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K-theory seminar
Talk #15
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Abstract

We begin higher Waldhausen K-theory. The main sources for this talk are the following.

- nLab.
- Charles Weibel's *The K-book: an introduction to algebraic K-theory*, Ch. IV.8.
- John Rognes's *Lecture Notes on Algebraic K-Theory*, Ch. 8.

For the original development, see Friedhelm Waldhausen's *Algebraic K-theory of spaces* (1985).

Let $\mathscr{C}$ be a Waldhausen category. Our goal is to construct the K-theory $K(\mathscr{C})$ of $\mathscr{C}$ as a based loop space ΩY endowed with a loop completion map $\iota : |w\mathscr{C}| \to K(\mathscr{C})$ where $w\mathscr{C}$ denotes the subcategory of weak equivalences. This will produce a function $\mathrm{ob}\,\mathscr{C} \to |w\mathscr{C}| \to \Omega Y$. Further, we'll require of $K(\mathscr{C})$ certain limit and coherence properties, thereby making $K(\mathscr{C})$ the underlying infinite loop space of a spectrum $\mathbf{K}(\mathscr{C})$, called the *algebraic K-theory spectrum* of $\mathscr{C}$.

Definition 1. A Waldhausen category $(\mathscr{C}, w)$ is *saturated* if whenever a composite fg is a weak equivalence, f is a weak equivalence iff g is.

Now, let us introduce the main concept to be generalized.

Definition 2. Let $\mathscr{C}$ be a category with cofibrations. Let the *extension category* $S_2\mathscr{C}$ have as objects the cofiber sequences in $(\mathscr{C}, \mathtt{co}\,\mathscr{C})$ and as morphisms the triples (f', f, f'') such that

$$\begin{tikzcd} X' \arrow[r, tail] \arrow[d, "f'"] & X \arrow[r, two heads] \arrow[d, "f"] & X'' \arrow[d, "f''"] \\ Y' \arrow[r, tail] & Y \arrow[r, two heads] & Y'' \end{tikzcd}$$

commutes. This is pointed at $* \rightarrowtail * \twoheadrightarrow *$.

Definition 3. Suppose an arbitrary triple (f', f, f'') as above has the property that whenever f' and f'' are weak equivalences, then so is f. In this case, we say $\mathscr{C}$ is *extensional* or *closed under extensions.*

Say that the morphism (f', f, f'') is a cofibration if f', f'', and $Y' \cup_{X'} X \to Y$ are cofibrations in $\mathscr{C}$. Say that the same triple is a weak equivalence if f', f, and f'' are weak equivalences in $\mathscr{C}$. This makes $S_2\mathscr{C}$ into a Waldhausen category.

Definition 4. Let $q \geq 0$. Let the *arrow category* $\mathrm{Ar}[q]$ *on* $[q]$ have as objects ordered pairs (i, j) with $i \leq j \leq q$ and as morphisms commutative diagrams of the form

$$\begin{tikzcd} i \arrow[r, "\leq"] \arrow[d, "\leq"'] & j \arrow[d, "\leq"] \\ i' \arrow[r, "\leq"'] & j' \end{tikzcd}.$$

We view $[q]$ as a full subcategory of $\mathrm{Ar}[q]$ via the embedding $[q] \xrightarrow{k\mapsto(0,k)} \mathrm{Ar}[q]$.

Note 5.

1. Any triple $i \leq j \leq k$ determines the morphisms $(i,j) \to (i,k)$ and $(i,k) \to (j,k)$. Conversely, any morphism in the arrow category is a composite of such triples.

2. $\mathrm{Ar}[q] \cong \mathbf{Fun}([1],[q])$ with each pair (i,j) identified with the functor satisfying $0 \mapsto i$ and $1 \mapsto j$.

Example 6. The category $\mathrm{Ar}[2]$ is generated by the commutative diagram

$$\begin{tikzcd} (0,0) \arrow[r] & (0,1) \arrow[r] \arrow[d] & (0,2) \arrow[d] \\ & (1,1) \arrow[r] & (1,2) \arrow[d] \\ & & (2,2) \end{tikzcd} .$$

Definition 7. Let $\mathscr{C}$ be a category with cofibrations and $q \geq 0$. Define $S_q\mathscr{C}$ as the full subcategory of $\mathbf{Fun}(\mathrm{Ar}[q],\mathscr{C})$ generated by $X : \mathrm{Ar}[q] \to \mathscr{C}$ such that

1. $X_{j,j} = *$ for each $j \in [q]$.

