F , so is φ^n . Hence the fixed points of φ^n form a subfield. This means that R_f is a subfield, hence a splitting field for f . Thus, $R_f \cong F$. \square

6.1 Lecture 13

Note 15. We write \mathbb{F}_q for the splitting field for $x^q - x \in \mathbb{F}_p[x]$.

Proposition 12.

1. The group \mathbb{F}_q^\times is a cyclic group of order $q - 1$

Proof. By the structure theorem for finite abelian groups, we get

$$\mathbb{F}_q^\times \cong \mathbb{Z}/p_1^{m_{11}} \times \mathbb{Z}/p_1^{m_{12}} \times \cdots \times \mathbb{Z}/p_1^{m_{1k}} \times \mathbb{Z}/p_2^{m_{21}} \times \mathbb{Z}/p_2^{m_{22}} \times \cdots \times \mathbb{Z}/p_2^{m_{2k}} \times \cdots \times \mathbb{Z}/p_s^{m_{s1}} \times \mathbb{Z}/p_s^{m_{s2}} \times \cdots \times \mathbb{Z}/p_s^{m_{sk}}.$$

Let $\alpha_i = p_1^{m_{1i}} p_2^{m_{2i}} \cdots p_s^{m_{si}}$ for each $i = 1, \dots, k$. Hence $|\mathbb{F}_q^\times| \mid d_1 d_2 \cdots d_k$ where $d_1 \mid d_2 \mid \cdots \mid d_k$. Hence every element in \mathbb{F}_q^\times has order dividing d_k . For any $a \in \mathbb{F}_q$, $a^{d_k+1} = a$, so that $|\mathbb{F}_q| = \deg x^{d_k+1} - x = d_k + 1$. Then $q \leq d_k + 1$, so that $q - 1 \leq d_k$. Since $d_k \mid q - 1$, we have that $d_k = q - 1$. Thus, $d_1 = d_2 = \cdots = d_{k-1} = 1$. Hence $\mathbb{F}_q^\times \cong \mathbb{Z}/p_1^{m_{1k}} \times \mathbb{Z}/p_2^{m_{2k}} \times \cdots \times \mathbb{Z}/p_s^{m_{sk}}$. Since the $p_i^{m_{ik}}$ are pairwise coprime, it follows that $\mathbb{F}_q^\times \cong \mathbb{Z}/p_1^{m_{1k}} p_2^{m_{2k}} \cdots p_s^{m_{sk}}$. \square

Corollary 13. Since \mathbb{F}_q^\times is cyclic, we know that $\mathbb{F}_q^\times = \langle \sigma \rangle$. Hence $\mathbb{F}_q = \mathbb{F}_p(\sigma)$.

2. $\text{Aut}(\mathbb{F}_q)$ is a cyclic group of order n . In fact, $\text{Aut}(\mathbb{F}_q) \cong \langle \varphi \rangle$ where φ denotes Frobenius.

Proof. We have arranged it so that \mathbb{F}_q is unique up to isomorphism over \mathbb{F}_p , so that each $\psi \in \text{Aut}(\mathbb{F}_q)$ restricts to the identity on $\mathbb{F}_p \subset \mathbb{F}_q$. This implies that $\text{Aut}(\mathbb{F}_q) \cong \text{Aut}(\mathbb{F}_q/\mathbb{F}_p)$, which is the subgroup of all $\psi : \mathbb{F}_q \xrightarrow{\cong} \mathbb{F}_q$ such that $\psi|_{\mathbb{F}_p} = \text{id}_{\mathbb{F}_p}$. [[Why?]]

Lemma 11. Let $L \supset K$ be a finite field extension of degree n . Then we have that $|\text{Aut}(L/K)| \leq n$.

Proof. Since $[L : K] = n < \infty$, we can construct L as a tower $K = K_0 \subset K_1 \subset \cdots \subset K_s = L$ where $K_{i+1} = K_i[\alpha_i]$ and α_i is a root of an irreducible $f_i(x) \in K_i[x]$. Consider $\varphi_0 : K \hookrightarrow L$ the natural inclusion. Applying the main lemma of Galois theory, we see that φ_0 extends to $\varphi_1 : K_1 \rightarrow L$ in finitely many ways such that the number of such φ_1 's equals the number of distinct roots of $f_0^{\varphi_0}$ in L . This quantity is $\leq \deg f_0 = [K_1 : K_0]$. Each φ_1 extends to a map $\varphi_2 : K_2 \rightarrow L$ in at most $\deg f_1 = [K_2 : K_1]$ ways. Therefore, φ_0 will extend to a map $\varphi_s : L \rightarrow L$ in $[K_1 : K_0][K_2 : K_1] \cdots [K_s : K_{s-1}]$ ways. It follows that $|\text{Aut}(L/K)| \leq \prod_{i=0}^{s-1} [K_{i+1} : K_i] = [L : K] = n$. \square

Corollary 14. If $f(x) \in K[x]$ and L is a splitting field for f and f has distinct roots in L , then $|\text{Aut}(L/K)| = [L : K]$.

We have that $|\text{Aut}(\mathbb{F}_q/\mathbb{F}_p)| = [\mathbb{F}_q : \mathbb{F}_p] = n$. If $\varphi \in \text{Aut}(\mathbb{F}_q/\mathbb{F}_p)$, then $\varphi^n = \text{id}$. Thus, it suffices to show that $\varphi^m \neq \text{id}$ for any $m < n$. Suppose that m has $\varphi^m = \text{id}$. Then $\varphi^m(a) = a$ for every $a \in \mathbb{F}_q$. Therefore, $q^{p^m} = a$ for each $a \in \mathbb{F}_q$, so that $p^n = q = |\mathbb{F}_q| \leq p^m$. Then $m \geq n$.

This completes our main proof. \square

3. There is a bijection (subfields of \mathbb{F}_q) \cong_{Set} (subgroups of $\text{Aut}(\mathbb{F}_q)$).

Proof. Let $F \subset \mathbb{F}_q$ be a subfield, so that $\mathbb{F}_p \subset F \subset \mathbb{F}_q$. We have that $|F| \mid |\mathbb{F}_q| = p^n$, so that $|F| \leq p^d$ for some $d \leq n$. Since $\mathbb{F}_q \supset F$, we have that \mathbb{F}_q is a vector space over F . If $[\mathbb{F}_q : F] = r$, then $\mathbb{F}_q \cong F^{\oplus r}$ as F -vector spaces. Note that $p^n = |\mathbb{F}_q| = |F|^r = (p^d)^r = p^{dr}$, which implies that $d \mid n$.

Since F is a finite field, it follows that F^\times is cyclic of order $p^d - 1$. Hence any $a \in F \subset \mathbb{F}_q$ satisfies $a^{p^d} = a$. But if $d \mid n$, then $x^q - x = x^{p^n} - x = (x^{p^d} - x)g(x)$ because $p^n - 1 = p^{dr} - 1 = (p^d)^r - 1 = (p^d - 1)m$ so that $x^{q-1} - 1 = (x^{p^d-1} - 1)g(x)$. But \mathbb{F}_q is the splitting field for $x^q - x$, and all roots of this are distinct. Thus, there are exactly p^d roots of $x^q - x$ that are the distinct roots of $x^{p^d} - x$. Therefore, $F = \mathbb{F}_{p^d}$ = (subfield of \mathbb{F}_q that is the splitting field for $x^{p^d} - x$) = (fixed subfield of φ^d).

Hence F is the fixed point subgroup of $\langle \varphi^d \rangle \trianglelefteq \text{Aut}(\mathbb{F}_q)$.

Let $\psi \in \text{Aut}(\mathbb{F}_q)$ with $\psi \notin \langle \varphi^d \rangle$. Then $\psi = \varphi^e$ for some $e \geq 0$ such that $d \nmid e$. If ξ generates F^\times and $\xi^{p^e} = \psi(\xi) = \xi$, then $p^d - 1 \mid p^e - 1$ since $|F^\times| = p^d - 1$. But this is impossible, which implies that $\psi \upharpoonright_F \neq \text{id}_F$. Therefore, $\langle \varphi^d \rangle = \text{Aut}(\mathbb{F}_q/F)$, and we have a bijection

$$\begin{aligned} (\text{subfields of } \mathbb{F}_q) &\cong_{\text{Set}} (\text{subgroups of } \text{Aut}(\mathbb{F}_q)) \\ F &\mapsto \text{Aut}(\mathbb{F}_q/F) \\ \mathbb{F}_q^G &\leftrightarrow G. \end{aligned}$$

□

6.2 Lecture 14

Proposition 13.

1. Let $\mathbb{F}_q^\times = \langle \theta \rangle$. Then $\mathbb{F}_q = \mathbb{F}_p(\theta)$, meaning that θ is a *primitive element* for the extension $\mathbb{F}_q \supset \mathbb{F}_p$. Further, if h denotes the minimal polynomial of θ over \mathbb{F}_p , then \mathbb{F}_q is the splitting field for h .

Proof. Every nonzero element of \mathbb{F}_q is a power of θ . Hence $\mathbb{F}_q = \mathbb{F}_p(\theta)$. Now, note that $\deg h = n$ because $[\mathbb{F}_q : \mathbb{F}_p] = n$. Write $h(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ where each $a_i \in \mathbb{F}_p$. If we view h over \mathbb{F}_q , then $\varphi(a_i) = a_i$ due to Fermat's little theorem. Hence $\varphi(h(x)) = h(\varphi(x))$ for any $x \in \mathbb{F}_q$, meaning that $\varphi(c)$ is a root of h whenever c is a root. Thus, we get n roots of h .

$$\theta, \theta^p, \theta^{p^2}, \dots, \theta^{p^{n-1}}$$

If K is the splitting field for h , then $\mathbb{F}_p \subset K \subset \mathbb{F}_q$. But $[K : \mathbb{F}_p] = n = [\mathbb{F}_q : \mathbb{F}_p]$, so that $K = \mathbb{F}_q$. □

2. Let $m \geq 0$ be any integer and $q = p^n$. Then there is some irreducible polynomial over \mathbb{F}_q of degree m

Proof. Let $\mathbb{F}_{p^{mn}}^\times = \langle \theta \rangle$. Then the minimal polynomial $p(x)$ of θ over \mathbb{F}_q has degree m , and $p(x)$ is irreducible since it is minimal. □

7 Cyclotomic fields

Remark 10. Let $q = p^n$ and $d > 0$ be any integer. Among the finitely many polynomials over \mathbb{F}_q of degree d , how many of these are irreducible? We have just shown that at least one is irreducible.

Definition. Define the *Möbius function* $\mu : \mathbb{Z}_{>0} \rightarrow \{-1, 0, 1\}$ by

$$n \mapsto \begin{cases} -1 & n = 1 \\ (-1)^k & n = p_1 \cdots p_k \text{ where the } p_i \text{ are pairwise distinct} \\ 0 & n \text{ is divisible by a square} \end{cases}$$

Proposition 14.

1. $\mu(k) \neq 0$ for some k .
2. $\mu(nm) = \mu(n)\mu(m)$ when $(n, m) = 1$.
3. $\sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$. (This completely characterizes the Möbius function.)

Proof. Let $n > 0$ be an integer and write $p_1^{k_1} \cdots p_k^{r_k}$ where the prime p_i are pairwise distinct. Let $n_0 = p_1 \cdots p_k$. Then $\sum_{d|n} \mu(d) = \sum_{d|n_0} \mu(d)$. If $d | n_0$, then $d = p_{i_1} \cdots p_{i_s}$, so that $\mu(d) = (-1)^s$. By the binomial theorem, it follows that

$$\begin{aligned} \sum_{d|n_0} \mu(d) &= \sum_{s=0}^k \binom{k}{s} (-1)^s \\ &= (1-1)^k = \begin{cases} 1 & k=0 \\ 0 & k>0 \end{cases}. \end{aligned}$$

$$\text{Thus, } \sum_{d|n} \mu(d) = \begin{cases} 1 & n=1 \\ 0 & n \neq 1 \end{cases}.$$

□

Corollary 15. For any $m \in \mathbb{Z}_{>0}$ and $d | m$, we have that

$$\sum_{d|n|m} \mu\left(\frac{m}{n}\right) = \begin{cases} 1 & d=m \\ 0 & d \neq m \end{cases}.$$

Lemma 12. Let A be an abelian group and $f, g : \mathbb{Z}_{>0} \rightarrow A$ be functions such that $f(n) = \sum_{d|n} g(d)$ for every n . Then we obtain the so-called *Möbius inversion formula*

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d).$$

If A is written multiplicatively, then this becomes

$$g(n) = \prod_{d|n} f(d)^{\mu\left(\frac{n}{d}\right)}.$$

Proof. We compute

$$\begin{aligned} \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) \sum_{k|d} g(k) \\ &= \sum_{d|n} \sum_{k|d} \mu\left(\frac{n}{d}\right) g(k) \\ &= \sum_{k|n} g(k) \sum_{d: k|d|n} \mu\left(\frac{n}{d}\right) \\ &= \sum_{k|n} g(k) \delta(k, n) \\ &= g(n). \end{aligned}$$

□

Definition. Define the *Euler (totient) function* $\varphi : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}$ by $\varphi(n) = \#\{m \in \mathbb{Z}_{>0} : m \leq n, (m, n) = 1\}$.

Note 16. If $n \in \mathbb{Z}_{>0}$, then $n = \sum_{d|n} \varphi(d)$. Therefore, if $f : \mathbb{Z}_{>0} \rightarrow \mathbb{Z}$ is given by $f(n) = n$ and $g := \varphi$, then we can apply the Möbius inversion formula to get

$$\begin{aligned} \varphi(n) = g(n) &= \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d) \\ &= \sum_{d|n} \mu\left(\frac{n}{d}\right) d \\ &= \sum_{m|n} \mu(m) \frac{n}{m} = \left(\sum_{m|n} \frac{\mu(m)}{m} \right) n. \end{aligned}$$

Lemma 13. If $n = p_1^{r_1} \cdots p_k^{r_k}$, then $\varphi(n) = n \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right)$.

Proof. Let $n_0 = p_1 \cdots p_k$. Then

$$\begin{aligned} \sum_{m|n} \frac{\mu(m)}{m} &= \sum_{m|n_0} \frac{\mu(m)}{m} \\ &= \underbrace{1}_{m=1} - \sum_{i=1}^k \frac{1}{p_i} \\ &\quad + \sum_{i < j} \frac{1}{p_i p_j} + \cdots + (-1)^s \sum_{i_1 < \cdots < i_s} \frac{1}{p_{i_1} \cdots p_{i_s}} \\ &\quad + \cdots + (-1)^k \frac{1}{p_1 \cdots p_k} \\ &= \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right). \end{aligned}$$

□

Example 12. Let $p(x) = x^n - 1 \in \mathbb{Q}[x]$ with $n > 0$. Let Γ_n be the splitting field for $p(x)$. We know that $\Gamma_n = \mathbb{Q}(\zeta_n)$, where ζ_n denotes a primitive n -th root of unity in \mathbb{C} . Let the set Prim_n consist of all the primitive n -th roots of unity.

We have that $\mu_n = \{1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1}\} = \prod_{d|n} \text{Prim}_d$. Define the d -th cyclotomic polynomial as

$$\Phi_d(x) = \prod_{\alpha \in \text{Prim}_d} (x - \alpha).$$

For example,

$$\begin{aligned} \Phi_1 &= x - 1 \\ \Phi_2 &= x + 1 \\ \Phi_3 &= x^2 + x + 1 \\ \Phi_4 &= x^2 + 1 \\ &\vdots \\ \Phi_p &= x^{p-1} + x^{p-2} + \cdots + x + 1 \text{ with } p \text{ prime.} \end{aligned}$$

Note that

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

Consider the functions $\Phi_{(-)} : \mathbb{Z}_{>0} \rightarrow \mathbb{C}(x)^\times$ and $f : \mathbb{Z}_{>0} \rightarrow \mathbb{C}(x)^\times$ where $f(n) = x^n - 1$. We can apply the Möbius inversion formula to get

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(\frac{n}{d})},$$

which is a rational function over \mathbb{Z} . We can write $\Phi_n(x) = \frac{a(x)}{b(x)}$ with $b(x)$ monic. Write $\Phi_n(x) = x^m + p_{m-1}X^{m-1} + \cdots + p_0$ and set $M = \text{lcm}\{c_i \mid p_i = \frac{t_i}{c_i}, i = 1, \dots, m-1\}$. Let $P_i = Mp_i$ for each $i = 1, \dots, m-1$ and $P_m = M$. Since $M\Phi_n(x)b(x) = Ma(x) \in \mathbb{Z}[x]$, we see that M divides each coefficient of $M\Phi_n(x)b(x)$. Suppose, towards a contradiction, that $M > 1$. Then there exists a prime divisor p of M . By our choice of M , there exists a maximal $0 \leq i_0 \leq m$ such that $p \nmid P_{i_0}$. If $\deg b(x) = s$, then the coefficient of X^{m+s} in $M\Phi_n(x)b(x)$ has the form $M + p \cdot t$ for some $t \in \mathbb{Z}$. But this is not divisible by p and thus not divisible by M , a contradiction. Thus, $M = 1$, so that $\Phi_n(x) \in \mathbb{Z}[x]$.

Moreover, since $\deg \Phi_n = \varphi(n)$, it follows that $[\Gamma_n : \mathbb{Q}] = \varphi(n)$.

7.1 Lecture 15

Note 17. Let $q = p^n$. Let $\psi_d(q) = \#\{p(x) \text{ irreducible over } \mathbb{F}_q \mid \deg p(x) = d\}$. If $f(x) \in \mathbb{F}_q[x]$ is irreducible, then $F = \mathbb{F}_q[x]/(f)$ is a finite field. Thus, $\#F = q^d = p^{nd}$, so that F is the splitting field for $x^{pnd} - x$ over \mathbb{F}_p . Also, F is just the set of roots of $x^{pnd} - x$. By construction, the polynomial $f(x) \in \mathbb{F}_q[x]$ has a root over F , and $x^{pnd} - x \in \mathbb{F}_q[x]$ has a root in F .

