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Homotopy and K-theory seminar
Talk #5
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Abstract

Even more basic category theory. The main sources for this talk are the following.

- *nLab*.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 4.
- Peter Johnstone's lecture notes for "Category Theory" (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4.

Definition. An object X of $\mathscr{C}$ is *initial* if for each $Y \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $f : X \to Y$. Moreover, we say that X is *terminal* if for each $Z \in \operatorname{ob}\mathscr{C}$, there is a unique morphism $g : Z \to X$. Either condition is called a *universal property* of X.

Definition. Any property P of $\mathscr{C}$ has a dual property P^{op} of $\mathscr{C}^{op}$ obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P. Then P is true of $\mathscr{C}$ iff P^{op} is true of $\mathscr{C}^{\mathrm{op}}$.

Lemma 1. Being initial and being terminal are dual properties.

Lemma 2. Any two initial objects of $\mathscr{C}$ are canonically isomorphic. The same holds for any two terminal objects of $\mathscr{C}$.

Proof. Compose the two unique morphisms to get an isomorphism between the two initial objects. Apply duality to get the second claim. ☐

Remark 1. Think of a universal property as follows. Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $X \in \operatorname{ob}\mathscr{C}$. A *universal arrow from X to F* is an ordered pair (Y, f) with $Y \in \operatorname{ob}\mathscr{D}$ and $f : X \to F(Y)$ a morphism of $\mathscr{C}$ with the property that for any $X' \in \operatorname{ob}\mathscr{D}$ and morphism $f' : X \to F(X')$ of $\mathscr{C}$, there exists a unique morphism $\hat{f} : Y \to X'$ of $\mathscr{D}$ such that $F(\hat{f}) \circ f = f'$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "f'"'] & F(Y) \arrow[d, dashed, "F(\hat{f})"] \\ & F(X') \end{tikzcd}$$

Dually, a *universal arrow from F to X* is an ordered pair (Y, f) with $Y \in \operatorname{ob}\mathscr{D}$ and $f : F(Y) \to X$ of $\mathscr{C}$ with the property that for any $X' \in \operatorname{ob}\mathscr{D}$ and morphism $f' : F(X') \to X$, there exists a unique morphism $\hat{f} : X' \to Y$ such that $f' = f \circ F(\hat{f})$.

$$\begin{tikzcd} F(X') \arrow[r, dashed, "F(\hat{f})"] \arrow[dr, "f'"'] & F(Y) \arrow[d, "f"] \\ & X \end{tikzcd}$$

Remark 2. To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor and $Y \in \operatorname{ob}\mathscr{D}$. The *slice* or *left fiber category*, denoted by (F/Y) or $(F \downarrow Y)$, has as objects pairs (X, f) where $f : F(X) \to Y$ and as morphisms from $f : F(X) \to Y$ to $f' : F(X') \to Y$ morphisms $g : X \to X'$ such that $f = f' \circ F(g)$.

Definition. The *coslice* or *right fiber category*, denoted by (Y/F) or $(Y \downarrow F)$, has as objects pairs (X, f) where $f : Y \to F(X)$ and as morphisms from $f : Y \to F(X)$ to $f' : Y \to F(X')$ morphisms $g : X \to X'$ such that $f' = F(g) \circ f$.

Remark 3. If $F^{op} : C^{op} \to D^{op}$ is opposite to the functor $F : \mathscr{C} \to \mathscr{D}$ and $Y \in \operatorname{ob} \mathscr{D}$, then $(Y/F)^{op} = F^{op}/Y$. Thus, the left and right fiber categories are dual in the sense that $P(Y, F)$ is true for any right fiber category Y/F iff $P^{op}(Y, F)$ is true for any left fiber category F/Y.

Proposition 1. Let $F : \mathscr{D} \to \mathscr{C}$ be a functor and $x \in \operatorname{ob} C$. Then $u : x \to Fr$ is a universal arrow from x to F iff it is initial object of the coslice $(x \downarrow F)$. Dually, $u' : Fr' \to x$ is a universal arrow from F to x iff it is a terminal object of the same category.

Proof. [[I messed this up during my talk. It should be correct as written now.]] Suppose that u is universal and $f : x \to Fy$ is another object of $(x \downarrow F)$. Then there is some unique $\hat{f} : r \to y$ such that $F(\hat{f}) \circ u = f$. Thus $F(\hat{f})$ is a unique morphism of the coslice.

