

Abstract

We introduce the concept of a universal property in category theory. The main sources for this talk are the following.

- nLab.
- John Rognes’s *Lecture Notes on Algebraic K-Theory*, Ch. 4.
- Peter Johnstone’s lecture notes for “Category Theory” (Mathematical Tripos Part III, Michaelmas 2015), Ch. 4.
- Steve Awodey’s *Category Theory*, Ch. 5.6.

1 Universal arrows

Definition 1.1. Let \mathcal{C} be a category.

1. An object X of \mathcal{C} is *initial* if for each $Y \in \text{ob } \mathcal{C}$, there is a unique morphism $f : X \rightarrow Y$.
2. We say that X is *terminal* if for each $Z \in \text{ob } \mathcal{C}$, there is a unique morphism $g : Z \rightarrow X$.

Either condition is called a *universal property* of X .

Any property P of \mathcal{C} has a dual property P^{op} of \mathcal{C}^{op} obtained by interchanging the source and target of any arrow as well as the order of any composition in the sentence expressing P . Then P is true of \mathcal{C} iff P^{op} is true of \mathcal{C}^{op} .

Example 1.2. Being initial and being terminal are dual properties.

Lemma 1.3. *Any two initial objects of \mathcal{C} are unique up to unique isomorphism. The same holds for any two terminal objects of \mathcal{C} .*

Proof sketch. Let X and X' be two initial objects. Compose the two unique morphisms $X \rightarrow X'$ and $X' \rightarrow X$ to get an isomorphism between X and X' . Apply duality to this argument for the case of terminal objects. \square

We can think of a universal property as follows. Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and $X \in \text{ob } \mathcal{C}$. A *universal arrow from X to F* is an ordered pair (Y, f) with $Y \in \text{ob } \mathcal{D}$ and $f : X \rightarrow F(Y)$ a morphism of \mathcal{C} with the property that for any $X' \in \text{ob } \mathcal{D}$ and morphism $f' : X \rightarrow F(X')$ of \mathcal{C} , there exists a unique morphism $\hat{f} : Y \rightarrow X'$ of \mathcal{D} such that $F(\hat{f}) \circ f = f'$.

$$\begin{array}{ccc} X & \xrightarrow{f} & F(Y) \\ & \searrow f' & \downarrow F(\hat{f}) \\ & & F(X') \end{array}$$

Dually, a *universal arrow from F to X* is an ordered pair (Y, f) with $Y \in \text{ob } \mathcal{D}$ and $f : F(Y) \rightarrow X$ of \mathcal{C} with the property that for any $X' \in \text{ob } \mathcal{D}$ and morphism $f' : F(X') \rightarrow X$, there exists a unique morphism $\hat{f} : X' \rightarrow Y$ such that $f' = f \circ F(\hat{f})$.

$$\begin{array}{ccc} F(X') & \xrightarrow{F(\hat{f})} & F(Y) \\ & \searrow f' & \downarrow f \\ & & X \end{array}$$

To see why this notion of universality agrees with the original one, we first generalize the notion of an arrow category.

Definition 1.4.

1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $Y \in \text{ob } \mathcal{D}$. The *slice* or *left fiber category*, denoted by (F/Y) or $(F \downarrow Y)$, has as objects pairs (X, f) where $f : F(X) \rightarrow Y$ and as morphisms from $f : F(X) \rightarrow Y$ to $f' : F(X') \rightarrow Y$ morphisms $g : X \rightarrow X'$ such that $f = f' \circ F(g)$.
2. The *coslice* or *right fiber category*, denoted by (Y/F) or $(Y \downarrow F)$, has as objects pairs (X, f) where $f : Y \rightarrow F(X)$ and as morphisms from $f : Y \rightarrow F(X)$ to $f' : Y \rightarrow F(X')$ morphisms $g : X \rightarrow X'$ such that $f' = F(g) \circ f$.

If $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ is opposite to the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and $Y \in \text{ob } \mathcal{D}$, then $(Y/F)^{\text{op}} = F^{\text{op}}/Y$. Thus, the left and right fiber categories are dual in the sense that $P(Y, F)$ is true of any right fiber category Y/F iff $P^{\text{op}}(Y, F)$ is true of any left fiber category F/Y .