Since $f(x)$ is irreducible in $\mathbb{F}_q[x]$, we see that $(f, x^{pnd} - x) \in \{1, f\}$ in $\mathbb{F}_q[x]$. But if $(f, x^{pnd} - x) = 1$, then $1 = a(x)f(x) + b(x)(x^{pnd} - x)$ with $a, b \in \mathbb{F}_q[x]$. If we write this as an equation in $F[x]$, then evaluating on $\alpha \in F$ a common root of $f(x)$ and $x^{pnd} - x$ will give us a contradiction. Hence $f(x) \mid x^{pnd} - x$ in $\mathbb{F}_q[x]$. Since all roots of $x^{pnd} - x$ are pairwise distinct, we see that any irreducible monic polynomial of degree d over \mathbb{F}_q appears exactly once in the decomposition of $x^{pnd} - x$ into irreducibles. Note that if $m = dr$, then $x^{q^d} - x \mid \underbrace{x^{q^m} - x}_{\text{has distinct roots}}$, then every irreducible monic polynomial over \mathbb{F}_q of degree $\mid m$ appears exactly once

in the irreducible decomposition of $x^{q^m} - x$. For each $d \geq 1$, let $f_{d,1}, f_{d,2}, \dots, f_{d,\psi_d(q)}$ be irreducible monic polynomials over \mathbb{F}_q of degree d . Then for any $m \geq 1$, we get

$$x^{q^m} - x = \prod_{d \mid m} \prod_{k=1}^{\psi_d(q)} f_{d,k}(x),$$

so that $q^m = \sum_{d \mid m} d\psi_d(q)$. Then

$$\psi_d(q) = \frac{1}{m} \sum_{d \mid m} \mu\left(\frac{m}{d}\right) q^d.$$

Example 13.

$$\begin{aligned} \psi_2(2) &= \frac{1}{2}(2^2 - 2) = 1. \\ \psi_3(2) &= \frac{1}{2}(2^3 - 2) = 2. \end{aligned}$$

Remark 11. A randomly chosen polynomial over \mathbb{F}_q of degree d will be irreducible with probability $\approx \frac{1}{d}$. Given a polynomial $p(x)$ over \mathbb{F}_q of degree d , there is no known algorithm that is polynomial in d for deciding whether $p(x)$ is irreducible.

8 Galois theory

Definition. If $L \supset K$ is a field extension, then define the *Galois group*

$$\text{Gal}(L/K) = \{\varphi \in \text{Aut}(L) : \varphi|_K = \text{id}_K\}.$$

Theorem 10. Let $L \supset K$ be a field extension of degree $n < \infty$. Let $G \leq \text{Gal}(L/K)$. Then the following hold.

- (a) $L^G = K \iff |G| = n$.
- (b) If $L^G = K$ and $K \subset P \subset Q \subset L$ is a chain of field extensions, then every homomorphism $\varphi : P \rightarrow L$ over K extends to a homomorphism $Q \rightarrow L$ in exactly $[Q : P]$ ways.

Proof.

- (a) For the (\Leftarrow) direction, note that if $G \leq \text{Aut}(L)$, then tautologically $G \leq \text{Gal}(L/L^G)$. Hence $|G| \leq |\text{Gal}(L/L^G)| = [L : L^G]$. If $G \leq \text{Gal}(L/K)$, then $L \subset L^G \subset L$, so that $[L : L^G] \leq [L : K] = n$. This means $|G| \leq n$.

Conversely, let $L^G = K$. Take $\alpha \in L$ and let $\text{Orb}_G(\alpha) = \{\lambda_1, \dots, \lambda_m\} \subset L$. Consider

$$f(x) = \prod_{i=1}^m (x - \lambda_i) \in L[x].$$

But the coefficients are symmetric polynomials in λ_i , and any $g \in G$ permutes the λ_i . In this case, g permutes the coefficients of $f(x)$. Hence $f(x) \in L^G[x] = K[x]$. By construction, α is a root of $f(x)$, the minimal polynomial of α . We can decompose $f(x)$ into linear factors in L .

Apply part (b) to $P = K$ and $Q = L$. In this case, (b) implies that if $L^G = K$, then $|\text{Gal}(L/K)| = [L : K] = n$. Thus, we must show that $G = \text{Gal}(L/K)$.

Let $\varphi \in \text{Gal}(L/K)$. Recall that $f(x)$ is the minimal polynomial of α over K . Note that $\varphi(x)$ is a root of $f^\varphi(x)$. Indeed, since $\varphi_K = \text{id}_K$, we have that $f^\varphi(x) = f(x)$. Hence $\varphi(\alpha) \in \text{Orb}_G(\alpha)$, so that there exists $g \in G$ such that $\varphi(\alpha) = g(\alpha)$. If L is a finite field, then we can take α to be the generator of L^\times . in which case $\varphi(\alpha) = g(\alpha) \implies \varphi(\alpha^k) = g(\alpha^k)$ for each $k \implies \varphi = g$. If L is infinite, then K is infinite and for any $g \in G$ we consider $L_g = \{a \in L \mid \varphi(a) = g(a)\} \subset L$. By definition, $L_g = L^{g^{-1} \circ \varphi}$ is a subfield in L . This contains K because $g^{-1} \circ \varphi \in \text{Gal}(L/K)$. Therefore, $K \subset L_g \subset L$ is a field extension, meaning that L_g is a K -vector subspace in L .

We have shown that $L = \bigcup_{g \in G} L_g$.

Lemma 14. If K is an infinite field and V is a finite-dimensional K -vector space and $V_1, V_2, \dots, V_g \subset V$ are subspaces, then $V = \bigcup_{i=1}^g V_i \implies V = V_k$ for some k .

Proof. Suppose that each $V_i \subsetneq V$ and that $V = \bigcup_{i=1}^g V_i$. Then there exists a linear map $f_i : V \rightarrow K$ such that $f_i|_{V_i} = 0$ and $f_i \neq 0$. Then $f : V \rightarrow K$ given by $f = \prod_{i=1}^k f_i$ is the function associates with a nonzero polynomial in $V = K^n$ of degree s . But f is the zero function since $V = \bigcup_{i=1}^k V_i$, a contradiction. \square

- (b) Suppose that $K \subset P \subset L$. Let $K \subset P \subset Q \subset L$ where $Q = P(\alpha) = P[\alpha]$ and α is a root of some irreducible $h(x) \in P[x]$. Let $f(x) = \prod_{i=1}^m (x - \lambda_i)$ where $\text{Orb}_G(\alpha) = \{\lambda_1, \dots, \lambda_m\} \subset L$. Then $f(x), g(x) \in P[x]$ have a common root, and $h(x)$ is irreducible. Hence $h \mid f$ in $P[x]$.

Let $\varphi : P \rightarrow L$ be any field homomorphism over K . Then $h^\varphi \mid \underbrace{f^\varphi}_{=f}$ in $L[x]$. But f decomposes into

distinct linear factors in $L[x]$. Hence h^φ equals a product of a subcollection of these factors. It follows that h^φ has $\deg h^\varphi = \deg h$ distinct roots in L . By the main lemma of Galois theory, since $Q \cong P[x]/(h)$, we see that φ extends in exactly $\deg h - [Q : P]$ ways.

This proves our result for simple field extensions. Since every finite extension is a tower of simple extensions, we are done by induction on the length of the tower. \square

Definition. A finite field extension $L \supset K$ is a *Galois extension* if $|\text{Gal}(L/K)| = [L : K]$.

Corollary 16. If $L \supset K$ is a Galois extension, then $K \subset P \subset L \implies L \supset P$ is Galois as well.

Proof. Take $Q = L$ and apply (b) then (a). \square

Definition. If K is a field and $f(x) \in K[x]$, then we say that f is *separable over K* if f has no repeated roots in any finite extension of K . Equivalently, f has not repeated roots in its splitting field.

8.1 Lecture 16

Proposition 15. A polynomial $f(x) \in K[x]$ is separable over K if and only if $(f, f') = 1$.

Proof. If $f, g \in K[x]$, then $(f, g) \in K[x]$. Suppose there exists $L \supset K$ such that f has a multiple root in L . Then there exists an irreducible polynomial $h(x) \in L[x]$ such that $h^2 \mid f$. This implies that $f = h^2 q$, so that $f' = 2hh'q + h^2q' = h(2h'q + hq')$. Hence $h \mid f'$ in $L[x]$. Then $h \mid (f, f')$ in $L[x]$, making $(f, f') \neq 1$.

Conversely, suppose that $(f, f') \neq 1$. Then there exists h irreducible in $K[x]$ such that $h \mid f'$ and $h \mid f$ in $K[x]$. We can write $f = hg$, so that $f' = h'g + hg'$. Either $h \mid g$ or $h' = 0$. In the former case, we

have that $h \mid g \implies h^2 \mid f \implies f$ has a double root in $L = K[x]_{(h)}$. In the latter case, we see that $\text{char}K = p > 0$ and $h(x) = a_0 + a_1x^p + a_2x^{2p} + \cdots + a_sx^{sp}$ with $a_s \neq 0$. Let $L \supset K$ be a finite field extension such that for any $i = 0, \dots, s$, we have b_i such that $b_i^p = q_i$. Then viewing $h(x) \in L[x]$, we get $h(x) = (b_0 + b_1x + b_2x^2 + \cdots + b_sx^s)^p$ since $b_s \neq 0$. Hence if $\varphi(x)$ is an irreducible factor of $b_0 + b_1x + \cdots + b_sx^s$ in $L[x]$, then if $F = L[x]_{(\varphi)}$, then φ has a root in F and h will have a root with multiplicity p in F . In this case, f has a root with multiplicity p in F . \square

Corollary 17. If K has $\text{char}K = 0$, then every irreducible $f(x) \in K[x]$ is separable.

Proof. If $\text{char}K = 0$, then $f \neq 0$ and f is irreducible. Since $\deg f > 0$, it follows that $f' \neq 0$. Hence (f, f') is a polynomial of degree ≥ 0 . Since f is irreducible, this means that $(f, f') \in \{1, f\}$. But $\deg f' < \deg f$, so that $(f, f') = 1$. \square

Corollary 18. If $f(x) \in K[x]$ is irreducible and $\text{char}K \nmid \deg f$, then f is separable.

Corollary 19. Every irreducible polynomial f over a finite field F is separable.

Proof. If f is irreducible and $f' \neq 0$, then apply a similar argument to the proof of Corollary 16. Suppose $f' = 0$. Then $f(x) = a_0 + a_1x^p + \cdots + a_sx^{sp}$ with $p = \text{char}F$. But as F is finite, we know that the Frobenius φ is an automorphism. Thus, any element in F has a p -th root in F . Hence there exists $b_i \in F$ such that $b_i^p = a_i$. This shows that $f(x) = (b_0 + b_1x + \cdots + b_sx^s)^p$, which contradicts that f is irreducible over F . \square

Example 14. There are irreducible polynomials over fields of characteristic > 0 that are not separable. For example, let $K = \mathbb{F}_p(t)$ and $f(x) = x^p - t$. This is irreducible in $K[x]$ but not separable over K .

Proof. If $L \supset K$ is such that f has a root α in L , then f splits in $L[x]$. We can write $f(x) = (x - \alpha)^p$. But if $0 < k < p$, then $\alpha^k \notin K$. This shows that f is irreducible in $F[x]$ but has a root of multiplicity p . \square

Theorem 11. If $f(x) \in K[x]$ and every irreducible factor of f is separable over K , then the splitting field L of f is Galois over K .

Proof. We constructed L as a tower

$$K = K_0 \subset K_1 \subset \cdots \subset K_s = L$$

where $K_{i+1} = K_i(\alpha_{i+1})$ and α_{i+1} is a root of some irreducible factor $f_{i+1}(x)$ of $f(x) \in K_i[x]$. Since f_{i+1} is irreducible in $K_i[x]$ and $f_{i+1} \mid f$ in $K_i[x]$, it follows that f_{i+1} must divide one of the irreducible factors of $f(x)$ in $K[x]$. But these are separable, which implies that f_{i+1} is separable for each i . By the main lemma of Galois theory, a field homomorphism $\varphi : K \rightarrow L$ extends to an isomorphism $\varphi : L \rightarrow L$ in $(\# \text{ of distinct roots in } f_1) \cdot (\# \text{ of distinct roots in } f_2) \cdots (\# \text{ of distinct roots in } f_{s-1})$ ways. Note that $(\# \text{ of distinct roots in } f_1) \cdot (\# \text{ of distinct roots in } f_2) \cdots (\# \text{ of distinct roots in } f_{s-1}) = \deg f_1 \cdot \deg f_2 \cdots \deg f_{s-1}$. Hence $|\text{Gal}(L/K)| = \deg f_1 \cdot \deg f_2 \cdots \deg f_{s-1} = [K_1 : K_0][K_2 : K_1] \cdots [K_s : K_{s-1}] = [L : K]$. \square

Note 18. If $f(x) \in K[x]$ and $L \supset K$ is the splitting field for f , then let $\alpha_1, \dots, \alpha_m$ denote the distinct roots of f in L . We have that $L = K(\alpha_1, \dots, \alpha_m)$ and any $\varphi \in \text{Gal}(L/K)$ sends $\{\alpha_1, \dots, \alpha_m\}$ to itself. This gives us a homomorphism $\text{Gal}(L/K) \rightarrow S_m$ that is injective by the main lemma of Galois theory. Therefore, $\text{Gal}(L/K) \subset S_m$.

Example 15.

1. Let K be a field and let $f(x) \in K[x]$ be irreducible of degree 2. Let L denote the splitting field for $f(x)$. Then $K[\sqrt{D}]$ where $D = \text{Discr}(f) \in K$. In this case, $[L : K] = 2$, and $\text{Gal}(L/K) \subset S_2$ since $D \neq 0$. Thus, f must have distinct roots in L . Note that $\text{Gal}(L/K) \neq \{\text{id}\}$, since these roots are not in K . This shows that $\text{Gal}(L/K) = \langle \sigma \rangle = S_2$ where $\sigma : L \rightarrow L$ is given by $a + b\sqrt{D} \mapsto a - b\sqrt{D}$.
2. Let $q = p^n$. Consider the extension $\mathbb{F}_q \supset \mathbb{F}_p$. Then $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p) = \langle \varphi \rangle \cong \mathbb{Z}/n$.

3. Recall that the cyclotomic field $\Gamma_n \supset \mathbb{Q}$ is the splitting field for $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \dots + x + 1)$. This polynomial has irreducible factors each of which is separable. Thus, $\Gamma_n \supset \mathbb{Q}$ is a Galois extension such that $\text{Gal}(\Gamma_n/\mathbb{Q}) \subset S_{\mu_n}$. Since any $g \in \text{Gal}(\Gamma_n/\mathbb{Q})$ respects multiplication in Γ_n , we see that $g \upharpoonright_{\mu_n}: \mu_n \rightarrow \mu_n$ is a group automorphism. It follows that $\text{Gal}(\Gamma_n/\mathbb{Q}) \subset \text{Aut}_{\mathbf{Grp}}(\mu_n) \cong (\mathbb{Z}/n)^\times$, which has order $\phi(n)$. We have shown that the minimal polynomial of a root of 1 over \mathbb{Q} is precisely $\Phi_n(x)$, where $\deg \Phi_n(x) = \phi(n)$. Hence $[\Gamma_n : \mathbb{Q}] = \phi(n)$, so that $\text{Gal}(\Gamma_n/\mathbb{Q}) = (\mathbb{Z}/n)^\times$.
4. Suppose that $\text{char} K \notin \{2, 3\}$. Let $f(x) \in K[x]$ be irreducible and monic of degree 3. Let $D \in K$ denote the discriminant of f . Let $L \supset K$ be the splitting field for f , so that $L \supset K$ is Galois. Then

$$|\text{Gal}(L/K)| = \begin{cases} 6 & D \notin K^2 \\ 3 & D \in K^2 \end{cases}.$$

But $\text{Gal}(L/K) \subset S_3$. This shows that

$$\text{Gal}(L/K) = \begin{cases} S_3 & D \notin K^2 \\ A_3 & D \in K^2 \end{cases}.$$

8.2 Lecture 17

Definition. Let k be a field and A be a finitely generated k -algebra. A collection $u_1, \dots, u_n \in A$ is a *transcendence basis* of A/k if

- the u_i are independent transcendentals over k and
- every $a \in A$ is algebraically dependent with $k[u_1, \dots, u_n]$.

Note 19. If A is a domain and u_1, \dots, u_n forms a transcendence basis of A/k , then they also form a transcendence basis of $\text{Frac}(A)$ over k . Observe that $x \in \text{Frac}(A)$ is algebraic over $k[u_1, \dots, u_n]$ if and only if it is algebraic over $k(u_1, \dots, u_n)$. Then $S := \{x \in \text{Frac}(A) \mid x \text{ is algebraic over } k[u_1, \dots, u_n]\}$ is a subfield. But $A \subset S \subset \text{Frac}(A)$, so that, by the universal property, $S = \text{Frac}(A)$. Hence $\text{Frac}(A)$ is algebraic over $k(u_1, \dots, u_n)$.

Let $A = k[u_1, \dots, u_n]$ and suppose that $\{u_1, \dots, u_d\}$ is a maximal subset of algebraically independent elements over k in $\{u_1, \dots, u_n\}$. Then u_1, \dots, u_d form a transcendence basis of A/k . Indeed, K equals the algebraic closure of $k(u_1, \dots, u_d)$ in $\text{Frac}(A)$. Thus, $u_1, \dots, u_n \in K$, so that $K = \text{Frac}(A)$. It follows that $K \supset A$.

As a result, if A is a finitely generated algebra without zero divisions, then A has a transcendence basis over k . Indeed, choose any system of generators of A/k and then choose a maximal subset of algebraically independent elements.

Lemma 15. Suppose that $\{u_1, \dots, u_n\}$ is a transcendence basis of A/k and that v is transcendental over $k[u_1, \dots, u_n]$. Then $\{v, u_2, u_3, \dots, u_n\}$ is also a transcendence basis of A/k .

