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Abstract

These notes are based on Jonathan Block's lectures for the course "Geometric Analysis and Topology II" at UPenn along with Allen Hatcher's *Algebraic Topology*. Any mistake in what follows is my own.

Contents

1 Basic geometric notions

1.1 Lecture 1

Example 1.1.1. There are usually many different (continuous) maps between spaces, but finding a homeomorphism can be difficult or impossible.

1. $\mathbb{R}^n \not\cong \mathbb{R}^m$ when $n \neq m$.
2. "X" $\not\cong$ "Y" since removing the intersection point of the letter "X" produces four components whereas removing the intersection point of "Y" produces three.
3. $\{a, b\} \not\cong \{a, b, c\}$.
4. (Alexander horned sphere) Let X denote the limit of the following iterative construction starting with the unit ball D^3.

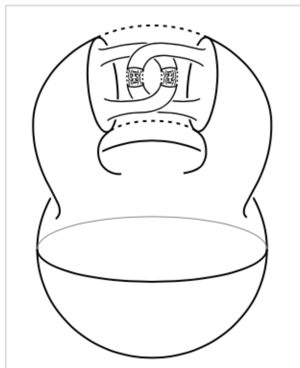


Figure 1: copied from Hatcher (171)

It turns out that $X \cong D^3$, so that $\partial X \cong S^2$.

Note 1.1.2. The following are types of spaces that we will care about.

1. Manifolds.

   Definition 1.1.3. A space X is *homogeneous* if for any $x, y \in X$ with $x \neq y$, there is some homeomorphism $f : X \to X$ such that $f(x) = y$ and $f(y) = x$.

   Proposition 1.1.4. *Any connected manifold is homogenous.*

   With respect to connected manifolds, we may thus restrict our attention to questions of global topology.

2. Algebraic varieties over $\mathbb{R}$ or $\mathbb{C}$.

   Example 1.1.5. Consider the affine variety $Z(xy) = \{(x, y) \in \mathbb{R}^2 : x = 0 \vee y = 0\}$. This is not homogenous, because of the singularity $(0, 0)$.

3. The Cantor set $\mathcal{C}$. This is the unique homeomorphism class of spaces that are compact, metrizable, and totally disconnected.

   Example 1.1.6. Given a prime p, complete $\mathbb{Q}$ endowed with the p-adic metric $|\cdot|_p$ to obtain the p-adic numbers $\mathbb{Q}_p$. Then the ring of p-adic integers $\mathcal{O}_p \subset \mathbb{Q}_p$ is the Cantor set.

4. CW-complexes (developed by J. H. C. Whitehead).[1]

   Recall that an n-cell is a space homeomorphic to $\operatorname{Int} D^n$ where $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$.

   Start with a discrete set X^0, called a 0-*skeleton*.

   By induction, define the n-*skeleton* as

$$X^n = {X^{n-1} \coprod_\alpha D^n_\alpha}\Big/{\sim}$$

   where $\varphi_\alpha : S^{n-1} \to X^{n-1}$ is an *attaching map* and $x \sim \varphi_\alpha(x)$ for each $x \in \partial D^n_\alpha$. Then

$$X^n = X^{n-1} \coprod_\alpha e^n_\alpha$$

   where each e^n_α is an n-cell.

   Set $X = \bigcup_n X^n$ and endow it with the *weak topology*: A is open in X if and only if $A \cap X^n$ is open in X^n for each n.

   Definition 1.1.7.

   - If X is a CW-complex, then the *dimension of* X is the maximum dimension of cells of X.
   - If X is a CW-complex consisting of only finitely many cells, then it is called a *finite CW-complex*.

   Each cell e^n_α has a *characteristic map* $\Phi_\alpha : D^n_\alpha \to X$ given by the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \coprod_\alpha D^n_\alpha \twoheadrightarrow X^n \hookrightarrow X.$$

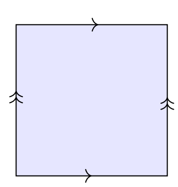
   This extends the attaching map φ_α and is a homeomorphism $\operatorname{Int} D^n_\alpha \to e^n_\alpha$.

Remark 1.1.8. If X is a CW-complex, then any function $f : X \to Y$ is continuous if and only if $f \restriction_{X^n}$ is continuous for each $n \geq 0$.

Example 1.1.9. The following are CW-complexes.

1. $\{p\}$
2. $S^0 = \{\pm 1\}$
3. Construct S^1 by adding semi-circles (i.e., 1-cells) above and below S^0.
4. Construct S^2 by adding hemispheres (i.e., 2-cells) above and below S^1.
5. Any orientable surface of genus g.

   Consider the case $g = 1$. Draw the torus $S^1 \times S^1$ as



   The frame and interior are homeomorphic to the 1-skeleton $S^1 \vee S^1$ and the 2-cell $\operatorname{Int} D^2$, respectively.

   Next, consider the case $g = 2$. Similarly, we can draw the two-holed torus as

[1] $\underbrace{\text{C}}_{\text{cell}}\underbrace{\text{W}}_{\text{weak}}$

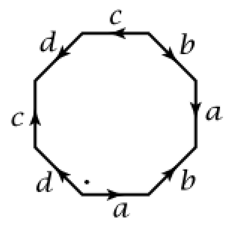


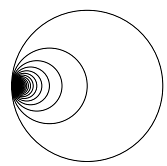
Figure 2: copied from Hatcher (5)

This is clearly a two-dimensional CW-complex.

Proposition 1.1.10. *Every manifold has a CW-structure.*

Example 1.1.11.

1. The Cantor set does not have a CW-structure, because it is totally disconnected.
2. Consider the subspace of $\mathbb{R}^2$ known as the Hawaiian earring.



If this had a CW-structure, then the sequence $\left(\frac{2}{n}, 0\right)_{n\in\mathbb{N}}$ would not converge to $(0,0)$, which is absurd.

Definition 1.1.12. Let $f, g : X \to Y$ be maps. A *homotopy from f to g* is a map $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in X$. We say that f is *homotopic to g*, written as $f \simeq g$.

Note 1.1.13. From now on, assume that every topological space is Hausdorff.

Definition 1.1.14. Let $A \subset X$ be a subspace. A *homotopy between f and g relative to A* is a homotopy F between f and g such that if $t \in [0,1]$, then $F(x,t) = f(x) = g(x)$ for each $x \in A$.

Example 1.1.15. Let $X := D^2 \backslash \{0\}$. Define $f : X \to X$ by $f(r,\theta) = (r,\theta)$ and $g : X \to X$ by $g(r,\theta) = (1,\theta)$. Then the map $F : X \times I \to X$ given by $F((r,\theta),t) = (t + (1-t)r, \theta)$ is a homotopy between f and g relative to $S^1 \subset X$.

Definition 1.1.16. A *deformation retraction of X onto $A \subset X$* is a map $F : X \times I \to X$ such that $F(x,0) = x$ for each $x \in X$, $F(a,1) = a$ for each $a \in A$, and $F(x,1) \in A$ for each $x \in X$.

Example 1.1.17. The solid torus $S^1 \times D^2$ deformation retracts onto S^1, as does the solid coffee mug.

Definition 1.1.18. Two spaces X and Y are *homotopy equivalent* ($\simeq$) if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathbb{1}_Y$ and $g \circ f \simeq \mathbb{1}_X$.

Definition 1.1.19. A space is *contractible* if it is homotopy equivalent to $\{*\}$.

Lemma 1.1.20. *Homotopy equivalence of maps is an equivalence relation.*

Proof. Both reflexivity and symmetry are obvious. To check transitivity, suppose that $F : f \simeq g$ and $G : g \simeq h$. Define $H(x,t) = \begin{cases} F(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$. Then $H : f \simeq h$, as required. □

Proposition 1.1.21. *Homotopy equivalence of spaces is an equivalence relation.*

Proof. We have a category $h\mathbf{Top}$ with spaces as objects and homotopy classes of maps $X \to Y$ as morphisms. Then two objects are isomorphic if and only if they are homotopy equivalent. But it's clear that any categorical isomorphism is an equivalence relation. □

1.2 Lecture 2

Example 1.2.1. Here are more CW-complexes.

1. Recall that $\mathbb{RP}^n \cong {S^n}/{\alpha}$ where α denotes the antipodal map. Thus, $\mathbb{RP}^n \cong {D^n}/{\sim}$ where $x \sim y$ iff $x, y \in \partial D^n$ and $x = -y$. This implies that $\mathbb{RP}^n \cong \mathbb{RP}^{n-1} \cup_\pi D^n$ where $\pi : \partial D^n = S^{n-1} \to \mathbb{RP}^{n-1}$ denotes projection. By induction, we have that
$$\mathbb{RP}^n \cong e^0 \cup e^1 \cup \cdots \cup e^n.$$

2. Recall that $\mathbb{CP}^n = \{ \underbrace{[z_0 : \cdots : z_n]}_{\text{homogeneous coordinates}} : z_i \in \mathbb{C},\ (z_0, \ldots, z_n) \neq 0\}$. Note that $\mathbb{CP}^n \cong {S^{2n+1}}/{U(1)}$. We see that $S^{2n+1} = \{(w, \sqrt{1 - |w|^2}) \in \mathbb{C}^n \times \mathbb{C} : |w| \leq 1\}$. When $(w, z) \in S^{2n+1}$ has $z \neq 0$, then $(w, z) \sim (x', y')$ for some unique $(x', y') \in D^{2n}$. Otherwise, $(w, z) \in S^{2n-1}$. This shows that $\mathbb{CP}^n \cong \mathbb{CP}^{n-1} \cup_\pi D^{2n}$ where $\pi : S^{2n-1} \to \mathbb{CP}^{n-1}$ denotes projection. By induction, we get
$$\mathbb{CP}^n \cong e^0 \cup e^2 \cup \cdots \cup e^{2n}.$$

Definition 1.2.2. A closed subspace A of a CW-complex X is a *subcomplex* if A equals some union of cells of X. In this case, the pair (X, A) is called a *CW pair.*

Proposition 1.2.3. *Let X and Y be pointed CW-complexes. Let $A \subset X$ be a subcomplex.*

1. *$X \coprod Y$ is a CW-complex.*
2. *$X \vee Y$ is a CW-complex.*
3. *$X \times Y$ is a CW-complex.*

   Note 1.2.4.

   (a) The topology of $X \times Y$ as a CW-complex may be finer than $X \times Y$ equipped with the product topology.

   (b) An uncountable product of CW-complexes under the product topology need not be a CW-complex.

4. *${X}/{A}$ is a CW-complex whose cells are precisely those of $X \setminus A$ together with the 0-cell $\pi(A)$ and attaching maps are precisely $\pi_{n-1} \circ \varphi_\alpha$ where $\pi_{n-1} : X^{n-1} \to {X^{n-1}}/{A^{n-1}}$ denotes projection and $\varphi_\alpha : S^{n-1} \to X^{n-1}$ is an attaching map.*

   Example 1.2.5. ${D^n}/{S^{n-1}} \cong S^2$ for any $n \geq 1$.

Definition 1.2.6. Given any space X, define the *cone of X* as
$$C(X) = {X \times I}/{(x, 1) \sim (y, 1)}.$$

Lemma 1.2.7. *$C(X)$ is contractible.*

Proof. Define $H((x, t), s) = \begin{cases} (x, (1-s)t) & t \neq 1 \\ (x, 1) & t = 1 \end{cases}.$ □

Definition 1.2.8. Given any space X, define the *suspension of X* as $S(X) = {X \times I}/{\sim}$ where $(x, 0) \sim (x', 0)$ and $(x, 1) \sim (x', 1)$.

Example 1.2.9. $S(S^n) = S^{n+1}$.

Proposition 1.2.10. *Both $C(-)$ and $S(-)$ preserve the property of being a CW-complex ("CW-hood").*

Definition 1.2.11. Let $f : X \to Y$ be a map of spaces. Define the *mapping cylinder of f* as

$$\mathrm{Cyl}(f) = {}^{(X \times I) \coprod Y}\big/_{(x,1) \sim f(x)}.$$

Note 1.2.12. $\mathrm{Cyl}(-)$ need not preserve CW-hood.

Lemma 1.2.13. $\mathrm{Cyl}(f) \simeq Y$.

Proof. Define

$$f : \mathrm{Cyl}(f) \to Y, \quad (x,t) \mapsto f(x) \quad y \mapsto y$$

and

$$g : Y \to \mathrm{Cyl}(f), \quad y \mapsto y.$$

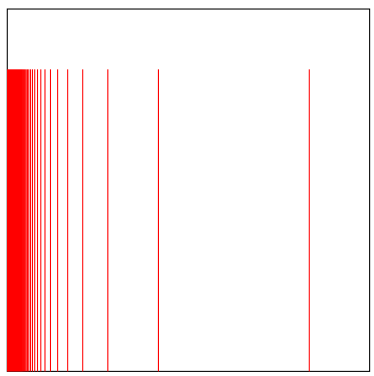
We see that $f \circ g = \mathbb{1}_Y$.

Exercise 1.2.14. *Prove that $g \circ f \simeq \mathbb{1}_{\mathrm{Cyl}(f)}$.*

□

Example 1.2.15.

1. Consider the following subspace of $\mathbb{R}^2$, called the comb space.



   This deformation retracts onto $(0,0)$ but not onto $(0,1)$.

2. Let X denote the comb space. Rotate X clockwise by 180 degrees to obtain the space A. Set $Y = X \cup A$ and note that A is closed in Y. Then both A and ${}^{Y}\big/_{A}$ are contractible, but Y is not.

Definition 1.2.16. Let the pair (X, A) consist of a space X and a subspace $A \subset X$. We say that (X, A) has the *homotopy extension property (HEP)* if we can fill the commutative diagram

$$\begin{tikzcd}
X \times \{0\} \arrow[rr, hook] \arrow[dr, "f"] & & X \times I \arrow[dl, dashed] \\
& Y & \\
A \times \{0\} \arrow[uu, hook] \arrow[rr, hook] & & A \times I \arrow[uu, hook] \arrow[ul, "H"']
\end{tikzcd}$$

of spaces where H is a given homotopy from $f \restriction_A$ to another map. The inclusion $\iota : A \to X$ is called a *cofibration* if (X, A) satisfies HEP.

1.3 Lecture 3

Lemma 1.3.1. *The pair (X, A) has HEP if and only if $A \times I \cup X \times \{0\}$ is a retract of $X \times I$.*

Proof.
$(\Longrightarrow)$ We have the commutative diagram

$$\begin{tikzcd}
A\times\{0\} \arrow[rr] \arrow[dd] & & A\times I \arrow[dl] \arrow[dd] \\
& A\times I\cup X\times\{0\} & \\
X\times\{0\} \arrow[rr] \arrow[ur] & & X\times I \arrow[ul, dashed, "\varphi"']
\end{tikzcd}.$$

Then φ is a retraction as desired.
$(\Longleftarrow)$ By hypothesis, there is some retraction $r : X \times I \to X \times \{0\} \cup A \times I$. This enables us to fill the diagram

$$\begin{tikzcd}
A\times\{0\} \arrow[rr] \arrow[dd] & & A\times I \arrow[dl, "H"'] \arrow[ddrr] & & \\
& Y & & & \\
X\times\{0\} \arrow[ur, "f"] \arrow[ddrrr] & & A\times I\cup X\times\{0\} \arrow[ul, dashed, "H\cup f"'] & & \\
& & & & \\
& & & X\times I \arrow[uul, "r"'] &
\end{tikzcd}.$$

If A is closed then $H \cup f$ is certainly continuous. If A is not closed, then our argument needs to be more careful. $\square$

Example 1.3.2.

(a) $S^{n-1} \hookrightarrow D^n$ is a cofibration.

Proof. We see that $S^{n-1} \times I \cup D^n \times \{0\}$ is a retract of $D^n \times I$ by the following radial projection from the point $(0, 2)$.

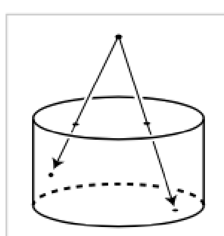


Figure 3: copied from Hatcher (15)

In fact, setting $r_t(x) = tr(x) + (1 - t)x$ for each $t \in I$ defines a deformation retraction r of $D^n \times I$ onto $S^{n-1} \times I \cup D^n \times \{0\}$. $\square$

(b) Let $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Then $A \hookrightarrow I$ is *not* a cofibration.

Proof. Suppose, for contradiction, that there is some retraction $r : I \times I \to A \times I \cup I \times \{0\}$. For each $n \geq 1$, the set $c_n := \left[\left(\frac{1}{n+1}, 1), (\frac{1}{n}, 1\right)\right]$ is connected, so that $r(c_n)$ is connected as well. Thus, there exists $(x_n, 1) \in c_n$ such that $r(x_n, 1) = (x_n, 0)$. But then $(x_n, 1) \to (0, 1)$ whereas $r(x_n, 1) \to (0, 0)$. As $r(0, 1) = (0, 1)$, this contradicts the continuity of r. □

(c) Let $f : B \to A$ be a map. Then $(\mathrm{Cyl}(f), B \times \{0\})$ satisfies HEP.

Lemma 1.3.3. *Any CW pair (X, A) satisfies HEP.*

Proof. Notice that $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching a certain number of copies of $D^n \times I$ along $D^n \times \{0\} \cup S^{n-1} \times I$. Recall the deformation retraction r from Example 1.3.2(b). Thus, we obtain a deformation retraction r_n of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$.

Define $U_n = X \times \{0\} \cup (X^n \cup A) \times I$ for each $n \geq -1$ with $X^{-1} = \emptyset$. Note that $U_n = (X^n \times I) \cup U_{-1}$. Extend each r_n to the homotopy $\hat{r}_n : U_n \times I \to U_n$ given by setting $\hat{r}_n(x) = x$ for each $x \in U_{-1}$. This is continuous since A is closed and $(X^n \times I) \cap U_{-1} = X^n \times \{0\} \cup A^n \times I \subset X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$. Then each $\hat{r}_n$ is a deformation retraction of U_n onto U_{n-1}. Perform $\hat{r}_n$ during the t-interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$. The infinite sequential concatenation R of the $\hat{r}_n$ is continuous at $t = 0$ when restricted to each $X^n \times I^2$ and is thus continuous on $\bigcup_n U_n = X \times I^2$. Therefore, R is a deformation retraction of $X \times I$ onto $U_{-1} = X \times \{0\} \cup A \times I$. □

Definition 1.3.4. Let X be a space. Define $\pi_0(X) = X/\sim$ where $x \sim y$ iff $\exists \varphi : I \to X$ such that $\varphi(0) = x$ and $\varphi(1) = y$.

This means that $\pi_0(X)$ is precisely the set of path components of X.

Definition 1.3.5. Let $\gamma, \hat{\gamma} : I \to X$ be paths in X such that $\gamma(0) = \hat{\gamma}(0)$ and $\gamma(1) = \hat{\gamma}(1)$. A *path homotopy from γ to $\hat{\gamma}$* is a homotopy $H : I \times I \to X$ such that $H(0, s) = \gamma(0)$ and $H(1, s) = \gamma(1)$ for each $s \in I$. In this case, we write $\gamma \simeq_p \hat{\gamma}$.

Proposition 1.3.6.

1. *If $\gamma_0 \simeq_p \gamma_1$ and $\eta_0 \simeq_p \eta_1$, then $\gamma_0 * \eta_0 \simeq_p \gamma_1 * \eta_1$.*
2. *$(\gamma_0 * \gamma_1) * \gamma_2 \simeq_p \gamma_0 * (\gamma_1 * \gamma_2)$.*
3. *Any map $f : X \to Y$ induces a map $f_* : \pi_0(X) \to \pi_0(Y)$. If $\gamma_0 \simeq_p \gamma_1$, then $f \circ \gamma_0 \simeq_p f \circ \gamma_1$.*
4. *For any path $\gamma : I \to X$, the path $\eta(t) := \gamma(1 - t)$ satisfies $\eta * \gamma \simeq_p c_{\gamma(1)}$ and $\gamma * \eta \simeq_p c_{\eta(1)}$ where c_x denotes the constant path at the point $x \in X$.*

Definition 1.3.7. Define the *fundamental groupoid of a space* X as the category $\Pi_1(X)$ with $\mathrm{ob}(\Pi_1(X)) = X$ and $\mathrm{Hom}_{\Pi_1(X)}(x, y) = \{[\gamma]_{\simeq_p} \mid \gamma \text{ is a path from } x \text{ to } y\}$. We make concatenation of paths the composition of morphisms.

Note 1.3.8. This is in fact a groupoid in the sense of category theory.

Definition 1.3.9. Given $x_0 \in X$, define the *fundamental group of the pointed space* (X, x_0) as

$$\pi_1(X, x_0) = \mathrm{Hom}_{\Pi_1(X)}(x_0, x_0).$$

This means that $\pi_1(X, x_0) = \{\gamma : S^1 \to X \mid \gamma(1, 0) = x_0\}/\simeq (\mathrm{rel}\ (1,0))$.

Proposition 1.3.10. *If there is some path p from x to y in X, then $\pi_1(X, x) \cong \pi_1(X, y)$.*

Proof. Define $\varphi_p : \pi_1(X, x) \to \pi_1(X, y)$ by $\gamma \mapsto p^{-1}\gamma p$. This is an isomorphism. □

Remark 1.3.11. Any map $f : X \to Y$ induces a functor $f_* : \Pi_1(X) \to \Pi_1(Y)$ that restricts to a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$.

Theorem 1.3.12. *If $n \geq 2$, then $\pi_1(S^n, x) = 0$.*

Proof. Note that if $\eta : I \to S^n$ satisfies $\operatorname{im}\eta = S^n \setminus \{p\}$ for some $p \in S^n$, then $\eta \simeq_p c_x$ since $S^n \setminus \{p\} \cong \mathbb{R}^n$. Thus, it suffices to prove the following lemma.

Lemma 1.3.13. *Every path γ in S^n is path-homotopic to some path η in S^n such that $\operatorname{im}\eta \neq S^n$.*

Proof. Write $\gamma(t) = (\gamma_0(t), \gamma_1(t), \dots, \gamma_n(t))$. The Weierstrass approximation theorem implies that we can approximate each γ_i by some smooth function. Hence we may find some smooth map $\tilde{\gamma}$ such that

$$|\gamma(t) - \tilde{\gamma}(t)| < \epsilon$$

for each $t \in I$. Now, there is some smooth retraction $r : D^{n+1} \setminus \{0\} \to S^n$. Define $H : I \times I \to S^n$ by

$$(t, s) \mapsto r(s\tilde{\gamma}(t) + (1-s)\gamma(t)).$$

This is a homotopy $\gamma \simeq_p r \circ \tilde{\gamma}$. But $r \circ \tilde{\gamma} : I \to S^n$ is smooth and $n > 1$. By Sard, it follows that $\operatorname{im}(r \circ \tilde{\gamma})$ has measure zero in S^n. Thus, $r \circ \tilde{\gamma}$ is not surjective, as desired. □

□

1.4 Lecture 4

Theorem 1.4.1. *If the pair (X, A) has HEP and A is contractible, then the natural projection $X \twoheadrightarrow X/A$ is a homotopy equivalence.*

Proof. There is some contraction $H : A \times I \to X$ of A onto, say, a_0. We can find some map $\widetilde{H}$ such that

$$\begin{tikzcd}
A \times \{0\} \arrow[rr, hook] \arrow[dd, hook] & & A \times I \arrow[dl, "H"'] \arrow[dd, hook] \\
& X & \\
X \times \{0\} \arrow[ur, hook] \arrow[rr, hook] & & X \times I \arrow[ul, dashed, "\widetilde{H}"']
\end{tikzcd}$$

commutes. Then $\widetilde{H}_0 = \mathbb{1}_X$, and $\widetilde{H}_t(A) \subset A$ for each $t \in I$. By the universal property of quotient spaces, we get some $\bar{H}_t$ such that

$$\begin{tikzcd}
X \arrow[r, "\widetilde{H}_t"] \arrow[d, two heads] & X \arrow[d, two heads] \\
X/A \arrow[r, dashed, "\bar{H}_t"'] & X/A
\end{tikzcd}$$

commutes for each t. Since $\widetilde{H}_1(a) = a_0$ for each $a \in A$, it follows that

$$\begin{tikzcd}
X \arrow[r, "\widetilde{H}_1"] \arrow[d, two heads] & X \arrow[d, two heads] \\
X/A \arrow[r, "\bar{H}_1"'] \arrow[ur, dashed, "p"] & X/A
\end{tikzcd}$$

commutes as well for some map p. If q denotes the natural projection, then

$$q \circ p(\bar{x}) = q \circ p \circ q(x) = \bar{H}_1 \circ q(x) = \bar{H}_1(\bar{x}).$$

Then p is homotopy inverse to q. □

Proposition 1.4.2. *If X is contractible, then $\pi_1(X, x_0) = 0$.*

Proof.

Lemma 1.4.3. *Let $F : I \times I \to X$ be any homotopy. Let $\gamma \coloneqq F\restriction_{\{0\}\times I}$, $\beta \coloneqq F\restriction_{I\times\{1\}}$, $\delta \coloneqq F\restriction_{\{1\}\times I}$, and $\alpha : F\restriction_{I\times\{0\}}$. Then $\beta \simeq_p \gamma^{-1} * \alpha * \delta$.*

Proof. Write $\gamma(1) = x_0$ and $\delta(1) = x_1$. Define the path homotopies

$$G(s,t) = \begin{cases} x_0 & s \le t \\ \gamma(1+t-s) & s \ge t \end{cases}$$

and

$$H(s,t) = \begin{cases} x_1 & 1-s \le t \\ \delta(s+t) & 1-s \ge t \end{cases}.$$

Then $G : c_{x_0} \simeq_p \gamma^{-1}$, and $H : c_{x_1} \simeq_p \delta$. Hence

$$\beta \simeq_p c_{x_0} * \beta * c_{x_1} \simeq_p \gamma^{-1} * \alpha * \delta,$$

as desired. □

By hypothesis, there is some contraction $F : X \times I \to X$ of X onto, say, the point x_0. Let $\gamma : I \to X$ be a loop at x_0. Then we get a homotopy $G : \gamma \simeq c_{x_0}$ where each G_t is a loop in X. Since $\eta \coloneqq G\restriction_{\{0\}\times I} = G\restriction_{\{1\}\times I}$, our lemma implies that $c_{x_0} \simeq_p \alpha^{-1} * c_{x_0} * \alpha \simeq_p \gamma$. □

2 Covering spaces

Definition 2.0.1. We say that a map $p : Y \to X$ is a *covering projection* if for each $x \in X$, there is some neighborhood $U \ni x$ such that

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[dr, "p"'] & U \times S \arrow[d, "\pi_1"] \\ & U \end{tikzcd}$$

commutes for some discrete set S. We call a triple of the form $(X \times S, X, \pi_1)$ a *trivial covering space.* If every S has $|S| = n$ for some $n \in \mathbb{N}$, then p is called an *n-fold cover of X*.

Proposition 2.0.2. *If $p : Y \to X$ is a covering projection, then Y is a manifold if and only if X is a manifold.*

Example 2.0.3.

1. $p : \mathbb{R} \to S^1$ given by $x \mapsto e^{2\pi i x}$.
2. $p : S^1 \subset \mathbb{C}^\times \to S^1$ given by $z \mapsto z^n$.
3. $p : S^n \to \mathbb{RP}^n$ given as the quotient map is a 2-fold cover of $\mathbb{RP}^n$.
4. Let X be a space and Γ be a discrete topological group.

   Definition 2.0.4. A *group action on a space* X is an injective group homomorphism $G \to \mathrm{Homeo}(X)$.

   Let Γ act on the space Y such that for every $y \in Y$, there is some open set $U \ni y$ such that $g \cdot U \cap U = \emptyset$ when $g \neq e$. We call such a group action *a covering space action* or *properly discontinuous.* In particular, this action is free. Proposition 1.40(a) of *Hatcher* states that $Y \twoheadrightarrow Y/\Gamma$ is a covering projection.

   Remark 2.0.5. If Y is simply connected, then $\pi_1\left(Y/\Gamma, y_0\right) \cong \Gamma$.

   Note 2.0.6. Let $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. We have that $S^1 \cong \mathbb{R}/{x \sim x+1} \cong \mathbb{R}/\mathbb{Z}$. Let $\mathbb{Z}$ act on S^1 by $n\cdot[x] = [x+n\alpha]$. But note that for any $x \in S^1$, the orbit of x is dense in S^1 since α is irrational. Thus, $S^1/\mathbb{Z}$ has the indiscrete topology, and $S^1 \twoheadrightarrow S^1/\mathbb{Z}$ is not a covering projection.

Example 2.0.7. The following, however, are covering projections.

1. $\mathbb{R}^2 \twoheadrightarrow {}^{\mathbb{R}^2}\!/_{\mathbb{Z}^2} \cong S^1 \times S^1$.

2. $\mathrm{SL}_2(\mathbb{R}) \twoheadrightarrow {}^{\mathrm{SL}_2(\mathbb{R})}\!/_{\mathrm{SL}_2(\mathbb{Z})}$.

3. Set $G = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$ and $\Gamma = \left\{ \begin{bmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$. Then Γ is a discrete subgroup of G, and $G \twoheadrightarrow {}^{G}\!/_{\Gamma}$ is a covering projection. We call ${}^{G}\!/_{\Gamma}$ an *Iwasawa manifold*. Since G is simply connected, we also have that $\pi_1\left({}^{G}\!/_{\Gamma}\right) \cong \Gamma$.

Exercise 2.0.8. $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Lemma 2.0.9. *Let $F : Z \times I \to X$ be any map and $p : Y \to X$ be a covering projection. Suppose that there is some $\widetilde{F}_0 : Z \times \{0\} \to Y$ such that $p \circ \widetilde{F}_0 = F_0$. Then there exists a unique map $\widetilde{F} : Z \times I \to Y$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F} \restriction_{Z \times \{0\}} = \widetilde{F}_0$.*

Corollary 2.0.10 (Path lifting property).

1. *If $\gamma : I \to X$ is a path and $y_0 \in Y$ with $p(y_0) = \gamma(0)$, then there exists a unique $\tilde{\gamma} : I \to Y$ such that $\tilde{\gamma}(0) = y_0$ and $p \circ \tilde{\gamma} = \gamma$.*

2. *Let γ_0, γ_1 be two paths in X and let $H : \gamma_0 \simeq_p \gamma_1$. Let $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ be respective lifts such that $\tilde{\gamma}_0(0) = \tilde{\gamma}_1(0)$. Then there exists a unique $\widetilde{H} : \tilde{\gamma}_0 \simeq_p \tilde{\gamma}_1$ such that $p \circ \widetilde{H} = H$.*

Proof.

1. Let $Z = *$

2. Let $Z = I$.

□

Theorem 2.0.11. $\pi_1(S^1, 1) \cong \mathbb{Z}$.

Proof. Let γ be a path in S^1 based at 1. We have a covering map $p : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t}$. By Corollary 2.0.10, there is some unique lift $\tilde{\gamma}$ of γ such that $\tilde{\gamma}(0) = 0$ and $p \circ \tilde{\gamma} = \gamma$. Since $p^{-1}(1) = \mathbb{Z}$, Corollary 2.0.10 gives a function $\psi : \pi_1(S^1, 1) \to \mathbb{Z}$ given by $[\gamma] \mapsto \tilde{\gamma}(1)$. Since $\mathbb{R}$ is simply connected, we have that ψ is bijective. It remains to verify that it's a homomorphism. Let $[f], [g] \in \pi_1(S^1, 1)$ and take their respective unique lifting $\tilde{f}$ and $\tilde{g}$. Write $n = \tilde{f}(1)$ and $m = \tilde{g}(1)$. Define the path $\tilde{\tilde{g}}(s) = n + \tilde{g}(s)$ in $\mathbb{R}$, which begins at n. Since $p(n + x) = p(x)$ for every $x \in \mathbb{R}$, we see that $\tilde{\tilde{g}}$ lifts g. Then $\tilde{f} * \tilde{\tilde{g}}$ lifts $f * g$ and ends at the point $n + m$. Hence $\psi([f] * [g]) = n + m = \psi([f]) + \psi([g])$, as required. □

Corollary 2.0.12. *There is no retraction of D^2 onto S^1.*

Proof. Suppose, for contradiction, that there is some retraction r. Then we get an induced sequence of group maps

$$\pi_1(S^1, 1) \xrightarrow{i_*} \pi_1(D^2, 1) \xrightarrow{r_*} \pi_1(S^1, 1) \ .$$

But $r_* \circ i_* = (r \circ i)_* = (\mathbb{1}_{S^1})_* = \mathbb{1}_{\pi_1(S^1)}$. Thus, r_* is surjective, which is impossible. □

Corollary 2.0.13 (Brouwer fixed point theorem in dimension 2). *If $\varphi : D^2 \to D^2$ is any map, then $\varphi(x_0) = x_0$ for some $x_0 \in D^2$.*

Proof. If not, then we may define a retraction r of D^2 onto S^1 as follows. For each $x \in D^2$, set $r(x)$ equal to the point on the circle that intersects the ray from $h(x)$ to x. □

2.1 Lecture 5

We give a proof of a result that we've already used.

Lemma 2.1.1 (Homotopy lifting property). *Let $F : Z \times I \to X$ be any map and $p : Y \to X$ be a covering projection. Suppose that there is some $\widetilde{F}_0 : Z \times \{0\} \to Y$ such that $p \circ \widetilde{F}_0 = F_0$. Then there exists a unique map $\widetilde{F} : Z \times I \to Y$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F} \restriction_{Z \times \{0\}} = \widetilde{F}_0$.*

Proof. Let $z_0 \in Z$.

Claim. *There exist a neighborhood $U_{z_0} \subset Z$ and a lift $\widetilde{F}$ of $F \restriction_{U_{z_0} \times I}$ such that $p \circ \widetilde{F} = F$ on $U_{z_0} \times I$.*

Proof. For any $(z_0, t) \in Z \times I$, note that $F(z_0, t)$ has some neighborhood $V_{z_0,t}$ such that

$$p^{-1}(V_{z_0,t}) = \coprod_{\alpha} V_{z_0,t,\alpha}$$

with $p : V_{z_0,t,\alpha} \to V_{z_0,t}$ a homeomorphism. Thus, $F^{-1}(V_{z_0,t})$ contains some set of the form $\underbrace{U_{z_0,t}}_{\text{nbhd of } z_0} \times (a_t, b_t)$, so that $F(U_{z_0,t} \times (a_t, b_t)) \subset V_{z_0,t}$. This makes $\{(a_t, b_t)\}_{t \in I}$ an open cover of I. As I is compact, there is some $k \in \mathbb{N}$ such that $\{(a_{t_i}, b_{t_i})\}_{i=1,\ldots,k}$ cover I. Set $U_{z_0} = \bigcap_{1 \le i \le k} U_{z_0,t_i}$, which must be open and contain z_0. We obtain a sequence $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $F(U_{z_0} \times [t_i, t_{i+1}])$ is contained in an evenly covered set of X. Now, $\widetilde{F}_0$ is contained in some unique sheet of $p^{-1}(U_{z_0,0}) = \coprod_j \widetilde{U}_{z_0,0,j} \subset Y$, say, j_0. Define $\widetilde{F} : U_{z_0} \times [0, t_1] \to \widetilde{U}_{z_0,0,j_0}$ as the composite $p_{j_0}^{-1} \circ F \restriction_{U_{z_0} \times [0,t_1]}$ where $p_{j_0} : \widetilde{U}_{z_0,0,j_0} \to U_{z_0,0}$ is some homeomorphism.

Suppose that we have extended $\widetilde{F}$ to $U_{z_0} \times [0, t_i]$. We can use a similar argument to define an extension $\widetilde{F}$ on $U_{z_0} \times [t_i, t_{i+1}]$. By induction, it follows that we can construct a lift $\tilde{F} : U_{z_0} \times I \to Y$ of F.

It remains to verify that such a lift is unique. For now, assume that $Z = *$. Suppose that $\widetilde{F}$ and $\widetilde{F}'$ are two lifts of $F : I \to X$ such that $\widetilde{F}(0) = \widetilde{F}'(0)$. Assume inductively that $\widetilde{F} = \widetilde{F}'$ on $[0, t_i]$. Since both $\widetilde{F}([t_i, t_{i+1}])$ and $\widetilde{F}'([t_i, t_{i+1}])$ are connected and $\widetilde{F}(t_i) = \widetilde{F}'(t_i)$, there is a single sheet over U_{i-1} in which both $\widetilde{F}([t_i, t_{i+1}])$ and $\widetilde{F}'([t_i, t_{i+1}])$ are contained. Since p is injective on this sheet and $p \circ \widetilde{F} = p \circ \widetilde{F}'$, it follows that $\widetilde{F} = \widetilde{F}'$ on $[t_i, t_{i+1}]$. We are done with our induction.