Proposition 1.5. *Let $F : \mathcal{D} \rightarrow \mathcal{C}$ be a functor and $x \in \text{ob } \mathcal{C}$. Then $u : x \rightarrow Fr$ is a universal arrow from x to F iff it is an initial object of the coslice $(x \downarrow F)$. Dually, $u' : Fr' \rightarrow x$ is a universal arrow from F to x iff it is a terminal object of the same category.*

Proof. Suppose that u is universal and $f : x \rightarrow Fy$ is another object of $(x \downarrow F)$. Then there exists a unique $\hat{f} : r \rightarrow y$ such that $F(\hat{f}) \circ u = f$. Thus $F(\hat{f})$ is a unique morphism of the coslice.

Conversely, suppose that u is initial. Then for any object $f : x \rightarrow Fy$ of $(x \downarrow F)$, there exists a unique arrow $Sg : Fr \rightarrow Fy$ such that $Sg \circ u = f$. Hence setting $\hat{f} = g$ makes u a universal arrow. \square

Corollary 1.6. *Any two universal arrows from x to F can be canonically identified by Lemma 1.3.*

2 (Co)limits

Definition 2.1. A *zero object* of \mathcal{C} is an object that is both initial and terminal.

Example 2.2. The unique initial object of **Set** is \emptyset , and the terminal objects are precisely the singleton sets. Hence there is no zero object. Moreover, there are no initial or terminal objects in $\text{iso}(\mathbf{Set})$.

Given $X \in \text{ob } \mathcal{C}$, the *undercategory* X/\mathcal{C} has as objects morphisms in \mathcal{C} of the form $i : X \rightarrow Y$ where X is fixed. Given $i : X \rightarrow Y$ and $i' : X \rightarrow Y'$ in $\text{ob } X/\mathcal{C}$, define the set of morphisms from i to i' as the morphisms $f : Y \rightarrow Y'$ where

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ & \searrow i' & \downarrow f \\ & & Y' \end{array}$$

commutes. (We call i the *structure morphism*.) Composition and identity carry over exactly from \mathcal{C} .

Likewise, given $x \in \text{ob } \mathcal{C}$, the *overcategory* \mathcal{C}/X has as objects morphisms in \mathcal{C} of the form $i : Y \rightarrow X$ where X is fixed. Given $i : Y \rightarrow X$ and $i' : Y' \rightarrow X$ in $\text{ob } \mathcal{C}/X$, define the set of morphisms from i to i' as the morphisms $f : Y \rightarrow Y'$ where

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ & \searrow i & \downarrow i' \\ & & X \end{array}$$

commutes. Composition and identity carry over exactly from \mathcal{C} .

Remark 2.3. If $X \in \text{ob } \mathcal{C}$, then $(X/\mathcal{C})^{\text{op}} = \mathcal{C}^{\text{op}}/X$. Thus, the under- and overcategories are dual in the sense that $P(X, \mathcal{C})$ is true of any undercategory X/\mathcal{C} iff $P^{\text{op}}(X, \mathcal{C})$ is true of any overcategory \mathcal{C}/X .

Lemma 2.4. *For any $X \in \mathcal{C}$, the identity morphism on X is an initial object X/\mathcal{C} . Dually, it is a terminal object in \mathcal{C}/X .*

Proof. Any morphism $i : X \rightarrow Y$ is itself the unique morphism from Id_X to i . □

Lemma 2.5. *Let X be an initial object of \mathcal{C} . The identity morphism on X is a zero object \mathcal{C}/X . Dually, if $Y \in \text{ob } \mathcal{C}$ is terminal, then Id_Y is a zero object in Y/\mathcal{C} .*

Proof. We already know that Id_X is terminal. If $p : Y \rightarrow X$ is an object in \mathcal{C}/X , then there is a unique morphism $f : X \rightarrow Y$. Then $f \circ p$ must equal Id_X . □

Example 2.6. Let (X, x) be a pointed set with $X = \{x\}$. Let \mathbf{Set}_* denotes the category of pointed sets with base point preserving functions. Since $\mathbf{Set}_* \cong X/\mathbf{Set}$, it follows that X is a zero object in \mathbf{Set}_* .