Proof. Note that v, u_2, \dots, u_n are algebraically independent over k whereas v, u_1, u_2, \dots, u_n are algebraically dependent. A nontrivial algebraic relation among these will be given by a polynomial $p(x)$ over k such that $p(x)$ includes a monomial involving u_1 with a nonzero coefficient. Then $p(x)$ can be viewed as a nonzero polynomial in $(k[v, u_2, \dots, u_n])[u_1]$ with $\deg \geq 1$ on u_1 . We have that u_1 is algebraic over $k[v, u_2, \dots, u_n]$. Thus, the algebraic closure of $k[v, u_2, \dots, u_n]$ in $\text{Frac}(A)$ contains u_1 , hence contains A . It follows that the algebraic closure of $k[v, u_2, \dots, u_n]$ equals $\text{Frac}(A)$.

This shows that any transcendence basis of A/k has the same cardinality. Indeed, let u_1, \dots, u_n and v_1, \dots, v_m be transcendence bases of A/k . Then at least one of the v_i 's must be transcendental over $k[u_2, \dots, u_n]$. This is because if each v_i is algebraic over $k[u_2, \dots, u_n]$, then $A \supset k[u_2, \dots, u_n]$ will be algebraic, in which case u_2, \dots, u_n is also a basis, a contradiction.

Say that v_1 is transcendental over $k[u_2, \dots, u_n]$. Then $A \supset k[v_1, u_2, \dots, u_n]$ is algebraic. . . . One of v_1, \dots, v_m

must be transcendental over $k[v_1, \dots, u_2, \dots, u_n]$. Hence $A \supset k[v_1, v_2, u_3, \dots, u_n]$ is algebraic. If $m \leq n$, then $A \supset k[v_1, v_2, \dots, v_m, u_{m+1}, \dots, u_n]$ is algebraic and $v_1, v_2, \dots, v_m, u_{m+1}, \dots, u_n$ are dependent. This is a contradiction unless $n = m$. \square

Corollary 20. If F is a field and $\tilde{F} \supset F$ is a field extension, then we can measure how far \tilde{F} is from being an algebraic extension of F by its *transcendence degree over F*

$$\text{trdeg}(\tilde{F}/F) := \text{card}(\text{independent transcendentals we need to add to } F \text{ to generate } \tilde{F}).$$

This is an invariant of the extension \tilde{F} .

Example 16. Let k be a field and a_1, a_2, \dots, a_n be indeterminates. Let $K := k(a_1, \dots, a_n)$. Consider $f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n \in K[x]$. Then $\text{Gal}(L/K) = S_n$ where L is the splitting field for f .

Proof. Let $x_1, x_2, \dots, x_n \in L$ denote the roots of f . Then $a_i = (-1)^i \sigma_i(x_1, \dots, x_n)$ where σ_i denotes the i -th elementary symmetric function. Hence $L = K(x_1, \dots, x_n) = k(x_1, \dots, x_n)$. Consider the chain of field extensions $L \supset K \supset k$. Note that $L \supset K$ is an algebraic extension and that $K \supset k$ is a transcendental extension because K is obtained from adding n independent transcendentals to k .

Since $\text{trdeg}(L/k) = \text{trdeg}(K/k) = n$ and $L = K(x_1, \dots, x_n)$, we see that x_1, \dots, x_n are algebraically independent over k . Therefore, there are pairwise distinct. This shows that $f(x) \in K[x]$ has distinct roots, so that $L \supset K$ is separable and thus a Galois extension. It follows that $\text{Gal}(L/K) = S_n$ and

$$L^{S_n} = (k(x_1, \dots, x_n))^{S_n} = K = k(\sigma_1, \dots, \sigma_n).$$

\square

Theorem 12. (Main theorem of Galois theory) Let $L \supset K$ be a Galois extension. Then the function given by $(K \subset P \subset L : P \text{ field}) \mapsto (G \leq \text{Gal}(L/K))$ and the function given by $L^G \mapsto G$ are inverse to each other. Furthermore, if $L \supset P \supset K$, then $P \supset K$ is a Galois extension of K if and only if $\text{Gal}(L/P) \trianglelefteq \text{Gal}(L/K)$.

Proof. Consider $K \subset P \subset L$ and $K \subset L^{\text{Gal}(L/P)} \subset L$. Then $L^{\text{Gal}(L/P)} \supset P$. From a theorem from two lectures ago, we have the following two results.

- (a) $[L : P] = |\text{Gal}(L/P)|$ for any $K \subset P \subset L$.
- (b) $[L : L^G] = |G|$ for any $G \leq \text{Gal}(L/K)$.

Therefore, $[K : L^{\text{Gal}(L/P)}] \cdot [L : P] = [L : P]$, so that $[L^{\text{Gal}(L/P)} : L] = 1$. Hence $L^{\text{Gal}(L/P)} = L$. Similarly, $\text{Gal}(L/L^G) \leq G$ satisfies $|\text{Gal}(L/L^G)| = |G|$, so that $\text{Gal}(L/L^G) = G$.

For the second part, note that any automorphism of P/K will extend to an automorphism of L/K . This shows that the map $\{\varphi \in \text{Gal}(L/K) \mid \varphi(P) \subset P\} \rightarrow \text{Gal}(P/K)$ given by $\varphi \mapsto \varphi|_P$ is surjective. Then $P \supset K$ will be Galois if and only if the elements of $\{\varphi \in \text{Gal}(L/K) \mid \varphi(P) \subset P\}$ induce $[P : K]$ distinct elements of $\text{Gal}(P/K)$.

We compute

$$\begin{aligned} |\text{Gal}(L/P)| &= [L : P] \\ [P : K] &= \frac{[L : K]}{[L : P]} = \frac{|\text{Gal}(L/K)|}{|\text{Gal}(L/P)|} \\ [P : K] &= [\text{Gal}(L/K) : \text{Gal}(L/P)]. \end{aligned}$$

Thus, $P \supset K$ is a Galois extension if and only if any element of $\text{Gal}(L/K)$ leaves P invariant. But $P = L^{\text{Gal}(L/P)}$, and $P = P^{\text{Gal}(L/P)}$. Hence any $g \in \text{Gal}(L/P)$ has $g(P) = g(L^{\text{Gal}(L/P)}) = L^{g \text{Gal}(L/P) g^{-1}}$. It follows that $g(P) = P \iff g \text{Gal}(L/P) g^{-1} = \text{Gal}(L/P)$. \square

8.3 Lecture 18

Example 17.

1. Let K be a field with $\text{char} K \notin \{2, 3\}$. Let f be an irreducible, monic, cubic polynomial over K . Let L be the splitting field for f . Let $D := \text{Discr } f \in K \setminus K^2$. Then $\text{Gal}(L/K) = S_3$. We get

$$L \supset L^{A_3} \supset K \\ \text{Gal}(L/K) \supseteq A_3 \supseteq \{e\}.$$

It follows that $L^{A_3} \supset K$ is Galois with $\text{Gal}(L^{A_3}/K) \cong \text{Gal}(L/K)/A_3 \cong C_2$. In fact, $L^{A_3} \cong K[\sqrt{D}]$.

2. Let $p > 2$ be prime. Consider the cyclotomic field $\Gamma_p \supset \mathbb{Q}$. We have that

$$\text{Gal}(\Gamma_p/\mathbb{Q}) \cong (\mathbb{Z}/p)^\times \cong \mathbb{Z}/(p-1).$$

Let $H \leq \text{Gal}(\Gamma_p/\mathbb{Q})$ be the unique subgroup of index 2. Then $[\Gamma_p^H : \mathbb{Q}] = 2$.

Let $\langle \varphi \rangle = \text{Gal}(\Gamma_p/\mathbb{Q})$. Then $\varphi|_{\mu_p} : \mu_p \rightarrow \mu_p$ is a group automorphism and uniquely determines φ , which in turn is uniquely determined by the image of ζ the positive p -th root of 1. Write $\varphi(\zeta) = 1\zeta^r$ for some $r \in \mathbb{Z}_{>0}$ such that $[r]_p \in \mathbb{Z}/p$ is a generator of $(\mathbb{Z}/p)^\times$.

Definition. Given $k \in \mathbb{Z}_{>0}$ and prime $p > 2$, define the *Legendre symbol*

$$\left(\frac{k}{p}\right) = \begin{cases} 1 & [k]_p \in ((\mathbb{Z}/p)^\times)^2 \\ -1 & [k]_p \notin ((\mathbb{Z}/p)^\times)^2 \end{cases}.$$

Consider $\alpha \in \Gamma_p$ given by

$$\alpha = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \zeta^{r^{k-1}} = \zeta - \zeta^r + \zeta^{r^2} - \cdots \zeta^{r^{p-2}}.$$

If $g \in \text{Gal}(\Gamma_p/\mathbb{Q})$, then

$$g(\alpha) = \begin{cases} \alpha & g \in H \\ -\alpha & g \notin H \end{cases}.$$

Then $\alpha \in \Gamma_p^H$. Also, α^2 is fixed by every element of $\text{Gal}(\Gamma_p/\mathbb{Q})$ and thus is rational. This implies that $\Gamma_p^H = \mathbb{Q}[\alpha]$.

Lemma 16. $\alpha^2 = (-1)^{\frac{p-1}{2}} p$, so that

$$\Gamma_p^H = \begin{cases} \mathbb{Q}[\sqrt{p}] & p \equiv 1 \pmod{4} \\ \mathbb{Q}[\sqrt{-p}] & p \not\equiv 1 \pmod{4} \end{cases}.$$

Proof. Let $L \supset K$ be a finite extension of fields. Then for any $u \in L$, we get a map $\text{mult}_u : L \rightarrow L$, which is linear over K . Applying trace determines a K -linear map $L \rightarrow K$ given by $u \mapsto \text{tr}(\text{mult}_u)$. This induces a symmetric bilinear map $\langle \cdot, \cdot \rangle : L \otimes_K L \rightarrow K$ given by $u \otimes v \mapsto \text{tr}(\text{mult}_u \circ \text{mult}_v)$. Note that if $u \neq 0$, then $\langle u, u^{-1} \rangle = \text{tr}(\text{mult}_{uu^{-1}}) = \text{tr}(\text{id}_L) = [L : K]$ since $(\text{char} K, [L : K]) = 1$. Now, the vector space Γ_p has a \mathbb{Q} -basis $\{1, \zeta, \zeta^2, \dots, \zeta^{p-2}\}$. Hence mult_{ζ^2} is a cyclic operator, and $\text{tr}(\text{mult}_1) = p-1$ and $\text{tr}(\text{mult}_{\zeta^k}) = -1$ for each $k = 1, \dots, p-2$. It follows that

$$\langle \zeta^k, \zeta^l \rangle = \begin{cases} p-1 & k+l \equiv 0 \pmod{p} \\ 1 & \text{otherwise} \end{cases}.$$

If $x = \sum_{i=0}^{p-1} x_i \zeta^i$ and $y = \sum_{i=0}^{p-1} y_i \zeta^i$ are two elements of Γ_p , then we can choose x_i and y_i such that $\sum x_i = 0$ and $\sum y_i = 0$. Thus,

$$\langle x, y \rangle = p(x_0 y_0 + \sum k = 1^{p-1} x_k y_{p-k}).$$

But $\alpha = \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \zeta^{r^{k-1}}$, so that

$$\begin{aligned} \alpha^2 &= \frac{1}{p-1} \langle \alpha, \alpha \rangle = \frac{1}{p-1} \sum_{k=1}^{p-1} p \left(\frac{k}{p}\right) \left(\frac{-k}{p}\right) \\ &= \frac{p}{p-1} \sum_{k=1}^p \left(\frac{k}{p}\right) \left(\frac{-k}{p}\right) \\ &= p \left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} p. \end{aligned}$$

□

Definition. Let $L \supset K$ be a field extension and $\alpha \in L$. We say that α can be expressed in radicals over K if it can be obtained from elements in K by applying $+$, \cdot , and $\sqrt[p]{}$, i.e., there exists a tower of subfields

$$K = K_0 \subset K_1 \subset \cdots \subset K_s \subset L$$

such that $K_{i+1} = K_i(\alpha_{i+1})$ where $\alpha_{i+1}^{r_{i+1}} \in K_i$ and $\alpha \in K_s$.

Proposition 16. If $f(x) \in K[x]$ is irreducible, $L \supset K$ is an extension, and α is a root of $f(x)$, then α can be expressed in radicals if and only if any root of f can be expressed in radicals in the splitting field for f .

Proof. If $L_1, L_2 \supset K$ are field extensions and $\alpha_1 \in L_1$ and $\alpha_2 \in L_2$ are roots of f , then by the main lemma there is a unique $\varphi : K(\alpha_1) \xrightarrow{\cong} K(\alpha_2)$ such that $\varphi(\alpha_1) = \alpha_2$. Now transport expressions by φ or φ^{-1} . □

9 Solvability in radicals

Definition. We say that $f(x) \in K[x]$ is *solvable in radicals* if every root of f can be expressed in radicals over K .

Note 20. This is equivalent to saying that L is a splitting field for f , then there is a tower of subfields $K = K_0 \subset K_1 \subset \cdots \subset K_s = L$ such that $K_{i+1} = K_i(\alpha_{i+1})$ where $\alpha_{i+1}^{r_{i+1}} \in K_i$.

Theorem 13. If K is a field with characteristic 0, $f(x) \in K[x]$ is irreducible, and $L \supset K$ is the splitting field for f , then f is solvable in radicals over K if and only if $\text{Gal}(L/K)$ is solvable.

Note 21.

1. A generic polynomial equation over K of $\deg \geq 5$ will not be solvable in radicals, since $\text{Gal} \cong S_n$.
2. If $f(x) \in \mathbb{Q}[x]$ is irreducible of degree 5, then f will not be solvable in radicals as soon as $\text{Gal}(L/\mathbb{Q}) \in \{S_5, A_5\}$. Suppose $f \in \mathbb{Q}[x]$ is such a polynomial and let $\alpha_1, \dots, \alpha_5$ be the roots of f . Note that $\text{Gal}(L/\mathbb{Q}) \subset S_5$. Since f is irreducible, it must be separable, which means that the α_i are pairwise distinct. Hence $5 \mid |\text{Gal}(L/\mathbb{Q})|$. Therefore, $\text{Gal}(L/\mathbb{Q})$ must contain an element of order 5, so that $\text{Gal}(L/\mathbb{Q})$ contains a 5-cycle. If we can choose f so that $\text{Gal}(L/\mathbb{Q})$ contains a transposition, then $\text{Gal}(L/\mathbb{Q}) = S_5$.

Choose f so that it has exactly three real roots. In this case, complex conjugation will belong to $\text{Gal}(L/\mathbb{Q})$, so that $\text{Gal}(L/\mathbb{Q}) = S_5$. Start with $x^5 - 16x = x(x-2)(x+2)(x^2+4)$, which has exact three real roots. To make this irreducible, shift its graph to obtain the polynomial $f(x) = x^5 - 16x + 2$.

9.1 Lecture 19

Theorem 14. If K is a field with characteristic 0, $f(x) \in K[x]$ is irreducible, and $L \supset K$ is the splitting field for f , then f is solvable in radicals over K if and only if $\text{Gal}(L/K)$ is solvable.

Proof.

(\Leftarrow) We have a series

$$G = G^{(0)} \supseteq G^{(1)} \supseteq \cdots \supseteq G^{(s)} \supseteq \{e\},$$

which we can refine to get a normal series

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_r = \{e\}$$

such that $G_{i+1}/G_i \cong \mathbb{Z}/n_i$. Letting $K_i = L^{n_i}$, we have a tower

$$K = K_0 \subset K_1 \subset \cdots \subset K_r = L.$$

Let F be the cyclotomic field that contains all roots of 1 of order $n = n_1 n_2 \cdots n_r$. Consider the tower

$$KF \subset K_1 F \subset \cdots \subset K_r F = LF.$$

Then $K_{n_i} F \supset K_1 F$ is a cyclic extension of degree dividing n_i .

Lemma 17. Let K be a field and let n have $\text{char} K \nmid n$. Suppose that $K \supset \mu_n$. Then

- (a) for any $\alpha \in K$, the extension $K(\sqrt[n]{\alpha}) \supset K$ is cyclic of order dividing n , and
- (b) for every $\tilde{K} \supset K$ Galois and cyclic of order n , there exists $\alpha \in K$ such that $\tilde{K} \cong K(\sqrt[n]{\alpha})$.

Note that (b) implies that $K_{i+1} F \supset K_i F$ for every i .

Proof.

- (a) By definition, $K(\sqrt[n]{\alpha})$ contains some root of $x^n - \alpha$. But K contains μ_n , so that $K(\sqrt[n]{\alpha})$ contains every root of $x^n - \alpha$. Thus, $K(\sqrt[n]{\alpha})$ is the splitting field for $x^n - \alpha$. Hence $K(\sqrt[n]{\alpha}) \supset K$ is Galois. If $\sigma \in \text{Gal}(K(\sqrt[n]{\alpha})/K)$, then $\sigma(\sqrt[n]{\alpha}) = \underbrace{\zeta_\sigma}_{\text{some } n\text{-th root of 1 depending on } \sigma} \sqrt[n]{\alpha}$. Then we get a map

$$\zeta : \text{Gal}(K(\sqrt[n]{\alpha})/K) \rightarrow \mu_n$$

given by $\sigma \mapsto \zeta_\sigma$. But since $K \supset \mu_n$, if $\sigma, \tau \in \text{Gal}(K(\sqrt[n]{\alpha})/K)$, then $\sigma(\tau(\sqrt[n]{\alpha})) = \sigma(\zeta_\tau \sqrt[n]{\alpha}) = \sigma(\zeta_\tau) \sigma(\sqrt[n]{\alpha})$. As $\zeta_\tau \in \mu_n \subset K$, we see that $\sigma(\zeta_\tau) = \zeta_\tau$. This implies that

$$\sigma(\tau(\sqrt[n]{\alpha})) = \zeta_\tau \zeta_\sigma \sqrt[n]{\alpha}.$$

But $\sigma(\tau(\sqrt[n]{\alpha})) = \zeta_{\tau\sigma} \sqrt[n]{\alpha}$ as well, so that $\zeta_{\tau\sigma} = \zeta_\tau \zeta_\sigma$. This shows that ζ is a homomorphism.

Moreover, if $\sigma \in \ker \zeta$, i.e., $\zeta_\sigma = 1$, then $\sigma(\sqrt[n]{\alpha}) = 1 \cdot \sqrt[n]{\alpha} = \sqrt[n]{\alpha}$. Since any $\sigma \in \text{Gal}(K(\sqrt[n]{\alpha})/K)$ preserving $\sqrt[n]{\alpha}$ must be the identity, it follows that ζ is injective. As a result, we get an embedding $\text{Gal}(K(\sqrt[n]{\alpha})/K) \leq \mu_n$.

