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Homotopy and K-theory seminar
Talk #10
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Abstract

We continue to do low-dimensional K-theory, finishing our description of $K_0(-)$ and then defining $K_1(-)$, and $K_2(-)$ for rings. The main sources for this talk are the following.

- nLab.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Chapters II and III.
- Eric M. Friedlander's *An Introduction to K-theory*, Chapter 1.

Definition 1. Let $\mathscr{C}$ be a category equipped with a "subcategory" $\mathtt{co}\mathscr{C}$ of morphisms called *cofibrations*. The pair $(\mathscr{C}, co)$ is a *category with cofibrations* if the following conditions hold.

1. (W0) Every isomorphism in $\mathscr{C}$ is a cofibration.
2. (W1) There is a base point 0 in $\mathscr{C}$ such that the unique morphism $0 \rightarrowtail A$ for every $A \in \operatorname{ob}\mathscr{C}$.
3. (W2) We have

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, dotted] \\ C \arrow[r, tail, dotted] & B \cup_A C \end{tikzcd}.$$

Remark 1. We see that $B \coprod C$ always exists as the pushout $B \cup_0 C$ and that the cokernel of any $i : A \rightarrowtail B$ exists as $B \cup_A 0$ along $A \to 0$. We call $A \rightarrowtail B \twoheadrightarrow {}^{B}\!/_{A}$ a *cofibration sequence*.

Definition 2. A *Waldhausen category* $\mathscr{C}$ is a category with cofibrations together with a subcategory $w(\mathscr{C})$ of morphisms called *weak equivalences* such that every isomorphism in $\mathscr{C}$ is a w.e. and the following "Glueing axiom" holds.

1. (W3) For any diagram

$$\begin{tikzcd} C \arrow[d, "\sim"'] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"'] & B \arrow[d, "\sim"'] \\ C' & A' \arrow[l] \arrow[r, tail] & B' \end{tikzcd},$$

   the induced map $B \cup_A C \to B' \cup_{A'} C'$ is a w.e.

Definition 3. Let $\mathscr{C}$ be a Waldhausen category. Define $K_0(\mathscr{C})$ as the abelian group generated by $[C]$ for each object C of $\mathscr{C}$ such that

1. $[C] = [C']$ if there some w.e. from C to C'
2. $[C] = [B] + \left[{}^{C}\!/_{B}\right]$ for every $B \rightarrowtail C \twoheadrightarrow {}^{C}\!/_{B}$
3. The weak equivalence classes of objects in $\mathscr{C}$ is a set.

Proposition 1.

1. $[0] = 0$.

2. $[B \coprod C] = [B] + [C]$.

3. $[B \cup_A C] = [B] + [C] - [A]$.

4. $[C] = 0$ *whenever* $0 \simeq C$.

Example 4. Let $\mathcal{R}_f(*)$ denote the category of finite CW complexes. Here, cofibrations and weak equivalences correspond to cellular inclusions and weak homotopy equivalences, respectively. One can show that $K_0(\mathcal{R}_f) \cong \mathbb{Z}$.

Definition 5. if $\mathscr{C}$ and $\mathscr{D}$ are Waldhausen, then a functor $F : \mathscr{C} \to \mathscr{D}$ is *exact* if

(a) preserves base points, cofibrations, and weak equivalences and

(b) for any $A \rightarrowtail B$, $FB \cup_{FA} FC \to F(B \cup_A C)$ is an isomorphism.

Note that F induces a group map $K_0(F) : K_0(\mathscr{C}) \to K_0(\mathscr{D})$.

Theorem 2. *Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor. Assume the following.*

(1) A morphism f is a w.e. iff $F(f)$ is a w.e.

(2) For any morphism $b : FA \to B$ in $\mathscr{B}$, there is some $a : A \rightarrowtail A'$ in $\mathscr{A}$ and a w.e. $b' : FA' \xrightarrow{\sim} B$ in $\mathscr{B}$ such that $b = b' \circ F(a)$. Moreover, we may choose a to be a w.e. whenever b is a w.e.

Then F induces an isomorphism $K_0(\mathscr{A}) \cong K_0(\mathscr{B})$.

