Perry Hart
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Abstract

We continue to look at low-dimensional K-theory, finishing our description of $K_0(-)$ and then defining $K_1(-)$, and $K_2(-)$ for rings. The main sources for this talk are the following.

- nLab
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Chapters II and III
- Eric M. Friedlander's *An Introduction to K-theory*, Chapter 1
- http://people.math.harvard.edu/~lurie/281notes/Lecture3-Whitehead.pdf

1 K_0 of a Waldhausen category

Definition 1.1. Let $\mathscr{C}$ be a category equipped with a "subcategory" $\mathtt{co}\mathscr{C}$ of morphisms called *cofibrations*. The pair $(\mathscr{C}, \mathtt{co})$ is a *category with cofibrations* $\rightarrowtail$ if the following conditions hold.

W0. Every isomorphism in $\mathscr{C}$ is a cofibration.

W1. There is a zero object $*$ in $\mathscr{C}$ such that the unique morphism $* \rightarrowtail A$ is a cofibration for any $A \in \mathrm{ob}\,\mathscr{C}$.

W2. $\mathscr{C}$ has all pushouts of the form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, dotted] \\ C \arrow[r, tail, dotted] & B \cup_A C \end{tikzcd}.$$

Terminology. The map $B \to B \cup_A C$ is known as the *cobase change* of $A \to C$ along $A \rightarrowtail B$.

Note that the coproduct $B \coprod C$ always exists as the pushout $B \cup_* C$ and that the cokernel of any $i : A \rightarrowtail B$ exists as the pushout $B \cup_A *$ along the unique map $A \to *$. We call $A \rightarrowtail B \twoheadrightarrow B/A$ a *cofiber sequence*.

Definition 1.2. A *Waldhausen category* $\mathscr{C}$ is a category with cofibrations together with a subcategory $w(\mathscr{C})$ of morphisms called *weak equivalences* $\xrightarrow{\sim}$ such that every isomorphism in $\mathscr{C}$ is a weak equivalence and the following "gluing axiom" holds.

W3. For any commutative diagram of the form

$$\begin{tikzcd} C \arrow[d, "\sim"'] & A \arrow[l] \arrow[r, tail] \arrow[d, "\sim"'] & B \arrow[d, "\sim"'] \\ C' & A' \arrow[l] \arrow[r, tail] & B' \end{tikzcd},$$

the induced map $B \cup_A C \to B' \cup_{A'} C'$ is a weak equivalence.

Definition 1.3. A Waldhausen category $(\mathscr{C}, w)$ is *saturated* if whenever a composite fg is a weak equivalence, f is a weak equivalence iff g is.

Let $\mathscr{C}$ be a Waldhausen category. Define $K_0(\mathscr{C})$ as the abelian group generated by $[C]$ for each object C of $\mathscr{C}$ such that

1. $[C] = [C']$ if there some weak equivalence from C to C'
2. $[C] = [B] + \left[{}^{C}\!/_{B}\right]$ for every $B \rightarrowtail C \twoheadrightarrow {}^{C}\!/_{B}$
3. The weak equivalence classes of objects in $\mathscr{C}$ is a set.

Proposition 1.4.

1. $[0] = 0$.

2. $[B \coprod C] = [B] + [C]$.

3. $[B \cup_A C] = [B] + [C] - [A]$.

4. $[C] = 0$ *whenever* $0 \simeq C$.

Example 1.5. Let $\mathcal{R}_f(*)$ denote the category of finite CW complexes. Here, cofibrations and weak equivalences correspond to cellular inclusions and weak homotopy equivalences, respectively. It is known that $K_0(\mathcal{R}_f) \cong \mathbb{Z}$.

Definition 1.6. Suppose that $\mathscr{C}$ and $\mathscr{D}$ are Waldhausen categories. A functor $F : \mathscr{C} \to \mathscr{D}$ is *exact* if

(a) it preserves base points, cofibrations, and weak equivalences and

(b) for any $A \rightarrowtail B$, the map $FB \cup_{FA} FC \to F(B \cup_A C)$ is an isomorphism.

In this case, F induces a group map $K_0(F) : K_0(\mathscr{C}) \to K_0(\mathscr{D})$.

