

Abstract

This is an introduction to natural transformations in category theory, including equivalences and adjunctions. The main sources for this talk are the following.

- nLab.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 3.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 1.

1 Natural transformations

Let \mathcal{C} and \mathcal{D} be categories and F and G be functors $\mathcal{C} \rightarrow \mathcal{D}$. A *natural transformation* $\phi : F \Rightarrow G$ is a function $A \mapsto f_A$ from $\text{ob } \mathcal{C}$ to $\text{mor } \mathcal{D}$ such that f_A is a map $F(A) \rightarrow G(A)$ and the following diagram commutes for any morphism $h : A \rightarrow B$ in \mathcal{C} .

$$\begin{array}{ccc} FA & \xrightarrow{Fh} & FB \\ f_A \downarrow & & \downarrow f_B \\ GA & \xrightarrow{Gh} & GB \end{array}$$

In symbols, this may be written as $f_B h_* = h_* f_A$, where f_A and f_B are called the *components* of ϕ .

Note 1.1. If every f_A is an isomorphism, then the $(f_A)^{-1}$ define a natural transformation between the same two functors.

If each f_A is an isomorphism, then we say that ϕ is a *natural isomorphism*. Note that if \mathcal{D} is a groupoid, then ϕ must be a natural isomorphism.

Let F , G , and H be functors $\mathcal{C} \rightarrow \mathcal{D}$. The *identity natural transformation* $\text{Id}_F : F \Rightarrow F$ is given by $A \mapsto \text{Id}_{F(A)}$. Moreover, given natural transformations $\phi : F \rightarrow G$ and $\psi : G \rightarrow H$, define the *composite natural transformation* $\psi \circ \phi$ by $A \mapsto (\psi \circ \phi)_A := \psi_A \circ \phi_A$.

Lemma 1.2. A natural transformation $\phi : F \Rightarrow G$ is a natural isomorphism iff it has an inverse $\phi^{-1} : G \Rightarrow F$.

Proof. This follows from Note 1.1 along with our definition of a composite natural transformation. □

Example 1.3.

1. Let R and S be commutative rings. Any ring homomorphism $f : R \rightarrow S$ induces a ring homomorphism $\text{GL}_n(f) : \text{GL}_n(R) \rightarrow \text{GL}_n(S)$ which satisfies $f(\det(A)) = \det(\text{GL}_n(f)(A))$. Viewing GL_n and $R \mapsto R^*$

as functors from **Ring** to **Grp** and $\det_R : \mathrm{GL}_n(R) \rightarrow R^*$ as a morphism in **Grp**, we see that \det_R defines a natural transformation $\phi : \mathrm{GL}_n \Rightarrow f^*$ where f^* denotes $f \downarrow_{R^*} R^* \rightarrow S^*$.

$$\begin{array}{ccc} \mathrm{GL}_n(R) & \xrightarrow{\mathrm{GL}_n(f)} & \mathrm{GL}_n(S) \\ \det_R \downarrow & & \downarrow \det_S \\ R^* & \xrightarrow{f^*} & S^* \end{array}$$

2. Consider the power set functor $\mathcal{P} : \mathbf{Set} \rightarrow \mathbf{Set}$, given on objects by $A \mapsto \mathcal{P}(A)$ and on morphisms g by $\mathcal{P}g(S) = g(S)$. Then the function $f_A : A \rightarrow \mathcal{P}(A)$ given by $a \mapsto \{a\}$ defines a natural transformation $\phi : \mathrm{Id}_{\mathbf{Set}} \Rightarrow \mathcal{P}$.
3. Set $\mathcal{C} = \mathcal{D} = \mathbf{Grp}$, $F = \mathrm{Id}_{\mathcal{C}}$, and $G = (-)^{\mathrm{ab}}$. Then given a group H , the natural projection $f : H \twoheadrightarrow H^{\mathrm{ab}}$ induces a natural transformation $\phi : F \Rightarrow G$.
4. We can view preorders (P, \leq) and (Q, \leq) as small categories and functors $F, G : P \rightarrow Q$ as order-preserving functions. Then there is a unique natural transformation $\phi : F \Rightarrow G$ iff $F(x) \leq G(x)$ for every $x \in P$.
5. The inversion isomorphism from a group G to its opposite group G^{op} defines a natural transformation $\phi : \mathrm{Id}_{\mathbf{Grp}} \Rightarrow ((-)^{\mathrm{op}} : \mathbf{Grp} \rightarrow \mathbf{Grp})$. In this sense, G is naturally isomorphic to G^{op} .

