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Homotopy and K-theory seminar
Talk #16
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**Abstract**

We continue doing higher Waldhausen $K$-theory. The main sources for this talk are the following.

- $n$Lab.
- Charles Weibel's *The $K$-book: an introduction to algebraic $K$-theory*, Ch. V.2.
- John Rognes's *Lecture Notes on Algebraic $K$-Theory*, Ch. 8.

Recall that $|wS_\bullet\mathscr{C}|$ is an $H$-space via the map

$$\coprod : |wS_\bullet\mathscr{C}| \times |wS_\bullet\mathscr{C}| \cong |wS_\bullet\mathscr{C} \times wS_\bullet\mathscr{C}| \to |wS_\bullet\mathscr{C}|.$$

This produces an $H$-space structure $(K(\mathscr{C}), +)$.

**Definition 1.** Let $\mathscr{B}$ and $\mathscr{C}$ be Waldhausen categories. We say that $F' \rightarrowtail F \twoheadrightarrow F''$ is a *short exact sequence* or *cofiber sequence of exact functors* if every $F'(B) \rightarrowtail F(B) \twoheadrightarrow F''(B)$ is a cofiber sequence and $F(A) \cup_{F'(A)} F'(B) \rightarrowtail F(B)$ is a cofibration in $\mathscr{C}$ for every $A \rightarrowtail B$ in $\mathscr{B}$.

Let $\mathscr{C}$ be a Waldhausen category. Let $(\eta) : A \rightarrowtail B \twoheadrightarrow C$ be an object in $S_2\mathscr{C}$. Define the source $s$, target $t$, and quotient $q$ functors $S_2\mathscr{C} \to \mathscr{C}$ by $s(\eta) = A$, $t(\eta) = B$, and $q(\eta) = C$. Then $s \rightarrowtail t \twoheadrightarrow q$ is a cofiber sequence of functors. Since defining a cofiber sequence of exact functors $\mathscr{B} \to \mathscr{C}$ is equivalent to defining an exact functor $\mathscr{B} \to S_2\mathscr{C}$, we may restrict our attention to $s \rightarrowtail t \twoheadrightarrow q$ when proving things about a given cofiber sequence of exact functors $\mathscr{B} \to \mathscr{C}$. We say that $S_2\mathscr{C}$ is universal in this sense.

**Theorem 1 (Extension theorem).** *Let $\mathscr{C}$ be Waldhausen. The exact functor $(s, q) : S_2\mathscr{C} \to \mathscr{C} \times \mathscr{C}$ induces a homotopy $K(S_2\mathscr{C}) \simeq K(\mathscr{C}) \times K(\mathscr{C})$. The functor $\coprod : (A, B) \to (A \rightarrowtail A \coprod B \twoheadrightarrow B)$ is a homotopy inverse.*

*Proof.* Let $\mathscr{C}^w_m$ denote the category of $m$-length sequences of weak equivalences. For each $n$, define $s_n\mathscr{C}^w_m$ as the commutative diagram

$$\begin{tikzcd}
X_1^0 \arrow[r, tail] \arrow[d, "\sim"'] & X_2^0 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X_n^0 \arrow[d, "\sim"'] \\
X_1^1 \arrow[r, tail] \arrow[d, "\sim"'] & X_2^1 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X_n^1 \arrow[d, "\sim"'] \\
\vdots \arrow[d, "\sim"'] & \vdots \arrow[d, "\sim"'] & & \vdots \arrow[d, "\sim"'] \\
X_1^m \arrow[r, tail] & X_2^m \arrow[r, tail] & \cdots \arrow[r, tail] & X_n^m
\end{tikzcd}.$$

This is naturally isomorphic to an $(m, n)$-bisimplex in $N_\bullet wS_\bullet\mathscr{C}$, which is thus isomorphic to the bisimplicial set $s_\bullet\mathscr{C}^w_{(-)}$. One can show that the source $s$ and quotient $q$ functors $S_2\mathscr{C} \to \mathscr{C}$ give a homotopy equivalence $s \times q : s_\bullet S_2(\mathscr{C}^w_m) \to s_\bullet\mathscr{C}^w_m \times s_\bullet\mathscr{C}^w_m$ for each $m$. Thus, we get a homotopy equivalence

$$s_\bullet S_2(\mathscr{C}^w_{(-)}) \simeq s_\bullet\mathscr{C}^w_{(-)} \times s_\bullet\mathscr{C}^w_{(-)}$$

between bisimplicial sets. But we already have that $s_\bullet\mathscr{C}^w_{(-)} \cong N_\bullet wS_\bullet\mathscr{C}$, completing the proof. □