Theorem 1.7. *Let* $F : \mathscr{A} \to \mathscr{B}$ *be an exact functor. Assume the following.*

(1) A morphism f *is a weak equivalence iff* $F(f)$ *is a weak equivalence.*

(2) For any morphism $b : FA \to B$ *in* $\mathscr{B}$*, there is some cofibration* $a : A \rightarrowtail A'$ *in* $\mathscr{A}$ *along with a weak equivalence* $b' : FA' \xrightarrow{\sim} B$ *in* $\mathscr{B}$ *such that* $b = b' \circ F(a)$*. Moreover, we may choose* a *to be a weak equivalence whenever* b *is a weak equivalence.*

Then F *induces an isomorphism* $K_0(\mathscr{A}) \cong K_0(\mathscr{B})$.

Proof. Apply condition (2) to any map $* \rightarrowtail B$ to get $FA' \xrightarrow{\sim} B$. If this is a weak equivalence, then there is some $A \xrightarrow{\sim} A'$. Hence there is a bijection between the set W of weak-equivalence classes of objects of $\mathscr{A}$ and that in $\mathscr{B}$.

The group $K_0(\mathscr{B})$ is given by the free abelian group $\mathbb{Z}[W]$ modulo the relation

$$[C] = [B] + \left[{}^{C}\!/_{B}\right].$$

Let $FA \xrightarrow{\sim} B$. Then applying condition (2) yields the diagram

$$\begin{tikzcd} * \arrow[d, no head] & FA \arrow[l] \arrow[r, tail] \arrow[d, "\sim"'] & FA' \arrow[d, "\sim"'] \\ * & B \arrow[l] \arrow[r, tail] & C \end{tikzcd}.$$

Apply the glueing axiom to see that $F\left({A'}\diagup_{A}\right) \to {C}\diagup_{B}$ is a weak equivalence. Hence $[C] = [B] + \left[{C}\diagup_{B}\right]$ holds iff $[A'] = [A] + \left[{A'}\diagup_{A}\right]$ holds. $\square$

2 K_1 for rings

Let R be a unital ring. Recall that direct limits in $\mathbf{Mod}_R$ always exist. Let

$$K_1(R) = \mathrm{GL}(R)^{\mathrm{ab}}$$

where $\mathrm{GL}(R) \equiv \mathrm{colim}_{n\in\mathbb{N}}\, \mathrm{GL}(n, R)$.

Note 2.1 (Universal property of K). The universal property of ab : $\mathbf{Grp} \to \mathbf{Ab}$ induces the universal property of K_1 that any homomorphism $f : \mathrm{GL}(R) \to H$ with H abelian has $f = g \circ \pi$ for some unique $g : K_1(R) \to H$.

Proposition 2.2. *Any ring map $f : R \to S$ induces a natural map $\mathrm{GL}(R) \to \mathrm{GL}(S)$. Hence K_1 is a functor* $\mathbf{Rng} \to \mathbf{Ab}$.

Thanks to Whitehead, we know that the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$ is equal to $E(R) = \bigcup_n E_n(R)$, the group of elementary matrices $E_{i,j}(r)$ where $r \in R$ and $i \neq j$. Thus, $K_1(R)$ can be viewed as the "stabilized" group of automorphisms of the trivial projective module modulo trivial automorphisms.

Example 2.3. If F is a field, then $K_1(F) = F^\times$.

Proof. It is each to check that $E_n(F) \cong \mathrm{SL}_n(F)$ for any $n \in \mathbb{N}$. Therefore, $E(F) \cong \mathrm{SL}(R)$. $\square$

Proposition 2.4. *Suppose R is a commutative ring. The sequence*

$$R^\times \cong \mathrm{GL}(1, R) \to \mathrm{GL}(R) \to K_1(R)$$

induces a natural split exact sequence.

$$1 \longrightarrow SK_1(R) \hookrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1,$$

where $SK_1(R) \coloneqq \ker(\det)$.

This means that $K_1(R) \cong R^\times \times SK_1(R)$.

Example 2.5. Suppose R is a Euclidean domain. Then $SK_1(R) = 1$, so that $K_1(R) \cong R^\times$.