Proof. Apply condition (2) to any $0 \rightarrowtail B$ to get $FA' \xrightarrow{\sim} B$. If this is a w.e., then there is some $A \xrightarrow{\sim} A'$. Hence there is a bijection between the set W of w.e. classes of objects of $\mathscr{A}$ and that in $\mathscr{B}$.

The group $K_0(\mathscr{B})$ is given by the free abelian group $\mathbb{Z}[W]$ modulo the relation

$$[C] = [B] + \left[{}^{C}\!/_{B}\right].$$

Let $FA \xrightarrow{\sim} B$. Then applying condition (2) yields the diagram

$$\begin{tikzcd}
0 & FA \arrow[l] \arrow[r, tail] \arrow[d, "\sim"'] & FA' \arrow[d, "\sim"'] \\
0 \arrow[u, no head] & B \arrow[l] \arrow[r, tail] & C
\end{tikzcd}.$$

Apply the Glueing axiom to see that $F\left({}^{A'}\!/_{A}\right) \to {}^{C}\!/_{B}$ is a w.e. Hence $[C] = [B] + \left[{}^{C}\!/_{B}\right]$ holds iff $[A'] = [A] + \left[{}^{A'}\!/_{A}\right]$ holds. ☐

Let R be a unital ring. Recall that direct limits in $\mathbf{Mod}_R$ always exist. Let

$$K_1 = \mathrm{GL}(R)^{\mathrm{ab}}$$

where $\mathrm{GL}(R) \equiv \operatorname{colim}_n \mathrm{GL}(n, R)$.

Remark 2. The universal property of $\mathrm{ab} : \mathbf{Grp} \to \mathbf{Ab}$ induces the universal property of K_1 that any homomorphism $f : \mathrm{GL}(R) \to H$ with H abelian has $f = g \circ \pi$ for some unique $g : K_1(R) \to H$.

Proposition 3. *Any ring map $f : R \to S$ induces a natural map $\mathrm{GL}(R) \to \mathrm{GL}(S)$. Hence K_1 is a functor* $\mathbf{Rng} \to \mathbf{Ab}$.

Thanks to Whitehead, we know that the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$ is equal to $E(R) = \bigcup_n E_n(R)$, the group of elementary matrices $E_{i,j}(r)$ where $r \in R$ and $i \neq j$. Thus, $K_1(R)$ can be viewed as the "stabilized" group of automorphisms of the trivial projective module modulo trivial automorphisms.

Example 6. If F is a field, then $K_1(F) = F^\times$.

Proof. It is each to check that $E_n(F) \cong \mathrm{SL}_n(F)$ for any $n \in \mathbb{N}$. Therefore, $E(F) \cong \mathrm{SL}(R)$. ☐

Proposition 4. *Suppose R is commutative. Consider the sequence $R^\times \cong \mathrm{GL}(1,R) \to \mathrm{GL}(R) \to K_1(R)$. This induces a natural split exact sequence.*

$$1 \longrightarrow SK_1(R) \hookrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1,$$

where $SK_1(R)$ denotes $\ker(\det)$. *Therefore, $K_1(R) \cong R^\times \times SK_1(R)$.*

Example 7. Suppose R is a Euclidean domain. Then $SK_1(R) = 1$, so that $K_1(R) \cong R^\times$.

Lemma 5. *Let D be a division ring. Then $K_1(D) \cong {}^{\mathrm{GL}_n(D)}\!\diagup_{E_n(D)}$ for any $n \geq 3$.*

Proof. Any invertible matrix over D is reducible (a la Gaussian elimination) to a diagonal matrix of the form $(r, 1, \ldots, 1)$. Moreover, $E_n(D) \trianglelefteq \mathrm{GL}_n(D)$ for each n. In particular, Dieudonné (1943) showed that ${}^{\mathrm{GL}_n(D)}\!\diagup_{E_n(D)} \cong {}^{D^\times}\!\diagup_{(D^\times)'}$ for any $n \neq 2$. ∎

Proposition 6 (Vaserstein). *Suppose R is Noetherian of dimension d, so that $E_n(R) \trianglelefteq \mathrm{GL}_n(R)$ for any $n \geq d+2$. Then $K_1(R) \cong {}^{\mathrm{GL}_n(R)}\!\diagup_{E_n(R)}$ for any $n \geq d+2$.*

Remark 3. Let D be a d-dimensional division algebra over the field $F \coloneqq Z(D)$. We know that $d = n^2$ for some integer n. By Zorn there is some maximal subfield $E \subset D$ such that $[E : F] = n$. Then $D \otimes_F E \cong M_n(E)$, where M_n denotes the n-dimensional matrix ring over E. Any field with this property is called a *splitting field* for D.