Definition 1.4. Let \mathcal{C} and \mathcal{D} be categories with \mathcal{C} small. The *functor category* $\mathbf{Fun}(\mathcal{C}, \mathcal{D}) := \mathcal{D}^{\mathcal{C}}$ has functors $F : \mathcal{C} \rightarrow \mathcal{D}$ as objects and natural transformations as morphisms.

Remark 1.5. Any Grothendieck universe models ZFC, in particular **Replacement**. This ensure that for any two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$, the class of natural transformation $\phi : F \Rightarrow G$ is a set. This means that $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ is locally small, a condition of our definition of a category.

Definition 1.6. Given a category \mathcal{C} , the *arrow category* $\mathrm{Ar}(\mathcal{C})$ of \mathcal{C} has as objects morphisms $f : X_0 \rightarrow X_1$ in \mathcal{C} and as morphisms $M : (f : X_0 \rightarrow X_1) \rightarrow (g : Y_0 \rightarrow Y_1)$ the pairs (M_0, M_1) of morphisms $M_0 : X_0 \rightarrow Y_0$ and $M_1 : X_1 \rightarrow Y_1$ such that

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ M_0 \downarrow & & \downarrow M_1 \\ Y_0 & \xrightarrow{g} & Y_1 \end{array}$$

commutes.

Note 1.7.

1. $\mathrm{Ar}(\mathcal{C}) \cong \mathbf{Fun}([1], \mathcal{C})$.
2. $\mathbf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathbf{Fun}(\mathcal{C}, \mathbf{Fun}(\mathcal{D}, \mathcal{E}))$.

2 Equivalences

Usually, it is useful to make our notion of *sameness* between categories weaker than *isomorphism*.

Definition 2.1. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an *equivalence* if there is a functor $G : \mathcal{D} \rightarrow \mathcal{C}$, called the *quasi-inverse* of F , such that $F \circ G \cong \text{Id}_{\mathcal{D}}$ and $G \circ F \cong \text{Id}_{\mathcal{C}}$. In this case, we say that F and G are *equivalent categories*. Moreover, we say that a property of \mathcal{C} is *categorical* if it is invariant under such equivalence.

Example 2.2. Let k be a field. Let the category \mathbf{Mat}_k have natural numbers as objects and morphisms $n \rightarrow p$ given by $p \times n$ matrices over k . Let \mathbf{fdMod} denote the category of finite-dimensional vector spaces with linear maps as morphisms. These two categories are equivalent. Indeed, send the natural number n to k^n in one direction and the space V to $\dim V$ in the other direction.

Definition 2.3. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is *essentially surjective* if for each object Z of \mathcal{D} , there is some object Y of \mathcal{C} such that $F(Y) \cong Z$.

Theorem 2.4. A functor is an equivalence iff it is full, faithful, and essentially surjective.¹

Definition 2.5. A *skeleton* of \mathcal{C} is a full subcategory $\mathcal{C}' \subset \mathcal{C}$ such that each element of $\text{ob } \mathcal{C}$ is isomorphic to exactly one element of $\text{ob } \mathcal{C}'$.

An application of Theorem 2.4 yields the following result.

Lemma 2.6. Let \mathcal{C}' be a skeleton of \mathcal{C} . Then the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an equivalence.

Lemma 2.7. Any two skeleta $\mathcal{C}', \mathcal{C}'' \subset \mathcal{C}$ are isomorphic.

Proof. Define $F : \mathcal{C}' \rightarrow \mathcal{C}''$ on objects by $F(X) = Y$ where $X \cong Y$ via a chosen map h_X and on morphisms $f \in \mathcal{C}(X, Y)$ by $F(f) = h_Y \circ f \circ (h_X)^{-1}$. To get F^{-1} , define $G : \mathcal{C}'' \rightarrow \mathcal{C}'$ by similarly choosing $(h_X)^{-1}$. \square

Remark 2.8. Both Lemma 2.6 and Lemma 2.7 are logically equivalent to the axiom of choice, as is the statement that every category admits a skeleton.