Conversely, suppose that u is initial. Then for any object $f : x \to Fy$ of $(x \downarrow F)$, there is some unique arrow $Sg : Fr \to Fy$ such that $Sg \circ u = f$. Hence setting $\hat{f} = g$ make u a universal arrow. $\square$

Corollary 1. Any two universal arrows from x to F can be canonically identified by Lemma 2.

Definition. A *zero object* of $\mathscr{C}$ is an object that is both initial and terminal. A *pointed category* is a category with a chosen zero object.

Example 1. The unique initial object of **Set** is $\emptyset$, and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in iso(**Set**).

Definition. Given $X \in \operatorname{ob} \mathscr{C}$, the *undercategory* $X/\mathscr{C}$ has as objects morphisms in $\mathscr{C}$ of the form $i : X \to Y$ where X is fixed. Given $i : X \to Y$ and $i' : X \to Y'$ in $\operatorname{ob} X/\mathscr{C}$, define the set of morphisms from i to i' as the morphisms $f : Y \to Y'$ where

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "i'"'] & Y \arrow[d, "f"] \\ & Y' \end{tikzcd}$$

commutes. We call i the *structure morphism.*

Composition and identity carry over exactly from $\mathscr{C}$.

Definition. Given $x \in \operatorname{ob} \mathscr{C}$, the *overcategory* $\mathscr{C}/X$ has as objects morphisms in $\mathscr{C}$ of the form $i : Y \to X$ where X is fixed. Given $i : Y \to X$ and $i' : Y' \to X$ in $\operatorname{ob} \mathscr{C}/X$, define the set of morphisms from i to i' as the morphisms $f : Y \to Y'$ where

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "i"'] & Y' \arrow[d, "i'"] \\ & X \end{tikzcd}$$

commutes. We again call i the *structure morphism.*

Composition and identity carry over exactly from $\mathscr{C}$.

Remark 4. If $X \in \operatorname{ob} \mathscr{C}$, then $(X/\mathscr{C})^{\mathrm{op}} = \mathscr{C}^{op}/X$. Thus, the under- and overcategories are dual in the sense that $P(X, \mathscr{C})$ is true for any undercategory $X/\mathscr{C}$ iff $P^{\mathrm{op}}(X, \mathscr{C})$ is true for any overcategory $\mathscr{C}/X$.

Lemma 3. For any $X \in \mathscr{C}$, the identity morphism on X is an initial object $X/\mathscr{C}$. Dually, it is a terminal object in $\mathscr{C}/X$.

Proof. Any $i : X \to Y$ is itself the unique morphism from Id_X to i. $\square$

Lemma 4. Let X be an initial object of $\mathscr{C}$. The identity morphism on X is a zero object $\mathscr{C}/X$. Dually, if $Y \in \operatorname{ob} \mathscr{C}$ is terminal, then Id_Y is a zero object in $Y/\mathscr{C}$.

Proof. We already know that Id_X is terminal. If $p : Y \to X$ is an object in $\mathscr{C}/X$, then there is a unique morphism $f : X \to Y$. Then $f \circ p$ must equal Id_X. $\square$

Example 2. Let (X, x) be a pointed set with $X = \{x\}$. Let $\mathbf{Set}_*$ denotes the category of pointed sets with base point preserving functions. Then since $\mathbf{Set}_* \cong X/\mathbf{Set}$, it follows that X is a zero object in $\mathbf{Set}_*$.

Definition. Given a morphism $\alpha : X \to Z$ in $\mathscr{C}$, define the *under-and-overcategory* $(X/\mathscr{C}/Z)_\alpha$ as having triples (Y, i, p) as obejcts where $i : X \to Y$ and $p : Y \to Z$ are morphisms in $\mathscr{C}$ such that $p \circ i = \alpha$. Define the set of morphisms from (Y, i, p) to (Y', u', p') as the morphisms $f : Y \to Y'$ such that

$$\begin{tikzcd} X \arrow[r, "i'"] \arrow[d, "i"'] \arrow[dr, "\alpha"] & Y' \arrow[d, "p'"] \\ Y \arrow[r, "p"] \arrow[ur, "f"] & Z \end{tikzcd}$$

commutes. If $\alpha = \mathrm{Id}_X$, then we call $(X/\mathscr{C}/X)_{\mathrm{Id}_X}$ the category of *retractive* objects over X as each triple (Y, i, p) is a retraction of Y onto X.