Definition 8. Let E' be a splitting field for D. For any $r \in \mathbb{N}$, the inclusions $D \hookrightarrow M_n(E')$ and $M_r(D) \hookrightarrow M_{nr}(E')$ induce maps $D^\times \subset \mathrm{GL}_n(E') \xrightarrow{\det} (E')^\times$ and $\mathrm{GL}_r(D) \to \mathrm{GL}_{nr}(E') \xrightarrow{\det} (E')^\times$ whose images are contained in F^*. The induced maps are called the *reduced norms* N_{red} for D.

Example 9. If $D = \mathbb{H}$, then N_{red} is the square of the usual norm. It induces an isomorphism $K_1(\mathbb{H}) \cong \mathbb{R}_+^\times$.

Proposition 7. *Let R be a commutative Banach algebra over $\mathbb{R}$ or $\mathbb{C}$. Recall that $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ are topological groups as subspaces of $\mathbb{R}^{n^2}$. We have that $E_n(R)$ is the path component of the identity matrix I_n for any $n \geq 2$.*

Corollary 8. *We may identify $SK_1(R)$ with the set $\pi_0 \,\mathrm{SL}(R)$.*

Proof. Note that $E(R) \leq \mathrm{SL}_(R)$. By the third isomorphism theorem, we get

$${}^{\mathrm{GL}(R)}\!\diagup_{E(R)}\Big/{}^{\mathrm{SL}(R)}\!\diagup_{E(R)} \cong {}^{\mathrm{GL}(R)}\!\diagup_{\mathrm{SL}(R)}.$$

Thus, we get the short exact sequence

$$1 \longrightarrow {}^{\mathrm{SL}(R)}\!\diagup_{E(R)} \longrightarrow {}^{\mathrm{GL}(R)}\!\diagup_{E(R)} \cong K_1(R) \longrightarrow {}^{\mathrm{GL}(R)}\!\diagup_{\mathrm{SL}(R)} \cong R^\times \longrightarrow 1$$

By the previous proposition, we know that ${}^{\mathrm{SL}(R)}\!\diagup_{E(R)} \cong \pi_0 \,\mathrm{SL}(R)$, giving the short exact sequence.

$$1 \longrightarrow \pi_0 \,\mathrm{SL}(R) \longrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1 \ .$$

∎

Example 10. If X is compact, then $SK_1(\mathbb{R}^X) \leftrightarrow [X, \mathrm{SL}(\mathbb{R})] \cong [X, \mathrm{SO}]$ and $SK_1(\mathbb{C}^X) \leftrightarrow [X, \mathrm{SL}(\mathbb{C})] \cong [X, \mathrm{SU}]$. In particular, $SK_1(\mathbb{R}^{S^1}) \leftrightarrow \pi_1 \,\mathrm{SO} \cong C_2$.

Remark 4. Let P be a finitely generated projective R-module. Each isomorphism $P \oplus Q \cong R^n$ induces a group map $\mathrm{Aut}(P) \to \mathrm{Aut}(P) \oplus \mathrm{Aut}(Q) \cong \mathrm{Aut}(R^n) \cong \mathrm{GL}_n(R)$. The group map $\mathrm{Aut}(P) \to \mathrm{GL}(R)$ is independent of the choice of isomorphism up to inner automorphism of $\mathrm{GL}(R)$. Therefore, there is a well-defined homomorphism $\Phi : \mathrm{Aut}(R) \to K_1(R)$.