As a result, the lifts $\{F \restriction_{U_z \times [0,t_1]}\}_{z \in Z}$ constructed above must agree with each other when $U_z \cap U_{z'} \neq \emptyset$. We may thus apply the gluing lemma to get a lift $\tilde{F} : Z \times I \to Y$ of F. This must be unique as it is unique when restricted to each segment $\{z\} \times I$. □

□

Corollary 2.1.2. *Let $p : (Y, y_0) \to (X, x_0)$ be a covering map. Then $p_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is injective.*

Proof. Let $[\gamma] \in \ker p_*$, so that $[p \circ \gamma] = [c_{x_0}]$. By the path lifting property, there must be some homotopy $\gamma \simeq_p c_{y_0}$. Thus, $[\gamma] = 0$. □

Theorem 2.1.3 (Fundamental theorem of algebra). *Any nonconstant $p(x) \in \mathbb{C}[x]$ has a root.*

Proof. We may assume that $p(x)$ is monic. Write $p(x) = z^n + a_1 z^{n-1} + \cdots + a_n$. Suppose that $p(x)$ has no roots. Then $p : \mathbb{C} \to \mathbb{C}^\times \simeq S^1$. For each real number $r \ge 0$, define the loop $f_r : I \to (S^1, 1)$ by

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}.$$

Note that $(f_r)_{r \ge 0}$ determines a path homotopy from the trivial loop, so that $f_r \simeq_p c_1$ for each r. Set $r' > 1 + |a_1| + \cdots + |a_n|$. If $|z| = r'$, then

$$\begin{aligned} |z|^n &> (|a_1| + \cdots + |a_n|)|z|^{n-1} \\ &\ge |a_1 z^{n-1}| + |a_2| + \cdots + |a_n| \\ &\ge |a_1 z^{n-1} + a_2 + \cdots + a_n|. \end{aligned}$$

This implies that if $0 \le t \le 1$, then $p_t(x) = z^n + t(a_1 z^{n-1} + a_2 + \cdots + a_n)$ is nonzero on the circle $|z| = r'$. Then the map

$$(s,t) \mapsto \frac{p_t(r'e^{2\pi i s})/p_t(r')}{|p_t(r'e^{2\pi i s})/p_t(r')|}$$

is a homotopy $e^{2\pi i n s} \simeq_p f_{r'}$. Hence $[1]^n = [e^{2\pi i n s}] = 0$. Since $\pi_1(S^1, 1) \cong \mathbb{Z}$, this makes $n = 0$. Therefore, $p(x)$ must be constant. $\square$

Remark 2.1.4. Let $\{G_\alpha\}$ be any collection of objects of **Grp**. Recall that the *free product* $*_\alpha G_\alpha$ *of the* G_α is the unique object satisfying the following universal property. For any collection of maps $(\varphi_\alpha : G_\alpha \to H)$ where H is a group, there exists a unique map $*\varphi_\alpha$ such that

$$\begin{tikzcd} *G_\alpha \arrow[r, "*\varphi_\alpha"] & H \\ G_\alpha \arrow[u, hook] \arrow[ur, "\varphi_\alpha"'] & \end{tikzcd}$$

commutes for each α.

Note 2.1.5. This is exactly the coproduct in **Grp**, which always exists.

Theorem 2.1.6 (van Kampen). *Write the space X as the union $\bigcup_{\alpha \in A} A_\alpha$ of path connected open subsets A_α . Assume that each A_α is path connected and that each $A_\alpha \cap A_\beta$ is path connected. Then*

$$\Phi : *_\alpha \pi_1(A_\alpha) \to \pi_1(X)$$

is surjective. Moreover, if each $A_\alpha \cap A_\beta \cap A_\delta$ is path connected, then $\ker \Phi$ *is precisely the normal subgroup N generated by every element of the form*

$$i_{\alpha\beta}(\gamma) i_{\beta\alpha}(\gamma^{-1}), \quad \gamma \in \pi_1(A_\alpha \cap A_\beta)$$

where $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$ denotes the map induced by inclusion. In this case, $\pi_1(X) \cong {*_\alpha G_\alpha}/{N}$ where $G_\alpha := \pi_1(A_\alpha)$. .

Remark 2.1.7. Equivalently, van Kampen says that functor $\pi_1(-) : \textbf{Top} \to \textbf{Grp}$ respects fibered coproducts (also known as *amalgamated free products*).

Corollary 2.1.8. *If $X = A \cup B$ where A and B are open in X and $\pi_0(A) = \pi_0(B) = \pi_0(A \cap B) = 0$, then*

$$\pi_1(X) \cong \pi_1(A) \underset{\pi_1(A\cap B)}{*} \pi_1(B).$$

Corollary 2.1.9. *Let $X = A \cup B$ such that both A and B are closed in X and path connected. Suppose that $\pi_0(A \cap B) = 0$. Further, suppose that $A \cap B$ is a deformation retract both of some open set U in A and of some open set V in B. Then $\pi_1(X) \cong \pi_1(A) \underset{\pi_1(A\cap B)}{*} \pi_1(B)$.*

Proof. Note that $U \setminus (\underbrace{A \cap B}_{=U\cap V})$ and $V \setminus (A \cap B)$ are open in $U \cup V$. Therefore, we may patch the given deformation retractions onto $A \cap B$ together to get a deformation retraction of $U \cup V$ onto $A \cap B$. Similarly, we can patch together our given deformation retractions with $\mathbb{1}_A$ and $\mathbb{1}_B$ to get deformation retractions of $A \cup V$ onto A and of $B \cup U$ onto B, respectively. This induces an isomorphism of diagrams (in particular, spans).

$$\begin{tikzcd} \pi_1(A) \arrow[d, "\cong"] & \pi_1(A \cap B) \arrow[l] \arrow[r] \arrow[d, "\cong"] & \pi_1(B) \arrow[d, "\cong"] \\ \pi_1(A \cup V) & \pi_1(U \cup V) \arrow[l] \arrow[r] & \pi_1(B \cup U) \end{tikzcd}$$

As a result, their pushouts (i.e., colimits) must be isomorphic.

Now, note that $(A \cup V)^c = B \setminus V$, which is closed in X. Likewise, $(B \cup U)^c$ is closed in X. We can apply van Kampen to get

$$\pi_1(X) \cong \pi_1(A \cup V) \underset{\pi_1(U \cup V)}{*} \pi_1(B \cup U).$$

By our argument above, this implies that

$$\pi_1(X) \cong \pi_1(A) \underset{\pi_1(A \cap B)}{*} \pi_1(B).$$

□

Example 2.1.10.

1. Let A and B denote the two circles forming the wedge sum $S_1 \vee S_1$. Note that $A \cap B = *$, so that $N = *$. Thus, $\pi_1(S_1 \vee S_1) \cong \mathbb{Z} * \mathbb{Z}$.

2. Recall that $\mathbb{RP}^2 \cong {D^2}/{\sim}$ where $x \sim -x$ when $x \in S^1$. Decompose ${D^2}/{\sim}$ into a small disk A around the origin and an annulus B so that $A \cap B$ is a smaller annulus bounded above by the boundary of A and below by the inner boundary of B. Then $\pi_1(A) = 0$, and $\pi_1(B) \cong \pi_1(\mathbb{RP}^1) \cong \mathbb{Z} \cong \pi_1(A \cap B)$. Write $\pi_(B) = \langle b \rangle$ and $\pi_1(A \cap B) = \langle \gamma \rangle$. Then $i_{AB}(\gamma) = 0$. Also, $i_{BA}(\gamma) = b^2$, as shown below.

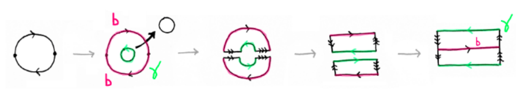


Figure 4: copied from *https://www.math3ma.com*

By van Kampen, it follows that $\pi_1(\mathbb{RP}^2) \cong \pi_1(A) \underset{\pi_1(A \cap B)}{*} \pi_1(B) \cong {\langle b \rangle}/{\langle b^2 \rangle} \cong \mathbb{Z}/2\mathbb{Z}$.

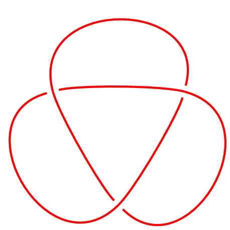
2.2 Lecture 6

Definition 2.2.1. A *knot* is a piecewise smooth embedding of S^1 into $\mathbb{R}^3$. Two knots K_1 and K_2 are *equivalent* if there is some homeomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\varphi(K_1) = K_2$.

Example 2.2.2.

1. The *unknot* is the standard embedding $S^1 \hookrightarrow \mathbb{R}^3$.

2. The following space is called the *trefoil knot*.



Lemma 2.2.3. *The knot group $\pi_1(\mathbb{R}^3 \setminus K)$ is isomorphic to $\pi_1(S^3 \setminus K)$.*

Proof. Recall that $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$. Write $S^3 \setminus K$ as the union of $\mathbb{R}^3 \setminus K$ and the open ball $B := (\mathbb{R}^3 \setminus D) \cup \{\infty\}$ where $D \supset K$ is a sufficiently large disk. Then both $B \cap (\mathbb{R}^3 \setminus K)$ and B are simply connected (the former being homeomorphic to $S^2 \times \mathbb{R}$). By van Kampen, $\pi_1(S^3 \setminus K) \cong \pi_1(\mathbb{R}^3 \setminus K)$. □

Note 2.2.4. We have that $S^3 \cong ST_1 \cup_T ST_2$ where ST_1 and ST_2 denote solid tori with common boundary a torus. Indeed,

$$
\begin{aligned}
S^3 &= \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\} \\
ST_1 &= \{(z_1, z_2) \in S^3 \mid |z_1|^2 \le \frac{1}{2}\} \cong D^2 \times S^1 \\
ST_1 &= \{(z_1, z_2) \in S^3 \mid |z_2|^2 \le \frac{1}{2}\} \cong S^1 \times D^2 \\
ST_1 \cap ST_2 &= \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 = |z_2|^2 = \frac{1}{2}\} \cong S^1 \times S^1.
\end{aligned}
$$

Definition 2.2.5. Let M^n be a manifold. A *foliation of* M is a collection $\{\mathscr{L}_\alpha\}_{\alpha \in A}$ of immersed submanifolds of fixed dimension l, called *leaves*, such that $M = \coprod_{\alpha \in A} \mathscr{L}_\alpha$ and for each $p \in M$, there is some smooth chart (U, φ) around p with $\varphi(\mathscr{L}_\alpha \cap U) \subset \mathbb{R}^l$ for each α.

Example 2.2.6.

1. The collection of circles $\{\{q\} \times S^1\}_{q \in S^1}$ forms a foliation of the torus.

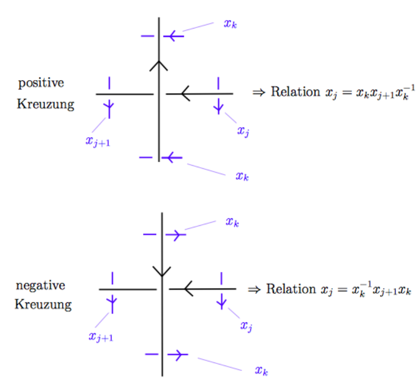
2. For each $\theta \in \mathbb{R}$, define the curve in the torus $\gamma_\theta(t) = (e^{it}, e^{i(\alpha t + \theta)})$. Then the collection of curves $\{\operatorname{im} \gamma_\theta\}_{\theta \in \mathbb{R}}$ forms another foliation of the torus. If $\theta \in \mathbb{Q}$, then each curve is an embedded circle. Otherwise, it is dense in the torus.

Theorem 2.2.7. *The torus is the only surface that admits a foliation.*

Remark 2.2.8. The so-called Reeb torus demonstrates that there are foliations on S^3.

Remark 2.2.9. If $M = \coprod_\alpha \mathscr{L}_\alpha$, then $\mathscr{L} := \bigcup_{x \in M} T_x \mathscr{L}_{\alpha_x}$ (where $x \in L_{\alpha_x}$) is a subbundle of TM such that $X, Y \in C^\infty(M, \mathscr{L}) \implies [X, Y] \in \mathscr{L}$. Frobenius states the converse of this.

Note 2.2.10 (Wirtinger presentation). Draw our knot K as follows. Take a finitely many arcs $\alpha_1, \ldots, \alpha_n$ such that each α_i is connected to $\alpha_{i+1} \mod n$. Orient the knot so that the arcs are labeled in the direction of the orientation. Draw a short arrow x_i passing under each α_i from right to left. The x_i represent loops starting at a base point going under the arc from its base to head and back to base. At each crossing, we get a relation among the x_i as follows.

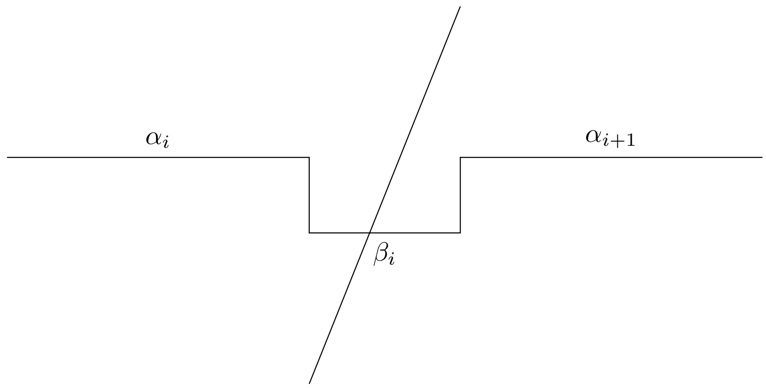


For each i, let r_i denote the relation $x_k x_i = x_{i+1} x_k$.

Proposition 2.2.11. $\pi_1(S^3 \setminus$ *unknot*$) \cong \mathbb{Z}$.

Theorem 2.2.12. $\pi_1(\mathbb{R}^3 \setminus K, *) \cong \langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$.

Proof. We may embed each arc α_i in the plane $z = 0$ except for a small vertical segment at each end of the arc, which will lie instead in the plane $z = -1$.



Let $A = \{z \geq 1\} \setminus K$. The lower boundary of A is the union of the n line segments with each β_i removed. For each $i = 1 \leq i \leq n$, let B_i equal the union of a solid rectangular box whose top lies on $z = -1$ surrounding but excluding β_i and an arc connection β_i to $*$. Make the B_i disjoint. Let C equal the closure of everything below the B_i. Then we can write

$$\mathbb{R}^3 \setminus K = A \cup B_1 \cup \cdots \cup B_n \cup C.$$

We see that $\pi_1(A, *) \cong \mathbb{F}_n$, $\pi_1(B_1) = 0$, and C is contractible. Now, $A \cap B_1$ equals a rectangle minus β_1 together with an arc connecting β_1 to $*$. Thus, $A \cap B_1 \simeq S^1$. Write $\pi_1(A \cap B_1) = \langle \gamma_1 \rangle \cong \mathbb{Z}$. But $i_{AB_1}(\gamma) = x_1 x_k^{-1} x_2^{-1} x_k$. By van Kampen, we get $\pi_1(A \cup B_1) \cong {}^{\mathbb{F}_n}\!/\!_{(r_1)}$. By induction, it follows that

$$\pi_1(A \cup B_1 \cup \cdots \cup B_n) \cong \mathbb{F}_n/(r_1, \ldots, r_n).$$

Finally, since C is contractible, we get $\pi_1(A \cup B_1 \cup \cdots \cup B_n \cup C) \cong \pi_1(A \cup B_1 \cup \cdots \cup B_n)$. $\square$

2.3 Lecture 7

Remark 2.3.1. If $G = \langle g_1, \ldots, g_n \mid w_1, \ldots, w_m \rangle$, then $G \cong {}^{\mathbb{F}(g_1, \ldots, g_n)}\!/\!_{N}$ where N denotes the normal subgroup generated by $w_1, \ldots, w_m$ and each w_i is a word in $g_1, \ldots, g_n$. Now, let $G_1 = \langle g_1, \ldots, g_n \mid w_1, \ldots, w_m \rangle$ and $G_2 = \langle h_1, \ldots, h_k \mid u_1, \ldots, u_l \rangle$ It is known that the following two problems are undecidable.

(a) (isomorphism problem) Is $G_1 \cong G_2$?

(b) (word problem) Given a word w over $\{g_1, \ldots, g_n\}$, is $w = e$?

Example 2.3.2. Let S_g denote the (closed) orientable surface of genus g. Note that $S_g \cong S_{g-1} \# T$. We can draw S_g as an oriented $4g$-gon with pairs of sides identified as follows.

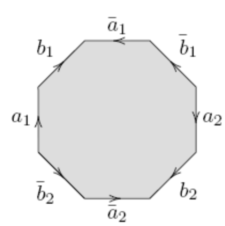


Figure 5: copied from *Manifold Atlas*

For example, $a_1 \sim \bar{a}_1$. Then we can decompose S_g into a small disk A around the origin and an annulus-like space B so that $A \cap B$ is a smaller annulus bounded above by the boundary of A and below by the inner boundary of B. Then $\pi_1(A \cap B) \cong \langle \gamma \rangle$ where $|\gamma| = \infty$. Further, $\pi_1(A) = 0$, and $\pi_1(B) \cong \mathop{\ast}_{\{1,2,\ldots,2g\}} \mathbb{Z}$ since B deformation retracts onto a bouquet of $2g$ circles. Observe that $i_{BA}(\gamma) = \prod_{i=1}^{g} [a_i, b_i]$, which implies that

$$\pi_1(S_g) \cong \Big\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \Big\rangle.$$

Note 2.3.3. The comb space is path connected but not locally path connected.

Lemma 2.3.4. *If $p : (Y, y_0) \to (X, x_0)$ is a covering map, then there is an isomorphism of sets*

$$p^{-1}(x_0) \xrightarrow{\cong} {\pi_1(X, x_0)}\big/{p_*\pi_1(Y, y_0)}.$$

Proof. Let γ be a loop in (X, x_0). Lift it to $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = y_0$. Set $\alpha(\gamma) = \tilde{\gamma}(1) \in p^{-1}(x_0)$. The map $\alpha : {\pi_1(X, x_0)}\big/{p_*\pi_1(Y, y_0)} \to p^{-1}(x_0)$ is well-defined since $\widetilde{h * \gamma} = \tilde{h} * \tilde{\gamma}$ when $[h] \in p_*\pi_1(Y, y_0)$. It is surjective because Y is path connected. If $\alpha(\gamma_1) = \alpha(\gamma_2)$, then $\tilde{\gamma}_1 * \tilde{\gamma}_2^{-1} = \widetilde{\gamma_1 * \gamma_2^{-1}}$ is a loop in (Y, y_0), so that $\gamma_1 * \gamma_2^{-1} \in p_*\pi_1(Y, y_0)$. This shows that α is injective as well. □

Lemma 2.3.5 (Lifting criterion). *Let $p : (Y, y_0) \to (X, x_0)$ be a covering map and $f : (Z, z_0) \to (X, x_0)$ a map with Z path connected and locally path connected. Then there exists some map $\tilde{f}$ fitting into the commutative diagram*

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ Z \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

if and only if $f_*\pi_1(Z, z_0) \subset p_*\pi_1(Y, y_0) \subset \pi_1(X, x_0)$.

Proof.
($\Longrightarrow$) Note that $f_* = (p\tilde{f})_* = p_*\tilde{f}_*$.

($\Longleftarrow$) Let $z \in Z$. Find some path $\gamma : z_0 \rightsquigarrow z$. Then $f\gamma$ is a path in X starting at x_0. There exists a unique lift $\widetilde{f\gamma}$ starting at y_0. Set $\tilde{f}(z) = \widetilde{f\gamma}(1)$. We must check that $\tilde{f}$ is well-defined. Let $\gamma' : z_0 \rightsquigarrow z$. Then $h_0 \coloneqq (f\gamma') * (f\gamma)^{-1}$ is a loop at x_0. Note that $[h_0] \in f_*\pi_1(Z, z_0) \subset p_*\pi_1(Y, y_0)$. Hence there is some homotopy $H : h_0 \simeq_p p \circ \tilde{h}_1$ for some loop $\tilde{h}_1$ at y_0. Apply the homotopy lifting property to get a homotopy $\widetilde{H} : \tilde{h}_0 \simeq_p \tilde{h}_1$, so that $\tilde{h}_0$ is a loop. By uniqueness of lifts, $\tilde{h}_0 = \widetilde{f\gamma'} * \widetilde{f\gamma}^{-1}$, which implies that $\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1 - 0) = \widetilde{f\gamma}(1)$, as required.

It remains to verify that $\tilde{f}$ is continuous. Find some path $\lambda : z_0 \rightsquigarrow z$. We know that there is some neighborhood $U \ni f(z)$ that is is evenly covered by p. Let $\widetilde{U}$ denote the sheet containing $\tilde{f}(z)$ and find some homeomorphism $p : \widetilde{U} \to U$. Find some path connected neighborhood $V \ni z$ such that $f(Z) \subset U$. Let $z' \in V$. Find some path $\eta : z \rightsquigarrow z'$. Then the path $(f\lambda) * (f\eta)$ lifts to $\left(\widetilde{f\lambda}\right) * \left(\widetilde{f\eta}\right)$ where $\widetilde{f\eta} = p^{-1}f\eta$. As z' was arbitrary in V, this implies that $\tilde{f} \restriction_V = p^{-1}f \restriction_V$, which is continuous. Hence $\tilde{f}$ is continuous as well. □

Proposition 2.3.6 (Unique lifting property). *Let $p : Y \to X$ be a covering map and $f : Z \to X$ a map with $\tilde{f}_1, \tilde{f}_2 : Z \to Y$ lifts of f. Let Z be connected. If there is some $z_0 \in Z$ such that $\tilde{f}_1(z_0) = \tilde{f}_2(z_0)$, then $\tilde{f}_1 = \tilde{f}_2$.*

Remark 2.3.7. Let X be a space with $\pi_0(X) = 0$. When is there a locally path connected covering space $p : \widetilde{X} \to X$ such that $\pi_1\left(\widetilde{X}\right) = 0$. Suppose that such a space exists. Let $x \in X$. Suppose that the open set $U \subset X$ with $x \in U$ is evenly covered. Ley $\widetilde{U} \subset \widetilde{X}$ be one sheet. If γ is a loop in U, then it lifts to a loop $\tilde{\gamma}$ at, say, $y \in \widetilde{U}$, which must be nullhomotopic. Find some homotopy $H : \tilde{\gamma} \simeq_p c_y$. Then $pH : \gamma \simeq_p c_x$. Therefore, for any $x \in X$, there is some neighborhood $U \ni x$ such that $\pi_1(U) = 0$.

Definition 2.3.8. Such a property of X is called being *semilocally simply connected.*

Theorem 2.3.9. *Suppose that X is path connected, locally path connected, and semilocally simply connected (hereafter "swell"). Then there is some covering map $p : \widetilde{X} \to X$ such that $\widetilde{X}$ is simply connected.*

Proof. Pick $x_0 \in X$. Let $P_{x_0} \coloneqq \{\gamma : I \to X \mid \gamma(0) = x_0\}$. Set $\widetilde{X} = {}^{P_{x_0}}\!/_{\simeq_p}$, endowed with the compact-open topology. $\square$

Lemma 2.3.10. *Let X be a swell space. If $Y_1, Y_2 \overset{p_1,p_2}{\longrightarrow} X$ are two simply connected covering spaces, then they are isomorphic in the following sense. There is some homeomorphism $\psi : Y_2 \to Y_1$ such that*

$$\begin{tikzcd} Y_1 \arrow[d, "p_1"'] & Y_2 \arrow[l, "\psi"'] \arrow[dl, "p_2"] \\ X & \end{tikzcd}$$

commutes (i.e., ψ is a bundle isomorphism).

Proof. By the lifting criterion, we can obtain lifts

$$\begin{tikzcd} & Y_2 \arrow[d, "p_2"] \\ Y_1 \arrow[r, "p_1"'] \arrow[ur, dashed, "\tilde{p}_1"] & X \end{tikzcd}.$$

$$\begin{tikzcd} & Y_1 \arrow[d, "p_1"] \\ Y_2 \arrow[r, "p_2"'] \arrow[ur, dashed, "\tilde{p}_2"] & X \end{tikzcd}$$

Then $\tilde{p}_2\tilde{p}_1 = \mathbb{1}_{Y_1}$ since p_1 has a unique lift. Similarly, $\tilde{p}_1\tilde{p}_2 = \mathbb{1}_{Y_2}$. Hence $\tilde{p}_2 : Y_2 \xrightarrow{\cong} Y_1$. $\square$

2.4 Lecture 8

Definition 2.4.1. If X is a swell space, then we call such a cover $p : \left(\widetilde{X}, \tilde{x}_0\right) \to (X, x_0)$ the *universal cover of X*. It is universal in that for any path connected covering space $p' : (Y, y_0) \to (X, x_0)$, there exists a unique covering map $p'' : \left(\widetilde{X}, \tilde{x}_0\right) \to (Y, y_0)$ such that $p''p' = p$.

Lemma 2.4.2. *Let X be swell. For any subgroup $H \leq \pi_1(X, x_0)$, there exists a path connected covering space $p_H : X_H \to X$ such that $p_H(\pi_1(X_H, h_H)) = H$.*

Proof. Define an equivalence relation $\sim$ on the universal cover $p : \widetilde{X} \to X$ as follows. Let $y_1 \sim y_2$ if $p(y_1) = p(y_2)$ and for any two paths γ_1 and γ_2 from x_0 to y_1 and y_2, respectively, $(p\gamma_1) * (p\gamma_2)^{-1} \in H$. Set $X_H = {}^{\widetilde{X}}\!/_{\sim}$ and note that the map $p_H : X_H \to X$ given by $[y] \mapsto p(y)$ is a covering space. Set $x_H = q(\tilde{x}_0)$ where $q : \widetilde{X} \to X_H$ denotes the natural projection. We know that $p_H\pi_1(X_H, x_H)$ consists of loops γ in (X, x_0) whose lifts to X_H are also loops. If $\tilde{\gamma}$ denotes the lift of γ to $\widetilde{X}$, then we see that

$$\tilde{\gamma}(1) \sim \tilde{x}_0 \iff p_H(\tilde{\gamma}^{-1}c_{\tilde{x}_0}) \in H,$$

meaning that $p_H \circ \tilde{\gamma}$ is a loop at x_0 if and only if γ belongs to H. $\square$

Lemma 2.4.3. *Let X be path connected and locally path connected. Suppose that $p_1 : (Y_1, y_1) \to (X, x_0)$ and $p_2 : (Y_2, y_2) \to (X, x_0)$ are two (path connected) covering maps. Then there exists an isomorphism $\varphi : (Y_1, y_1) \to (Y_2, y_2)$ if and only if $p_{1*}\pi_1(Y_1, y_1) = p_{2*}\pi_1(Y_2, y_2)$.*

Proof. The ($\Longrightarrow$) direction follows from the fact that $p_1 = p_2 f$ and $p_2 = p_1 f^{-1}$.

($\Longleftarrow$) Apply the lifting criterion twice to lifts $\tilde{p}_1$ and $\tilde{p}_2$ such that $p_2\tilde{p}_1 = p_1$ and $p_1\tilde{p}_2 = p_2$. By the unique lifting property, $\tilde{p}_1\tilde{p}_2 = \mathbb{1}$ and $\tilde{p}_2\tilde{p}_1 = \mathbb{1}$. Hence $\tilde{p}_1$ and $\tilde{p}_2$ are inverse isomorphisms. $\square$

Corollary 2.4.4. *Let X be swell. There is a bijection between the set of isomorphism classes of path connected covering spaces $p : (Y, y) \to (X, x_0)$ of X and the set of subgroups of $\pi_1(X, x_0)$ given by*

$$(Y, y) \longleftrightarrow p_*\pi_1(Y, y).$$

Definition 2.4.5. Let $p : \widetilde{X} \to X$ be a covering space. The set of isomorphisms $\widetilde{X} \to \widetilde{X}$, called *deck transformations*, forms a group $G\left(\widetilde{X}\right)$ under composition.

Remark 2.4.6. This corresponds to the Galois group.

Note 2.4.7. By the unique lifting property, if $\widetilde{X}$ is path connected, then any $f \in G\left(\widetilde{X}\right)$ is entirely determined by its value at a single point.

Definition 2.4.8. A covering space $p : Y \to X$ is *normal* if $G(Y)$ acts transitively on the set $p^{-1}(x)$ for each $x \in X$.

Lemma 2.4.9. *Let $p : (Y, y_0) \to (X, x_0)$ be a path connected covering space with X swell. Let $H \coloneqq p_*\pi_1(Y, y_0)$. Then*

1. *p is normal if and only if $H \trianglelefteq \pi_1(X, x_0)$.*
2. *$G(Y) \cong {}^{N(H)}\!/_{H}$.*

Hence $G(Y) \cong {}^{\pi_1(X, x_0)}\!/_{H}$ when p is normal.

Proof.

1. For any $y_1 \in p^{-1}(x_0)$, there is some path $\eta : y_0 \rightsquigarrow y_1$. Then $\pi_1(Y, y_0) \cong \pi_1(Y, y_1)$ via the map $[\gamma] \mapsto \eta^{-1}\gamma\eta$. Note that $p\eta \in \pi_1(X, x_0)$ and that $p_*\pi_1(Y, y_0) = [p\eta]p_*\pi_1(Y, y_1)[p\eta]^{-1}$. Hence $[\gamma] \in N(H) \iff p_*\pi_1(Y, y_0) = p_*\pi_1(Y, y_1)$ where γ lifts to a path $y_0 \rightsquigarrow y_1$. By the lifting criterion together with the unique lifting property, it follows that $[\gamma] \in N(H)$ if and only if there is some deck transformation $Y \to Y$ mapping y_0 to y_1. Hence $N(H) = \pi_1(X, x_0)$ if and only if p is normal.
2. Define $\varphi : N(H) \to G(Y)$ by mapping $[\gamma]$ to the deck transformation sending y_0 to y_1. The preceding argument shows that φ is surjective. Further, $\ker\varphi$ consists of loop at x_0 whose lifts are loops in Y. Thus, $\ker\varphi = H$.

$\square$

Example 2.4.10. If Γ acts freely and properly discontinuously on the space Y, then the quotient map $Y \to {}^{Y}\!/_{\Gamma}$ is a normal covering space such that $\Gamma \cong G(Y)$. Hence $\Gamma \cong {}^{\pi_1(X, x_0)}\!/_{p_*\pi_1(Y, y_0)}$ where $X \coloneqq {}^{Y}\!/_{\Gamma}$.

Definition 2.4.11. A *graph* is a 1-dimensional CW-complex.

Lemma 2.4.12. *Let G be a graph. Then $\pi_1(G)$ is a free group.*

Proof. Since every tree is contractible, we see that G is homotopy equivalent to a wedge sum of circles by noting that $\bigvee_\alpha S^1_\alpha \simeq {}^{G}\!/_{T} \simeq G$ where T is a maximal tree contained in G. $\square$

Note 2.4.13. If X is a graph and $p : Y \to X$ is a covering space, then Y is also a graph.

Theorem 2.4.14. *Let Γ be a free group and $H \leq \Gamma$. Then H is free.*

Proof. We have that $\Gamma \cong \pi_1(X)$ where $X \coloneqq \bigvee_\alpha S^1_\alpha$. Also, H corresponds to some covering space $p : Y \to X$. Therefore, Y is a graph, and $\pi_1(Y)$ is free. This implies that $H = p_*\pi_1(Y)$ is free since p_* is injective. $\square$

3 Homology

Definition 3.0.1.

1. The *standard k-simplex* is the set
$$\Delta^k = \{(t_0, \ldots, t_k) \mid \sum t_i = 1,\ t_i \geq 0\} = \{(x_1, \ldots, x_n) \mid 0 \leq x_1 \leq x_2 \leq \cdots \leq x_k \leq 1\}.$$

2. For each $i = 0, \ldots, k+1$, define the *face map* $\partial^i : \Delta^k \to \Delta^{k+1}$ by $(t_0, \ldots, t_k) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_k)$.

3. An *(abstract) simplicial complex* is a set K together with a collection of finite subsets $\Sigma \subset P(K)$ such that if $\sigma \in \Sigma$ and $\tau \subset \sigma$, then $\tau \in \Sigma$. If $\sigma \in \Sigma$ and $|\sigma| = k$, then σ is called a *$(k-1)$-simplex*. Any subset of a simplex is called a *face*.

4. An *ordered simplex* is a simplex equipped with an ordering of its vertices.

Definition 3.0.2. Let (K, Σ) be a simplicial complex. Let S denote the set of ordered k-simplices in K. Define
$$C_k(K, \Sigma) = \bigoplus_{s \in S} \mathbb{Z} \big/ {\sim}$$
where $v_0 \cdots v_k \sim (-1)^\tau v_{\tau(0)} \cdots v_{\tau(k)}$ for any $\tau \in S_{k+1}$.

Note 3.0.3. $C_k(K, \Sigma)$ is a free abelian group.

Definition 3.0.4. Define the *differential* $\partial_k : C_k(K, \Sigma) \to C_{k-1}(K, \Sigma)$ by
$$\partial_k(v_0 \cdots v_k) = \sum_{i=0}^{k} (-1)^i v_0 \cdots \widehat{v_i} \cdots v_k,$$
which we extend linearly.

Exercise 3.0.5. *Show that* $\partial^2 = 0$.

Example 3.0.6. $v_0v_1v_2 \overset{\partial_2}{\mapsto} v_1v_2 - v_0v_2 + v_0v_1 \overset{\partial_1}{\mapsto} v_2 - v_1 - (v_2 - v_0) + v_1 - v_0 = 0.$

Definition 3.0.7. Define the *k-th homology group of* (K, Σ) as
$$H_k(K, \Sigma) = \ker \partial_k \big/ \operatorname{im} \partial_{k+1}.$$
We call $Z_k(K, \Sigma) := \ker \partial_k$ *cycles* and $B_k(K, \Sigma) := \operatorname{im} \partial_{k+1}$ *boundaries*.

3.1 Lecture 9

Note 3.1.1. The category of simplicial complexes Δ has as morphisms functions $f : (K, \Sigma_K) \to (L, \Sigma_L)$ such that $f(\sigma) \in \Sigma_L$ when $\sigma \in \Sigma_K$. We call these *simplicial maps*.

Definition 3.1.2. A *subcomplex of* K is a set $L \subset K$ such that inclusion is a simplicial map.

Example 3.1.3. If K is a simplicial complex, then $K^p := \{\sigma \in \Sigma_K : |\sigma| \leq p + 1\}$.

Definition 3.1.4. If K is a simplicial complex, then define $|K|$ as the set
$$\{\alpha : K \to I \mid \sum_{v \in K} \alpha(v) = 1 \wedge \{v \mid \alpha(v) \neq 0\} \in \Sigma_K\}.$$
Let $|K|_d$ denote this set endowed with the topology induced by the metric given by
$$d(\alpha, \beta) = \sqrt{\sum_{v \in K} |\alpha(v) - \beta(v)|^2}.$$

Example 3.1.5. Let $K = \{0, 1, \dots, n\}$ and let Σ consist of all finite subsets of K. Then

$$|K|_d = \{(\alpha(0), \dots, \alpha(n)) \mid \alpha(0) + \cdots + \alpha(n) = 1\} \cong_{\mathbf{Set}} \Delta^n,$$

which recovers the Euclidean metric.

Note 3.1.6.

1. When $\sigma \in \Sigma_K$ is a q-simplex, there exists a natural map $|\sigma^q|_d : \Delta^q \to |K|_d$ given by mapping x to the function α where $\alpha(\sigma) = x$ and $\alpha(v) = 0$ for any $v \notin \sigma$.
2. Define a new topology on $|K|$ where $A \subset |K|$ is closed (resp. open) if $(|\sigma|_d)^{-1}(A)$ is closed (resp. open) in $\Delta^{|\sigma|-1}$ for each $\sigma \in \Sigma_K$. From now on, $|K|$ is assumed to have this topology.
3. Given a simplicial map $f : K \to L$, define the map $|f| : |K| \to |L|$ by $\alpha \mapsto (w \mapsto \sum_{v \in f^{-1}(w)} \alpha(v))$. Thus, we get a functor $|\cdot| : \Delta \to \mathbf{Top}$.

