Complex Network Science / Dynamics:

Non-trivial topology

Complex systems $\Longleftrightarrow$ Networks

complex systems' units $\Longleftrightarrow$ Nods in networks

interaction between units $\Longleftrightarrow$ links / edges in networks

chaos in deterministic systems is the result of studying coupled systems in mathematics

Neural networks is another example for the usage of complex system study.

Self-organization is a well-known behaviour/phenomena ~~concluding~~ which is the conclusion of ~~complex~~ interactions between system units

Network dynamics: The changes of network topology and the interaction between nodes during time

To look at network from complex system perspective: nodes and links in a network are representative of system's units and interactions. Each unit can have its own linear/nonlinear dynamics and then in the connection/interaction with other units, we ~~could~~ can observe pattern-formation processes which is called self-organization and one well-known example of it is synchronization.

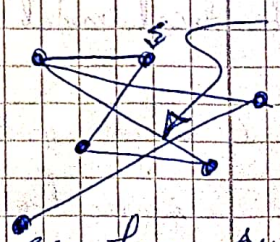
Synchronization will be the main topic of our discussion in the rest of the ours. → Examples: neural networks, AC-electrical network, fire-flies ...

Reference: * Synchronization: A universal concept in non-linear sciences
Kurths, Pikorsky, Rosenblum

* Synchronization in complex networks, physics Reports 469 (2008) 93-153

Let's look again to ~~one~~ a network with dynamical nodes in the connection of each other. Imagine each node is an autonomous oscillators. In connection to each other and critical coupling strength they can form synchronized pattern

with or without leader → conductor of an orchestra



① Introduce KM to describe above system

② To measure synchronization

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③ what will happen if the network expose to internal or external noises / perturbations?

④ The modification of the KM with an additional inertial terms.

① Kuramoto Model:

Reference: Lecture notes in physics, International symposium on Mathematical Problems in theoretical physics (1975), Yoshiki Kuramoto

To continue → Kuramoto's assumption: ① weak coupling between nodes

Winfree approach

② nodes are identical or nearly identical

③ the periodic / sinusoidal phase differences between each pair of nodes.

→ Historical view to Synchronization:

- 1665 Huygens: Mathematician and Physicist, inventor of the pendulum clock. He observed / discovered the synchronization in two pendulum clocks suspended side by side of each other.
- 1948 Wiener: He gave the explanation about the synchronization without leader which occurs between neurons or fire flies ... He did not explain mathematically this process.
- 1967 Winfree: studies the nonlinear dynamics of a large population of weakly coupled limitcycle oscillators (More information with intrinsic frequencies distributed about some mean value). He did not solve analytically the problem by himself → all-to-all network

Winfree model: $\dot{\theta}_i = \omega_i + \left(\sum_{j=1}^{N} X(\theta_j)\right) Z(\theta_i) \quad i = 1 \dots N$ ($Z(\theta_i)$: sensitivity function)

- 1998 Watts and Strogatz: no new contribution in Synchronization but pioneer in complex network. → To review the first part of the lecture the structure of WS and degree distribution

Mathematical model of Kuramoto model: → To watch Kuramoto's talk in youtube

$$\dot{\theta}_i = \omega_i + \sum_j^{N} a_{ij} \sin(\theta_j - \theta_i) \quad ; \quad a_{ij} = \boxed{K}/\boxed{N}$$

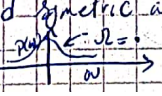
ω_i: intrinsic frequency; K: coupling strength; N: for the thermodynamic limit when $N \to \infty$ the equation converge.

Reference: Y. Kuramoto, lecture notes in physics, vol. 39, Springer, New-York

in Kuramoto's assumption $g(\omega)$ was unimodal (Gaussian distribution) and symetric about it mean, Ω →

$g(\Omega + \omega) = g(\Omega - \omega)$

$\theta_i \to \theta_i + \Omega t$ going to a rotating frame at Ω $\quad g(\omega) = g(-\omega)$



Mean to d probl

locked oscilla

Drifting oscilla still stat dis o

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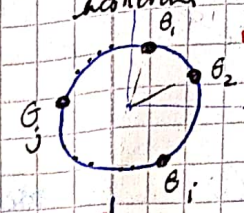
macroscopic complex order parameter to measure synchrony:

$$\left(r(t)\, e^{i\Phi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \right) \times e^{-i\theta_i} \quad ; \quad 0 \leq r(t) \leq 1$$

average phase

$$r(t)\, e^{i(\Phi(t) - \theta_i)} = \frac{1}{N} \sum_{j=1}^{N} e^{i(\theta_j - \theta_i)}$$

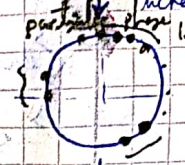
incoherent behavior



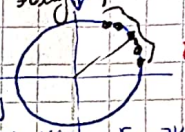
$r(t) \sim 0$

increase coupling

partially phase locking



fully phase locking



$r(t) \sim 1$

Re part : $r(t) \cos(\Phi(t) - \theta_i) = \frac{1}{N} \sum \cos(\theta_j - \theta_i)$

Im part : $r(t) \sin(\Phi(t) - \theta_i) = \frac{1}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i)$

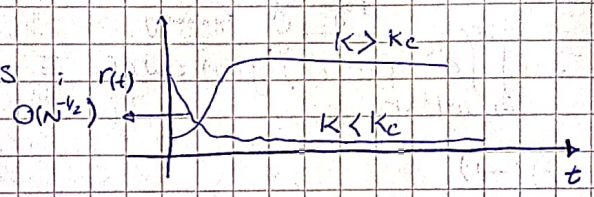
$\rightarrow$ KM: $\dot{\theta}_i = \omega_i + \frac{K}{N} \sum \sin(\theta_j - \theta_i)$

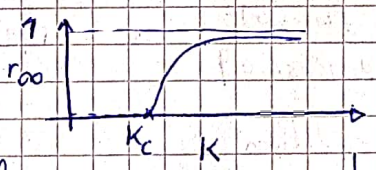
Mean-field theory: to decrease the problem of many particles/bodies to single one $\longrightarrow$ $\dot{\theta}_i = \omega_i + K r(t) \sin(\Phi(t) - \theta_i)$ $\rightarrow$ Oscillator i is interacting with other oscillators via r, ψ $\leftrightarrow$ mean-field quantities.

mean field quantities

$$\left.\begin{array}{l} \lim_{t \to \infty} r(t)(K) = r(K) \\ \text{Assumption } \Phi = 0 \text{ around } \Omega \end{array}\right\} \rightarrow \dot{\theta}_i = \omega_i - \underbrace{Kr}_{\text{coupling}} \sin(\theta_i)$$

Simulation results :





long term behaviour of governing equation, i.e.