- (b) Suppose that $\text{Gal}(\tilde{K}/K)$ is cyclic of order $d \mid n$. We want to show that there exists $\alpha \in K$ such that $\tilde{K} \cong K(\sqrt[d]{\alpha})$.

Let $\alpha \in \tilde{K}$ and $\xi \in \mu_d \subset \mu_n \subset K$. The *Lagrange resolvent* of (α, ξ) is the element

$$l(\alpha, \xi) = \alpha + \xi \sigma(\alpha) + \xi^2 \sigma^2(\alpha) + \cdots + \xi^{d-1} \sigma^{d-1}(\alpha)$$

of \tilde{K} where $\sigma \in \text{Gal}(\tilde{K}/K)$ is a generator.

Note that $\sigma(l(\alpha, \xi)) = \xi^{-1} l(\alpha, \xi)$, so that $\sigma(l(\alpha, \xi)^2) = \xi^{-k} l(\alpha, \xi)^k$.

Suppose that ξ is a primitive d -th root of unity. We see that $\text{id} + \xi\sigma + \xi^2\sigma^2 + \cdots + \xi^{d-1}\sigma^{d-1}$ is a linear combination of operators $L \rightarrow L$ viewed as a K -vector space. But in $\text{End}_K(\tilde{K})$ the generators are linearly independent. Therefore, module this statement, we have that

$$\sum_{k=0}^{d-1} \xi^k \sigma^k \neq 0$$

in $\text{End}_K(\tilde{K})$. Hence there exists $\alpha \in \tilde{K}$ such that $l(\alpha, \xi) = \sum_{k=0}^{d-1} \xi^k \sigma^k(\alpha) \neq 0$. But for each $i = 0, \dots, d-1$, we see that $\sigma^i(l(\alpha, \xi)) = \xi^{-i} l(\alpha, \xi)$. This implies that

$$l(\alpha, \xi), \sigma(l(\alpha, \xi)), \sigma^2(l(\alpha, \xi)), \dots, \sigma^{d-1}(l(\alpha, \xi))$$

are pairwise distinct in \tilde{K} . Therefore, $l(\alpha, \xi) \in \tilde{K}$ does not belong to any proper subfield of \tilde{K} . It follows that $\tilde{K} = K(l(\alpha, \xi))$. But $\sigma(l(\alpha, \xi)^d) = \underbrace{\xi^{-1}}_1 l(\alpha, \xi)^d = l(\alpha, \xi)^d$. Hence

$$l(\alpha, \xi)^d = \tilde{K}^{\text{Gal}(\tilde{K}/K)} = K.$$

This proves our lemma modulo the statement that $\text{id}, \sigma, \sigma^2, \dots, \sigma^{d-1}$ are linearly independent linear operators.

Note 22. The σ^i belong to $\text{End}_K(\tilde{K})$ and commute with each other. They can be simultaneously diagonalized over $L \supset K$ the splitting field for $f(x) = \det(\sigma - x \cdot \text{id})$. Writing a linear combination of the σ^i and evaluating it on a basis of eigenvectors will produce a homogenous linear system with a Vandermonde coefficient matrix. Then one needs to show that σ has distinct eigenvalues. □

(\implies) See the next lecture. □

9.2 Lecture 20

Definition. If G is a group and k a field, then a k -character of G is a group homomorphism $\chi : G \rightarrow \text{GL}_1(k) = k^\times$.

Note 23. Each k -character χ of G can be viewed as a function with values in k .

Lemma 18. (Dedekind) If χ_1, \dots, χ_s are pairwise distinct k -characters of G , then they are linearly independent in $\text{Fun}(G, k)$.

Proof. We induct on s . If $s = 1$, then χ_1 must be linearly independent since $\chi_1 \neq 0$. Suppose, inductively, that any collection $\sigma_1, \dots, \sigma_t$ of characters with $t \leq s$ is linearly independent. Suppose that χ_1, \dots, χ_s are linearly dependent. Then there are $a_1, \dots, a_s \in k$ such that $a_1 \chi_1 + \dots + a_s \chi_s$ is the zero function. By our IH, each a_i must be nonzero, say, a_s . Let $b_i = -\frac{a_i}{a_s}$. Then

$$\sum_{i=1}^{s-1} b_i \chi_i = \chi_s.$$

If $g, h \in G$, then

$$\chi_s(h) \chi_s(g) = \sum_{i=1}^{s-1} b_i \chi_i(h) \chi_i(g),$$

in which case $\chi_s(g) = \sum_{i=1}^{s-1} (b_i \frac{\chi_i(h)}{\chi_s(h)}) \chi_i(g)$. Fix $h \in G$, so that

$$\chi_s = \chi_s = \sum_{i=1}^{s-1} (b_i \frac{\chi_i(h)}{\chi_s(h)}) \chi_i$$

and $\chi_s = \sum_{i=1}^{s-1} b_i \chi_i$. It follows that $0 = \sum_{i=1}^{s-1} (b_i \frac{\chi_i(h)}{\chi_s(h)} - b_i) \chi_i$. By our IH, we see that $b_i \frac{\chi_i(h)}{\chi_s(h)} - b_i = 0$. But $b_i \neq 0$ for any i . We have that $\chi_i(h) = \chi_s(h)$ for any $i = 1, \dots, s-1$. This proves that $\chi_i = \chi_s$ for any $i = 1, \dots, s-1$. This contradicts the assumption that the χ_1, \dots, χ_s are pairwise distinct. □

Definition. If $K_1, K_2 \subset L$, then the *composite* of K_1 and K_2 in L is the field

$$K_1 K_2 = \bigcap_{\text{subfield}} \{P \mid P \subset L, K_1, K_2 \subset P\}.$$

Note 24. Let K_1 and K_2 be finite extensions of k , so that $K_1 = k(a_1, \dots, a_s)$ and $K_2 = k(b_1, \dots, b_t)$. Then the field $k(a_1, \dots, a_s, b_1, \dots, b_t)$ both contains $K_1 K_2$ and is contained in some L such that $K_1, K_2 \subset L$. Hence

$$K_1 K_2 = k(a_1, \dots, a_s, b_1, \dots, b_t).$$

Lemma 19. Suppose that K and F are two finite field extensions of k . Then

- (a) If $K \supset k$ is Galois, then so is $KF \supset F$.
- (b) $\text{Gal}(KF/F) = \text{Gal}(K/K \cap F)$.

Proof.

- (a) If $K \supset k$ is Galois, then K is the splitting field of some separable polynomial $f(x) \in k[x]$. Thus, KF is the splitting field of $f(x)$ viewed over F . But if f is separable over k , then it is separable over F . Therefore, $KF \supset F$ is Galois.
- (b) Consider the tower of extensions $k \subset K \subset KF$. The main theorem of Galois theory says that $\text{Gal}(K/k) \trianglelefteq \text{Gal}(KF/k)$ since $K \supset k$ is assumed to be Galois. Thus, if $\sigma \in \text{Gal}(KF/k)$, then $\sigma(K) \subset K$. Indeed, $\sigma(K) = K$ as a subfield in KF if and only if $\sigma(K) = (KF)^{\text{Gal}(KF/K)}$. Let $g \in \text{Gal}(KF/K) \subset \text{Gal}(KF/k)$. Then $g(x) = x$ for any $x \in K$.

Let $x \in K$. Consider $\sigma(x) \in KF$. We must show that $g(\sigma(x)) = \sigma(x)$ for any $g \in \text{Gal}(KF/K)$, i.e., $(\sigma^{-1}g\sigma)(x) = x$ for any g . But since $\sigma^{-1}g\sigma \in \sigma^{-1}\text{Gal}(KF/K)\sigma$, we see that $\sigma^{-1}g\sigma(x) = x$ for any $x \in K$. Hence we get a natural homomorphism $\rho : \text{Gal}(KF/F) \rightarrow \text{Gal}(K/k)$ given by $\sigma \mapsto \sigma \upharpoonright_K$. Note that

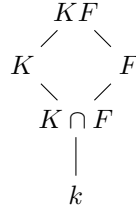
$$\begin{aligned} \ker \rho &= \{\sigma \in \text{Gal}(KF/F) \mid \sigma \upharpoonright_K = \text{id}_K\} \\ &= \{\sigma \in \text{Gal}(KF/k) \mid \sigma \upharpoonright_K = \text{id}_K, \sigma \upharpoonright_F = \text{id}_F\}. \end{aligned}$$

But KF is generated by K and F , so that $\sigma = \text{id}_{KF}$. This shows that $\ker \rho = \{\text{id}_{KF}\}$. We see that $\text{Gal}(KF/F) \subset \text{Gal}(K/k)$.

Let $H := \text{im } \rho \subset \text{Gal}(K/k)$ and consider its fixed subfield K^H . Note that (b) is equivalent to saying that $K^H = K \cap F$. We have that $K^H \supset K \cap F$ because $K^H = \{x \in K \mid \forall \sigma \in \text{Gal}(KF/F), \sigma(x) = x\}$. Moreover, if we view K^H as subfield of KF , then $k \subset K^H \subset KF$ and $k \subset F \subset KF$. Since $\text{Gal}(KF/F)$ fixes K^H and F (pointwise), it follows that $\text{Gal}(KF/F)$ fixes $K^H F$. Therefore, $K^H F \subset KF^{\text{Gal}(KF/F)} = F$, so that $K^H \subset H$. This proves that $K^H \subset F \cap K$.

□

Corollary 21. If both K and F are Galois field extensions of k , then



is a lattice of Galois field extensions.

Theorem 15. If K is a field with characteristic 0, $f(x) \in K[x]$ is irreducible, and $L \supset K$ is the splitting field for f , then f is solvable in radicals over K if and only if $\text{Gal}(L/K)$ is solvable.

Proof.

(\Leftarrow) This was proven in the previous lecture.

(\implies) For any root α of f , we can find an extension $K_\alpha \supset K$ such that $\alpha \in L_\alpha \subset L$ and there exists a tower of radical extensions

$$K = K_0 \subset K_1 \subset \cdots \subset K_s = K_\alpha \subset L$$

with $K_{i+1} = K_i(\alpha_{i+1})$ and $\alpha_{i+1}^{n_{i+1}} \in L_i$.

Claim 5. Without loss of generality, we may assume that K_α satisfies the following properties.

- $\alpha \in K_\alpha$.
- $K_\alpha \supset K$ is Galois.
- Each step of K_α (viewed as our tower of radical extensions) is Galois and cyclic.

Proof. Since $K_\alpha \supset K$ is a finite extension, we can find a K -basis e_1, \dots, e_n of K_α . Let $f_i \in K[x]$ denote the minimal polynomial of e_i . Let S_i denote the splitting field of f_i . Then $S_i \supset K$ is a Galois extension and contains e_i . Note that the composite of the S_i contains each e_i . Let $L_\alpha = S_1 S_2 \cdots S_n$. Then $K \subset K_\alpha \subset L_\alpha$. (We call L_α the *Galois closure* of K_α .) Consider the tower $K = K_0 \subset K_1 \subset \cdots \subset K_s = K_\alpha$ where $K_{i+1} \supset K_i$ is a radical extension of degree n_i . If $\sigma \in \text{Gal}(L_\alpha/K)$, then $K = \sigma K \subset \sigma K_1 \subset \cdots \subset \sigma K_\alpha$ is still a tower of radical extensions.

By taking the composites $K_1 \sigma K_1 \subset \cdots \subset K_1 \sigma K_s$ and $K_2 K_1 \sigma K_1 \cdots$, we get a composite of all $\{\sigma K_\alpha\}_{\sigma \in \text{Gal}(L_\alpha/K)}$, which will be a tower of radical extensions. But $K \subset \prod_\sigma \sigma K_\alpha \subset L_\alpha$, and L_α is generated by all σK_α . Hence $L_\alpha = \prod_\sigma \sigma K_\alpha = L$.

We still must prove that each step in our radical tower is Galois and cyclic. Let $n = n_1 n_2 \cdots n_k$. Let $F = K[\mu_n]$. If the tower $K = K_0 \subset K_1 \subset \cdots \subset K_t = L_\alpha$ has $K_i = K_{i-1}[\sqrt[n_i]{a_i}]$, then we can pass to composites

$$K \subset K_0 F \subset K_1 F \subset \cdots \subset K_t F = L_\alpha F.$$

We see that $LF \supset K$ is radical and Galois as the splitting field for $x^n - 1$ and that $K_i F \supset K_{i+1} F$ is radical of degree n_i and contains μ_{n_i} . Thus, $K_i F \supset K_{i+1} F$ is Galois and cyclic of degree dividing n_i by Lemma 16(a).

We have constructed an extension $LF \supset K$ such that

- $\alpha \in LF$,
- $LF \supset L$ is Galois, and
- LF is a tower of radical, cyclic, Galois extensions.

It follows that $\text{Gal}(LF/K)$ is solvable. But $LF \supset L \supset K$ where $L \supset K$ is Galois. Hence $\sigma(L) \subset L$ for any $\sigma \in \text{Gal}(LF/K)$, so that $\text{Gal}(L/K) < \text{Gal}(LF/K)$. This proves that $\text{Gal}(L/K)$ is solvable. \square

\square

9.3 Lecture 21

Definition. Let K be a field and $f(x) \in K[x]$. We say that f is *solvable in quadratic radicals* if the splitting field L for f is a tower

$$K = K_0 \subset K_1 \subset \cdots \subset K_s = L$$

such that $K_i = K_{i-1}[\sqrt{a_i}]$ for some $a_i \in K_{i-1}$.

Theorem 16. Let K be a field with $\text{char} K \neq 2$ and $f(x) \in K[x]$ be irreducible. Then f is solvable in quadratic radicals if and only if $[L : K] = 2^n$ for some n where L denotes the splitting field for f .

Proof.

(\implies) We have that $L \supset K$ is a tower of quadratic extensions. Hence $[L : K] = 2^n$ for some n .

(\impliedby) We have that $[L : K] = 2^n$ for some $n \geq 0$ and $\deg f = [K(\alpha) : K] \mid [L : K]$ where α is a root of $f(x)$. Thus, $[K(\alpha) : K]$ equals a power of 2, so that f is separable. This shows that $L \supset K$ is Galois and thus that $G := \text{Gal}(L/K)$ has order 2^n . It follows that there is some normal series

$$G = G^0 \supseteq G^1 \supseteq \cdots \supseteq G^s = \{e\}$$

such that $G^i/G^{i+1} \cong \mathbb{Z}/2$. This induces a tower of field extensions

$$K = L^{G^0} \subset L^{G^1} \subset \cdots \subset L^{G^s} = L$$

such that $[L^{G^{i+1}} : L^{G^i}] = 2$. □

Remark 12. (The construction problem) Given a unit measure and segments of lengths a_1, \dots, a_k , we want to construct a segment of length α using ruler and compass. Elementary geometry shows that such a construction is possible if and only if α can be expressed in quadratic radicals over $\mathbb{Q}(a_1, \dots, a_k)$.

If α is transcendental over $\mathbb{Q}(a_1, \dots, a_k)$, then our construction is impossible.

Example 18. We see that π cannot be constructed over \mathbb{Q} , i.e., we cannot square the circle.

Moreover, if α is algebraic over $\mathbb{Q}(a_1, \dots, a_k)$, then α can be constructed by our previous theorem if and only if the minimal polynomial of α has degree power of 2.

Example 19.

- (a) Doubling the cube. Given a segment of length one, construct a segment of length $\sqrt[3]{2}$. Since the minimal polynomial of $\sqrt[3]{2}$ is $x^3 - 2$, such a construction is impossible.
- (b) Trisecting an angle φ . Given a segment of length $\cos \varphi$, construct a segment of length $\cos(\frac{\varphi}{3})$. The minimal polynomial of $\cos(\frac{\varphi}{3})$ over $\mathbb{Q}(\cos \varphi)$ is $4x^3 - 3x - \cos \varphi$. In general, this is irreducible, in which case our construction is impossible.
- (c) Constructing regular n -gons. Given a segment of length i , construct a segment of length $\cos(\frac{2\pi}{n})$. This is possible if and only if $e^{\frac{2\pi i}{n}}$ is expressible in quadratic radicals over \mathbb{Q} . In turn, this happens if and only if $[\Gamma_n : \mathbb{Q}] = 2^s$. For example, if p is prime, then we can construct a regular p -gon if and only if $1 + 2^k$ for some k . Currently, the largest known such p is 65,537.

10 Further applications of Galois theory

10.1 Lecture 22

Remark 13. (Fundamental theorem of algebra) The following statements are true.

- If $f(x) \in \mathbb{R}[x]$ has odd degree, then it has a real root.
- Every $\alpha \in \mathbb{C}$ has a square root in \mathbb{C} .

Now, suppose that $K \supsetneq \mathbb{R}$ is a finite field extension. If $[K : \mathbb{R}]$ is odd and $\alpha \in K \setminus \mathbb{R}$, then $K \supset \mathbb{R}(\alpha) \supset \mathbb{R}$, in which case $\deg f \mid [K : \mathbb{R}]$ where f denotes the minimal polynomial of α over \mathbb{R} . In this case, f has odd degree and thus has a root in \mathbb{R} , so that $\mathbb{R}(\alpha) = \mathbb{R}$, a contradiction. This proves that $[K : \mathbb{R}]$ is odd.

We want to show that any $f(x) \in \mathbb{C}[x]$ has a root in \mathbb{C} . Note that if c is a complex root of $f(x)\overline{f(x)} \in \mathbb{R}[x]$, then either c or \bar{c} is a root of $f(x)$. Thus, it suffices to show that any polynomial over \mathbb{R} has a root in \mathbb{C} .