3 Adjunctions

Definition 3.1.

1. Let $Z \in \text{ob } \mathcal{C}$. Define the contravariant functor $\mathcal{Y}_Z : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ on objects by $Y \mapsto \mathcal{C}(Y, Z)$ and on morphisms by sending $f : X \rightarrow Y$ in \mathcal{C} to the map $f^* : \mathcal{C}(Y, Z) \rightarrow \mathcal{C}(X, Z)$ given by $g \mapsto gf$. We call $\mathcal{C}(-, Z) := \mathcal{Y}^Z$ the set-valued functor *represented by* Z in \mathcal{C} .
2. Let $X \in \text{ob } \mathcal{C}$. Define the functor $\mathcal{Y}^X : \mathcal{C} \rightarrow \mathbf{Set}$ on objects by $Y \mapsto \mathcal{C}(X, Y)$ and on morphisms by sending $g : Y \rightarrow Z$ to the map $g_* : \mathcal{C}(X, Y) \rightarrow \mathcal{C}(X, Z)$ given by $f \mapsto gf$. We call $\mathcal{C}(X, -) := \mathcal{Y}^X$ the set-valued functor *corepresented by* X in \mathcal{C} .

A functor of the form $\mathcal{C} \times \mathcal{C}' \rightarrow \mathcal{D}$ is called a *bifunctor*. In particular, define $\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$ on objects by $(X, X') \mapsto \mathcal{C}(X, X')$ and on morphisms by sending $(f, f') : (X, X') \rightarrow (Y, Y')$ to the map $\mathcal{C}(f, f') : \mathcal{C}(X, X') \rightarrow \mathcal{C}(Y, Y')$ given by $g \mapsto f'gf$.

¹Theorem 3.2.10 (Rognes).

Definition 3.2 (Kan). Let \mathcal{C} and \mathcal{D} be categories and $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors. Consider the set-valued bifunctors $\mathcal{D}(F(-), -), \mathcal{C}(-, G(-)) : \mathcal{C}^{\text{op}} \times \mathcal{D} \rightarrow \mathbf{Set}$. An *adjunction from F to G* is a natural isomorphism

$$\phi : \mathcal{D}(F(-), -) \Rightarrow \mathcal{C}(-, G(-)).$$

If such a ϕ exists, then we say that (F, G) is an *adjoint pair* of functors.

Note that ϕ is natural in the sense that for any map $c : X' \rightarrow X$ in \mathcal{C} and $d : Y \rightarrow Y'$ in \mathcal{D} , the square

$$\begin{array}{ccc} \mathcal{D}(FX, Y) & \xrightarrow{\phi_{X,Y}} & \mathcal{C}(X, GY) \\ c^* d_* \downarrow & & \downarrow c^* d_* \\ \mathcal{D}(FX', Y') & \xrightarrow{\phi_{X',Y'}} & \mathcal{C}(X', GY') \end{array}$$

commutes

Proposition 3.3. *Left and right adjoints are unique up to unique isomorphism.*

Terminology. We call F the *left adjoint* to G and G the *right adjoint* to F . In symbols, $F \dashv G$.

Example 3.4. The forgetful functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ has a left adjoint $F : \mathbf{Set} \rightarrow \mathbf{Grp}$ sending a set to the free group generated by A .

Example 3.5. Let R be a ring. The forgetful functor $U : R\text{-}\mathbf{Mod} \rightarrow \mathbf{Set}$ has a left adjoint $R(-)$ sending a set S to $\bigoplus_{s \in S} R$, the free R -module generated by S .

The forgetful functor has no right adjoint in either Example 3.4 or Example 3.5. It does, however, have one in the following setting.

Example 3.6. The forgetful functor $U : \mathbf{Top} \rightarrow \mathbf{Set}$ has a left adjoint that sends a set to the same set equipped with the discrete topology. It also has a right adjoint that sends a set to the same set equipped with the indiscrete topology.