Definition 3.1.7. A *triangulation of a space* X is a pair $((K, \Sigma), \varphi)$ such that $\varphi : |K| \to X$ is a homeomorphism.

Remark 3.1.8. The Cantor set has no triangulation.

Definition 3.1.9. The category of *chain complexes* **Ch** has as objects sequences of abelian groups of the form

$$A_0 \xleftarrow{\partial_1} A_1 \xleftarrow{\partial_2} A_2 \xleftarrow{\partial_3} A_3 \xleftarrow{\partial_3} \cdots$$

such that $\partial_i \partial_{i+1} = 0$ for each $i \in \mathbb{N}$. Its morphisms are commutative diagrams of the form

$$\begin{tikzcd}
A_0 \arrow[d, "f_0"] & A_1 \arrow[l, "\partial_1"'] \arrow[d, "f_1"] & A_2 \arrow[l, "\partial_2"'] \arrow[d, "f_2"] & A_3 \arrow[l, "\partial_3"'] \arrow[d, "f_3"] & \cdots \arrow[l, "\partial_4"'] \\
B_0 & B_1 \arrow[l, "\partial_1'"] & B_2 \arrow[l, "\partial_2'"] & B_3 \arrow[l, "\partial_3'"] & \cdots \arrow[l, "\partial_4'"]
\end{tikzcd},$$

so that $\partial_{i-1} f_i = f_{i-1} \partial_{i-1}$ for each i. In this case, we call $f \coloneqq (f_i)_{i \in \mathbb{Z}}$ a *chain map.*

Definition 3.1.10.

1. Given a chain complex (A, ∂), define its *i-th homology group* as

$$H_i(A, \partial) = {}^{Z(A_i)}\!\big/_{B(A_i)}$$

where $\underbrace{Z_i \coloneqq \ker \partial_i}_{cycles}$ and $\underbrace{B_i \coloneqq \operatorname{im} \partial_{i+1}}_{boundaries}$.

2. Any morphism $f : (A, \partial) \to (B, \partial')$ induces a map $H_q(f) : H_q(A) \to H_q(B)$. We say that f is a *quasi-isomorphism* if $H_q(f)$ is an isomorphism for each q.

Note 3.1.11. Any chain map preserves both cycles and boundaries.

Note 3.1.12. We get a functor $C_\bullet : \Delta \to \mathbf{Ch}$ given by $(K, \Sigma) \mapsto (C_q(K, \Sigma), \partial_q)_{q \geq 0}$. As a result, we find ourselves with the following diagram of functors.

$$\begin{tikzcd}
\Delta \arrow[r, "|\cdot|"] \arrow[d, "C_\bullet"'] & \mathbf{Top} \\
\mathbf{Ch} \arrow[r, "H_\bullet"] & \mathbf{Ab}
\end{tikzcd}$$

Remark 3.1.13 (Singular simplex). The *Hauptvermutung* is the conjecture that if X is a nice topological space such as a manifold, then for any two triangulations of X, you can get from one to the other in a combinatorial way. It turns out that this is false. Hence we cannot fill our above diagram with a functor $\mathbf{Top} \to \Delta$.

Eilenberg and Steenrod, however, created the following approach to obtain certain reverse arrows in our diagram. Let X be a space. A *singular q-simplex on X* is a continuous map $\Delta^q \to X$. Let $\mathrm{Sing}_q(X)$ denote the set of all singular q-simplices on X. Let $C_q(X)$ be the free abelian group on $\mathrm{Sing}_q(X)$. Define $\partial_q : C_q(X) \to C_{q-1}(X)$ by

$$\sigma \mapsto \sum_{i=0}^{q} (-1)^i \sigma \restriction_{[v_0,\ldots,\widehat{v_i},\ldots,v_q]}$$

where $\Delta^q \cong [v_0, \ldots, v_q]$. Then $\partial_{q-1}\partial_q = 0$. We now have a functor $\mathbf{Top} \to \mathbf{Ch}$ that maps each map $f : X \to Y$ of spaces to the chain map (f_n) where $f_n(g : \Delta^n \to X) = f \circ g_n$. Note that the induced homology functor $\mathbf{Top} \to \mathbf{Ab}$ is a topological invariant.

Note 3.1.14. Suppose that $F : X \times I \to Y$ is a homotopy from f to g. Then we want to show that $H_q(f) = H_q(g)$ for each q. Although $\Delta^q \times I$ is not a simplex, we can write $\Delta^q \times \{0\} = [v_0, \ldots, v_q]$ and $\Delta^q \times \{1\} = [v'_0, \ldots, v'_q]$ and consider the following decomposition of $\Delta^q \times I$ into simplices.

$$\begin{gathered}
[v_0, v_1, \ldots, v_q] \\
[v_0, \ldots, v_q, v'_q] \\
[v_0, \ldots, v_{q-1}, v'_{q-1}, v'_q] \\
\vdots \\
[v'_0, v'_1, \ldots, v'_q]
\end{gathered}$$

Definition 3.1.15. Let (A, ∂) and (B, ∂) be two chain complexes and $f, g : (A, \partial_A) \to (B, \partial_B)$ two chain maps. A *chain homotopy between f and g* is a map $H : A_q \to B_{q+1}$ for each q such that $\partial_B H + H\partial_A = f - g$.

Proposition 3.1.16. *If f and g are chain homotopic, then $H_q(f) = H_q(g)$.*

Note 3.1.17. Let H be a chain homotopy between f and g. If $x \in A_q$ and $\partial x = 0$, then $f(x) - g(x) = (\partial H + H\partial)(x) = \partial H(x)$, so that $f(x) - g(x)$ is a boundary.

3.2 Lecture 10

Lemma 3.2.1. *Let $F : X \times I \to Y$ be a homotopy between f and g. Decompose $\Delta^k \times I$ into $k + 1$ simplices. Label the vertices of $\Delta^k \times \{0\}$ and $\Delta^k \times \{1\}$ by $v_0, \ldots, v_k$ and $w_0, \ldots, w_k$, respectively. Then $\bigcup_{i=0}^{k} \underbrace{[v_0, v_1, \ldots, v_i, w_i, \ldots, w_k]}_{\text{convex span}}$. For each k, define the map $p : C_k(X) \to C_{k+1}(Y)$ by*

$$p\sigma = \sum_i (-1)^i F \circ (\sigma \times \mathbb{1}_I \restriction_{[v_0,\ldots,v_i,w_i,\ldots,w_k]}).$$

Then p is a chain homotopy between f_* and g_*.

Proof. We just verify some low-dimensional cases. First, consider a simplex $\sigma : \Delta^1 \to X$. Then

$$\begin{aligned}
\partial p\sigma &= (F \circ \sigma \times \mathbb{1})[v_0, w_0] + (F \circ \sigma \times \mathbb{1})[w_0, w_1] - (F \circ \sigma \times \mathbb{1})[v_1, w_1] - (F \circ \sigma \times \mathbb{1})[v_0, v_1]. \\
p\partial\sigma &= -(F \circ \sigma \times \mathbb{1})[v_0, w_0] + (F \circ \sigma \times \mathbb{1})[v_1, w_1].
\end{aligned}$$

Thus,

$$\partial p\sigma + p\partial\sigma = (F \circ \sigma \times \mathbb{1})[w_0, w_1] - (F \circ \sigma \times \mathbb{1})[v_0, v_1] = g\sigma - f\sigma.$$

Next, consider a simplex $\sigma : \Delta^2 = [v_0, v_1, v_2] \to X$ Note that

$$p\sigma = [v_0, w_0, w_1, w_2] - [v_0, v_1, w_1, w_2] + [v_0, v_1, v_2, w_2].$$

From this we compute

$$\begin{aligned}\partial p\sigma = {} & [w_0, w_1, w_2] - [v_0, w_1, w_2] + [v_0, w_0, w_2] - [v_0, w_0, w_1] \\ & - [v_1, w_1, w_2] + [v_0, w_1, w_2] - [v_0, v_1, w_2] + [v_0, v_1, w_1] \\ & + [v_1, v_2, w_2] - [v_0, v_2, w_2] + [v_0, v_1, w_2] - [v_0, v_1, v_2].\end{aligned}$$

Moreover,

$$\partial\sigma = \sigma\restriction_{[v_1,v_2]} - \sigma\restriction_{[v_0,v_2]} + \sigma\restriction_{[v_0,v_1]},$$

so that

$$p\partial\sigma = \sigma\restriction_{[v_1,w_1,w_2]} - \sigma\restriction_{[v_1,v_2,w_2]} - \sigma\restriction_{[v_0,w_0,w_2]} + \sigma\restriction_{[v_0,v_2,w_2]} + \sigma\restriction_{[v_0,w_0,w_1]} - \sigma\restriction_{[v_0,v_1,w_1]}.$$

We conclude that $\partial p\sigma + p\partial\sigma = (F \circ \sigma \times \mathbb{1})[w_0, w_1, w_2] - (F \circ \sigma \times \mathbb{1})[v_0, v_1, v_2] = g\sigma - f\sigma$. □

Corollary 3.2.2. *If $X \simeq Y$, then $H_*(X) \cong H_*(Y)$.*

Proof. Lemma 3.2.1 shows that $H_*(-)$ is actually a functor **Htpy** $\to$ **Ab**. Hence our result follows from the fact that functors preserve the identity morphism. □

Example 3.2.3. Let $X = \{x_0\}$, so that $\mathrm{Sing}_k(X) = \{c_{x_0} : \Delta^k \to \{x_0\}\}$. Then

$$C_*(X) = \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \xleftarrow{\partial} \mathbb{Z} \xleftarrow{\partial} \cdots.$$

Note that $\partial c_{x_0} = c_{x_0}([v_1]) - c_{x_0}([v_0]) = 0$ when $c_{x_0} : \Delta^1 \to X$. If $c_{x_0} : \Delta^k \to X$, then

$$\partial c_{x_0} = \sum_{i=0}^{k} (-1)^i \underbrace{c_{x_0}\restriction_{[v_0,\ldots,\hat{v}_i,\ldots,v_k]}}_{c_{x_0}:\Delta^{k-1}\to X}.$$

Hence $\partial_k : C_k(X) \to C_{k-1}(X)$ equals 0 if k is odd and $\mathbb{1}$ if k is even. As a result, we get a sequence in homology

$$\mathbb{Z} \qquad 0 \qquad 0 \qquad 0 \qquad \cdots$$

Corollary 3.2.4. *If X is contractible, then* $H_*(X) = \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * \neq 0 \end{cases}$.

Lemma 3.2.5. $H_*(X) \cong \bigoplus_{X_\alpha \in \pi_0(X)} H_*(X_\alpha)$.

Proof. Note that

$$\begin{aligned}\mathrm{Map}(\Delta^k, X) &\cong_{\mathbf{Set}} \mathrm{Map}(\Delta^k, \coprod_\alpha X_\alpha) \\ &\cong_{\mathbf{Set}} \coprod_\alpha \mathrm{Map}(\Delta^k, X_\alpha)\end{aligned}$$

because Δ^k is path connected. Therefore, $C_*(X) \cong \bigoplus_\alpha C_*(X_\alpha)$. □

Corollary 3.2.6. *The functor $C_*(-)$:* **Top** $\to$ **Ch** *preserves coproducts.*

Lemma 3.2.7. *The functor $H_*(-)$:* **Ch** $\to$ **Ab** *preserves coproducts.*

Proof. Let S be a set and (A_s, ∂) be a chain complex for each $s \in S$. Note that $\bigoplus_S H_n(A) = \bigoplus_S {}^{\ker \partial_n}\!/\!_{\mathrm{im}\, \partial_{n+1}}$ and $H_n(\bigoplus_S A) = {}^{\ker \bigoplus_S \partial_n}\!/\!_{\mathrm{im} \bigoplus_S \partial_{n+1}}$. But **Ch** and **Ab** have arbitrary direct sums, and the bifunctor $\bigoplus(-,-)$ commutes with kernels. Since $\bigoplus$ always commutes with images and with quotients, it follows that $\bigoplus_S H_n(A) = H_n(\bigoplus_S A)$, as desired. □

Example 3.2.8. $H_*(S^0) = \begin{cases} \mathbb{Z}\times\mathbb{Z} & * = 0 \\ 0 & * \neq 0 \end{cases}.$

Definition 3.2.9 (Reduced homology). Define $\epsilon : C_0(X) \to \mathbb{Z}$ by $\sum a_x \cdot x \mapsto \sum a_x$. Note that

$$(\epsilon \circ \partial)(\sigma : \Delta^1 \to X) = \epsilon(1 \cdot \sigma(v_1) - 1 \cdot \sigma(v_0)) = 1 - 1 = 0.$$

This induces a map $\epsilon : H_0(X) \to \mathbb{Z}$. Let

$$\widetilde{C}_k(X) = \begin{cases} C_k(X) & k > 0 \\ \ker \epsilon & k = 0 \end{cases}.$$

Define

$$\widetilde{H}_k(X) = H_k\left(\widetilde{C}_k(X)\right).$$

Note that $\widetilde{H}_k(X) = H_k(X)$ for any $k > 0$.

Note 3.2.10.

1. $\widetilde{H}_*(\{x_0\}) = 0$.
2. Both $\widetilde{C}(-)$ and $\widetilde{H}(-)$ are functors and homotopy invariant.

Example 3.2.11. $\widetilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}.$

Proposition 3.2.12. $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$

Proof. We have a commutative diagram

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & B_0(X) \arrow[r] \arrow[d] & B_0(X) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \epsilon \arrow[r, hook] \arrow[d] & C_0(X) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \widetilde{H}_0(X) \arrow[r, hook] \arrow[d] & H_0(X) \arrow[r, "\epsilon"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}.$$

The three columns and the top two rows are exact. The nine lemma implies that the bottom row is exact. ☐

Corollary 3.2.13. $\widetilde{H}_0(X) = 0$ *whenever* X *is path connected.*

Definition 3.2.14 (Relative homology). Given a pair (X, A) with $A \subset X$, define $C_*(X, A) = {C_*(X)}/{C_*(A)}$. Since $\partial : C_*(A) \to C_*(A)$, this descends to a map $\partial : C_*(X, A) \to C_*(X, A)$. Define $H_*(X, A) = H_*(C_*(X, A))$.

Example 3.2.15.

1. Consider the pair $(\mathbb{R}, [0, 3])$. Let $\sigma : \Delta^1 \to [0, 3] \subset \mathbb{R}$. Then $\partial\sigma = [3] - [0]$, so that $[0] = [3]$ in $C_0(\mathbb{R}, [0, 3])$.
2. Consider the pair $(\mathbb{R}, \mathbb{R} \setminus (0, 1))$. Choose $[x_0] \in C_0(A)$ and $[x_1] \in C_0((0, 1))$. Let $\sigma : \Delta^1 \xrightarrow{\cong} [x_1, 1]$. Then $\partial\sigma = [1] - [x_1]$. As $[1] \in C_0(A)$, we see that $[x_1] = \partial\sigma$ in $C_0(\mathbb{R}, \mathbb{R} \setminus (0, 1))$.

Definition 3.2.16. We say that a pair (X, A) is *good* if there is some neighborhood $U \subset X$ of A such that U deformation retracts onto A.

Theorem 3.2.17. *If (X, A) is a good pair, then $H_*(X, A) \cong \widetilde{H}_*(X/A)$.*

Example 3.2.18. $H_*(D^n, S^{n-1}) \cong \widetilde{H}_*(S^n)$.

Lemma 3.2.19. *Let X be a path connected space and $A \subset X$.*

1. *$H_0(X) \cong \mathbb{Z}$.*
2. *The map $i_* : H_0(A) \to H_0(X)$ induced by inclusion is surjective.*

Proof.

1. Consider $x \in X$ as a vertex. For any vertex $y \in X$, there is some path $c : x \rightsquigarrow y$. Note that $c \in C_1(X)$ and that $\partial c = y - x$. Hence $[y] = [x]$ in $H_0(X)$. This shows that $H_0(X) \cong \langle [x] \rangle \cong \mathbb{Z}$.
2. Let $[x]$ generate $H_0(X)$ and let $y \in Y$. We can find a path $i(y) \rightsquigarrow x$. Hence $x - i(y) \in B_0(X)$, so that $[x] = [i(y)] = i_*([y])$. This shows that i_* is surjective.

□

Lemma 3.2.20 (Snake). *Suppose that A, B, and C are three chain complexes and $i : A \to B$ and $j : B \to C$ are chains maps such that for each k, the sequence $0 \to A_k \to B_k \to C_k \to 0$ is exact. Then there exists a long exact sequence in homology*

$$\begin{tikzcd} \cdots \arrow[r] & H_k(A) \arrow[r] & H_k(B) \arrow[r] & H_k(C) \arrow[dll, "\partial'"'] \\ H_{k-1}(A) \arrow[r] & H_{k-1}(B) \arrow[r] & H_{k-1}(C) \arrow[r] & \cdots \end{tikzcd}.$$

Proof. We have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & A_{k+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{k+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{k+1} \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & A_k \arrow[r, "i"] \arrow[d, "\partial"] & B_k \arrow[r, "j"] \arrow[d, "\partial"] & C_k \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & A_{k-1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{k-1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{k-1} \arrow[r] \arrow[d, "\partial"] & 0 \\ 0 \arrow[r] & A_{k-2} \arrow[r, "i"] & B_{k-2} \arrow[r, "j"] & C_{k-2} \arrow[r] & 0 \end{tikzcd}.$$

Let $c \in C_k$ such that $\partial c = 0$. By exactness, there exists $b \in B_k$ such that $j(b) = c$. Then $\partial b \in B_{k-1}$ such that $j\partial b = \partial j b = \partial c = 0$. Hence there exists a unique $a \in A_{k-1}$ such that $i(a) = \partial b$. Then $i\partial a = \partial i a = \partial\partial b = 0$. Since i is injective, this implies that $\partial a = 0$. Define the map $\partial' : H_k(C) \to H_{k-1}(A)$ by $[c] \mapsto [a]$.

Exercise 3.2.21.

1. *Show that $\partial : H_k(C) \to H_{k-1}(A)$ by $c \mapsto a$ is a well-defined homomorphism.*

   Proof. Suppose that $j(b') = c$. Then $b - b' \in \ker j = \operatorname{im} i$, so that $i(u) = b - b'$ for some $u \in A_k$. Then

$$\begin{aligned} &i(a) - \partial b' \\ &= \partial b - \partial b' \\ &= \partial b - b' \\ &= \partial i u \\ &= i \partial u. \end{aligned}$$

Therefore, $\partial b' = i(a - \partial u)$. Since $[a] = [a - \partial u]$ in $H_{k-1}(A)$, we see that a is independent of our choice of b. From now on, we will denote such an a by a_c. It is clear that ∂' is also independent of our choice of $c \in \ker \partial$ and thus is well-defined. Finally, it is straightforward to check that it is a homomorphism. □

2. *Verify that the given long sequence is exact both at $H_k(C)$ and at $H_{k-1}(A)$.*

   Proof.
   $\underline{H_k(C)}$: We must show that $\operatorname{im} j(\bullet) = \ker \partial'$.

   Let $[f] \in \operatorname{im} j(\bullet)$, so that $[f] = [j(g)]$ for some $g \in Z(B_k) = \ker \partial$. Note that $i(a_c) = \partial g = 0$. As i is injective, it follows that $\partial([f]) = [a_c] = 0$, i.e., $\operatorname{im} j(\bullet) \subset \ker \partial'$.

   Conversely, suppose that $[f] \in \ker \partial'$. Then $a_c \in \operatorname{im} \partial$, so that $a_c = \partial(\tilde{f})$ for some $\tilde{f}$. Moreover, $f = j(g)$ for some g. Letting $y \coloneqq i(\tilde{f})$, we get

   $$\partial(y) = \partial(i(\tilde{f})) = i(\partial(\tilde{f})) = i(a_c)$$

   and

   $$j(g - y) = j(g) - j(y) = f - \underbrace{j(i(\tilde{f}))}_{ji=0} = f.$$

   Therefore, $\partial(g - y) = i(a_c) - i(a_c) = 0$, so that $g - y \in \ker \partial = Z(B_k)$. Since $[j(g - y)] = [f]$, it follows that $[f] \in \operatorname{im} j(\bullet)$, i.e., $\operatorname{im} j(\bullet) \supset \ker \partial'$.

   $\underline{H_{k-1}(A)}$: We must show that $\ker i(\bullet) = \operatorname{im} \partial'$. The fact that $\ker i(\bullet) \supset \operatorname{im} \partial'$ is evident from our construction of ∂'.

   Conversely, let $[f] \in \ker i(\bullet)$, so that $[if] = 0$. Then $if = \partial g$ for some $g \in B_k$. Hence $f = a_c$ for some $c \in j^{-1}(g)$. This implies that $\partial'([c]) = [a_c] = [f]$, so that $[f] \in \operatorname{im} \partial'$. □

□

Corollary 3.2.22. *Note that we have a canonical short exact sequence $0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$. Therefore, for any pair (X, A), there exists a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & H_{k+1}(X,A) \arrow[r] & H_k(A) \arrow[r] & H_k(X) \arrow[r] & H_k(X,A) \arrow[dlll] \\
& H_{k-1}(A) \arrow[r] & H_{k-1}(X) \arrow[r] & H_{k-1}(X,A) \arrow[r] & \cdots \arrow[dlll] \\
& H_0(A) \arrow[r] & H_0(X) \arrow[r] & H_0(X,A) \arrow[r] & 0
\end{tikzcd}.$$

Note 3.2.23. By Lemma 3.2.19, the bottom row

$$H_0(A) \longrightarrow H_0(X) \longrightarrow H_0(X, A) \longrightarrow 0$$

is precisely $\mathbb{Z} \to \mathbb{Z} \to 0 \to 0$ when X and A are path connected.

Definition 3.2.24. A *triangle of spaces* is a triple (X, A, B) such that $X \supset A \supset B$.

Corollary 3.2.25. *Let (X, A, B) be a triangle. The short exact sequence $0 \to C_*(A, B) \to C_*(X, B) \to C_*(X, A) \to 0$ induces a long exact sequence*

$$\cdots \to H_k(A, B) \to H_k(X, B) \to H_k(X, A) \to H_{k-1}(A, B) \to \cdots.$$

3.3 Lecture 11

Theorem 3.3.1 (Excision). *If (X, A) is a pair and $Z \subset A$ such that $\mathrm{cl}(Z) \subset \mathring{A}$. Then you can excise Z in that the inclusion of pairs $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism of homology.*

Corollary 3.3.2. *If (X, A) is a good pair, then the natural projection $q : (X, A) \to (X/A, A/A)$ induces an isomorphism on homology*

$$H_*(X, A) \to H_*(X/A, A/A) \cong \widetilde{H}_*(X/A).$$

Proof. There is some neighborhood U of A that deformation retracts onto A. For each k, the diagram

$$\begin{tikzcd}
H_k(X,A) \arrow[r, "1"] \arrow[d, "5"'] & H_k(X,U) \arrow[d, "6"'] & H_k(X \setminus A, U \setminus A) \arrow[l, "2"'] \arrow[d, "7"'] \\
H_k(X/A, A/A) \arrow[r, "3"'] & H_k(X/A.U/A) & H_k(X/A \setminus A/A, U/A \setminus A/A) \arrow[l, "4"]
\end{tikzcd}$$

commutes. Arrow 1 is an isomorphism due to the exactness of the sequence

$$\underbrace{H_k(U, A)}_{0} \to H_k(X, A) \to H_k(X, U) \to \underbrace{H_{k-1}(U, A)}_{0}$$

obtained from Corollary 3.2.22. Similarly, arrow 3 is an isomorphism. Arrows 2 and 4 are isomorphisms by excision. Arrow 7 is an isomorphism since $(X \setminus A, U \setminus A) \hookrightarrow (X/A \setminus A/A, U/A \setminus A/A)$ is a homeomorphism of pairs. Therefore, arrow 5 is an isomorphism since out diagram commutes. . ☐

Definition 3.3.3. A based space (X, x_0) is *nondegenerate* if the pair $(X, \{x_0\})$ is good.

Corollary 3.3.4. *Let $\{(X_\alpha, x_\alpha)\}_\alpha$ be a set of nondegenerate bases spaces. Then $\widetilde{H}_*(\bigvee_\alpha X_\alpha) \cong \bigoplus_\alpha \widetilde{H}_*(X_\alpha)$.*

Proof. Note that $\bigvee_\alpha X_\alpha = {\coprod_\alpha X_\alpha}/{\coprod_\alpha \{x_\alpha\}}$. Now apply Corollary 3.2.25 together with the fact that the functor $H_*(-,-) : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Ab}$ preserves coproducts. ☐

Definition 3.3.5. Let (K, Σ) be a simplicial complex and $L \subset K$ a subcomplex. Then ${C_*(K)}/{C_*(L)} = C_*(K, L)$. Recall that

$$|K| = \{f : K \to I \mid \{v \in K \mid f(v) \neq 0\} \in \Sigma, \sum_{v \in K} f(v) = 1\}.$$

Define $\varphi : C_k(K) \to C_k(|K|)$ by $[v_0, v_1, \dots, v_k] \mapsto (\sigma_{[v_0,\dots,v_k]} : \Delta^k \to |K|)$ where

$$\sigma_{[v_0,\dots,v_k]}(t_0, \dots, t_k)(v \in K) = \begin{cases} t_i & v = v_i \\ 0 & v \neq v_i \end{cases},$$

extended by linearity. This induces a map $\varphi : C_*(K) \to C_*(|K|)$.

Exercise 3.3.6. *Check that $\varphi\partial = \partial\varphi$.*

Definition 3.3.7. For each I, define the *i-skeleton K^i of K* to be the simplicial complex (K, Σ_{K^i}) where

$$\Sigma^j_{K^i} = \begin{cases} \Sigma^j_K & j \leq i \\ \emptyset & j > i \end{cases}.$$

Note 3.3.8. We have that $C_i(K^0) = \begin{cases} \mathbb{Z}[\Sigma^0_K] & i = 0 \\ 0 & i > 0 \end{cases}$. Since $|K^0| = K^0$, it follows that $H_*(|K^0|) = \mathbb{Z}[K^0]$.

Lemma 3.3.9. *$\varphi : H_*(K) \to H_*(|K|)$ is an isomorphism.*

Proof. Induct on the k-skeleton of K. The case where $k = 0$ is obvious. Suppose that $\varphi : H_*(K^i) \xrightarrow{\cong} H_*(|K^i|)$. Note that

$$C_*(K^{i+1}, K^i) = {C_*(K^{i+1})}\big/{C_*(K^i)} \cong \begin{cases} \mathbb{Z}[\Sigma^{i+1}_{K^{i+1}} = \Sigma^{i+1}_K] & * = i+1 \\ 0 & * \neq i+1 \end{cases}.$$

Therefore, $H_{i+1}(K^{i+1}, K^i) \cong \mathbb{Z}[\Sigma^{i+1}_K]$. Moreover, observe that $K^{i+1}/K^i \cong \bigvee_{\sigma \in \Sigma^{i+1}_K} S^{i+1}_\sigma$. From this we get

$$H_*(|K^{i+1}|, |K^i|) \cong \widetilde{H}_*(|K^{i+1}|/|K^i|) \cong \begin{cases} \mathbb{Z}[\Sigma^{i+1}_{K^{i+1}} = \Sigma^{i+1}_K] & * = i+1 \\ 0 & * \neq i+1 \end{cases}.$$

We now have a commutative diagram with exact rows

$$\begin{tikzcd}
H_{i+2}(K^{i+1}, K^i) \arrow[r] \arrow[d, "\varphi\cong"] & H_{i+1}(K^i) \arrow[r] \arrow[d, "\varphi\cong"] & H_{i+1}(K^{i+1}) \arrow[r] \arrow[d, "\varphi"] & H_{i+1}(K^{i+1}, K^i) \arrow[r] \arrow[d, "\varphi\cong"] & H_i(K^i) \arrow[r] \arrow[d, "\varphi\cong"] & H_i(K^{i+1}) \arrow[r] \arrow[d, "\varphi"] & H_i(K^{i+1}, K^i) \arrow[d, "\varphi\cong"] \\
H_{i+2}(|K^{i+1}|, |K^i|) \arrow[r] & H_{i+1}(|K^i|) \arrow[r] & H_{i+1}(|K^{i+1}|) \arrow[r] & H_{i+1}(|K^{i+1}|, |K^i|) \arrow[r] & H_i(|K^i|) \arrow[r] & H_i(|K^{i+1}|) \arrow[r] & H_i(|K^{i+1}|, |K^i|)
\end{tikzcd}.$$

Lemma 3.3.10 (Five lemma). *Consider the following commutative diagram in an abelian category with exact rows.*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

If f_2 and f_4 are isos, f_1 is epi, and f_5 is mono, then f_3 is iso.

As a result, $\varphi : H_*(K^{i+1}) \cong H_*(|K^{i+1}|)$. By induction, we are done. ☐

Note 3.3.11. If $i \geq j+1$, then $C_j(K) = C_j(K^i)$, so that $H_j(K) \cong H_j(K^i)$.

Proposition 3.3.12. *A map $\psi : C \to |K|$ with C a compact space has $\operatorname{im}\psi \subset |K^i|$ for some i.*

Corollary 3.3.13. $H_*(|K|) = \underbrace{\varinjlim H_*(|K^i|)}_{\text{direct limit}}$.

Proof. Any cycle $c \in C_i(|K|)$ is finite linear combination of singular simplices. Proposition 3.3.12 implies that there exists $j \geq 0$ small enough so that $c \in C_i(K^j|)$ and $[c] \neq 0 \in H_i(|K^j|)$. ☐

3.4 Lecture 12

Theorem 3.4.1 (Excision). *If (X, A) is a pair and $Z \subset A$ such that $\mathrm{cl}(Z) \subset \mathring{A}$. Then you can excise Z in that the inclusion of pairs $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism of homology.*

Lemma 3.4.2. *The excision theorem holds if and only if whenever $X = \mathring{A} \cup \mathring{B}$, the inclusion of pairs $(B, A \cap B) \to (X, A)$ induces an isomorphism on homology $H_*(B, A \cap B) \to H_*(X, A)$.*

Proof. For the forward direction, set $Z = X \setminus B$. Conversely, set $B = X \setminus Z$. ☐

Definition 3.4.3. Let X be a space. Let $\mathcal{U} = \{\mathring{U}_\alpha\}$ be a cover of X with each $U_\alpha \subset X$.

1. Define $\mathrm{Sing}^{\mathcal{U}}_n(X) = \{\sigma : \Delta^n \to X \mid \operatorname{im}\sigma \subset U_\alpha \text{ for some } \alpha\}$. We call elements of this set *small simplices relative to $\mathcal{U}$.*

2. Define $C^{\mathcal{U}}_n(X) = \mathbb{Z}[\mathrm{Sing}^{\mathcal{U}}_n(X)]$.

Theorem 3.4.4. *The inclusion map $i : C_*^{\mathcal{U}}(X) \to C_*(X)$ is a chain homotopy equivalence, i.e., there exists $\rho : C_*(X) \to C_*^{\mathcal{U}}(X)$ such that both ρi and $i\rho$ are chain homotopic to $\mathbb{1}$.*

Definition 3.4.5.

1. Given an n-simplex $\sigma = [v_0, v_1, \dots, v_n]$, the *barycenter of* σ is the point $b \coloneqq \sum_{i=0}^{n} t_i v_i$ where $t_i = \frac{1}{n+1}$ for each $i = 0, \dots, n$.

2. Define the *barycentric subdivision* B_σ *of* σ recursively as follows.
   - If $\sigma = [v_0]$, then let $B_\sigma = \sigma$.
   - If $\sigma = [v_0, v_1, \dots, v_n]$, then let B_σ consist of the n-simplices $[b, w_0, \dots, w_{n-1}]$ where $[w_0, \dots, w_{n-1}]$ is an $(n-1)$ simplex in the barycentric subdivision of a face of $[v_0, v_1, \dots, v_n]$.

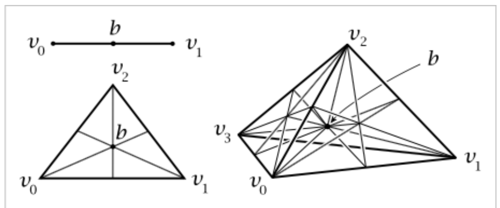


Figure 6: copied from Hatcher (120)

Note 3.4.6. The vertices of simplices in the barycentric subdivision of σ are precisely the barycenters of all the k-dimensional faces of σ for each $0 \le k \le n$.

Definition 3.4.7. The *diameter of* $[v_0, \dots, v_n]$ is $\max\limits_{0 \le i,j \le n} |v_i - v_j|$.

Lemma 3.4.8. *If d denotes the diameter of $\sigma \coloneqq [v_0, \dots, v_n]$, then the diameter of any n-simplex of B_σ is at most $\frac{nd}{n+1}$.*

Proof. For each i, we have that

$$\begin{aligned} |b - v_i| &= |\frac{v_0 + \cdots + v_n}{n+1} - v_i| \\ &= |\frac{v_0 + \cdots + v_n}{n+1} - \frac{(n+1)v_i}{n+1}| \end{aligned}$$

But $|v_i - v_i| = 0$, and $|v_k - v_i| \le d$ for each k. Hence $|b - v_i| \le \frac{nd}{n+1}$. We are done by induction on n. $\square$

Definition 3.4.9. Let (K, Σ_K) be a simplicial complex. Define a new simplicial complex $(K', \Sigma_{K'})$ such that $K' = \Sigma_K$ and $\Sigma_{K'}^n = \{\sigma_0 \subset \sigma_1 \subset \cdots \subset \cdots \subset \sigma_n\}$ wheere each $\sigma_i \in \Sigma_K$.

Definition 3.4.10. Let $\sigma : \Delta^n \to X$, which induces a map $\sigma_* : C_*(\Delta^n) \to C_*(X)$. Let $Y \subset \mathbb{R}^n$ be convex (so that we can apply barycentric subdivision to it.) For each $k \ge 0$, define the group of *linear chains* $LC_k(Y)$ as the free abelian group on the set of linear maps $\Delta^k \to Y$. For convenience, let $LC_{-1}(Y) \coloneqq \mathbb{Z}[\emptyset] \cong \mathbb{Z}$.

Note 3.4.11. Any linear chain $\lambda : \Delta^k \to Y$ is determined by the values $w_i \coloneqq \lambda(v_i)$ where v_i denotes the i-th vertex of Δ^k. For each $b \in Y$, we have a group homomorphism $b : LC_k(Y) \to LC_{k+1}(Y)$ given by $[w_0, \dots, w_k] \mapsto [b, w_0, \dots, w_k]$. Note that

$$\begin{aligned} \partial b[w_0, \dots, w_k] &= \partial[b, w_0, \dots, w_k] \\ &= [w_0, \dots, w_k] - \sum_{i=0}^{k} (-1)^i [b, w_0, \dots, \hat{w}_j, \dots, w_k] \\ &= [w_0, \dots, w_k] - b\partial[w_0, \dots, w_k]. \end{aligned}$$

This shows that $\partial b + b\partial = \mathbb{1}$, so that b is a chain homotopy between $\mathbb{1}$ and and 0.

Definition 3.4.12. Let b_λ denote the image under λ of the barycenter of Δ^n. Define the *subdivision map* $S : LC_*(Y) \to LC_*(Y)$ recursively by

- $S(\emptyset) = \emptyset$
- $S\lambda = b_\lambda S(\partial\lambda)$.

Note 3.4.13. If λ is an embedding with $\operatorname{im}\lambda = [w_0, \ldots, w_n]$, then $S\lambda =$ (sum of the n-simplices (up to sign) in the barycentric subdivision of $[w_0, \ldots, w_n]$).