Given a morphism $\alpha : X \rightarrow Z$ in \mathcal{C} , define the *under-and-overcategory* $(X/\mathcal{C}/Z)_\alpha$ as having triples (Y, i, p) as objects where $i : X \rightarrow Y$ and $p : Y \rightarrow Z$ are morphisms in \mathcal{C} such that $p \circ i = \alpha$. Define the set of morphisms from (Y, i, p) to (Y', i', p') as the set of morphisms $f : Y \rightarrow Y'$ such that

$$\begin{array}{ccc} X & \xrightarrow{i'} & Y' \\ i \downarrow & \searrow f & \downarrow p' \\ Y & \xrightarrow{p} & Z \end{array}$$

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commutes. If $\alpha = \text{Id}_X$, then we call $(X/\mathcal{C}/X)_{\text{Id}_X}$ the category of *retractive* objects over X , with each triple (Y, i, p) being a retraction of Y onto X .

Example 2.7. If $F : \mathcal{C} \rightarrow \mathcal{C}$ is the identity functor, then the undercategory Y/\mathcal{C} equals the right fiber category Y/F , and the overcategory \mathcal{C}/Y equals the left fiber category F/Y .

Let \mathcal{J} be a category. A *diagram of shape \mathcal{J} in \mathcal{C}* is a functor $F : \mathcal{J} \rightarrow \mathcal{C}$.

Definition 2.8. Given a functor $F : \mathcal{J} \rightarrow \mathcal{C}$ and $X \in \text{ob } \mathcal{C}$, a *cone over F* consists of an *apex* $X \in \text{ob } \mathcal{C}$ and *legs* $f_j : X \rightarrow F(j)$ for each $j \in \text{ob } \mathcal{J}$ such that for any morphism $\alpha : j \rightarrow j'$, the triangle

$$\begin{array}{ccc} X & \xrightarrow{f_j} & F(j) \\ & \searrow f_{j'} & \downarrow F\alpha \\ & & F(j') \end{array}$$

commutes.

This is simply a natural transformation $\Delta_{\mathcal{J}} X \Rightarrow F$ where $\Delta_{\mathcal{J}} X$ denotes the constant functor on \mathcal{J} at X . If \mathcal{J} is small, then $\Delta_{\mathcal{J}}$ is a functor from \mathcal{C} to $\mathbf{Fun}(\mathcal{J}, \mathcal{C})$.

Definition 2.9.

1. The *category of cones over F* is the right fiber category X/F .
2. The *category of cocones under F* is the left fiber category F/X .

Definition 2.10 (Colimit). Let \mathcal{C} and \mathcal{D} be categories and $g : Y \rightarrow Z$ a morphism in \mathcal{D} . Let $\Delta_{\mathcal{C}} g : \Delta_{\mathcal{C}} Y \Rightarrow \Delta_{\mathcal{C}} Z$ be the natural transformation with components $X \mapsto g$.

1. A *colimit* $\text{colim}_{\mathcal{C}} F$ of the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an object Y of \mathcal{D} together with a natural transformation $i : F \Rightarrow \Delta_{\mathcal{C}} Y$ such that for any $Z \in \text{ob } \mathcal{D}$ and any natural transformation $j : F \Rightarrow \Delta_{\mathcal{C}} Z$, there is a unique morphism $g : Y \rightarrow Z$ such that $j = \Delta_{\mathcal{C}} g \circ i$.
2. We say that \mathcal{D} *admits/has \mathcal{C} -shaped colimits* if each functor $G : \mathcal{C} \rightarrow \mathcal{D}$ has a colimit.
3. We say that \mathcal{D} is *cocomplete* if each functor $G : \mathcal{C} \rightarrow \mathcal{D}$ with \mathcal{C} small has a colimit.

If \mathcal{C} is small, then a colimit of $F : \mathcal{C} \rightarrow \mathcal{D}$ is just an initial object in the right fiber category $F/\Delta_{\mathcal{C}}$, which has as objects pairs $(Z, j : F \rightarrow \Delta Z)$ and as morphisms from (Y, i) to (Z, j) the morphisms $g : Y \rightarrow Z$ in \mathcal{D} such that $\Delta g \circ i = j$.

Example 2.11. If \mathcal{C} is the empty category, then the empty functor $F : \mathcal{C} \rightarrow \mathcal{D}$ satisfies $F/\Delta_{\mathcal{C}} \cong \mathcal{D}$, so that the colimit of F is exactly the initial object of \mathcal{D} .