**Theorem 2 (The additivity theorem).** *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors $\mathscr{B} \to \mathscr{C}$. Then $F_* \simeq F'_* + F''_*$ as maps $K(\mathscr{B}) \to K(\mathscr{C})$. Hence $F_* = F'_* + F''_*$ as maps $K_i(\mathscr{B}) \to K_i(\mathscr{C})$.*

*Proof.* As $S_2\mathscr{C}$ is universal, it suffices to prove that $t_* \simeq s_* + q_*$. Notice that the two composites

$$\mathscr{C} \times \mathscr{C} \xrightarrow{\coprod} S_2\mathscr{C} \underset{s \coprod q}{\overset{t}{\rightrightarrows}} \mathscr{C}$$

are the same. The extension theorem implies that $K(\coprod) : K(\mathscr{C}) \times K(\mathscr{C}) \to K(S_2\mathscr{C})$ is a homotopy equivalence. Since the $H$-space structure on $K(\mathscr{C})$ is induced by $\coprod$, we get $t_* \simeq s_* + q_*$. $\square$

**Definition 2.** Let $\mathscr{C}$ be Waldhausen. We say that a sequence $* \to A_n \to \cdots \to A_0 \to *$ is *admissibly exact* if each morphism in the sequences can be written as a cofiber sequence $A_{i+1} \twoheadrightarrow B_i \rightarrowtail A_i$.

**Corollary 3.** *Suppose that $* \to F^0 \to F^1 \to \cdots \to F^n \to *$ is an admissibly exact sequence of exact functors $\mathscr{B} \to \mathscr{C}$. Then $\sum_i (-1)^i F^i_* = 0$ as maps $K_i(\mathscr{B}) \to K_i(\mathscr{C})$.*

*Proof.* Induct on $n$. $\square$

**Corollary 4.** *Let $F' \rightarrowtail F \twoheadrightarrow F''$ be a short exact sequence of exact functors $\mathscr{B} \to \mathscr{C}$. Then*

$$F''_* \simeq F_* - F_* \simeq 0.$$

*This implies that the homotopy fiber of $F''_* : K(\mathscr{B}) \to K(\mathscr{C})$ is homotopy equivalent to $K(\mathscr{B}) \vee \Omega K(\mathscr{C})$.*

**Definition 3.** Let $\mathscr{C}$ be a Waldhausen category. Recall the arrow category $\mathrm{Ar}(\mathscr{C})$ of $\mathscr{C}$ consisting of morphisms in $\mathscr{C}$ as objects and commutative squares as morphisms. Let $s$ and $t$ denote the source and target functors $\mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$, respectively.

A functor $T : \mathrm{Ar}(\mathscr{C}) \to \mathscr{C}$ is a *(mapping) cylinder functor* on $\mathscr{C}$ if it comes equipped with natrual transformations $j_1 : s \Rightarrow T$, $j_2 : t \Rightarrow T$, and $p : T \Rightarrow t$ such that for any $f : A \to B$, we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "j_1"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] & B \arrow[l, "j_2"'] \arrow[dl, "="] \\ & B & \end{tikzcd} .$$

Moreover, $T$ must satisfy the following axioms.

1. $T$ sends every initial morphism $* \to A$ to $A$ for any $A \in \mathrm{ob}\,\mathscr{C}$.

2. $j_1 \coprod j_2 : A \coprod B \rightarrowtail T(f)$ is a cofibration for any $f : A \to B$.

3. Given a morphism $(a, b) : f \to f'$ in $\mathrm{Ar}(\mathscr{C})$, if both $a$ and $b$ are w.e. in $\mathscr{C}$, then so is $T(f) \to T(f')$.

4. Given a morphism $(a, b) : f \to f'$ in $\mathrm{Ar}(\mathscr{C})$, if both $a$ and $b$ are cofibrations in $\mathscr{C}$, then so is $T(f) \to T(f')$. Also, the map $A' \coprod_A T(f) \coprod_B B' \to T(f')$ induced by axiom 2 is a cofibration in $\mathscr{C}$.

5. (Cylinder Axiom) The map $p : T(f) \to B$ is a w.e. in $\mathscr{C}$.