Lemma 3.4.14. *$S\partial = \partial S$.*

Proof. We see that $S \restriction_{LC_{-1}(Y)} = \mathbb{1}$. Also, on $LC_0(Y)$ we have that $S([v]) = b_{[v]}S(\partial[v]) = b_{[v]}(\emptyset) = [v]$. Thus, $S \restriction_{LC_0(Y)} = \mathbb{1}$ as well. Hence the equation $S\partial = \partial S$ holds on $LC_0(Y)$. If $* > 0$, then apply the fact that $\partial b_\lambda + b_\lambda \partial = \mathbb{1}$ to get

$$\begin{aligned}
\partial S\lambda &= \partial b_\lambda S)\partial\lambda \\
&= (1 - b_\lambda\partial)S(\partial\lambda) \\
&= S(\partial\lambda) - b_\lambda \partial S(\partial\lambda) \\
&= S(\partial\lambda) - \underbrace{b_\lambda S(\partial\partial\lambda)}_{\text{by induction}} \\
&= S(\partial\lambda).
\end{aligned}$$

□

Remark 3.4.15. Define $T : LC_n(Y) \to LC_{n+1}(Y)$ a chain homotopy between $\mathbb{1}$ and S recursively by

- $T(\emptyset) = 0$
- $T\lambda = b_\lambda(\lambda - T\partial\lambda)$.

Then the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & LC_2(Y) \arrow[r, "\partial"] \arrow[d, "S"] & LC_1(Y) \arrow[r, "\partial"] \arrow[d, "S"] \arrow[dl, "T"'] & LC_0(Y) \arrow[r, "\partial"] \arrow[d, "\mathbb{1}"] \arrow[dl, "T"'] & LC_{-1}(Y) \arrow[r, "\partial"] \arrow[d, "\mathbb{1}"] \arrow[dl, "T"'] & 0 \\
\cdots \arrow[r, "\partial"] & LC_2(Y) \arrow[r, "\partial"] & LC_1(Y) \arrow[r, "\partial"] & LC_0(Y) \arrow[r, "\partial"] & LC_{-1}(Y) \arrow[r, "\partial"] & 0
\end{tikzcd}$$

commutes.

Claim. $\partial T + T\partial = \mathbb{1} - S$.

Proof. When $n = -1$, this is immediate as $T = 0$ and $S = \mathbb{1}$. If $n > 0$, we compute

$$\begin{aligned}
\partial T\lambda &= \partial b_\lambda(\lambda - T\partial\lambda) \\
&= (\mathbb{1} - b_\lambda\partial)(\lambda - T\partial\lambda) \\
&= \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \partial T\partial\lambda) \\
&= \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \underbrace{(\mathbb{1} - S - T\partial)}_{\text{by induction}}\partial\lambda) \\
&= \lambda - T\partial\lambda - b_\lambda(\partial\lambda - \partial\lambda + S\partial\lambda + T\partial\partial\lambda) \\
&= \lambda - T\partial\lambda - b_\lambda S\partial\lambda \\
&= \lambda - T\partial\lambda - S\lambda.
\end{aligned}$$

□

Remark 3.4.16. We extend S and T to maps $C_n(X) \to C_n(X)$ and $C_n(X) \to C_{n+1}(X)$, respectively. Let $\sigma \in C_n(X)$.

(a) Define $S\sigma = \sigma_* S \mathbb{1}_{\Delta^n}$.

(b) Define $T\sigma = \sigma_* T \mathbb{1}_{\Delta^n}$.

Some easy calculations show that S^m is a chain map for any $m \geq 0$ and that T is a chain homotopy between $\mathbb{1}$ and S.

(c) For each $m \geq 1$, define the map $T_m = \sum_{j=0}^{m-1} TS^j$.

Then T_m is a homotopy between $\mathbb{1}$ and S^m. Indeed,

$$\begin{aligned}\partial T_m + T_m \partial &= \partial \sum_{j=0}^{m-1} TS^j + \sum_{j=0}^{m-1} TS^j \partial \\ &= \sum_{j=0}^{m-1} \partial TS^j + TS^j \partial \\ &= \sum_{j=0}^{m-1} \partial TS^j + T\partial S^j \\ &= \sum_{j=0}^{m-1} (\mathbb{1} - S)S^j = \mathbb{1} - S^m.\end{aligned}$$

Lemma 3.4.17 (Lebesgue covering). *Let Z be a compact metric space. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of Z. Then $\exists \delta > 0$ (called a Lebesgue number for $\mathcal{U}$) such that for any set S with diam$(S) < \delta$, S is contained in U_α for some α.*

Note 3.4.18. Let X be any space. Let $\mathcal{U} = \{\mathring{U}_\alpha\}$ be a cover of X. If $\sigma : \Delta^n \to X$, then consider the open cover $\{\sigma^{-1}(U_\alpha)\}$ of Δ^n. Any n-simplex in the m-th iteration of the barycentric subdivision of Δ^n has diameter $d' \leq \left(\frac{n}{n+1}\right)^m d$ where d denotes the diameter of Δ^n. Then d' is less than a Lebesgue number for $\{\sigma^{-1}(U_\alpha)\}$ when m is large enough, in which case $S^m(\sigma) \in C_n^{\mathcal{U}}(X)$.

Define $m(\sigma)$ as the smallest $m \in \mathbb{N}$ such that $S^m(\sigma) = C_n^{\mathcal{U}}(X)$. Define the map $D : C_n(X) \to C_{n+1}(X)$ by $D\sigma = T_{m(\sigma)}\sigma$.

3.5 Lecture 13

Lemma 3.5.1. *Define the function $\rho : C_*(X) \to C_*(X)$ by $\rho = \mathbb{1} - \partial D - D\partial$.*

1. *ρ is a chain map.*
2. *$\operatorname{im}\rho \subset C_*^{\mathcal{U}}(X)$.*
3. *ρ is a chain homotopy inverse to the inclusion map $\iota : C_*^{\mathcal{U}}(X) \to C_*(X)$.*

Proof.

1. We compute

$$\begin{aligned}\partial\rho\sigma &= \partial(\sigma - \partial D\sigma - D\partial\sigma) \\ &= \partial\sigma - \partial D\partial\sigma \\ &= \partial\sigma - (\mathbb{1} - D\partial - \rho)\partial\sigma \\ &= \rho\partial\sigma.\end{aligned}$$

2. We compute

$$\begin{aligned}\rho(\sigma) &= \sigma - \partial D\sigma - D\partial\sigma \\ &= \sigma - \partial T_{m(\sigma)}\sigma - D\partial\sigma \\ &= S^{m(\sigma)}\sigma + T_{m(\sigma)}\partial\sigma - D\partial\sigma\end{aligned}$$

since $\partial T_m + T_m \partial = \mathbb{1} - S^m$. Note that $S^{m(\sigma)}\sigma \in C_n^{\mathcal{U}}$ and that the term $T_{m(\sigma)}\partial\sigma - D\partial\sigma$ is a linear combination of terms $T_{m(\sigma)}(\partial\sigma_j) - T_{m(\sigma_j)}(\sigma_j)$ where each σ_j is the restriction of σ to a face of Δ^n. Since $m(\sigma_j) \leq m(\sigma)$ for each σ_j, it follows that $T_{m(\sigma)}\partial\sigma - D\partial\sigma$ is a linear combination of terms $TS^i(\sigma_j)$ such that $i \geq m(\sigma_j)$. Each of these terms belongs to $C_n^{\mathcal{U}}$ because T preserves small simplices relative to $\mathcal{U}$.

3. We know that $\partial D + D\partial = \mathbb{1} - \iota\rho$. Also, if σ is small relative to $\mathcal{U}$, then $m(\sigma) = 0$, so that $D\sigma = 0$. Thus, $\rho\iota = \mathbb{1}$.

$\square$

Excision theorem. We prove the equivalent statement expressed in Lemma 3.4.2. Write $X = \mathring{A} \cup \mathring{B}$ with $A, B \subset X$. Let $\mathcal{U} = \{A, B\}$. Note that $C_*(A)$ is a subcomplex of $C_*^{\mathcal{U}}(X)$. The diagram

$$\begin{tikzcd} 0 \arrow[r] & C_*(A) \arrow[r] \arrow[d, "\text{q-iso}"] & C_*^{\mathcal{U}}(X) \arrow[r] \arrow[d, "\text{q-iso}"] & C_*^{\mathcal{U}}(X)/C_*(A) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X)/C_*(A) \arrow[r] & 0 \end{tikzcd}$$

commutes. The five lemma implies that the third vertical arrow is a quasi-isomorphism as well. Finally, note that the inclusion map ${C_*(B)}\big/{C_*(A\cap B)} \to {C_*^{\mathcal{U}}(X)}\big/{C_*(A)}$ is a quasi-isomorphism since both the domain and codomain are free groups with basis the singular n-simplices in B that are outside A. $\square$

Corollary 3.5.2 (Mayer-Vietoris (MV) sequence). *Let $X = \mathring{A} \cup \mathring{B}$. Then there exists a LES*

$$\cdots \longrightarrow H_k(A\cap B) \longrightarrow H_k(A) \oplus H_k(B) \longrightarrow H_k(X) \xrightarrow{\partial} H_{k-1}(A\cap B) \longrightarrow \cdots.$$

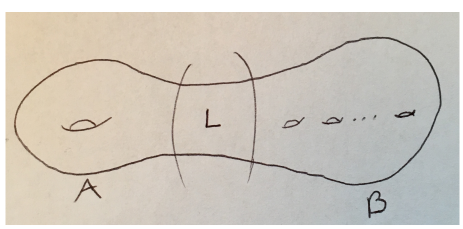
Proof. Consider the sequence

$$0 \longrightarrow C_*(A\cap B) \xrightarrow{\varphi} C_*(A) \oplus C_*(B) \xrightarrow{\psi} C_*(X) \longrightarrow 0$$

where $\varphi : c \mapsto (c, -c)$ and $\psi : (c_1, c_2) \mapsto c_1 + c_2$. It's easy to check that this is exact, inducing our desired LES.

We can describe the boundary map $\partial : H_k(X) \to H_{k-1}(A\cap B)$. Let $[c] \in H_k(X)$. Then $[c] = [x+y]$ where $x : \Delta^k \to A \subset X$ and $y : \Delta^k \to B \subset X$. Since $\partial(x+y) = 0$, we have that $\partial x = -\partial y$. Then $\partial[c] = [\partial x] = -[\partial y]$. $\square$

Example 3.5.3. Depict the orientable surface S_g of genus g as

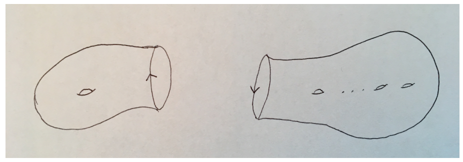


so that $A\cap B$ is a cylinder L of finite height. Then $A \cap B = S^1$, $A \simeq S^1 \vee S^1$, and $B \simeq \underbrace{S^1 \vee \cdots \vee S^1}_{2g-2 \text{ copies}}$.

We have the MV sequence

$$\begin{tikzcd} H_2(A) \oplus H_2(B) \arrow[r] & H_2(S_g) \arrow[r] & H_1(A\cap B) \arrow[r] & H_1(A) \oplus H_1(B) \arrow[r] & H_1(S_g) \arrow[dllll] \\ H_0(A\cap B) \arrow[r] & H_0(A) \oplus H_0(B) \arrow[r] & H_0(S_g) & & \end{tikzcd}.$$

Note that $H_1(A\cap B)$ is generated by the attaching map γ of the 2-cell, which consists of the loop positively traversing $A\cap B$ and the loop negatively traversing $A\cap B$.



Hence γ is homologous to 0. Moreover, both L and S_g are path connected. Hence our previous sequence becomes

$$\begin{tikzcd} 0\oplus 0 \arrow[r] & H_2(S_g) \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z}^2\oplus\mathbb{Z}^{2g-2} \arrow[r] & H_1(S_g) \arrow[llllld, "0"'] \\ \mathbb{Z} \arrow[r, "n\mapsto(n,-n)"'] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "(n,m)\mapsto n+m"'] & \mathbb{Z} \end{tikzcd}.$$

This implies that $H_1(S_g)\cong\mathbb{Z}^{2g}$ and $H_2(S_g)\cong\mathbb{Z}$.

Example 3.5.4. Let X be space and recall that $\Sigma X = {}^{X\times I}\!/\!_{\sim}$ where $(x,0)\sim(x',0)$ and $(x,1)\sim(x',1)$. Let p_0 denote the bottom point of the suspension and p_1 the top point. We can decompose this space as

$$\Sigma X = C_+X\cup_X C_-X$$

where $C_+X := \Sigma X\setminus\{p_0\}$ and $C_-X := \Sigma X\setminus\{p_1\}$. Then both C_+X and C_-X are homotopy equivalent to the cone space CX and thus are contractible. Note that $C_+X\cap C_-X\simeq X$. Consider the MV sequence

$$\begin{tikzcd} H_k(X) \arrow[r] & H_k(C_+X)\oplus H_k(C_-X) \arrow[r] & H_k(\Sigma X) \arrow[lld] \\ H_{k-1}(X) \arrow[r] & H_{k-1}(C_+X)\oplus H_{k-1}(C_-X) \end{tikzcd}.$$

When $k-1>0$, this becomes

$$\begin{tikzcd} H_k(X) \arrow[r] & 0\oplus 0 \arrow[r] & H_k(\Sigma X) \arrow[r] & H_{k-1}(X) \arrow[r] & 0\oplus 0 \end{tikzcd},$$

in which case $H_k(\Sigma X)\cong H_{k-1}(X)$. Moreover, the exact sequence

$$\begin{tikzcd} H_1(C_+X)\oplus H_1(C_-X) \arrow[r] & H_1(\Sigma X) \arrow[ld, "0"'] \\ H_0(X) \arrow[r] & H_0(C_+X)\oplus H_0(C_-X) \arrow[r] & H_0(\Sigma X) \end{tikzcd}$$

shows that $H_1(\Sigma X)=0$.

Theorem 3.5.5 (Brouwer's invariance of domain). *Let $U\subset\mathbb{R}^n$ be open and $V\subset\mathbb{R}^m$ be open. If $U\cong V$, then $n=m$.*

Proof. Let $x\in U$ and consider $H_*(U\setminus\{x\})$. By excision, we get $H_*(U,U\setminus\{x\})\cong H_*(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\})$. By Corollary 3.2.13, we also get a LES

$$\begin{tikzcd} H_k(\mathbb{R}^n\setminus\{x\}) \arrow[r] & H_k(\mathbb{R}^n) \arrow[r] & H_k(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\}) \arrow[r] & H_{k-1}(\mathbb{R}^n\setminus\{x\}) \arrow[r] & H_{k-1}(\mathbb{R}^n) \end{tikzcd}.$$

Let $k\geq 2$. Then $H_k(\mathbb{R}^n,\mathbb{R}^n\setminus\{x\})\cong H_{k-1}(\mathbb{R}^n\setminus\{x\})\cong H_{k-1}(S^{n-1})$. It follows that

$$H_k(U,U\setminus\{x\})\cong\begin{cases}\mathbb{Z} & k=n\\ 0 & k\neq n\end{cases}.$$

If $n>m\geq 1$, then $H_n(U,U\setminus\{x\})\cong\mathbb{Z}$ whereas $H_n(V,V\setminus\{x'\})\cong 0$ for any $x'\in V$. In this case, $U\ncong V$. $\square$

3.6 Lecture 14

Lemma 3.6.1. *Recall that $H_n(D^n, S^{n-1}) \cong H_n(S^n) \cong \mathbb{Z}$. Note that $(D^n, S^{n-1}) \cong (\Delta^n, \partial\Delta^n)$ and that $i_n : \Delta^n \hookrightarrow \Delta^n$ is a cycle in $C_n(\Delta^n, \partial\Delta^n)$. We claim that i_n generates $H_n(\Delta^n, \partial\Delta^n)$.*

Proof. We do induction on n. For the base case, it is obvious that $H_0(\Delta^0, \underbrace{\partial\Delta^0}_{\emptyset})$ is generated by i_0. Let the subspace $\wedge \subset \partial\Delta^n$ consist of all but one of the faces of $\partial\Delta^n$. Consider the triple $(\Delta^n, \partial\Delta^n, \wedge)$. This induces the LES

$$H_n(\partial\Delta^n, \wedge) \longrightarrow H_n(\Delta^n, \wedge) \longrightarrow H_n(\Delta^n, \partial\Delta^n) \longrightarrow H_{n-1}(\partial\Delta^n, \wedge) \longrightarrow H_{n-1}(\Delta^n, \wedge)\ .$$

Since $H_n(\Delta^n, \wedge) = 0 = H_{n-1}(\Delta^n, \wedge)$, it follows that $\alpha : H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\cong} H_{n-1}(\partial\Delta^n, \wedge)$. Moreover, obtain an isomorphism $\beta : H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \xrightarrow{\cong} H_{n-1}(\partial\Delta^n, \wedge)$ from the homeomorphism of pairs $(\Delta^{n-1}, \partial\Delta^{n-1}) \to (\partial\Delta^n, \wedge)$ that maps Δ^{n-1} to the face missing in $\wedge$. The inductive step now holds because $\alpha i_n = \partial i_n = \pm\beta i_{n-1}$. □

Corollary 3.6.2. *Consider the sphere*

$$S^n = D^n \cup_{S^{n-1}} D^n = \Delta^n \cup_{\partial\Delta^n} \Delta^n$$

and the inclusion maps

$$\Delta_1^n : \Delta^n \to \mathit{first}(\Delta^n)$$
$$\Delta_2^n : \Delta^n \to \mathit{second}(\Delta^n).$$

Then $\Delta_1^n - \Delta_2^n$ is a cycle. We claim that $\Delta_1^n - \Delta_2^n$ generates $\widetilde{H}_n(S^n)$.

Proof. We have that $\widetilde{H}_n(S^n) \xrightarrow{\cong} H_n(S^n, \operatorname{im}\partial_2^n) \xleftarrow{\cong} H_n(\operatorname{im}\Delta_1^n, \partial \operatorname{im}\Delta_1^n)$ where $\Delta_1^n - \Delta_2^n \mapsto \Delta_1^n \hookleftarrow i_n$. □

Definition 3.6.3. Let $f : S^n \to S^n$ be a map, inducing the endomorphism $f_* : \widetilde{H}_n(S^n) \cong \mathbb{Z} \to \widetilde{H}_n(S^n) \cong \mathbb{Z}$. Then $f_*(g) = \alpha g$ for some $\alpha \in \mathbb{Z}$. We call such an α the *degree of f*, written as $\deg f$.

Proposition 3.6.4.

1. $\deg \mathbb{1}_{S^n} = 1$.
2. *If $f : S^n \to S^n$ is not surjective, then $\deg f = 0$.*

   Proof. Let $x \in S^n \setminus \operatorname{im} f$. Since $S^n \setminus \{x\} \cong \mathbb{R}^n$, we have that

$$\begin{tikzcd} \widetilde{H}_n(S^n) \arrow[r, "f_*"] \arrow[d] & \widetilde{H}_n(S^n) \\ \underbrace{\widetilde{H}_n(S^n \setminus \{x\})}_{0} \arrow[ur] & \end{tikzcd}$$

   □

3. *If $f \simeq g$, then $\deg f = \deg g$ since $f_* = g_*$.*
4. *By functorality, we get $\deg fg = \deg f \cdot \deg g$.*

   Corollary 3.6.5. *If f is a homotopy equivalence, then $\deg f = \pm 1$.*

5. *Define the map $i_k : S^n \to S^n$ by $(x_0, \dots, x_n) \mapsto (x_0, \dots, -x_k, \dots, x_n)$. Then $\deg i_k = -1$.*

Proof. Recall that $\Delta_1^n - \Delta_2^n$ generates $\widetilde{H}_n(S^n)$. Situate things so that $\partial\Delta_1^n = \partial\Delta_2^n$ is in the hyperplane $x_k = 0$. Then i_k fixes both $\partial\Delta_1^n$ and $\partial\Delta_2^n$ and interchanges the two hemispheres. Hence $i_k(\Delta_1^n - \Delta_2^n) = \Delta_2^n - \Delta_1^n$. $\square$

6. *Let $i : S^n \to S^n$ be the antipodal map. Then $\deg i = (-1)^{n+1}$ since $i = i_0 \circ i_1 \circ \cdots \circ i_n$.*

7. *Suppose that $f : S^n \to S^n$ has no fixed point. Then $\deg f = (-1)^{n+1}$.*

   Proof. If $f(x) \neq x$ for every $x \in S^n$, then the line segment $(1-t)f(x) - tx$ never passes through the origin. Therefore, the maps
   $$H_t = \frac{(1-t)f(x) - tx}{|(1-t)f(x) - tx|}$$
   define a homotopy between f and the antipodal map i. $\square$

Note 3.6.6. Let $f : S^n \to S^n$ and $x \in S^n$ such that $f^{-1}(x) = \{x_1, \ldots, x_m\}$. Let $U_1, \ldots, U_m$ be pairwise disjoint neighborhoods of $x_1, \ldots, x_m$, respectively, where we can find a neighborhood V of x such that each U_i is mapped by f into V. Then $f(U_i \setminus \{x_i\}) \subset V \setminus \{x\}$. For each $i = 1, \ldots, m$, we have a commutative diagram

$$\begin{tikzcd}
 & H_n(U_i, U_i \setminus \{x_i\}) \arrow[r, "f_*"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & H_n(V, V \setminus \{x\}) \arrow[d, "\cong"] \\
H_n(S^n, S^n \setminus \{x_i\}) & H_n(S^n, S^n \setminus \{f^{-1}(x)\}) \arrow[l, "p_i"'] \arrow[r, "f_*"] & H_n(S^n, S^n \setminus \{x\}) \\
 & H_n(S^n) \arrow[r, "f_*"] \arrow[u, "j"] \arrow[ul, "\cong"] & H_n(S^n) \arrow[u, "\cong"]
\end{tikzcd}$$

where the two upper isomorphisms come from excision and the two lower isomorphisms come from the LES of Corollary 3.2.13. Therefore, the homomorphism $f_* : H_n(U_i, U_i \setminus \{x_i\}) \to H_n(V, V \setminus \{x\})$ can be viewed as a homomorphism $f_* : \mathbb{Z} \to \mathbb{Z}$, so that $f_*(g) = \alpha g$ for some $\alpha \in \mathbb{Z}$. We call α the *local degree of f at x*, written as $\deg_{x_i} f$.

Lemma 3.6.7. $\deg f = \sum_{i=1}^m \deg_{x_i} f$.

Proof. By excision, we get

$$H_n(S^n, S^n \setminus f^{-1}(x)) \cong H_n\Big(\coprod_{i=1}^m U_i, \coprod_{i=1}^m U_i \setminus \{x_i\}\Big) \cong \bigoplus_{i=1}^m H_n(U_i, U_i \setminus \{x_i\}).$$

By the naturality of excision, we see that k_i corresponds to the i-th inclusion map and that p_i corresponds to the i-th projection map. By a straightforward diagram chase, we are done. $\square$

3.7 Lecture 15

Lemma 3.7.1. *Let X be a CW-complex with skeleta X^n.*

1. $H_k(X^n, X^{n-1}) = \begin{cases} 0 & k \neq n \\ \mathbb{Z}[n\text{-cells of } X] & k = n \end{cases}.$

2. *$H_k(X^n) = 0$ when $k > n \geq 0$. In particular, $H_k(X) = 0$ when $k > \dim X$.*

3. *The inclusion $i : X^n \to X$ induces an isomorphism $i_* : H_k(X^n) \xrightarrow{\cong} H_k(X)$ when $k < n$.*

Proof.

1. Since (X^n, X^{n-1}) is a good pair, we see that
$$H_k(X^n, X^{n-1}) \cong \widetilde{H}_k(X^n/X^{n-1}) \cong \widetilde{H}_k\left(\bigvee_{n\text{-cells of X}} S^n\right).$$
2. Assume that $k > n$. We have a LES
$$\cdots \longrightarrow \underbrace{H_{k+1}(X^n, X^{n-1})}_{0} \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow \underbrace{H_k(X^n, X^{n-1})}_{0} \longrightarrow \cdots .$$
Hence the map $H_k(X^{n-1}) \to H_k(X^n)$ is an isomorphism. From this we get a chain of isomorphisms
$$H_k(X^0) \xrightarrow{\cong} H_k(X^1) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^{k-1})$$
induced by inclusion. We are done because $H_k(X^0) = 0$.
3. If $k < n$, then the LES from part 2 produces a chain of isomorphisms
$$H_k(X^n) \cong H_k(X^{n+1}) \cong H_k(X^{n+2}) \cong \cdots .$$
Any $c \in C_k(X)$ has compact image and thus intersects at most finitely many cells of X. Thus, $[c] \in H_k(X^m)$ for some $m > k$. By our chain of isomorphisms, it follows that $[c] = [\tilde{c}]$ for some $[\tilde{c}] \in H_k(X^n)$. This proves that i_* is surjective. Moreover, if $[ic] = 0$ in $H_k(X)$, then there exists $m > n$ such that $c = \partial\tilde{c}$ for some chain $\tilde{c}$ in X^m. By out chain of isomorphism, it follows that $[c] = 0$ in $H_k(X^n)$. This proves that i_* is surjective.

□

Definition 3.7.2 (Cellular homology). Let X be a CW-complex. Consider the three pairs (X^{n+1}, X^n), (X^n, X^{n-1}), and (X^{n-1}, X^{n-2}). We have a commutative diagram

$$\begin{tikzcd}
 & 0 & & & \\
0 \arrow[dr] & H_n(X^{n+1}) \cong H_n(X) \arrow[u] & & & \\
 & H_n(X^n) \arrow[u] \arrow[dr, "j_n"] & & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1}, X^n) \arrow[u, "\partial_{n+1}"] \arrow[r, "d_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[r, "d_n"] \arrow[dr, "\partial_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
 & \vdots \arrow[u] & & H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"'] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

where $d_n \coloneqq j_{n-1}\partial_n$, called a *cellular boundary map*. It is clear that $d^2 = 0$, so that the horizontal row is a chain complex $(H_n(X^n, X^{n-1}), d_n)$. This gives rise to the *cellular homology of* X, written as $H_*^{CW}(X)$.

Theorem 3.7.3. *For any CW-complex X, $H_n^{CW}(X) \cong H_n(X)$.*

Proof. Note that $H_n(X) \cong H_n(X^n)/\operatorname{im}\partial_{n+1}$. Since j_n is injective,
$$\operatorname{im} d_{n+1} = \operatorname{im} j_n\partial_{n+1} = j_n(\operatorname{im}\partial_{n+1}) \cong \operatorname{im}\partial_{n+1}$$

. Since j_{n-1} is injective as well, we get

$$H_n(X^n) \cong j_n(H_n(X^n)) = \operatorname{im} j_n = \ker \partial_n = \ker d_n.$$

Thus, $j_n : H_n(X^n) \xrightarrow{\cong} \ker d_n$ such that $j_n(\operatorname{im} \partial_{n+1}) = \operatorname{im} d_{n+1}$, which implies that

$$H_n(X^n) / \operatorname{im} \partial_{n+1} \cong \ker d_n / \operatorname{im} d_{n+1} = H_n^{CW}(X).$$

□

Corollary 3.7.4. *If X is connected and contains only one 0-cell, then $d_1 : H_1(X^1, X^0) \to H_0(X^0)$ is the zero map.*

Corollary 3.7.5. *If X is a CW-complex with no n-cells, then $H_n(X) = 0$.*

Example 3.7.6. Recall that $\mathbb{CP}^n = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}$. Hence each map $d_i = 0$, so that $H_k^{CW}(\mathbb{CP}^n) \cong H_k(X^k, X^{k-1})$. Therefore,

$$H_k(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & k \text{ even and } k \leq 2n \\ 0 & \text{otherwise} \end{cases}.$$

Proposition 3.7.7 (Cellular boundary formula). *Let X be a CW-complex and $n > 1$. Let $d_{\alpha\beta}$ denote the degree of the composition*

$$S_\alpha^{n-1} \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q_\beta q} S_\beta^{n-1}$$

where φ_α denotes the attaching map of e_α^n, $q : X^{n-1} \to X^{n-1} / X^{n-2}$ denotes the quotient map, and $q_\beta : X^{n-1} / X^{n-2} \to S_\beta^{n-1}$ denotes the map collapsing $X^{n-1} \setminus e_\beta^{n-1}$ to a point. Then

$$d_n(e_\alpha^n) = \sum_\beta d_{\alpha\beta} e_\beta^{n-1}.$$

Note that this summation contains only finitely many terms since φ_α has compact image and thus intersects only finitely cells e_β^{n-1}.

Proof. Let $\Delta_{\alpha\beta} \coloneqq q_\beta q \varphi_\alpha$. Then we have a commutative diagram

$$\begin{tikzcd}
H_n(D_\alpha^n, \partial D_\alpha^n) \arrow[r, "\partial"] \arrow[d, "\Phi_{\alpha *}"] & \widetilde{H}_{n-1}(\partial D_\alpha^n) \arrow[r, "\Delta_{\alpha\beta *}"] \arrow[d, "\varphi_{\alpha *}"] & \widetilde{H}_{n-1}(S_\beta^{n-1}) \\
H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) \arrow[u, "q_{\beta *}"'] \arrow[d, "\cong"] \\
 & H_{n-1}(X^{n-1}, X^{n-2}) \arrow[r, "\cong"] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2})
\end{tikzcd}.$$

The map $\Phi_{\alpha\beta *}$ sends any generator $[D_\alpha^n]$ to the generator e_α^n. From this we se that

$$d_n(e_\alpha^n) = j_{n-1} \varphi_{\alpha *} \partial [D_\alpha^n].$$

Also, the map $q_{\beta *}$ is precisely the projection map onto the copy of $\mathbb{Z}$ corresponding to the basis element e_β^{n-1}. A simple diagram chase yields our desired formula. □

Example 3.7.8.

1. The closed orientable surface S_g of genus g has one 0-cell, $2g$ 1-cells, and one 2-cell. Thus, we get the cellular chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \longrightarrow 0 .$$

We know that $d_1 = 0$ because S_g is connected and has exactly one 0-cell. Moreover, the maps $\Delta_{\alpha\beta}$ are homotopic to constant maps, which implies that $d_2 = 0$. It follows that

$$H_n(S_g) \cong \begin{cases} \mathbb{Z} & n \in \{0,2\} \\ \mathbb{Z}^{2g} & n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

2. Recall that $\mathbb{RP}^n = e^0 \cup e^1 \cup \cdots \cup e^n$ with attaching maps the two-sheeted covering projections $\varphi : S^{k-1} \to \mathbb{RP}^{k-1}$. If $q : \mathbb{RP}^{k-1} \to \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} = S^{k-1}$ denotes the quotient map, then the composition $q\varphi$ is a homeomorphism when restricted to each of the two components of $S^{k-1} \setminus S^{k-2}$, one being the identity and the other being the antipodal map. In particular, these two homeomorphisms are obtained from each other by precomposing with the antipodal map $S^{k-1} \to S^{k-1}$, which has degree $(-1)^k$. Therefore, $\deg q\varphi = \deg(\mathbb{1}) + \deg(-\mathbb{1}) = 1 + (-1)^k$ by our local-degree formula. It follows that

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \qquad n \text{ even}$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0 \qquad n \text{ odd.}$$

Hence

$$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \text{ odd} \\ \mathbb{Z}/2 & 0 < k < n,\ k \text{ odd} \\ 0 & \text{otherwise} \end{cases}.$$

3. Let $m > 1$ be an integer and $l_1, \ldots, l_n$ be relatively prime to m. Let ρ be the action of $\mathbb{Z}/m$ on $S^{2n-1} \subset \mathbb{C}$ generated by the homeomorphism

$$(z_1, \ldots, z_n) \mapsto (e^{\frac{2\pi i l_1}{m}} z_1, \ldots, e^{\frac{2\pi i l_n}{m}} z_n).$$

The orbit space $S^{2n-1}/\mathbb{Z}_m$ is called the *lens space* $L \coloneqq L_m(l_1, \ldots, l_n)$. Note that the projection $S^{2n-1} \to L$ is a covering space since ρ is free.

Definition 3.7.9.

1. Let G be a finitely generated abelian group. We can write G uniquely as $\mathbb{Z}^m \times T_1 \times \cdots \times T_s$ where each T_i is a finite cyclic group. Define $\operatorname{rank} G = m$.

2. Let X be a space whose singular chain complex is finite. The *Euler characteristic* $\chi(X)$ *of* X is $\sum_n (-1)^n \operatorname{rank} H_n(X)$, which is a finite sum by Corollary 3.7.4.

Note 3.7.10. Let (C_*, d) be a chain complex of finitely generated abelian groups. The induced homology groups H_* are finitely generated abelian groups as well.

Exercise 3.7.11. *If* $0 \to A \to B \to C \to 0$ *is an exact sequence of finitely generated abelian groups, then* $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$.

Lemma 3.7.12. *Let* $(C_*(X), d)$ *be a finite singular chain complex. Then* $\chi(X) = \sum_n (-1)^n \operatorname{rank} C_n$.

Proof. We can write out chain complex as

$$0 \longrightarrow C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0\ .$$

We have short exact sequences $0 \to Z_n \to C_n \to B_{n-1} \to 0$ and $0 \to B_n \to Z_n \to H_n \to 0$. This shows that $\operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}$ and $\operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n$. It follows that

$$\operatorname{rank} C_n = \operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1}.$$

Hence $\sum_n (-1)^n \operatorname{rank} C_n(X) = \sum_n (-1)^n \operatorname{rank} H_n(X)$. $\square$

Corollary 3.7.13. *Let* X *be a finite CW-complex. Then*

$$\chi(X) = \sum_n (-1)^n c_n$$

where c_n *denotes the number of* n*-cells of* X.

3.8 Lecture 16

Theorem 3.8.1. *Let X be a path connected space. There exists a surjective map $h : \pi_1(X) \to H_1(X)$ with* $\ker h = [\pi_1(X), \pi_1(X)]$. *In this case,*

$$\pi_1(X)_{ab} \cong H_1(X).$$

Proof. Define $h : \pi_1(X) \to H_1(X)$ by $\gamma \mapsto \sigma_\gamma$ where $\sigma_\gamma(t) = \gamma(t)$. $\square$

Example 3.8.2. We get surjective maps $\pi_1(S^1 \vee S^1) \cong \mathbb{F}_2 \to H_1(S^1 \vee S^1) \cong \mathbb{Z}^2$ and $\pi_1(S_g) \to H_1(S_g) \cong \mathbb{Z}^{2g}$.

Definition 3.8.3. Let R be a (unital) ring. Let M be an R-module and N be a right R-module. The *tensor product* $(N \otimes_R M, \psi)$ consists of an R-module $N \otimes_R M$ and a R-bilinear map $\psi : N \times M \to N \otimes_R M$ such that for any R-bilinear map $f : N \times M \to F$, there is a unique R-linear map $g : N \otimes_R M \to F$ such that $g\psi = f$.

Proposition 3.8.4.

1. $N \otimes_R R \cong N$.
2. $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{Z}^m \cong \mathbb{Z}^{nm}$.
3. $\mathbb{Z}/n \otimes_{\mathbb{Z}} \mathbb{Z}/m \cong \mathbb{Z}/(n,m)$.

Definition 3.8.5 (Homology with coefficients). Let G be an abelian group and X be a space. Define $(C_*(X;G), \partial)$ as $(C_* \otimes_{\mathbb{Z}} G, \partial \otimes \mathbb{1})$ and $H_*(X;G)$ as $H_*(C_*(X;G))$.

Note 3.8.6. It it *not* true that if C_* is a chain complex, then $H_*(C_* \otimes G) \cong H_*(C_*) \otimes G$.

Remark 3.8.7.

1. Let M be an R-module. Consider the functor $N \mapsto N \otimes_R M$ from $R^{\mathrm{op}}-\mathbf{Mod}$ to $\mathbf{Ab}$. This is right exact and thus has *left derived functors* denoted by $\mathrm{Tor}_i^R(-, M)$. Specifically, from any projective resolution $\cdots \to P^2 \to P^2 \to P^0 \to N \to 0$ in $R^{\mathrm{op}}-\mathbf{Mod}$, construct C_* the chain complex

$$\cdots \to P^2 \otimes_R M \to P^1 \otimes_R M \to P^0 \otimes_R M \to 0.$$

   Then $\mathrm{Tor}_i^R(N, M) = H_i(C_*)$.

   Note 3.8.8. The definition of Tor_i^R is independent of the choice of projective resolution.