$$\dot{\theta}_i = \omega_i - Kr \sin\theta_i$$

locked oscillators $\leftarrow$ 1/ For oscillators with frequency around the center of $g(\omega)$ in steady state

i.e. $\dot{\theta}_i = 0 \rightarrow \omega_i = Kr \sin\theta_i$

$|\omega_i| \leq Kr \quad ; \quad |\theta_i| \leq \pi/2$

Drifting oscillators still with stationarity distribution on the circle $\leftarrow$ 2/ For oscillators with frequency on the tails of $g(\omega)$ or

$|\omega_i| > Kr$

Distribution of drifting oscillators:

$$f(\theta, \omega) = \frac{C}{|\omega - Kr \sin\theta|} \rightarrow \text{normalization constant}$$

$$\int_{-\pi}^{\pi} f(\theta, \omega)\, d\theta = 1 \rightarrow \int \frac{C\, d\theta}{|\omega - Kr \sin\theta|} = 1$$

$$\rightarrow C = \frac{1}{2\pi} \sqrt{\omega^2 - (Kr)^2} \quad \text{for each } \omega$$

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Self-consistency condition

$$\langle e^{i\theta}\rangle = \langle e^{i\theta}\rangle_{loc} + \langle e^{i\theta}\rangle_{drift}$$

$$re^{i\varphi} = r = \underbrace{\langle e^{i\theta}\rangle_{loc}}_{①} + \underbrace{\langle e^{i\theta}\rangle_{drift}}_{②}$$

① locked oscillators : $\omega_i = Kr\sin\theta_i^* \rightarrow \sin\theta_i^* = \omega_i / Kr$

$$\langle e^{i\theta}\rangle_{loc} = \langle \cos\theta + \cancel{i\sin\theta}\rangle_{loc} = \langle \cos\theta\rangle_{loc}$$

$\Rightarrow$ for symmetry

$$\langle \cos\theta\rangle_{loc} = \int_{-Kr}^{Kr} \cos\theta(\omega)\, g(\omega)\, d\omega$$

$$\omega = Kr\sin\theta \rightarrow$$

$$d\omega = Kr\cos\theta\, d\theta$$

$$\langle \cos\theta\rangle_{loc} = \int_{-Kr}^{Kr} \cos\theta\, g(Kr\sin\theta)\, Kr\cos\theta\, d\theta = Kr\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(Kr\sin\theta)\, d\theta$$

② Drifting oscillar :

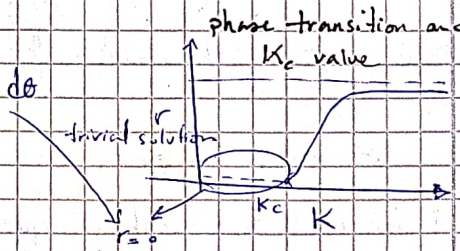
$$\langle e^{i\theta}\rangle_{drift} = \int_{-\pi}^{\pi}\int_{|\omega|>Kr} e^{i\theta}\, \overbrace{\rho(\theta,\omega)}^{\text{symmetry}}\, \overbrace{g(\omega)}^{\text{symmetry}}\, d\omega\, d\theta \equiv 0$$

$$g(\omega) = g(-\omega)$$

$$\rho(\omega,\theta) = \frac{\frac{1}{2\pi}\sqrt{\omega^2-(Kr)^2}}{|\omega - Kr\sin\theta|} \rightarrow \rho(\theta+\pi,-\omega) = \rho(\theta,\omega)$$

$\rightarrow$ Self consistency condition

$$r = Kr\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(Kr\sin\theta)\, d\theta$$



$$1 = K\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(Kr\sin\theta)\, d\theta$$

$$1 = K\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(\cdot)\, d\theta = K\, g(\cdot)\, \frac{\frac{\sin 2\theta}{2} + \theta}{2}\Bigg|_{-\pi/2}^{\pi/2}$$

$$= K\, g(\cdot)\left\{\pi/4 + \pi/4\right\} = \frac{\pi}{2} K\, g(\cdot)$$

$$K_c = \frac{2}{\pi g(\cdot)}$$

exp. for Gaussian distribution with unit variance $\rightarrow K_c \sim 1.6$
in all-to-all network

Near the K_c, the order parameter r, obeys the square-root scaling law for mean-field models, i.e.

$$r \sim (K - K_c)^{\beta} \quad ; \quad \beta = 1/2$$

for $g(\omega) = \frac{\gamma}{\pi(\gamma^2+\omega^2)}$ = Cauchy density

$$\rightarrow 1 = K\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(\omega)\, d\theta$$

$$\Rightarrow r = \sqrt{1 - \frac{K_c}{K}}$$

* finite-size scaling (Home work)

a linear

Continuum limit of KM:

For each natural frequency ω, a continuum of oscillators distributed on the circle.

$f(\theta,t,\omega)$ = a fraction of oscillators lie between θ and $\theta+d\theta$ at t

$$\int_0^{2\pi} \underbrace{f(\theta,t,\omega)}_{2\pi\text{-periodic in }\theta}\, d\theta = 1 \qquad \text{for all } \omega \text{ and } t$$

continuity equation:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \theta}(f v)$$

$$v(\theta,t,\omega) = \omega + Kr\sin(\varphi-\theta) = \omega + Kr\,\mathrm{Im}\left(e^{i(\varphi-\theta)}\right) = \omega + K\,\mathrm{Im}\left(re^{i\varphi}e^{-i\theta}\right)$$

$$\left(re^{i\varphi} = \int_0^{2\pi}\int_{-\infty}^{\infty} e^{i\theta'} f(\theta',\omega',t)\, g(\omega')\, d\omega' d\theta' \right) \times e^{-i\theta}$$

$$re^{i(\varphi-\theta)} = \iint e^{i(\theta'-\theta)} f(\theta',\omega',t)\, g(\omega')\, d\omega' d\theta'$$

$$r\sin(\varphi-\theta) = \iint \sin(\theta'-\theta)\, f(\theta',\omega',t)\, g(\omega')\, d\omega' d\theta'$$

$$\rightarrow v(\theta,t,\omega) = \omega + K\iint \sin(\theta'-\theta)\, f(\theta',\omega',t)\, g(\omega')\, d\omega' d\theta'$$

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \theta}\left[f\left(\omega + K\iint \ldots\right)\right] \rightarrow$$ it can answer systematically all questions about stability, bifurcation, ...