**Definition 4.** Let $T$ be a cylinder functor on $\mathscr{C}$.

1. We call $T(A \to *)$ the *cone* of $A$, denoted by $\mathrm{cone}(A)$.

2. We call ${}^{\mathrm{cone}(A)}\!/\!_{A}$ the *suspension* of $A$, denoted by $\Sigma A$.

**Corollary 5.** *The induced suspension map $\Sigma : K(\mathscr{C}) \to K(\mathscr{C})$ is a homotopy inverse for the $H$-space $K(\mathscr{C})$.*

*Proof.* Note that axiom 3 gives us a cofiber sequence $A \rightarrowtail \mathrm{cone}(A) \twoheadrightarrow \Sigma A$. Therefore, $1 \rightarrowtail \mathrm{cone} \twoheadrightarrow \Sigma$ is an exact sequence of functors. By the cylinder axiom, we know that cone is null-homotopic. It follows by the additivity theorem that $\Sigma_* + 1 = \mathrm{cone}_* = *$. □

Let $\mathscr{C}$ be a category with cofibrations. Equip it with two Waldhausen subcategories $v(\mathscr{C})$ and $w(\mathscr{C})$ of weak equivalences such that $v(\mathscr{C}) \subset w(\mathscr{C})$. Assume that $(\mathscr{C}, w)$ admits a cylinder functor. Suppose that $w(\mathscr{C})$ is saturated and closed under extensions.

*Notation.* Let $\mathscr{C}^w$ denote the Waldhausen subcategory of $(\mathscr{C}, v)$ consisting of any $A$ where $* \to A$ is in $w(\mathscr{C})$.

Are the initial morphisms the only w.e.?

**Theorem 6 (Waldhausen localization theorem).** *The sequence*

$$K(A^w) \to K(\mathscr{C}, v) \to K(\mathscr{C}, w)$$

*is a homotopy fibration sequence.*

*Proof.* Recall that a small bicategory is a bisimplicial set such that each row/column is the nerve of a category. Note that $v_{(-)}w_{(-)}\mathscr{C}$ is a bicategory whose bimorphisms are commutative squares of the form

$$\begin{tikzcd} (-) \arrow[r, "w'"] \arrow[d, "v"'] & (-) \arrow[d, "v'"] \\ (-) \arrow[r, "w"] & (-) \end{tikzcd} .$$

It turns out that treating $w\mathscr{C}$ as a bicategory with a single vertical morphism proves that $w\mathscr{C} \simeq v_{(-)}w_{(-)}\mathscr{C}$. This gives $wS_n\mathscr{C} \simeq v_{(-)}w_{(-)}S_n\mathscr{C}$ for each $n$.

Now, let $v_{(-)}\mathrm{co}w_{(-)}\mathscr{C}$ denote the subcategory of the above squares where the horizontal maps are also cofibrations. One can show that the inclusion $v_{(-)}\mathrm{co}w_{(-)}\mathscr{C} \subset v_{(-)}w_{(-)}\mathscr{C}$ is a homotopy equivalence. Since each $S_n\mathscr{C}$ inherits a cylinder functor from $\mathscr{C}$, we simplicial bi-subcategory $v_{(-)}\mathrm{co}w_{(-)}S_\bullet\mathscr{C}$ such that the inclusion intro $v_{(-)}w_{(-)}S_\bullet\mathscr{C}$ is a homotopy equivalence. We have now obtained the following diagram.

$$\begin{tikzcd} vS_\bullet C^w \arrow[r] & vS_\bullet C \arrow[r] \arrow[d] & v_{(-)}\mathrm{co}w_{(-)}S_\bullet C \arrow[d, "\simeq"] \\ & wS_\bullet C \arrow[r, "\simeq"] & v_{(-)}w_{(-)}S_\bullet C \end{tikzcd}$$

What about the left vertical morphism?

It therefore suffices to show that the top row is a fibration. You can do this by using the relative $K$-theory space construction. See Weibel, IV.8.5.3 and V.2.1. □

**Definition 5.** Let $\mathscr{A}$ be an exact category embedded in an abelian category $\mathscr{B}$ and let $\mathbf{Ch}^b(\mathscr{A})$ denote the category of bounded chain complexes in $\mathscr{A}$. One can verify that $\mathbf{Ch}^b(\mathscr{A})$ is Waldhausen where the cofibrations $A_\bullet \rightarrowtail B_\bullet$ are precisely the degree-wise admissible monomorphisms (i.e., those giving a short exact sequence $A_n \to B_n \to {}^{B_n}\!/_{A_n}$ in $\mathscr{A}$ for each $n$) and the w.e. are precisely the chain maps which are quasi-isomorphisms of complexes in $\mathbf{Ch}(\mathscr{B})$.