Proposition 2.12. *There is a natural bijection $\mathcal{D}(Y, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta Z)$ if and only if $Y = \text{colim}_{\mathcal{C}} F$.*

Lemma 2.13. *Any two colimits are unique up to unique isomorphism.*

Proof. When \mathcal{C} is small, this follows immediately from Lemma 1.3. Notice, however, that our proof of Lemma 1.3 does *not* require that \mathcal{C} be locally small (a property which Rognes stipulates of any category). \square

Remark 2.14. Assume that \mathcal{D} has \mathcal{C} -shaped colimits and that \mathcal{C} is small. Then a (possibly global) choice function $\text{colim}_{\mathcal{C}} : \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}$ given by choosing a colimit for each functor determines a functor that is left adjoint to the constant diagram functor $\Delta_{\mathcal{C}} : \mathcal{D} \rightarrow \mathbf{Fun}(\mathcal{C}, \mathcal{D})$. Indeed, for any functor $F : \mathcal{C} \rightarrow \mathcal{D}$, there is a bijection $\mathcal{D}(\text{colim}_{\mathcal{C}} F, Z) \cong \mathbf{Fun}(\mathcal{C}, \mathcal{D})(F, \Delta_{\mathcal{C}} Z)$.

Definition 2.15 (Limit). A *limit* of the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a colimit of $F^{\text{op}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$.

Explicitly, a limit for $F : \mathcal{C} \rightarrow \mathcal{D}$ is an object Z of \mathcal{D} along with a natural transformation $p : \Delta_{\mathcal{C}} Z \Rightarrow F$ such that for any $Y \in \text{ob } \mathcal{D}$ and any natural transformation $q : \Delta_{\mathcal{C}} Y \Rightarrow F$, there is a unique morphism $g : Y \rightarrow Z$ such that $q = p \circ \Delta_{\mathcal{C}} g$.

Note that the colimit of a functor F is exactly the limit of F^{op} . Hence *limit* and *colimit* are dual properties, and our results so far for colimits can be dualized for limits.

Definition 2.16 ((Co)product). Let \mathcal{J} be a discrete small category. Consider a diagram $\{A_i\}_{i \in \text{ob } \mathcal{J}}$ of shape \mathcal{J} .

1. The limit of $\{A_i\}_i$ is called the *product* $\prod_i A_i$, equipped with projections $\pi_i : \prod_i A_i \rightarrow A_i$ such that for every $f_i : U \rightarrow A_i$ there exists a unique map $f := (f_i) : U \rightarrow \prod_i A_i$ satisfying $\pi_i \circ f = f_i$.
2. The colimit of $\{A_i\}_i$ is called the *coproduct* $\coprod_i A_i$, equipped with inclusions $u_i : A_i \rightarrow \coprod_i A_i$ such that for any $f_i : A_i \rightarrow Y$, there exists a unique map $f := (f_i) : \coprod_i A_i \rightarrow Y$ satisfying $f_i = f \circ u_i$.

Familiar examples of limits include cartesian products and direct products, whereas familiar examples of colimits include disjoint unions and free products.

Example 2.17.

- (1) Consider any small diagram $F : \mathcal{J} \rightarrow \mathbf{Set}$. On the one hand,

$$\text{colim}_{\mathcal{J}} F_j \cong \left(\coprod_{j \in \text{ob } \mathcal{J}} F_j \right) / \sim$$

where \sim is the smallest equivalence relation such that $F_j \ni f_j \sim f_{j'} \in F_{j'}$, whenever $F(\psi)(f_j) = f_{j'}$ for some $\psi : j \rightarrow j'$.

On the other hand,

$$\lim_{\mathcal{J}} F_j \cong \left\{ (f_j)_j \in \prod_{j \in \text{ob } \mathcal{J}} F_j \mid \forall \psi : j \rightarrow j' \text{ in } \mathcal{J}, F(\psi)(f_j) = f_{j'} \right\}.$$

We have shown that \mathbf{Set} is both complete and cocomplete.