Lemma 2.6. *Let D be a division ring. Then $K_1(D) \cong {\mathrm{GL}_n(D)}\diagup_{E_n(D)}$ for any $n \geq 3$.*

Proof. Any invertible matrix over D is reducible (à la Gaussian elimination) to a diagonal matrix of the form $(r, 1, \ldots, 1)$. Moreover, $E_n(D) \trianglelefteq \mathrm{GL}_n(D)$ for each n. In particular, Dieudonné (1943) proved that ${\mathrm{GL}_n(D)}\diagup_{E_n(D)} \cong {D^\times}\diagup_{(D^\times)'}$ for any $n \neq 2$. $\square$

Now, suppose that R is Noetherian of dimension d, so that $E_n(R) \trianglelefteq \mathrm{GL}_n(R)$ for any $n \geq d+2$.

Proposition 2.7 (Vaserstein). *$K_1(R) \cong \mathrm{GL}_n(R)/E_n(R)$ for any $n \geq d+2$.*

Let D be a d-dimensional division algebra over the field $F \coloneqq Z(D)$. We know that $d = n^2$ for some integer n. By Zorn's lemma, there is some maximal subfield $E \subset D$ such that $[E : F] = n$. Then $D \otimes_F E \cong M_n(E)$, where M_n denotes the n-dimensional matrix ring over E. Any field with this property is called a *splitting field* for D.

Let E' be a splitting field for D. For any $r \in \mathbb{N}$, the inclusions $D \hookrightarrow M_n(E')$ and $M_r(D) \hookrightarrow M_{nr}(E')$ induce maps $D^\times \subset \mathrm{GL}_n(E') \xrightarrow{\det} (E')^\times$ and $\mathrm{GL}_r(D) \to \mathrm{GL}_{nr}(E') \xrightarrow{\det} (E')^\times$ whose images are contained in F^*. The induced maps are called the *reduced norms* N_{red} for D.

Example 2.8. If $D = \mathbb{H}$, then N_{red} is the square of the usual norm. It induces an isomorphism $K_1(\mathbb{H}) \cong \mathbb{R}_+^\times$.

Let R be a commutative Banach algebra over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ (i.e., a Banach space equipped with a commutative bilinear multiplication map $m : R \times R \to R$ such that $\|m(a,b)\| \leq \|a\| \cdot \|b\|$). Recall that both $\mathrm{GL}_n(R)$ and $\mathrm{SL}_n(R)$ are topological groups as subspaces of $\mathbb{R}^{n^2}$.

Proposition 2.9. *We have that $E_n(R)$ is the path component of the identity matrix I_n for any $n \geq 2$.*

Corollary 2.10. *We may identify $SK_1(R)$ with the set $\pi_0 \mathrm{SL}(R)$.*

Proof. Note that $E(R) \leq \mathrm{SL}(R)$. By the third isomorphism theorem, we get

$$\mathrm{GL}(R)/E(R) \big/ \mathrm{SL}(R)/E(R) \cong \mathrm{GL}(R)/\mathrm{SL}(R).$$

Thus, we get the short exact sequence

$$1 \longrightarrow \mathrm{SL}(R)/E(R) \longrightarrow \mathrm{GL}(R)/E(R) \cong K_1(R) \longrightarrow \mathrm{GL}(R)/\mathrm{SL}(R) \cong R^\times \longrightarrow 1$$

By Proposition 2.9, we know that $\mathrm{SL}(R)/E(R) \cong \pi_0 \mathrm{SL}(R)$, yielding a short exact sequence.

$$1 \longrightarrow \pi_0 \mathrm{SL}(R) \longrightarrow K_1(R) \xrightarrow{\det} R^\times \longrightarrow 1 .$$

□

Example 2.11. If X is compact, then

$$SK_1(\mathbb{R}^X) \leftrightarrow [X, \mathrm{SL}(\mathbb{R})] \cong [X, \mathrm{SO}]$$
$$SK_1(\mathbb{C}^X) \leftrightarrow [X, \mathrm{SL}(\mathbb{C})] \cong [X, \mathrm{SU}] .$$

In particular, $SK_1(\mathbb{R}^{S^1}) \leftrightarrow \pi_1 \mathrm{SO} \cong C_2$.

