Short Introduction to Dynamical Systems

March 31, 2021

Introduction to Dynamical Systems

Why Dynamical Systems?

- Ordinary differential equations (ODEs) are ubiquitous in a wide variety of fields
$$\dot{x} = F(x)$$

Why Dynamical Systems?

- Ordinary differential equations (ODEs) are ubiquitous in a wide variety of fields

$$\dot{x} = F(x)$$

- Sometimes we can find analytical solutions for our system of interest

$$\dot{x} = cx \qquad x(t) = Ce^{ct}$$

Why Dynamical Systems?

- Ordinary differential equations (ODEs) are ubiquitous in a wide variety of fields

$$\dot{x} = F(x)$$

- Sometimes we can find analytical solutions for our system of interest

$$\dot{x} = cx \qquad x(t) = Ce^{ct}$$

- But in general, analytical solutions for ODE systems, can not be found

$$C_m \dot{V} = -I_L(V) - I_{Na}(V) - I_K(V, n) + I_{app},$$
$$\dot{n} = \frac{n_\infty(V) - n}{\tau_n(V)},$$

Why Dynamical Systems?

- Ordinary differential equations (ODEs) are ubiquitous in a wide variety of fields
$$\dot{x} = F(x)$$
- Sometimes we can find analytical solutions for our system of interest
$$\dot{x} = cx \qquad x(t) = Ce^{ct}$$
- But in general, analytical solutions for ODE systems, can not be found
$$C_m \dot{V} = -I_L(V) - I_{Na}(V) - I_K(V, n) + I_{app},$$
$$\dot{n} = \frac{n_\infty(V) - n}{\tau_n(V)},$$
- Nevertheless, Dynamical Systems show which is the asymptotic ($t \to \infty$) behavior of systems in the form $\dot{x} = F(x)$

Example

- Consider the system $\dot{x} = F(x)$

$$\dot{x} = F(x) \qquad x \in \mathbb{R}^n$$

Example

- Consider the system $\dot{x} = F(x)$

$$\dot{x} = F(x) \qquad\qquad x \in \mathbb{R}^n$$

- Example in $\mathbb{R}^2$, $\quad x = (x_1, x_2) \in \mathbb{R}^2$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

Example

- Consider the system $\dot{x} = F(x)$

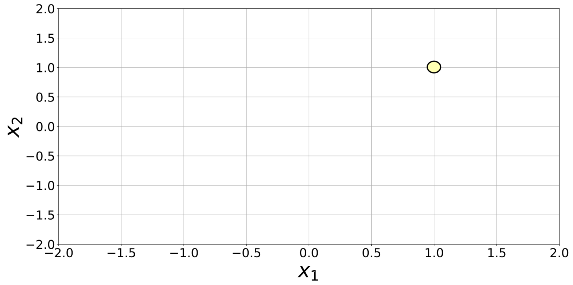
$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

Example

- Consider the system $\dot{x} = F(x)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

- Consider $\beta = -1$, take $x = (1, 1)$

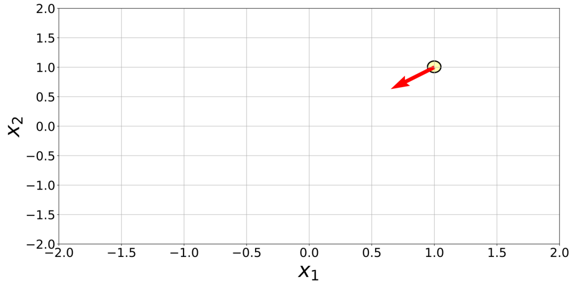


Example

- Consider the system $\dot{x} = F(x)$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

- Consider $\beta = -1$, take $x = (1, 1) \rightarrow F(1, 1) = (-4, -2)$

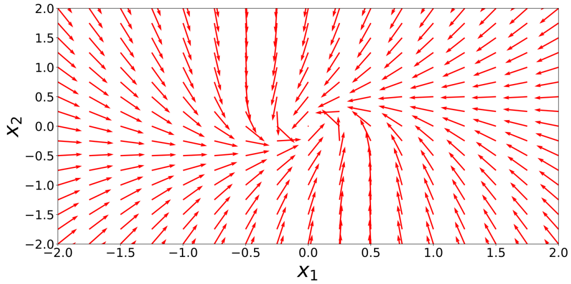


Example

- Consider the system $\dot{x} = F(x)$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

- Consider $\beta = -1$

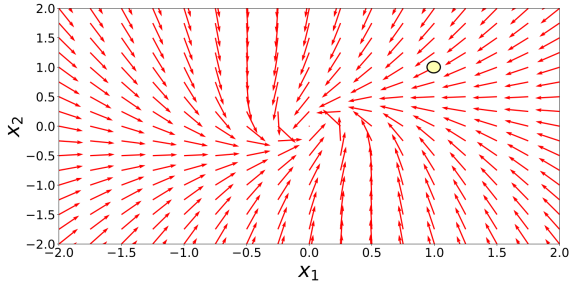


Example

- Consider the system $\dot{x} = F(x)$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