Let $g(x) \in \mathbb{R}[x]$ be non-constant and irreducible. Let L denote the splitting field for g . Then $[L : K] = |\text{Gal}(L/\mathbb{R})|$ is even, so that there is some nontrivial 2-Sylow subgroup $H \leq \text{Gal}(L/\mathbb{R})$. This means that the

intermediate extension $L \supset L^H \supset \mathbb{R}$ has odd degree. But then $L^H = \mathbb{R}$. This means that $L \supset L^H$ is Galois, so that

$$[L : \mathbb{R}] = [L : L^H] = |\text{Gal}(L/L^H)| = |H| = 2^n$$

for some n . By Theorem 16, it follows that $g(x)$ is solvable in quadratic radicals. Therefore, $L = \mathbb{C}$ since $[\mathbb{C} : \mathbb{R}] = 2$.

Theorem 17. (Primitive element theorem) Suppose that $L \supset K$ is a finite field extension. This has a *primitive element*, i.e., $L = K(\theta)$ for some $\theta \in L$, if and only if there are at most finitely many intermediate fields $K \subset F \subset L$.

Proof. If K is finite, then L is a finite group with cyclic multiplicative group $\langle \theta \rangle$. In this case, we have shown that $L = K(\theta)$.

(\Leftarrow) For any $\alpha, \beta \in L$, consider the collection of intermediate fields

$$K \subset K(\alpha + c\beta) \subset L$$

where $c \in K$. Thus, $\exists c, c' \in K$ such that $E := K(\alpha + c\beta) = K(\alpha + c'\beta)$. Hence $(c - c')\beta \in E$, and $c - c' \in K \setminus \{0\}$. Then $\beta \in E$, so that $\alpha \in E$. This shows that $E \supset K(\alpha, \beta)$. It's clear that $E \subset K(\alpha, \beta)$. Hence $E = K(\alpha, \beta)$. But $L \supset K$ is a finite extension, which implies that $L = K(\alpha_1, \dots, \alpha_n)$ for some $\alpha_1, \dots, \alpha_n$. By induction on n , we can find elements $c_2, \dots, c_n \in K$ such that

$$K(\alpha_1, \dots, \alpha_n) = K(\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + \dots + c_n\alpha_n).$$

(\Rightarrow) We have that $L = K(\theta)$. Let $f(x) \in K[x]$ denote the minimal polynomial of θ over K . Let $K \subset F \subset L$ be an intermediate field extension. Let $g_F(x) \in F[x]$ denote the minimal polynomial over F . This proves that $g_F(x) \mid f(x)$ in $F[x]$. We get a map

$$(\text{intermediate field extensions } K \subset F \subset L) \rightarrow (\text{divisors of } f(x))$$

given by $F \mapsto g_F(x)$. Since there are at most finitely many divisors of $f(x)$, it suffices to check that this map is injective.

Suppose that $K \subset F \subset L$. Let $F_0 \subset F$ be the subfield obtained from K by adjoining the coefficients of $g_F(x)$. It is enough to show that $F_0 = F$. Note that $g_F(x)$ is irreducible in $F[x]$, so that $g_F(x)$ is irreducible in $F_0[x]$. Therefore, $g_F(x) \in F_0[x]$, which means that $g_F(x)$ is the minimal polynomial of θ over F_0 . Then $[L : F_0] = \deg g_F = [L : F]$, so that $F_0 = F$. \square

Corollary 22. If $L \supset K$ is a (finite) separable extension, then L has a primitive element.

Proof. It suffices to show that if $\alpha, \beta \in L$ are separable over K , then $K(\alpha, \beta) = K(\theta)$ for some θ . If K is finite, then we're done. Suppose that K is infinite. Let $\varphi_1, \dots, \varphi_n$ denote the distinct embeddings of $K(\alpha, \beta)$ in \overline{K} over K . Consider

$$f(x) = \prod_{i \neq j} (\varphi_i(\alpha) + x\varphi_i(\beta) - \varphi_j(\alpha) - x\varphi_j(\beta)).$$

Since this is not the zero polynomial, there is some $c \in K$ such that $f(c) \neq 0$. It follows that the $\varphi_i(\alpha + c\beta)$ are pairwise distinct in \overline{K} . Then $[K(\alpha + c\beta) : K] \geq n$. But $[K(\alpha, \beta) : K] = n$, so that $K(\alpha, \beta) = K(\alpha + c\beta)$. \square

Theorem 18. Let K be a field and $f(x) \in K[x]$ be a monic separable polynomial. Let L denote the splitting field of f , so that $L \supset K$ is Galois. Let $G_f := \text{Gal}(L/K) \subset S_n$ where $n = \deg f$. Let $\text{char} K \neq 2$. Then $L^{G_f \cap A_n} = K(\Delta(f))$ where $\Delta(f) = \prod_{i < j} (\lambda_i - \lambda_j)$ and $\lambda_1, \dots, \lambda_n$ denote the distinct roots of f . Note that $\Delta(f)$ is a square root of $\text{Discr}(f) \in K$.

Proof. Consider x_1, \dots, x_n purely transcendental elements over K . Let $K(x_1, \dots, x_n) \supset K$ be the corresponding extension. There is a group homomorphism $\Phi : S_n \rightarrow \text{Gal}(K(x_1, \dots, x_n)/K)$ given by $\sigma \mapsto \Phi_\sigma$ where

$$\Phi_\sigma(f)(x_1, \dots, x_n) = f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

This is injective, and $K(x_1, \dots, x_n)^{S_n} = K(\sigma_1, \dots, \sigma_n)$ where $\sigma_1, \dots, \sigma_n \in K[x_1, \dots, x_n]$ are the alternating symmetric polynomials. Further, $\text{Gal}(K(x_1, \dots, x_n)/K(\sigma_1, \dots, \sigma_n)) = S_n$. Let $\Delta_n = \prod_{i < j} (x_i - x_j) \in K(x_1, \dots, x_n)$. Then $\Phi_\sigma(\Delta_n) = \text{sgn}(\sigma)\Delta_n$, and $\Delta_n \notin K(\sigma_1, \dots, \sigma_n)$.

Define $\text{ev} : K(x_1, \dots, x_n) \rightarrow L$ by $x_i \mapsto \lambda_i$. Then $\text{ev} \circ \Phi_\sigma = \sigma^{-1} \circ \text{ev}$. Thus, $\text{ev}(\Delta(f)) = \text{ev}(\Phi_\sigma(\Delta_n)) = \sigma^{-1}(\Delta(f))$. This shows that the subgroup in G_f fixing $\Delta(f)$ is precisely $G_f \cap A_n$. \square

Corollary 23. If $\text{char} K \neq 2$ and $f(x)$ is monic and separable over K , then $G_f \subset A_n \iff \text{Discr}(f) \in K^2$.

10.2 Lecture 23

Theorem 19. Suppose that K is a field and $f(x) \in K[x]$ is separable. Then f is irreducible if and only if the Galois group G_f acts transitively on the set of roots of f .

Proof.

(\implies) For any two roots λ_i, λ_j of f , we have that $K(\lambda_i) \cong K(\lambda_j)$ as fields over K because both $\text{ev}_{\lambda_i} : K[x] \rightarrow K(\lambda_i)$ and $\text{ev}_{\lambda_j} : K[x] \rightarrow K(\lambda_j)$ induces isomorphisms with $K[x]_{(f)}$. By the main lemma of Galois theory, we can extend this isomorphisms to an automorphism $\sigma : L \rightarrow L$ of the splitting field L for f . Thus, $\sigma \in \text{Gal}(L/K)$ with $\sigma(\lambda_i) = \lambda_j$.

(\impliedby) Let $\{\lambda_1, \dots, \lambda_n\}$ denote the set of roots of f . Let $f(x) = g(x)h(x)$ where $\deg g \geq 1$ and g is irreducible. We must show that h is constant. Let λ be any root of g . Then there exists $\sigma_i \in G_f$ such that $\sigma_i(\lambda) = \lambda_i$ for each $i = 1, \dots, n$. Note that

$$g(\lambda_i) = g(\sigma_i(\lambda)) = \sigma_i(g(\lambda)) = 0$$

, so that each λ_i is a root of g . Hence $f \mid g$, which implies that h is constant. \square

Theorem 20. Suppose that p is prime and that $f(x) \in \mathbb{Q}[x]$ is monic and irreducible with $\deg f = p$. Suppose that f has exactly two non-real roots in \mathbb{C} . Then $G_f = S_p$.

Proof. Let L be the splitting field for $f(x)$. Write $f(x) = \prod_{i=1}^p (x - x_i)$ with each $\lambda_i \in \mathbb{C}$. Then $\mathbb{Q}(\lambda_1, \dots, \lambda_p) \subset \mathbb{C}$. We see that

$$\mathbb{Q} \subset \mathbb{Q}(\lambda_i) \subset \mathbb{Q}(\lambda_1, \dots, \lambda_p) \subset \mathbb{C},$$

so that $[\mathbb{Q}(\lambda_i) : \mathbb{Q}] \mid [L : \mathbb{Q}]$. Since $p \mid [L : \mathbb{Q}] = |G_f| \subset S_p$, it follows from Sylow that G_p contains an element of order p , i.e., that G_f contains a p -cycle. Also, the element in G_f that switches the roots is the complex conjugate pair of a transposition. \square

Theorem 21. (Brouwer) For any prime $p \geq 5$, there are infinitely many polynomials in $\mathbb{Q}[x]$ of degree p with Galois group S_p .

Proof. Let k be an odd integer and let $0 \leq m, n_1 \leq n_2 < \dots < n_{k-2}$ be even integers. Consider

$$g(x) = (x^2 + m)(x - n_1)(x - n_2) \cdots (x - n_{k-2}).$$

This polynomial has $\frac{k-3}{2}$ local maxima. Also, for each odd $h \in \mathbb{Z}$, $|g(h)| > 2$. Hence if c denotes a local maximum of g , then $g(c) > 2$. This shows that if $f(x) = g(x) - 2$, then there are

- $\frac{k-3}{2}$ positive local maxima in $[n_1, n_{k-2}]$ and
- $\frac{k-3}{2}$ negative local maxima in $[n_1, n_{k-2}]$.

It follows that $f(x)$ has $k-3$ real roots in $[n_1, n_{k-2}]$ with $f(n_{k-2}) = -2$ and $\lim_{x \rightarrow \infty} f(x) > 0$. Therefore, we have another real roots $> n_{k-2}$. Hence $f(x)$ has at least $k-2$ real roots. Let $\lambda, \dots, \lambda_n \in \mathbb{C}$ denote the distinct roots of f . Then

$$\prod_{i=1}^k (x - \lambda_i) = f(x) = (x^2 + m)(x - n_1)(x - n_2) \cdots (x - n_{k-2}) - 2$$

, and $-\sum_{i=1}^k \lambda_i = -\sum_{i=1}^{k-2} n_i$. From this, we compute

$$\begin{aligned} \sum_{i < j} \lambda_i \lambda_j &= m + \sum_{a < b} n_a n_b \\ \sum_{i=1}^k \lambda_i^2 &= \left(\sum_{i=1}^k \lambda_i \right)^2 = \sum_{i < j} \lambda_i \lambda_j \\ &= \left(\sum_{i=1}^{k-2} n_i \right)^2 - 2m - 2 \left(\sum_{a < b} n_a n_b \right) \\ &= \sum_{i=1}^{k-2} n_i^2 - 2m. \end{aligned}$$

Choose $m \gg \sum n_i^2$, so that $\sum_{i=1}^k \lambda_i^2 < 0$. This implies that there exists a non-real root. Hence we must have exactly two real roots. Further, we can write $f(x) = x^k + a_1 x^{k-1} + \cdots + a_{k-1} x + a_k$ with each $a_i \in 2\mathbb{Z}$. Since $a_k = f(0) = g(0) - 2$, we see that $2 \mid a_{k-1}$ but $4 \nmid a_{k-1}$. By Eisenstein's criterion, f must be irreducible. We thus get infinitely many f 's such that $G_f = S_p$. \square

11 Chain complexes and chain maps

Remark 14. (Homological algebra) The originators of homological algebra include Betti, Poincaré, and Riemann. The main goal of this subject is to extract invariants from topological spaces. Decompose X into contractible pieces (such as cells or simplices) to reduce X to combinatorial data. Specifically, reduce X to a collection of pieces of various dimensions where the boundary of a piece of dimension n is glued to a sub-collection of pieces of dimension $n-1$.

Emmy Noether introduced groups of chains $C_i(X)$, a free abelian group generated by the collection of i -dimensional pieces, equipped with boundary relations $\partial_i : C_i(X) \rightarrow C_{i-1}(X)$. From this, we obtain abelian groups $H_i(X) \equiv \ker \partial_i / \text{im } \partial_{i+1}$, which are algebraic invariants of X .

Hilbert wanted to extract numerical invariants from a module. Specifically, if k is a field and $K := k[x_1, \dots, x_n]$, then he wanted to understand the complexity of a module over K (or, more generally, any graded module over k).

Note 25. A typical graded module over R will be a module of the form MR/I where $I \trianglelefteq R$ is a homogeneous ideal. By the Hilbert basis theorem, $I \trianglelefteq R$ is generated by finitely many homogenous polynomials f_1, f_2, \dots, f_{r_0} . Thus, we have surjective map $\psi : R^{\oplus r_0} \rightarrow I$ given by $(a_1, \dots, a_{r_0}) \mapsto \sum a_i f_i$. But, there generators are not, in general, independent. Therefore, we consider the module of relations $Z_0(I) \equiv \ker \psi$ among the f_i . Note that $Z_0(I)$ is finitely generated. We can choose generators and get a map $\psi' : R^{\oplus r_1} \rightarrow Z_0(I)$. Then

$$R^{\oplus r_1} \rightarrow R^{\oplus r_0} \rightarrow I \rightarrow 0$$

is an exact sequence of graded R -modules. If $Z_1(I) \equiv \ker \psi'$ is not zero, then choose generators again to get a map $\psi'' : R^{\oplus r_2} \rightarrow Z_1(I)$. Continuing in this way, we get an exact sequence

$$\cdots \rightarrow R^{\oplus r_2} \rightarrow R^{\oplus r_1} \rightarrow R^{\oplus r_0} \rightarrow I \rightarrow 0.$$

The *length* of this sequence is defined to be $\max\{i \mid r_i \neq 0\}$. This is an invariant of I and of R/I .

Theorem 22. (Hilbert's syzygy theorem) Hilbert's syzygy theorem states that $Z_{n-1}(I)$ is free, i.e., that there is an exact sequence of graded R -modules

$$0 \rightarrow R^{\oplus r_n} \rightarrow R^{\oplus r_{n-1}} \rightarrow \dots \rightarrow R^{\oplus r_0} \rightarrow I \rightarrow 0.$$

11.1 Lecture 24

Definition.

1. A *chain complex* (in \mathbf{Ab}) is a pair $(M_\bullet, \partial_\bullet)$ where $M_\bullet = \{M_i\}_{i \in \mathbb{Z}}$ is a set of abelian groups and $\partial_\bullet = \{\partial_i\}_{i \in \mathbb{Z}}$ is a set of morphisms in \mathbf{Ab} such that the i -th differential $\partial_i : M_i \rightarrow M_{i-1}$ satisfies $\partial_{i-1} \circ \partial_i = 0$. We call $Z_n \equiv \ker \partial_n$ the *group of degree n cycles* and $B_n \equiv \text{im } \partial_{n+1}$ the *group of degree n boundaries*. Finally, we call $H_n \equiv Z_n/B_n$ the *degree n homology group*.
2. A *((co)chain) complex* (in \mathbf{Ab}) is a pair (M^\bullet, d^\bullet) where $M^\bullet = \{M^i\}_{i \in \mathbb{Z}}$ is a set of abelian groups and $d^\bullet = \{d^i\}_{i \in \mathbb{Z}}$ is a set of morphisms in \mathbf{Ab} such that the i -th differential $d^i : M^i \rightarrow M^{i+1}$ satisfies $d^{i+1} \circ d^i = 0$. We call $Z^n \equiv \ker d^n$ the *group of degree n cocycles* and $B^n \equiv \text{im } d^{n-1}$ the *group of degree n coboundaries*. Finally, we call $H^n \equiv Z^n/B^n$ the *degree n cohomology group*.

Definition. Let (A^\bullet, d_A^\bullet) and (B^\bullet, d_B^\bullet) be complexes. A *chain map* $f^\bullet : (A^\bullet, d_A^\bullet) \rightarrow (B^\bullet, d_B^\bullet)$ consists of group homomorphisms $f^i : A^i \rightarrow B^i$ for each $i \in \mathbb{Z}$ such that $d_B^i \circ f^i = f^{i+1} \circ d_A^i$.

Note 26. Any chain map $f^\bullet : (A^\bullet, d_A^\bullet) \rightarrow (B^\bullet, d_B^\bullet)$ restricts term-wise to maps $f^i : Z^i(A^\bullet) \rightarrow Z^i(B^\bullet)$ and maps $f^i : B^i(A^\bullet) \rightarrow B^i(B^\bullet)$. Thus, it induces a map $f^* : H^i(A^\bullet) \rightarrow H^i(B^\bullet)$.

Note 27. We have a natural isomorphism $\mathbf{Ch}(\mathbf{Ab}) \rightarrow \mathbf{CoCh}(\mathbf{Ab})$ given by $N_i \mapsto M^{-i}$ and $\partial_i \mapsto d^{-i}$.

Definition. We say that (A^\bullet, d^\bullet) is *bounded above* if there is some N such that $A^n = 0$ for any $n \geq N$. We define *bounded below* similarly. We say that (A^\bullet, d^\bullet) is *bounded* if it is both bounded above and bounded below. We get the subcategories $\mathbf{CoCh}^-(\mathbf{Ab})$, $\mathbf{CoCh}^+(\mathbf{Ab})$, and $\mathbf{CoCh}^b(\mathbf{Ab})$, respectively.

Note 28. If $C^\bullet = \bigoplus_{i \in \mathbb{Z}} C^i$ is a graded abelian group, then it induces a natural complex $(\underline{C}^\bullet, 0)$ where $\underline{C}^i \equiv C^i$. In particular, any abelian group may be viewed as a complex.

Conversely, given a complex (M^\bullet, d^\bullet) , we can form the graded abelian group $M^\bullet \equiv \bigoplus_{i \in \mathbb{Z}} M^i$ and package the differential d^i into a single group map $D : M^\bullet \rightarrow M^\bullet$ such that $D|_{M^i} = d^i$ and $D^2 = 0$. We can write D as the block diagonal matrix

$$\begin{bmatrix} 0 & & & & \\ d^i & 0 & & & \\ & d^{i+1} & 0 & & \\ & & d^{i+2} & 0 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

As a result, we obtain the *cochain functor* given by $(A^\bullet, d^\bullet) \rightarrow \bigoplus_{i \in \mathbb{Z}} A^i$ and $f^\bullet \mapsto (f^i)_{i \in \mathbb{Z}}$.

