MAT1856/APM466 Assignment 1

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February 3, 2020

Fundamental Questions - 25 points

1.

  (a) Government bonds can assist in (1)funding deficits in the federal budget, (2)raising capital for various projects such as infrastructure spending and (3)controlling the nation's money supply by the Central Bank.

  (b) 1) When deciding fiscal policies, government need to consider the market expectation on the future interest rate changes and economic activity, which can be shown by the slope (trend) of the yield curve. 2) When deciding the monetary policies by issuing government bonds, the yield of government bonds are viewed as the benchmark for the financial market, thus matter.

  (c) The government can reduce the money supply by asking the Central Bank to sell the government bonds.

2. I select 10 bonds as follows:

ISIN	Coupon	Maturity date
D929	1.50%	3/1/2020
E596	0.75%	9/1/2020
F254	0.75%	3/1/2021
F585	0.75%	9/1/2021
G328	0.50%	3/1/2022
*ZU15	2.75%	6/1/2022
H490	1.75%	3/1/2023
*A610	1.50%	6/1/2023
J546	2.25%	3/1/2024
J967	1.50%	9/1/2024
→ *K528*	*1.25%*	*3/1/2025*

I choose them because

- *Good timing*: They are the most evenly distributed bonds in maturity among the 32 bonds within 2020 Jan to 2025 Jan. (8 of them exactly mature on March/September form) And none of them is out of the 5 year range. (Closest one is 3/1/2020, Farthest one is 9/1/20204)
- *Similar coupon rate*: All of them have the similar small coupon rate.

Remark:

- Since we do not have data point for 9/1/2022 and 9/1/2023, I use the bonds(*) with small coupon rate which mature at 6/1/2022 and 6/1/2024 to substitute them.

- Since we want graphs from 202x-2025/1/1, I would use the extra 11^{th} bond(→) to estimate the point at 2025/1/1.

3. Let $X = (X_1, X_2, \ldots)^T$ be the stochastic processes. Then the i^{th} largest eigenvalue λ_i is the maximum of the $Var(X \cdot \vec{u}_i)$, where u_i is a normalized vector, and the corresponding normalized eigenvector $\vec{v}_i$ is the vector to maximize the $Var(X \cdot \vec{u}_i$.

Empirical Questions - 75 points

4. Before answering the questions, I made few simple assumptions about the bonds for calculation convenience.

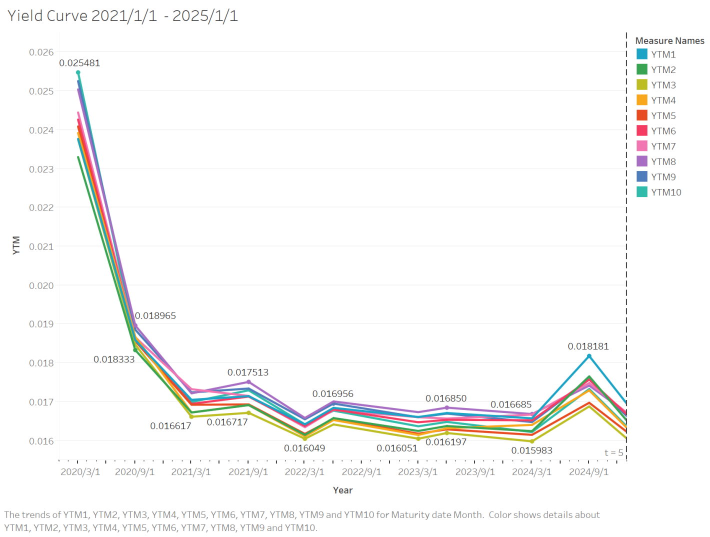
   - I defined that 1 year = 360 days, 1 month = 30 days.
   - I defined that 2020/1/1 as the t = 0 in the graph, and set a today vector as

   $$today = (2, 3, 6, 7, 8, 9, 10, 13, 14, 15).$$

   In practice, the graph (e.g. YTM_t) at day t can be obtained by shift my coordinates to the right by $today_t$ unit.

   - Data set and values are scrapped and calculated via myself at https://github.com/PINGXIANG/APM466_A1 (Tianhao, 2020).

   (a) The Yield curve over 5 years is as follows.



   Step 1: Calculate DP. Since the dollar amount dictated by the market through the bid ask process is *Clean price*(DP_i), I first calculate the *Accrued interest*(AI_i) and compute the *Dirty price*(DP_i) for each bond i as

   $$DP_i = AI_i + CP_i = \frac{4 \times 30 + (today_t - 1)}{360} \cdot C_i + CP_i \quad w.l. \quad C_i = Coupon_i 100\%.$$

   The AI_6 and AI_8 are modified as $\frac{30+(today_t-1)}{360} \cdot C_i$, for each day t.

   Step 2: Calculate YTM.

- Define the cash flow for each bond i at day t noted as

$$cash\_flow = c(-PV, C, ..., FV)$$

.