Stationary state: $\frac{\partial f}{\partial t} = 0 \rightarrow f v = C(\omega)$

$$f = \frac{C(\omega)}{v} = \begin{cases} f_{drift} & C(\omega) = \text{const} \\ f(\omega) = f_{loc} & C(\omega) = 0 \end{cases}$$

Reference: From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators, Steven H. Strogatz

Stability of the incoherent state (linear stability)

$$f(\theta,t,\omega) = \frac{1}{2\pi} + \varepsilon \underbrace{\eta(\theta,t,\omega)}_{\text{perturbation}} \quad : \quad f_0(\theta,\omega) = \frac{1}{2\pi}$$

A Fourier series of $\eta(\theta,t,\omega)$ in θ

$$\eta(\theta,t,\omega) = c(t,\omega)e^{i\theta} + \underbrace{c.c.}_{\text{complex conjugate}} + \eta^{\perp}(\theta,t,\omega)$$ Higher harmonics of η

$$f = \frac{1}{2\pi} + \varepsilon c(t,\omega)e^{i\theta} + \varepsilon\, c.c.$$

$$\frac{\partial}{\partial t} f = \frac{\partial}{\partial \theta}(f v)$$

$$\Rightarrow \frac{\partial c}{\partial t} = -i\omega c + \frac{K}{2}\int_{-\infty}^{\infty} c(t,\omega')\, g(\omega')\, d\omega'$$

a linear operator

$$\frac{\partial c}{\partial t} = -i\omega c + \ldots$$

Graph theoretical formulation of KM:

B = incidence matrix $N \times e$ = nodes × edges

$L = BB^T$ = Laplacian matrix $N \times N$

- semi positive definite
- there is i where $\lambda_i = 0$
- $\{\lambda_i(L)\}$ captures many topological properties of the graph
- first non-zero λ_i gives a measure of connectedness

$L_W = BWB^T$ = weighted edges

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum \sin(\theta_j - \theta_i) \rightarrow$$

$$\dot{\theta} = \omega - \frac{K}{N} B \sin(B^T\theta)$$

$$r^2 = \frac{N - 2e + 2\mathbb{1}_e^T \cos(B^T\theta)}{N^2}$$; e ≡ edges and $N \times e$ matrix

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References: on the stability of the KM of coupled nonlinear oscillators,

Continue: Stability of incoherent state:

$$\frac{\partial c}{\partial t} = -i\omega c + \frac{K}{2}\int_{-\infty}^{\infty} c(t,\omega')\, g(\omega')\, d\omega' \rightarrow Ac = -i\omega c + \frac{K}{2}\int_{-\infty}^{\infty} c(t,\omega')\, g(\omega')\, d\omega'$$

To consider continous and discrete spectrum of A

* The evolution of c depends on all other frequencies through integral on c

Discrete spectrum of the above equation:

$$c(t,\omega) = b(\omega)\, e^{\lambda t}$$; λ = eigenvalue

$$b(\omega)\lambda e^{\lambda t} = -i\omega b(\omega) e^{\lambda t} + \frac{K}{2}\int_{-\infty}^{\infty} b(\omega') e^{\lambda t} g(\omega')\, d\omega'$$

$$b(\omega)\lambda = -i\omega b(\omega) + \underbrace{\frac{K}{2}\int b(\omega') g(\omega')\, d\omega'}_{B}$$

$$b\lambda = -i\omega b + B \rightarrow (\lambda + i\omega) b = B$$

$$b = \frac{B}{\lambda + i\omega}$$

$$B = \frac{K}{2}\int b g\, d\omega = \frac{K}{2}\int \frac{B g}{\lambda + i\omega}\, d\omega$$

$$1 = \frac{K}{2}\int \frac{g(\omega)}{\lambda + i\omega}\, d\omega$$

Assumptions: $g(\omega) = g(-\omega)$, $\omega \geq \nu \rightarrow g(\omega) < g(\nu)$ $\Big\} \rightarrow$ at least one solution for λ which is real

$$1 = \frac{K}{2}\int \frac{g(\omega)(\lambda - i\omega)}{\lambda^2 + \omega^2}\, d\omega = \frac{K}{2}\left\{\int \frac{\lambda g(\omega)}{\lambda^2 + \omega^2}\, d\omega - i\int \frac{\omega g(\omega)}{\lambda^2 + \omega^2}\, d\omega\right\}$$

$$\overline{1 = \frac{K}{2}\int \frac{\lambda g(\omega)}{\lambda^2 + \omega^2}\, d\omega}$$ so $\lambda \geq 0 \rightarrow$ * Incoherent state of KM can never be asymptotically stable: it is either unstable or neutrally stable

for $\lambda \to 0^+ \Rightarrow \frac{\lambda}{\lambda^2+\omega^2} = \pi\delta(\omega)$

$$1 = \frac{K_c}{2}\pi g(0) \to K_c = \frac{2}{\pi g(0)}$$

Stability of the synchronization state:

$$\dot{\theta}_i = \underbrace{\omega_i + \frac{K}{N}\sum_{j=1}^{n} \sin(\theta_j - \theta_i)}_{X_i}$$

① $X_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)$; X_i is a component i of a vector X

$$\bar{X} = -\bar{\nabla} F \to F = -\int \bar{X} \cdot d\bar{\theta}$$

$$F(\theta_1, \dots, \theta_N) = -\sum_{i=1}^{N} \omega_i \theta_i - \frac{K}{2N}\sum_{i,j=1}^{N} \cos(\theta_j - \theta_i)$$

$$re^{i\varphi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \xrightarrow{\text{mean field}} ② \dot{\theta}_i = \omega_i - Kr\sin(\theta_i)$$

$$r^2 = \frac{1}{N^2}\sum_{i,j=1}^{N} e^{i(\theta_j - \theta_i)} = \frac{1}{N^2}\sum_{i,j=1}^{N} \cos(\theta_j - \theta_i) \to ③ F = -\sum_i \omega_i \theta_i - \frac{KNr^2}{2}$$

locked states: $X_i = 0 \to \frac{\omega_i}{Kr} = \sin(\theta_i)$

$$\cos(\theta_i) = \pm\sqrt{1 - \frac{\omega_i^2}{(Kr)^2}}$$

for $\varphi = 0 \to r = \frac{1}{N}\sum_{j=1}^{N}(\cos\theta_j + i\cancel{\sin\theta_j}) = \frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j)$

$$r = \frac{1}{N}\sum_{j=1}^{N} \pm\sqrt{1 - \frac{\omega_i^2}{(Kr)^2}} \to 1 = \frac{1}{N}\sum_{i=1}^{N} \underbrace{\pm\sqrt{r^2 - \frac{\omega_i^2}{K^2}}}_{2^N \ \pm \text{ signs combinati}}$$

* Among 2^N $\pm$ sign combination only ones with positive sign are accepted.