Lemma 9. *Suppose that R is commutative and T is an R-algebra. Then $K_1(T)$ has a natural module structure over $K_0(R)$.*

Proof. By the previous remark, for any $P \in \mathbf{P}(R)$ and $m \in \mathbb{N}$, there is a homomorphism $\Phi : \mathrm{Aut}(P \otimes T^m) \to K_1(R \otimes T)$. For any $\beta \in \mathrm{GL}_m(T)$, define $[P] \cdot \beta = \Phi(1_P \otimes \beta)$. This action factors through $K_0(R)$ and $K_1(T)$, inducing an operation $K_0(R) \times K_1(T) \to K_1(R \otimes S)$. Now, since T is an R-algebra, there is a ring map $R \otimes T \to T$. The induced composite $K_0(R) \times K_1(T) \to K_1(R \otimes T) \to K_1(T)$ is the desired module structure. □

Theorem 10. *One can show that $K_1(R)$ is determined by the category $\mathbf{P}(R)$. Thus, if R and S are Morita equivalent, then $K_1(R) \cong K_1(R)$.*

Let π be a finitely generated group. Define the *first Whitehead group $Wh_1(\pi)$* of π as the cokernel of the map $\pi \times \{\pm 1\} \to K_1(\mathbb{Z}[\pi])$ given by $(g, \pm 1) \mapsto (\pm g)$.

Theorem 11. *A homotopy equivalence of finite CW-complexes with fundamental group π is a simple homotopy equivalence iff it vanishes under the Whitehead torsion τ, which is a certain function from continuous maps to $Wh_1(\pi)$.*

Theorem 12 (The s-cobordism theorem). *Suppose that W, M, and N are compact PL-manifolds and that W is a cobordism of M and N. Then if $\dim(M) \geq 5$, it follows that $(W, M, N) \cong (M \times [0,1], M \times 0, M \times 1)$ iff $\tau = 0$.*

Corollary 13. *Let A denote the disjoint union of W, CM, and CN. Then N is PL-homeomorphic to ΣM iff $\tau = 0$ (even though they are homeomorphic as spaces).*

Corollary 14. *The Generalized Poincaré Conjecture holds.*

Definition 11. Let I is an ideal in R. Define $GL(I)$ as the kernel of the map $\mathrm{GL}(R) \to \mathrm{GL}\left({}^{R}\!/\!_{I}\right)$. Moreover, define $E(R, I)$ as the smallest normal subgroup of $E(R)$ that contains $E_{i,j}(x)$ for $r \in I$ and $i \neq j$.

Proposition 15. $[\mathrm{GL}(I), \mathrm{GL}(I)] \subset E(R, I) \trianglelefteq \mathrm{GL}(I)$

Definition 12. The *relative group* $K_1(R, I)$ is the the abelian group ${}^{\mathrm{GL}(I)}\!/\!_{E(R,I)}$.

Remark 5. Swan has shown that a ring homomorphism $f : R \to S$ that maps the ideal I isomorphically to the ideal J need not induce an isomorphism $K_1(R, I) \to K_1(S, J)$.

Proposition 16. *There is an exact sequence*

$$K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1\left({}^{R}\!/\!_{I}\right) \longrightarrow K_0(I) \longrightarrow K_0(R) \longrightarrow K_0\left({}^{R}\!/\!_{I}\right).$$ [1]

Definition 13. Let $n \geq 3$ and R be a ring. The *Steinberg group $St_n(R)$* is the group generated by the symbols $x_{ij}(r)$ with $1 \leq i \neq j \leq n$ and $r \in R$ that satisfy the following relations.

1. $$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$$

2. $$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & j \neq k, \quad i \neq l \\ x_{il}(rs) & j = k, \quad i \neq l \\ x_{kj}(-sr) & j \neq k, \quad i = l \end{cases}$$

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[1] Section III.2.3 (Weibel).

There is a natural group surjection $\phi_n : St_n(R) \to E_n(R)$ given by $x_{ij}(r) \mapsto E_{ij}(r)$. Moreover, there is a group map $St_n(R) \hookrightarrow St_{n+1}(R)$. Note that $St(R) \coloneqq \mathrm{colim}_n\, St_n(R)$ exists. The ϕ_n thus form a group epimorphism $\phi : St(R) \to E(R)$.