Definition. We say that (A^\bullet, d^\bullet) is *acyclic* or *exact* if $H^\bullet(A^\bullet, d^\bullet) = 0$.

Definition. If A is an abelian group, then a *left resolution* of A is an exact complex $(C^\bullet, d^\bullet) \in \text{ob } \mathbf{CoCh}^{\leq 0}(\mathbf{Ab})$ of the form

$$\dots \rightarrow C^{i-1} \rightarrow C^i \rightarrow \dots \rightarrow C^0 \rightarrow A \rightarrow 0.$$

Example 20. If $I \trianglelefteq k[x_1, \dots, x_n]$ is a homogenous ideal, then Hilbert's syzygy theorem says that I has a left resolution of length $n+1$ with $n+1$ terms free finitely generated R -modules.

Definition. Let $a \in \mathbb{Z}$. Define the *shift functor*

$$-[a] : \mathbf{CoCh}(\mathbf{Ab}) \rightarrow \mathbf{CoCh}(\mathbf{Ab})$$

as follows. Let (M^\bullet, d_M^\bullet) be a complex. Form the pair $(M^\bullet[a], d_{M[a]}^\bullet)$ where $(M^\bullet[a])^n \equiv M^{a+n}$ and $(d_{M[a]}^\bullet)^n \equiv (-1)^a d_M^{a+n}$. If f^\bullet is a chain map, then let $(f^\bullet[a])^n \equiv f^{a+n}$.

Proposition 17. The shift functor is an equivalence that preserves $\mathbf{CoCh}^-(\mathbf{Ab})$, $\mathbf{CoCh}^+(\mathbf{Ab})$, and $\mathbf{CoCh}^b(\mathbf{Ab})$.

Definition. Let $f : M \rightarrow N$ be a chain map. Form $\text{cone}(f)$ the *cone of f* as a new complex where $\text{cone}(f)^\bullet \equiv N \oplus M[1]$ and $d_{\text{cone}(f)}^\bullet \equiv \begin{bmatrix} d_N & f \\ 0 & d_{M[1]} \end{bmatrix}$.

Note 29. We see that

$$\text{cone}(f)^n = N^n \oplus M^{n+1}$$

and $d_{\text{cone}(f)}^n : N^n \oplus M^{n+1} \rightarrow N^{n+1} \oplus M^{n+2}$ with

$$d_{\text{cone}(f)}^n = \begin{bmatrix} d_N^n & f^{n+1} \\ 0 & -d_M^{n+1} \end{bmatrix}.$$

Exercise 12. Show that $d_{\text{cone}(f)}^{i+1} \circ d_{\text{cone}(f)}^i = 0$.

Definition.

1. A *double complex* is a pair $(A^{\bullet,\bullet}, d^\bullet, \delta^\bullet)$ where $A^{i,j} = \{A^{i,j}\}_{(i,j) \in \mathbb{Z}^2}$ and both $d : A^{\bullet,\bullet} \rightarrow A^{\bullet+1,\bullet}$ and $\delta : A^{\bullet,\bullet} \rightarrow A^{\bullet,\bullet+1}$ are homomorphisms such that $d\delta = \delta d$ and $d^2 = \delta^2 = 0$. As a commutative diagram, this has the form

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ \dots & \xrightarrow{\delta} & A^{p,q} & \xrightarrow{\delta} & A^{p,q+1} & \xrightarrow{\delta} & A^{p,q+2} \xrightarrow{\delta} \dots \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ \dots & \xrightarrow{\delta} & A^{p+1,q} & \xrightarrow{\delta} & A^{p+1,q+1} & \xrightarrow{\delta} & A^{p+1,q+2} \xrightarrow{\delta} \dots \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ \dots & \xrightarrow{\delta} & A^{p+2,q} & \xrightarrow{\delta} & A^{p+2,q+1} & \xrightarrow{\delta} & A^{p+2,q+2} \xrightarrow{\delta} \dots \\ & & \downarrow d & & \downarrow d & & \downarrow d \\ & & \vdots & & \vdots & & \vdots \end{array}$$

2. The *total complex of $(A^{\bullet,\bullet}, d^\bullet, \delta^\bullet)$* is the complex $\text{Tot}(A)$ where $\text{Tot}(A)^n \equiv \bigoplus_{p+q=n} A^{p,q}$ and $d_{\text{Tot}(A)} \upharpoonright_{A^{p,q}} \equiv d + (-1)^p \delta$.

Proposition 18. Any chain map $f : M \rightarrow N$ induces a double complex

$$\begin{array}{ccccccc} M^{i-1,0} & \xrightarrow{d_M} & M^{i,0} & \xrightarrow{d_M} & M^{i+1,0} & \xrightarrow{d_M} & M^{i+2,0} \xrightarrow{d_M} \dots \\ \downarrow f & & \downarrow f & & \downarrow f & & \downarrow f \\ N^{i-1,1} & \xrightarrow{d_N} & N^{i,1} & \xrightarrow{d_N} & N^{i+1,1} & \xrightarrow{d_N} & N^{i+2,1} \xrightarrow{d_N} \dots \end{array}$$

The total complex of this is precisely $\text{cone}(f)$.

Exercise 13.

Let N and C be complexes. Suppose that $C \xrightarrow{\iota} N$ is a chain map where each $\iota^n : N^n \rightarrow C^n$ is injective. Let $s^n : C^n \rightarrow N^n$ be a group homomorphism such that $s^n \circ \iota^n = \text{id}_{N^n}$. Then $M := (C/N, d_{C/N})$ is a complex. Our choice of s^n produces a splitting $C^\bullet \cong N^\bullet \oplus M^\bullet[1]$ in the category of graded abelian groups.

Thus, we have the map $d_C = \begin{bmatrix} d_N & f \\ 0 & d_{M[1]} \end{bmatrix}$ where $f : M \rightarrow N$ is a map of graded abelian groups. Show that f is a chain map and $C \cong \text{cone}(f)$.

11.2 Lecture 25

Definition. Let $f, g : A^\bullet \rightarrow B^\bullet$ be two chain maps. A *homotopy between f and g* is a map of graded abelian groups $h : A^\bullet \rightarrow B^{\bullet-1}$ such the

$$d_B h + h d_A = f - g.$$

We say that f and g are *homotopy equivalent* (written as $f \sim g$) if there is a homotopy between them.

Proposition 19.

1. Homotopy is an equivalence relation.
2. The class $\text{mor}^{\sim 0} \mathbf{CoCh}(\mathbf{Ab})$ of all chain maps homotopic to 0 is a two-sided ideal in $\text{mor} \mathbf{CoCh}(\mathbf{Ab})$.
3. If $f \simeq g : A^\bullet \rightarrow B^\bullet$, then $H^\bullet(f) = H^\bullet(g)$.
4. If $f \simeq g$ and c is a cocycle, then $f(c) - g(c) = d_B h(c)$, which is a coboundary.

Definition. Let $\mathcal{C}(\mathbf{Ab})$ denote the category with complexes as objects and homotopy classes of chain maps as morphisms.

Note 30.

1. We have that $\text{Hom}_{\mathcal{C}(\mathbf{Ab})}(A, B) = \text{Hom}_{\mathbf{CoCh}(\mathbf{Ab})}(A, B) / \text{Hom}_{\mathbf{CoCh}(\mathbf{Ab})}^{\sim 0}(A, B)$.
2. H^\bullet descends to a well-defined functor in the sense that the diagram

$$\begin{array}{ccc} \mathbf{CoCh}(\mathbf{Ab}) & \longrightarrow & \mathcal{C}(\mathbf{Ab}) \\ H^\bullet \downarrow & \swarrow H^\bullet & \\ \mathbf{grAb} & & \end{array} .$$

commutes.

Definition. A *short exact sequence of complex* is a sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ of complexes such that each sequence

$$0 \rightarrow A^n \xrightarrow{f^n} B^n \xrightarrow{g^n} C^n \rightarrow 0$$

is exact in \mathbf{Ab} .

Definition. Let

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

be a short exact sequence of complexes. Consider the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A^{n-1} & \xrightarrow{f^{n-1}} & B^{n-1} & \xrightarrow{g^{n-1}} & C^{n-1} \longrightarrow 0 \\ & & d_A^{n-1} \downarrow & & \downarrow d_B^{n-1} & & \downarrow d_C^{n-1} \\ 0 & \longrightarrow & A^n & \xrightarrow{f^n} & B^n & \xrightarrow{g^n} & C^n \longrightarrow 0 \\ & & d_A^n \downarrow & & \downarrow d_B^n & & \downarrow d_C^n \\ 0 & \longrightarrow & A^{n+1} & \xrightarrow{f^{n+1}} & B^{n+1} & \xrightarrow{g^{n+1}} & C^{n+1} \longrightarrow 0 \end{array} .$$

Define a collection of *edge homomorphisms* $\{\delta^n : H^n(C) \rightarrow H^{n+1}(A)\}_{n \in \mathbb{Z}}$ as follows. Let $c \in C^n$ with $d_C^n(c) = 0$. By exactness, there is some $b \in B^n$ such that $g^n(b) = c$. But then $d_B^n(b) \in \ker g^{n+1} = \text{im } f^{n+1}$. Since f^{n+1} is injective, this means that there is a unique $a \in A^{n+1}$ such that $f^{n+1}(a) = d_B^n(b)$. Define $\delta^n([c]) = [a]$.

Exercise 14. Check that δ^n is a homomorphism and that it is independent both of our choice of c and of our choice of b .

Lemma 20. (Snake) Any short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

complexes induces a *long exact sequence in cohomology*

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & H^{n-1}(C) & & \\ & & \delta^{n-1} & \nearrow & & & \\ H^n(A) & \xleftarrow{f^*} & H^n(B) & \xrightarrow{g^*} & H^n(C) & \longrightarrow & \cdots \\ & & \delta^n & \nwarrow & & & \\ H^{n+1}(A) & \xleftarrow{f^*} & H^{n+1}(B) & \longrightarrow & \cdots & & \end{array}$$

Proof.

Exactness at $H^n(B)$: We have that $0_{H^n(C)} = H^n(0) = H^n(g \circ f) = H^n(g) \circ H^n(f)$. Hence $\text{im } H^n(f) \subset \ker H^n(g)$.

For the reverse inclusion, let $[b] \in \ker H^n(g) \subset H^n(B)$. Then $g(b) \in C^n$ must be a coboundary, so that there is some $c \in C^{n-1}$ such that $g(b) = d_C c$. Choose a lift $b_1 \in B^{n-1}$ of c , meaning that $g(b_1) = c$. Then $b - d_B b_1 \in Z^n(B)$, and $[b] = [b - d_B b_1]$. But

$$g(b - d_B b_1) = g(b) - g(d_B b_1) = g(b) - d_C g(b_1) = g(b) - d_C c = 0.$$

Hence $b - d_B b_1 \in \ker g \subset B^n$. This implies that there exists a unique $a \in A^n$ such that $b - d_B b_1 = f(a)$. Also,

$$f(d_A a) = d_B(f(a)) = d_B(b - d_B b_1) = 0.$$

Since f is injective, we see that $d_A a = 0$, i.e., $a \in Z^n(A)$. Thus, $H^n(f)([a]) = [f(a)] = [b - d_B b_1] = [b]$. This proves that $[b] \in \text{im } H^n(f)$.

Exactness at $H^n(C)$: Let $[b] \in H^n(B)$. Note that $\delta^n(H^n(g)([b])) = [a]$ where $a \in A^{n+1}$ denotes the unique element such that $f(a) = d_B b$. Since $d_B b = 0$ and f is injective, it follows that $a = 0$. Hence $\text{im } H^n(g) \subset \ker \delta^n$.

Conversely, let $[c] \in \ker \delta^n$. Choose $b \in B^n$ such that $g(b) = c$ and then the unique $a \in A^{n+1}$ such that $f(a) = d_B b$. Thus, $\delta^n([c]) = [a] = 0$, so that $a \in B^{n+1}(A)$, i.e., $d_A a_1 = a$ for some $a_1 \in A^n$. Note that $g(b - f(a_1)) = g(b) - g(f(a_1)) = c - 0 = c$. Further,

$$\begin{aligned} d_B(b - f(a_1)) &= d_B(b) - d_B(f(a_1)) \\ &= f(a) - f(d_A a_1) \\ &= f(a) - f(a) \\ &= 0. \end{aligned}$$

This shows that $b - f(a_1)$ is a cocycle. Thus, $H^n(g)([b - f(a_1)]) = [g(b - f(a_1))] = [c]$, so that $[c] \in \text{im } H^n(g)$.

Exactness at $H^{n+1}(A)$: Let $[c] \in H^n(C)$ and find $[a] = \delta^n([c])$, where

$$\begin{array}{ccc} b & \xrightarrow{g} & c \\ \downarrow & & \\ a & \xrightarrow{f} & d_B b \end{array}$$

Then $H^{n+1}(f)([a]) = [f(a)] = [d_B b] = 0$. It follows that $\text{im } \delta^n \subset \ker H^{n+1}(f)$.

Conversely, let $[a] \in \ker H^{n+1}(f)$, so that $H^{n+1}(f)([a]) = [f(a)] = 0$. This means that $f(a) = d_B b$ for some $b \in B^n$. Then $\delta^n([g(b)]) = [a]$. This shows that $\text{im } \delta^n \supset \ker H^{n+1}(f)$. \square

12 Additive categories

Definition. 1. A category \mathcal{C} is *enhanced over \mathbf{Ab}* if $\text{Hom}_{\mathcal{C}}(a, b)$ is an abelian group for any $a, b \in \text{ob } \mathcal{C}$ and

$$\text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) \xrightarrow{\circ} \text{Hom}_{\mathcal{C}}(x, z)$$

is bilinear for any $x, y, z \in \text{ob } \mathcal{C}$.

2. A category \mathcal{C} is called *additive* if it is enhanced over \mathbf{Ab} and has finite products.

Example 21. The following are additive categories.

1. \mathbf{Ab} .
2. $R\text{-Mod}$.

Note 31.

1. Let \mathcal{C} be category with finite products. The product of the empty diagram is the terminal object in \mathcal{C} since it is the initial object in \mathbf{Set} .
2. If \mathcal{C} is additive and $*$ is the terminal object in \mathcal{C} , then $\text{Hom}_{\mathcal{C}}(*, *)$ consists of a single element, which must equal the group identity element.

Exercise 15. Verify the following statements.

1. If \mathcal{C} is a additive, then its terminal object is also initial and thus is a *zero object* in \mathcal{C} .
2. A zero object $0_{\mathcal{C}}$ satisfies $\text{Hom}_{\mathcal{C}}(x, 0_{\mathcal{C}}) = 0$ and $\text{Hom}_{\mathcal{C}}(0_{\mathcal{C}}, x) = 0$ for any $x \in \text{ob } \mathcal{C}$.
3. Any additive category has finite coproducts that are equal to finite products.

12.1 Lecture 26

Definition. Let \mathcal{C} be an additive category. Let $f : x \rightarrow y$ be a morphism in \mathcal{C} .

1. A *kernel (object)* for f is a pair (k, q) where $k \in \text{ob } \mathcal{C}$ and $q : k \rightarrow x$ such that for any $z \in \text{ob } \mathcal{C}$, the natural sequence

$$\text{Hom}(z, k) \xrightarrow{q \circ -} \text{Hom}(z, x) \xrightarrow{f \circ -} \text{Hom}(z, y)$$

is exact.

2. A *cokernel (object)* for f is a pair (c, p) where $c \in \text{ob } \mathcal{C}$ and $p : y \rightarrow c$ such that for any $z \in \text{ob } \mathcal{C}$, the natural sequence

$$\text{Hom}(c, z) \xrightarrow{- \circ p} \text{Hom}(y, z) \xrightarrow{- \circ f} \text{Hom}(x, z)$$

is exact.

Definition. We say that a category \mathcal{A} is *abelian* if

1. \mathcal{A} is additive and
2. for any morphism $f : x \rightarrow y$ in \mathcal{A} , there exists a sequence $k \xrightarrow{q} x \xrightarrow{a} i \xrightarrow{b} y \xrightarrow{p} c$ in \mathcal{A} such that
 - (k, q) is a kernel for f ,
 - (c, p) is a cokernel for f ,
 - (c, a) is a cokernel for q , and
 - (i, b) is a kernel for p .

We call i the *image of f* .

Definition. If \mathcal{A} is a abelian, then a sequence $x \xrightarrow{f} y \xrightarrow{g} z$ in \mathcal{A} is *exact* if $\text{im } f = \ker g$.

Example 22.

1. **Ab.**
2. **$R\text{-Mod}$.**
3. **PreShAb_X** where X is a space.

Note 32. Our notion of and results for cohomology for complexes of abelian groups hold for complexes of objects in an abelian category.

Theorem 23. (Freyd-Mitchell) Every abelian category admits a fully faithful embedding in $R\text{-Mod}$ for some ring R .

Remark 15. It is not, in general, possible to complete an additive category \mathcal{C} to an abelian one. Still, we can always add enough images to \mathcal{C} to get cones of maps of complexes.

Example 23. (Idempotent completion) Let \mathcal{C} be additive. A map $e : x \rightarrow x$ in \mathcal{C} is an *idempotent* if $e^2 = e$. Let $\mathcal{C} = \mathbf{Vect}_k$. Then an idempotent $e : x \rightarrow x$ is a projection map, i.e., $x = x_1 \oplus x_2$ such that $e = i_1 \circ p_1$.

Definition. If \mathcal{C} is additive and $e : x \rightarrow x$ is idempotent in \mathcal{C} , then we say that e *has an image in \mathcal{C}* if there exists a decomposition $x = x_1 \oplus x_2$ such that

$$e = \begin{bmatrix} \text{id}_{x_1} & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to this decomposition. We say that x_1 is the *image of e* .