Definition 3.7. A subcategory $\mathcal{C} \subset \mathcal{D}$ is *reflective* if the inclusion functor is a right adjoint and is *coreflective* if the inclusion functor is a left adjoint.

Example 3.8.

1. The full subcategory $\mathbf{Ab} \subset \mathbf{Grp}$ is reflexive as the inclusion functor is right adjoint to $(-)^{\text{ab}}$.
2. Let $\mathbf{Ab}_T \subset \mathbf{Ab}$ denote the subcategory of torsion groups. This is coreflective as the inclusion functor is right adjoint to the functor sending an abelian group to its torsion subgroup.

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors.

Definition 3.9. Given an adjunction $\phi : \mathcal{D}(F(-), -) \Rightarrow \mathcal{C}(-, G(-))$, define the *unit morphism*

$$\eta_X = \phi_{X, FX} (\text{Id}_{FX}) \in \mathcal{C}(X, GF(X))$$

and the *counit morphism*

$$\epsilon_Y = \phi_{GY, Y}^{-1} (\text{Id}_{GY}) \in \mathcal{D}(FG(Y), Y).$$

Lemma 3.10. *The unit morphisms $(\eta_X)_{X \in \text{ob } \mathcal{C}}$ define a natural transformation $\eta : \text{Id}_{\mathcal{C}} \Rightarrow GF$, and the counit morphisms $(\epsilon_Y)_{Y \in \text{ob } \mathcal{D}}$ define a natural transformation $\epsilon : FG \Rightarrow \text{Id}_{\mathcal{D}}$.*

Proof. For simplicity, let us just prove that ϵ is a natural transformation. We must check that

$$\begin{array}{ccc} FG(Y) & \xrightarrow{FG(y)} & FG(Y') \\ \epsilon_Y \downarrow & & \downarrow \epsilon_{Y'} \\ Y & \xrightarrow{y} & Y' \end{array}$$

commutes for any map $y : Y \rightarrow Y'$ in \mathcal{D} . By the naturality of ϕ , we have that

$$\begin{aligned} y \circ \epsilon_Y &= y \circ \phi^{-1}(\text{Id}_{GY}) \\ &= \phi^{-1}(Gy \circ \text{Id}_{GY}) \\ &= \phi^{-1}(\text{Id}_{GY'} \circ Gy) \\ &= \phi^{-1}(\text{Id}_{GY'}) \circ FG(y) \\ &= \epsilon_{Y'} \circ FG(y), \end{aligned}$$

as required. □

Moreover, one can verify that the unit and counit of ϕ satisfy the *triangle identities*,

$$\begin{aligned} \epsilon_{FC} \circ F\eta_C &= 1_{FC} \\ G\epsilon_D \circ \eta_{GD} &= 1_{GD}, \end{aligned}$$

for any $C \in \text{ob } \mathcal{C}$ and $D \in \text{ob } \mathcal{D}$.

Conversely, suppose that F and G come equipped with two natural transformations

$$\begin{aligned} \eta : \text{Id}_{\mathcal{C}} &\Rightarrow GF \\ \epsilon : FG &\Rightarrow \text{Id}_{\mathcal{D}} \end{aligned}$$

satisfying the triangle identities. Then we get an adjunction ϕ from F to G with the component $\phi_{X,Y} : \mathcal{D}(FX, Y) \rightarrow \mathcal{C}(X, GY)$ defined by

$$f \mapsto Gf \circ \eta_X.$$

Even so, F and G need *not* be equivalent, as η and ϵ may not be isomorphisms. Further, a given equivalence $\mathcal{C} \xrightleftharpoons[R]{L} \mathcal{D}$ of categories need *not* be an adjunction, as its associated natural transformations

$$\begin{aligned} \eta' : \text{Id}_{\mathcal{C}} &\Rightarrow RL \\ \epsilon' : LR &\Rightarrow \text{Id}_{\mathcal{D}} \end{aligned}$$

may not satisfy the triangle inequalities. Nevertheless, (L, R) is an adjoint pair with unit η' and counit another natural transformation defined in terms of η' and ϵ' . By symmetry, (R, L) is also an adjoint pair.