- Consider $\beta = -1$

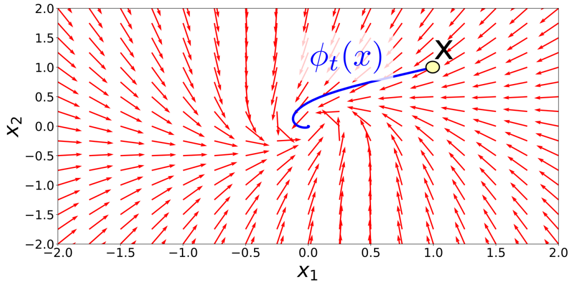


Example

- Consider the system $\dot{x} = F(x)$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

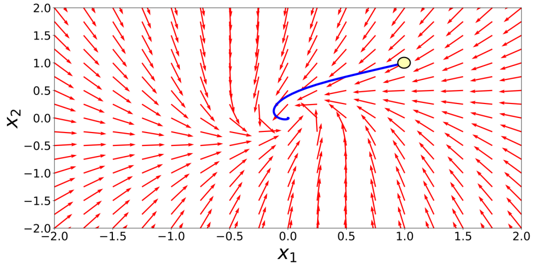
- Consider $\beta = -1$



the trajectory of a given point x corresponds to the flow $\phi_t(x)$ generated by the vector field $F(x)$

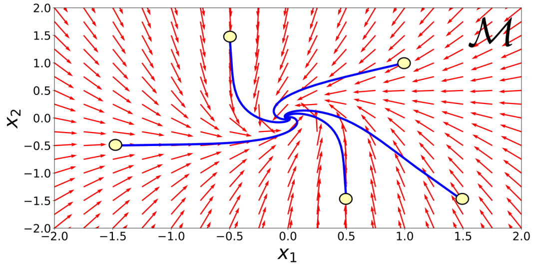
Example

- Although we not solve $\dot{x} = F(x)$ we obtain the asymptotic behavior



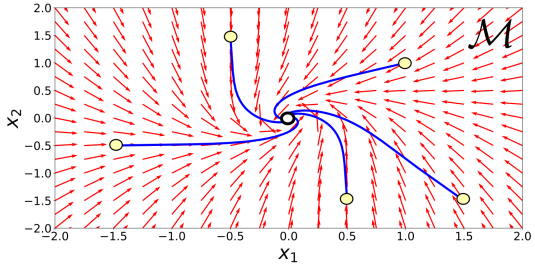
Example

- Although we not solve $\dot{x} = F(x)$ we obtain the asymptotic behavior



Example

- Although we not solve $\dot{x} = F(x)$ we obtain the asymptotic behavior

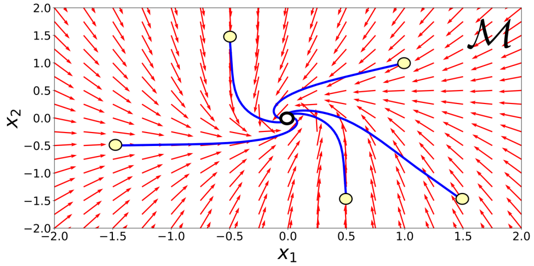


- Why points in $\mathcal{M}$ approach $x^* = (0,0)$?

$$\dot{x} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix}\Bigg|_{(x_1,x_2)=0} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}\Bigg|_{(x_1,x_2)=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Example

- Although we not solve $\dot{x} = F(x)$ we obtain the asymptotic behavior



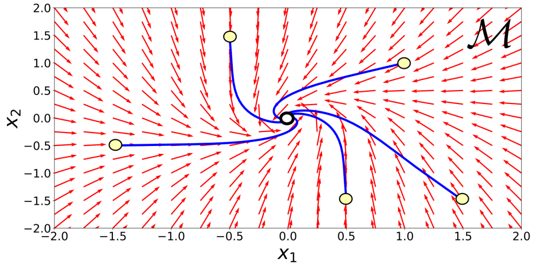
- Why points in $\mathcal{M}$ approach $x^* = (0, 0)$?

$$\dot{x} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix}\Bigg|_{(x_1,x_2)=0} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}\Bigg|_{(x_1,x_2)=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- $x^* = (0, 0)$ it is an equilibrium

Example

- Although we not solve $\dot{x} = F(x)$ we obtain the asymptotic behavior



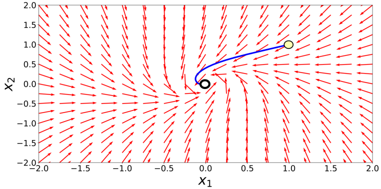
- Why points in $\mathcal{M}$ approach $x^* = (0,0)$?

$$\dot{x} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix}\Bigg|_{(x_1,x_2)=0} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}\Bigg|_{(x_1,x_2)=0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- $x^* = (0,0)$ it is an equilibrium
- It is a stable equilibrium

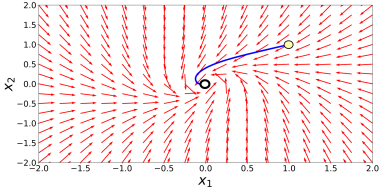
Example

- What does it mean that the attractor of our system is a point?