Let $K_2(R) = \ker \phi$. We have an exact sequence

$$1 \longrightarrow K_2(R) \longrightarrow St(R) \xrightarrow{\phi} GL(R) \longrightarrow K_1(R) \longrightarrow 1 \ .$$

Lemma 17. *$K_2(R) = Z(St(R))$.*

Proof. That $K_2(R) \supset Z(St(R))$ follows from the fact that $Z(E(R))$ is trivial. The reverse containment is easy but more tedious to prove. See III.5.2.1 (Weibel). ☐

Note that $K_2(-) : \mathbf{Rng} \to \mathbf{Ab}$ is a functor.

Example 14. A sort of Euclidean algorithm yields the following computations.

1. $K_2(\mathbb{Z}) \cong C_2$
2. $K_2(\mathbb{Z}[i]) = 1$
3. $K_2(F) \cong K_2(F[t])$ when F is a field

Theorem 18. *Let $K_2(n, R) = \ker \phi_n$. Suppose that R is Noetherian of dimension d. Then $K_2(n, R) \cong K_2(R)$ for any $n \geq d + 3$.*

Theorem 19. *One can show that $K_2(R)$ is determined by the category $\mathbf{P}(R)$. Thus, if R and S are Morita equivalent, then $K_2(R) \cong K_2(R)$.*

Example 15. R and $S \coloneqq M_n(R)$ are Morita equivalent for any $n \geq 1$, so that $K_i(R) \cong K_i(M_n(R))$ for $i = 0, 1, 2$. Such an equivalence is given as follows. In one direction, define $F : M \mapsto M^n$. In the other direction, define $G : M \mapsto e_{11}M$ where e_11 denotes the matrix with 1 in position $(1, 1)$ and 0 elsewhere. Define the natural isomorphism $\mathrm{Id}_{\mathbf{Mod}_R} \Rightarrow G \circ F$ by the components $f_M : M \to \{(m, 0, \ldots, 0) : m \in M\}$. Further, define the natural isomorphism $\mathrm{Id}_{\mathbf{Mod}_S} \Rightarrow F \circ G$ by the components $g_M : M \to (e_{11}M)^n$ given by $m \mapsto (e_{11}m, \ldots, e_{1n}m)$. Hence $\mathbf{Mod}_R$ and $\mathbf{Mod}_S$ are equivalent, hence Morita equivalence as they are preadditive.

Lemma 20. *Let R be a commutative Banach algebra. Then there is a surjection from $K_2(R)$ onto $\pi_1\, \mathrm{SL}(R)$.*

Proof. See III.5.9 (Weibel). ☐

Example 16. There is a surjection $K_2(\mathbb{R}) \to \pi_1\, \mathrm{SL}_(\mathbb{R}) \cong \pi_1\, \mathrm{SO} \cong C_2$. Hence $K_2(\mathbb{R})$ is nontrivial.

Theorem 21 (Matsumoto 1969). *Let F be a field. Then $K_2(F)$ is isomorphic to the free abelian group with system of generators $\{a, b\}$ satisfying the following relations.*

1. $\{ac, b\} = \{a, b\}\{c, b\}$
2. $\{a, bd\} = \{a, b\}\{a, d\}$
3. $\{a, 1 - a\} = 1$ *when* $a \neq 1 \neq 1 - a$.

Terminology. The $\{a, b\}$ are called *Steinberg symbols.*

Suppose that $A, B \in E(F)$ commute. Write $\phi(a) = A$ and $\phi(b) = B$. Then define

$$A \bigstar B = [a, b] \in K_2(R).$$

If $a, b \in F$, then we can alternatively define the Steinberg symbol

$$\{a, b\} = \begin{bmatrix} r & & \\ & r^{-1} & \\ & & 1 \end{bmatrix} \bigstar \begin{bmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{bmatrix}.$$

Corollary 22. *$K_2(\mathbb{F}_p^n) = 1$ for any prime p and $n \geq 1$.*

Proof. The proof is entirely computational. See III.6.1.1 (Weibel). ☐

Proposition 23. *If $F \supset \mathbb{Q}(t)$, then $|K_2(F)| = |F|$.*