locked phase: $X_i = 0 \to -\nabla F = 0$

stable point: $\frac{\partial \dot{X}_i}{\partial \theta_i} < 0 \to \frac{\partial F}{\partial \theta^2} \geq$

$$re^{i\varphi} = \frac{1}{N}\sum_{j=1} e^{i\theta_j}$$

$$re^{i(\varphi - \theta_i)} = \frac{1}{N}\sum_{j=1} e^{i(\theta_j - \theta_i)}$$

$$r\cos(\varphi - \theta_i) = \frac{1}{N}\sum_j \cos(\theta_j - \theta_i)$$

$$-\frac{\partial F}{\partial \theta_i} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)$$

$$-\frac{\partial^2 F}{\partial \theta_i^2} = 0 + \frac{K}{N}\sum_{j \neq i}^{N} \cos(\theta_j - \theta_i)$$

$$\frac{\partial^2 F}{\partial \theta_i^2} = \frac{K}{N}\sum_{j \neq i}^{N} \cos(\theta_j - \theta_i) = \frac{K}{N}\left[\sum_{j=1}^{N} \cos(\theta_j - \theta_i) - 1\right] =$$

$$Kr\cos(\varphi - \theta_i) - K/N$$

$$\frac{\partial^2 F}{\partial \theta_i^2} \geq 0 \to Kr\cos(\theta_i) \geq \frac{K}{N} \to \boxed{r\cos(\theta_i) \geq \frac{1}{N}}$$ some of them stable

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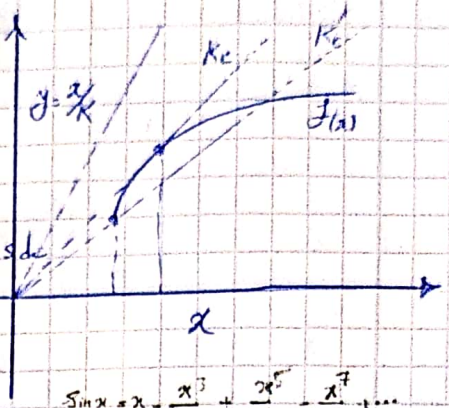
Come back to self-consistency equation:

$$\left.\begin{aligned} Kr^2 &= \frac{1}{N}\sum_{i=1}^{N}\sqrt{(Kr)^2-\omega_i^2} \\ (Kr)^2 &= x \end{aligned}\right\} \rightarrow f(x) = \frac{1}{N}\sum_{i=1}^{N}\sqrt{x-\omega_i^2}$$



$y = f(x)$

separately plot $f(x)$ and right hand side and then considering the crossing points

- $0 < K < K_c \rightarrow$ no solution
- $K = K_c \rightarrow$ one solution
- $K_c < K < K_c' \rightarrow$ 2 solutions
- $K > K_c' \rightarrow$ one solution

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots$$

Linear stability of locked states:

linearization: $\theta_i = \theta_i^* + x_i$

$\dot\theta_i = \dot x_i$

$$\dot\theta_i = \omega_i + \frac{K}{N}\sum \sin(\theta_j - \theta_i)$$

$$\dot x_i = \omega_i + \frac{K}{N}\sum \sin(\overbrace{\theta_j^* - \theta_i^*}^{A} + \overbrace{x_j - x_i}^{B})$$

$$= \omega_i + \frac{K}{N}\sum\left\{\sin(\theta_j^* - \theta_i^*)\overbrace{\cos(x_j - x_i)}^{1} + \cos(\theta_j^* - \theta_i^*)\overbrace{\sin(x_j - x_i)}^{x_j - x_i}\right\}$$

$$\underbrace{\omega_i + \underbrace{\frac{K}{N}\sum \sin(\theta_j^* - \theta_i^*)}_{-Kr\sin\theta_i^*}}_{\text{phase locked: } \omega_i = Kr\sin\theta_i^*} + \frac{K}{N}\sum \cos(\theta_j^* - \theta_i^*)(x_j - x_i)$$

$$\dot x_i = \frac{K}{N}\sum \cos(\theta_j^* - \theta_i^*)(x_j - x_i)$$

To define matrix J

$$J_{ij} = \frac{\partial \dot X_i}{\partial \theta_j} = -\frac{\partial^2 F}{\partial \theta_i \partial \theta_j} = \begin{cases} \frac{K}{N}\cos(\theta_j - \theta_i) & \text{for } i \neq j \\ \frac{K}{N} - Kr\cos\theta_i & \text{for } i = j \end{cases}$$

* J is symmetric, therefore there exists an orthogonal basis of eigenvectors J

As J is symmetric then all eigenvalue are negative, one is zero

$$\sum_{j=1}^{N} J_{ij} = \sum_{j\neq i}\left\{\frac{K}{N}\cos(\theta_j - \theta_i)\right\} + \sum_{j=1}\left(\frac{K}{N} - Kr\cos\theta_i\right)$$

$$= \frac{K}{N}\sum_{j=1}\cos(\theta_j - \theta_i) - Kr\cos\theta_i + Kr\cos\theta_i - Kr\cos\theta_i = 0$$

$\rightarrow$ Then there is eigenvector $(1, 1, \dots)$ with $\lambda = 0$

$$J = \frac{K}{N}C - KrD$$

$$C_{ij} = \overbrace{\cos(\theta_j - \theta_i)}^{\cos\theta_j\cos\theta_i + \sin\theta_j\sin\theta_i} \rightarrow C_{ij} = \overset{\cos\theta_i}{c_i}c_j + \overset{\sin\theta_i}{s_i}s_j$$

D = Diagonal matrix with entries $\cos\theta_i$

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J is symmetric therefore $J\bar{x}\cdot\bar{x} < 0$ is the stability condition:

$\rightarrow \|x\| \geq 0$ and $\sum_{i=1}^{n} x_i = 0$ i.e. $\bar{x}$ is orthogonal to $(1\,1\,\dots)$

Example / Problem: find self-consistency equation for continuum actual coupling

$$\dot{\theta}_j = \omega_j + \sum_{k=1}^{N} a_{jk}(\theta_k - \theta_j) \quad ; \quad a_{jk} = \frac{\alpha}{N}\sin(\theta_k - \theta_j)$$

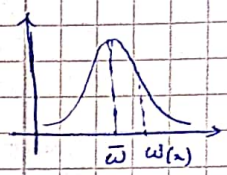
$$\dot{\theta}_j = \omega_j + \frac{\alpha}{N}\sum_{k=1}^{N\to\infty}\sin(\theta_k - \theta_j) \rightarrow \dot{\theta}_j = \omega_j(x) + \alpha\int_0^1 \sin(\theta(x',t) - \theta(x,t))\,dx'$$

$$\lim_{N\to\infty}\sum_{k=1}^{N} f(a + k\Delta x)\Delta x = \int_a^b f(x)\,dx$$

$$\frac{d\bar{\theta}}{dt} = \bar{\omega} \quad ; \quad \bar{\omega} = \int_0^1 \omega(x)\,dx \quad ; \quad \bar{\theta} = \int_0^1 \theta(x,t)\,dx$$