- Define the payment time for each bond i at day t as

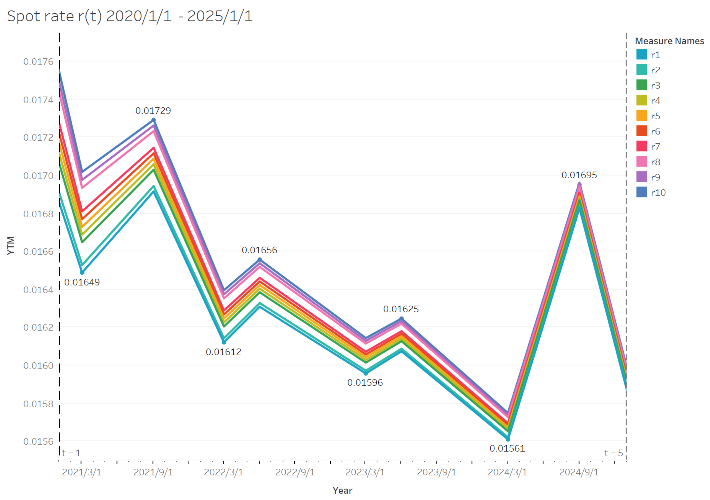
$$pmt\_time^{1} = c(0, t_0, t_0 + \frac{1}{2}, ..., t_0 + \frac{n-1}{2})$$

.

- By R-Package "jrvFinance", calculate the YTM by plugging the $cash\_flow$ and pmt_time into the $irr()$ method.

Need to note that the $cash\_flow$ and pmt_time for the bond 6 and bond 8 have been separately modified. Formulas can be found in my code.

(b) The Spot rate curve for 1-5 years is as follows.



The trends of r1, r2, r3, r4, r5, r6, r7, r8, r9 and r10 for Maturity date Month. Color shows details about r1, r2, r3, r4, r5, r6, r7, r8, r9 and r10.

Calculate Short rate r.

- Since the bond -D929 maturing in less than six months have no intermediate coupon, I can use the definition of yield to calculate in a straightforward manner the yield curve for

$$r_1(t_0) = -\frac{\ln(DP_1/FV_1)}{t_0} \quad w.l. \quad FV_1 = 100(1 + \frac{1}{2}C_1).$$

- Do bootstrapping for the first 5 bonds. That is, given the series of yield $\{r_1, r_2, \cdots, r_{i-1}\}$ calculated, the $(i)^{th}$ yield r_i can be calculated via solving the following equation

$$DP_i = \frac{1}{2}C_i \cdot e^{-r_1 \cdot t} + \sum_{j=2}^{i-1} \frac{1}{2}C_i \cdot e^{-r_j \cdot (\frac{j-1}{2} + t_0)} + FV_i \cdot e^{-r_i \cdot (\frac{i-1}{2} + t_0)}.$$

- Estimate the r_6 for bond 6 at day t via linear interpolation(Will, 2018). That is, estimate the new_r for July/December by taking the mid-point from the r calculated above, and then calculate the r_6 by the similar calculation.

[1] For expression convenience, $t_0 = \frac{60-(today_t - 1)}{360}$, which is the time to wait for the first payment at day t in the unit of year.

- Estimate the short rate r_6^* for $t = 2022/9/1$ via linear extrapolation (Will, 2018). That is,

$$\widehat{r}(t^*) = r(t_1) + \frac{r(t_1) - r(r_2)}{t_1 - t_2} \cdot (t_1 - t_2)$$

where t_1 and t_2 are the closest and the second closet data point to t^*.
- Given the estimated r_6^*, short rate for bond 7 can be calculated via process 2. Similar estimation for bond 8 can be done as well. Then, the short rate for bond 9 and 10 can be calculated by the 1^{st} and 2^{nd} process.

(c) *Calculate the one year forward rate f.*

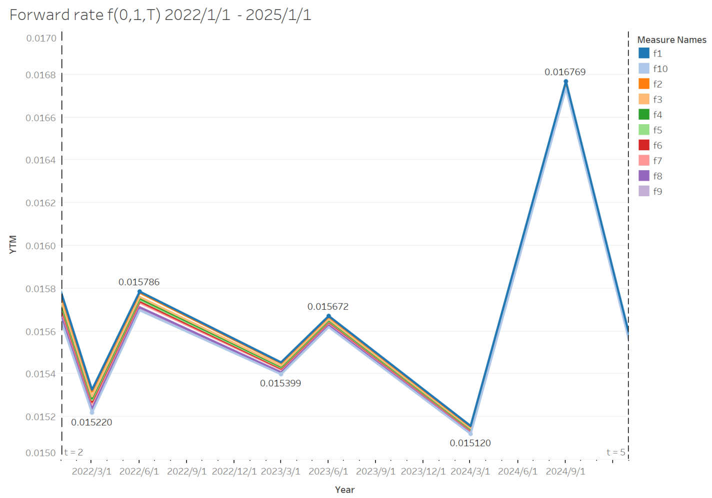
- To derive the one year forward curve, I would use the formula

$$e^{f \cdot (T_i - T_1)} = \frac{e^{r_i \cdot (T_i - t_0)}}{e^{r_1 \cdot (T1 - t_0)}},$$

where $T_1 = 1$ (year) and $T_2 \in [2, 5]$ is the terms, and r_1 is the estimated 1 year spot rate while r_2 is the spot rate for term T_2. Thus, we have

$$f(t = 0, 1, T_2) = \frac{r_i \cdot (T_i - t_0) - r_1 \cdot (1 - t_0)}{T_i - 1}.$$

- After obtain one year forward points from the formula via the spot rate I obtain (the points with March/September style, but not the estimated July/December style), I can get estimate the 1yr-1yr, 1yr-2yr, 1yr-3yr, 1yr-4yr forward rate via interpolation, or via the graph.
The 1 year forward curve for 2-5 years is as follows.