Example 3. If $F : \mathscr{C} \to \mathscr{C}$ is the identity functor, then the undercategory $Y/\mathscr{C}$ equals the right fiber category Y/F while the overcategory $\mathscr{C}/Y$ equals the left fiber category F/Y.

Definition. Let $\mathscr{J}$ be a category. A *diagram of shape* $\mathscr{J}$ *in* $\mathscr{C}$ is a functor $F : \mathscr{J} \to \mathscr{C}$.

Definition. Given a functor $F : \mathscr{J} \to \mathscr{C}$ and $X \in \operatorname{ob} \mathscr{C}$, a *cone over* F consists of an *apex* $X \in \operatorname{ob} \mathscr{C}$ and legs $f_j : X \to F(j)$ for each $J \in \operatorname{ob} \mathscr{J}$ such that for any $\alpha : j \to j'$,

$$\begin{tikzcd} X \arrow[r, "f_j"] \arrow[dr, "f_{j'}"'] & F(j) \arrow[d, "F\alpha"] \\ & F(j') \end{tikzcd}$$

commutes. This is just a natural transformation $\Delta_{\mathscr{J}} X \Rightarrow F$ where $\Delta_{\mathscr{J}} X$ denotes the constant functor on $\mathscr{J}$ at X. If $\mathscr{J}$ is small, then $\Delta_{\mathscr{J}}$ is just a functor from $\mathscr{C}$ to $\mathbf{Fun}(\mathscr{J}, \mathscr{C})$.

Definition. The *category of cones over* F is the right fiber category X/F. The *category of cones under* F is the left fiber category F/X.

Definition. Let $\mathscr{C}$ and $\mathscr{D}$ be categories and $g : Y \to Z$ a morphism in $\mathscr{D}$. Let $\Delta_{\mathscr{C}} g : \Delta_{\mathscr{C}} Y \Rightarrow \Delta_{\mathscr{C}} Z$ be the natural transformation with components $X \mapsto g$. A *colimit* for the functor $F : \mathscr{C} \to \mathscr{D}$ consists of an object Y of $\mathscr{D}$ and a natural transformation $i : F \Rightarrow \Delta_{\mathscr{C}} Y$ such that for any $Z \in \operatorname{ob} \mathscr{D}$ and natural transformation $j : F \Rightarrow \Delta_{\mathscr{C}} Z$, there is a unique morphism $g : Y \to Z$ such that $j = \Delta_{\mathscr{C}} g \circ i$. We write $Y = \operatorname{colim}_{\mathscr{C}} F$.

Definition. We say that $\mathscr{D}$ admits *all* $\mathscr{C}$*-shaped colimits* if each functor $G : \mathscr{C} \to \mathscr{D}$ has a colimit and that $\mathscr{D}$ is *cocomplete* if each functor $G : \mathscr{C} \to \mathscr{D}$ with $\mathscr{C}$ small has a colimit.

Remark 5. If $\mathscr{C}$ is small, then a colimit of $F : \mathscr{C} \to \mathscr{D}$ is just an initial object in the right fiber category $F/\Delta_{\mathscr{C}}$, which has as objects pairs $(Z, j : F \to \Delta Z)$ and as morphisms from (Y, i) to (Z, j) the morphisms $g : Y \to Z$ in $\mathscr{D}$ such that $\Delta g \circ i = j$.

Remark 6. Notice that there is a natural bijection $\mathscr{D}(Y, Z) \cong \mathbf{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta Z)$ iff $Y = \operatorname{colim}_{\mathscr{C}} F$.

Proposition 2. Any two colimits are canonically isomorphic.

Proof. When $\mathscr{C}$ is small, this is immediate from Lemma 2. But note that the proof of Lemma 2 does not require that $\mathscr{C}$ be locally small (a property which Rognes stipulates of any category). $\square$

Lemma 5. Assume that $\mathscr{D}$ admits all $\mathscr{C}$-shapes colimits and that $\mathscr{C}$ is small. Then a (possibly global) global choice function $\operatorname{colim}_{\mathscr{C}} : \mathbf{Fun}(\mathscr{C}, \mathscr{D}) \to \mathscr{D}$ given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor $\Delta_{\mathscr{C}} : \mathscr{D} \to \mathbf{Fun}(\mathscr{C}, \mathscr{D})$.

Proof. For any functor $F : \mathscr{C} \to \mathscr{D}$, there is a bijection $\mathscr{D}(\operatorname{colim}_{\mathscr{C}} F, Z) \cong \mathbf{Fun}(\mathscr{C}, \mathscr{D})(F, \Delta_{\mathscr{C}} Z)$. $\square$

Definition. A limit of the functor $F : \mathscr{C} \to \mathscr{D}$ is the colimit of $F^{op} : \mathscr{C}^{op} \to \mathscr{D}^{op}$.