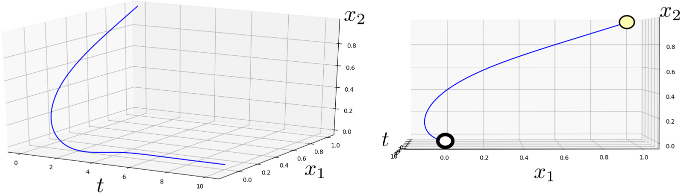


Example

- What does it mean that the attractor of our system is a point?

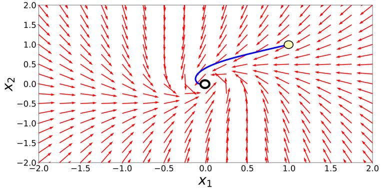


- Lets take $x = (1, 1)$ and integrate it using an ODE solver

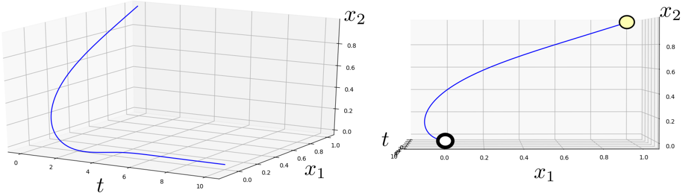


Example

- What does it mean that the attractor of our system is a point?



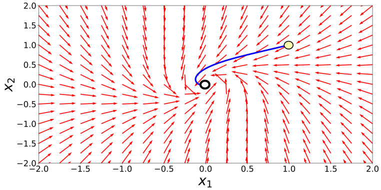
- Lets take $x = (1, 1)$ and integrate it using an ODE solver



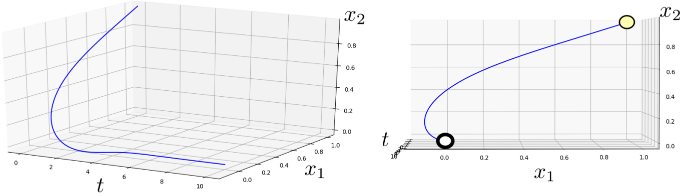
The phase space is precisely the time projection of the trajectories

Example

- What does it mean that the attractor of our system is a point?



- Lets take $x = (1, 1)$ and integrate it using an ODE solver



Fixed points of ODEs correspond to trajectories approaching a constant value

Example

- Lets come back to the system $\dot{x} = F(x)$

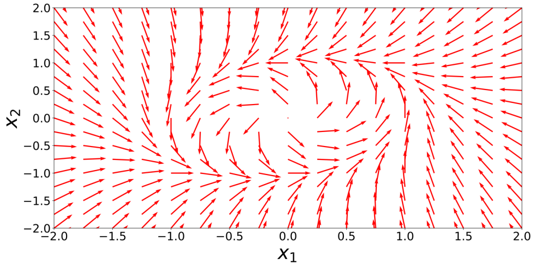
$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

Example

- Lets come back to the system $\dot{x} = F(x)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

- Now consider $\beta = 1$

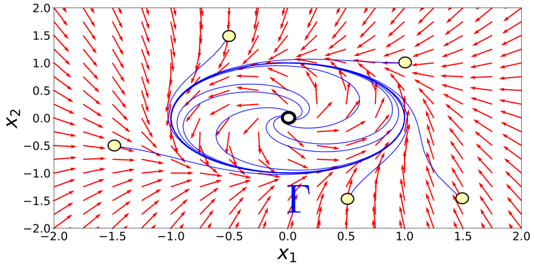


Example

- Lets come back to the system $\dot{x} = F(x)$

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} F_{x_1} \\ F_{x_2} \end{pmatrix} = \begin{pmatrix} \beta x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ x_1 + \beta x_2 - x_2(x_1^2 + x_2^2) \end{pmatrix}$$