In phase-locking: $\theta(x,t) - \theta(y,t) = \underbrace{\theta(x) - \theta(y)}_{\text{independent of time}}$

$$\boxed{\theta(x,t) = \bar{\omega}t + \varphi(x)}$$



$$\omega(x) - \bar{\omega} = \hat{\gamma}\Delta(x) \quad ; \quad \max_{0\leq x\leq 1}|\Delta(x)| = 1$$

$$\bar{\omega} = \omega(x) + \alpha\int_0^1 \sin(\varphi(x') - \varphi(x))\,dx'$$

$$-\hat{\gamma}\Delta(x) = -\alpha\int_0^1 \sin(\varphi(x) - \varphi(x'))\,dx' \longrightarrow \text{Solution of this eq.: } \sin\varphi(x) = \frac{\gamma\Delta(x)}{c}$$

$$\gamma\Delta(x) = \int_0^1 [\sin\varphi(x)\cos\varphi(x') - \cos\varphi(x)\sin\varphi(x')]\,dx'$$

$$= \sin\varphi(x)\int_0^1 \cos\varphi(x')\,dx' - \cos\varphi(x)\int_0^1 \sin\varphi(x')\,dx'$$

$$= \sin\varphi(x)\int_0^1 \sqrt{1 - \frac{\gamma^2\Delta^2(x')}{c^2}}\,dx' - \cos\varphi(x)\underbrace{\int_0^1 \frac{\gamma\Delta(x')}{c}\,dx'}_{=0}$$

as $\langle\omega(x)\rangle - \bar{\omega} = \hat{\gamma}\Delta(x)$

$$\gamma\Delta(x) = \sin\varphi(x)\int_0^1 \sqrt{1 - \frac{\gamma^2\Delta^2(x')}{c^2}}\,dx'$$

$$\gamma\Delta(x) = \frac{\gamma\Delta(x)}{c}\int \sqrt{1 - \frac{\gamma^2\Delta^2(x')}{c^2}}\,dx'$$

$$c = \int_0^1 \sqrt{1 - \frac{\gamma^2\Delta^2(x')}{c^2}}\,dx' \quad ; \quad k = \frac{c}{\gamma}$$

$$k\gamma = \int_0^1 \sqrt{1 - \frac{\gamma^2\Delta^2(x')}{\gamma^2 k^2}}\,dx' = \frac{1}{k}\int_0^1 \sqrt{k^2 - \Delta^2(x')}\,dx' \rightarrow k \geq 1$$

$$\gamma = \frac{1}{k^2}\int_0^1 \sqrt{k^2 - \Delta^2(x')}\,dx'$$

$$\gamma = F(k) \leq \underbrace{\frac{1}{k^2}\int_0^1 \sqrt{k^2 - 0}\,dx'}_{1/k} < 1$$

The relation between variance and coupling strength

Therefore there is no solution for $\gamma > 1$ / $\gamma/\alpha > 1$

$$\omega - \bar{\omega} = \hat{\gamma}\Delta(x) = \gamma\alpha\Delta(x) \rightarrow \frac{\omega - \bar{\omega}}{\alpha} = \gamma\Delta(x)$$

$$\max\left|\frac{\omega - \bar{\omega}}{\alpha}\right| = \gamma\,\underbrace{\max|\Delta(x)|}_{1} = \gamma$$

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KM with external forces : ① periodic force
② random noise

Reference : H. Sakaguchi, Progress of theoretical physics, vol 79, No. 1 (1988)

① periodic force :

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_j \sin(\theta_i - \theta_j) - b\sin(\theta_i - \omega_f t)$$

* ω_f is close to mean value of natural frequency

To consider different states:

(i) For $K = 0 \rightarrow \dot{\theta}_i = \omega_i - b\sin(\theta_i - \omega_f t)$

$$\psi_i = \theta_i - \omega_f t \rightarrow \dot{\psi}_i = \dot{\theta}_i - \omega_f$$

$$\dot{\psi}_i + \omega_f = \omega_i - b\sin(\psi_i)$$

$$\dot{\psi}_i = \omega_i - \omega_f - b\sin(\psi_i)$$

$$\dot{\psi}_i = 0 \rightarrow \omega - \omega_f = b\sin(\psi_i)$$

* oscillator i is entrained to the frequency of external force

(ii) For $K \neq 0$ and $b = 0 \rightarrow \dot{\theta}_i = \omega_i - \frac{K}{N}\sum_j \sin(\theta_i - \theta_j)$

$K > K_c$ for phase locking

coupling frequency with external force

(iii) For K and $b \neq 0 \rightarrow \dot{\theta}_i = \omega_i - \frac{K}{N}\sum \sin(\theta_i - \theta_j) - b\sin(\theta_i - \omega_f t)$

For this case we investigate a steadily rotating state, so the complex order parameter rotates steadily at the frequency of the external force

$$\sigma e^{i(\overbrace{\varphi_0 + \omega_f t}^{\varphi(t)})} = \frac{1}{N}\sum_j e^{i\theta_j}$$

$$\sigma\sin(\varphi_0 + \omega_f t - \theta_i) = \frac{1}{N}\sum_j \sin(\theta_j - \theta_i)$$

$$\begin{cases} b' = (b^2 + K^2\sigma^2 - 2K\sigma b\sin\varphi_0) \\ \tan\psi_0 = K\sigma\dfrac{\sin\varphi_0}{b + K\sigma\cos\varphi_0} \end{cases}$$

$$\psi_i = \theta_i - \omega_f t \rightarrow \dot{\psi}_i = \dot{\theta}_i - \omega_f$$

$$\Rightarrow \dot{\psi}_i = \omega_i - \omega_f - b'\sin(\psi_i - \psi_0)$$

fixed points (phase locking) : $\dot{\psi}_i = 0 \rightarrow \omega_i - \omega_f = -b'\sin(\psi_i - \psi_0)$ $\quad \left|\frac{\omega_i - \omega_f}{b'}\right| \leq 1$

$$\psi_i = \sin^{-1}\left(\frac{\omega_i - \omega_f}{b'}\right) + \psi_0$$

No phase locking $\rightarrow \psi_i = \tilde{\omega}_i t + \psi_0 + \underbrace{h(\tilde{\omega}_i t)}_{\text{periodic function}}$ $\quad \left|\frac{\omega_i - \omega_f}{b'}\right| > 1$

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$$\sigma e^{i(\varphi_0+\omega_f t)} = \frac{1}{N}\sum e^{i\theta_j}$$

$$\sigma e^{i\varphi_0} = \frac{1}{N}\sum e^{i(\theta_j-\omega_f t)} = \frac{1}{N}\sum e^{i\psi_j}$$

$$\sigma e^{i\varphi_0} = \int_0^{2\pi} n(\psi)e^{i\psi}\,d\psi = \int_{-\infty}^{\infty} g(\omega)\,d\omega \int_0^{2\pi} n(\psi;\omega)e^{i\psi}\,d\psi$$