2. Consider a functor F from $\mathbf{Top}$. It may be that FX depends on more than the homotopy type of X. To "extend" F to the homotopy category, we replace X by an equivalent cofibrant space or CW-complex. Since projective modules are precisely the cofibrant spaces in $\mathbf{Mod}$, we see that

$$\cdots \to P^1 \to P^0$$

   is a *cofibrant replacement* of N. This is why we remove N when constructing our new chain complex C_* above.

Example 3.8.9.

1. We have a free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ of $\mathbb{Z}/n$ over $\mathbb{Z}$, from which we form the complex

$$\mathbb{Z} \otimes \mathbb{Z} \xrightarrow{n \otimes \mathbb{1}} \mathbb{Z} \otimes \mathbb{Z} \to 0.$$

   This is precisely $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0$. Hence

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & i = 0 \\ 0 & i > 0 \end{cases}.$$

2. We have a trivial free resolution $0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ of $\mathbb{Z}$ over $\mathbb{Z}$. From this we form the complex

$$0 \to \mathbb{Z} \otimes \mathbb{Z}/n \to 0,$$

which becomes $0 \to \mathbb{Z}/n \to 0$. Hence

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n) \cong \begin{cases} \mathbb{Z}/n & i = 0 \\ 0 & i > 0 \end{cases}.$$

3. We have a free resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n \to 0$ of $\mathbb{Z}/n$ over $\mathbb{Z}$, from which we form the complex

$$\mathbb{Z} \otimes \mathbb{Z}/m \xrightarrow{n \otimes \mathbb{1}} \mathbb{Z} \otimes \mathbb{Z}/m \to 0.$$

This is precisely $\mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m \to 0$. Hence

$$\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/n, \mathbb{Z}/m) \cong \begin{cases} \ker(\mathbb{Z}/m \xrightarrow{n} \mathbb{Z}/m) \cong \mathbb{Z}/(n,m) & i = 1 \\ 0 & i \neq 1 \end{cases}.$$

Remark 3.8.10. Suppose that (C_*, ∂) is a chain complex and that $0 \to P^1 \to P^0 \to H \to 0$ is a projective resolution. For each $n \in \mathbb{N}$, we get a (non-canonical) split short exact sequence

$$0 \to H_n(X) \otimes G \to H_n(X; G) \to \mathrm{Tor}_1(H_{n-1}(X), G) \to 0,$$

called the *universal coefficient sequence (for homology)*. Therefore,

$$H_n(X; G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}_1(H_{n-1}(X), G).$$

Example 3.8.11. Let $X = \mathbb{RP}^n$ and $G = \mathbb{Z}/2$. By the universal coefficient sequence, it is straightforward to show that

$$H_n(\mathbb{RP}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$$

for every n. For example, we have that

$$\begin{aligned} H_1(\mathbb{RP}^n, \mathbb{Z}/2) \cong H_1(\mathbb{RP}^n; \mathbb{Z}/2) &\cong (H_1(X) \otimes \mathbb{Z}/2) \oplus \mathrm{Tor}_1(H_0(\mathbb{RP}^n), \mathbb{Z}/2) \\ &\cong (\mathbb{Z}/2 \otimes \mathbb{Z}/2) \oplus \mathrm{Tor}_1(\mathbb{Z}, \mathbb{Z}/2) \\ &\cong \mathbb{Z}/2 \oplus 0 \\ &\cong \mathbb{Z}/2. \end{aligned}$$

3.9 Lecture 17

Definition 3.9.1. We say that a space M is an *n-dimensional manifold* if it is Hausdorff and for any $x \in M$, there exist an open set $U \ni x$ and a homeomorphism $U \to \mathbb{R}^n$. In this case, we call U a *coordinate ball around* x.

Definition 3.9.2. Let X be an n-manifold and $x \in X$. Let U be a coordinate ball around x. Then

$$\begin{aligned} H_k(X, X \setminus x) &\cong H_k(U, U \setminus x) \\ &\cong \widetilde{H}_{k-1}(U \setminus x) \\ &\cong \widetilde{H}_{k-1}(S^{n-1}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z} & k = n \end{cases}. \end{aligned}$$

There are precisely two generators of $\widetilde{H}_n(U, U \setminus x)$, a choice α_x of which is called a *local orientation of X at x*.

Lemma 3.9.3. *Given $\alpha_x \in H_n(X, X \setminus x)$, there exist a neighborhood U of x and $\alpha_U \in H_n(X, X \setminus U)$ such that $j_x^U(\alpha_U) = \alpha_x$ where $j_x^U : H_n(X, X \setminus U) \to H_n(X, X \setminus x)$.*

Proof. Let c be a relative cycle representing α_x. Then $\operatorname{supp} \partial c \subset X \setminus x$. Let $U = X \setminus \operatorname{supp} \partial c$ and $\alpha_U = [c] \in H_n(X, X \setminus U)$. □

Lemma 3.9.4. *Every neighborhood W of x contains some neighborhood U of x such that for each $y \in U$, $j_y^U : H_n(X, X \setminus U) \to H_n(X, X \setminus y)$ is an isomorphism.*

Proof. Let V be a coordinate neighborhood of x that is contained in W. Then there is a homeomorphism $\psi : V \to \mathbb{R}^n$. Let U be a unit ball via ψ. We get a commutative diagram

$$\begin{tikzcd}
H_n(X, X \setminus U) \arrow[d, "j_y^U"'] & H_n(V, V \setminus U) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \widetilde{H}_{n-1}(V \setminus U) \arrow[d, "\cong"] \\
H_n(X, X \setminus y) & H_n(V, V \setminus y) \arrow[l, "\cong"] \arrow[r, "\cong"'] & \widetilde{H}_{n-1}(V \setminus y)
\end{tikzcd},$$

which shows that j_y^U is an isomorphism for any $y \in U$. □

Definition 3.9.5. An n-manifold X is *orientable* if for any $x \in X$, we can choose a local orientation $\alpha_x \in H_n(X, X \setminus x)$ such that for any $x \in X$, we can find an open set $U \ni x$ and $\alpha_U \in H_n(X, X \setminus U)$ such that $j_y^U(\alpha_U) = \alpha_y$ for every $y \in U$. We call such a choice of local orientations *locally compatible*. If we specify the local orientations α_x, then we say that X is *oriented*.

Remark 3.9.6. Let X be an n-manifold. Any inclusion $U \subset V$ of open sets induces a map $H_n(X, X \setminus V) \to H_n(X, X \setminus U)$. Thus,

$$U \mapsto H_n(X, X \setminus U)$$

defines a presheaf $\mathcal{O}_n$ on X, called the *orientation sheaf of* X. By Lemma 3.9.3 and Lemma 3.9.4 along with excision, this is a *locally constant sheaf* in that for any $x \in X$, there is some open set $V \ni x$ such that $U \subset V \mapsto H_n(X, X \setminus U)$ is constant.

Definition 3.9.7. Let $x \in X$. The *stalk of $\mathcal{O}_n$ at x* is

$$\varinjlim_{U \ni x} H_n(X, X \setminus U).$$

Note 3.9.8. The stalk of $\mathcal{O}_n$ at x is isomorphic to $H_n(X, X \setminus x) \cong \mathbb{Z}$.

Note 3.9.9. Let X be an n-manifold.

1. Let $X^{or} \coloneqq \{(x, \alpha) \mid x \in X,\ \alpha \in H_n(X, X \setminus x) = \langle \alpha \rangle\}$. Topologize this by letting a basic neighborhood of (x, α) look like

   $$\widetilde{U} \coloneqq \{(y, j_y^U(\alpha_U)) \mid y \in U\}$$

   where U is a neighborhood of x such that j_y^U is an isomorphism for each $y \in U$.

   The projection map $p : X^{or} \to X$ is two-to-one and thus a two-fold covering of X, called the *orientation cover*. A (continuous) section $o : X \to X^{or}$ of p is an orientation of X.

   Note that X^{or} has a canonical orientation $(x, \alpha_x) \mapsto \alpha_x$ since α_x generates

   $$H_n(X^{or}, X^{or} \setminus (x, \alpha_x)) \cong H_n\left(\widetilde{U}, \widetilde{U} \setminus (x, \alpha_x)\right) \cong H_n(X, X \setminus x).$$

Proposition 3.9.10. *If X is connected, then X is orientable if and only if X^{or} has precisely two components.*

Corollary 3.9.11. *If X is simply connected, then X is orientable.*

Proof. If X is simply connected, then it has no subgroup of index 2. Since every nontrivial two-sheeted covering space of X corresponds to an index-2 subgroup of $\pi_1(X)$, it follows that p is trivial. Thus, $X^{or} \cong X \coprod X$. $\square$

2. Let $X^{Zor} := \{(x, \alpha) \mid x \in X,\ \alpha \in H_n(X, X \setminus x)\}$. Topologize this by letting a basic neighborhood of (x, α) look like

$$\widetilde{U} := \{(y, j_y^U(\alpha_U)) \mid y \in U\}$$

where U is a basic neighborhood of x but j_y^U need not be an isomorphism.

The projection $p : X^{Zor} \to X$ is an $\aleph_0$-sheeted covering projection, with each sheet corresponding to an element of $\mathbb{Z}$. Let $\Gamma(A)$ denote the space of sections $s : A \to X^{Zor}$ over an open subset $A \subset X$. Note that $\Gamma(A)$ inherits an abelian group operation from $\mathbb{Z}$.

Theorem 3.9.12. *Let X be an n-manifold and $A \subset X$ be compact.*

(a) $H_q(X, X \setminus A) = 0$ when $q > n$.

(b) Define the group homomorphism $j_A : H_n(X, X \setminus A) \to \Gamma(A)$ by $\alpha \mapsto (x \mapsto \alpha_x)$ where $\alpha_x = j_x^A(\alpha)$. This is an isomorphism.

Proof. First of all, note that the case where $A = \emptyset$ is obvious. There are four other cases to consider.

Step 1: Suppose that our theorem is true of the compact sets A, B, and $A \cap B$ in X. Then our theorem is true from $A \cap B$.

Proof. Consider the MV sequence

$$H_{q+1}(X, X \setminus (A \cap B)) \to H_q(X, X \setminus (A \cup B)) \to H_q(X, X \setminus A) \oplus H_q(X, X \setminus B) \to H_q(X, X \setminus (A \cap B)).$$

If $q > n$, then our hypotheses immediately imply that $H_q(X, X \setminus (A \cup B)) = 0$. This verifies part (a). For part (b), we can extend our last sequence to a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H_n(X, X \setminus (A \cup B)) \arrow[r] \arrow[d, "j_{A\cup B}"] & H_n(X, X \setminus A) \oplus H_n(X, X \setminus B) \arrow[r] \arrow[d, "j_A \oplus j_B (\cong)"] & H_n(X, X \setminus (A \cap B)) \arrow[d, "j_{A\cap B} (\cong)"] \\
0 \arrow[r] & \Gamma(A \cup B) \arrow[r] & \Gamma(A) \oplus \Gamma(B) \arrow[r] & \Gamma(A \cap B)
\end{tikzcd}.$$

By the five lemma, it follows that $j_{A \cup B}$ is an isomorphism. $\square$

Step 2: Suppose that A is contained in a coordinate chart U evenly covered by $p : X^{Zor} \to X$. Also, suppose that we can write A as a finite union of parallelepipeds $A_1, A_2, \ldots, A_m$ such that each face of A_i is parallel to some coordinate axis. Then our theorem holds for A.

Proof. We apply induction on m. If $m = 1$, it's enough to observe that

$$\begin{aligned} H_q(X, X \setminus A) &\cong H_q(U, U \setminus A) \\ &\cong \widetilde{H}_{q-1}(U \setminus A) \cong \widetilde{H}_{q-1}(S^{n-1}). \end{aligned}$$

Suppose, inductively, that our theorem is true of $m \in \mathbb{N}$. Let $B = A_1 \cup A_2 \cup \cdots \cup A_m$. By our IH, our theorem holds for B and A_{m+1}. Further,

$$B \cap A_{m+1} = (A_1 \cap A_{m+1}) \cup \cdots \cup (A_m \cap A_{m+1}).$$

Each factor of this union is a parallelepiped where each face is parallel to some coordinate axis. By induction, it follows that our theorem holds for $B \cap A_{m+1}$. By Step 1, it holds for $B \cup A_{m+1}$ as well. $\square$

Step 3: Suppose that A is contained in a coordinate chart U evenly covered by $p : X^{Zor} \to X$. Then our theorem holds for A.

Proof. Let $s \in \Gamma(A)$. Without loss of generality, we may assume that $\operatorname{im} s$ is contained in a single sheet over U. Thus, s extends to some $s^* \in \Gamma(U)$.

For each $x \in A$, find some parallelepiped $P_x \subset U$ with $x \in \operatorname{Int} P_x$ such that each wall of P_x is parallel to some coordinate axis. Let $A' = \bigcup_{x \in A} P_x$. Since A is compact, we may write $A' = P_{x_1} \cup \cdots \cup P_{x_k}$. We have a commutative square

$$\begin{tikzcd} H_n(X, X \setminus A') \arrow[r, "j_{A'}"] \arrow[d] & \Gamma(A') \arrow[d] \\ H_n(X, X \setminus A) \arrow[r, "j_A"] & \Gamma(A) \end{tikzcd}.$$

Note that the right arrow is surjective. By Step 2, we have that $j_{A'}$ is an isomorphism. It follows that j_A is surjective.

Let $c \in H_q(X.X \setminus A)$ with $q \geq n$. If $q = n$, suppose that $j_A(c) = 0$. We must show that $c = 0$. Let z denote a relative cycle representing c, so that $\partial z \subset X \setminus A$. Then $V := X \;\; \operatorname{cl}(\partial z)$ is an open set containing A. Let $c' := [z]$ in $H_q(X, X \setminus V)$. If $q = n$, then $j_x^V(c') = j_x^A(c) = 0$ for each $x \in A$.. Hence there exists an open set $A \subset V' \subset V$ such that $j_x^V(c') = 0$ for each $x \in V'$. Now, construct A' as before, so that $j_{A'}(c') = 0$ by Step 2. It follows that $c = j_A^{A'}(j_{A'}^V(c')) = 0$. ☐

Step 4: Suppose that A is compact. Then our theorem holds for A.

Proof. Note that A can be written as a union of coordinate charts $U_1, U_2, \ldots, U_m$ that are evenly covered by p. Thus, we can apply induction on m. Our base case follows directly from Step 3, and our inductive step follows by a similar argument to Step 2. ☐

☐

Corollary 3.9.13. *Let X be an n-manifold with $A \subset X$ compact. Let $x \mapsto \alpha_x \in H_n(X, X \setminus x; R)$ be an R-orientation on X. Then there exists an element $\alpha_A \in H_n(M, M \setminus A; R)$ whose image in $H_n(X, X \setminus x; R)$ equals α_x for every $x \in A$.*

Corollary 3.9.14. *If X is a closed n-manifold, then $H_q(X) = 0$ for any $q > n$.*

Corollary 3.9.15. *If X is a closed connected orientable manifold, then*

$$H_n(X) \cong \Gamma(X) \cong \mathbb{Z}.$$

In this case, we call a generator $[X]$ of $H_n(X)$ a fundamental or orientation class for X. Note that $[X]$ is precisely a section of $X^{or} \to X$ and thus determines an orientation of X. We denote the opposite orientation to this by $-[X]$.

Note 3.9.16. If X is not orientable, then $H_n(X) = 0$.

Remark 3.9.17. Let $M \subset X$ be a closed oriented n-submanifold. Then $i_*[M] \in H_n(X)$. But it is not true that any homology class in $H_n(X)$ can be represented by the fundamental class for a submanifold.

Definition 3.9.18. Let X be any space. Define the *open cone on X* as

$$C^0 X = {}^{[0,1) \times X}\!\Big/_{(x,0) \sim (x',0)}.$$

Definition 3.9.19. Let X be a paracompact Hausdorff space. Let

$$X = X_n \supset X_{n-1} \supset X_{n-2} \supset \cdots \supset X_1 \supset X_0 \supset X_{-1} = \emptyset$$

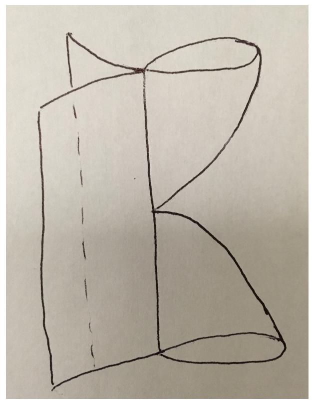
be a filtration of X such that for any $x \in X_i \setminus X_{i-1}$, there exist

- a neighborhood N of x,
- a compact $(n-i-1)$-dimensional stratified space L

  $$L = L_{n-i-1} \supset L_{n-i-2} \supset \cdots \supset L_0,$$

  and
- a homeomorphism $\varphi : \mathbb{R}^i \times C^0 L \to N$ that restricts to a homeomorphism $R^i \times C^0 L_j \to N \cap X_{i+j+1}$.

We call $S_i \coloneqq X_i \setminus X_{i-1}$ the *i-dimensional stratum of* X and L a *link* of this stratum.



Note 3.9.20. The i-dimensional stratum of X is an i-manifold.

Example 3.9.21.

1. Complex algebraic varieties.
2. Complex analytic varieties.
3. Real algebraic varietes.
4. Real analytic varieties.
5. Real semi-algebraic varieties.
6. Subanalytic spaces.
7. Triangulated spaces.

Definition 3.9.22. An n-dimensional stratified space is called a *pseudomanifold* if $S_{n-1} = \emptyset$.

Example 3.9.23.

1. Complex algebraic varieties.
2. Complex analytic varieties.
3. Triangulated spaces such that any $(n-1)$-simplex is the face of exactly two n-simplices.

Remark 3.9.24. Every homology class can be represented by a pseudomanifold.

3.10 Lecture 18

Definition 3.10.1. Let X be an n-manifold. Let R be a commutative ring. Note that $H_n(X, X \setminus x; R) \cong R$ for any $x \in X$. An *R-orientation of* X is a locally compatible mapping $x \mapsto u_x$ such that $(u_x) = H_n(X, X \setminus x; R)$, i.e., u_x is a unit.

Example 3.10.2. Let $R = \mathbb{Z}/2$. Then there is exactly one choice of generator of $H_n(X, X \setminus x; R)$. It's easy to show that the orientation cover of X must be X itself. This implies that any manifold has a $\mathbb{Z}/2$-orientation. It follows that $H_n(X; \mathbb{Z}/2) \cong \mathbb{Z}/2$ when X is closed and connected.

4 Cohomology

Definition 4.0.1. Let (C_*, ∂) be a chain complex of free abelian groups and G be an abelian group. Let

$$C_G^n \coloneqq \mathrm{Hom}_{\mathbb{Z}}(C_n, G).$$

Define the *coboundary map* $\delta : C_G^n \to C_G^{n+1}$ by $(\varphi : C_n \to G) \mapsto (c \mapsto \varphi(\partial c))$. Then $\delta^2 = 0$. so that we get a *cochain complex*

$$C_G^0 \xrightarrow{\delta} C_G^1 \xrightarrow{\delta} C_G^2 \xrightarrow{\delta} \cdots \xrightarrow{\delta} C_G^k \xrightarrow{\delta} C_G^{k+1} \xrightarrow{\delta} \cdots ..$$

Define the group of *k-cocycles* as

$$Z_G^k = \ker(\delta : C_G^k \to C_G^{k+1})$$

and the group of *k-coboundaries* as

$$B_G^k = \mathrm{im}(\delta : C_G^{k-1} \to C_G^k).$$

The *k-th cohomology group of C (with coefficients in G)* is

$$H^k(C; G) = {Z_G^k}/{B_G^k}.$$

Note 4.0.2.

1. Let $\varphi \in Z_G^n$, so that $\delta\varphi = 0$. Then $\varphi\partial = 0$, which means that $B_n \subset \ker\varphi$. This induces a map $\tilde{\varphi} : H_n(C) \to G$. We thus have an additive map $h : \varphi \mapsto \tilde{\varphi}$.

   To see that h is well-defined, suppose that $\varphi = \delta\psi$ for some $\psi : C_{n-1} \to G$. Then $\varphi = \psi\partial$, which implies that φ vanishes on Z_n. Hence $\tilde{\varphi} = 0$. This proves that $h : H^n(C; G) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(C); G)$ is well-defined.

2. Let A and B be abelian groups. There is a surjective map $f : F_0 \to A$ where F_0 is a free abelian group. Form a free resolution F of A

   $$0 \to F_1 \xrightarrow{i} F_0 \xrightarrow{f} A \to 0$$

   where $F_1 \coloneqq \ker f$. Consider the induced map $i^* : \mathrm{Hom}(F_0, B) \to \mathrm{Hom}(F_1, B)$. Let

   $$\mathrm{Ext}(A, B) \coloneqq H^1(F; B) = \mathrm{coker}\, i^*,$$

   which is independent of our choice of F.

Example 4.0.3.

1. If A is free, then $\mathrm{Ext}(A, B) = 0$.

2. If $A = \mathbb{Z}/n$, then consider the free resolution $0 \to \mathbb{Z} \xrightarrow{n\cdot -} \mathbb{Z} \to \mathbb{Z} \to 0$. This induces a map

   $$\mathrm{Hom}(\mathbb{Z}, B) \cong B \xrightarrow{n\cdot -} \mathrm{Hom}(\mathbb{Z}, B) \cong B,$$

   so that $\mathrm{Ext}(\mathbb{Z}/n, B) \cong B/nB$.

Theorem 4.0.4 (Universal coefficient theorem (for cohomology)). *There exists a non-canonical split short exact sequence*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C), G) \longrightarrow H^n(C; G) \xrightarrow{h} \mathrm{Hom}(H_n(C), G) \longrightarrow 0 .$$

Note 4.0.5.

1. Let X be a space and $C_*(X)$ our singular chain complex. Then $C^n(X; G) = \mathrm{Hom}(C_n(X), G)$. For any *singular n-cochain* φ, we have that

   $$\delta\varphi(\sigma : \Delta^{n+1} \to X) = \sum_{k=0}^{n+1} (-1)^k \varphi(\sigma \restriction_{[v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}]}).$$

2. Any map $f : X \to Y$ of spaces induces a chain map $f_* : C_*(X) \to C_*(Y)$, which in turn induces a cochain map $f^* : C^*(Y;G) \to C^*(X;G)$ such that $\delta f = f\delta$. We thus get a map on cohomology $f^* : H^n(Y;G) \to H^n(X;G)$, called the *pullback of f*.

3. **(Reduced cohomology)** Define the map $f : G \to C^n(X;G)$ by $g \mapsto c_g$ where c_g denotes the constant map at g. Let $\widetilde{C}^*(X;G) \coloneqq \operatorname{coker} f$, thereby defining $\widetilde{H}^*(X;G)$.

   Then $\widetilde{H}^n(X;G) = H^n(X;G)$ when $n > 0$. Note that $C^0(X;G)$ is precisely the group of functions $X \to G$. For any function $\varphi : X \to G$, we have that $\delta\varphi(\sigma) = \varphi(\sigma(1)) - \varphi(\sigma(0))$ for any $\sigma \in \operatorname{Sing}_1(X)$. This shows that φ is a cocycle if and only if it is constant on each path component of X. Thus,

$$H^0(X;G) \cong \prod_{\pi_0(X)} G.$$

   It follows that $\widetilde{H}^0(X;G)$ is precisely the group of all functions $X \to G$ constant on each path component of X modulo the group of globally constant functions $X \to G$.

4. **(Relative cohomology)** Let (X,A) be a pair. Note that $C_n(X,A)$ is isomorphic to the free abelian group with basis $\{\varphi : \Delta^n \to X \mid \operatorname{im}\varphi \subset X \setminus A\}$. Applying $\operatorname{Hom}(-,G)$ to the short exact sequence $\eta : 0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$ yields the sequence

$$0 \longleftarrow C^n(A;G) \longleftarrow C^n(X;G) \longleftarrow C^n(X,A;G) \longleftarrow 0.$$

   This is exact because η splits. Thus,

$$C^n(X,A;G) \cong \{\varphi : \operatorname{Sing}_n(X) \to G \mid \sigma : \Delta^n \to A \implies \varphi(\sigma) = 0\}.$$

Proposition 4.0.6.

1. ***(Homotopy invariance)*** *If $f \simeq g$ as maps $X \to Y$, then $f^* = g^*$ as maps $H^*(Y;G) \to H^*(X;G)$.*

2. ***(Snake)*** *Let (X,A) be pair. There exists a long exact sequence in cohomology*

$$\begin{tikzcd} \cdots \arrow[r] & H^n(X,A) \arrow[r] & H^n(X) \arrow[r] & H^n(A) \arrow[dlll] \\ H^{n+1}(X,A) \arrow[r] & \cdots & & \end{tikzcd}.$$

3. ***(Excision)*** *Let $Z \subset A \subset X$ such that $\operatorname{cl} Z \subset \operatorname{Int} A$. Then $H^*(X,A;G) \cong H^*(X \setminus Z, A \setminus Z; G)$.*

4. ***(MV sequence)*** *Let $X = \operatorname{Int} A \cup \operatorname{Int} B$. Then there exists a LES*

$$\begin{tikzcd} H^n(X) \arrow[r] & H^n(A) \oplus H^n(B) \arrow[r] & H^n(A \cap B) \arrow[dll] \\ H^{n+1}(X) \arrow[r] & \cdots & \end{tikzcd}.$$

Definition 4.0.7. Let X be a space and R be a ring. Let $k, l \in \mathbb{N}$. Define the *cup product* $C^k(X;R) \times C^l(X;R) \xrightarrow{\smile} C^{k+l}(X;R)$ by

$$\varphi \smile \psi(\sigma : \Delta^{k+l} \to X) = \varphi(\sigma \restriction_{[v_0,\ldots,v_k]}) \cdot_R \psi(\sigma \restriction_{v_k,\ldots,v_{k+l}}).$$

Note 4.0.8. The same definition works for relative singular cochain complexes. In general, if $f \in C^k(X,A;R)$ and $g \in C^l(X,B;R)$, then $f \smile g \in C^{k+l}(X, A \cup B; R)$.

Proposition 4.0.9. *Let $\mathbb{1} \in C^0(X;R)$ denote the constant map at 1_R. Then $\mathbb{1} \smile \varphi = \varphi \smile \mathbb{1} = \varphi$.*

Lemma 4.0.10. $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$.

Proof. We do the case where $\varphi, \psi \in C^1(X; R)$. If $\sigma \in \mathrm{Sing}_3(X)$, then

$$\begin{aligned}(\delta\varphi \smile \psi - \varphi \smile \delta\psi)(\sigma) &= \delta\varphi([v_0, v_1, v_2])\psi([v_2, v_3]) - \varphi([v_0, v_1])\delta\psi([v_1, v_2, v_3]) \\ &= \varphi([v_1, v_2])\psi([v_2, v_3]) - \varphi([v_0, v_2])\psi([v_2, v_3]) + \varphi([v_0, v_1])\psi([v_2, v_3]) \\ &\quad - \varphi([v_0, v_1])\psi([v_2, v_3]) + \varphi([v_0, v_1])\psi([v_1, v_3]) - \varphi([v_0, v_1])\psi([v_1, v_2]) \\ &= \delta(\varphi \smile \psi)(\sigma)\end{aligned}$$

□

Corollary 4.0.11. *The cup product descends to an operation*

$$H^k(X, A; R) \times H^l(X, A; R) \xrightarrow{\smile} H^{k+l}(X, A; R).$$

4.1 Lecture 19

Example 4.1.1.

1. Let $X = S_g$. By the universal coefficient theorem, we have that

   $$0 \to \mathrm{Ext}(H_0(X), \mathbb{Z}) \to H^1(X; \mathbb{Z}) \to \mathrm{Hom}(H_1(X), \mathbb{Z}) \to 0.$$

   Since $H_0(X) \cong \mathbb{Z}$ and $H_1(X) \cong \mathbb{Z}^{2g}$, it follows that $H^1(X; \mathbb{Z}) \cong \mathbb{Z}^{2g}$.

2. Let $X = \mathbb{RP}^n$.

k	H_k	$\mathrm{Ext}(H_{k-1}, \mathbb{Z})$	$\mathrm{Hom}(H_k, \mathbb{Z})$	$H^k(X; \mathbb{Z})$	$\mathrm{Ext}(H_{k-1}, \mathbb{Z}/2)$	$\mathrm{Hom}(H_k, \mathbb{Z}/2)$	$H^k(X; \mathbb{Z}/2)$
0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
1	$\mathbb{Z}/2$	0	0	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$
2	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
3	$\mathbb{Z}/2$	0	0	0			$\mathbb{Z}/2$
4	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$			
5	$\mathbb{Z}/2$	0	0				
6	0						
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$		$\mathbb{Z}/2$		0			
$n-1$	$\mathbb{Z}/2$	0		$\mathbb{Z}/2$			
n	$\begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$	$\begin{cases} \mathbb{Z}/2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$	$\mathbb{Z},\ n$ odd	$\mathbb{Z},\ n$ odd			$\mathbb{Z}/2,\ n$ odd

3. Let $0 \leq k \leq 2n$. Then $H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$.

   Proposition 4.1.2. *If X has torsion-free homology, then $H^n(X; \mathbb{Z}) \cong \mathrm{Hom}(H_n(X; \mathbb{Z}), \mathbb{Z})$ for every n.*

   Therefore,

   $$H^k(\mathbb{CP}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}.$$

Theorem 4.1.3. *If R is commutative and $\varphi, \psi \in H^*(X, A; R)$, then $\varphi \smile \psi = (-1)^{|\varphi||\psi|}\psi \smile \varphi$ where $|\cdot|$ denotes the degree of a cocycle. That is, $\smile$ is graded commutative.*

Remark 4.1.4. Yet, $\smile$ is *not* graded commutative at the chain level. By contrast, the wedge product $\wedge$ of differential forms is graded commutative at the chain level.

Example 4.1.5. Let $X = S_g$. One can show that $H_1(X)$ has $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ as a basis. Since $H^1(X;\mathbb{Z}) \cong \mathrm{Hom}(H_1(X), \mathbb{Z})$ by the universal coefficient theorem, there exists a dual basis $(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ for $H^1(X;\mathbb{Z})$.

For each $1 \leq i \leq g$, we want to represent each α_i by a cocycle φ_i and each β_i by a cocycle ψ_i. To this end, let

$$\varphi_i(\sigma : \Delta^1 \to X) = \begin{cases} 1 & \sigma \cap \alpha_i \neq \emptyset \\ 0 & \sigma \cap \alpha_i = \emptyset \end{cases}.$$

Define ψ_i similarly. Letting $g = 2$, it is straightforward to verify that both $\delta\varphi_i$ and $\delta\psi_i$ vanish on each 2-simplex inside S_2.

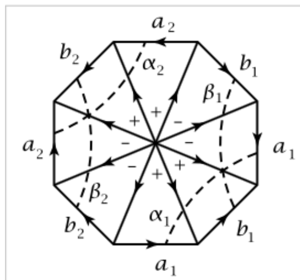


Figure 7: copied from Hatcher (207)

It is also straightforward to verify that $\varphi_1 \smile \psi_1$ vanishes on each 2-simplex inside S_2 except the lower one marked by b_1, on which it equals 1. It follows that $\varphi_1 \smile \psi_1(c) = 1$ where c denotes the linear combination of the 2-chains comprising S_2 with the indicated coefficients ± 1. This shows that $[c]$ generates $H_2(X) \cong \mathbb{Z}$ and that $\varphi_1 \smile \psi_1$ represents the dual generator γ of $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$. Hence $\alpha_1 \smile \beta_1 = [\gamma]$. In general, we can compute the relations

$$\begin{aligned} \alpha_i \smile \beta_j &= \left\{ \begin{matrix} \gamma & i = j \\ 0 & i \neq j \end{matrix} \right\} = -(\beta_i \smile \alpha_j) \\ \alpha_i \smile \alpha_j &= 0 \\ \beta_i \smile \beta_j &= 0. \end{aligned}$$

These completely determine $H^1(X;\mathbb{Z}) \times H^1(X;\mathbb{Z}) \xrightarrow{\smile} H^2(X;\mathbb{Z})$ because $\smile$ is distributive.

Definition 4.1.6. Let X and Y be spaces. The *external product* is the R-bilinear map $H^k(X;R) \times H^l(Y;R) \xrightarrow{\times} H^{k+l}(X \times Y;R)$ given by

$$a \times b \equiv p_1^* a \smile p_2^* b$$

where p_i denotes the i-th projection map for each $i = 1, 2$. This indues a unique linear map

$$H^k(X;R) \otimes_R H^l(Y;R) \to H^{k+l}(X \times Y;R).$$

Note 4.1.7. The same definition works for $H^k(X, A;R) \times H^l(Y, B;R) \xrightarrow{\times} H^{k+l}(X \times Y, X \times B \cup A \times Y;R)$.

Note 4.1.8. We can view $H^*(X;R)$ as the *graded ring* $(\bigoplus_{k \geq 0} H^k(X;R), +, \smile)$.

Proposition 4.1.9. *The cup product is a graded ring homomorphism.*

Theorem 4.1.10 (Künneth (special case)). *Suppose that X and Y are CW-complexes such that $H^n(Y;R)$ is a finitely generated free R-module for each n. Then $\times$ induces a ring isomorphism $H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R)$, i.e.,*

$$\bigoplus_{p+q=n} H^p(X;R) \otimes_R H^q(Y;R) \cong H^n(X \times Y;R)$$

for each n.

Remark 4.1.11. We could drop the assumption that X and Y are CW-complexes.

Definition 4.1.12. A *cohomology theory* h^* on CW-pairs is a sequence of contravariant functors from CW-pairs to graded abelian groups that satisfies the following properties.

1. If $f \simeq g$ as maps $X \to Y$, then $f^* = g^*$ as maps $h^*(Y) \to h^*(X)$.
2. $h^*(X, A) \cong h^*(X/A, \mathsf{pt})$.
3. For any CW-pair (X, A), there exists a LES
$$\cdots \longrightarrow h^n(X, A) \longrightarrow h^n(X) \longrightarrow h^n(A) \longrightarrow h^{n+1}(X, A) \longrightarrow \cdots .$$
4. $h^*(\bigvee_\alpha X_\alpha) \cong \prod_\alpha h^*(X_\alpha)$.

Lemma 4.1.13. *Let $\mu : h_1^* \to h_2^*$ be a natural transformation of cohomology theories on CW-pairs. If $\mu(\mathsf{pt}, \emptyset) : h_1^*(\mathsf{pt}, \emptyset) \to h_2^*(\mathsf{pt}, \emptyset)$ is an isomorphism, then μ is an isomorphism for any CW-pair.*

Proof. Due to the five lemma, it's enough to prove that for any CW-complex X, $\mu_X : h_1^*(X) \to h_2^*(X)$. Assuming that X is finite-dimensional, we do this by induction on the dimension of X. Since X^0 is a discrete set of points, our base case holds almost by assumption.

Suppose, inductively, that $\mu_{X^{n-1}}$ is an isomorphism. Since μ is a natural transformation, we get a commutative diagram with exact rows

$$\begin{tikzcd}
h_1^k(X^n, X^{n-1}) \arrow[r] \arrow[d, "\mu"] & h_1^k(X^n) \arrow[r] \arrow[d, "\mu"] & h_1^k(X^{n-1}) \arrow[r] \arrow[d, "\mu(\cong)"] & h_1^{k+1}(X^n, X^{n-1}) \arrow[r] \arrow[d, "\mu"] & \cdots \\
h_2^k(X^n, X^{n-1}) \arrow[r] & h_2^k(X^n) \arrow[r] & h_2^k(X^{n-1}) \arrow[r] & h_2^{k+1}(X^n, X^{n-1}) \arrow[r] & \cdots
\end{tikzcd}.$$

By the five lemma, we must need to show that $\mu : h_1^k(X^n, X^{n-1}) \to h_2^k(X^n, X^{n-1})$ is an isomorphism. We have another commutative diagram

$$\begin{tikzcd}
h_1^k(X^n, X^{n-1}) \arrow[r, "\mu"] \arrow[d, dashed, "\cong"] & h_2^k(X^n, X^{n-1}) \arrow[d, dashed, "\cong"] \\
h_1^k(X^n/X^{n-1}) \arrow[r, "\mu"] \arrow[d, dashed, "\cong"] & h_2^k(X^n/X^{n-1}) \arrow[d, dashed, "\cong"] \\
h_1^k(\bigvee_\alpha S^n) \arrow[r, "\mu"] \arrow[d, dashed, "\cong"] & h_2^k(\bigvee_\alpha S^n) \arrow[d, dashed, "\cong"] \\
\prod_\alpha h_1^k(S^n) \arrow[r, "\mu"] & \prod_\alpha h_2^k(S^n)
\end{tikzcd}.$$

Thus, we want to show that bottommost μ is an isomorphism. Since S^{n-1} is an $(n-1)$-dimensional CW-complex and $D^n \simeq \mathsf{pt}$, applying the five lemma together with out IH to the LES for (D^n, S^{n-1}) proves that $h_1^k(S^n) \cong h_2^k(S^n)$.