- Consider $\beta = 1$

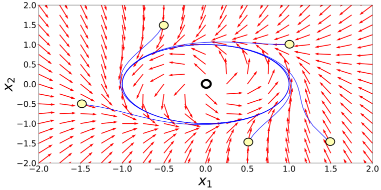


The portrait of the system dramatically changed

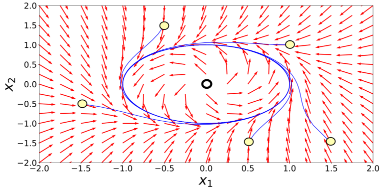
- x^* is an unstable equilibrium
- Trajectories approach a closed curve Γ

Example

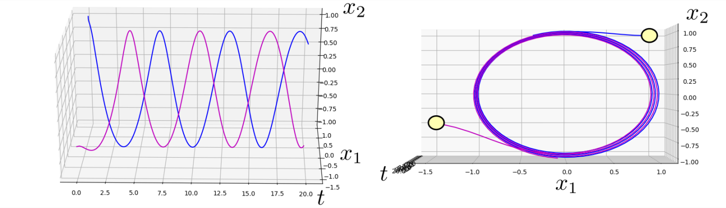
- What does it mean the attractor of our system is a closed curve Γ?



- What does it mean the attractor of our system is a closed curve Γ?



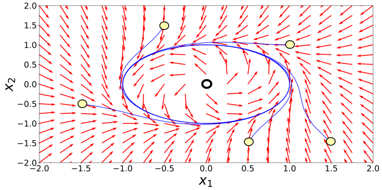
- Lets take some points and integrate them



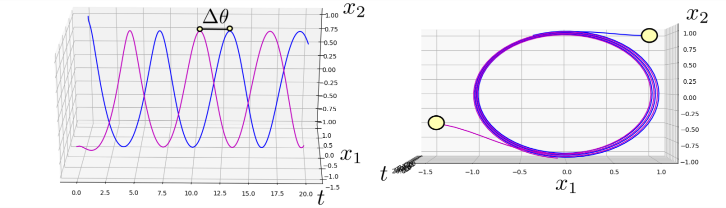
the system has periodic dynamics

Example

- What does it mean the attractor of our system is a closed curve Γ?



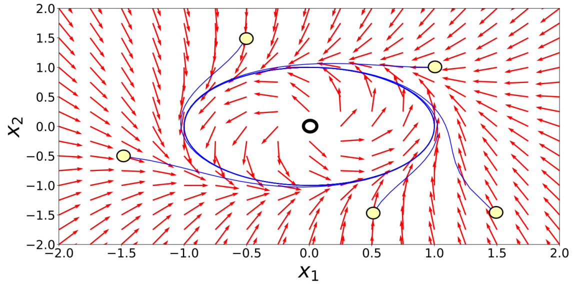
- Lets take some points and integrate them



different points approach Γ with different phases θ

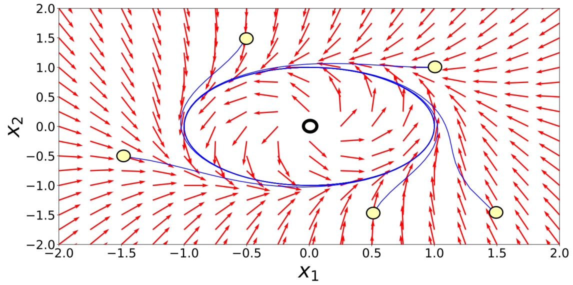
Phases and Isochrons

- As different points on $\mathcal{M}$ approach the cycle Γ with different phases



Phases and Isochrons

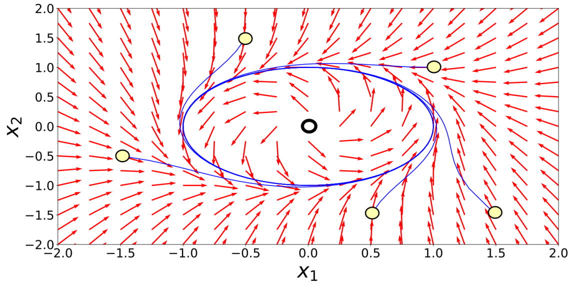
- As different points on $\mathcal{M}$ approach the cycle Γ with different phases



we can assign a phase θ to each point of Γ

Phases and Isochrons

- As different points on $\mathcal{M}$ approach the cycle Γ with different phases

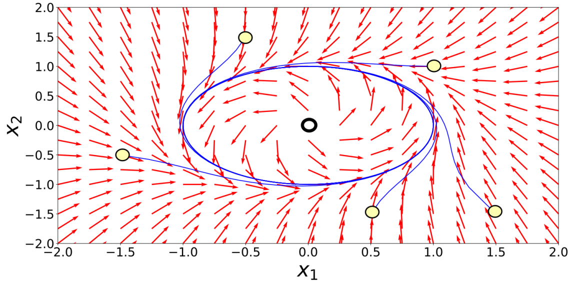


we can assign a phase θ to each point of Γ

- We can define a parameterization $x = \gamma(\theta) \quad$ for $x \in \Gamma$

Phases and Isochrons

- As different points on $\mathcal{M}$ approach the cycle Γ with different phases