Note 33. If $e : x \rightarrow x$ be an idempotent. Then $\text{id}_x e : x \rightarrow x$ is also an idempotent. Indeed,

$$(\text{id}_x - e)^2 = \text{id}_x^2 - \text{id}_x e - e \text{id}_x + e^2 = \text{id}_x - e = \text{id}_x - e.$$

If $x = x_1 \oplus x_2$ has $e = \begin{bmatrix} \text{id}_{x_1} & 0 \\ 0 & 0 \end{bmatrix}$, then $\text{id}_x - e = \begin{bmatrix} 0 & 0 \\ 0 & \text{id}_{x_2} \end{bmatrix}$, so that $\text{id}_x - e$ has x_2 as an image.

Definition. A category \mathcal{C} is *idempotent complete* or *Karoubian* if \mathcal{C} is additive and any idempotent in \mathcal{C} has an image in \mathcal{C} .

Exercise 16. For any additive category \mathcal{C} , there exists a unique (up to unique isomorphism) category \mathcal{C}^{Kor} together with a functor $F : \mathcal{C} \rightarrow \mathcal{C}^{\text{Kor}}$ such that

1. \mathcal{C}^{Kor} is idempotent complete,
2. F is fully faithful, and
3. every object in \mathcal{C}^{Kor} is an image of an idempotent in \mathcal{C} .

Definition. A *graded additive category* is an additive category \mathcal{C} such that for any $x, y \in \text{ob } \mathcal{C}$, $\text{Hom}(x, y)$ is a graded abelian group, i.e., $\text{Hom}(x, y) \cong \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(x, y)$ and $\text{Hom}(x, y) \times \text{Hom}(y, z) \xrightarrow{\circ} \text{Hom}(x, z)$ satisfies $\text{Hom}^n(x, y) \times \text{Hom}^m(y, z) \xrightarrow{\circ} \text{Hom}^{n+m}(x, z)$ where \circ is bilinear.

Definition. A graded additive category \mathcal{C} is a *differential graded category* if for any $x, y \in \text{ob } \mathcal{C}$, the graded group $\text{Hom}(x, y)$ is equipped with with a homomorphism $d : \text{Hom}(x, y) \rightarrow \text{Hom}(x, y)$ such that

- $d : \text{Hom}^n(x, y) \rightarrow \text{Hom}^{n+1}(x, y)$,
- $d^2 = 0$, and

- d satisfies the *graded Leibniz rule*, i.e., if $f \in \text{Hom}^n(x, y)$ and $g \in \text{Hom}(a, x)$, then

$$d(f \circ g) = df \circ g + (-1)^n f \circ dg.$$

Proposition 20. Let \mathcal{C} be a category.

1. If \mathcal{C} is additive, then for any $x \in \text{ob } \mathcal{C}$, $\text{Hom}(x, x)$ is a ring (in fact, a \mathbb{Z} -algebra).
2. If \mathcal{C} is a graded additive category, then for any $x \in \text{ob } \mathcal{C}$, $\text{End}(x)$ is a graded ring.
3. If \vec{x} is differential graded category, then $\text{End}(x)$ is a differential graded algebra.

Definition. If \mathcal{C} is a differential graded category, then the *homotopy category of \mathcal{C}* is the category $\text{Ho}(\mathcal{C})$ (or $[\mathcal{C}]$) given by

$$\begin{aligned} \text{ob } \text{Ho}(\mathcal{C}) &= \text{ob } \mathcal{C} \\ \text{Hom}_{\text{Ho}(\mathcal{C})}(x, y) &= H^0(\text{Hom}_{\mathcal{C}}(x, y), d) \\ &= \frac{\ker(\text{Hom}_{\mathcal{C}}^0(x, y) \xrightarrow{d} \text{Hom}_{\mathcal{C}}^1(x, y))}{\text{im}(\text{Hom}_{\mathcal{C}}^{-1}(x, y) \xrightarrow{d} \text{Hom}_{\mathcal{C}}^0(x, y))}. \end{aligned}$$

Note 34. Let B be an additive category. Define the category $\mathbf{Compl}(B)$ of complexes in B by

$$\begin{aligned} \text{ob } \mathbf{Compl}(B) &= (\text{complexes of objects in } B) \\ \text{mor } \mathbf{Compl}(B) &= (\text{morphisms of complexes}). \end{aligned}$$

This is an additive category. We can also refine this definition by incorporating degree-shifting maps to get a differential graded category of complexes in B . Define the category $\mathbf{Compl}^\bullet(B)$ by

$$\begin{aligned} \text{ob } \mathbf{Compl}^\bullet(B) &= (\text{complexes of objects in } B) \\ \text{Hom}_{\mathbf{Compl}^\bullet(B)}(M, N) &= \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(M, N) \end{aligned}$$

where

$$\text{Hom}^n(M, N) \equiv \prod_{a \in \mathbb{Z}} \text{Hom}_B(M^a, N^{a+n}).$$

The composition is obtained component-wise from the composition in B . Define $d : \text{Hom}^n(M, N) \rightarrow \text{Hom}^{n+1}(M, N)$ by

$$(f_a)_{a \in \mathbb{Z}} \rightarrow (d_N \circ f_a + (-1)^n f_{a+1} \circ d_M)_{a \in \mathbb{Z}}.$$

This makes $\mathbf{Compl}^\bullet(B)$ a differential graded category.

Let $M, N \in \text{ob } \mathbf{Compl}^\bullet(B)$. Then

$$\begin{aligned} Z^0(\text{Hom}_{\mathbf{Compl}^\bullet(B)}(M, N)) &= \ker(\text{Hom}^0 \xrightarrow{d} \text{Hom}^1) \\ &= \text{Hom}_{\mathbf{Compl}(B)}(M, N) \\ B^0(\text{Hom}_{\mathbf{Compl}^\bullet(B)}(M, N)) &= \text{im}(\text{Hom}^{-1} \xrightarrow{d} \text{Hom}^0) \\ &= (\text{homotopies of 0-maps of complexes}). \end{aligned}$$

Also, we have that

$$H^0(\text{Hom}_{\mathbf{Compl}^\bullet(B)}(M, N)) = (\text{maps of complexes}) / (\text{homotopies}).$$

Example 24. $\text{Ho}(\mathbf{Comp}^\bullet(\mathbf{Ab})) = \mathcal{C}(\mathbf{Ab})$, and $Z^0(\mathbf{Comp}^\bullet(\mathbf{Ab})) = \mathbf{CoCh}(\mathbf{Ab})$.

12.2 Lecture 27

Definition. Let \mathcal{C} be a category. For any $x \in \text{ob } \mathcal{C}$, define $x[n]$ as the object, if it exists, in \mathcal{C} that represents the shift functor on morphisms $\text{Hom}_{\mathcal{C}}(-, x)[n] : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Comp}(\mathbf{Ab})$. If $f : x \rightarrow y$ is a morphism in \mathcal{C} , then define the *cone* $\text{cone}(f)$ of f to be the object, if it exists, in \mathcal{C} that represents the functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Comp}(\mathbf{Ab})$ given by $z \mapsto \text{cone}(\text{Hom}_{\mathcal{C}}(z, x) \xrightarrow{f \circ -} \text{Hom}_{\mathcal{C}}(z, y))$.

Definition. A category \mathcal{C} is called *strongly pre-triangulated* if every object in \mathcal{C} has shifts in \mathcal{C} and every morphism in \mathcal{C} has cones in \mathcal{C} . We call \mathcal{C} *pre-triangulated* if every object in \mathcal{C} has shifts in $\text{Ho}(\mathcal{C})$ and every morphism in \mathcal{C} has cones in $\text{Ho}(\mathcal{C})$.

Note 35. Both the assignment $x \mapsto x[n]$ and the assignment $f \mapsto \text{cone}(f)$ are functorial.

Definition. Given a differential graded category \mathcal{C} , we define $\text{Ho}^{\bullet}(\mathcal{C})$ as the graded additive category with

$$\begin{aligned} \text{ob } \text{Ho}^{\bullet}(\mathcal{C}) &= \text{ob } \mathcal{C} \\ \text{Hom}_{\text{Ho}^{\bullet}(\mathcal{C})}(x, y) &= H^{\bullet}(\text{Hom}_{\mathcal{C}}(x, y)). \end{aligned}$$

Note 36. If \mathcal{C} is strongly pre-triangulated, then $\text{Ho}^{\bullet}(\mathcal{C})$ and $\text{Ho}(\mathcal{C})$ contain the same information. Indeed, $\text{Ho}(\mathcal{C})$ is precisely the degree zero piece of $\text{Ho}^{\bullet}(\mathcal{C})$. Conversely, if $x, y \in \text{ob } \mathcal{C}$, then

$$\text{Hom}_{\text{Ho}^{\bullet}(\mathcal{C})}(x, y) = \bigoplus_{a \in \mathbb{Z}} \text{Hom}_{\text{Ho}^{\bullet}(\mathcal{C})}^a(x, y)$$

where $\text{Hom}_{\text{Ho}^{\bullet}(\mathcal{C})}^a(x, y) = \text{Hom}_{\text{Ho}(\mathcal{C})}(x, y[a]) = \text{Hom}_{\text{Ho}(\mathcal{C})}(x, y)[a]$. From now on, if \mathcal{C} is strongly pre-triangulated, then we write $\text{Ho}(\mathcal{C})$ for the graded homotopy category.

Definition. If \mathcal{C} is strongly pre-triangulated, then a *triangle* \triangle in $\text{Ho}(\mathcal{C})$ is a sequence of degree zero maps $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} x[1]$. We represent this as

$$\begin{array}{ccccc} x & \xrightarrow{u} & y & & \\ & \nwarrow w & \downarrow v & & \\ & & z & & \end{array} .$$

Note 37. Let \mathcal{C} be strongly pre-triangulated. Given a triangle

$$\begin{array}{ccccc} x & \xrightarrow{u} & y & & \\ & \nwarrow w & \downarrow v & & \\ & & z & & \end{array} .$$

we get a long sequence of maps

$$\begin{array}{ccccccc} x[-1] & \xrightarrow{u[-1]} & y[-1] & \xrightarrow{v[-1]} & z[-1] & & \\ & & \searrow w[-1] & & & & \\ x & \xleftarrow{u} & y & \xrightarrow{v} & z & & \\ & & \nwarrow w & & & & \\ x[1] & \xleftarrow{u[1]} & y[1] & \xrightarrow{v[1]} & z[1] & \xrightarrow{w[1]} & \cdots \end{array} .$$

in \mathcal{C} .

Definition. Let \mathcal{C} be strongly pre-triangulated. We say that a triangle in $\text{Ho}(\mathcal{C})$ is *exact* if it is isomorphic to the triangle

$$x \xrightarrow{u} y \xrightarrow{\text{"inclusion"}} \text{cone}(u) \xrightarrow{\text{"projection''}} x[1] .$$

13 Triangulated categories

Definition. A graded additive category \mathcal{D} is *triangulated* if \mathcal{D} is equipped with a shift functor $[1] : \mathcal{D} \rightarrow \mathcal{D}$ and a collection of *distinguished triangles* such that the following axioms hold.

0. Every triangle that is isomorphic to a distinguished triangle is distinguished.
1. For any object x in \mathcal{D} , the triangle $x \xrightarrow{\text{id}_x} x \rightarrow 0 \rightarrow x[1]$ is distinguished.
2. (*rotation invariance*) The shift rotation of a triangle \triangle is distinguished if and only if \triangle is, i.e., the triangle $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} x[1]$ is distinguished if and only if the triangle $y \xrightarrow{v} z \xrightarrow{w} x[1] \xrightarrow{-u[1]} y[1]$ is distinguished.
3. Every morphism $u : x \rightarrow y$ can be included in a distinguished triangle $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} x[1]$, and every commutative square

$$\begin{array}{ccc} x & \xrightarrow{u} & y \\ f \downarrow & & \downarrow g \\ x' & \xrightarrow{u'} & y' \end{array}$$

can be completed to a commutative diagram of distinguished triangles, i.e.,

$$\begin{array}{ccccccc} x & \xrightarrow{u} & y & \xrightarrow{v} & z & \xrightarrow{w} & x[1] \\ f \downarrow & & \downarrow g & & \downarrow h & & \downarrow f[1] \\ x' & \xrightarrow{u'} & y' & \xrightarrow{v'} & z' & \xrightarrow{w'} & x'[1] \end{array}$$

4. (*octahedron axiom*) Given any two distinguished triangles $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} x[1]$ and $y \xrightarrow{f} y' \xrightarrow{g} q \xrightarrow{h} y[1]$, we can complete them to a commutative diagram

$$\begin{array}{ccccccc} x & \xrightarrow{u} & y & \xrightarrow{v} & z & \xrightarrow{w} & x[1] \\ \parallel & & \downarrow f & & \downarrow a & & \parallel \\ x & \longrightarrow & y' & \longrightarrow & z' & \longrightarrow & x[1] \\ & & \downarrow g & & \downarrow b & & \downarrow u[1] \\ & & q & \xlongequal{\quad} & q & \longrightarrow & y[1] \\ & & \downarrow h & & \downarrow c & & \\ & & y[1] & \longrightarrow & z[1] & & \end{array}$$

where each new triangle is distinguished.

Note 38. The octahedron axiom is the formal transplant of the second isomorphism theorem for $\mathbf{Comp}(\mathbf{Ab})$,¹ which states that given two complexes L and M , an inclusion $f : L \hookrightarrow M$, and a subcomplex N of L and of M , we have that $M/L \cong (M/N)/(L/N)$, i.e., if

$$\begin{array}{ccccccc} & N & & N & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & M/L \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L/N & \longrightarrow & M/N & \longrightarrow & (M/N)/(L/N) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

¹The second isomorphism theorem holds in some form for any abelian category.

has exact rows and exact left two columns, then the third column is also exact.

Lemma 21. Suppose that \mathcal{C} is strongly pre-triangulated and let $\alpha : M \rightarrow N$ be a morphism in \mathcal{C} such that α is injective (i.e., $\ker \alpha$ exists and is trivial) with $d\alpha = 0$ and α is split (i.e., there exists $\beta : N \rightarrow M$ with $\beta \circ \alpha = \text{id}_M$). We call such an α a *split monomorphism* in \mathcal{C} . Then the following statements hold.

- (a) The map $\text{cone}(\alpha) \rightarrow N/M$ is a homotopy equivalence.
- (b) Any morphism in \mathcal{C} is homotopy equivalent to a split mono, i.e., given $f : M \rightarrow L$, we can construct a natural diagram

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ & \searrow f & \downarrow g \\ & & L \end{array}$$

in \mathcal{C} such that α is a split mono and g is an iso in $\text{Ho}(\mathcal{C})$.

Proof. For (b), take $N = L \oplus \text{cone}(\text{id}_M)$. □

Theorem 24. If \mathcal{C} is a strongly pre-triangulated differential graded category and $\mathcal{D} = \text{Ho}(\mathcal{C})$, then \mathcal{D} is triangulated with exact triangles as the distinguished triangles.

Proof.

Verifying axioms (0) and (1) is trivial.

For axiom (2), if $x \rightarrow y \rightarrow z \rightarrow x[1]$ is a triangle, then we can use our previous lemma to rewrite it as a homotopy equivalent triangle $M \rightarrow N \rightarrow L \rightarrow M[1]$ where $M \xrightarrow{\alpha} N$ is a split mono. In this case, we can check that $N \rightarrow L \rightarrow M[1] \rightarrow N[1]$ is exact by using the splitting.

For axiom (3), note that any $u : x \rightarrow y$ is included in $x \rightarrow y \rightarrow \text{cone}(u) \rightarrow x[1]$. Moreover, if

$$\begin{array}{ccc} x & \xrightarrow{u} & y \\ f \downarrow & & \downarrow g \\ x' & \xrightarrow{u'} & y' \end{array}$$

is commutative in $\text{Ho}(\mathcal{C})$ and we lift f, g, u , and u' to maps $\tilde{\cdot}$ in \mathcal{C} , then we get a diagram

$$\begin{array}{ccccccc} x & \xrightarrow{\tilde{u}} & y & \longrightarrow & \text{cone}(u) & \longrightarrow & x[1] \\ \tilde{f} \downarrow & & \downarrow \tilde{g} & & M \downarrow & & \downarrow \tilde{f}[1] \\ x' & \xrightarrow{\tilde{u}'} & y' & \longrightarrow & \text{cone}(u') & \longrightarrow & x'[1] \end{array}$$

in \mathcal{C} where $M \equiv \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}$, $\delta \in \text{Hom}^{-1}(x, y')$, and $\tilde{g} \circ \tilde{u} - \underbrace{\tilde{u}' \circ \tilde{f}}_{d(\delta)} \sim 0$.

For axiom (4), given a distinguished triangle $M \rightarrow N \rightarrow L \rightarrow M[1]$, we apply our previous lemma twice to get a homotopy equivalent distinguished triangle $M \rightarrow N' \rightarrow L'' \rightarrow M[1]$ where each map in this is a split mono. We are done after an application of the second isomorphism theorem. □

Remark 16. Such reasoning can be applied to complete any differential graded category to a triangulated one.

13.1 Lecture 28

Definition. If \mathcal{A} and \mathcal{B} are differential graded categories, then a *differential graded functor* $F : \mathcal{A} \rightarrow \mathcal{B}$ satisfies

- F is additive, i.e., $F : \text{Hom}_{\mathcal{A}}(x, y) \rightarrow \text{Hom}_{\mathcal{B}}(F(x), F(y))$ is a group homomorphism for any $x, y \in \text{ob } \mathcal{A}$, and
- F respects differentials, i.e., if $x, y \in \text{ob } \mathcal{A}$, then $F : \text{Hom}_{\mathcal{A}}(x, y) \rightarrow \text{Hom}_{\mathcal{B}}(F(x), F(y))$ is a map of complexes.