2. $X_{i,j} \rightarrowtail X_{i,k} \twoheadrightarrow X_{j,k}$ is a cofiber sequence for any $i < j < k$ in $[q]$. Equivalently, if $i \leq j \leq k$ in $[q]$, then the square

$$\begin{tikzcd} X_{i,j} \arrow[r, tail] \arrow[d, two heads] & X_{i,k} \arrow[d, two heads] \\ X_{j,j} = * \arrow[r, tail] & X_{j,k} \end{tikzcd}$$

is a pushout.

This is pointed at the constant diagram at $*$.

Note 8. A generic object in $S_q\mathscr{C}$ looks like

$$\begin{tikzcd} * \arrow[r, tail] & X_1 \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & X_{q-1} \arrow[r, tail] \arrow[d, two heads] & X_q \arrow[d, two heads] \\ & * \arrow[r, tail] & \cdots \arrow[r, tail] & X_{q-1}/X_1 \arrow[r, tail] \arrow[d, two heads] & X_q/X_1 \arrow[d, two heads] \\ & & \ddots & \vdots \arrow[d, two heads] & \vdots \arrow[d, two heads] \\ & & & * \arrow[r, tail] & X_q/X_{q-1} \arrow[d, two heads] \\ & & & & * \end{tikzcd} . \tag{†}$$

where X_q corresponds to $X_{0,q}$ and X_j/X_i to $X_{i,j}$ for any $1 \leq i \leq j \leq q$.

Definition 9. Let $(\mathscr{C}, \mathsf{co}\,\mathscr{C})$ be a category with cofibrations. Let $\mathsf{co}\, S_q\mathscr{C} \subset S_q\mathscr{C}$ consist of the morphisms $f : X \rightarrowtail Y$ of $\mathrm{Ar}[q]$-shaped diagrams such that for each $1 \leq j \leq q$ we have

$$
\begin{tikzcd}
X_{0,j-1} \arrow[r, tail] \arrow[d, "f_{0,j-1}"'] & X_{0,j} \arrow[d] \arrow[ddr, bend left, "f_{0,j}"] & \\
Y_{0,j-1} \arrow[r, tail] \arrow[drr, tail, bend right] & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} \arrow[dr, tail, dashed] & \\
 & & Y_{0,j}
\end{tikzcd}
.
$$

Proposition 10. *If $f : X \to Y$ is a cofibration of $S_q\mathscr{C}$, then*

$$
\begin{tikzcd}
X_{i,j} \arrow[r, tail] \arrow[d, tail, "f_{i,j}"'] & X_{i,k} \arrow[d, tail, "f_{i,k}"] \\
Y_{i,j} \arrow[r, tail] & Y_{i,k}
\end{tikzcd}
$$

for any $i \leq j \leq k$ in $[q]$.[1]

Lemma 11. *$(S_q\mathscr{C}, \mathsf{co}\, S_1\mathscr{C})$ is a category with cofibrations.*

Proof. First notice that the composite of two cofibrations $g \circ f : X \to Y \to Z$ is a cofibration because we have

$$
\begin{tikzcd}
X_{0,j-1} \arrow[r, tail] \arrow[d, "f_{0,j-1}"'] & X_{0,j} \arrow[d] \arrow[dr, bend left, "f_{0,j}"] & & \\
Y_{0,j-1} \arrow[r, tail] \arrow[d, "g_{0,j-1}"'] & X_{0,j} \cup_{X_{0,j-1}} Y_{0,j-1} \arrow[r, tail] \arrow[d] & Y_{0,j} \arrow[d] \arrow[dr, bend left, "f_{0,j}"] & \\
Z_{0,j-1} \arrow[r, tail] & X_{0,j} \cup_{X_{0,j-1}} Z_{0,j-1} \arrow[r, tail] & Y_{0,j} \cup_{Y_{0,j-1}} Z_{0,j-1} \arrow[r, tail] & Z_{0,j}
\end{tikzcd}
.
$$

It's clear that any isomorphism or initial morphism in $S_q\mathscr{C}$ is a cofibration.