The case where X is infinite-dimensional reduces to the case where it's finite-dimensional. We omit the details. ☐

Proposition 4.1.14. *Let Y be a CW-complex and $H^*(Y; R)$ be free over R. Consider the functors*

$$h_1^*(X, A) \equiv H^*(X, A; R) \otimes_R H^*(Y; R)$$
$$h_2^*(X, A) \equiv H^*(X \times Y, A \times Y; R)$$

where X is a CW-complex. If $A = \emptyset$, then the map $\mu : h_1^*(X) \to h_2^*(X)$ given by $a \otimes b \mapsto a \times b$ defines a natural transformation $h_1^*(-) \to h_2^*(-)$.

Künneth. It suffices to show that $\mu : h_1^*(X) \to h_2^*(X)$ is an isomorphism when $X = \mathsf{pt}$. But, in this case, μ is precisely the map $R \otimes_R H^n(Y; R) \to H^n(Y; R)$ given by $r \otimes y \mapsto ry$, which is a well-known isomorphism. ☐

Example 4.1.15. $H^*(S^n) \otimes H^*(S^m) \cong H^*(S^n \times S^m)$.

4.2 Lecture 20

Theorem 4.2.1 (Künneth (special case)). *Let (X, A) and (Y, B) be CW-pairs. If $H^k(Y, B; R)$ is a finitely generated free R-module for each k, then*

$$H^*(X, A; R) \otimes_R H^*(X, B; R) \xrightarrow{\times} H^*(X \times Y, A \times Y \cup X \times B; R)$$

is a ring isomorphism.

Definition 4.2.2. Let (X, x_0) and (Y, y_0) be pointed spaces. The *smash product* $X \wedge Y$ is

$$(X \times Y)/(X \times \{y_0\} \cup \{x_0\} \times Y).$$

Remark 4.2.3. The smash product is a monoidal product in $\mathbf{Top}_*$.

Example 4.2.4.

1. $S^n \wedge S^m \cong S^{n+m}$.

   Proof. It suffices to show that if (X, x_0) and (Y, y_0) are compact manifolds, then $X \wedge Y \cong ((X \setminus \{x_0\}) \times (Y \setminus \{y_0\}))^*$ where $(-)^*$ denotes the one-point compactification. Let $Z := (X \setminus \{x_0\}) \times (Y \setminus \{y_0\})$. Note that Z is noncompact but locally compact.

   We see that $\psi := \pi \circ i : Z \to X \wedge Y$ is injective with $\operatorname{im} \psi = X \wedge Y \setminus \{(x_0, y_0)\}$. Let $U \subset Z$ be open. Then U is open in $X \times Y$. Since $\pi^{-1}(\pi(U)) = U$, it follows that $\pi(U)$ is open in $X \wedge Y$. Thus, ψ is an open map, so that ψ is a topological embedding. Since $X \wedge Y$ is compact, it follows that ψ is a Hausdorff compactification. □

2. $S^1 \wedge X \cong (X \times I)/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)$, called the *reduced suspension* ΣX *of* X.

Example 4.2.5.

1. Let (X, x_0) and (Y, y_0) be pointed spaces with $H^*(Y, y_0)$ free and finitely generated over $\mathbb{Z}$. Then $H^*(X, x_0) \otimes H^*(Y, y_0) \cong H^*(X \times Y, X \times \{y_0\} \cup Y \times \{x_0\}) \cong H^*(X \wedge Y)$.

2. $\widetilde{H}(S^n) \times \widetilde{H}^m(S^m) \cong H^{n+m}(S^{n+m})$.

Note 4.2.6. Any polynomial ring is graded since $R[x] = \bigoplus_{n \in \mathbb{N}} Rx^n$.

Theorem 4.2.7. *Let $X = \mathbb{CP}^n$ (resp. $\mathbb{RP}^n$) and $R = \mathbb{Z}$ (resp. $\mathbb{Z}/2$), then $H^*(X; R) \cong R[\tilde{x}]/(\tilde{x}^{n+1})$ where $|\tilde{x}| = 2$ (resp. 1).*

Proof. See *Hatcher*, Theorem 3.19 for a proof of the case where $X = \mathbb{RP}^n$. We will assume that $X = \mathbb{CP}^n$.

The inclusion $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$ induces an isomorphism $H^k(\mathbb{CP}^{n-1}) \to H^k(\mathbb{CP}^n)$ when $k < 2n - 1$. Thus, it suffices to show that if $\omega \in H^{2i}(\mathbb{CP}^n)$ is a generator and $\omega' \in H^{2n-2i}(\mathbb{CP}^n)$ is a generator where $0 \le i \le n$ is even, then $\omega \smile \omega'$ generates $H^{2n}(\mathbb{CP}^n)$. Set $j = n - i$. Then $\mathbb{CP}^j = \{[z_0, \dots, z_n] \mid z_0 = z_1 = \cdots = z_{i-1} = 0\}$, and $\mathbb{CP}^\cap \mathbb{CP}^j = p := [0, \dots, 0, \underbrace{1}_{i\text{-th spot}}, 0 \dots, 0]$. Note that $\mathbb{CP}^i \cap U \cong \mathbb{C}^i$ and $\mathbb{CP}^j \cap U \cong \mathbb{C}^j$ where $U = \{[x_0, \dots, z_n] \mid z_i \neq 0\}$. Consider the commutative diagram

$$\begin{tikzcd}
H^{2i}(\mathbb{CP}^n) \times H^{2j}(\mathbb{CP}^n) \arrow[r, "\smile"] & H^{2n}(\mathbb{CP}^n) \\
H^{2i}(\mathbb{CP}^n, \mathbb{CP}^n \setminus \mathbb{C}^j) \times H^{2j}(\mathbb{CP}^n, \mathbb{CP}^n \setminus \mathbb{C}^i) \arrow[r, "\smile"] \arrow[u, "\varphi_1"] \arrow[d, "\varphi_2"'] & H^{2n}(\mathbb{CP}^n, \mathbb{CP}^n \setminus \{p\}) \arrow[u, "\cong"'] \arrow[d, "\cong"] \\
H^{2i}(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{C}^j) \times H^{2j}(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{C}^i) \arrow[r, "\smile"] & H^{2n}(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\})
\end{tikzcd},$$

where the top right isomorphism comes from the five lemma applied to the LES's for $(\mathbb{CP}^n, \mathbb{CP}^n \setminus \{p\})$ and $(\mathbb{CP}^n, \mathbb{CP}^{n-1})$ and the bottom right isomorphism comes from excision.

Claim. *Both φ_1 and φ_2 are isomorphisms.*

Proof. We have a commutative diagram

$$\begin{tikzcd}
H^{2i}(\mathbb{CP}^n) \arrow[d, "\cong"'] & H^{2i}(\mathbb{CP}^n, \mathbb{CP}^{i-1}) \arrow[l, "\cong"'] \arrow[d, "\cong"'] & H^{2i}(\mathbb{CP}^n, \mathbb{CP}^n \setminus \mathbb{CP}^j) \arrow[l, "\psi"'] \arrow[r, "\cong"] \arrow[d] & H^{2i}(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{C}^j) \arrow[d] \\
H^{2i}(\mathbb{CP}^i) & H^{2i}(\mathbb{CP}^i, \mathbb{CP}^{i-1}) \arrow[l, "\cong"] & H^{2i}(\mathbb{CP}^i, \mathbb{CP}^i \setminus \{p\}) \arrow[l, "\gamma"] \arrow[r, "\cong"'] & H^{2i}(\mathbb{C}^i, \mathbb{C}^i \setminus \{0\})
\end{tikzcd},$$

where the isomorphisms of the leftmost square come from cellular cohomology and those from the rightmost square come from excision. It suffices to show that each map in this diagram is an isomorphism. Note that γ is an isomorphism because $\mathbb{CP}^i \setminus \{p\}$ deformation retracts onto $\mathbb{CP}^{i-1}$. Hence it suffices to observe that ψ is an isomorphism because $\mathbb{CP}^n \setminus \mathbb{CP}^j$ deformation retracts onto $\mathbb{CP}^{i-1}$. Indeed, $\mathbb{CP}^n \setminus \mathbb{CP}^j = \{[z_0, \ldots, z_n] \mid \exists s \in \{0, \ldots, i-1\}$ such that $z_s \neq 0\}$, so that the maps given by $f_t([z_0, \ldots, z_n]) = [z_0, \ldots, z_{i-1}, tz_i, \ldots, tz_n]$ $(1 \xrightarrow{t} 0)$ define a suitable deformation retraction. □

Moreover, the map $H^{2i}(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{C}^j) \times H^{2j}(\mathbb{C}^n, \mathbb{C}^n \setminus \mathbb{C}^i) \xrightarrow{\smile} H^{2n}(\mathbb{C}^n, \mathbb{C}^n \setminus \{0\})$ is equivalent to the map $H^{2i}(I^{2i}, \partial I^{2i}) \times H^{2j}(I^{2j}, \partial I^{2j}) \xrightarrow{\smile} H^{2n}(I^{2n}, \partial I^{2n})$, which maps any pair of generators to a generator by the Künneth theorem. By commutativity, it follows that $H^{2i}(\mathbb{CP}^n) \times H^{2j}(\mathbb{CP}^n) \xrightarrow{\smile} H^{2n}(\mathbb{CP}^n)$ also maps any pair of generators to a generator. □

Corollary 4.2.8.

1. *If $\mathbb{R}^n$ has a division algebra structure over $\mathbb{R}$, then $n = 2^l$ for some $l \in \mathbb{Z}_{\geq 0}$.*
2. *If $\mathbb{C}^n$ has a division algebra structure over $\mathbb{C}$, then $n = 1$.*

Proof.

1. Suppose that $\mathbb{R}^n$ is a division algebra over $\mathbb{R}$. Then its bilinear product induces a continuous map $h : \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \to \mathbb{RP}^{n-1}$ such that $h \restriction_{\mathbb{RP}^{n-1} \times \{y\}}$ and $h \restriction_{\{x\} \times \mathbb{RP}^{n-1}}$ are homeomorphisms for any $x, y \in \mathbb{RP}^{n-1}$. The induced homomorphism $h^* : \mathbb{Z}_2[\tilde{x}] / (\tilde{x}^n) \to \mathbb{Z}_2[\tilde{x}_1, \tilde{x}_2] / (\tilde{x}_1^n, \tilde{x}_2^n)$ is determined by the value $\tilde{x} \mapsto k_1\tilde{x}_1 + k_2\tilde{x}_2$. Since h restricts to a homeomorphism on each copy of $\mathbb{RP}^{n-1}$, it follows that $k_1\tilde{x}_1$ and $k_2\tilde{x}_2$ are nonzero. Hence $k_1 = 1 = k_2$. But

   $$0 = \tilde{x}^n = (\tilde{x}_1 + \tilde{x}^2)^n = \sum_{k=1}^{n-1} \binom{n}{k} \tilde{x}_1^k \tilde{x}_2^{n-k}.$$

   This implies that $\binom{n}{k} \equiv 0 \mod 2$ for each $1 \leq k \leq n-1$.

   Exercise 4.2.9. *Use elementary number theory to prove this happens only if n equals a power of 2.*

2. By a similar argument, we deduce that $\binom{n}{k} = 0$ for each $1 \leq k \leq n-1$. This implies that $n = 1$.

□

Definition 4.2.10. Let X be any space.

1. We say that X *has category* n (written as $\mathrm{cat}(X) = n$) if X can be written as a union of n contractible open subsets but not $n-1$.
2. The *cup length* cup-length(X) *of* X is $\max\{c \in \mathbb{Z} \mid \exists x_1, \ldots, x_c \in H^*(X, \mathbb{Z})$ such that $|x_i| \geq 1$ and $x_i \cdots x_c \neq 0\}$.

Example 4.2.11. We know that $\mathbb{CP}^n = \bigcup_{i=0}^n U_i$ where $U_i \coloneqq \{[z_0, \ldots, z_n] \mid z_i \neq 0\} \cong \mathbb{C}^n$. Is it possible to write $\mathbb{CP}^n$ as a union of n contractible subspaces?

Theorem 4.2.12. *If X is a space, then* $\mathrm{cat}(X) - 1 \geq$ cup-length(X).

Proof. Let $c := \mathrm{cat}(X)$. Then $X = \bigcup_{i=1}^c$ with each U_i open and contractible. By LES in cohomology, we get $H^j(X) \cong H^j(X, U_i)$ for any i and any $j \geq 1$. Suppose that $x_i \in H^{j_i}(X)$ for each $1 \leq i \leq c$ where $|x_i| \geq 1$. Then $x_1 \cdots x_c \in H^{j_1+\cdots+j_c}(X, \bigcup_{i=1}^c U_i) = H^{j_1+\cdots+j_c}(X, X) = 0$. Thus, $x_1 \cdots x_c = 0$. □

Corollary 4.2.13. $\mathrm{cat}(\mathbb{CP}^n) \geq \text{cup-length}(\mathbb{CP}^n) + 1 = n + 1$.

Definition 4.2.14. Let $k \geq l$. The *cap product* is the map $C_k(X; R) \times C^l(X; R) \xrightarrow{\frown} C_{k-l}(X; R)$ defined by

$$\sigma \frown \varphi = \varphi(\sigma \restriction_{[v_0,\ldots,v_l]})\sigma \restriction_{[v_l,\ldots,v_k]} .$$

Lemma 4.2.15. *$\partial(\sigma \frown \varphi) = (-1)^l(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$.*

Proof. We compute

$$\begin{aligned}
\partial\sigma \frown \varphi &= \sum_{i=0}^{k} (-1)^i \sigma \restriction_{[v_0,\ldots,\hat{v}_i,\ldots,v_k]} \frown \varphi \\
&= \sum_{i=0}^{l} (-1)^i \varphi(\sigma \restriction_{[v_0,\ldots,\hat{v}_i,\ldots,v_{l+1}]})\sigma \restriction_{[v_{l+1},\ldots,v_k]} \\
&\quad + \sum_{i=l+1}^{k} (-1)^i \varphi(\sigma \restriction_{[v_0,\ldots,v_l]})\sigma \restriction_{[v_l,\ldots,\hat{v}_i,\ldots,v_k]} \\
\sigma \frown \delta\varphi &= \delta\varphi(\sigma \restriction_{[v_0,\ldots,v_{l+1}]})\sigma \restriction_{[v_{l+1},\ldots,v_k]} \\
&= \sum_{i=0}^{l+1} (-1)^i \varphi(\sigma \restriction_{[v_0,\ldots,\hat{v}_i,\ldots,v_{l+1}]})\sigma \restriction_{[v_{l+1},\ldots,v_k]} \\
\partial(\sigma \frown \varphi) &= \partial(\varphi(\sigma \restriction_{[v_0,\ldots,v_l]})\sigma \restriction_{[v_l,\ldots,v_k]}) \\
&= \sum_{i=l}^{k} (-1)^{i-l} \varphi(\sigma \restriction_{[v_0,\ldots,v_l]})\sigma \restriction_{[v_l,\ldots,\hat{v}_i,\ldots,v_k]} .
\end{aligned}$$

It's easy to check that $(-1)^l\partial(\sigma \frown \varphi) = \partial\sigma \frown \varphi - \sigma \frown \delta\varphi$. □

Corollary 4.2.16. *The cap product induces a map $H_k(X; R) \times H^l(X; R) \xrightarrow{\frown} H_{k-l}(X; R)$.*

Proof. If σ is a cycle and φ a cocycle, then clearly $\sigma \frown \varphi$ is a cycle. Also, if $\sigma = \partial d$ and $\delta\varphi = 0$, then $\sigma \frown \varphi = \partial d \frown \varphi = \pm\partial(d \frown \varphi)$. Hence the cap product of a boundary and cocycle is a boundary. Similarly, the cap product of a cycle and coboundary is a boundary. □

Theorem 4.2.17 (Poincaré duality (PD)). *Let X be a closed R-oriented n-manifold. Then the map $H^l(X; R) \to H_{n-l}(X; R)$ given by $\varphi \mapsto [X] \frown \varphi$ is an isomorphism for every l.*

Remark 4.2.18. PD is a global consequence of one local condition (manifold-hood) and two global conditions (orientability and closedness).

4.3 Lecture 21

Note 4.3.1. We have the following relative forms of the cap product.

$$\begin{aligned}
H_k(X, A \cup B) \times H^l(X, A) &\xrightarrow{\frown} H_{k-l}(X, B) \\
H_k(X, A) \times H^l(X) &\xrightarrow{\frown} H_{k-l}(X, A) \\
H_k(X, A) \times H^l(X, A) &\xrightarrow{\frown} H_{k-l}(X)
\end{aligned}$$

Proposition 4.3.2 (Projection formula). *Let $f : X \to Y$ be a map of spaces. Let $[\sigma] \in H_k(X)$ and $[\varphi] \in H^l(Y)$. Then*

$$f_*([\sigma]) \frown [\varphi] = f_*([\sigma] \frown f^*([\varphi]))$$

in $H_{k-l}(Y)$.

Definition 4.3.3. Let X be a locally compact CW-complex. Let $C_c^i \equiv \{\varphi \in C^i(X;R) \mid \exists K_\varphi \subset X$ compact such that $\varphi(c) = 0$ for any $c \in C_i(X \setminus K_\varphi)\}$. This induces the *cohomology group with compact support* $H_c^i(X;R)$.

Note 4.3.4. The groups $C_c^i(X;R)$ consisting of cochains φ that are nonzero only on i-simplices contained in some compact set K_φ do not form a subcomplex of the singular cochain complex. Indeed, if $\varphi \in C^0(\mathbb{R})$ has $\varphi(x_0) = 1$ and $\varphi(x) = 0$ for any $x \neq x_0$, then $\delta\varphi(\sigma : \underbrace{y \rightsquigarrow x_0}_{y \neq x_0}) = \varphi(\sigma(1)) - \varphi(\sigma(0)) \neq 0$.

Example 4.3.5. Let X be a simplicial complex and let $C_c^i(X;G)$ denote the subgroup of compactly supported i-cochains. View $\mathbb{R}$ as a simplicial complex with vertices at the integer points. If $\varphi \in C_c^0$ satisfies $\delta\varphi = 0$, then φ must be constant since $\delta\varphi([n, n+1]) = \varphi(n+1) - \varphi(n)$ for every $n \in \mathbb{Z}$. In this case, φ must be identically zero since it is compactly supported. Thus, $H_c^0(\mathbb{R}, \mathbb{Z}) = 0$.

Define $\epsilon : C_c^1(\mathbb{R};\mathbb{Z}) \to \mathbb{Z}$ by $\epsilon(\psi) = \sum_{n\in\mathbb{Z}} \psi([n, n+1])$, which makes sense since ψ is compactly supported. We have that

$$\epsilon(\delta\phi) = \sum_{n\in\mathbb{Z}} \delta\varphi([n, n+1]) = \sum_{n\in\mathbb{Z}} \varphi(n+1) - \varphi(n) = 0.$$

Thus, ϵ vanishes on any coboundary. Since $\epsilon \neq 0$, it follows that there are 1-cocycles that are not coboundaries, so that $H_c^1(\mathbb{R};\mathbb{Z}) \neq 0$.

Exercise 4.3.6. *Show that the induced map $\epsilon : H_c^1(\mathbb{R};\mathbb{Z}) \to \mathbb{Z}$ is injective, i.e., if $\epsilon(\psi) = 0$, then $\psi = \delta\phi$ for some ϕ.*

Definition 4.3.7.

1. Let i be a poset. We say that I is a *directed set* if for any $\alpha, \beta \in I$, there exists $\gamma \in I$ such that $\alpha, \beta \leq \gamma$.

   Let G_α be an abelian group for each $\alpha \in I$. Suppose that for any $\alpha \leq \beta$, there is some homomorphism $f_{\alpha\beta} : G_\alpha \to G_\beta$ such that

   - $f_{\alpha\alpha} = \mathbb{1}_{G_\alpha}$ for any $\alpha \in I$ and
   - $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$ when $\alpha \leq \beta \leq \gamma$.

   Then $\{G_\alpha, f_{\alpha\beta}\}$ is called a *directed system* of groups.

2. Suppose that $\{G_\alpha, f_{\alpha\beta}\}$ is a directed system of groups. The *direct limit* group is

   $$\varinjlim G_\alpha \equiv \bigoplus_{\alpha \in I} G_\alpha \big/ N$$

   where N is the subgroup generated by all elements of the form $a - f_{\alpha\beta}(a)$ with $a \in G_\alpha$.

   Equivalently, $\varinjlim G_\alpha$ has as its underlying set $\coprod_{\alpha\in I} G_\alpha \big/ \sim$ where $x_\alpha \simeq x_\beta$ if there exists $\gamma \in I$ such that $\alpha, \beta \leq \gamma$ and $f_\alpha\beta(x_\alpha) = f_{\beta\gamma}(x_\beta)$. We equip this with the group structure

   $$[x_\alpha] + [x_\beta] = [f_{\alpha\gamma}(x_\alpha) + f_{\beta\gamma}(x_\beta)]$$

   where $\alpha, \beta \leq \gamma$. (This is well-defined because I is a poset by hypothesis.)

Aside. Let $\mathbf{Ab}^I$ denote the category of directed systems (viewed as functors $I \to \mathbf{Ab}$), so that a morphism in $\mathbf{Ab}^I$ is precisely a collection of homomorphisms $\{w_\alpha\} : \{G_\alpha, f_{\alpha\beta}\} \to \{H_\alpha, g_{\alpha\beta}\}$ such that the square

$$\begin{tikzcd} G_\alpha \arrow[r, "f_{\alpha\beta}"] \arrow[d, "w_\alpha"'] & G_\beta \arrow[d, "w_\beta"] \\ H_\alpha \arrow[r, "g_{\alpha\beta}"'] & H_\beta \end{tikzcd}$$

commutes for any $\alpha \leq \beta$. Let $a_\alpha : G_\alpha \to \varinjlim G_\alpha$ and $b_\alpha : H_\alpha \to \varinjlim H_\alpha$ denote the canonical maps. Then, by the universal property of colimits, the maps $b_\alpha \circ w_\alpha : G_\alpha \to \varinjlim H_\alpha$ induce a unique map $w : \varinjlim G_\alpha \to \varinjlim H_\alpha$ such that $w \circ a_\alpha = b_\alpha \circ w_\alpha$. The functor $\varinjlim : \mathbf{Ab}^I \to \mathbf{Ab}$ sends any morphism $\{w_\alpha\}$ to w.

Example 4.3.8. We have that $C^i_c(X;R) = \bigcup_{K\subset X} C^i(X, X\setminus K; R)$ with K compact in X. Note that the set K_X of compact sets in X is a directed set both under inclusion $\subset$ (for a directed system of homology groups) and under reverse inclusion $\supset$ (for a directed system of cohomology groups).

Proposition 4.3.9.

1. *Let I be a directed set. Let $X = \bigcup_{\alpha\in I} X_\alpha$ such that any compact set in X is contained in some X_α. Then $\varinjlim H_i(X_\alpha; G) \cong H_i(X;G)$ for each i.*

2. *$\varinjlim_{K_X} H^i(X, X\setminus X_\alpha; R) \cong H^i_c(X;R)$.*

Remark 4.3.10 (De Rham homology). Let X be a smooth n-manifold and consider its de Rham cochain complex $(\Omega^i(X), d)$. The *de Rham homology* $H_i(X)$ arises from the chain complex given by

$$\Omega_i(X) \equiv \Omega^i(X)^\vee = \mathbb{R}[\{f : \Omega^i(X) \to \mathbb{R} \mid f \text{ continuous}\}]$$

and $\partial\xi \equiv \xi d$. Note that $\Omega^i(X)^\vee$ is compactly supported.

Suppose that X is closed and oriented. The linear functional $\Omega^n(X) \to \mathbb{R}$ given by $\omega \mapsto \int_X \omega$ is called an *n-current*. Let $S \subset X$ be a closed k-submanifold. Then $\eta \mapsto \int_S d\eta \restriction_S = \int_{\partial S} \eta \restriction_S = 0$ defines a $(k-1)$-current $\Omega^{k-1}(X) \to \mathbb{R}$. We can view an element of $\Omega_i(X)$ as a distribution-valued form.

Further, we can take the topological dual of $(\Omega^*_c(X), d)$, but it won't be compactly supported. This dual $\Omega^{\mathrm{BM}}_*(X)$ induces the so-called *Borel-Moore homology of X*. Another version of PD states that if X is an oriented n-manifold, then

$$\begin{aligned} H^k_{\mathrm{dR},c}(X) &\cong H^{\mathrm{dR}}_{n-k}(X) \\ H^k_{\mathrm{dR}}(X) &\cong H^{\mathrm{BM}}_{n-k}(X). \end{aligned}$$

Note 4.3.11. Let X be an n-manifold with orientation $x \mapsto \alpha_x$. Let $K \subset L \subset X$ be a chain of compact subsets. We have a commutative diagram

$$\begin{tikzcd} H_n(X, X\setminus L; R) \times H^k(X, X\setminus L; R) \arrow[r, "\frown"] \arrow[d, bend right, "i_*"'] & H_{n-k}(X;R) \\ H_n(X, X\setminus K; R) \times H^k(X, X\setminus K; R) \arrow[u, bend right, "i^*"'] \arrow[ur, "\frown"'] & \end{tikzcd}.$$

There exist unique $\mu_K \in H_n(X, X\setminus K)$ and $\mu_L \in H_n(X, X\setminus L)$ such that $\mu_K(x) = \alpha_x$ and $\mu_L(y) = \alpha_y$ for each $x \in K$ and each $y \in L$. By uniqueness, $i_*(\mu_L) = \mu_K$. By naturality of $\frown$,

$$\mu_K \frown x = i_*(\mu_L) \frown x = \mu_L \frown i^*(x)$$

for any $x \in H^k(X, X\setminus K)$. Therefore, the direct limit (over K_X) of the maps $g_K : H^k(X, X\setminus K) \to H_{n-k}(X)$ given by $x \mapsto \mu_K \frown x$ induces the *duality homomorphism*

$$D_X : H^k_c(X) \to H_{n-k}(X).$$

4.4 Lecture 22

Remark 4.4.1. Any proper map of locally compact CW-complexes $f : X \to Y$ induces a homomorphism $f^* : H^*_c(Y;R) \to H^*_c(X;R)$. It also induces a map $C_c(Y) \to C_c(X)$ where $C_c(-)$ where $C_c(\cdot)$ denotes the space of compactly supported maps $\cdot \to \mathbb{R}$.

But we also have a *wrong way functor* or an *umkehr map* in that the inclusion $i : U \to V$ of open sets in X induces a homomorphism $i_! : H^k_c(U;R) \to H^k_c(V;R)$. It also induces a map $C_c(U) \to C_c(V)$ since every CW-complex is paracompact and thus always admits subordinate partitions of unity.

Note 4.4.2. Let $U \subset V$ be an inclusion of open sets in X. By applying excision twice, we see that $\varinjlim_{K_U} H^k(X, X\setminus K) = \varinjlim_{K_V} H^k(X, X\setminus K)$.

Proposition 4.4.3. *Let X be an R-oriented n-manifold. Let U and V be open in X with $X = U \cup V$. Then there exists a commutative diagram up to sign*

$$\begin{tikzcd}
\cdots \arrow[r] & H_c^k(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H_c^k(U)\oplus H_c^k(V) \arrow[r] \arrow[d, "D_U\oplus -D_V"] & H_c^k(X) \arrow[r] \arrow[d, "D_X"] & H_c^{k+1}(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(X) \arrow[r] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}.$$

Theorem 4.4.4. *Let X be an R-oriented n-manifold. Then $D_X : H_c^k(X) \to H_{n-k}(X)$ is an isomorphism for each i.*

Proof.

Step 1: If $X = U \cup V$ with U and V open and D_U, D_V, and $D_{U\cap V}$ are isomorphisms, then D_X is an isomorphism.

Proof. Apply the five lemma to the diagram of Proposition 4.4.3. □

Step 2: Suppose that X equals the union of a sequence of open sets $U_1 \subset U_2 \subset \cdots$. If each $D_{U_i} : H_c^k(U_i) \to H_{n-k}(U_i)$ is an isomorphism, then so is D_X.

Proof. By excision, we have that

$$H_c^k(U_i) \cong \varinjlim_{K_{U_i}} H^k(U_i, U_i \setminus K) \cong \varinjlim_{K_{U_i}} H^j(X, X \setminus K)$$

for each i. From this, we see that there are natural maps $H_c^k(U_i) \to H_c^k(U_{i+1})$. Hence we can take $\varinjlim H_c^k(U_i) \cong H_c^k(X)$. Also, Proposition 4.3.9(1) implies that $\varinjlim H_{n-k}(U_i) \cong H_{n-k}(X)$, so that $D_X = \varinjlim D_{U_i}$. The direct limit preserves any isomorphism since it is a functor. Therefore, D_X is an isomorphism. □

Note 4.4.5. Our proof of Step 2 works so long as the directed set is totally ordered.

Step 3: If $X = \mathbb{R}^n$, then D_X is an isomorphism.

Proof. Note that $\mathbb{R}^n \cong \operatorname{Int}\Delta^n$. Therefore, $H_c^k(\mathbb{R}^n) \cong H^k(\Delta^n, \partial\Delta^n)$. We can thus can view the map D_X as the map

$$\tau : H^k(\Delta^n, \partial\Delta^n) \to H_{n-k}(\Delta^n)$$

given by $x \mapsto [\Delta^n] \frown x$ where Δ^n denotes the identity map $\Delta^n \to \Delta^n$, which represents a generator of $H_n(\Delta^n, \partial\Delta^n)$. If $k \neq n$, then τ is the automorphism of the trivial group. Suppose that $k = n$. Note that $H^n(\Delta^n, \partial\Delta^n) \cong \operatorname{Hom}(H_n(\Delta^n, \partial\Delta^n), R)$ by the universal coefficient theorem, where any generator of $H^n(\Delta^n, \partial\Delta^n)$ is represented by a cocycle φ such that $\varphi([\Delta^n]) = 1$. But then

$$\tau(\varphi) = [\Delta^n] \frown \varphi = \varphi(\sigma \restriction_{[v_0,\ldots,v_n]})\,\mathrm{id}_{v_n} = \mathrm{id}_{v_n},$$

which represents a generator of $H_0(\Delta^n)$. Thus, τ is an isomorphism. □

Step 4: If U is any open set in $\mathbb{R}^n$, then D_U is an isomorphism.

Proof. Since $\mathbb{R}^n$ is second countable, we can write $U = \bigcup_{i\in\mathbb{N}} U_i$ where each $U_i \subset \mathbb{R}^n$ is an open ball. Let $V_i = \bigcup_{j\leq i} U_j$, so that $V_1 \subset V_2 \subset \cdots$.

We claim that each each D_{V_i} is an isomorphism. To do this, we prove the slightly more general claim that if Z_s is any finite union $\bigcup_{k=1}^s B_k$ of open balls in $\mathbb{R}^n$, then D_{Z_s} is an isomorphism. The base case follows automatically from Step 3. The inductive step holds because both Z_s and $Z_s \cap B_{s+1}$ equal the union of s open balls, in which case Step 1 implies that $D_{Z_s\cup B_{s+1}} = D_{Z_{s+1}}$ is an isomorphism.

Since $\bigcup_{i=1}^\infty V_i = U$ and each D_{V_i} is an isomorphism, it follows from Step 2 that D_U is an isomorphism. □

Step 5: If X is any finite or countable union of open sets each of which is homeomorphic to $\mathbb{R}^n$, then D_X is an isomorphism.

Proof. Use a nearly identical proof to that of Step 4. $\square$

Step 6: If X is any noncompact manifold, then D_X is an isomorphism.

Proof. Let $\mathcal{U}$ denote the set of open $U \subset X$ such that D_U is an isomorphism. Order $\mathcal{U}$ by inclusion. Any chain in $\mathcal{U}$ has an upper bound due to Step 2. By Zorn's lemma, it follows that $\mathcal{U}$ has some maximal element U_0. If there exists $x \in X \setminus U_0$, then find a neighborhood U_x of x that is homeomorphic to $\mathbb{R}^n$. In this case, D_{U_x} is an isomorphism due to Step 3 and $D_{U_x \cap U_0}$ is an isomorphism due to Step 4, so that $D_{U_x \cup U_0}$ is an isomorphism due to Step 1. But this is impossible since U_0 is maximal. Thus, $U_0 = X$. $\square$

$\square$

Corollary 4.4.6. *If X is a closed R-oriented manifold of odd dimension n, then $\chi(X) = 0$.*

Proof. We have that

$$\begin{aligned}\chi(X) &= \sum_i (-1)^i \operatorname{rank} H_i(X) \\ &= \sum_i (-1)^i \operatorname{rank} H^{n-i}(X) \\ &= \sum_i (-1)^i \operatorname{rank} H_{n-i}(X) = 0.\end{aligned}$$

$\square$

Exercise 4.4.7. *Let X be a closed manifold. Use the universal coefficient theorem to show that the value $\sum_i (-1)^i \operatorname{rank} H_i(X;\mathbb{Z})$ remains the same when $\mathbb{Z}$ is replaced by any R.*

Corollary 4.4.8. *Since any manifold X is $\mathbb{Z}/2$-oriented, $\chi(X) = 0$ whenever X is closed and $\dim X$ is odd.*

Remark 4.4.9. Take any finite-dimensional simplicial complex X and embed X into some $\mathbb{R}^N$. Take a tubular neighborhood U of X in $\mathbb{R}^N$. Then U is a non-compact manifold such that $U \simeq X$. This suggests that the (co)homology of compact manifolds is much richer than that of non-compact ones.

Example 4.4.10. Let X be a closed R-oriented 4-manifold that is simply connected. By PD, we quickly get

$$H^k(X) \cong \begin{cases} \mathbb{Z} & k = 0, 4 \\ \pi_1(X)_{ab} = 0 & k = 3 \end{cases}.$$

By the universal coefficient theorem, it follows that $H^1(X) = 0$. Finally, the same theorem shows that $H_2(X) \cong H^2(X) \cong \operatorname{Hom}(H_2(X), \mathbb{Z})$, which is a free abelian group since $H_2(X)$ is finitely generated.

Proposition 4.4.11. *If X and Y are oriented n-manifolds, then*

$$H^k(X \# Y) \cong \begin{cases} H^k(X) \oplus H^k(Y) & 0 < k < n \\ \mathbb{Z} & k = 0, n \end{cases}.$$

Definition 4.4.12. Let X be a closed R-orientable n-manifold. Consider the bilinear form

$$B : H^k(X;R) \times H^{n-k}(X;R) \to R$$

given by $(f, g) \mapsto (f \smile g)[X]$. We say that B is *nondegenerate* or *nonsingular* if the induced maps $H^k(X;R) \to \operatorname{Hom}_R(H^{n-k}(X;R), R)$ and $H^{n-k}(X;R) \to \operatorname{Hom}_R(H^k(X;R), R)$ are both isomorphisms.