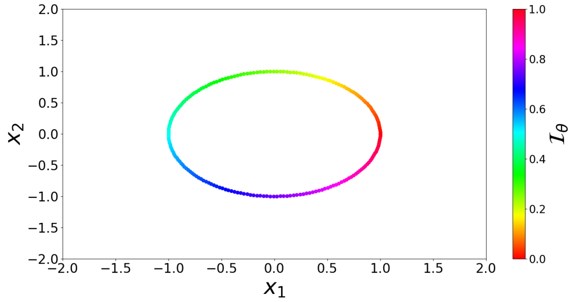


we can assign a phase θ to each point of Γ

- We can define a parameterization $x = \gamma(\theta) \quad$ for $x \in \Gamma$
- Next slides will be devoted to explain the isochrons $\rightarrow$ geometrical interpretation of the distribution of phases in $\mathcal{M}$

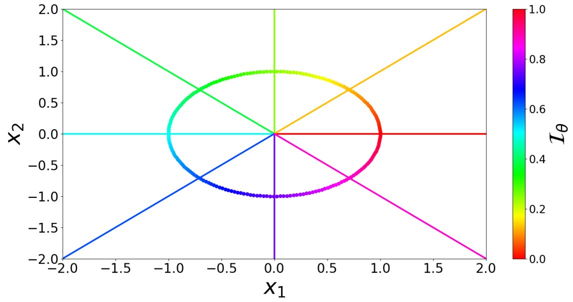
Phases and Isochrons

- We assign a phase θ to each point of Γ



Phases and Isochrons

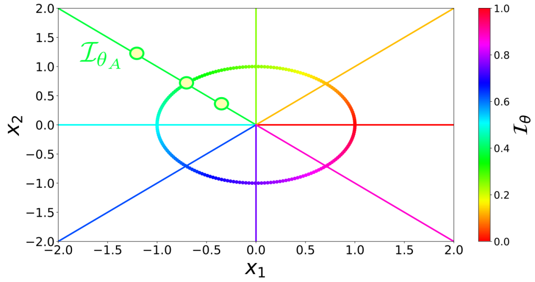
- We assign a phase θ to each point of Γ



and extend the concept of phase θ to $\mathcal{M}$ by the isochrons $\mathcal{I}_\theta$

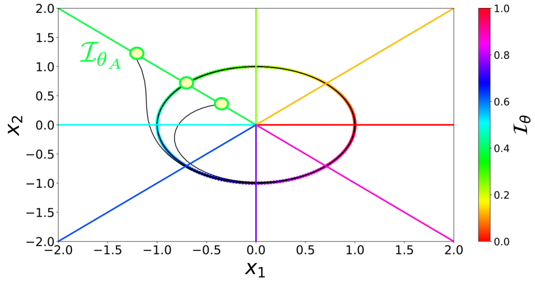
Phases and Isochrons

- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ



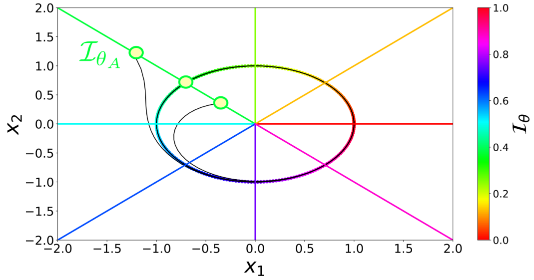
Phases and Isochrons

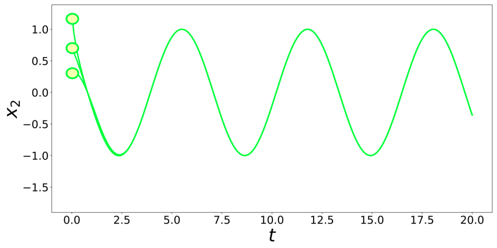
- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ



Phases and Isochrons

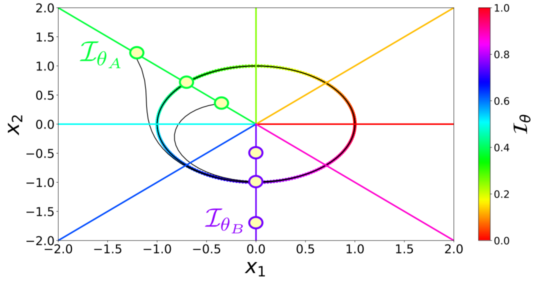
- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ

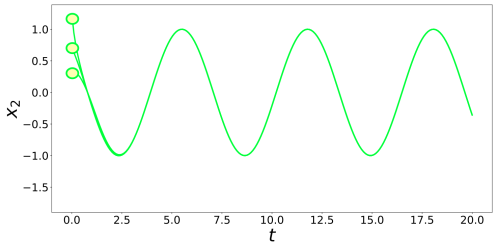




Phases and Isochrons

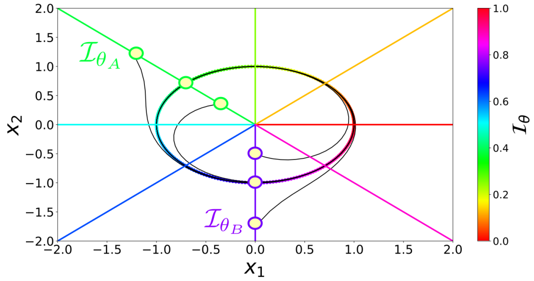
- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ

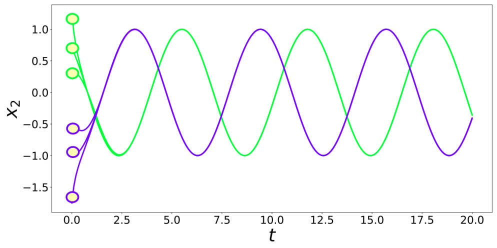




Phases and Isochrons

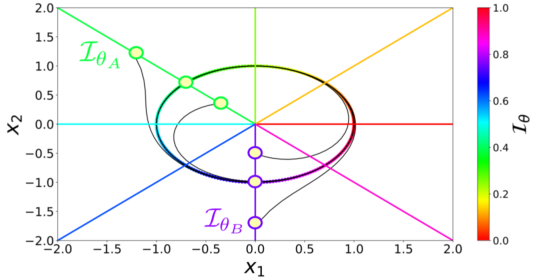
- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ

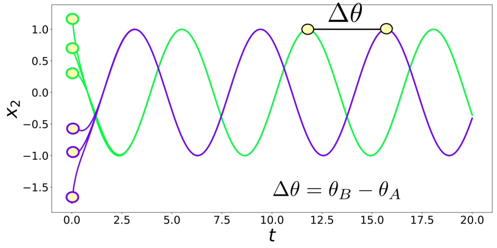




Phases and Isochrons

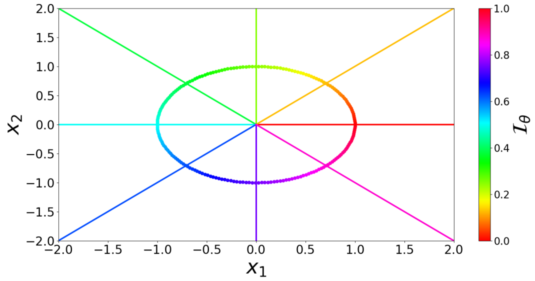
- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ





Phases and Isochrons

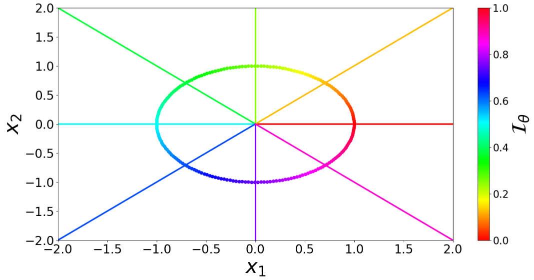
- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ



Therefore, the computation of the isochrons $\mathcal{I}_\theta$ illustrates the distribution of asymptotic phases for points in $\mathcal{M}$

Phases and Isochrons

- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ

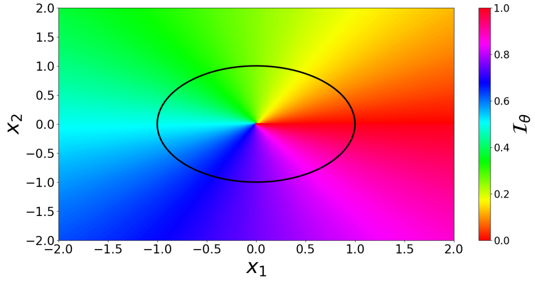


Therefore, the computation of the isochrons $\mathcal{I}_\theta$ illustrates the distribution of asymptotic phases for points in $\mathcal{M}$

- For the sake of illustraton just 8 different isochrons were plotted

Phases and Isochrons

- The $\mathcal{I}_\theta$ are the sets of points reaching Γ with the same phase θ