To see that (W2) is satisfied, let $f : X \to Y$ and $g : X \to Z$ be morphisms in $S_q\mathscr{C}$. It's easy to verify that each component $f_{i,j} : X_{i,j} \to Y_{i,j}$ is a cofibration. Thus, each pushout $W_{i,j} \coloneqq Y_{i,j} \cup_{X_{i,j}} Z_{i,j}$ exists. These form a functor $W : \mathrm{Ar}[q] \to \mathscr{C}$. If $i < j < k$, then we have $W_{i,j} \rightarrowtail W_{i,k} \twoheadrightarrow W_{j,k}$ because the left morphism factors as the composite of two cofibrations

$$
\begin{tikzcd}
Z_{i,j} \arrow[r, tail] \arrow[d, "f_{i,j} \cup \mathrm{Id}"'] & Z_{i,k} \arrow[d, "f_{i,j} \cup \mathrm{Id}"] & \\
Y_{i,j} \cup_{X_{i,j}} Z_{i,j} \arrow[r, tail] & Y_{i,j} \cup_{X_{i,j}} Z_{i,k} \arrow[r, tail] & Y_{i,k} \cup_{X_{i,k}} Z_{i,k} \\
 & Y_{i,j} \cup_{X_{i,j}} X_{i,k} \arrow[r, tail] \arrow[u, "\mathrm{Id} \cup g_{i,k}"] & Y_{i,k} \arrow[u, "\mathrm{Id} \cup g_{i,k}"']
\end{tikzcd}
.
$$

[1] Lemma 8.3.12 (Rognes).

The fact that colimits commute with each other ensures that $W_{j,k} \cong {}^{W_{i,k}}\!/_{W_{i,j}}$ Hence W is the pushout of f and g. To verify that this is a cofibration, we must check that the pushout map $W_{0,j-1} \cup_{Z_{0,j-1}} Z_{0,j} \to W_{0,j}$ is a cofibration. But this follows from the pushout square

$$\begin{tikzcd} Y_{0,j-1} \cup_{X_{0,j-1}} X_{0,j} \arrow[r, tail] \arrow[d] & Y_{0,j} \arrow[d] \\ Y_{0,j-1} \cup_{X_{0,j-1}} Z_{0,j} \arrow[r, tail] & Y_{0,j} \cup_{X_{0,j}} Z_{0,j} \end{tikzcd}.$$

$\square$

Definition 12. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Let $wS_q\mathscr{C} \subset S_q\mathscr{C}$ consist of the morphisms $f : X \xrightarrow{\sim} Y$ of $\mathrm{Ar}[q]$-shaped diagrams such that the component $f_{0,j} : X_{0,j} \to Y_{0,j}$ is a weak equivalence in $\mathscr{C}$ for each $1 \leq j \leq q$.

Proposition 13. *Let f be a weak equivalence in $S_q\mathscr{C}$. Each component $f_{i,j} : X_{i,j} \to Y_{i,j}$ is a weak equivalence in $\mathscr{C}$.*

Proof. Apply the Gluing axiom to the diagram

$$\begin{tikzcd} X_{0,j} \arrow[d, "\cong"'] & X_{0,i} \arrow[l, tail] \arrow[r] \arrow[d, "\cong"'] & * \arrow[d, "="'] \\ Y_{0,j} & Y_{0,i} \arrow[l, tail] \arrow[r] & * \end{tikzcd}.$$

Then $X_{i,j} \cong X_{0,j} \cup_{X_{0,i}} * \xrightarrow{\sim} Y_{0,j} \cup_{Y_{0,i}} * \cong Y_{i,j}$, as desired. $\square$

Lemma 14. *$(S_q\mathscr{C}, wS_q\mathscr{C})$ is a Waldhausen category.*

Definition 15. Let $\mathscr{C}$ be a category with cofibrations. If $\alpha : [p] \to [q]$, then define $\alpha^* : S_q\mathscr{C} \to S_p\mathscr{C}$ by

$$\alpha^*(X : \mathrm{Ar}[q] \to \mathscr{C}) = X \circ \mathrm{Ar}(\alpha) : \mathrm{Ar}[p] \to \mathrm{Ar}[q] \to \mathscr{C}.$$

It's easy to check that this satisfies the two conditions of a diagram in $S_p\mathscr{C}$. Moreover, the face maps d_i are obtained by deleting the row $X_{i,-}$ and the column containing X_i in (†) and then reindexing as necessary. The degeneracy maps s_i are given by duplicating X_i and then reindexing such that $X_{i+1,i} = 0$.

Not sure that the s_i work.