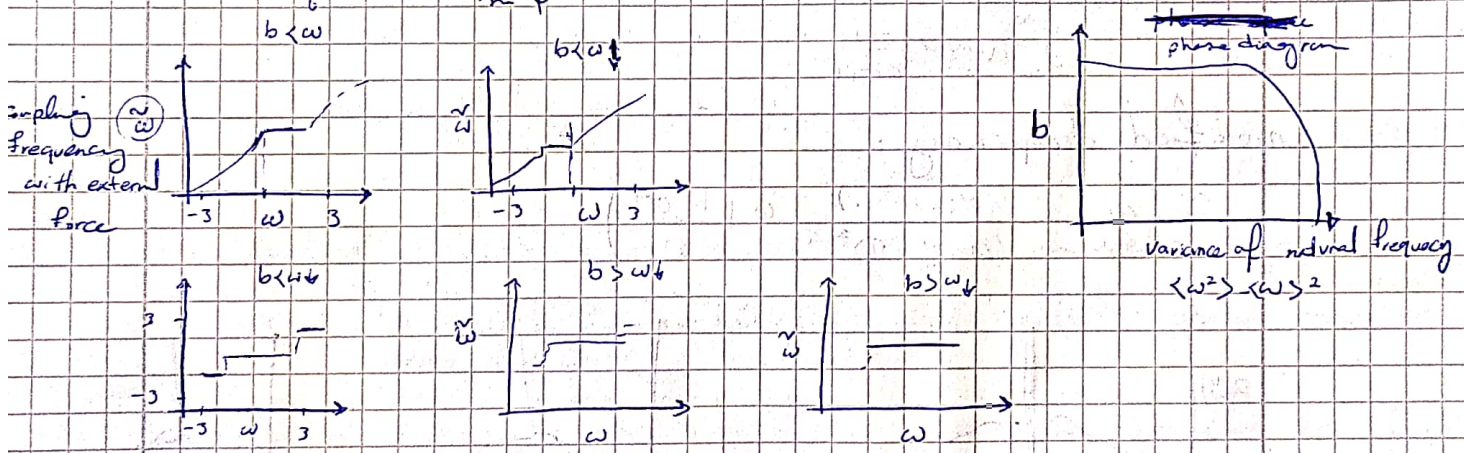
$$n(\psi) = n_s(\psi) + n_{ds}(\psi)$$

$$n_s(\psi) = g(\omega_f + b'\sin(\psi-\psi_0)) \quad ; \quad d\omega = b'\cos(\psi-\psi_0)$$

$$n_{ds}(\psi) = \frac{1}{2\pi}\int_{|\omega-\omega_f|>b'} \frac{g(\omega)\,c}{\omega-\omega_f-b'\sin(\psi-\psi_0)}\,d\omega = \frac{1}{2\pi}\int d\omega\, g(\omega)\,\frac{\sqrt{(\omega-\omega_f)^2-b'^2}}{\omega-\omega_f-b'\sin(\psi-\psi_0)}$$

$$\rightarrow \sigma e^{i(\varphi_0-\psi_0)} = b'\left\{\int_{-\pi/2}^{\pi/2} d\psi\, g(\omega_f+b'\sin\psi)\cos\psi\, e^{i\psi} + iJ\right\}$$

$$J = \int_0^{\pi/2} d\psi\,\frac{\cos\psi(1-\cos\psi)}{\sin^3\psi}\left\{g\left(\omega_f+\frac{b'}{\sin\psi}\right) - g\left(\omega_f-\frac{b'}{\sin\psi}\right)\right\}$$



② random noise,

Gaussian white noise

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_j \sin(\theta_i-\theta_j) + f_i(t)$$

$$\langle f_i(t)\rangle = 0$$

$$\langle f_i(t)f_j(t')\rangle = \delta_{ij}\delta(t-t')$$

$$\sigma e^{i\varphi} = \frac{1}{N}\sum_j e^{i\theta_j} \rightarrow \sigma e^{i(\varphi_0+\omega_0 t)} = \frac{1}{N}\sum e^{i\theta_j}$$

$$\sigma\sin(\varphi_0+\omega_0 t-\theta_i) = \frac{1}{N}\sum \sin(\theta_j-\theta_i)$$

$$\dot{\theta}_i = \omega_i + K\sigma\sin(\varphi_0+\omega_0 t-\theta_i) + f_i(t)$$

$$\psi = \theta_i - \omega_0 t \rightarrow \dot{\psi} = \dot{\theta} - \omega_0$$

$$\omega = \omega_i - \omega_0$$

$$\psi = \theta_i - \varphi_0 - \omega_0 t$$

$$\rightarrow \dot{\psi} = \omega - K\sigma\sin(\psi) + f_i(t) \quad \text{Langevin eq.}$$

Fokker-Planck eq.: $\dfrac{\partial n(\psi;\omega)}{\partial t} = -\dfrac{\partial}{\partial\psi}\left[(\omega-K\sigma\sin\psi)\,n(\psi;\omega)\right] + D\dfrac{\partial^2}{\partial\psi^2}n(\psi;\omega)$

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$$\dot{P}(x,t) = \partial_x \left[D^{(1)}(x,t) + \partial_x D^{(2)}(x,t) \right] P(x,t)$$

$$P^{st}(x,t) = \exp\left[-\varphi(x,k) - G(k) \right]$$

$$\varphi(x,k) = -\int_{-\infty}^{x} \frac{D^{(1)}(\tilde{x},k)}{D^{(2)}(\tilde{x},k)}\, d\tilde{x} \quad ; \quad G(k) = -\ln \int \exp\left[-\varphi(x,k) \right] dk$$

$$\rightarrow \boxed{n(\psi;\omega)} = \exp\left(\frac{-K\sigma^2 + \omega\psi + K\sigma^2 \cos\psi}{D} \right) n(0;\omega) \left\{ 1 + \frac{\left(e^{-2\pi\omega/D} - 1\right) \int_0^{\psi} e^{(-\omega\psi - K\sigma^2 - K\sigma^2\cos\psi)/D}\, d\psi}{\int_0^{2\pi} e^{(-\omega\psi - K\sigma^2 - K\sigma^2\cos\psi)/D}\, d\psi} \right\}$$

