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Abstract

These notes are based on Scott Weinstein's "Model Theory" lectures at UPenn along with David Marker's *Model Theory: An Introduction*. Any mistake in what follows is my own.

Contents

1 Introduction

1.1 Lecture 1

Recall the structure $\mathbb{N} \coloneqq \langle \omega, \mathtt{S}, \mathtt{0} \rangle$ where

- ω denotes the set of natural numbers $\{0, 1, 2, \ldots, \}$,
- $\mathtt{S}$ is interpreted as the successor function $\omega \to \omega$, and
- the constant symbol $\mathtt{0}$ is interpreted as the natural number 0.

The formal language $\mathcal{L}$ for which $\mathbb{N}$ is a structure consists of the first-order (FO) logical symbols

$$\forall,\ \exists,\ \wedge,\ \neg,\ \vee,\ \rightarrow,\ =$$

along with non-logical symbols such as $\mathtt{0}$, $\mathtt{S}^n\mathtt{0} \coloneqq \underbrace{\mathtt{S} \cdots \mathtt{S}}_{n\text{-copies}} \mathtt{0}$, and $\mathtt{S}^n x$. Let FO denote the set of all (first-order) $\mathcal{L}$-sentences.

The *theory of* $\mathbb{N}$ is

$$\mathrm{Th}(\mathbb{N}) \coloneqq \{\varphi \in \mathsf{FO} \mid \mathbb{N} \models \varphi\},$$

which consists of all sentences satisfied by $\mathbb{N}$. Further, for any $\sigma \in \mathsf{FO}$, consider the set

$$\mathrm{Cn}(\sigma) \coloneqq \{\varphi \in \mathsf{FO} \mid \sigma \models \varphi\}$$

of consequences of σ.

Question. Can we find a sentence σ such that $\mathrm{Cn}(\sigma) = \mathrm{Th}(\mathbb{N})$?

Let $\Delta = \{\forall x\, (\mathtt{S}x \neq \mathtt{0}),\ \forall x \forall y\, (\mathtt{S}x = \mathtt{S}y \rightarrow x = y),\ \forall x\, (x \neq \mathtt{0} \rightarrow \exists y\, (\mathtt{S}y = x))\}$. Each element of Δ is clearly true in $\mathbb{N}$, i.e., $\mathbb{A} \models \Delta$. But is it the case that $\mathrm{Cn}(\Delta) = \mathrm{Th}(\mathbb{N})$? No, provided that we allow ourselves access to monadic second-order sentences. Specifically, consider the *induction axiom* IA:

$$\forall P\, ((P(0) \wedge \forall x\, (P(x) \rightarrow P(\mathtt{S}x)) \rightarrow \forall x\, (P(x)))). \tag{$*$}$$

This is clearly true in $\mathbb{N}$. Consider, however, a new structure $\mathbb{A} \coloneqq \langle \omega \cup \mathbb{Z}, \mathtt{S}, \mathtt{0} \rangle$. Then $\Delta \subset \mathrm{Th}(\mathbb{A})$, and we have a $\mathbb{Z}$*-chain* in $\mathbb{A}$ (pretending, for the moment, that $|\mathbb{A}|$ has the usual order $<$):

$$\begin{tikzcd} \cdots \arrow[r, bend left] & -(n+1)^{\mathbb{A}} \arrow[r, bend left] & -n^{\mathbb{A}} \arrow[r, bend left] & \cdots \arrow[r, bend left] & -1^{\mathbb{A}} \arrow[r, bend left] & \mathtt{0}^{\mathbb{A}} \arrow[r, bend left] & 1^{\mathbb{A}} \arrow[r, bend left] & \cdots . \end{tikzcd}$$

The second-order sentence ($*$) with P instantiated by the "initial segment" $\mathbb{Z}_{\geq -1}$ is not true in $\mathbb{A}$, so that $\mathbb{A} \not\models \mathsf{IA}$. In this case, $\mathsf{IA} \in \mathrm{Th}(\mathbb{N}) \setminus \mathrm{Cn}(\sigma)$.

Sill, we want to restrict ourselves to FO. Recall that two structures $\mathbb{B}$ and $\mathbb{C}$ are *elementarily equivalent* if $\mathrm{Th}(\mathbb{B}) = \mathrm{Th}(\mathbb{C})$.

Question. Are $\mathbb{A}$ and $\mathbb{N}$ elementarily equivalent?

If we can find some sentence belonging to $\mathrm{Th}(\mathbb{N}) \setminus \mathrm{Th}(\mathbb{A})$, then $\mathrm{Cn}(\Delta) \neq \mathrm{Th}(\mathbb{N})$.

Definition 1.1.1. A sentence σ is *categorical* if for any structures $\mathbb{B}$ and $\mathbb{C}$, if $\mathbb{B} \models \sigma$ and $\mathbb{C} \models \sigma$, then $\mathbb{B} \cong \mathbb{C}$.

Example 1.1.2. $\Delta' \coloneqq \Delta + \mathsf{IA}$ is categorical.

Perhaps exhibiting that the usual order $<$ on ω is definable in $\mathbb{N}$ would reveal that $\mathbb{A} \not\equiv \mathbb{N}$. For this, we must find a (well-formed) formula $\theta(x, y)$ such that for every $n, m \in \omega$,

$$m < n \iff \mathbb{N} \models \theta[n, m].$$

Thanks to Lagrange's four square theorem, we could define $<$ on the positive integers. But it's unclear how to proceed further.

Theorem 1.1.3. *If $\mathbb{B}$ is infinite, then for every infinite cardinal κ, there is some $\mathbb{C}$ such that $\mathbb{C} \equiv \mathbb{B}$ and card$(\mathbb{C}) = \kappa$.*

Corollary 1.1.4. *If $\mathbb{B}$ is infinite, then there exists a $\mathbb{C}$ such that $\mathbb{C} \equiv \mathbb{B}$ and $\mathbb{C} \not\cong \mathbb{B}$.*

Therefore, Δ does *not* categorically describe $\mathbb{N}$. Now, consider the structure $\widetilde{\mathbb{N}}$ obtained from $\mathbb{N}$ by adding a single point $\bullet$ fixed by S. Then the sentence $\forall x\,(Sx \neq x)$ is true in $\mathbb{N}$ but not in $\widetilde{\mathbb{N}}$. Moreover, $\Delta \subset \mathrm{Th}(\widetilde{\mathbb{N}})$, which proves that

$$\mathrm{Cn}(\Delta) \neq \mathrm{Th}(\mathbb{N}).$$

With this in mind, let $\Sigma = \Delta \cup \{\forall x\,(\mathsf{S}^n x \neq x) \mid n \in \omega\}$. To show that $\mathrm{Cn}(\Sigma) = \mathrm{Th}(\mathbb{N})$, it suffices to show that for any $\mathbb{B}$, if $\mathbb{B} \models \Sigma$, then $\mathbb{B} \equiv \mathbb{N}$. To this end, for any cardinal κ, let $\mathbb{A}_\kappa = \mathbb{N} \cup (\kappa \times \mathbb{Z})$, which is precisely the structure obtained form $\mathbb{N}$ by adding κ many disjoint $\mathbb{Z}$-chains.

Question. How many structures that

(a) satisfy Σ and

(b) are of cardinality κ

are there up to isomorphism?

If $\kappa < \omega$, then card$(\mathbb{A}_\kappa) = \aleph_0$. Also, if $\kappa > \omega$, then card$(\mathbb{A}_\kappa) = \kappa$, so that Σ is *κ-categorical*, i.e., every structure satisfying (a) and (b) is isomorphic to $\mathbb{A}_\kappa$.