Proposition 16. *Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Each functor $\alpha^* : S_q\mathscr{C} \to S_p\mathscr{C}$ is exact, so that $(S_\bullet\mathscr{C}, wS_\bullet\mathscr{C})$ is a simplicial Waldhausen category.*

The nerve $N_\bullet wS_\bullet\mathscr{C}$ is a bisimplicial set with (p,q)-bisimplices the diagrams of the form

$$\begin{tikzcd}
* \arrow[r, tail] & X_1^0 \arrow[r, tail] \arrow[d, "\sim"'] & X_2^0 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X_q^0 \arrow[d, "\sim"'] \\
* \arrow[r, tail] & X_1^1 \arrow[r, tail] \arrow[d, "\sim"'] & X_2^1 \arrow[r, tail] \arrow[d, "\sim"'] & \cdots \arrow[r, tail] & X_q^1 \arrow[d, "\sim"'] \\
& \vdots \arrow[d, "\sim"'] & \vdots \arrow[d, "\sim"'] & & \vdots \arrow[d, "\sim"'] \\
* \arrow[r, tail] & X_1^p \arrow[r, tail] & X_2^p \arrow[r, tail] & \cdots \arrow[r, tail] & X_q^p
\end{tikzcd}$$

such that $X_{i,j}^k \cong X_j^k / X_i^k$ for every $i \leq j \leq q$ and $k \in [p]$.

Lemma 17. *There is a natural map $N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 \to N_\bullet wS_\bullet\mathscr{C}$, which automatically induces a based map $\sigma : \Sigma\,|w\mathscr{C}| \to |wS_\bullet\mathscr{C}|$ of classifying spaces.*

Proof. We can treat $N_\bullet wS_\bullet\mathscr{C}$ as the simplicial set $[q] \mapsto N_\bullet wS_q\mathscr{C}$. This defines a right skeletal structure on $N_\bullet wS_\bullet\mathscr{C}$.

If $q = 0$, then $wS_0\mathscr{C} = S_0\mathscr{C} = *$, so that $N_\bullet wS_0\mathscr{C} = *$ as well. If $q = 1$, then $wS_1\mathscr{C} \cong w\mathscr{C}$. Thus, the right 1-skeleton is equal to $N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1$, which in turn must be equal to the image I of the canonical map

$$\coprod_{q\leq 1} N_\bullet wS_q\mathscr{C} \times \Delta_\bullet^q \to N_\bullet wS_\bullet\mathscr{C}.$$

Now, the degeneracy map s_0 collapses $\{*\} \times \Delta_\bullet^1$, and the face maps d_0 and d_1 collapse $N_\bullet w\mathscr{C} \times \partial\Delta_\bullet^1$. Therefore, I must equal

$$N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 = \frac{N_\bullet w\mathscr{C} \times \Delta_\bullet^1}{\{*\} \times \Delta_\bullet^1 \cup N_\bullet w\mathscr{C} \times \partial\Delta_\bullet^1}.$$

We have defined a natural inclusion map $\lambda : N_\bullet w\mathscr{C} \wedge \Delta_\bullet^1 \to N_\bullet wS_\bullet\mathscr{C}$.

Since $\Delta_\bullet^1$ is isomorphic to the unit interval and the map λ agrees on the endpoints, we can pass to S^1 during the suspension. Hence λ induces the desired map σ.[2] $\square$

Note 18. The axiom (W3) implies that $w\mathscr{C}$ is closed under coproducts, making $|wS_\bullet\mathscr{C}|$ into an H-space via the map

$$\coprod : |wS_\bullet\mathscr{C}| \times |wS_\bullet\mathscr{C}| \cong |wS_\bullet\mathscr{C} \times wS_\bullet\mathscr{C}| \to |wS_\bullet\mathscr{C}|\,.$$

Definition 19. Let $(\mathscr{C}, w\mathscr{C})$ be a Waldhausen category. Define the *algebraic K-theory space*

$$K(\mathscr{C}, w) = \Omega\,|N_\bullet wS_\bullet\mathscr{C}|\,.$$

[2]This is a tentative explanation due to Thomas Brazelton.

Then we have a right adjoint $\iota : |w\mathscr{C}| \to K(\mathscr{C}, w)$ to the based map σ.

Moreover, let $F : (\mathscr{C}, w\mathscr{C}) \to (\mathscr{D}, w\mathscr{D})$ be an exact functor. Let

$$K(F) = \Omega\, |wS_\bullet F| : K(\mathscr{C}, w) \to K(\mathscr{D}, w).$$

This is the *algebraic K-theory functor* $K : \mathbf{Wald} \to \mathbf{Top}_*$.

Note that any exact category $\mathscr{A}$ is a Waldhausen category with cofibrations the admissible exact sequences and weak equivalences the isomorphisms. Waldhausen showed that $|iS_\bullet\mathscr{A}|$ (where $i(-)$ denotes the isomorphism category) and $BQ\mathscr{A}$ are homotopy equivalent. Therefore, our current definition of higher algebraic K-theory agrees with Quillen's.

Example 20. Let R be a ring. Define the *algebraic K-theory space of R* as

$$K(R) = K(\mathbf{P}(R), i)$$

where the weak equivalences are precisely the injective R-linear maps with projective cokernel and the cofibrations are precisely the R-linear maps.