**Theorem 7 (Gillet-Waldhausen).** *Let $\mathscr{A}$ be an exact category closed under kernels of surjections. Then the exact inclusion $\mathscr{A} \to \mathbf{Ch}^b(\mathscr{A})$ induces a homotopy equivalence $K(\mathscr{A}) \simeq K\,\mathbf{Ch}^b(\mathscr{A})$. Hence*

$$K_i(\mathscr{A}) = K_i\,\mathbf{Ch}^b(\mathscr{A})$$

*for every $i$.*

*Proof.* Apply the localization theorem. See Weibel, V.2.2. □

**Definition 6.** Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between Waldhausen categories. We say that $F$ satisfies the *approximate lifting property* if for any map $b : F(A) \to B$ in $\mathscr{B}$, there is some map $a : A \to A'$ in $\mathscr{A}$ and some w.e. $b' : F(A') \simeq B$ in $\mathscr{B}$ so that

$$\begin{tikzcd} F(A') \arrow[r, dashed, "\simeq"] & B \\ F(A) \arrow[u, dashed, "F(a)"] \arrow[ur, "b"'] & \end{tikzcd}.$$

commutes. In this way, we can lift to a w.e.

**Proposition 8.** *Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between Waldhausen categories such that the following hold.*

1. *$F$ satisfies the approximate lifting property.*
2. *$\mathscr{A}$ admits a cylinder functor.*
3. *A morphism $f$ in $\mathscr{A}$ is a w.e. iff $F(f)$ is a w.e. in $\mathscr{B}$.*

*Then $wF : w\mathscr{A} \to w\mathscr{B}$ is a homotopy equivalence.*

**Corollary 9 (Waldhausen approximation theorem).** *With the same conditions as before, we have*

$$K(\mathscr{A}) \simeq K(\mathscr{B}).$$

*Proof.* One can show that each functor $S_n\mathscr{A} \to S_n\mathscr{B}$ is exact and also has the approximate lifting property. The previous proposition thus gives degree-wise homotopy equivalence between the bisimplicial map $wS_\bullet\mathscr{A} \to wS_\bullet\mathscr{B}$, which is enough. □

**Definition 7.** Let $\mathscr{A}$ be an abelian category $\mathbf{Ch}(\mathscr{A})$ denote the category of chain complexes over $\mathscr{A}$. We say that a complex $C_\bullet$ is *homologically bounded* if only finitely many $H_i(C_j)$ are nonzero. Let $\mathbf{Ch}^{hb}_{\pm}$ denote the subcategory of bounded below (respectively, bounded above) complexes.

**Example 8.** Let $\mathscr{A}$ be an abelian category. By homology theory, we have that $\mathbf{Ch}^b(\mathscr{A}) \subset \mathbf{Ch}^{hb}_-(\mathscr{A})$ and $\mathbf{Ch}^{hb}_+(\mathscr{A}) \subset \mathbf{Ch}^{hb}(\mathscr{A})$ have the approximate lifting property. We also have that $\mathbf{Ch}^b(\mathscr{A}) \subset \mathbf{Ch}^{hb}_+(\mathscr{A})$ and $\mathbf{Ch}^{hb}_+(\mathscr{A}) \subset \mathbf{Ch}^{hb}(\mathscr{A})$ satisfy the dual of the approximate lifting property. Thus, we can apply the approximation theorem and Gillet-Waldhausen to see that

$$K(\mathscr{A}) \simeq K\,\mathbf{Ch}^b(\mathscr{A}) \simeq K\,\mathbf{Ch}^{hb}_- \simeq K\,\mathbf{Ch}^{hb}_+(\mathscr{A}) \simeq K\,\mathbf{Ch}^{hb}(\mathscr{A}).$$

**Definition 9.** A *symmetric spectrum* $\mathbf{X}$ in topological spaces in a sequence of based $\Sigma_n$-spaces $(X_n)$ endowed with structure maps $\sigma : X_n \wedge S^1 \to X_{n+1}$ such that $\sigma^k : X_n \wedge S^k \to X_{n+k}$ is $(\Sigma_n \times \Sigma_k)$-equivariant for any $n, k \geq 0$, where $S^k \coloneqq \underbrace{S^1 \wedge \cdots \wedge S^1}_{k\text{-times}}$. A map $\mathbf{f} : \vec{x} \to \mathbf{Y}$ of symmetric spectra is a sequence $(f_n : X_n \to Y_n)$ of based $\Sigma_n$-equivariant maps such that for each $n \geq 0$, the square

$$\begin{tikzcd} X_n \wedge S^1 \arrow[r, "f_n \wedge \mathrm{Id}"] \arrow[d, "\sigma"'] & Y_n \wedge S^1 \arrow[d, "\sigma"] \\ X_{n+1} \arrow[r, "f_{n+1}"] & Y_{n+1} \end{tikzcd}$$

commutes. Let $\mathrm{Sp}^\Sigma$ denote the category of symmetric spectra in topological spaces.