- (2) Let A be any set. Define the *cumulative hierarchy* $V_n(A)$ of rank $n < \omega$ over A along with a countable sequence

$$V_0 \xrightarrow{v_0} V_1 \xrightarrow{v_1} V_2 \longrightarrow \cdots \longrightarrow V_n \xrightarrow{v_n} V_{n+1} \longrightarrow \cdots$$

of maps recursively by

$$\begin{aligned} V_0(A) &= A \\ V_{n+1}(A) &= A \coprod \mathcal{P}(V_n(A)) \\ v_0 : A &\hookrightarrow A \coprod \mathcal{P}(A), & a &\mapsto a \\ v_{n+1} : A \coprod \mathcal{P}(V_n(A)) &\rightarrow A \coprod \mathcal{P}(V_{n+1}(A)), & (\text{Id}_A, \mathcal{P}(V_n(A))) & \end{aligned}$$

Let $V_\omega(A) = \text{colim}_{n < \omega} V_n(A)$, which exists by part (1). Then $V_\omega(\emptyset)$ is exactly the set of all hereditarily finite sets. To see that $V_\omega(-)$ is a functor $\mathbf{Set} \rightarrow \mathbf{Set}$, let $f : A \rightarrow B$ be a function. Then we can build a cocone

$$\begin{array}{ccccccc} V_0(A) & \xrightarrow{v_0} & V_1(A) & \longrightarrow & \cdots & \longrightarrow & V_n(A) & \xrightarrow{v_n} & V_{n+1}(A) \\ f_0 \equiv f \downarrow & & \downarrow (f, \mathcal{P}(f)) & & & & \downarrow (f, \mathcal{P}(f_{n-1})) & & \downarrow (f, \mathcal{P}(f_n)) \\ V_0(B) & \longrightarrow & V_1(B) & \longrightarrow & \cdots & \longrightarrow & V_n(B) & \longrightarrow & V_{n+1}(B) & \longrightarrow & V_\omega(B) \end{array}$$

there is a unique mediating map $B \cup_C A \rightarrow Z$ fitting into it.

Example 2.21.

1. The pullback in **Set** of $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ is precisely $\{(x, y) \in X \times Y : f(x) = g(y)\}$, called the *fibred product* of X and Y over Z .
2. The pushout in **Set** of $f : Z \rightarrow X$ and $g : Z \rightarrow Y$ is precisely the quotient of $X \amalg Y$ by the equivalence relation \sim generated by the formula $(\forall z \in Z) (f(z) \sim g(z))$. We call $X \amalg Y / \sim$ the *fibred sum* of X and Y under Z .

Example 2.22. Let $\mathbf{FinSet}_{\text{mono}}$ denote the category of all finite sets with injective functions as arrows. The category of *nominal sets* consists of all pullback-preserving functors $\mathbf{FinSet}_{\text{mono}} \rightarrow \mathbf{Set}$ with natural transformations as arrows. These correctly encode the syntax of functional programming languages modulo renaming of bound variables (which is necessary for substitution).

Proposition 2.23. *The pullback of a monomorphism in a category \mathcal{C} is again a monomorphism in \mathcal{C} .*

Proof. Consider any pullback square

$$\begin{array}{ccc} B \times_C A & \xrightarrow{\pi_2} & A \\ \pi_1 \downarrow & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

in \mathcal{C} where f is monic. We must show that π_1 is monic. Let $h_1, h_2 : B' \rightarrow B \times_C A$ be morphisms in \mathcal{C} such that

$$\begin{aligned} \pi_1 \circ h_1 &= \pi_1 \circ h_2 \\ \Downarrow \\ f \circ \pi_2 \circ h_1 &= g \circ \pi_1 \circ h_1 = g \circ \pi_1 \circ h_2 = f \circ \pi_2 \circ h_2. \end{aligned}$$

Since f is monic by assumption, it follows that $\pi_2 \circ h_1 = \pi_2 \circ h_2$. As a result, the universal property of pullbacks implies that $h_1 = h_2$, as required. \square

Our next two results are quite useful and follow directly from the universal property of pullback (dually, pushout) squares.

Proposition 2.24. *Let $f : X \rightarrow Y$ be a morphism in a category \mathcal{C} .*

1. *The commutative square*

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \parallel & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback if and only if f is a monomorphism.

2. *The commutative square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f \downarrow & & \parallel \\ Y & \xlongequal{\quad} & Y \end{array}$$

is a pushout if and only if f is an epimorphism.

Proposition 2.25 (Pasting law). *Consider a commutative diagram of the form*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

in a category \mathcal{C} .