Remark 7. Explicitly, a limit for $F : \mathscr{C} \to \mathscr{D}$ is an object Z of $\mathscr{D}$ and a natural transformation $p : \Delta_{\mathscr{C}} Z \Rightarrow F$ such that for any $Y \in \operatorname{ob} \mathscr{D}$ and natural transformation $q : \Delta_{\mathscr{C}} Y \Rightarrow F$, there is a unique morphism $g : Y \to Z$ such that $q = p \circ \Delta_{\mathscr{C}} g$.

Remark 8. The colimit of a functor F is the limit of F^{op}. Hence limit and colimit are dual properties, and the above results for colimits can be dualized.

Example 4. If $\mathscr{C}$ is the empty category, then the empty functor $F : \mathscr{C} \to \mathscr{D}$ has $F/\Delta_{\mathscr{C}} \cong \mathscr{D}$, so that the colimit is an initial object of $\mathscr{D}$.

Definition. Let $\mathscr{J}$ be a discrete small category. A diagram of shape $\mathscr{J}$ is a family $\{A_i\}_{i \in J}$. A limit for this diagram is the *product* $\prod_i A_i$ equipped with projections $\pi_i : \prod_i A_i \to A_i$ such that for every $f_i : U \to A_i$ there is some unique $f : U \to \prod_i A_i$ with $\pi_i \circ f = f_i$.

Dually, a colimit for the diagram is the *coproduct* $\sum_i A_i$ equipped with inclusions $u_i : A_i \to \sum_i A_i$ such that for any $f_i : A_i \to Y$, there is some unique $f : \sum_i A_i \to Y$ with $f_i = f \circ u_i$.

Example 5. Familiar examples include disjoint unions, free products, cartesian products, and direct products.

Definition. Let $\mathscr{J}$ be the category $\bullet \rightrightarrows \bullet$. Then a diagram of shape $\mathscr{J}$ looks like $A \overset{f}{\underset{g}{\rightrightarrows}} B$. A cone over this with apex C and legs $f_1 : C \to A$ and $f_2 : C \to B$ satisfies $ff_1 = f_2 = gf_1$. If such an object C together with f_1 is the limit of the diagram, then we say it is the *equalizer* of f and g. Dually, a colimit is called the *coequalizer*.

Example 6. The equalizer in **Set** of $f, g : X \to Y$ is the subset $X' \coloneqq \{x \in X : f(x) = g(x)\}$ together with the inclusion $X' \hookrightarrow X$. The coequalizer of (f, g) if $Y/\sim$ together with the quotient map on B where $\sim$ is the smallest equivalence relation under which $f(x) \sim g(x)$ for every x.

Example 7. The same idea applies to **Grp**. The relation $\sim$ just becomes a particular minimal normal subgroup.

Definition. Let $\mathscr{J}$ be the category $\bullet \to \bullet \leftarrow \bullet$. Then a diagram of this shape looks like $B \xrightarrow{f} D \xleftarrow{g} A$, while a cone over this diagram looks like

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, "\alpha"'] & A \arrow[d, "g"] \\ B \arrow[r, "f"] & D \end{tikzcd}$$

If such an object C together with i and j is the limit of this diagram, then we call it the *pullback* of f and g, denoted by $B \times_D A$.

Definition. We can perform an analogous construction for $\mathscr{J}^{op}$. Then the colimit of the resulting diagram is called the *pushout*, denoted by $B \cup_D A$.

Example 8. The pullback in **Set** of $f : X \to Z$ and $g : Y \to Z$ is the subset $\{(x, y) \in X \times Y : f(x) = g(y)\}$, called the *fiber product* of X and Y over Z.

Theorem 1. (Freyd)

1. If $\mathscr{C}$ has equalizers and all small (resp. finite) products, then it has all small (resp. finite) limits.
2. If $\mathscr{C}$ has pullbacks and a terminal object, then it has all finite limits.

Proof.