$$\int_0^{2\pi} n(\psi;\omega)\, d\psi = 1$$

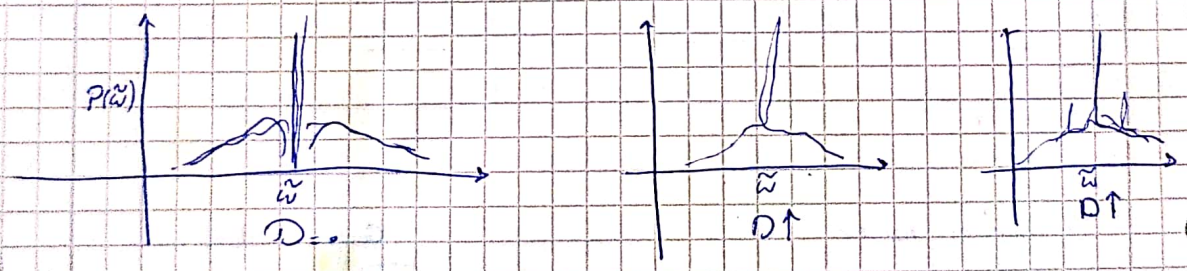
$$\sigma e^{i\varphi} = \int_0^{2\pi} d\psi \exp(i\psi) = \int_{-\infty}^{\infty} d\omega\, g(\omega) \int_0^{2\pi} d\psi\, n(\psi;\omega)\, e^{i\psi}$$

$$\sigma = K\sigma \left[\frac{1}{2} \int_{-\infty}^{\infty} g(D\omega + \omega_0) \frac{d\omega}{\omega^2 + 1} - \frac{K^2 \sigma^2}{4D^2} \int_{-\infty}^{\infty} g(D\omega + \omega_0) \left\{ \frac{1}{\omega^2 + 4} - \frac{\omega}{(\omega^2+1)^2} \right\} d\omega + O\left(\left(\frac{K\sigma}{D} \right)^4 \right) \right]$$

$$K_c(D) = \frac{2}{\int_{-\infty}^{\infty} g(D\omega + \omega_0) \frac{d\omega}{\omega^2 + 1}}$$

modified frequency:

$$\tilde{\omega} = \omega - \frac{1}{2} \frac{K\sigma(\omega - \omega_0)}{(\omega - \omega_0)^2 + D^2} + O\left(\frac{K^4 \sigma^4}{D^4} \right)$$



Kuramoto Model + Inertia (or extended version of KM)

B. Ermentrout (1991): The observation shows that pteroptyx malaccae (fireflies in Southeast Asia) alter firing frequencies as much as 15% to higher or lower frequency in response to the external signals, i.e:

$$\omega_{min} \leftarrow \omega_{nat} \text{ (desired frequency)} \rightarrow \omega_{max}$$

This adaption is considered to be responsible for the synchronous firing with very small phase lags

Ref: An adaptive model for synchrony in the firefly pteroptyx malaccae

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Ermentrout, therefore, introduced the adaptive effect of the firing frequency to the phase model

$\dot{\theta}_i = \omega_i$ KM: $\dot{\theta}_i = \omega_i + \sum_j a_{ji}(\theta_j - \theta_i)$

$\dot{\omega}_i = \epsilon(\omega_{i0} - \omega_i) + \sum H_{ji}(\theta_j - \theta_i)$

if ϵ is small and ω_i is in the range of adaptive frequency then H_{ji} is sinusoidal and one can simulate the synchrony observed in southeast Asian fireflies.

Later Tanaka-Lichtenberg-Oishi (1997) showed that the above eq. is same as

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} h(\theta_j - \theta_i)$$

if $t_{old} \to \epsilon t_{new}$

$\Omega_i \equiv \epsilon\omega_{i0}$

$H_{ji}(\theta_j - \theta_i) \equiv \frac{K}{N} h(\theta_j - \theta_i)$

$m \equiv \epsilon^{-2}$

$$\Rightarrow \frac{d\dot{\theta}_i}{\epsilon^2 dt_{new}} = \epsilon\omega_{i0} - \epsilon\frac{d\theta_i}{\epsilon dt_{new}} + \sum \frac{K}{N} h_{ji}(\theta_j - \theta_i)$$

$$m\ddot{\theta}_i = \Omega_i - \dot{\theta}_i + \frac{K}{N}\sum h_{ji}(\theta_j - \theta_i)$$

Ref: Self-synchronization of coupled oscillators with hysteretic responses, TLO (1997)

First order phase transition resulting from finite inertia in coupled oscillator system, TLO (1997)

In mean field theory and by applying phase order parameter $re^{i\Theta} = \frac{1}{N}\sum_j e^{i\theta_j}$

$$m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N}\sum \sin(\theta_j - \theta_i) \to m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i - Kr\sin\theta_i$$

$\hookrightarrow$ damped driven pendulum/oscillator,

$\ddot{\phi} + \underbrace{2\beta\dot{\phi}}_{\text{damping term}} = \gamma\omega_0^2\cos\omega t - \omega_0^2\sin\phi$

For a single damped driven pendulum $\to$

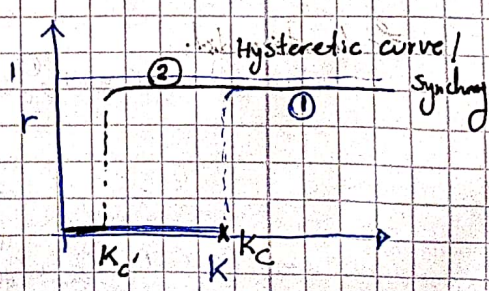
equilibrium state: $\Omega_0 = Kr$

* The pendulum continue to rotate for $\Omega > \Omega_0$ even if $\Omega < \Omega_0$ because of the inertia, the pendulum continues rotating till reach Ω_p and then because of damping factor $\langle\dot{\theta}\rangle$ goes to zero

Self-consistent equation for extended KM:

Case ①: The coupling strength K is increased from the incoherent $r \sim 0$ state

Case ②: The coupling K is decreased from the coherent state with $r \sim 1$



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Both cases include locked and drifting oscillators. In case ① Ω_p is the frequency of locked oscillators and drifting oscillators with $\Omega < \Omega_p$ will be entrained to Ω_p. In case ② locked oscillators become desynchronized when $\Omega > \Omega_D$

For both case ① and ② we consider first locked oscillators similar to what we did in KM.

$$r^{②}_{loc} = \int n_s(\theta)\, d\theta\, e^{i\theta} = \int_{|\Omega|<\Omega_D} g(\Omega)\, e^{i\theta}\, d\Omega = \int_{-\pi/2}^{\pi/2} \cos\theta\, g(kr\sin\theta)\, kr\cos\theta\, d\theta$$

$$= kr\int_{-\pi/2}^{\pi/2} \cos^2\theta\, g(kr\sin\theta)\, d\theta$$

$$r^{①}_{loc} = \int n_s(\theta)\, d\theta\, e^{i\theta} = \int_{|\Omega|<\Omega_p} e^{i\theta}\, g(kr\sin\theta)\, kr\cos\theta\, d\theta = kr\int_{-\theta_p = \sin^{-1}\left(\frac{\Omega_p}{kr}\right)}^{\theta_p} \cos^2\theta\, g(kr\sin\theta)\, d\theta$$

$$r^{①,②}_{drift} = \int n_D(\theta,\Omega)\, g(\Omega)\, d\theta\, d\Omega$$

$$n_D(\theta,\omega) = \frac{\tilde{T}^{-1}}{|\dot\theta|} = \frac{\tilde{\Omega}}{2\pi|\dot\theta|}$$; $\tilde{T}$ and $\tilde{\Omega}$ are respectively period and frequency of θ

$$\rightarrow r^{②,①}_{drift} = \int_{|\Omega|>\Omega_{p,D}} \int_0^{\tilde{T}} e^{i\theta(t,\Omega)} \frac{\tilde{\Omega}}{2\pi|\dot\theta|}\, g(\Omega)\, d\Omega\, d\theta\, \dot\theta\, dt$$