1. *Suppose that the righthand square is a pullback. Then the total rectangle is a pullback if and only if the lefthand square is one.*
2. *Suppose that the lefthand square is a pushout. Then the total rectangle is a pushout if and only if the righthand square is one.*

Corollary 2.26. *The operations of forming pullbacks and forming pushouts are associative up to isomorphism.*

All coequalizers $A \begin{smallmatrix} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{smallmatrix} B \xrightarrow{h} C$ can be obtained from taking binary coproducts and pushouts as follows.

$$\begin{array}{ccc} A \amalg A & \xrightarrow{(f,g)} & B \\ (\text{Id}_A, \text{Id}_A) \downarrow & \lrcorner & \downarrow h \\ A & \longrightarrow & C \end{array}$$

Therefore, any category with binary coproducts and pushouts has coequalizers.

Moreover, any colimit of a sequence of the form

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots \quad (*)$$

is precisely the coequalizer of

$$\prod_n X_n \begin{array}{c} \xrightarrow{\text{Id}} \\ \xrightarrow{(u_{n+1} \circ f_n)_n} \end{array} \prod_n X_n.$$

Therefore, any category with coequalizers and small coproducts has colimits of diagrams like $(*)$. This fact can be generalized as follows.

Theorem 2.27 (Freyd).

- (i) *If \mathcal{C} has equalizers and small (resp. finite) products, then it has small (resp. finite) limits.*
- (ii) *If \mathcal{C} has pullbacks and a terminal object, then it has finite limits.*

Proof sketch.

1. Let $F : \mathcal{J} \rightarrow \mathcal{C}$ be any diagram with \mathcal{J} small. Consider the following two morphisms in \mathcal{C} :

$$\begin{aligned} f, g : \prod_{j \in \text{ob } \mathcal{J}} F_j &\rightarrow \prod_{\alpha : i \rightarrow j} F_j \\ \pi_{\alpha : i \rightarrow j} \circ f &\equiv \pi_j \\ \pi_{\alpha : i \rightarrow j} \circ g &\equiv F(\alpha) \circ \pi_i. \end{aligned}$$

Then $\lim_{\mathcal{J}} F$ is precisely the equalizer of f and g .

2. Thanks to part (i), it suffices to show that \mathcal{C} has equalizers and finite products. By assumption, there is some terminal object 1 . Then any product $A_1 \times A_2$ can be realized as the pullback of $A_1 \rightarrow 1 \leftarrow A_2$. By induction, it follows that \mathcal{C} has finite products. Moreover, for any morphisms $f, g : A \rightarrow B$, note that any cone over the diagram

$$A \xrightarrow{(\text{Id}_A, g)} A \times B \xleftarrow{(\text{Id}_A, f)} A$$

yields morphisms $h : A \rightarrow C$ and $k : C \rightarrow A$ such that $h = k$ and $fk = gh$. As a result, the pullback for this cospan is an equalizer of f and g , and thus our proof is complete. \square

We may view Example 2.17(1) as an instance of Theorem 2.27.

Next, let us show that adjoints interact nicely with (co)limits under mild conditions.

Proposition 2.28 (Left adjoints preserve colimits). *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be functors such that (F, G) is an adjoint pair. Let \mathcal{E} be small category. If $X : \mathcal{E} \rightarrow \mathcal{C}$ is a functor whose colimit exists, then*

$$\text{colim}_{\mathcal{E}}(F \circ X) \cong F \left(\text{colim}_{\mathcal{E}} X \right).$$

Dually, if $Y : \mathcal{E} \rightarrow \mathcal{D}$ is a functor whose limit exists, then

$$\lim_{\mathcal{E}}(G \circ Y) \cong G \left(\lim_{\mathcal{E}} Y \right).$$

Proof. We have the following chain of natural bijections in $Y \in \text{ob } \mathcal{D}$:

$$\begin{aligned} \mathcal{D} \left(F \left(\text{colim}_{\mathcal{E}} X \right), Y \right) &\cong \mathcal{C} \left(\text{colim}_{\mathcal{E}} X, G(Y) \right) \\ &\cong \lim_{\mathcal{E}} \mathcal{C}(X(-), G(Y)) \\ &\cong \lim_{\mathcal{E}} \mathcal{D}(F(X(-)), Y) \\ &\cong \mathbf{Fun}(\mathcal{E}, \mathcal{D})(F \circ X, \Delta Y). \end{aligned}$$

The second bijection exists because both sets can be identified with the components of all natural transformations $X \Rightarrow \Delta G(Y)$. \square

3 Fibers and Fibrations

Definition 3.1. Suppose \mathcal{C} has a terminal object 1 . Let $f : X \rightarrow Y$ be a morphism in \mathcal{C} .