Example 21. Assume that $\mathscr{C}$is a small Waldhausen category where $w\mathscr{C}$ consists of the isomorphisms in $\mathscr{C}$. If $s_n\mathscr{C}$ denotes the set of objects of $S_n\mathscr{C}$, then we get a simplicial set $s_\bullet\mathscr{C}$. Waldhausen showed that the inclusion map $|s_\bullet\mathscr{C}| \hookrightarrow |iS_\bullet\mathscr{C}|$ is a homotopy equivalence. This makes $\Omega|s_\bullet\mathscr{C}|$ into a so-called simplicial model for $K(\mathscr{C}, w)$.

Remark 22. Since $wS_0\mathscr{C} = *$ and every simplex of degree $n > 0$ is attached to $*$, it follows that the classifying space $|wS_\bullet\mathscr{C}|$ is connected. Therefore, we preserve any homotopical information when passing to the loop space.

Definition 23. The *i-th algebraic K-group* is $K_i(\mathscr{C}, w) \equiv \pi_i K(\mathscr{C}, w)$ for each $i \geq 0$.

Proposition 24. *$\pi_1\, |wS_\bullet\mathscr{C}| \cong K_0(\mathscr{C}, w)$.*

Lemma 25. *The group $K_0(\mathscr{C}, w)$ is generated by all elements $[X]$ such that*

- *$[X'] + [X''] = [X]$ for every cofiber sequence $X' \rightarrowtail X \twoheadrightarrow X''$ and*
- *$[X] = [Y]$ for every weak equivalence $X \xrightarrow{\sim} Y$.*

Proof. In light of Proposition 24, it suffices to compute $\pi_1\, |N_\bullet wS_\bullet\mathscr{C}|$ based at the $(0, 0)$-bisimplex $*$. For this, just notice the CW structure of $|N_\bullet wS_\bullet\mathscr{C}|$, with 1-cells the $(0, 1)$-bisimplices and 2-cells the $(0, 2)$-bisimplices $X' \rightarrowtail X \twoheadrightarrow X''$ and the $(1, 1)$-bisimplices $X \xrightarrow{\sim} Y$, which are attached to the 1-cells X and Y. Any cell of dimension $n > 2$ is irrelevant to computing π_1. □

As a result, we obtain functors

$$K_i : \mathbf{Wald} \to \mathbf{Top}_* \to \mathbf{Ab}$$

known as the *algebraic K-group functors*. Indeed, thanks to Proposition 24, we know that

$$K_i(\mathscr{C}, w) = \pi_{i+1} |wS_\bullet\mathscr{C}| ,$$

which is abelian for $i \geq 1$. Moreover, note that if $X' \rightarrowtail X' \vee X'' \twoheadrightarrow X''$ and $X'' \rightarrowtail X' \vee X'' \twoheadrightarrow X'$ are cofiber sequences, then Lemma 25 implies that

$$[X'] + [X''] = [X' \vee X''] = [X'' + X'] .$$

Hence $K_0(\mathscr{C}, w)$ is also abelian.

Example 26. Let X be a CW complex and $\mathcal{R}(X)$ denote the category of CW complexes Y obtained by attaching at least one cell to X so that X is a retract of Y. Equip this with cofibrations in the form of cellular inclusions fixing X and weak equivalence in the form of homotopy equivalences. This makes $\mathcal{R}(X)$ into a Waldhausen category.

If $\mathcal{R}_f(X)$ denotes the subcategory of those Y obtained by attaching finitely many cells, then we denote $K(\mathcal{R}_f(X))$ by $A(X)$.

Proposition 27. *$A_0(X) \cong \mathbb{Z}$.*

Definition 28. If $\mathscr{B}$ is a Waldhausen subcategory of $\mathscr{C}$, then it is *cofinal in $\mathscr{C}$* if for any $X \in \operatorname{ob}\mathscr{C}$, there is some $X' \in \operatorname{ob}\mathscr{C}$ such that $X \coprod X' \in \operatorname{ob}\mathscr{B}$.

Theorem 29. *Let $(\mathscr{B}, w)$ be cofinal in $(\mathscr{C}, w)$ and closed under extensions. Assume that $K_0(\mathscr{B}) = K_0(\mathscr{C})$. Then $wS_\bullet\mathscr{B} \to wS_\bullet\mathscr{C}$ is a homotopy equivalence.*

It follows that $K_i(\mathscr{B}) \cong K_i(\mathscr{C})$ for every $i \geq 0$.