Let P be a finitely generated projective R-module. Any choice of isomorphism $P \oplus Q \cong R^n$ induces a group map

$$\mathrm{Aut}(P) \to \mathrm{Aut}(P) \oplus \mathrm{Aut}(Q) \cong \mathrm{Aut}(R^n) \cong \mathrm{GL}(n, R).$$

The group map $\mathrm{Aut}(P) \to \mathrm{GL}(R)$ is independent of our choice of isomorphism up to inner automorphism of $\mathrm{GL}(R)$. Therefore, there is a well-defined homomorphism $\Phi : \mathrm{Aut}(R) \to K_1(R)$.

Lemma 2.12. *Suppose that R is commutative and T is an R-algebra. Then $K_1(T)$ has a natural module structure over $K_0(R)$.*

Proof. For any $P \in \mathbf{P}(R)$ and $m \in \mathbb{N}$, consider the homomorphism $\Phi : \mathrm{Aut}(P \otimes T^m) \to K_1(R \otimes T)$. For any $\beta \in \mathrm{GL}_m(T)$, let

$$[P] \cdot \beta = \Phi(1_P \otimes \beta).$$

This action factors through $K_0(R)$ and $K_1(T)$, inducing an operation $K_0(R) \times K_1(T) \to K_1(R \otimes S)$. Now, since T is an R-algebra, there is a ring map $R \otimes T \to T$. The induced composite $K_0(R) \times K_1(T) \to K_1(R \otimes T) \to K_1(T)$ is the desired module structure. □

As it turns out, $K_1(R)$ is completely determined by the category $\mathbf{P}(R)$. This means that K_1 is invariant under Morita equivalence, just as K_0 is.

Theorem 2.13. *If R and S are Morita equivalent, then $K_1(R) \cong K_1(R)$.*

For an application of K_1 to manifold theory, let π be a finitely generated group. Define the *Whitehead group* $\mathrm{Wh}(\pi)$ of π as the cokernel of the map $\pi \times \{\pm 1\} \to K_1(\mathbb{Z}\pi)$ given by $(g, \pm 1) \mapsto \left[\pm g\right]$.

Definition 2.14. Suppose that W, M, and N are compact manifolds (possibly smooth or piecewise-linear). Suppose that M and N are without boundary. Let $\dim(M) = \dim(N) = n$ and $\dim(W) = n + 1$.

1. We say that W is a *cobordism of M and N* if $\partial W \cong M \coprod N$.
2. We say that W is an *h-cobordism of M and N* if it is a cobordism of M and N and the inclusion maps $i_M : M \hookrightarrow \partial W$ and $i_N : N \hookrightarrow \partial W$ are homotopy equivalences.

Let R be a ring. A *based chain complex over R* is a bounded chain complex

$$\cdots \longrightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \longrightarrow \cdots$$

of finitely generated free R-modules together with a choice B_n of basis (ordered in a predetermined way) for each F_n. The *Euler characteristic of* (F_*, d_n) is the finite sum

$$\chi(F_*) \equiv \sum_n (-1)^n |B_n| .$$

If F_* is acyclic, then it is contractible, so that there is some map $h : F_* \to F_{*+1}$ such that $dh + hd = \mathrm{Id}_{F_*}$. In this case, one can check that

$$d + h : \bigoplus_n F_{2n} \to \bigoplus_n F_{2n+1}.$$

is an isomorphism of free R-modules. If $\chi(F_*) = 0$, then this yields an element $\underbrace{\rho(F_*) \coloneqq [d + h]}_{\textit{Reidemeister torsion}}$ of $K_1(R)/\{\pm 1\}$, which is independent of our choice of null-homotopy h.

Suppose that $f : X_* \to Y_*$ is a quasi-isomorphism of based chain complexes over R. Then $\mathrm{cone}(f)$ is an acyclic based chain complex over R. Further, if $\chi(X_*) = \chi(Y_*)$, then $\chi(\mathrm{cone}(f)) = 0$, in which case we may define the *torsion of f* as the element $\rho(\mathrm{cone}(f))$ of $K_1(R)/\{\pm 1\}$.