1. Given a global element $p : 1 \rightarrow Y$ of Y , the *fiber* $f^{-1}(p)$ of f at p is the pullback of the cospan $1 \rightarrow Y \leftarrow X$.
2. The *cofiber* Y/X of f is the pushout of the span $1 \leftarrow X \rightarrow Y$.

For any functor $F : \mathcal{C} \rightarrow \mathcal{D}$, the *fiber category* $F^{-1}(Y)$ is the full subcategory of \mathcal{C} generated by those objects X such that $F(X) = Y$.

For each $Y \in \text{ob } \mathcal{D}$, there is a full and faithful functor $F^{-1}(Y) \rightarrow F/Y$ given by $X \mapsto (X, \text{Id}_Y)$. We say that \mathcal{C} is a *precofibered category* over \mathcal{D} if F has a left adjoint given by

$$(Z, g : F(Z) \rightarrow Y) \mapsto g_*(Z).$$

Further, there is a full and faithful functor $F^{-1}(Y) \rightarrow Y/F$. We say that \mathcal{C} is a *prefibered category* over \mathcal{D} if this functor has a right adjoint given by $(Z, g : Y \rightarrow F(Z)) \mapsto g_*(Z)$.

Definition 3.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor.

1. Let $f : c' \rightarrow c$ be a morphism in \mathcal{C} . We say f is *cartesian* if for any morphism $f' : c'' \rightarrow c$ in \mathcal{C} and any morphism $g : F(c'') \rightarrow F(c')$ in \mathcal{D} such that $Ff \circ g = Ff'$, there exists a unique morphism $\phi : c'' \rightarrow c$ such that $f' = f \circ \phi$ and $F\phi = g$.

In pictures,

$$\begin{array}{ccc} F(c'') & \xrightarrow{g} & F(c') \\ & \searrow Ff' & \downarrow Ff \\ & & F(c) \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} c'' & \xrightarrow{\exists! \phi} & c' \\ & \searrow f' & \downarrow f \\ & & c \end{array}, \quad \phi \xrightarrow{F} g.$$

2. We say that F is a *fibration* if for any $c \in \text{ob } \mathcal{C}$ and morphism $f : d \rightarrow Fc$ in \mathcal{D} , there is a cartesian morphism $\phi_f : c' \rightarrow c$ such that $F\phi_f = f$. Such a ϕ_f is called a *cartesian lifting* of f to c .

In this case, assuming the axiom of choice, we obtain a mapping $f \mapsto \phi_f$, known as a *cleavage* of F . If this respects the identity map and composition, then we call F a *normal* and *split* fibration, respectively.

Intuitively, if F is a fibration, then the fibers $F^{-1}(Y)$ depend functorially on $Y \in \text{ob } \mathcal{D}$.

Example 3.3.

1. Let the category **Mod** consist of pairs (R, M) as objects where R is a ring and M is a left R -module and pairs (f, \tilde{f}) as morphisms where $f : R \rightarrow R'$ is a ring homomorphism and $\tilde{f} : M \rightarrow M'$ is an R -linear map with M' viewed as an R -module via f . Then the forgetful functor $U : \mathbf{Mod} \rightarrow \mathbf{Ring}$ is a fibration.
2. For any category \mathcal{C} with pullbacks, consider the arrow category $\text{Ar}(\mathcal{C})$ along with the codomain functor $\text{cod} : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$ defined by

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ a' & \xrightarrow{\quad} & b' \end{array} \mapsto b \rightarrow b'.$$

This is a fibration. Indeed, for any object $x \rightarrow y$ in $\text{Ar}(\mathcal{C})$ and any morphism $z \rightarrow y$ in \mathcal{C} , the cartesian lifting of $z \rightarrow y$ to $x \rightarrow y$ is given by the pullback square

$$\begin{array}{ccc} z \times_y x & \longrightarrow & x \\ \downarrow & & \downarrow \\ z & \longrightarrow & y \end{array}.$$