PSet 2

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1) a) The eigenvalues of a real anti-Hermitian matrix $A = -A^T$ must be imaginary or 0

$A^* = A^T$

$Ax = \lambda x \Rightarrow x^* A x = x^* \lambda x \Rightarrow (x^* A x)^T = x^* \lambda x \Rightarrow (x^* A^* x)^T = x^* \lambda x$

$(x^*)^T = x$ reversing laws

$\Rightarrow (-Ax)^* x = x^* \lambda x \Rightarrow (-\lambda x)^* x = x^* \lambda x \Rightarrow -x^* \lambda^* x = x^* \lambda x \Rightarrow$

replace Ax

$\Rightarrow -\bar{\lambda} x^* x = \lambda x^* x \Rightarrow (\bar{\lambda} + \lambda) x^* x = 0 \Rightarrow \bar{\lambda} + \lambda = 0 \Rightarrow \bar{\lambda} = -\lambda$

$x^* x$ is positive

if λ is imaginary or 0

b) Since A is real-symmetric, $A = A^T$ and $A^T A$ share the same eigenvectors and the eigenvalues of $A^T A$ are the squares of eigenvalues of A

V = eigenvectors of $A^T A$ = eigenvectors of $A = (v_1, v_2, v_3)$

Σ = eigenvalues of $A^T A$ = eigenvalues of A: $\lambda_1^2, \lambda_2^2, \lambda_3^2$

$A = U \Sigma V^T \Rightarrow A^T A = V \Sigma^T U^T U \Sigma V^T = V \Sigma^2 V^T$

Σ^2 = eigenvalues of $A^T A$ = eigenvalues of A^2

similarly $\Rightarrow U$ = eigenvectors of AA^T = eigenvectors of $A = (u_1, u_2, u_3)$

$$\Rightarrow A = \underbrace{(e_1, e_2, e_3)}_{U} \underbrace{\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}}_{V^T} \Rightarrow (e_1, e_2, e_3) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

c) From the results I see that the singular values of A are the positives of the eigenvalues of B.

$$B = \begin{pmatrix} 0 & U\Sigma V^T \\ V\Sigma^T U & 0 \end{pmatrix} \Rightarrow Bv - \lambda v = 0 \Rightarrow |B - \lambda I| = 0 \Rightarrow$$

$$\Rightarrow \left| \begin{bmatrix} 0 & U\Sigma V^T \\ V\Sigma^T U & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \Rightarrow \begin{vmatrix} -\lambda & U\Sigma V^T \\ V\Sigma^T U & -\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\Rightarrow \lambda^2 - U\Sigma V^T V \Sigma^T U = 0 \Rightarrow \lambda^2 = U \Sigma \Sigma^T U^T \Rightarrow \lambda^2 = U \Sigma^2 U^T$$

This shows that the eigenvalues of B are the positive and negative singular values of A since the singular values are only positive in a real matrix

2 a) $\|Q_1 A Q_2\|_F = \max_{x \neq 0} \frac{\|Q_1 A Q_2 x\|_2}{\|x\|_2} = \max_{x \neq 0} \frac{\|A Q_2 x\|_2}{\|Q_2 x\|_2} = \max_{y \neq 0} \frac{\|Ay\|_2}{\|y\|_2} = \max_{y \neq 0} \|Ay\|$

$\|Q_1 x\| = \|x\|$

$\|Q_1 A Q_2\|_F = \sqrt{\operatorname{trace}(Q_2^T A^T \cancel{Q_1^T Q_1} A Q_2)} = \sqrt{\operatorname{trace}(Q_2^T A^T A Q_2)} = \sqrt{\operatorname{trace}(A^T A)} = \|A\|_F$

through cyclic property

b) $\|U \Sigma V^T\|_2 = \max_{x \neq 0} \frac{\|U \Sigma V^T x\|_2}{\|x\|_2} \Rightarrow \max_{x \neq 0} \frac{\|\Sigma V^T x\|_2}{\|x\|} \Rightarrow \max_{y \neq 0} \frac{\|\Sigma y\|_2}{\|V y\|} \Rightarrow$

$\Rightarrow \max_{y \neq 0} \frac{\|\Sigma y\|_2}{\|y\|} \Rightarrow \max_{y \neq 0} \|\Sigma\|_2 = \sigma_1$

$\|U \Sigma V^T\|_F = \sqrt{tr(V \Sigma^T U^T U \Sigma V^T)} = \sqrt{tr(V \Sigma^T \Sigma V^T)} \Rightarrow$

$\Rightarrow \sqrt{tr(\Sigma^T \Sigma)} = \sqrt{\sum_k \sigma_k^2}$

3. a) $A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow (4-\lambda)(4-\lambda) = 0 =$

$\Rightarrow$ eigenvals $A^T A$ = (eigenvals A)^2 $\Rightarrow$ eigenvals of $A = (2, 2, \pm 2)$

$A = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{U = I} \underbrace{\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}}_{\text{rotation}} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{rotate } V^T} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow$

$\Rightarrow \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

b) $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow A^T A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \Sigma = I$

$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow V = I$

$\Rightarrow A = U \Sigma V^T \Rightarrow AV = U\Sigma$

$\Rightarrow A = U \Rightarrow A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} I \Rightarrow \begin{pmatrix} \cos\theta & 0 \\ 0 & 0 \end{pmatrix}$

4) a) $x^T x = c Q^T Q c = c^2 = c_1^2 + c_2^2 + \dots c_n^2$

$x^T S x = c Q^T S Q c \Rightarrow c \Lambda c = \lambda_1 c_1^2 + \dots \lambda_n c_n^2$

$S = Q \Lambda Q^T \Rightarrow \Lambda = Q^T S Q$

b) If we use the answers above and plug them into the quotient we get:

$$R(x) = \frac{x^T S x}{x^T x} = \frac{\lambda_1 c_1^2 + \dots \lambda_n c_n^2}{c_1^2 + \dots c_n^2}$$

which is a weighted average and we know that the $\operatorname{mean}(x_k)$ for $x \in K^n$ is $\frac{1}{n} \sum_{k=1}^{n} x_k \leq \max(x_k)$ so if the mean is equal to the largest element, then all the values are equal to the max so it is maximal. Since λ_1 is the max value $R(q_1)$ must be maximized

c) From above we saw that if we have a real-symmetric matrix and it's eigenvalues, we are able to find the maximum of the average (in norm) by setting x to the largest eigenvalue, which is σ_1. To get a real symmetric matrix if we do $A^T A$ so $\|A^T A\|_2 = \max \frac{\|A^T A x\|_2}{\|x\|_2}$ and so if we put that all together we see $\|A^T A\|_2 = \sigma^2$ so for $\|A\| = \sigma_1$.

5 a) A is an $N-M \times M$ matrix whose entries are the terms used to predict X_{t+1}.

b is a $N-M \times 1$ vector whose entries are the actual values of X_{t+1}

b) See code

$A = U \Sigma V^T$

$A^+ = V \Sigma^+ U^T$

6. a) $AA^+A = U\Sigma \cancel{V^T V}^{I} \Sigma^+ \cancel{U^T U}^{I} \Sigma V^T = U \Sigma \cancel{\Sigma^+ \Sigma} V^T = U \Sigma V^T = A$

$A^+ A A^+ = V \Sigma^+ \cancel{U^T U}^{I} \Sigma V^T \cancel{V}^{I} \Sigma^+ U^T = V \cancel{\Sigma^+ \Sigma}^{I} \Sigma^+ U^T = V \Sigma^+ U^T = A^+$

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a) See code