The trends of f1, f10, f2, f3, f4, f5, f6, f7, f8 and f9 for Maturity date Month. Color shows details about f1, f10, f2, f3, f4, f5, f6, f7, f8 and f9.

Remark: Need to note that my 1-yr forward curve looks more converge than my peers. One potential explanation is that

- Given the same data set, the slight difference in the calculation of *Dirty Price*, i.e., my assumption that 1year = 360 days and 1 month = 30 days, will causes the slight difference in the calculation of *Spot rate*.
- The slight difference in the *Spot rate* will be amplified in the Bootstrapping step by step.
- Given the estimation of bond 6 and bond 8, the *Spot rate* will be amplified again.

- Given the compounding interest rate e^{rT}, any slight difference would be amplified by the ln(∗) function since our results are closing to 0 and thus $\Delta \ln x$ is large.

This has been shown after I and one of my peer checked our result. We test each other's result by step-by-step debugging. However, after the amplification, we have a really different result in terms of the AM and convergence of the 1-yr forward curve, which causes significant difference when calculating the covariance matrix for forward return (**Since my 1-yr forward curve converge, the variance of each time series become super small**).

5. The covariance matrix for the daily log-returns of yield is

	X_1	X_2	X_3	X_4	X_5
X_1	1.453863e-04	4.008501e-05	7.492042e-05	6.851507e-05	4.062467e-05
X_2	4.008501e-05	1.660256e-04	5.334003e-05	1.126977e-04	1.127724e-04
X_3	7.492042e-05	5.334003e-05	9.820674e-05	1.155942e-04	1.137111e-04
X_4	6.851507e-05	1.126977e-04	1.155942e-04	1.697478e-04	1.926809e-04
X_5	4.062467e-05	1.127724e-04	1.137111e-04	1.926809e-04	3.185210e-04

Table 1: log-returns of yield.

The covariance matrix for the daily log-returns of 1-yr forward rate is

	X_1	X_2	X_3	X_4
X_1	3.822845e-07	1.801201e-07	1.159496e-07	8.166261e-08
X_2	1.801201e-07	8.497714e-08	5.468584e-08	3.849423e-08
X_3	1.159496e-07	5.468584e-08	3.519500e-08	2.477755e-08
X_4	8.166261e-08	3.849423e-08	2.477755e-08	1.744748e-08

Table 2: log-returns 1-yr forward rate

6. The eigenvalues and eigenvecotrs for the log-returns of yield covariance matrix are

λ	6.028690e-04	1.532101e-04	1.048382e-04	3.445196e-05	2.518134e-06
$\vec{v}$	$\begin{bmatrix}-0.2232\\-0.3653\\-0.3386\\-0.5148\\-0.6610\end{bmatrix}$	$\begin{bmatrix}0.8256\\0.0376\\0.2768\\0.0580\\-0.4867\end{bmatrix}$	$\begin{bmatrix}0.1682\\-0.8989\\0.2199\\-0.0152\\0.3391\end{bmatrix}$	$\begin{bmatrix}0.4789\\0.1247\\-0.5691\\-0.4871\\0.4402\end{bmatrix}$	$\begin{bmatrix}-0.1038\\0.2037\\0.6605\\-0.7028\\0.1315\end{bmatrix}$

Table 3: eigen of yield rate

The eigenvalues and eigenvecotrs for the log-return of forward covariance matrix are

Given the corresponding normalized eigen vector $\vec{v}$, the largest eigenvalue λ_i give the maximum $Var(X \cdot \vec{v}_i)$, where $X = (X_1, X_2, ...)^T$ is the time series process.

λ	5.197989e-07	1.049994e-10	2.261770e-13	2.369024e-18
$\vec{v}$	$\begin{bmatrix} 0.8575 \\ 0.4041 \\ 0.2601 \\ 0.1832 \end{bmatrix}$	$\begin{bmatrix} 0.4931 \\ -0.7928 \\ -0.354 \\ -0.057 \end{bmatrix}$	$\begin{bmatrix} 0.1433 \\ 0.3654 \\ -0.4939 \\ -0.7758 \end{bmatrix}$	$\begin{bmatrix} -0.0293 \\ 0.2730 \\ -0.7505 \\ 0.6010 \end{bmatrix}$

Table 4: eigen of 1-yr forward rate

References

Tianhao, Wang (2020), "Apm466 spider for assignment 1." https://github.com/PINGXIANG/APM466_A1.

Will, Kenton (2018), "Interpolation." https://www.investopedia.com/terms/i/interpolation.asp.