$\theta(t,-\Omega) = -\theta(t,\Omega)$ $\quad g(\Omega) = g(-\Omega)$

$$= \frac{1}{2\pi}\int_{|\Omega|>\Omega_{p,D}} \int_0^{\tilde{T}} e^{i\theta(t,\Omega)}\, |\tilde{\Omega}|\, g(\Omega)\, d\Omega\, dt$$

$\tilde{\Omega}(-\Omega) = -\tilde{\Omega}(\Omega)$

$$= \frac{1}{2\pi}\int_{\Omega>\Omega_{p,D}} \int_0^{\tilde{T}} \left[\cos\theta(t,\Omega) + \cos\theta(t,-\Omega)\right] |\tilde{\Omega}|\, g(\Omega)\, dt\, d\Omega$$

$$+ \frac{1}{2\pi}\cancel{\int\int \left[\sin\theta(t,\Omega) + \sin\theta(t,-\Omega)\right]} |\tilde{\Omega}|\, g(\Omega)\, dt\, d\Omega$$

$$= \frac{1}{\pi}\int_{\Omega>\Omega_{p,D}} \int_0^{\tilde{T}} \cos\theta(t,\Omega)\, |\tilde{\Omega}|\, g(\Omega)\, dt\, d\Omega$$

$$\Rightarrow r^{①,②} = r^{①,②}_{loc} + r^{①,②}_{drift}$$

for Lorentzian dist.

Case ①: $r = r^{①}_{loc} + r^{①}_{drift} = \frac{Kr}{m^2}\left(\frac{1}{\pi d^3}\ln\frac{\sqrt{d^2+\Omega_p^2}}{\Omega_p} - \frac{1}{2\pi d\,\Omega_p^2}\right) + \frac{2}{\pi kr}\left[\sqrt{d^2+k^2r^2}\,\tan^{-1}\left(\sqrt{d^2+k^2r^2}\,\tan\theta_p\right) - d\theta_p\right]$

Case ②: $r = r^{②}_{loc} + r^{②}_{drift} = \frac{Kr}{m^2}\left(\frac{1}{\pi d^3}\ln\frac{\sqrt{d^2+k^2r^2}}{kr} - \frac{1}{2\pi k^2r^2}\right) + \left(\sqrt{d^2+k^2r^2} - d\right)/kr$

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Kuramoto model on complex networks:

$$\dot{\theta}_i = \omega_i + \sum K_{ij} \sin(\theta_j - \theta_i) \quad ; \quad K_{ij} = \sigma_{ij}\, a_{ij}$$

coupling strength

connectivity matrix

① $\sigma_{ij} = \frac{K}{N}$ in complex network and for $N \to \infty$ means that coupling part will be zero except for nodes with degree $\sim N$. These nodes are available in networks with power-law degree distributions

② $\sigma_{ij} = K/k_i$, where k_i is the degree of node i. To solve the paradox of heterogeneity. this coupling strength can impose a dynamic homogenity that masks heterogenity

③ $\sigma_{ij} = K/\text{const}$ the coupling part could diverge in the thermodynamic limit $N \to \infty$

④ $\sigma_{ij} = K/k_{max}$ for comparison between networks one can scale coupling strength

network A $k_{max} = k_A$, network B $k_{max} = k_B$ $\Longrightarrow$

$$\sigma = \frac{K_A}{k_A} = \frac{K_B}{k_B}$$

Then:

$$\dot{\theta}_i = \omega_i + \sigma \sum_j a_{ij} \sin(\theta_j - \theta_i)$$

Onset of synchronization in complex network:

There are lots of paper regarding synchronization process in complex networ, and most work are numerical. Rare analytical asumption with strong assumption.

For example network structures on synchronization, such as clustering coefficient and average shortest path length.

order parameter for complex network:

local order parameter: $r_i e^{i\phi_i} = \sum_j a_{ij} \langle e^{i\theta_j} \rangle_t$; $\langle \dots \rangle_t$ = time average

$$\left(r_i e^{i\phi_i} = \sum_j a_{ij} \langle e^{i\theta_j} \rangle_t \pm \sum a_{ij} e^{i\theta_j} \right) \times e^{-i\theta_i}$$

global order parameter: $r = \frac{\sum_i r_i}{\sum k_i}$

$$r_i e^{i(\phi_i - \theta_i)} = \left\{ \sum a_{ij} \langle e^{i\theta_j} \rangle_t e^{-i\theta_i} - \sum a_{ij} e^{i(\theta_j - \theta_i)} \right\} + \sum a_{ij} e^{i(\theta_j - \theta_i)}$$

$$\dot{\theta}_i = \omega_i - \sigma r_i \sin(\theta_i - \phi_i) - \sigma h_i(t)$$

$$h_i(t) = \mathrm{Im}\left\{ e^{-i\theta_i} \sum_{j=1}^{N} a_{ij} \left(\langle e^{i\theta_j} \rangle_t - e^{i\theta_j} \right) \right\}$$

Imaginary: $r_i \sin(\phi_i - \theta_i) = \mathrm{Im}\{-\} + \sum a_{ij} \sin(\theta_j - \theta_i)$

$$\sum a_{ij} \sin(\theta_j - \theta_i) = r_i \sin(\phi_i - \theta_i) - \mathrm{Im}\{-\}$$

In the case that $h_i(t) \ll r_i$ then:

$$\dot{\theta}_i = \omega_i - \sigma r_i \sin(\theta_i - \phi_i) \longrightarrow \text{Stationary solution} \quad \dot{\theta}_i = 0, \quad \omega_i = \sigma r_i \sin(\theta_i - \phi_i)$$

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Similar to KM for all-to-all network, we have contribution of locked and drifting oscillators.

$$r_i = \sum_{j=1}^{N} a_{ij} \langle e^{i(\theta_j - \phi_i)} \rangle_t = \sum_{|\omega_j| < \sigma r_j} a_{ij} e^{i(\theta_j - \phi_i)} + \sum_{|\omega_j| > \sigma r_j} a_{ij} \langle e^{i(\theta_j - \phi_i)} \rangle_t$$

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