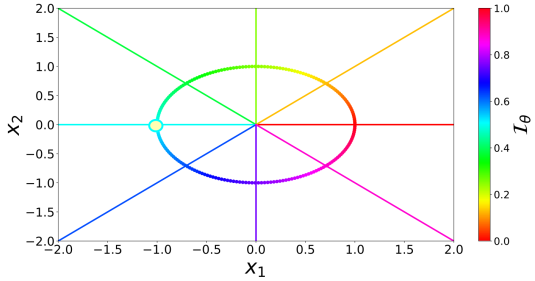


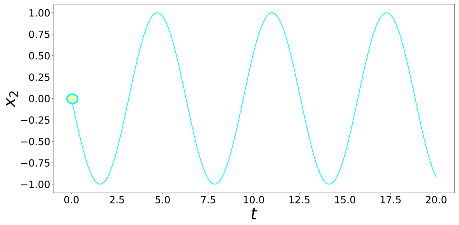
  Therefore, the computation of the isochrons $\mathcal{I}_\theta$ illustrates the distribution of asymptotic phases for points in $\mathcal{M}$

- For the sake of illustraton just 8 different isochrons were plotted
- The complete set of Isochrons folliates the whole basin of attraction $\mathcal{M}$ of Γ

Phases and Isochrons

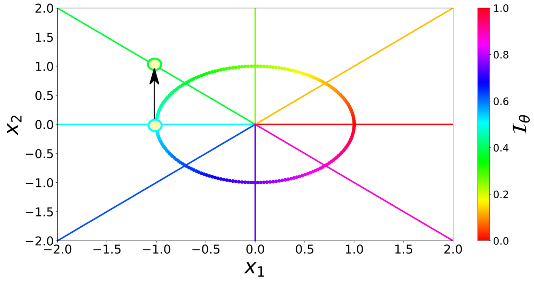
- The computation of $\mathcal{I}_\theta$ provides full understanding of the system under perturbations

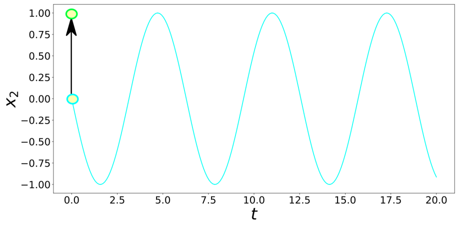




Phases and Isochrons

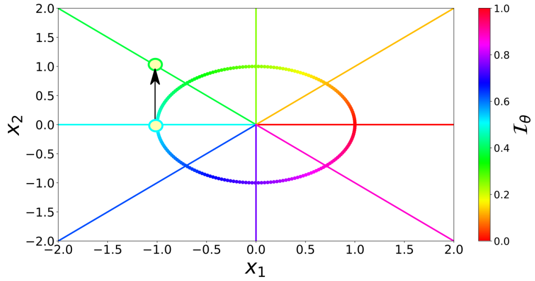
- The computation of $\mathcal{I}_\theta$ provides full understanding of the system under perturbations

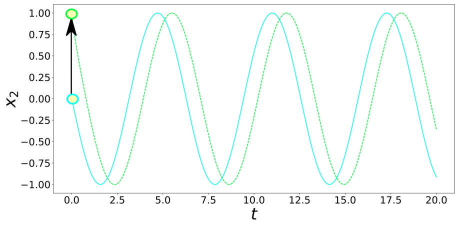




Phases and Isochrons

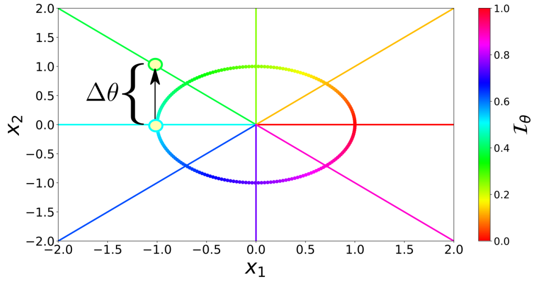
- The computation of $\mathcal{I}_\theta$ provides full understanding of the system under perturbations

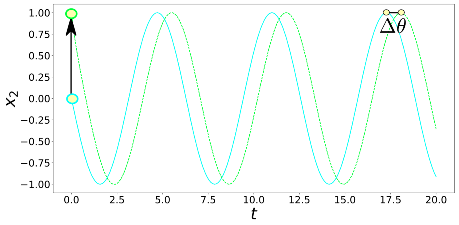




Phases and Isochrons

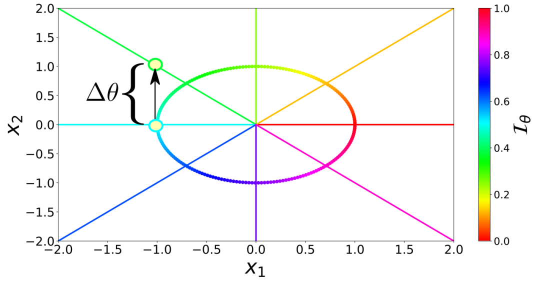
- The computation of $\mathcal{I}_\theta$ provides full understanding of the system under perturbations





Phases and Isochrons

- The computation of $\mathcal{I}_\theta$ provides full understanding of the system under perturbations