1. See Johnstone, Theorem 4.9.

2. By part 1, it suffices to show that $\mathscr{C}$ has equalizers and all finite products. By assumption there is some terminal object 1. Then any product $A_1 \times A_2$ can be realized as the pullback of $A_1 \to 1 \leftarrow A_2$. By induction $\mathscr{C}$ has all finite products. Moreover, for morphisms $f, g : A \to B$, note that any cone over the diagram $A \underset{(1_A,g)}{\longrightarrow} A \times B \underset{(1_A,f)}{\longleftarrow} A$ admits morphisms $h : A \to C$ and $k : C \to A$ such that $h = k$ and $fk = gh$. Thus the pullback for this diagram is an equalizer for (f, g), completing the proof.

□

Corollary 2. Both **Set** and **Grp** are complete and cocomplete (or *bicomplete*).

Remark 9. It turns out that adjoints interact nicely with (co)limits under mild conditions.

Proposition 3. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ be an adjoint pair and $\mathscr{E}$ small. If $X : \mathscr{E} \to \mathscr{C}$ is a functor with $\operatorname{colim}_{\mathscr{E}} X$, then

$$\operatorname{colim}_{\mathscr{E}}(F \circ X) = F(\operatorname{colim}_{\mathscr{E}} X).$$

Dually, if $Y : \mathscr{E} \to \mathscr{D}$ is a functor with $\lim_{\mathscr{E}} Y$, then

$$\lim_{\mathscr{E}}(G \circ Y) = G(\lim_{\mathscr{E}} Y).$$

Proof. We have the following chain of natural bijections for each $Y \in \mathscr{D}$:

$$\mathscr{D}(F(\operatorname{colim}_{\mathscr{E}}), Y) \cong \mathscr{C}(\operatorname{colim}_{\mathscr{E}} X, G(Y)) \cong \lim_{\mathscr{E}} \mathscr{C}(X(-), G(Y)) \cong \lim_{\mathscr{E}} \mathscr{D}(F(X(-)), Y) \cong \mathbf{Fun}(\mathscr{E}, \mathscr{D})(F \circ X, \Delta Y).$$

The second bijection follows from the fact that both sets can be identified with the components of the natural transformations from X to $\Delta G(Y)$.

The second claim follows by duality. □

Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. The *fiber category* $F^{-1}(Y)$ is the full subcategory of $\mathscr{C}$ generated by the objects X with $F(X) = Y$.

Definition. Suppose $\mathscr{C}$ has terminal object 1. A *cofiber* of a morphism $f : X \to Y$ is a pushout of the diagram $1 \leftarrow X \to Y$. We write Y/X. Further, given a morphism $p : 1 \to Y$, the *fiber* of f at p is a pullback of $1 \to Y \leftarrow X$. We write $f^{-1}(p)$.

Definition. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. For each $Y \in \operatorname{ob}\mathscr{D}$, there is a full and faithful functor $F^{-1}(Y) \rightarrowtail F/Y$ given by $X \mapsto (X, \mathrm{Id}_Y)$. We say that $\mathscr{C}$ is a *precofibered category* over $\mathscr{D}$ if this functor admits a left adjoint given by $(Z, g : F(Z) \to Y) \mapsto g_*(Z)$.

Moreover, there is a full and faithful functor $F^{-1}(Y) \rightarrowtail Y/F$ defined in the same way. We say that $\mathscr{C}$ is a *prefibered category* over $\mathscr{D}$ if this functor admits a right adjoint given by $(Z, g : Y \to F(Z)) \mapsto g_*(Z)$.

Definition. Let $F ; \mathscr{C} \to \mathscr{D}$ be a functor and $f : c' \to c$ be a morphism in $\mathscr{C}$. We say f is *cartesian* if for any morphism $f' : c'' \to c$ in $\mathscr{C}$ and $g : F(c'') \to F(c')$ in $\mathscr{D}$ such that $Ff \circ g = Ff'$, there exists a unique $\phi : c'' \to c$ such that $f' = f \circ \phi$ and $F\phi = g$. In other words, any filling of the following diagram can be lifted to a filling in $\mathscr{D}$.

$$\begin{tikzcd} c'' \arrow[r, dashed, "\exists!"] \arrow[dr, "f'"'] & c' \arrow[d, "f"] \\ & c \end{tikzcd}$$

Definition. We say that F is a *fibration* if for any $c \in \mathscr{C}$ and morphism $f : d \to Fc$, there is a cartesian $\phi : c' \to c$ such that $F\phi = f$. Such ϕ is called a *cartesian lifting* of f to c.

Example 9. Let **Mod** denote the category of left R-modules where R is a ring. Then the forgetful functor $U : \mathbf{Mod} \to \mathbf{Ring}$ is a fibration.