**Definition 10.** Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. The *external $n$-fold $S_\bullet$-construction* on $\mathscr{C}$ is the $n$-multisimplicial Waldhausen category

$$(S_\bullet \cdots S_\bullet \mathscr{C}, wS_\bullet \cdots S_\bullet \mathscr{C}).$$

It multidegree $(q_1, \ldots, q_n)$, it has as objects the diagrams $X : \mathrm{Ar}[q_1] \times \cdots \times \mathrm{Ar}[q_n] \to \mathscr{C}$ such that

1. $X((i_1, j_1), \ldots, (i_n, j_n)) = *$ if $i_k = j_k$ for some $1 \leq k \leq n$.
2. $X(\ldots, (i_t, j_t), \ldots) \rightarrowtail X(\ldots, (i_t, k_t), \ldots) \twoheadrightarrow X(\ldots, (j_t, k_t), \ldots)$ is a cofiber sequence in the $(n-1)$-fold iterated $S_\bullet$-construction for any $i_t \leq j_t \leq k_t$ in $[q_t]$.

**Definition 11.** Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. The *internal $n$-fold $S_\bullet$-construction* on $\mathscr{C}$ is the simplicial Waldhausen category

$$(S_\bullet^{(n)} \mathscr{C}, wS_\bullet^{(n)} \mathscr{C}).$$

It has as $q$-simplices the functor categories $(S_q \cdots S_q \mathscr{C}, wS_q \cdots S_q \mathscr{C})$ whose objects are the $(\mathrm{Ar}[q])^n$-shaped diagrams $X : (\mathrm{Ar}[q])^n \to \mathscr{C}$ such that

1. $X((i_1, j_1), \ldots, (i_n, j_n)) = *$ if $i_k = j_k$ for some $1 \leq k \leq n$.
2. $X(\ldots, (i_t, j_t), \ldots) \rightarrowtail X(\ldots, (i_t, k_t), \ldots) \twoheadrightarrow X(\ldots, (j_t, k_t), \ldots)$ is a cofiber sequence in the $(n-1)$-fold iterated $S_\bullet$-construction for any $i_t \leq j_t \leq k_t$ in $[q]$.

Note that $\Sigma_n$ acts on $S_\bullet^{(n)} \mathscr{C}$ by $(\pi \cdot X)(\ldots, (i_t, j_t), \ldots) = X(\ldots, (i_{\pi^{-1}(t)}, j_{\pi^{-1}(t)}), \ldots)$.

**Definition 12.** The *(symmetric) algebraic $K$-theory spectrum* $\mathbf{K}(\mathscr{C}, w)$ of a small Waldhausen category $(\mathscr{C}, w\mathscr{C})$ has $n$-th space $K(\mathscr{C}, w)_n = |wS_\bullet^{(n)} \mathscr{C}|$ based at $*$. There is a $\Sigma_n$-action on $K(\mathscr{C}, w)_n$ induced by permuting the order of the internal $S_\bullet$-constructions. Moreover, we have

$$|wS_\bullet^{(n)} \mathscr{C}| \wedge S^1 \cong |wS_\bullet^{(n)} S_\bullet \mathscr{C}|^{(1)} \subset |wS_\bullet^{(n)} S_\bullet \mathscr{C}| \cong |wS_\bullet^{(n+1)} \mathscr{C}|$$

, where $^{(1)}$ denotes the 1-skeleton with respect to the last simplicial direction. This determines the structure map $\sigma$. Then $\sigma^k$ is $(\Sigma_n \times \Sigma_k)$-invariant.

**Theorem 10.** *For any $i \geq 0$, we have that $K_i(\mathscr{C}, w) = \pi_{i+1} K(\mathscr{C}, w)_1 \cong \pi_i \mathbf{K}(\mathscr{C}, w)$.*

*Proof.* See Rognes, Lemma 8.7.4. ∎

In this way, we encode our algebraic $K$-theory in an infinite loop space.