Lemma 4.4.13. *If $\varphi \in C_k(X;R)$, $\psi \in C^l(X;R)$, and $\alpha \in C_{k+l}(X;R)$, then $\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$.*

Proof. We compute

$$\begin{aligned}\psi(\alpha \frown \varphi) &= \psi(\varphi(\sigma \restriction_{[v_0,\dots,v_k]})\sigma \restriction_{[v_k,\dots,v_{k+l}]}) \\ &= \varphi(\sigma \restriction_{[v_0,\dots,v_k]})\psi(\sigma \restriction_{[v_k,\dots,v_{k+l}]}) = (\varphi \smile \psi)(\alpha).\end{aligned}$$

□

Theorem 4.4.14. *Our bilinear form B is nondegenerate modulo torsion.*

Proof. Consider the composition

$$H^{n-k}(X;R) \xrightarrow{h} \operatorname{Hom}_R\left(H_{n-k}(X;R), R\right) \xrightarrow{D^*} \operatorname{Hom}_R\left(H^k(X;R), R\right).$$

The map h is obtained from the universal coefficient theorem and is an isomorphism modulo torsion. Moreover, D^*h maps each $f \in H^{n-k}(X;R)$ to the homomorphism given by $g \mapsto f([X] \frown g) = (f \smile g)[X]$. Since D is an isomorphism, it follows that B is nondegenerate in its second factor. It is also nondegenerate in its first factor because of the commutativity of the cup product. □

Remark 4.4.15. For any $4k$-manifold X, both rank H^{2k} and the signature of $B : H^{2k} \times H^{2k} \to H^{4k}$ are topological invariants over $\mathbb{R}$ where the *signature of B* is #(positive eigenvalues of B) − #(negative eigenvalues of B). The former, however, is the only such invariant over $\mathbb{C}$.

Theorem 4.4.16 (Friedman). *Any oriented closed simply connected $4k$-manifold is classified up to homeomorphism by the signature of B (along with, sometimes, a $\mathbb{Z}/2$-valued invariant).*

4.5 Lecture 23

Definition 4.5.1. Let $\mathbb{H}^n \coloneqq \{(x^1,\dots,x^n) \in \mathbb{R}^n : x_n \geq 0\}$. An *$n$-dimensional manifold with boundary M* is a Hausdorff space such that for any $p \in M$, there exist a neighborhood U of p and a homeomorphism $\varphi : U \to \mathbb{H}^n$. Any point $p \in M$ is called a *boundary point* if it belongs to $\varphi^{-1}(\{x_n = 0\})$.

Proposition 4.5.2. *For any compact manifold with boundary M, there exists a collar neighborhood of ∂M, i.e., an open $U \supset \partial M$ such that $U \cong \partial M \times [0,1)$.*

Definition 4.5.3. A manifold with boundary is *R-oriented* if $M \setminus \partial M = \operatorname{Int} M$ is R-oriented.

Proposition 4.5.4. *Let M be a compact n-manifold with boundary. Find $\partial M \times [0,1)$ a collar neighborhood of ∂M. Then*

$$H_n(M, \partial M; R) \cong H_n(M \setminus \partial M, \partial M \times (0,\epsilon)).$$

Corollary 4.5.5. *If M is R-oriented, then we obtain a relative fundamental class $[M] \in H_n(M, \partial M; R)$, which restricts to a given orientation at every point in $M \setminus \partial M$.*

Theorem 4.5.6 (Lefschetz duality). *Let M^n be an R-oriented compact manifold with boundary. Then the map $D_M : H^k(M, \partial M; R) \to H_{n-k}(M;R)$ given by $D\varphi = [M] \frown \varphi$ is an isomorphism.*

Proof. We will suppress our notation for the coefficient ring R. Let C denote the directed set of compact sets K contained in a complement of collar neighbohood of ∂M. Observe that

$$\begin{aligned}H^k_c(\operatorname{Int} M) &\cong \varinjlim_{K_{\operatorname{Int} M}} H^k(\operatorname{Int} M, \operatorname{Int} M \setminus K) \\ &\cong \varinjlim_{C} H^k(M, M \setminus K) \cong \varinjlim_{\epsilon\text{-collars } U} H^k(M, U) \\ &\cong H^k(M, \partial M).\end{aligned}$$

By PD, it follows that

$$H^k(M, \partial M) \cong H^k_c(\operatorname{Int} M) \cong H_{n-k}(\operatorname{Int} M) \cong H_{n-k}(M).$$

□

Definition 4.5.7.

1. A *double (cochain) complex* is a commutative diagram

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
A^{0,2} \arrow[r, "\delta"] \arrow[u, "d"] & A^{1,2} \arrow[r, "\delta"] \arrow[u, "d"] & A^{2,2} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
A^{0,1} \arrow[r, "\delta"] \arrow[u, "d"] & A^{1,1} \arrow[r, "\delta"] \arrow[u, "d"] & A^{2,1} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
A^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & A^{1,0} \arrow[r, "\delta"] \arrow[u, "d"] & A^{2,0} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

   in **Ab** such that $d^2 = 0 = \delta^2$ and $d\delta + \delta d = 0$.

2. If $(A^{\bullet,\bullet}, d^\bullet, \delta^\bullet)$ denotes a double complex, then the *total complex* $\mathrm{Tot}(A)$ *of* A is the single complex with $\mathrm{Tot}(A)^n \equiv \bigoplus_{p+q=n} A^{p,q}$ and $d_{\mathrm{Tot}(A)} \restriction_{A^{p,q}} \equiv \delta + (-1)^p d$.

3. If $(A^{\bullet,\bullet}, d^\bullet, \delta^\bullet)$ denotes a double complex, then define the *cohomology of* A as

$$H^*(A, d, \delta) = H^*(\mathrm{Tot}(A), d_{\mathrm{Tot}(A)}).$$

Lemma 4.5.8. *Suppose that*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] \arrow[u] & A^2 \arrow[r, "\epsilon"] \arrow[u, "d"] & A^{0,2} \arrow[r, "\delta"] \arrow[u, "d"] & A^{1,2} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] \arrow[u] & A^1 \arrow[r, "\epsilon"] \arrow[u, "d"] & A^{0,1} \arrow[r, "\delta"] \arrow[u, "d"] & A^{1,1} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
0 \arrow[r] \arrow[u] & A^0 \arrow[r, "\epsilon"] \arrow[u, "d"] & A^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & A^{1,0} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
A^{0,1} \arrow[r, "\delta"] \arrow[u, "d"] & A^{1,1} \arrow[r, "\delta"] \arrow[u, "d"] & A^{2.1} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
A^{0,0} \arrow[r, "\delta"] \arrow[u, "d"] & A^{1,0} \arrow[r, "\delta"] \arrow[u, "d"] & A^{2,0} \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
A^{0} \arrow[r, "\delta"] \arrow[u, "\epsilon"] & A^{1} \arrow[r, "\delta"] \arrow[u, "\epsilon"] & A^{2} \arrow[r, "\delta"] \arrow[u, "\epsilon"] & \cdots \\
0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

are augmented double complexes with exact rows and exact columns, respectively. Then there exists a cochain map $(A^\bullet, d) \to (\mathrm{Tot}(A^{\bullet,\bullet}), d_{\mathrm{Tot}})$ *that induces an isomorphism* $H^*(A^\bullet, d) \to H^*(A^{\bullet,\bullet}, d, \delta)$.

Proof. We just consider the case where the rows of $A^{\bullet,\bullet}$ are augmented. Note that $d_{\mathrm{Tot}}\epsilon = (\delta + d)\epsilon = d\epsilon = \epsilon d$. Hence $\epsilon^\bullet$ is a cochain map, thereby inducing a map $\epsilon^* : H^*(A^\bullet, d) \to H^*(A^{\bullet,\bullet}, d, \delta)$. We want to prove that ϵ^* is bijective.

Let $[c] \in H^*(A^{\bullet,\bullet}, d, \delta)$, so that $D(c) = 0$. Write $c = (c_0, \dots, c_n)$. Since each row is exact by hypothesis, there is some s such that $\delta(s) = c_0$. We have that $c - d_{\mathrm{Tot}}(s) = (0, c_1', \dots, c_n')$ for some c_i'. By repeating this procedure enough times, we get $[c] = [(0, \dots, 0, v_n)]$ for some v_n. Note that $\delta(v_n) = 0 = d(v_n)$. Hence there is some z such that $\epsilon(z) = v_n$. But $\epsilon(d(z)) = d(v_n) = 0$, and ϵ is injective. Thus, $d(z) = 0$. This proves that

$$\epsilon([z]) = [(0, \dots, 0, \epsilon(z))] = [(0, \dots, 0, v_n)] = [c],$$

so that ϵ^* is surjective.

Suppose that $\epsilon^*([v]) = 0$, so that $\epsilon(v) = d_{\mathrm{Tot}}(p) = \delta \pm d(p)$ for some p. Note that $\delta(p) = 0$. Thus, as before, we get $[p] = [(0, \dots, 0, p_n)]$ for some p_n, so that $\epsilon(v) = D(0, \dots, 0, p_n)$. This implies that $\delta(p_n) = 0$. Since each row is exact, there is some y such that $\epsilon(y) = p_n$. Then

$$\begin{aligned}\epsilon(v) &= \delta(p_n) + d(p_n)\\ &= 0 + d(\epsilon(y))\\ &= \epsilon(d(y)).\end{aligned}$$

Since ϵ is injective, it follows that v is a coboundary, i.e., $[v] = 0$. Therefore, ϵ^* is injective. $\square$

Definition 4.5.9. Let M be a smooth n-manifold and Λ be a countable poset. An open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of X is a *good cover* if for any chain $\alpha_0 \leq \cdots \leq \alpha_k$ in Λ, we have that

$$U_{\alpha_0 \dots \alpha_k} \coloneqq U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$$

is either empty or diffeomorphic to $\mathbb{R}^n$.

Note 4.5.10. It is well known that every paracompact smooth n-manifold M admits a Riemannian metric. Further, one can show that M can be covered by finitely many geodesically convex open sets. Thus, M has a finite good cover.

Definition 4.5.11. Let M be a paracompact smooth n-manifold, so that it has some finite good cover $\mathcal{U} \coloneqq \{U_{\alpha_i}\}$. The *Čech-de-Rham complex* $\check{C}^\bullet(\mathcal{U}; \Omega^\bullet)$ of $\mathcal{U}$ is the double complex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
\prod_{\alpha_0} \Omega^2(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1} \Omega^2(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1\alpha_2} \Omega^2(U_{\alpha_0\alpha_1\alpha_2}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1\alpha_2\alpha_3} \Omega^2(U_{\alpha_0\alpha_1\alpha_2\alpha_3}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\prod_{\alpha_0} \Omega^1(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1} \Omega^1(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1\alpha_2} \Omega^1(U_{\alpha_0\alpha_1\alpha_2}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1\alpha_2\alpha_3} \Omega^1(U_{\alpha_0\alpha_1\alpha_2\alpha_3}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots \\
\prod_{\alpha_0} \Omega^0(U_{\alpha_0}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1} \Omega^0(U_{\alpha_0\alpha_1}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1\alpha_2} \Omega^0(U_{\alpha_0\alpha_1\alpha_2}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0\alpha_1\alpha_2\alpha_3} \Omega^0(U_{\alpha_0\alpha_1\alpha_2\alpha_3}) \arrow[r, "\delta"] \arrow[u, "d"] & \cdots
\end{tikzcd}.$$

where

$$(\delta\omega)_{\alpha_0 \dots \alpha_{p+1}} \equiv \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}} \restriction_{U_{\alpha_0 \dots \alpha_{p+1}}}.$$

Example 4.5.12. Let $\omega \in \prod_{\alpha_0} \Omega^q(U_{\alpha_0})$. Then $(\delta\omega)_{\alpha_0\alpha_1} = \omega_{\alpha_1} \restriction_{U_{\alpha_0\alpha_1}} - \omega_{\alpha_0} \restriction_{U_{\alpha_0\alpha_1}}$. From this, we compute

$$\begin{aligned}(\delta\delta\omega)_{\alpha_0\alpha_1\alpha_2} &= (\delta\omega)_{\alpha_1\alpha_2} \restriction_{U_{\alpha_0\alpha_1\alpha_2}} - (\delta\omega)_{\alpha_0\alpha_2} \restriction_{U_{\alpha_0\alpha_1\alpha_2}} + (\delta\omega)_{\alpha_0\alpha_1} \restriction_{U_{\alpha_0\alpha_1\alpha_2}}\\ &= \omega_{\alpha_2} \restriction_{U_{\alpha_0\alpha_1\alpha_2}} - \omega_{\alpha_1} \restriction_{U_{\alpha_0\alpha_1\alpha_2}} - \omega_{\alpha_2} \restriction_{U_{\alpha_0\alpha_1\alpha_2}}\\ &+ \omega_{\alpha_0} \restriction_{U_{\alpha_0\alpha_1\alpha_2}} + \omega_{\alpha_1} \restriction_{U_{\alpha_0\alpha_1\alpha_2}} - \omega_{\alpha_0} \restriction_{U_{\alpha_0\alpha_1\alpha_2}}\\ &= 0.\end{aligned}$$

If X is a space, then let $C_{lc}(X, \mathbb{R})$ denote the additive group of locally constant functions $X \to \mathbb{R}$. Now, let $\underline{\check{C}^\bullet(\mathcal{U}; \Omega^\bullet)}$ denote the augmented Čech-de-Rham complex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[u] \arrow[r] & \Omega^2(M) \arrow[u, "d"'] \arrow[r, "\epsilon"] & \prod_{\alpha_0} \Omega^2(U_{\alpha_0}) \arrow[u, "d"'] \arrow[r, "\delta"] & \prod_{\alpha_0\alpha_1} \Omega^2(U_{\alpha_0\alpha_1}) \arrow[u, "d"'] \arrow[r, "\delta"] & \prod_{\alpha_0\alpha_1\alpha_2} \Omega^2(U_{\alpha_0\alpha_1\alpha_2}) \arrow[u, "d"'] \arrow[r, "\delta"] & \cdots \\
0 \arrow[u] \arrow[r] & \Omega^1(M) \arrow[u, "d"'] \arrow[r, "\epsilon"] & \prod_{\alpha_0} \Omega^1(U_{\alpha_0}) \arrow[u, "d"'] \arrow[r, "\delta"] & \prod_{\alpha_0\alpha_1} \Omega^1(U_{\alpha_0\alpha_1}) \arrow[u, "d"'] \arrow[r, "\delta"] & \prod_{\alpha_0\alpha_1\alpha_2} \Omega^1(U_{\alpha_0\alpha_1\alpha_2}) \arrow[u, "d"'] \arrow[r, "\delta"] & \cdots \\
0 \arrow[u] \arrow[r] & \Omega^0(M) \arrow[u, "d"'] \arrow[r, "\epsilon"] & \prod_{\alpha_0} \Omega^0(U_{\alpha_0}) \arrow[u, "d"'] \arrow[r, "\delta"] & \prod_{\alpha_0\alpha_1} \Omega^0(U_{\alpha_0\alpha_1}) \arrow[u, "d"'] \arrow[r, "\delta"] & \prod_{\alpha_0\alpha_1\alpha_2} \Omega^0(U_{\alpha_0\alpha_1\alpha_2}) \arrow[u, "d"'] \arrow[r, "\delta"] & \cdots \\
& & \prod_{\alpha_0} C_{lc}(U_{\alpha_0}, \mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \prod_{\alpha_0\alpha_1} C_{lc}(U_{\alpha_0\alpha_1}, \mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \prod_{\alpha_0\alpha_1\alpha_2} C_{lc}(U_{\alpha_0\alpha_1\alpha_2}, \mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \cdots \\
& & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

where $(\epsilon\omega)_{\alpha_0} \equiv \omega \restriction_{U_{\alpha_0}}$. The cohomology of the bottom row of $\underline{\check{C}^\bullet(\mathcal{U}; \Omega^\bullet)}$ is known as the *Čech cohomology of $\mathcal{U}$ with coefficients in* $\mathbb{R}$ and is denoted by $\check{H}^*(\mathcal{U}; \mathbb{R})$.

Lemma 4.5.13. *The rows of $\underline{\check{C}^\bullet(\mathcal{U}; \Omega^\bullet)}$ are exact.*

Proof. It is clear that $\delta\epsilon = 0$. Since $\delta^2 = 0$, it just remains to check that $\ker\delta \subset \operatorname{im}\delta$ and $\ker\delta \subset \operatorname{im}\epsilon$. Let $\omega \in \prod_{\alpha_0\dots\alpha_p} \Omega^q(U_{\alpha_0\dots\alpha_p})$ such that $\omega \in \ker\delta$. Then $(\delta\omega)_{\alpha\alpha_0\dots\alpha_p} = \omega_{\alpha_0\dots\alpha_p} + \sum_{i=0}^{p}(-1)^{i+1}\omega_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} = 0$, so that

$$\omega_{\alpha_0\dots\alpha_p} = \sum_{i=0}^{p}(-1)^i\omega_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p}.$$

Now, choose a partition of unity $\{\lambda_\alpha\}$ subordinate to $\mathcal{U}$. Define $s\omega \in \prod_{\alpha_0\dots\alpha_{p-1}} \Omega^q(U_{\alpha_0\dots\alpha_{p-1}})$ by

$$(s\omega)_{\alpha_0\dots\alpha_{p-1}} = \sum_\alpha \lambda_\alpha\omega_{\alpha\alpha_0\dots\alpha_{p-1}}.$$

We have that

$$\begin{aligned}
(\delta s\omega)_{\alpha_0\dots\alpha_p} &= \sum_{i=0}^{p}(-1)^i(s\omega)_{\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} \\
&= \sum_{i=0}^{p}\sum_\alpha(-1)^i\lambda_\alpha\omega_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} \\
&= \sum_\alpha\sum_{i=0}^{p}(-1)^i\lambda_\alpha\omega_{\alpha\alpha_0\dots\hat{\alpha}_i\dots\alpha_p} \\
&= \sum_\alpha \lambda_\alpha\omega_{\alpha_0\dots\alpha_p} \\
&= \omega_{\alpha_0\dots\alpha_p}.
\end{aligned}$$

This shows that $\delta s\omega = \omega$, so that $\omega \in \operatorname{im}\delta$, as desired.

The fact that $\ker\delta \subset \operatorname{im}\epsilon$ is seen by gluing differential forms on U_α together to get a form on M. $\square$

Proposition 4.5.14. *Let V be a smooth vector field and ω, η be differential forms. Let $\mathcal{L}$ denote the Lie derivative.*

1. $i_V(\omega \wedge \eta) = i_V(\omega) + \eta + (-1)^{|\omega|}\omega \wedge i_V\eta$.

2. ***(Cartan's magic formula)*** $\mathcal{L}_V\omega = i_V d(\omega) + d(i_V\omega)$.

Lemma 4.5.15 (Poincaré). *Let $F : M \times I \to N$ be a smooth map where N is a smooth manifold. Let i_{-} denote interior multiplication. For any $t \in I$, define $j_t : M \to M \times I$ by $j_t(p) = (p, t)$. For any $q \in \mathbb{N}$, define $\sigma : \Omega^q(N) \to \Omega^{q-1}(M)$ by*

$$(\sigma\omega)_p = (x \mapsto \int_0^1 j_t^* i_{\frac{\partial}{\partial t}}(F^*\omega)(x)dt)$$

where $x \in \bigwedge^{q-1}(T_p^*M)$. Then $(d\sigma + \sigma d)\omega = F_1^*\omega - F_0^*\omega$. That is, σ is a cochain homotopy $F_0^* \Rightarrow F_1^*$.

Proof. Using Proposition 4.5.14, we compute

$$\begin{aligned}
d\sigma\omega(x) + \sigma d\omega(x) &= \int_0^1 dj_t^* i_{\frac{\partial}{\partial t}}(F^*\omega)(x)dt + \int_0^1 j_t^* i_{\frac{\partial}{\partial t}}(F^*d\omega)(x)dt \\
&= \int_0^1 j_t^* di_{\frac{\partial}{\partial t}}(F^*\omega)(x)dt + \int_0^1 j_t^* i_{\frac{\partial}{\partial t}}(dF^*\omega)(x)dt \\
&= \int_0^1 j_t^* \mathcal{L}_{\frac{\partial}{\partial t}} F^*\omega(x)dt.
\end{aligned}$$

Note that the flow of $(0, \frac{\partial}{\partial t})$ is given by $\theta_t(p, s) = (p, t + s)$. Hence $F \circ \underbrace{\theta_t \circ j_0}_{j_t} = F_t$. It follows that

$$\begin{aligned}
j_t^* \mathcal{L}_{\frac{\partial}{\partial t}} F^*\omega &= j_0^* \theta_t^* \mathcal{L}_{\frac{\partial}{\partial t}} F^*\omega \\
&= j_0^* \frac{d}{dt} \theta_t^* F^*\omega \\
&= \frac{d}{dt} j_0^* \theta_t^* F^*\omega \\
&= \frac{d}{dt} F_t^*\omega.
\end{aligned}$$

As a result, we can use Stokes' theorem to get

$$\begin{aligned}
d\sigma\omega(x) + \sigma d\omega(x) &= \int_0^1 j_t^* \mathcal{L}_{\frac{\partial}{\partial t}} F^*\omega(x)dt \\
&= \int_0^1 \frac{d}{dt} F_t^*\omega(x)dt \\
&= \int_0^1 dF_t^*\omega(x) \\
&= F_1^*\omega(x) - F_0^*\omega(x).
\end{aligned}$$

$\square$

Corollary 4.5.16. *If $U \cong \mathbb{R}^n$, then the sequence*

$$\mathbb{R} \to \Omega^0(U) \to \Omega^1(U) \to \Omega^2(U) \to \cdots$$

is exact. Therefore, the columns of $\underline{\check{C}^\bullet(\mathcal{U}; \Omega^\bullet)}$ are also exact.

Proof. It is easy to check that our sequence is exact at $\Omega^0(U)$. We must show that it is exact at any $\Omega^i(U)$ with $i \geq 1$. There is some homotopy $F : U \times I \to U$ such that $F_0 = c_x$ and $F_1 = \mathrm{id}_U$. Let $\omega \in \Omega^i(U)$. We have that $(d\sigma + \sigma d)\omega = F_1^*\omega - F_0^*\omega = \omega - 0 = \omega$. Hence $d\sigma + \sigma d = \mathbb{1}_{\Omega^i(U)}$. $\square$

Corollary 4.5.17. *If $\mathcal{U}$ is a finite good cover of M, then $H^*_{\mathrm{dR}}(M) \cong \check{H}^*(\mathcal{U}; \mathbb{R})$.*

Theorem 4.5.18 (De Rham). *Let M be a smooth manifold and let $(\Omega^*(M), d)$ denote the cochain complex of differential forms. Then $H^*_{\mathrm{dR}}(M) \equiv H^*(\Omega^*(M), d)$. If M is paracompact, then*

$$H^*_{\mathrm{dR}}(M) \cong H^*(M; \mathbb{R})$$

as graded rings.

Proof. Choose a finite good cover $\mathcal{U}$ of M. We have just proven that $H^*_{\mathrm{dR}}(M) \cong \check{H}^*(\mathcal{U}; \mathbb{R})$. By a similar argument, we can also show that $H^*(M; \mathbb{R}) \cong \check{H}^*(\mathcal{U}; \mathbb{R})$. □

4.6 Lecture 24

5 Sheaves on spaces

Definition 5.0.1. Let X be a space and $\mathbf{Op}(X)$ denote the category of open sets in X. A *presheaf on X* is a functor $F : \mathbf{Op}(X)^{\mathrm{op}} \to \mathbf{Ab}$ such that $F(\emptyset) = 0$. If $V \subset X$ is open, then any $s \in F(V)$ is called a *section of U*. If $i : U \hookrightarrow V$, then $F(i) : F(V) \to F(U)$ is called a *restriction morphism* and is denoted by r_{UV}. Let $r_{UV}(s) \coloneqq s \restriction_U$. Let $\mathbf{PreSh}_X$ denote the category of presheaves on X.

Remark 5.0.2 (Čech cohomology). We can build a cohomology theory of presheaves as follows. Let X be a space with $\mathcal{U}$ an open cover. Let F be a presheaf on X. Let $\check{C}(\mathcal{U}, F)$ denote the chain complex

$$\prod_{\alpha_0} F(U_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0\alpha_1} F(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod_{\alpha_0\alpha_1\alpha_2} F(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots$$

where for any $f = (\underbrace{f_{\alpha_0\ldots\alpha_p}}_{\in F(U_{\alpha_0\ldots\alpha_p})})$, we take $(\delta f)_{\alpha_0\ldots\alpha_{p+1}} \equiv \sum_{i=0}^{p+1} (-1)^i f_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_{p+1}} \restriction_{U_{\alpha_0\cdots\alpha_{p+1}}}$. Define the *$p$-th Čech cohomology group* as

$$\check{H}^p(X, F) = \varinjlim_{\mathcal{R}} \check{H}^p(\mathcal{U}, F)$$

where $\mathcal{R}$ denotes the set of all open covers of X ordered by refinement.

Definition 5.0.3. A presheaf F on X is a *sheaf* if for any open $U \subset X$ and any open cover $\{U_\alpha\}$ of U, we have an exact sequence

$$0 \to F(U) \to \prod_{\alpha} F(U_\alpha) \xrightarrow{\tau} \prod_{\alpha_0\alpha_1} F(U_{\alpha_0\alpha_1})$$

where $(\tau f)_{\alpha_0\alpha_1} \equiv f_{\alpha_1} \restriction_{U_{\alpha_0\alpha_1}} - f_{\alpha_0} \restriction_{U_{\alpha_0\alpha_1}}$. Let $\mathbf{Sh}_X$ denote the category of sheaves on X, which is a full subcategory of $\mathbf{PreSh}_X$.

Example 5.0.4.

1. If X is a manifold, then the assignment $U \mapsto \Omega^q(U)$ defines a sheaf. Indeed, let $g, f \in \Omega^q(U)$ such that $f \restriction_{U_\alpha} = f \restriction_{U_\alpha}$ for any α. Then $g = f$. Moreover, if $f_\alpha \in \Omega^q(U_\alpha)$ satisfies $f_\alpha \restriction_{U_{\alpha\beta}} = f_\beta \restriction_{U_{\alpha\beta}}$ for every β, then there is some $f \in \Omega^q(U)$ such that $f \restriction_{U_\alpha} = f_\alpha$.

2. Given an abelian group A, the *constant presheaf with value A* is given by $F(U) = A$. This is not a sheaf. Let U and V be any two spaces. Then $\{U, V\}$ is a cover for $U \coprod V$, but

$$0 \to F(U \coprod V) \to F(U) \times F(V) \to F(U \cap V)$$

is not exact.

3. Given a set A endowed with the discrete topology, the *constant sheaf with value A* is given by

$$U \mapsto \{f : U \to A \mid f \text{ locally constant}\} = \{f : U \to A \mid f \text{ continuous}\}.$$

This is denoted by $\underline{A}$.

4. If X is a holomorphic manifold, then we have a sheaf $\mathcal{O}(-)$ of holomorphic functions on open subsets of X.
5. The assignment sending each U to the group $C(U)$ of functions $U \to \mathbb{R}$ is a sheaf.
6. The assignment sending each U to the group $C_b(U)$ of bounded continuous functions $U \to \mathbb{R}$ is a presheaf but not a sheaf. Indeed, for each $i \in \mathbb{Z}$, define $f_i : U_i \to \mathbb{R}$ by $f_i(x) = x$ where $U_i \coloneqq (i-1, i+1) \subset \mathbb{R}$. Each $f_i \in C_b(U_i)$, but you cannot glue the f_i to a global bounded continuous function.

Definition 5.0.5. Let F be a presheaf on X and $x \in X$. Define *the stalk F_x of F at x* as $\varinjlim_{U \ni x} F(U)$.

Example 5.0.6.

1. For any $x \in X$, $\underline{A}_x = A$.
2. The *germs of continuous functions at x* are precisely the elements of
$$C_x = {\{(U,f) \mid x \in U,\ f \in C(U)\}}/{\sim}$$
where $(U,f) \sim (V,g)$ if $f = g$ on $U \cap V$.
3. Let $X = \mathbb{R}$. Let $C^\infty(-)$ denote the sheaf of smooth functions on X. Then we have an exact sequence
$$0 \to (\text{flat functions}) \to C_0^\infty \to \mathbb{R}[\![x]\!] \to 0.$$
4. Let $\mathcal{A}$ denote the sheaf of real analytic functions. Then $\mathcal{A}_0 \cong \mathbb{R}\{x\}$, the ring of real analytic functions.

Definition 5.0.7. Let F be a presheaf on X. Let $U \subset X$ be open and $s \in F(U)$. Define $\bar{s} : U \to \coprod_{x \in X} F_x$ by $x \mapsto (x, (U,s))$. Define the *étalé space of F* as
$$\text{Ét}(F) = \left(\coprod_{x \in X} F_x, \tau \right)$$
where
$$\tau = \{S \subset \text{Ét}(F) \mid \bar{s}^{-1}(S) \text{ is open},\ s \in F(U),\ U \subset X\}.$$
Equivalently, τ is generated by sets of the form $\text{Ét}(U,s) \coloneqq \{(U, s_x) \mid x \in U\}$.

Proposition 5.0.8. *The natural projection $\pi : \text{Ét}(F) \to X$ is a local homeomorphism.*

Note 5.0.9. We have a functor $\mathbf{PreSh}_X \xrightarrow{+} \mathbf{Sh}_X$ given by $F \mapsto F^+$ where F^+ is given by
$$U \mapsto \{s : U \to \text{Ét}(F) \mid \pi s(x) = x \text{ and } s \text{ is continuous}\}.$$
This is left adjoint to the forgetful functor $\mathbf{Sh}_X \to \mathbf{PreSh}_X$.

5.1 Lecture 25

Lemma 5.1.1. *Let $F, G \in \operatorname{ob} \mathbf{Sh}_X$ and $f, g : F \to G$.*

(a) If $f_x = g_x$ for every $x \in X$ where $g_x, f_x : F_x \to G_x$, then $f = g$.

(b) $f_x : F_x \to G_x$ is injective for every x if and only if $f_U : F(U) \to G(U)$ is injective for every open $U \subset X$.

(c) $f_x : F_X \to G_x$ is an isomorphism if and only if f is an isomorphism.

Proof.

(a) Consider the commutative square

$$\begin{array}{ccc} F(U) & \xrightarrow{\alpha} & \prod_{x\in U} F_x \\ \downarrow & & \downarrow{\scriptstyle \prod_{x\in U} f_x = \prod_{x\in U} g_x} \\ G(U) & \xrightarrow[\beta]{} & \prod_{x\in U} G_x \end{array}.$$

Note that α and β are injective by sheaf-hood. Since $f_x = g_x$ for every $x \in U$, we can replace $f(U)$ by $G(U)$ and still make our square commute.

(b) This follows from our commutative square.

(c) Part (b) shows that $f(U) : F(U) \to G(U)$ is injective. Hence we must show that it's surjective. Let $s \in G(U)$. For any $x \in U$, there exist $U_x \ni x$ and $s^x \in F(U_x)$ such that $f(s^x)_x = s_x$. Now use sheafiness to glue the s^x together.

$\square$

Proposition 5.1.2. *There exists a natural transformation $\mathbb{1} \xrightarrow{U} +$ that is an isomorphism on stalks, i.e., $U_x : F_x \to F_x^+$ is an isomorphism for every x.*

Definition 5.1.3. Given any $f : F \to G$ with $F, G \in \operatorname{ob} \mathbf{Sh}_X$, let $\ker f(U) \equiv \ker(f(U) : F(U) \to G(U))$ for each open $U \subset X$. Then $\ker f$ is a sheaf. Let $\operatorname{coker}^- f(U) \equiv \operatorname{coker}(f(U) : F(U) \to G(U))$. This induces a presheaf $\operatorname{coker}^- f$. Finally, let $\operatorname{coker} f \equiv \operatorname{coker}^- f^+$.

Proposition 5.1.4. *$\mathbf{Sh}_X$ is an abelian category.*

Exercise 5.1.5. *There are no projectives in $\mathbf{Sh}_X$.*

Proposition 5.1.6 (Baer). *Let I be a right R-module. Then I is injective if and only if for any right ideal $J \trianglelefteq R$, any map $\alpha : J \to I$ extends to a map $\tilde{\alpha} : R \to I$.*

Proof. This is an application of Zorn's lemma. $\square$

Corollary 5.1.7. *There are enough injectives in $R^{\mathrm{op}}-\mathbf{Mod}$, i.e., any object can be embedded in an injective object.*

Proposition 5.1.8. *There are enough injectives in $\mathbf{Sh}_X$.*

Proof. If $\{I_x\}_{x\in X}$ denotes a family of abelian groups , then define $\underline{I}(U) = \prod_{x\in U} I_x$ and $\operatorname{Hom}_{\mathbf{Sh}}(F, \underline{I}) = \prod_{x\in X} \operatorname{Hom}(F_x, I_x)$. As a result, if I_x is injective (i.e., divisible), then I will be injective. Given any sheaf F, choose I_x injective such that $F_x \hookrightarrow I_x$. By doing this, we can form an injective object I so that $0 \to F \to I$ is exact. $\square$

Definition 5.1.9. Given any diagram of sheaves

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0,$$

we say that this is *exact* if $0 \to A_x \xrightarrow{i_x} B_x \xrightarrow{j_x} C_x \to 0$ is exact for each $x \in X$.

Proposition 5.1.10. *If $0 \to A \to B \to C \to 0$ is exact in $\mathbf{Sh}_X$, then $0 \to A(X) \to B(X) \to C(X)$ is exact in $\mathbf{Ab}$.*

Example 5.1.11. Let $X = \mathbb{C}^\times$. Then

$$0 \to \underline{\mathbb{Z}} \xrightarrow{i} \mathcal{O}_X \xrightarrow{g} \mathcal{O}_X^X \to 0$$

is exact where $\mathcal{O}_X^X(U) \equiv \{\varphi : U \to \mathbb{C} \text{ holomorphic} \mid \varphi(x) \neq 0 \text{ for any } x \in U\}$ and $g : f \mapsto e^{2\pi i f}$.

Proof. Clearly, i is injective. If $f \in \mathcal{O}_X(U)$ and $e^{2\pi i f} = 1$, then $f : U \to \mathbb{C}$ is locally constant and $f(x) \in \mathbb{Z}$. If $\varphi \in \mathcal{O}^X_{X,x}$, then there exist a small open simply connected set U and $\tilde{\varphi} \in \mathcal{O}^X_X(U)$ such that $\tilde{\varphi}_x = \varphi$ and $\log \tilde{\varphi}_x$ exists. Globally, however, $z \in \mathcal{O}^X_X(\mathbb{C}^\times)$ cannot be hit by $e^{2\pi i \varphi}$. □

Definition 5.1.12. Let F be a sheaf on X. A *resolution of* F is an exact sequence of the form

$$0 \to F \to F^0 \to F^1 \to F^2 \to \cdots .$$

Example 5.1.13.

1. Let X be a smooth manifold. Then

$$0 \to \mathbb{R} \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \to \cdots$$

is a resolution of $\mathbb{R}$.

Proof. This is precisely Poincaré's lemma. □

2. Let X be locally contractible. Define $C^p(U)$ as the singular $\mathbb{Z}$-valued p-cochains in the open $U \subset X$. This defines a presheaf on X. Let C^p_X denote its sheafification. Then

$$0 \to \underline{\mathbb{Z}} \to C^0_X \xrightarrow{\delta} C^1_X \xrightarrow{\delta} C^2_X \to \cdots$$

is a resolution.

Definition 5.1.14 (Sheaf cohomology). To derive $\mathrm{Ext}^*_{\mathbf{Sh}}(F, G)$, take an injective resolution $I^\bullet$ of G and then form the complex

$$\mathrm{Hom}_{\mathbf{Sh}}(F, I^0) \to \mathrm{Hom}_{\mathbf{Sh}}(F, I^1) \to \cdots .$$

Then define $\mathrm{Ext}^*_{\mathbf{Sh}_X}(F, G)$ as the cohomology of this complex. Note that $\mathrm{Hom}_{\mathbf{Sh}}(\underline{\mathbb{Z}}, I^\bullet) = I^\bullet(X)$. We define the cohomology $H^*(X, G)$ of G as $\mathrm{Ext}_{\mathbf{Sh}}(\underline{\mathbb{Z}}, G)$. Standard homological algebra shows that this si independent of our choice of injective resolution.

Definition 5.1.15. A sheaf F on X is *flabby* if for any U, $F(X) \to F(U)$ is surjective. Equivalently, a sheaf F on X is flabby if $F(V) \to F(U)$ is surjective for any inclusion $U \subset V$.

