Dynamics on Networks

22. November 2021

Newman treats processes on networks in chapters 16-19, the dynamical systems we will look at are chapter 19. Random walks are in 6.14, with an easy review of the core results. Besides this the main reference for this part of the course is [1] which is also available at `https://arxiv.org/abs/1612.03281` and gives an excellent if terse overview of random walks on networks.

Important: In the stochastic system literature transition matrices are defined differently from these notes. We have T_{ij} denoting the transition from j to i, the random walk literature usually has it the other way around.

1 Discrete time random walk

Take a network with adjacency matrix A_{ij}, and degrees $d_i = \sum_j A_{ij}$ and $D_{ij} = \delta_{ij} d_i$. a random walker has transition matrix given by

$$T_{i \leftarrow j} = \frac{A_{ij}}{d_j} = A \cdot D^{-1} \tag{1}$$

The stationary distribution of this random walker is defined by:

$$p_i^* = \sum_j T_{i \leftarrow j} p_j^* = \sum_j \frac{A_{ij} p_j^*}{d_j} \tag{2}$$

Ansatz:

$$p_i^* = c d_i \tag{3}$$
$$c d_i = \sum_j \frac{A_{ij} c d_j}{d_j} \tag{4}$$
$$c d_i = c \sum_j A_{ij} \tag{5}$$

To understand the dynamics we need the spectrum and basis of $A \cdot D^{-1}$. First an observation:

$$(1, 1, 1, 1, \dots)^T \cdot A = (d_1, d_2, d_3, d_4, \dots)^T \tag{6}$$
$$(1, 1, 1, 1, \dots)^T \cdot A \cdot D^{-1} = (1, 1, 1, 1, \dots)^T \tag{7}$$

So we have an eigenvalue 1 with right eigenvector p^* and left eigenvector $(1, 1, 1, 1, \dots)^T$.

If the graph is strongly connected the Perron Frobenius theorem gives us a lot of information about A, in particular we know that this is a simple eigenvalue and all others have equal or smaller modulus.

Supplemental reading: Perron–Frobenius theorem for irreducible non-negative matrices. Wikipedia article is a very good summary

Note that:

$$\tilde{A} = D^{-1/2} \cdot A \cdot D^{-1/2} \tag{8}$$
$$T = D^{1/2} \cdot \tilde{A} \cdot D^{-1/2} \tag{9}$$

Thus T and $\tilde{A}$ are similar, and have the same spectrum (and easily related eigenvectors), the eigenvectors we already found are transformed into:

$$\lambda^1 = 1 \tag{10}$$
$$v^1 = (d_1^{0.5}, d_2^{0.5}, d_3^{0.5}, d_4^{0.5}, \dots)^T \tag{11}$$

The matrix $\tilde{A}$ is symmetric and thus has real eigenvalues which we order $\lambda^i > \lambda^{i+1}$.

The slowest process to die off is that given by λ^2, the Cheeger constant / conductance tells us something here:

Consider a graph partitions S, $\overline{S}$ with $Vol(S) < Vol(\overline{S})$ where

$$Vol(S) = \sum_{i \in S} d_i \tag{12}$$

is the sum of d_i in S, call $b(S)$ the number of links from S to $\overline{S}$, then

$$h = \min_S \frac{b(S)}{|S|} \tag{13}$$

then we have:

$$\frac{h^2}{2} < 1 - |\lambda^2| < 2h \tag{14}$$

Proof is omitted here (but I might give a partial one next time...). This material is discussed in [1] 3.2.3, which gives many nice references.

Supplemental reading: We can design dynamical processes to understand properties of the graph. The most famous one is PageRank, section 5.2.1 in [1].

2 General dynamics on a network

We need to say what we mean by a dynamical system on a network, one possibility is this:

$$\dot{x}_i = f(x_i) + \sum_j A_{ij} g(x_i - x_j) \tag{15}$$

Assume we have a fix point x^*, then we can look at the linear behaviour of the system:

$$
\begin{aligned}
x_i(t) &= x_i^* + \epsilon z_i(t) && (16)\\
x_{ij}^* &= x_i^* - x_j^* && (17)\\
\dot{x}_i^* &= 0 = f(x_i^*) + \sum_j A_{ij} g(x_{ij}^*) && (18)\\
\epsilon \dot{z}_i &= f(x_i^*) + \epsilon f'(x_i^*) z_i + \sum_j A_{ij} g(x_{ij}^*) + \epsilon \sum_j A_{ij} g'(x_{ij}^*)(z_i - z_j) + O(\epsilon^2) && (19)\\
\dot{z}_i &= f'(x_i^*) z_i + \sum_j A_{ij} g'(x_{ij}^*)(z_i - z_j) && (20)\\
& && (21)
\end{aligned}
$$

Consider the weighted adjacency matrix $A_{ij}^w = A_{ij} g'(x_{ij}^*)$ and

Literatur

[1] Naoki Masuda, Mason A Porter und Renaud Lambiotte. „Random walks and diffusion on networks“. In: *Physics reports* 716 (2017), S. 1–58.