Now, suppose that $f : X \to Y$ is a homotopy equivalence of finite connected CW complexes. Since these are locally contractible, they admit respective universal covering spaces $\widetilde{X}$ and $\widetilde{Y}$. If f is a cellular map, then it induces a map

$$\lambda_f : C_*(\widetilde{X}; \mathbb{Z}) \to C_*(\widetilde{Y}; \mathbb{Z})$$

of cellular chain complexes, which must be a quasi-isomorphism since f is assumed to be a homotopy equivalence. Note that $C_*(\widetilde{X}; \mathbb{Z})$ and $C_*(\widetilde{Y}; \mathbb{Z})$ may be viewed as based chain complexes over $\mathbb{Z}\pi_1(Y)$. In this case, the *Whitehead torsion* $\tau(f)$ *of* f is the image of the torsion of λ_f under the natural projection $K_1(\mathbb{Z}\pi_1(Y))/\{\pm 1\} \twoheadrightarrow \mathrm{Wh}(\mathbb{Z}\pi_1(Y))$.

Theorem 2.15 (s-cobordism). *Suppose that W, M, and N are compact manifolds and that W is an h-cobordism of M and N. If $\dim(M) \geq 5$, then $(W, M, N) \cong (M \times [0,1], M \times 0, M \times 1)$ iff $\tau(i_M)$ vanishes.*

Corollary 2.16 (Generalized Poincaré conjecture). *Let M be an n-manifold that is homotopy equivalent to S^n. If $n \geq 5$, then M is homeomorphic to S^n.*

Definition 2.17. Let I be an ideal in R. Define $\mathrm{GL}(I)$ as the kernel of the map $\mathrm{GL}(R) \to \mathrm{GL}\left(R/I\right)$. Moreover, define $E(R, I)$ as the smallest normal subgroup of $E(R)$ that contains $E_{i,j}(r)$ for any $r \in I$ and $i \neq j$.

Proposition 2.18. $[\mathrm{GL}(I), \mathrm{GL}(I)] \subset E(R, I) \trianglelefteq \mathrm{GL}(I)$

Definition 2.19. The *relative group* $K_1(R, I)$ is the the abelian group $\mathrm{GL}(I)/E(R, I)$.

Remark 2.20. Swan has shown that a ring homomorphism $f : R \to S$ mapping the ideal I isomorphically to the ideal J need *not* induce an isomorphism $K_1(R, I) \to K_1(S, J)$.

Proposition 2.21. *We have an exact sequence*

$$K_1(R, I) \longrightarrow K_1(R) \longrightarrow K_1\left(R/I\right) \longrightarrow K_0(I) \longrightarrow K_0(R) \longrightarrow K_0\left(R/I\right).$$ [1]

3 K_2 for rings

Definition 3.1. Let $n \geq 3$ and R be a ring. The *Steinberg group* $\mathrm{St}_n(R)$ is the group generated by the symbols $x_{ij}(r)$ with $1 \leq i \neq j \leq n$ and $r \in R$ that satisfy the following relations.

(i)

$$x_{ij}(r) x_{ij}(s) = x_{ij}(r + s)$$

(ii)

$$[x_{ij}(r), x_{kl}(s)] = \begin{cases} 1 & j \neq k,\ i \neq l \\ x_{il}(rs) & j = k,\ i \neq l \\ x_{kj}(-sr) & j \neq k,\ i = l \end{cases}.$$

[1] Section III.2.3 (Weibel).

We have a natural group surjection $\phi_n : \mathrm{St}_n(R) \to E_n(R)$ given by $x_{ij}(r) \mapsto E_{ij}(r)$. Moreover, there is a group map $\mathrm{St}_n(R) \hookrightarrow \mathrm{St}_{n+1}(R)$. Since $\mathrm{St}(R) \coloneqq \mathrm{colim}_n \mathrm{St}_n(R)$ exists, the ϕ_n form a group epimorphism $\phi : \mathrm{St}(R) \to E(R)$. Let

$$K_2(n, R) = \ker \phi_n$$
$$K_2(R) = \ker \phi.$$

Note that $K_2(-)$ is a functor $\mathbf{Rng} \to \mathbf{Ab}$. Furthermore, we have an exact sequence

$$1 \longrightarrow K_2(R) \longrightarrow \mathrm{St}(R) \xrightarrow{\phi} \mathrm{GL}(R) \longrightarrow K_1(R) \longrightarrow 1 .$$

Lemma 3.2. $K_2(R) \cong Z(\mathrm{St}(R))$.