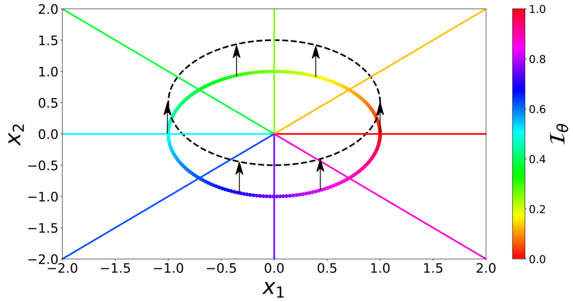
perturbations produce a phase shift $\Delta\theta$ because they change trajectories from one isochron to other

Phase Response Curves

- More importantly, $\mathcal{I}_\theta$ illustrate the phasic dependence of the system under perturbations

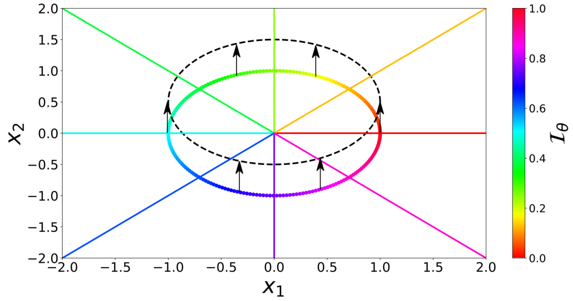
Phase Response Curves

- More importantly, $\mathcal{I}_\theta$ illustrate the phasic dependence of the system under perturbations
- The same perturbation applied at different phases θ produce a different phase shift $\Delta\theta$



Phase Response Curves

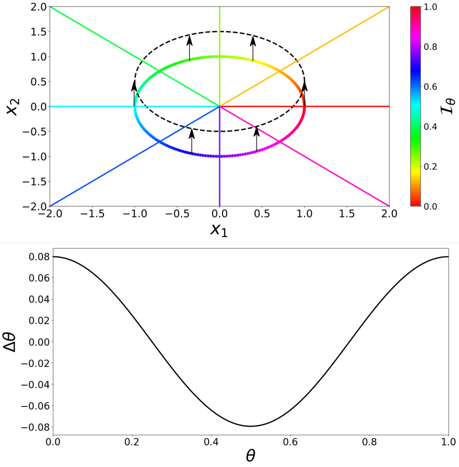
- More importantly, $\mathcal{I}_\theta$ illustrate the phasic dependence of the system under perturbations
- The same perturbation applied at different phases θ produce a different phase shift $\Delta\theta$



- That is exactly what the Phase Response Curves (PRCs) measure

Phase Response Curves

- The Phase Response Curves (PRC) measure which is the dependency between the phase θ at which the perturbation is applied and the corresponding phase shift $\Delta\theta$



Summary

- Systems $\dot{x} = F(x)$
  - $F(x)$ is a vector field defining a flow $\phi_t(x)$ (trajectories)
  - Trajectories are plotted in the phase space
  - Point and periodic attractors

Summary

- Systems $\dot{x} = F(x)$
  - $F(x)$ is a vector field defining a flow $\phi_t(x)$ (trajectories)
  - Trajectories are plotted in the phase space
  - Point and periodic attractors
- We studied the phase θ of the oscillation

Summary

- Systems $\dot{x} = F(x)$
  - $F(x)$ is a vector field defining a flow $\phi_t(x)$ (trajectories)
  - Trajectories are plotted in the phase space
  - Point and periodic attractors
- We studied the phase θ of the oscillation
- We linked the phase θ of the oscillation with isochrons $\mathcal{I}_\theta$ (sets of points with the same asymptotic phase)

Summary

- Systems $\dot{x} = F(x)$
  - $F(x)$ is a vector field defining a flow $\phi_t(x)$ (trajectories)
  - Trajectories are plotted in the phase space
  - Point and periodic attractors
- We studied the phase θ of the oscillation
- We linked the phase θ of the oscillation with isochrons $\mathcal{I}_\theta$ (sets of points with the same asymptotic phase)
- We have seen the relationship between the isochrons $\mathcal{I}_\theta$ and the phase shifts $\Delta\theta$ due to perturbations

Summary

- Systems $\dot{x} = F(x)$
  - $F(x)$ is a vector field defining a flow $\phi_t(x)$ (trajectories)
  - Trajectories are plotted in the phase space
  - Point and periodic attractors
- We studied the phase θ of the oscillation
- We linked the phase θ of the oscillation with isochrons $\mathcal{I}_\theta$ (sets of points with the same asymptotic phase)
- We have seen the relationship between the isochrons $\mathcal{I}_\theta$ and the phase shifts $\Delta\theta$ due to perturbations
- Finally, we understood how this phasic dependence is illustrated by means of the PRCs