Example 5.1.16 (Godement's construction). Given any sheaf G, define a new sheaf $\mathrm{Gd}(G)$ by

$$\mathrm{Gd}(G)(U) = \prod_{x \in U} G_x$$
$$\mathrm{Gd}(G)(i : U \hookrightarrow V) = (s \mapsto s \restriction_V)$$
$$.$$

Then we have an exact sequence $0 \to G \to \mathrm{Gd}(G) \to H_0 \to 0$ where $H_0 \equiv \mathrm{coker}(G \to \mathrm{Gd}(G))$. From this, we can form the exact sequence $0 \to H_0 \to \mathrm{Gd}(H_0) \to H_1 \to 0$ where $H_1 \equiv \mathrm{coker}(H_0 \to \mathrm{Gd}(H_0))$, and so on. What results is a resolution

$$0 \to \mathrm{Gd}(G) \to \mathrm{Gd}(H_0) \to \mathrm{Gd}(H_1) \to \cdots .$$

Note that $\mathrm{Gd}(G)$, as well as each sheaf $\mathrm{Gd}(H_i)$, is flabby.

Definition 5.1.17. A sheaf F on X is *soft* if for any closed set C, the map $F(X) \to F(C)$ is surjective in X where $F(C) \equiv \varinjlim_{U \supset C} F(U)$.

Example 5.1.18. Ω^0_X is soft but not flabby.

5.2 Lecture 26

Lemma 5.2.1.

1. *Any injective sheaf is flabby.*
2. *If $0 \to E \to F \to G \to 0$ is an exact sequence of sheaves and E is flabby, then $0 \to E(X) \xrightarrow{\iota} F(X) \xrightarrow{j} G(X) \to 0$ is exact. (Note that $0 \to E(X) \to F(X) \to G(X)$ is exact automatically.)*

Proof.

1. Let I be injective, We can embed I in a flabby sheaf F via a map i. Since I is injective, we get a map ρ such that $\rho i = \mathbb{1}_I$. For any open $U \subset X$, we have a commutative diagram

$$\begin{tikzcd} F(X) \arrow[r, "r_{UX}"] \arrow[d, "\rho_X"'] & F(U) \arrow[d, "\rho_U"] \\ I(U) \arrow[r] & I(U) \end{tikzcd} .$$

Note that r_{UX} is surjective because F is flabby and that both ρ_X and ρ_U are surjective because $\rho i = \mathbb{1}_I$. Hence s_{UX} is surjective.

2. Let $g \in G(X)$ We want to show that there exists $f \in F(X)$ with $\iota f = g$. Let $\mathcal{A} \coloneqq \{(U, u) \mid U \subset X$ open, $u \in F(U)$ with $g(u) = g \restriction_U\}$. We see that $\mathcal{A}$ is partially ordered by $\leq$ where $(U_1, u_1) \leq (U_2, u_2)$ if $U_1 \subset U_2$ and $u_1 = u_2 \restriction_{U_1}$. It also satisfies the hypotheses of Zorn's lemma. so that there is some maximal element (U, u) of $\mathcal{A}$. Suppose, towards a contradiction, that $U \neq X$. Choose any $x \in X \setminus U$ and choose a neighborhood $V \ni x$ and $x \in F(V)$ with $j(v) = s \restriction_V$. Then we have that

$$j(\underbrace{r_{V\cap U,U}(u) - r_{V\cap U,V}(v)}_{\tilde{r}}) = 0$$

in $G(U \cap V)$. Thus, $\tilde{r}$ comes from $E(U \cap V)$. Since E is flabby, $\tilde{r}$ extends to a section $\tilde{\tilde{r}}$ in $E(V)$. Consider $v + \tilde{\tilde{r}}$. Observe that $u \restriction_{U\cap V} = v + \tilde{\tilde{r}} \restriction_{U\cap V}$, and thus these piece together by the sheaf property to a section $\tilde{u}$ in $F(U \cup V)$ such that $j(\tilde{u}) = g \restriction_{\tilde{U}}$. This is a contradiction.

$\square$

Proposition 5.2.2. *If $0 \to E \to F \to G \to 0$ is exact and both E and F are flabby, then G is flabby.*

Lemma 5.2.3. *If F is flabby, then $H^n(X, F) = 0$ for each $n \geq 1$.*

Proof. Let $0 \to F \to I \to G \to 0$ be exact where I is injective (hence flabby). Then G is flabby. We have a LES

$$H^0(X, F) \to H^0(X, I) \to H^0(X, G) \to H^1(X, F) \to H^1(X, I),$$

where the map $H^0(X, I) \to H^0(X, G)$ is surjective since F is flabby and $H^1(X, I) = 0$ since I is injective. Therefore, $H^1(X, F) = 0$.

Assume, inductively, that we have shown that $H^k(X, F) = 0$ where $0 < k < n$ for any flabby sheaf F. Then $H^{n-1}(X, G) = 0$. since G is flabby. Also, $H^n(X, I) = 0$ since I is injective. In light of the exact sequence $H^{n-1}(X, G) \to H^n(X, F) \to H^n(X, I)$, it follows that $H^n(X, F) = 0$. $\square$

Lemma 5.2.4. *If E is a sheaf on X and*

$$0 \to E \to F^0 \to F^1 \to F^2 \to \cdots$$

is a flabby resolution, then $H^*(X, E) \cong H^*(\Gamma(X, F^\bullet))$ where $\Gamma(X, -)$ denotes the global section functor.

Proof. We can form an augmented double complex of sheaves

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
 & I^{0,2} \arrow[u] \arrow[r] & I^{1,2} \arrow[u] \arrow[r] & I^{2,2} \arrow[u] \arrow[r] & \cdots \\
 & I^{0,1} \arrow[u] \arrow[r] & I^{1,1} \arrow[u] \arrow[r] & I^{2,1} \arrow[u] \arrow[r] & \cdots \\
 & I^{0,0} \arrow[u] \arrow[r] & I^{1,0} \arrow[u] \arrow[r] & I^{2,0} \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & E \arrow[r] & F^0 \arrow[u] \arrow[r] & F^1 \arrow[u] \arrow[r] & F^2 \arrow[u] \arrow[r] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

such that each $I^{i,\bullet}$ is an injective resolution of F^i. After applying the global section functor to this complex, we obtain a new augmented double complex

$$\begin{tikzcd}
 & & \vdots & \vdots & \vdots & \\
 & & I^{0,2}(X) \arrow[u] \arrow[r] & I^{1,2}(X) \arrow[u] \arrow[r] & I^{2,2}(X) \arrow[u] \arrow[r] & \cdots \\
 & & I^{0,1}(X) \arrow[u] \arrow[r] & I^{1,1}(X) \arrow[u] \arrow[r] & I^{2,1}(X) \arrow[u] \arrow[r] & \cdots \\
 & & I^{0,0}(X) \arrow[u] \arrow[r] & I^{1,0}(X) \arrow[u] \arrow[r] & I^{2,0}(X) \arrow[u] \arrow[r] & \cdots \\
0 \arrow[r] & E(X) \arrow[r] & F^0(X) \arrow[u] \arrow[r] & F^1(X) \arrow[u] \arrow[r] & F^2(X) \arrow[u] \arrow[r] & \cdots \\
 & & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}.$$

The columns of our new complex are acyclic.

Claim. $\mathrm{Tot}(I^{\bullet,\bullet}, d_{\mathrm{Tot}})$ *forms an injective resolution of* E.

Proof. Note that the direct sum of two injective abelian groups is injective.[2] Consider the sequence

$$0 \longrightarrow E(X) \xrightarrow{g\circ f} I^{0,0} \xrightarrow{\delta+d} I^{1,0} \oplus I^{0,1} \longrightarrow I^{2,0} \oplus I^{1,1} \oplus I^{0,2} \longrightarrow \cdots$$

where $f : E(X) \to F^0(X)$ and $g : F^0 \to I^{0,0}$. Clearly $g \circ f$ is injective. Moreover, since $\mathrm{Tot}(I^{\bullet,\bullet})$ is bounded and has exact columns, it follows that each term of our sequence after $I^{0,0}$ is exact. Thus, it remains to verify that out sequence is exact at $I^{0,0}$.

Suppose that $x \in \ker \delta + d$. Then $(\delta(x), d(x)) = (0, 0)$, so that $g(y) = x$ for some $y \in F^0(X)$. Since g is injective and

$$g(\delta(y)) = \delta(g(y)) = \delta(x) = 0$$

[2]One can find online that the coproduct of two injective R-modules is injective if and only if R is Noetherian.

, we see that $\delta(y) = 0$. Hence there is some $y' \in E(X)$ such that $f(y') = y$. This implies that $g(f(y')) = x$, which thus belongs to $\operatorname{im} g \circ f$.

Conversely, suppose that $x = g(f(y))$ for some $y \in E(X)$. Then

$$\begin{aligned} \delta + d(g(f(y))) &= (\delta(g(f(y))), d(g(f(y)))) \\ &= (g(\delta(f(y))), d(g(f(y)))) \\ &= (0, 0) \end{aligned}$$

since $\delta \circ f = 0 = d \circ g$. Therefore, $x \in \ker \delta + d$. □

It follows that

$$H^*(X, E) \cong H^*(I^{\bullet,\bullet}, d, \delta) \cong H^*(\Gamma(X, F^\bullet)).$$

□

Remark 5.2.5. Let X be a space and $\underline{R}$ be a sheaf of rings over X. Let $\mathbf{Sh}_R(X)$ denote the category of sheaves of R-modules on X. Let M be a sheaf of R-modules and consider the commutative diagram

$$\begin{array}{ccc} \underline{R}(U) \times M(U) & \longrightarrow & M(U) \\ \downarrow & & \downarrow \\ \underline{R}(V) \times M(V) & \longrightarrow & M(V) \end{array} .$$

Flabbiness still suffices to compute $H^*(X, M) \equiv \operatorname{Ext}_{\mathbf{Sh}_R(X)}(R, M)$. Nevertheless, it does not suffice to compute $\operatorname{Ext}_{\mathbf{Sh}_R(X)}(N, M)$.

Proposition 5.2.6. *Let E be soft and $0 \to E \to F \to G \to 0$ be exact. Then $0 \to E(X) \to F(X) \to G(X) \to 0$ is exact.*

Definition 5.2.7. Let X be paracompact. A sheaf E on X is *fine* if for any locally finite cover $\{U_\alpha\}$, there exists a family of homomorphisms $\lambda_\alpha : E \to E$ such that

- $\sum_\alpha \lambda_\alpha = 1$ and
- for each α, there is some neighborhood N of ${U_\alpha}^c$ such that $\lambda_\alpha(x) = 0$ whenever $x \in N$.

Proposition 5.2.8.

1. *Any flabby sheaf is soft.*
2. *If $0 \to E \to F \to G \to 0$ is exact and both E and F are soft, then G is soft.*
3. *If E is soft, then $H^k(X, E) = 0$ for any $k > 0$.*
4. *Let*
$$0 \to E \to F^0 \to F^1 \to \cdots$$
be a resolution where each F^i is soft. Then $H^*(X, E) \cong H^*(F^\bullet(X))$.

Example 5.2.9.

1. The sheaf on X of continuous functions is soft.
2. If X is a smooth manifold, then the sheaf on X of smooth functions is soft due to the existence of bump functions.

Corollary 5.2.10. *The de Rham theorem.*

Proof. We know that each Ω^i_X is soft. Moreover, the sequence

$$0 \to \underline{\mathbb{R}}_X \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2(X) \to \cdots$$

is exact by the Poincaré lemma. $\square$

Note 5.2.11. Let $f : X \to Y$ be a continuous function. Let E be a sheaf on X and G be a sheaf on Y. We define $f_*E \in \mathbf{Sh}_Y$ by

$$(f_*E)(U) = E(f^{-1}(U)).$$

Also, we define $f^*G \in \mathbf{Sh}_X$ by

$$f^*G(V) = (\varinjlim_{U \supset f(V)} G(U))^+.$$

Consider the commutative square

$$\begin{tikzcd} X \times_Y \acute{\mathrm{E}}\mathrm{t}(G) \arrow[r] \arrow[d, "\pi"'] & \acute{\mathrm{E}}\mathrm{t}(G) \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

Then $f^*G(V)$ equals the section over V of $\pi : X \times_Y \acute{\mathrm{E}}\mathrm{t}(G) \to V$.

Proposition 5.2.12. $(f^*G)_x \cong G_{f(x)}$.

Corollary 5.2.13. *If* $0 \to E \to F \to G \to 0$ *is an exact sequence in* $\mathbf{Sh}_Y$*, then* $0 \to f^*E \to f^*F \to f^*G \to 0$ *is exact in* $\mathbf{Sh}_X$*.*

Proposition 5.2.14. (f^*, f_*) *is an adjoint pair.*

Corollary 5.2.15. f_* *takes injectives to injectives.*

Proof. Let I be injective. Let $0 \to E \to F \to G \to 0$ be an exact sequence in $\mathbf{Sh}_Y$. We must show that

$$0 \to \mathrm{Hom}_{\mathbf{Sh}_Y}(G, f_*I) \to \mathrm{Hom}_{\mathbf{Sh}_Y}(F, f_*I) \to \mathrm{Hom}_{\mathbf{Sh}_Y}(E, f_*I) \to 0$$

is exact. But by adjointness, we see that

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathbf{Sh}_Y}(G, f_*I) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbf{Sh}_Y}(F, f_*I) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathbf{Sh}_Y}(E, f_*I) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathrm{Hom}_{\mathbf{Sh}_Y}(f^*G, I) \arrow[r] & \mathrm{Hom}_{\mathbf{Sh}_Y}(f^*F, I) \arrow[r] & \mathrm{Hom}_{\mathbf{Sh}_Y}(f^*E, I) \arrow[r] & 0 \end{tikzcd}$$

is commutative. Since I is injective, it follows that the bottom row is exact. Hence the top row is exact as well, as required. $\square$

Corollary 5.2.16. *Our proof shows that any right adjoint preserves injectives.*

Remark 5.2.17. Let $f : X \to \mathsf{pt}$ be a map. Then $f_*E = \Gamma(X, E)$ for any sheaf E on X. If $f : X \to B$ is a fiber bundle, then

$$f_*E(U \subset B) = E(\underbrace{f^{-1}(U)}_{U \times B}).$$

5.3 Lecture 27

Definition 5.3.1. Let R be a sheaf of rings on X. A *sheaf F of R-modules* is a sheaf on X such that for any inclusion $V \subset U$ of open sets in X,

- $F(U)$ is an $R(U)$-module and
- if $f \in R(U)$ and $g \in F(U)$, then $(fg)\restriction_V = f \restriction_V g \restriction_V$.

Proposition 5.3.2. *Let Z and W be closed in the space X. Let F be sheaf on X. If $s \in F(Z)$ and $t \in F(W)$ satisfy $s \restriction_{Z\cap W} = t \restriction_{Z\cap W}$, then there exists a unique $w \in F(Z \cup W)$ extending s and t.*

Lemma 5.3.3. *If R is a soft sheaf of rings on X, then any sheaf F of R-modules is soft.*

Proof. Let $C \subset X$ be closed and let $s \in F(C)$. There exist a neighborhood U of C and a section $s' \in F(U)$ that represents s.

Let $L \subset U$ be a compact set in X such that $\mathrm{Int}(L) \supset C$. Let t be the section equal to 1 on C and t' be the section equal to 0 on ∂L. By Proposition 5.3.2, we can glue these to get a section w over $C \cup \partial L$. By softness, we can extend w to a section $w' \in R(L)$. Again, we can glue w and the section equal to 0 on $\mathrm{cl}(U \setminus L)$ to get a section $m \in R(U)$.

Note that

$$(m \cdot s') \restriction_C = m \restriction_C \cdot s' \restriction_C = s.$$

Now, glue $(m \cdot s') \restriction_L$ and the section y which is equal to 0 on $\mathrm{cl}(C \setminus L)$ to get a section $y' \in F(X)$ such that $y' \restriction_C = s$. This proves that the homomorphism $F(X) \to F(C)$ is surjective. $\square$

Corollary 5.3.4. *If X is a smooth manifold, then the sheaf $\Omega^i(X)$ of i-forms is soft because it is a module over the sheaf of smooth functions.*

Note 5.3.5. Let X be a smooth manifold. We have that $0 \to \underline{\mathbb{R}} \to \Omega^\bullet_X$ is a soft resolution and that $\underbrace{H^*(X, \underline{\mathbb{R}})}_{\text{Čech cohom.}} \cong H^*(\Omega^\bullet(X), d)$.

Note 5.3.6. Let A be a ring. For each $p \in \mathbb{N}$, define the presheaf $U \subset X \mapsto \{f : \mathrm{Sing}_p(U) \to A\}$. Sheafify this to get $\mathfrak{S}^p(A)$. Define the map of sheaves $\delta : \mathfrak{S}^p(A) \to \mathfrak{S}^{p+1}(A)$ by

$$\delta_U(f : \mathrm{Sing}_p(U) \to A)(\sigma : \Delta^{p+1} \to A) = \sum_{i=0}^{p+1} (-1)^i f\sigma \restriction_{\partial_i \Delta^{p+1}}$$

where $\partial_i \Delta^p = [e_0, \ldots, e_{i-1}, \hat{e}_i, e_{i+1}, \ldots, e_p]$. This induces a resolution $0 \to \underline{A} \to \mathfrak{S}^\bullet(A)$.

Likewise, if X is a smooth manifold, then define the sheaf $U \mapsto \{f : \mathrm{Sing}_p^\infty(U) \to A\}$ where $\mathrm{Sing}_p^\infty(U)$ denotes the free abelian group on the set of smooth maps $\sigma : \Delta^p \subset \mathbb{R}^{p+1} \to U$, i.e., all σ with a smooth extension to some neighborhood of Δ^p. Sheafify this to get $\mathfrak{S}^p_\infty(A)$. We still have a resolution

$$0 \to \underline{A} \to \mathfrak{S}^\bullet_\infty(A).$$

Lemma 5.3.7. *Let X be a smooth manifold. Define the map $I^p : \Omega^p_X \to \mathfrak{S}^p_\infty(\mathbb{R})$ by*

$$I^p(\omega \in \Omega^p(U))(\sigma \in \mathrm{Sing}_p^\infty(U)) = \int_{\Delta^p} \sigma^* \omega.$$

Then $I^\bullet$ is a chain map $\Omega^\bullet_X \to \mathfrak{S}^\bullet_\infty(\mathbb{R})$.

Proof. Let $\omega \in \Omega^p(U)$ and $\sigma \in \mathrm{Sing}^\infty_{p+1}(U)$, We can apply Stokes' theorem to get

$$\begin{aligned} I(d\omega)(\sigma) &= \int_{\Delta^{p+1}} \sigma^* d\omega \\ &= \int_{\Delta^{p+1}} d\sigma^* \omega \\ &= \sum_{i=0}^{p+1} (-1)^i \int_{\partial_i \Delta^{p+1}} \sigma^* \omega \\ &= \delta(I(\omega)). \end{aligned}$$

□

Theorem 5.3.8 (De Rham). *$I^\bullet$ induces an isomorphism $I : H^p(\Omega^\bullet(X)) \to H^p(\mathfrak{S}^\bullet_\infty(X;\mathbb{R}))$ for each p.*

Proof. It suffices to show that $\mathfrak{S}^\bullet_\infty(X;\mathbb{R})$ is soft. We show that it is, in fact, flabby. Since each $\mathfrak{S}^p_\infty(X;\mathbb{R})$ is a module over $\mathfrak{S}^0_\infty(X;\mathbb{R})$, it suffices to show that $\mathfrak{S}^0_\infty(X;\mathbb{R})$ is flabby. Note that if $U \subset X$ is open, then $\mathfrak{S}^0_\infty(U;\mathbb{R}) = \{f : \mathrm{Sing}^\infty_0(U) \to \mathbb{R}\}$, which is precisely the set of all maps $\mathbb{Z}[U] \to \mathbb{R}$. Since we can extend any such function to a map $\mathbb{Z}[X] \to \mathbb{R}$, we see that $\mathfrak{S}^0_\infty(X;\mathbb{R})$ is flabby. □

Theorem 5.3.9. *Let X be paracompact and let F be a sheaf on X. Suppose that $\mathcal{U} \coloneqq \{U_\alpha\}$ is an open cover of X such that $U_{\alpha_0\alpha_1\ldots\alpha_p}$ either is empty or has $H^k(U_{\alpha_0\alpha_1\ldots\alpha_p}, F) = 0$ for any $k > 0$. Then $H^*(X,F) \cong \check{H}^*(\mathcal{U}, F)$.*

Corollary 5.3.10. *If X is locally contractible and paracompact, then $\underbrace{H^*(X;\mathbb{Z})}_{singular} \cong H^*(X, \underline{\mathbb{Z}})$.*

Note 5.3.11. Let $f : X \to Y$ be a map of spaces and F be a sheaf on X. Define the *i-th right derived functor* as

$$\mathcal{R}^i f_* F = (U \subset Y \mapsto H^i(f^{-1}(U), F))^+.$$

This is equivalent to the sheaf $\underline{H}^i(f_* I) \equiv (U \subset Y \mapsto H^i(\Gamma(U, f_* I)))$ where $I^\bullet$ is an injective resolution of F. We can view $\mathcal{R} f_* F$ as a complex

$$f_* I^0 \to f_* I^1 \to f_* I^2 \to \cdots$$

of injective sheaves on Y.

Furthermore, given a complex of sheaves $(C^\bullet, \delta)$ on X, the *cohomology sheaf* $\underline{H}^i(C^\bullet)$ is given by the sheafification of $U \mapsto H^i(C^\bullet(U), \delta)$. Note that $\underline{H}^i(C^\bullet)_x = H^i(C^\bullet_x, \delta)$ for any $x \in X$.

Consider the augmented double complex

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ I^{0,1} & \longrightarrow & I^{1,1} & \longrightarrow & I^{2,1} & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ I^{0,0} & \longrightarrow & I^{1,0} & \longrightarrow & I^{2,0} & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ C^o & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots \end{array}$$

where each $C^i \to I^{i,\bullet}$ is an injective resolution. Define the *i-th cohomology group* as

$$\mathbb{H}^i(X, C^\bullet) = H^i(\mathrm{Tot}\,\Gamma(X, I^{\bullet,\bullet})).$$

We have that

$$\mathbb{H}^*(Y, \mathcal{R} f_* F) \cong H^*(X, F).$$

6 Vector bundles

Definition 6.0.1. Let X be a space. A *vector bundle on* X is a triple (X, E, π) such that

- E is a space (called the *total space*),
- $\pi : E \to X$ is continuous surjection,
- for any $x \in X$, there exist an open $U \ni x$ in X and a homeomorphism $\varphi_U : \pi^{-1}(U) \to U \times E_0$ (called a *local trivialization*) where E_0 is a vector space,
- $\varphi_U \restriction_x : E_x := \pi^{-1}(x) \to \{x\} \times E_0$ is an isomorphism of vector spaces for each $x \in X$, and
- if $p_1 : U \times E_0 \to U$ denotes the first projection map, then

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi_U"] \arrow[dr, "\pi"'] & U \times E_0 \arrow[d, "p_1"] \\ & U \end{tikzcd}$$

  commutes.

We call the dimension of E_0 the *rank of the bundle.* If π is smooth and each φ_U is a diffeomorphism, then we say that E is a *smooth vector bundle on* X.

Example 6.0.2.

1. $X \times E_0$ is called the *trivial vector bundle.*
2. The Möbius strip, i.e., ${[0,1] \times \mathbb{R}}/{\sim}$ where $(0, -t) \sim (1, t)$.
3. Let X be a smooth manifold. Then the tangent bundle TX is a vector bundle.
4. If G is a Lie group, then $TG \cong G \times T_0G$ via $(L_g)_*\xi \leftarrow\!\shortmid (g, \xi)$ where L_g denotes left translation by g.

Note 6.0.3.

1. Let (X, E, π) be a vector bundle. Given two distinct points $x, y \in X$, we can find neighborhoods $U \ni x$, $V \ni y$ and homeomorphisms φ_U, φ_V. Consider the composition

$$(U \cap V) \times E_0 \xrightarrow{\varphi_V^{-1}} \pi^{-1}(U \cap V) \xrightarrow{\varphi_U} (U \cap V) \times E_0.$$

   Let $g_{UV}(x, e) = \varphi_U(\varphi_V^{-1}(x, e))$ for each $(x, e) \in (U \cap V) \times E_0$. Thus, we can write $g_{UV}(x, e) = (x, g_{UV}(x)(e))$ where $g_{UV}(x) : E_0 \to E_0$ is a linear isomorphism. Note that $g_{UU}(x) = \mathbb{1}_{E_0}$ for any $x \in U$ and that $g_{UV}g_{VW} = g_{UW}$. We call g_{UV} a *transition function for* E.

2. Let X be a smooth manifold and $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X. Let the transition function $g_{\alpha\beta} : U_{\alpha\beta} \to \mathcal{L}(E_0)$ be smooth for any $\alpha, \beta \in I$ such that $g_{\alpha\alpha} = \mathbb{1}$ and $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$ on $U_{\alpha\beta\gamma}$. Then there exists a smooth vector bundle on X, namely

$${\coprod_\alpha U_\alpha \times E_0}/{\sim}$$

   where $U_{\alpha_0\alpha_1} \times E_0 \ni (x, e) \sim (x, g_{\alpha_0\alpha_1}(x)(e)) \in U_{\alpha_0} \times E_0$. We call $\{U_\alpha\}$ a *trivializing cover.*

6.1 Lecture 28

Notation.

1. Let $\pi : E^{k/\mathbb{F}} \to X$ denote an $\mathbb{F}$-vector bundle of rank k.

2. Let $C^\infty(X; E)$ denote the space of smooth sections of a given smooth bundle (X, E, π).

Exercise 6.1.1. *Let $\pi : E \to X$ be a smooth vector bundle.*

1. *Let $\{U_\alpha\}$ be an open cover of X and $\{g_{\alpha\beta}\}$ a collection of transition functions. Show that $C^\infty(X; E) \cong \{s_\alpha : U_\alpha \to \mathbb{R}^k \mid g_{\alpha\beta} s_\beta = s_\alpha\}$.*

2. *Consider $\mathbb{CP}^n$. There exists a rank one tautological bundle $\gamma \xrightarrow{\pi} \mathbb{CP}^n$, also known as the complex line bundle. We have that $\mathbb{CP}^n \equiv \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times =$ (lines through 0 in $\mathbb{C}^{n+1}$). If $l \subset \mathbb{C}^{n+1}$ is such a line, then let $\alpha_l = l$. Compute $g_{01} : U_{01} \to \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$. It should look like z^{-1}.*

3. *Consider the bundle γ^* on $\mathcal{O}(n)$, where the transition function is precisely z^n. Show that meromorphic functions on $\mathbb{CP}^1$ correspond to algebraic sections of $\mathcal{O}(n)$, which are precisely homogenous polynomial of degree n.*

Note 6.1.2. We want to differentiate sections of a vector bundle. Let $s : X \to E$ be a section. Let

$$d_v s_x = \lim_{h \to 0} \frac{s(x + hv) - s(x)}{h}.$$

We need a way to compare points in different fibers. This leads to the notation of a *connection*. Define $\nabla : C^\infty(X; E) \to \Omega^1(X; E) \cong C^\infty(X; T^*X \otimes E)$ by $fs \mapsto df \otimes s + f\nabla s$ where $s \in C^\infty(X; E)$ and $f \in C^\infty(X)$. We have that $\nabla_v(s) = i_v \nabla s \in \Omega^1(X; E)$ and $\nabla_{fv}(s) = f\nabla_v(s)$.

On a trivial bundle, connections exist. Indeed, given $X \times V$ and $s : X \to V$, we see that $\nabla_0(s) = ds \in \Omega^1(X) \otimes V$. More generally, if $\omega \in \omega^1(X) \otimes \mathrm{End}(V)$, then $\nabla_0 + w$ is also a connection.

On e general vector bundle $\pi : E \to X$, take a local trivialization $E \restriction_{U_\alpha} \to U_\alpha \times \mathbb{R}^k$ and put a connection ∇_α on $E \restriction_{U_\alpha}$. Let $\{\lambda_\alpha\}$ be a patron of unity subordinate to the U_α. Set $\nabla = \sum_\alpha \lambda_\alpha \nabla_\alpha$. Thus, connections always exist.

Let $\pi : E^{k/\mathbb{R}} \to X$ be a vector bundle with a connection ∇. Let $x, y \in X$ and let $p(t)$ be a path from x to y. Let $e(t)$ be a section of E along $p(t)$. We say that $e(t)$ is *covariantly constant* or *parallel* if $\nabla_{\mathring{p}(t)} e(t) = 0$. Consider $E_x \ni e_0$. The we have a first-order ODE with initial condition $e(0) = e_0$ and $\nabla_{\mathring{p}(t)} e(t) = 0$. By Picard, there exists a unique solution $e(t)$. Thus, we have defined a map given by $h(p)(e_0) = e(1)$, so that $h(p) : E_x \to E_y$ is linear. We call h the *holonomy map*. We get

$$\nabla_v s(x) = \lim_{t \to 0} \frac{h(p_t)^{-1} s(p(t)) 0 s(x)}{t}$$

where p is any path with $p'(0) = v$.

Given $\pi : E \to X$ and $\nabla : \Omega^0(X; E) \to |omega^1(X; E)$, we can extend ∇ to a map $\nabla : \Omega^k(X; E) \to \Omega^{k+1}(X; E)$ by taking

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^{|\omega|} \omega \wedge ds.$$

Consider the composition $\Omega^0(X; E) \xrightarrow{\nabla} \Omega^1(X; E) \xrightarrow{\nabla} \Omega^2(X; E)$. Observe that

$$\begin{aligned} \nabla^2(fs) &= \nabla(\nabla(fs)) \\ &= \nabla(df \cdots s + f\nabla_s) \\ &= d^2 f \cdot s - df\nabla_s + df\nabla_s + f\nabla^2 s \\ &= f\nabla^2 s. \end{aligned}$$

Hence $F := \nabla^2$ is a tensor and belongs to $\Omega^2(X; \mathrm{End}(E))$. Note that F is precisely the *curvature*.

Definition 6.1.3. We say that ∇ is *flat* if $F_\nabla = 0$.

Proposition 6.1.4. *If p_0 and p_1 are path homotopic and ∇ is flat, then $h(p_0) = h(p_1)$. (In this case, we call the holonomy the monodromy.)*

Definition 6.1.5. Let $\pi^{k/\mathbb{R}} \to X$ be a vector bundle and $\{U_\alpha\}$ a trivializing cover with local trivializations φ_α. A *homomorphism $\psi : E \to F$ of smooth vector bundles on X* is a smooth map such that $\psi_x \coloneqq \psi \restriction_{E_x}: E_x \to F_x$ is a homomorphism for each $x \in X$. It is called an *isomorphism* if each ψ_x is an isomorphism.

Note 6.1.6. Suppose that ψ is an isomorphism. Let $g_{\alpha\beta} = \varphi_\alpha\varphi_\beta^{-1}$ and $h_{\alpha\beta} = \tau_\alpha\tau_\beta^{-1}$, which are maps $U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{R})$. Note that

$$\begin{tikzcd} U_\alpha \times \mathbb{R}^k \arrow[r, "\varphi_\alpha^{-1}"] & E\restriction_{U_\alpha} \arrow[d, "\psi"] \\ & F\restriction_{U_\alpha} \arrow[ul, "\tau_\alpha"] \end{tikzcd} .$$

Then, on $U_{\alpha\beta}$, we have that

$$\begin{aligned}
\tau_\alpha\psi\varphi_\alpha^{-1} &= c_\alpha \\
\tau_\alpha\psi &= c_\alpha\varphi_\alpha \\
\tau_\alpha &= c_\alpha\varphi_\alpha\psi^{-1} \\
g_{\alpha\beta} &= \varphi_\alpha\varphi_\beta^{-1} \\
h_{\alpha\beta} &= \tau_\alpha\tau_\beta^{-1} \\
&= (c_\alpha\varphi_\alpha\psi^{-1})(c_\beta\varphi_\beta\psi^{-1})^{-1} \\
&= c_\alpha\varphi_\alpha\psi^{-1}\psi\varphi_\beta^{-1}c_\beta^{-1} \\
&= c_\alpha\varphi_\alpha\varphi_\beta^{-1}c_\beta^{-1} \\
&= c_\alpha g_{\alpha\beta}c_\beta^{-1},
\end{aligned}$$

i.e., $h_{\alpha\beta}c_\beta = c_\alpha g_{\alpha\beta}$.

Proposition 6.1.7. *E and F are isomorphic if and only if there exist a common trivializing cover $\{U_\alpha\}$ with respective transition functions $g_{\alpha\beta}$ and $h_{\alpha\beta}$ and a function $c_\alpha : U_\alpha \to \mathrm{GL}_k(\mathbb{R})$ such that $h_{\alpha\beta}c_\beta = c_\alpha g_{\alpha\beta}$.*

Definition 6.1.8. We say that a vector bundle $\pi : E \to X$ is *flat* if there exist U_α and $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{R})$ where each $g_{\alpha\beta}$ is locally constant. We define the flat structures $g_{\alpha\beta}$ and $h_{\alpha\beta}$ to be *equivalent* if there is some $c_\alpha : U_\alpha \to \mathrm{GL}_k(\mathbb{R})$ that is locally constant and has $h_{\alpha\beta}c_\beta = c_\alpha g_{\alpha\beta}$.

Theorem 6.1.9. *Let X be a path connected smooth compact manifold. The following object are in 1-1 correspondence up to isomorphism.*

1. *A vector bundle $E \to X$ with a flat structure.*
2. *A vector bundle $E \to X$ with a flat connection.*
3. *A homomorphism $\beta : \pi_1(X, x_0) \to \mathrm{GL}_k(\mathbb{R})$.*

Proof.

$\underline{(3) \implies (1)}$ Let $\beta : \pi_1(X, x_0) \to \mathrm{GL}_k(\mathbb{R})$ be a homomorphism. Let $p : \widetilde{X} \to X$ be a universal cover. We can form $E_p \coloneqq \widetilde{X} \times_{\pi_1(X,x_0)} \mathbb{R}^k$, which equals $\widetilde{X} \times \mathbb{R}^k/\sim$ where $(x\gamma, v) \sim (x, p(\gamma)v)$. Consider the map $T : E_p \to X$ given by $(x, v) \mapsto p(x)$. Then each fiber is isomorphic to V.

Claim. *T is a flat vector bundle.*

Proof. There exists a trivializing cover of $\widetilde{X}$ with $\varphi_\alpha : \widetilde{X} \restriction_{U_\alpha} \to U_\alpha \times \Gamma$ where $\Gamma \cong \pi_1(X, x_0)$ as $\pi_1(X, x_0)$ modules. Then $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{Aut}(\Gamma)$ with Γ discrete. The $g_{\alpha\beta}$ for E_p are precisely the $p(\tilde{g}_{\alpha\beta})$, so that there are locally constant. ∎

(1) $\implies$ (2) Let $\pi : E \to C$ be a flat vector bundle. Then each $g_{\alpha\beta} : U_{\alpha\beta} \to \mathrm{GL}_k(\mathbb{R})$ is locally constant. A section $s_\alpha : U_\alpha \to \mathbb{R}^k$ has $\nabla_0 s_\alpha = ds_\alpha$. Note that $s \in C^\infty(X; E)$ corresponds to s_α where $g_{\alpha\beta} s_\beta = s_\alpha$. We have that

$$\nabla(s_\alpha) = \nabla(g_{\alpha\beta} s_\beta) = dg_{\alpha\beta} s_\beta + g_{\alpha\beta} \nabla(s_\beta).$$

But this equals 0 because $g_{\alpha\beta}$ is locally constant. Since $\nabla_0(s_\alpha) = g_{\alpha\beta} \nabla_0(s_\beta)$, it follows that the s_α transform correctly to form a global section. Further, it's clear that $\nabla_0^2 = 0$.

(2) $\implies$ (3) Given (E, ∇) with ∇ flat and γ a path based at $x_0 \in X$, let $h(\gamma) : E_{x_0} \to E_{x_0}$. This is an isomorphism that depends only on the homotopy class of γ. □

Theorem 6.1.10 (De Rham theorem for local systems). *Let (E, ∇) be flat and $\nabla^2 = 0$. Then $(\Omega^\bullet(X; E), \nabla)$ is a complex, and*

$$H^*(\Omega^\bullet(X; E), \nabla) \cong H^*(X; \underline{E})$$

where $\underline{E}$ denotes the locally constant sheaf that is defined by the flat structure that E obtains from ∇, i.e., $\underline{E}(U) = \{s_\alpha : U \cap U_\alpha \to \mathrm{GL}_k(\mathbb{R}) \mid s_\alpha$ is locally constant and $s_\alpha = g_{\alpha\beta} s_\beta\}$.