Proof. The fact that $K_2(R) \supset Z(\mathrm{St}(R))$ follows from the fact that $Z(E(R))$ is trivial. The reverse containment is easy but more tedious to prove. See III.5.2.1 (Weibel). □

Example 3.3. A certain sort of Euclidean algorithm yields the following computations.

1. $K_2(\mathbb{Z}) \cong C_2$
2. $K_2(\mathbb{Z}[i]) = 1$
3. $K_2(F) \cong K_2(F[t])$ when F is a field

Theorem 3.4. *Suppose that R is Noetherian of dimension d. Then $K_2(n, R) \cong K_2(R)$ for any $n \geq d + 3$.*

Theorem 3.5. *If R and S are Morita equivalent, then $K_2(R) \cong K_2(R)$.*

Example 3.6. Let $n \in \mathbb{Z}_{\geq 1}$. Let R be any ring and let $S = M_n(R)$. These are Morita equivalent, so that

$$K_i(R) \cong K_i(M_n(R))$$

for each $i = 0, 1, 2$. Indeed, in one direction, define $F : M \mapsto M^n$. In the other direction, define $G : M \mapsto e_{11}M$ where e_11 denotes the matrix with 1 in position $(1, 1)$ and 0 elsewhere. Define the natural isomorphism $\mathrm{Id}_{\mathbf{Mod}_R} \Rightarrow G \circ F$ by the components $f_M : M \to \{(m, 0, \ldots, 0) : m \in M\}$. Further, define the natural isomorphism $\mathrm{Id}_{\mathbf{Mod}_S} \Rightarrow F \circ G$ by the components $g_M : M \to (e_{11}M)^n$ given by $m \mapsto (e_{11}m, \ldots, e_{1n}m)$. Hence $\mathbf{Mod}_R$ and $\mathbf{Mod}_S$ are equivalent, hence Morita equivalence as they are preadditive.

Lemma 3.7. *Let R be a commutative Banach algebra. Then there is a surjection from $K_2(R)$ onto $\pi_1 \mathrm{SL}(R)$.*[2]

Example 3.8. There is a surjection $K_2(\mathbb{R}) \to \pi_1 \mathrm{SL}_(\mathbb{R}) \cong \pi_1 \mathrm{SO} \cong C_2$. Hence $K_2(\mathbb{R})$ is nontrivial.

Theorem 3.9 (Matsumoto 1969). *Let F be a field. Then $K_2(F)$ is isomorphic to the free abelian group with system of generators $\{a, b\}$ satisfying the following relations.*

(i) $\{ac, b\} = \{a, b\} \{c, b\}$

(ii) $\{a, bd\} = \{a, b\} \{a, d\}$

[2] III.5.9 (Weibel).

(iii) $\{a, 1-a\} = 1$ *when* $a \neq 1 \neq 1-a$.

Terminology. The $\{a, b\}$ are called *Steinberg symbols.*

Suppose that $A, B \in E(F)$ commute. Write $\phi(a) = A$ and $\phi(b) = B$. Then define

$$A \bigstar B = [a, b] \in K_2(R).$$

If $a, b \in F$, then we can alternatively define the Steinberg symbol

$$\{a, b\} = \begin{bmatrix} r & & \\ & r^{-1} & \\ & & 1 \end{bmatrix} \bigstar \begin{bmatrix} s & & \\ & 1 & \\ & & s^{-1} \end{bmatrix}.$$

Corollary 3.10. $K_2(\mathbb{F}_p^n) = 1$ *for any prime* p *and any integer* $n \geq 1$.

Proof. The proof is entirely computational. See III.6.1.1 (Weibel). $\square$

Proposition 3.11. *If* $F \supset \mathbb{Q}(t)$*, then* $|K_2(F)| = |F|$.