Definition. If $F, G : \mathcal{A} \rightarrow \mathcal{B}$ are two differential graded functors between differential graded categories, then define, for each $n \in \mathbb{Z}$, the group

$$\text{Hom}^n(F, G) \equiv \{\varphi_x \mid \varphi_x : F(x) \rightarrow G(x) \text{ in } \text{Hom}_{\mathcal{B}}^n(F(x), G(x)), x \in \text{ob } \mathcal{A}\}.$$

A map $F \rightarrow G$ is a natural transformation $F \rightarrow G$ such that each component $\varphi_x : F(x) \rightarrow G(x)$ belongs to $\text{Hom}_{\mathcal{B}}^n(F(x), G(x))$. The differential on $\prod_{x \in \text{ob } \mathcal{A}} \text{Hom}^\bullet(F(x), G(x))$ induces a differential on

$$\text{Hom}^\bullet(F, G) \equiv \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n(F, G).$$

This produces a complex of maps between F and G , and we get a differential graded category $\mathbf{dgFun}(\mathcal{A}, \mathcal{B})$.

Exercise 17.

1. If $F : \mathcal{A} \rightarrow \mathcal{B}$ is a differential graded functor, then $H^0(F) : H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$ is an additive functor.
2. If $F, G : \mathcal{A} \rightarrow \mathcal{B}$ are differential graded functors, then there is an embedding $H^0(\text{Hom}(F, G)) \subset \text{Hom}(H^0(F), H^0(G))$.

Definition. If \mathcal{A} is a differential graded category, then a *left \mathcal{A} -module* is a differential graded functor $\mathcal{A} \rightarrow \mathbf{Compl}(\mathbf{Ab})$ and a *right \mathcal{A} -module* is a differential graded functor $\mathcal{A}^{\text{op}} \rightarrow \mathbf{Compl}(\mathbf{Ab})$.

Exercise 18.

1. If \mathcal{A} is a differential graded category with a single object $*$, then $\mathcal{A} \longleftrightarrow R := \text{Hom}_{\mathcal{A}}(*, *)$, which is precisely the complex of abelian groups equipped with a multiplication \cdot such that λ satisfies the graded Leibniz rule for \cdot . Then a module over \mathcal{A} is precisely the data of a complex x of abelian groups together with a differential graded algebra homomorphism $R \rightarrow \text{Hom}_{\mathbf{Compl}(\mathbf{Ab})}(x, x)$.
2. Given a differential graded category \mathcal{A} , we have respective categories of left and right modules over \mathcal{A} that are linear over a field k , namely

$$\begin{aligned} \mathcal{A}\text{-}\mathbf{dgmod}_k &\equiv \mathbf{dgFun}(\mathcal{A}, \mathbf{Compl}(k\text{-}\mathbf{Vect})) \\ \mathbf{dgmod}_k\text{-}\mathcal{A} &\equiv \mathbf{dgFun}(\mathcal{A}^{\text{op}}, \mathbf{Compl}(k\text{-}\mathbf{Vect})). \end{aligned}$$

Show that the differential graded functors

$$\begin{aligned} h^\bullet : \mathcal{A}^{\text{op}} &\rightarrow \mathcal{A}\text{-}\mathbf{dgmod}_k \\ x &\mapsto h^\times \equiv \text{Hom}_{\mathcal{A}}(x, -) \\ h_\bullet : \mathcal{A} &\rightarrow \mathbf{dgmod}_k\text{-}\mathcal{A} \\ h_\times &\equiv \text{Hom}_{\mathcal{A}^{\text{op}}}(x, -) = \text{Hom}_{\mathcal{A}}(-, x) \end{aligned}$$

are fully faithful differential graded functors.

Lemma 22. If \mathcal{A} is a small differential graded category, then $H^0(\mathcal{A}^{\text{op}}\text{-}\mathbf{dgmod}_k)$ is triangulated.

Claim 6. If \mathcal{A} is a pre-triangulated differential graded category, then the fully faithful functor $H^0(h_\bullet) : H^0(\mathcal{A}) \rightarrow H^0(\mathcal{A}^{\text{op}}\text{-}\mathbf{dgmod}_k)$ gives a triangulated structure on $H^0(\mathcal{A})$.

Definition. We say that an object F in $\mathcal{A}^{\text{op}}\text{-}\mathbf{dgmod}_k$ is *compact* or *perfect* if $F : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Compl}(k\text{-}\mathbf{Vect})$ commutes with arbitrary coproducts.

Note 39. h^\times is compact for any $x \in \text{ob } \mathcal{A}$.

Definition. We say that a k -linear differential graded category \mathcal{A} is *triangulated* if every compact object in $\mathcal{A}^{\text{op}}\text{-}\mathbf{dgmod}_k$ is representable.

Note 40. A triangulated differential graded category is automatically strongly pre-triangulated, and $H^0(\mathcal{A})$ is triangulated.

Exercise 19.

1. Suppose that \mathcal{D} is a triangulated additive category. Let $M \rightarrow N \rightarrow C \rightarrow M[1]$ be a distinguished triangle. Show that for every $L \in \text{ob } \mathcal{D}$, the sequence

$$\cdots \rightarrow \text{Hom}(L, M) \rightarrow \text{Hom}(L, N) \rightarrow \text{Hom}(L, C) \rightarrow \text{Hom}(L, M[1]) \rightarrow \text{Hom}(L, N[1]) \rightarrow \text{Hom}(L, C[1]) \rightarrow \cdots$$

is a long exact sequence of abelian groups.

2. Suppose that \mathcal{D} is triangulated. Show that the sum $\Delta_1 \oplus \Delta_2$ of two triangles in \mathcal{D} is distinguished if and only if both Δ_1 and Δ_2 are distinguished.

Definition. If \mathcal{D}_1 and \mathcal{D}_2 are triangulated additive categories, then a *triangulated* (or *exact*) *functor* $F : \mathcal{D}_1 \rightarrow \mathcal{D}_2$ is an additive functor such that

- F is equipped with an isomorphism $\sigma : F \circ [1] \rightarrow [1] \circ F$ and
- F sends distinguished triangles to distinguished triangles.

A *morphism of two triangulated functors* (F, θ_F) and (G, θ_G) is a morphism $f : F \rightarrow G$ of additive functors such that f intertwines θ_F and θ_G . We get a category of triangulated functors $\mathcal{D}_1 \rightarrow \mathcal{D}_2$.

Note 41. If \mathcal{A} and \mathcal{B} are differential graded categories and $F : \mathcal{A} \rightarrow \mathcal{B}$ is a differential graded functor, then we have a natural differential graded functor $\mathcal{A}^{\text{op}}\text{-}\mathbf{dgmod}_k \xrightarrow{F} \mathcal{B}^{\text{op}}\text{-}\mathbf{dgmod}_k$ so that $H^0(F)$ is triangulated.

Definition. If \mathcal{D} is a triangulated category and \mathcal{A} is an abelian category, then a *cohomological functor* is a functor $H : \mathcal{D} \rightarrow \mathcal{A}$ such that

- H is additive and
- H sends distinguished Δ 's in \mathcal{D} into long exact sequences in \mathcal{A} .

Example 25.

1. If $\mathcal{C}(\mathbf{Ab})$ denotes the triangulated category of homotopy classes of complexes of abelian groups, then $H^\bullet : \mathcal{C}(\mathbf{Ab}) \rightarrow \mathbf{grAb}$ is a cohomological functor.
2. If \mathcal{D} is a triangulated category and $L \in \text{ob } \mathcal{D}$, then $h^L : \mathcal{D} \rightarrow \mathbf{Ab}$ given by $M \mapsto Z^0(\text{Hom}_{\mathcal{D}}(L, M))$ is a cohomological functor.

13.2 Lecture 29

Note 42. Let \mathcal{D} be a triangulated category and $\mathcal{V} \subset \mathcal{D}$ a triangulated subcategory (i.e., the inclusion functor is triangulated). We wish to construct a quotient category \mathcal{D}/\mathcal{V} , i.e., a triangulated category \mathcal{D}/\mathcal{V} together with a triangulated functor $q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{V}$ such that $q(x) = 0$ for any $x \in \text{ob } \mathcal{V}$ and for any triangulated functor $f : \mathcal{D} \rightarrow \mathcal{D}'$ with $f(x) = 0$ for any $x \in \text{ob } \mathcal{V}$, we have $g \circ q = f$.

In the triangulated category of triangulated categories with exact functors, the triangle $\mathcal{V} \rightarrow \mathcal{D} \rightarrow \mathcal{D}/\mathcal{V} \rightarrow \mathcal{V}[1]$ is exact.

Note 43. If \mathcal{D} is triangulated and $u : x \rightarrow y$ is a morphism in \mathcal{D} , then there exists an object $\text{cone}(u)$ in \mathcal{D} that is unique up to a non-unique isomorphism. This is the third term in a distinguished \triangle completing u .

Exercise 20. If $u : x \rightarrow y$ is a map in \mathcal{D} , then it is an isomorphism in \mathcal{D} if and only if $\text{cone}(u) = 0$.

Definition. If \mathcal{D} is a triangulated category and $\mathcal{V} \subset \mathcal{D}$ a triangulated subcategory, then a morphism $u : x \rightarrow y$ in \mathcal{D} is a \mathcal{V} -quasi-isomorphism if $\text{cone}(u) \in \text{ob}(\mathcal{V})$.

Exercise 21. Let $\mathcal{V} \subset \mathcal{D}$ be a pair of triangulated categories. Use the octahedron axiom to show that if f and g are composable morphisms in \mathcal{D} , then every morphism in $\{f, g, g \circ f\}$ is a \mathcal{V} -quasi-isomorphism if and only if at least two morphisms in it are \mathcal{V} -quasi-isomorphisms.

Remark 17. One may define \mathcal{D}/\mathcal{V} as the localization of \mathcal{D} in the set of all \mathcal{V} -quasi-isomorphisms. But doing so required a lot of work.

Definition. Suppose that \mathcal{I} is a small category. We say that \mathcal{I} is a *directed category* if it satisfies the following properties.

1. If $x_1, x_2 \in \text{ob } \mathcal{I}$, then there is some diagram

$$\begin{array}{ccc} x_1 & \longrightarrow & x_3 \\ & \nearrow & \\ x_2 & & \end{array}$$

of maps in \mathcal{I} .

2. If

$$\begin{array}{ccc} x_1 & \longrightarrow & x_2 \\ & \searrow & \\ & & x_3 \end{array}$$

is a diagram of maps in \mathcal{I} , then there exist maps $x_2 \rightarrow x_4, x_3 \rightarrow x_4$ in \mathcal{I} such that

$$\begin{array}{ccccc} x_1 & \longrightarrow & x_2 & \longrightarrow & x_4 \\ & \searrow & & \nearrow & \\ & & x_3 & & \end{array}$$

commutes in \mathcal{I} .

3. For any two parallel maps $f, g : x \rightarrow y$, there exists a map $h : y \rightarrow z$ such that $h \circ f = h \circ g$.

Exercise 22. Let \mathcal{I} be small. There is a well-defined functor $\text{colim} : \text{Fun}(\mathcal{I}, \mathbf{Ab}) \rightarrow \mathbf{Ab}$, but this need not be exact even though both \mathbf{Ab} and $\text{Fun}(\mathcal{I}, \mathbf{Ab})$ are abelian. Show, however, that if \mathcal{I} is directed, then colim is an exact functor.

Definition. Let $\mathcal{V} \subset \mathcal{D}$ be a pair of triangulated categories. Let $x \in \text{ob } \mathcal{D}$ and let \mathcal{D}/\vec{x} be the full subcategory of \mathcal{D}/\mathcal{V} consisting of morphisms $y \rightarrow x$ that are \mathcal{V} -quasi-isomorphisms. Similarly, let x/\mathcal{Q} be the full subcategory of x/\mathcal{D} consisting morphisms $x \rightarrow z$ that are \mathcal{V} -quasi-isomorphisms.

Exercise 23.

1. Both x/\mathcal{D} and $(\mathcal{D}/x)^{\text{op}}$ are directed categories.
2. Any map in x/\mathcal{D} or \mathcal{D}/x is automatically a \mathcal{V} -quasi-isomorphism.

Definition. Define the *Verdier quotient* of \mathcal{D} by \mathcal{V} is the category \mathcal{D}/\mathcal{V} with $\text{ob } \mathcal{D}/\mathcal{V} \equiv \text{ob } \mathcal{D}$ and $\text{Hom}_{\mathcal{D}/\mathcal{V}}(a, b) \equiv \text{colim}_{a' \in (\mathcal{D}/a)^{\text{op}}} \text{Hom}_{\mathcal{D}}(a', b)$.

Remark 18. There exists a canonical isomorphism $\operatorname{colim}_{a' \in (\mathcal{D}/a)^{\operatorname{op}}} \operatorname{Hom}_{\mathcal{D}}(a', b) \cong \operatorname{colim}_{b' \in (b/\mathcal{D})} \operatorname{Hom}_{\mathcal{D}}(a, b')$. For this, we must check that given a top triangle

$$\begin{array}{ccc} a' & \xrightarrow{\text{q-iso}} & a \\ & \searrow & \downarrow \\ & & b \end{array},$$

we can form a commutative double triangle

$$\begin{array}{ccccc} a' & \xrightarrow{\text{q-iso}} & a & \longrightarrow & b' \\ & \searrow & \downarrow & \nearrow & \\ & & b & & \end{array} \quad \text{q-iso}.$$

As a result, we get $q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{V}$.

Lemma 23. If $x \in \operatorname{ob} \mathcal{D}$ has $q(x) = 0$ in \mathcal{D}/\mathcal{V} , then x is a direct summand of an object in \mathcal{V} .

Proof. We have that $q(x) = 0 \iff$ there is some $y \in \mathcal{D}$ such that $\varphi : y \rightarrow x$ is a \mathcal{V} -quasi-isomorphism. In this case, $\underbrace{\operatorname{cone}(\varphi)}_{y[1] \oplus x} \in \mathcal{V}$. \square

Definition. A triangulated subcategory $\mathcal{V} \subset \mathcal{D}$ is *thick* if any object in \mathcal{D} that is isomorphic to a direct summand of an object in \mathcal{V} is an object in \mathcal{V} .

Note 44. If \mathcal{V} is a strict full thick triangulated subcategory of \mathcal{D} , then $q : \mathcal{D} \rightarrow \mathcal{D}/\mathcal{V}$ kills all and only objects in \mathcal{V} .

Definition. If \mathcal{D} is triangulated and $\mathcal{U}, \mathcal{V} \subset \mathcal{D}$ are strict full triangulated subcategories, then $(\mathcal{U}, \mathcal{V})$ is an *admissible pair of subcategories* if

- (a) $\operatorname{Hom}_{\mathcal{D}}(x, y) = 0$ for any $x \in \operatorname{ob} \mathcal{U}$ and $y \in \operatorname{ob} \mathcal{V}$ and
- (b) any object $z \in \operatorname{ob} \mathcal{D}$ fits in a distinguished triangle $x \rightarrow z \rightarrow y \rightarrow x[1]$ with $x \in \operatorname{ob} \mathcal{U}$ and $y \in \operatorname{ob} \mathcal{V}$.

Exercise 24.

1. The \triangle in condition (b) is unique up to a unique isomorphism and is functorial in z .
2. The functor $\mathcal{D} \rightarrow \mathcal{U}$ given by $z \mapsto x(z)$ is triangulated and is right adjoint to $\mathcal{U} \hookrightarrow \mathcal{D}$.
Dually, the functor $\mathcal{D} \rightarrow \mathcal{V}$ given by $z \mapsto y(z)$ is triangulated and is left adjoint to $\mathcal{V} \hookrightarrow \mathcal{D}$.
3. Each of \mathcal{U} and \mathcal{V} determines the other. Specifically, $\mathcal{V} = \mathcal{U}^{\perp} \equiv \underbrace{\{y \in \operatorname{ob} \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(x, y) = 0, x \in \operatorname{ob} \mathcal{U}\}}_{\text{full subcategory}}$,
and $\mathcal{U} = {}^{\perp}\mathcal{V} \equiv \underbrace{\{x \in \operatorname{ob} \mathcal{D} \mid \operatorname{Hom}_{\mathcal{D}}(x, y) = 0, y \in \operatorname{ob} \mathcal{V}\}}_{\text{full subcategory}}$. In particular, both \mathcal{U} and \mathcal{V} are thick subcategories.
4. The natural compositions $\mathcal{U} \hookrightarrow \mathcal{D} \rightarrow \mathcal{D}/\mathcal{V}$ and $\mathcal{V} \hookrightarrow \mathcal{D} \rightarrow \mathcal{D}/\mathcal{U}$ are triangulated equivalences.

Definition. An additive pair $(\mathcal{U}, \mathcal{V})$ is called a *semiorthogonal decomposition of \mathcal{D} into \mathcal{U} and \mathcal{V}* .

Proposition 21. If $\mathcal{U} \subset \mathcal{D}$ is a strict full triangulated thick subcategory, then TFAE.

1. The inclusion $\mathcal{U} \hookrightarrow \mathcal{D}$ has a left adjoint.
2. The quotient $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{U}$ has a right adjoint.
3. $(\mathcal{U}, \mathcal{U}^{\perp})$ is admissible.

Definition. If \mathcal{A} is an abelian category, then the *derived category of \mathcal{A}* is the triangulated category

$$\mathcal{D}(\mathcal{A}) \equiv \mathcal{C}(\mathcal{A}) / \mathcal{C}(\mathcal{A})^{\text{acyclic}},$$

where $\mathcal{C}(\mathcal{A})^{\text{acyclic}}$ is the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of those x with zero cohomology.

Remark 19. To do computations in $\mathcal{D}(\mathcal{A})$, we must understand when $\mathcal{D}(a)$ can be embedded in $\mathcal{C}(\mathcal{A})$ so that $(\mathcal{C}(\mathcal{A})^{\text{acyclic}}, \mathcal{D}(\mathcal{A}))$ is an adjoint pair. This requires $(\mathcal{C}(\mathcal{A})^{\text{acyclic}})^{\perp}$ to be large.

Definition. Define ${}^{\perp}\mathcal{C}(\mathcal{A})^{\text{acyclic}}$ as the category of *homotopically projective objects* in $\mathcal{C}(\mathcal{A})$ and $(\mathcal{C}(\mathcal{A})^{\text{acyclic}})^{\perp}$ as the category of *homotopically injective objects* in $\mathcal{C}(\mathcal{A})$.

Proposition 22. Every bounded-above complex of projectives is a homotopically projective object in $\mathcal{C}(\mathcal{A})$. Moreover, any bounded-below complex of injectives is a homotopically injective object in $\mathcal{C}(\mathcal{A})$.