This report contains our progress studying the AR(2) model

$$x_t = \beta_2 x_{t-1} + \beta_1 x_{t-2} + \epsilon_t \tag{1}$$

reproducing data from V1.

This model can be rewritten as

$$\begin{aligned} \dot{I} &= w_{ie} E, \\ \dot{E} &= w_{ee} E + w_{ei} I + \epsilon_t \end{aligned} \tag{2}$$

where $w_{ee} = -(1+\beta_1)$, $w_{ei} = \beta_2 + \beta_1 - 1$ and $w_{ie} = 1$. For this study we will use the set of parameters $\beta_1, \beta_2 = \{-0.9606, 1.8188\}$, for which $w_{ee}, w_{ei}, w_{ie} = \{-0.0394, 0.1418, 1\}$.

In absence of noise, the origin is a fixed point. For our choice of parameters , its eigenvalues are given by

$$\mu_{\pm} = \lambda \pm i\omega = -0.0197 \pm 0.376i \tag{3}$$

so it is a stable focus very near a Hopf bifurcation.

Following unpublished work, we can compute a phase function, denoted as $\Theta(x)$, for system (2). For a linear focus, the expected value of $\Theta(x)$ evolves as

$$\frac{d}{dt}\mathbb{E}[\Theta(X(t))] = \omega, \qquad \text{for } \omega \text{ in } (3) \tag{4}$$

which is formally analogous to the deterministic phase dynamics.

Therefore, for an arbitrary perturbation of amplitude A, we propose

$$PRC(\theta, A) = \mathbb{E}[\Theta(X_{\text{pert}}(t))] - \mathbb{E}[\Theta(X_0(t))] \tag{5}$$

as a way of computing PRCs for stochastic linear focus.

Next we show preliminary results applying a pulse of amplitude A in the E direction. The numerical simulation used the following protocol:

- First, we interpolate the phase values produced by $\Theta(x)$ to get a function $f(E, I)$ that associates a phase θ to any point of the phase space (cf Fig. 1).
- Next, we took a hundred points on a circle of radius 0.3, which we denote as Γ. We assume we can parameterize Γ by means of θ, that is, $x = \gamma(\theta)$
- Then, for each $x_{ini} = \gamma(\theta) \in \Gamma$:
  - We compute $x_{pert} = x_{ini} + A$, where $A = 0.1$ is a perturbation in the E direction (see Fig. 1).
  - Next, using a Heun integration algorithm, we compute $N = 10000$ realisations of the T-time integration of both, x_{pert} and x_{ini}. Then, for each realisation, we compute the phase of their final points using the function $f(E, I)$.

– Finally, we computed the mean perturbed (unperturbed) phase θ_{pert} (θ_{ini}) averaging the individual perturbed (unperturbed) phases. We then compute the mean phase shift as $\Delta\theta = \theta_{pert} - \theta_{ini}$

- Repeating this process for all the points in Γ we compute the PRCs in Fig. 2.

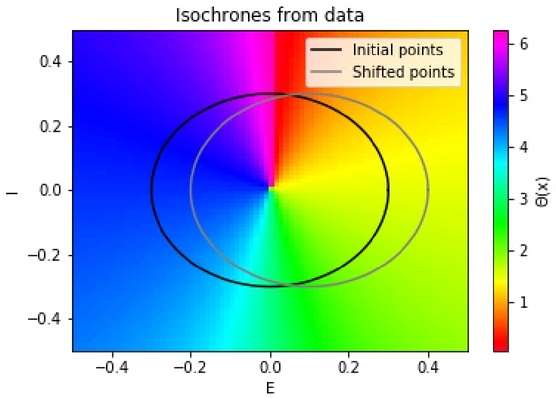

Figure 1: Phase function $\Theta(x)$ and the curves Γ and $\Gamma + A$ used to compute the PRCs

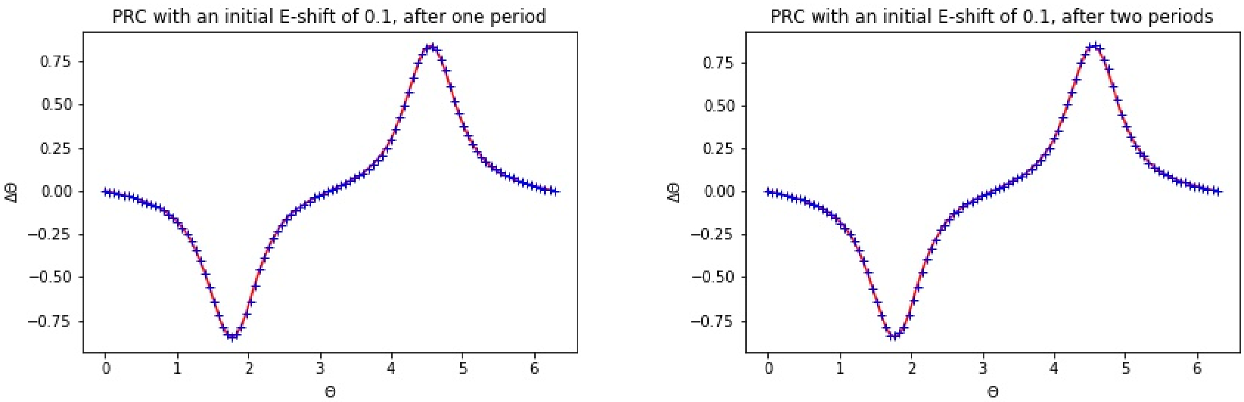

Figure 2: Phase response curves: in red, results at time 0; in blue, results after one and two periods respectively

We obtain PRCs with bimodal shape, probably a consequence of system (2) being near a Hopf. The result remains numerically stable through time.